RESOLUTIONS AND TOR ALGEBRA STRUCTURES FOR TRIVARIATE MONOMIAL IDEALS

by

JARED LAFAYETTE PAINTER

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ABSTRACT

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JARED LAFAYETTE PAINTER, Ph.D.

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Supervising Professor: David A. Jorgensen

In this manuscript we explore properties of minimal free resolutions and their relationship to the Tor-algebra structure for trivariate monomial ideals.

We begin with an in-depth analysis of minimal free resolutions of $S = R/I$, where $R = \Bbbk[x, y, z]$ is a polynomial ring over a field $\Bbbk$, and I is a monomial ideal that is primary to the homogeneous maximal ideal $\mathfrak{m}$ of R. We will define a special form of the minimal free resolution of S, and then determine when we get nonzero elements from I as entries in the matrices of the resolution. We find a complete answer to this question for the second matrix of our special resolution for all trivariate monomial ideals. For the third matrix, we provide a complete answer for generic monomial ideals. We also observe differences for resolutions of generic monomial ideals in comparison to non-generic monomial ideals.

We will find that our results on free resolutions relate to the Tor-algebra structure for S. In [4] Avramov describes the Tor-algebra structure $A = \operatorname{Tor}^R(\Bbbk, S)$, for rings of codepth 3. His description of this structure is comprised of 5 categories. We will explore this structure, and will determine which of the 5 categories can be real-

ized by monomial ideals. We will also learn how to describe the Tor-algebra structure from the minimal free resolution of S. Finally, we will find classes of monomial ideals with the desired Tor-algebra structure, and give a complete classification for generic monomial ideals.

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CHAPTER 1

INTRODUCTION

The study of free resolutions is one of the most highly studied topics in the field of commutative algebra. In [16] Eisenbud makes a most elegant interpretation of the information obtained from free resolutions saying, "A free resolution may be thought of as the result of fully solving a system of linear equations with polynomial coefficients." There is currently a lot known about free resolutions of modules over polynomial rings $R = \Bbbk[x_1, \ldots, x_r]$ over a field $\Bbbk$. One of the most famous results is Hilbert's syzygy theorem, which states that every finitely generated module over R, has a minimal free resolution of length at most r. Hilbert initiated the field of commutative algebra and the study of free resolutions.

A free finitely generated R-module F is isomorphic to R^n, where n is a positive integer, and a free resolution of an R-module M, is an exact sequence of free R-modules. One may think of the differentials, or maps, in a free resolution of M as matrices. This notion is analogous to linear transformations between vector spaces. Exact information about the matrices of free resolutions is known in only a few cases, and in general free resolutions are hard to compute. In projective dimension 2 a precise description of the matrices in the resolution is given by the Hilbert-Burch theorem, see [10]. Buchsbaum and Eisenbud give some descriptions of these for special cases in projective dimension 3 [11, 12]. Also A. Brown describes these matrices for a special class of ideals in [9]. There are also nice ways to represent minimal free resolutions of rings obtained from quotients of monomial ideals, see [8, 19–21].

When studying free resolutions we commonly like to explore the simplest examples first. It seems only natural to begin our studies of these resolutions by exploring resolutions of $S = R/I$, where I is a monomial ideal. Macaulay showed that resolutions of monomial ideals can provide us with some information for resolutions of finitely generated R-modules in general. His theorem states that the numerical data obtained from free resolutions of monomial ideals matches that of resolutions of ideals in general, e.g. the ranks of the free modules in both resolutions can be the same.

Resolutions of monomial ideals have been studied significantly over the past 15 years. In [8] it is shown how resolutions of monomial ideals can be represented by simplicial complexes, and how resolutions of generic monomial ideals are represented by the Scarf complex. It is also shown in [19–21], that if $R = \Bbbk[x, y, z]$ is a polynomial ring in three variables (a trivariate polynomial ring) over a field $\Bbbk$ and I is a monomial ideal of R, then we can represent the minimal free resolution of S by some planar graph. We can use these methods to describe minimal free resolutions of rings given by monomial ideals in great detail. While these results give us nice ways to think about resolutions of monomial ideals, they do not give us specific information about the entries of the matrices for these resolutions in general.

In Chapter 3 we will give a precise description of the entries of these matrices using a special form of the free resolution for monomial ideals. In general all minimal free resolutions are isomorphic and thus the ranks of the free modules involved are uniquely determined. However, the entries in the matrices can change somewhat without changing the minimality of the resolution. To ensure there is no confusion in what our minimal free resolutions look like, we will use a unique form of the resolution, defined in Definition 2.2.6. We are primarily interested in when we can get nonzero elements of I as entries in the matrices of our minimal free resolution. We would also like to know what is the maximum number of such entries we can have in the resolution. We find a complete answer to this question for the second matrix of the resolution in Theorem 3.2.1. For generic monomial ideals we will answer this question for all matrices in the resolution, Theorem 3.3.11. In Section 3.4 we will even learn how to tell whether or not I is generic by looking at our special free resolution of S.

We will also study the Tor-algebra structure for rings given by quotients of monomial ideals. Avramov gives a classification of the Tor-algebra structure for rings of codepth 3, which is comprised of 5 categories. We want to determine which of these 5 categories can be realized by monomial ideals, and we would like to find classes of monomial ideals with the desired Tor-algebra structure. One of the ways we can describe the Tor-algebra structure is through the Bass series. The Bass numbers of a ring were first introduced by Hyman Bass in [7], though they were not referred to as "Bass numbers" until a later date. These have been studied significantly over the last fifty years. In particular the growth of the Bass numbers has been studied recently, see [4–6, 13, 18]. Avramov's goal in [4] was to determine if the Bass sequence for commutative local rings of codepth 3 is always increasing. He gives a complete answer to this question by finding closed form expressions for the Bass series of S using the Poincaré series. These expressions are uniquely related to the Tor-algebra classification for S and the minimal free resolution of S. Because of this we can use the fact that when S is zero dimensional the Bass numbers of S are equal to the Betti numbers of the canonical module of S, to obtain some information about the Tor-algebra structure for S.

In Chapter 4 we will classify the Tor-algebra structure for monomial ideals, based on Avramov's results in [4]. We do this by relating the Tor-algebra structure to results we obtain in Chapter 3 for our unique free resolution of S. Specifically we prove that we can relate some of the invariants for the the Tor-algebra to the number of nonzero entries from I we have in the matrices of the free resolution of S, in Theorem 4.1.5 and Proposition 4.1.8. In most cases we find we can give a complete description of the Tor-algebra structure for S by simply looking at our unique minimal free resolution for S. Using this we will find several classes of monomial ideals with the desired Tor-algebra structure. In Theorem 4.3.2 we will give a complete classification for generic monomial ideals. We will also find a new class of examples in one of our categories for the Tor-algebra, in Theorem 4.3.4. Previously the only examples known in this class were the examples given by A. Brown in [9], and were constructed so that the rank of the last free module in the minimal free resolution was 2. Our class of examples will consist of monomial ideals generated by $\rho+3$ monomials for any $\rho \geq 2$, so that the rank of the last free module in the minimal free resolution is $\rho \geq 2$.

Chapter 2 will consist of the necessary information needed to describe our results. We begin with an introduction to free resolutions, and will define our special form of these resolutions in Definition 2.2.6. We give a description of minimal and non-minimal second syzygies for S in Lemma 2.2.7 and Remark 2.2.8. Then, in Section 2.3 we describe Tor and how it has an induced graded $\Bbbk$-algebra structure from the algebra structure on the Kosuzl complex of a minimal set of generators for the maximal ideal of S. We also introduce general information about the Tor-algebra provided by Avramov in [4]. In Section 2.3.4 we will learn how we can relate the lower Bass numbers of S to the number of rows in the last matrix of our special free resolution which have only entries from I. We then give the closed form expressions for the Poincaré and Bass series for S, from [4].

CHAPTER 2

PRELIMINARY CONCEPTS

2.1 Assumptions and Notation

For the duration of this paper we will assume $R = \Bbbk[x, y, z]$ is a trivariate polynomial ring over a field $\Bbbk$, and that I is a monomial ideal that is primary to the homogeneous maximal ideal $\mathfrak{m} = \langle x, y, z\rangle$. We will also assume that $I \subseteq \mathfrak{m}^2$ throughout. A monomial ideal is simply an ideal which is minimally generated by monomials $m_1, \ldots, m_n$. The assumption that I is $\mathfrak{m}$-primary means that I is minimally generated by at least three monomials, such that three of the minimal generators are x^a, y^b, and z^c with $a, b, c > 0$. We will let $S = R/I$ which is in fact local and Artinian. We will also assume that S is not Gorenstein. From the previous assumptions it is known that S is Gorenstein if and only if $I = \langle x^a, y^b, z^c\rangle$ with $a, b, c > 0$, so we will add the condition that I is minimally generated by at least four monomials. We note that some of our results will hold in more generality, but we will assume the above hold unless stated otherwise.

We will denote the least common multiple of monomials $m_1, \ldots, m_r$ by $m_{1\ldots r}$. In particular, $m_{ij} = [m_i, m_j]$ denotes the least common multiple of m_i and m_j. The greatest common divisor of monomials m_i and m_j will be denoted by (m_i, m_j). Throughout this paper the monomial m_i will be represented by $x^{a_i}y^{b_i}z^{c_i}$. We will also say that a monomial m' *strongly divides* a monomial m, denoted $m' \| m$, if m' divides m/x_i for all variables x_i dividing m. If a monomial m' *strictly divides* a monomial m we will write $m'|_< m$. At times it may be more efficient to represent the variables of the polynomial ring generally with x_i, $1 \leq i \leq 3$. For this reason we will assume that the variables x, y, and z are interchangeable with x_1, x_2, and x_3 respectively.

Many of our results will rely on when the least common multiples between one or more monomials generating I are divided by other generators of I or other least common multiples of generators of I. The following lemma is a simple fact, which we will use frequently.

Lemma 2.1.1. *Let m_i, m_j, and m_k be distinct minimal generators of I, then*

1. *$m_k | m_{ij}$ if and only if both m_{ik} and m_{jk} divide m_{ij}*
2. *$m_{ik} | m_{ij}$ if and only if $m_{jk} | m_{ij}$*
3. *if $m_k \| m_{ij}$ then $m_{ik} |_< m_{ij}$ and $m_{jk} |_< m_{ij}$.*

Proof: To prove this we only need consider the exponents on one of the variables in these monomials. It is easy to see we can extend our argument to the rest of the variables.

(1): If $m_k | m_{ij}$ then $a_k \leq \max\{a_i, a_j\}$, which implies that $\max\{a_i, a_k\}$ and $\max\{a_j, a_k\}$ are both less than or equal to $\max\{a_i, a_j\}$. Thus both m_{ik} and m_{jk} divide m_{ij}. The reverse direction is similar.

(2): If $m_{ik} | m_{ij}$ then $\max\{a_i, a_k\} \leq \max\{a_i, a_j\}$, which implies that $\max\{a_j, a_k\} \leq \max\{a_i, a_j\}$, thus $m_{jk} | m_{ij}$. The reverse direction is the same.

(3): Since m_i and m_j are both minimal generators of I, then without loss of generality we may assume that $a_i > a_j$ and $b_j > b_i$. By definition, if $m_k \| m_{ij}$, then $a_k < \max\{a_i, a_j\} = a_i$ and $b_k < \max\{b_i, b_j\} = b_j$. This implies that $\max\{a_j, a_k\} < \max\{a_i, a_j\}$ and $\max\{b_i, b_k\} < \max\{b_i, b_j\}$. Thus $m_{ik} |_< m_{ij}$ and $m_{jk} |_< m_{ij}$. $\square$

2.2 Free Resolutions

In Chapter 3 we will be discussing minimal free resolutions of S. In our case it is known that the projective dimension of S is 3. Much is known about minimal free resolutions of trivariate monomial ideals, see [8, 19–21]. Here we will discuss some of what is known about these resolutions and in Chapter 3 we will give more precise descriptions of the behavior of these resolutions. We will begin by defining free resolutions in general and refer the reader to [15, 16] for further information on free resolutions.

2.2.1 Computing Resolutions

Definition 2.2.1. A free resolution of an R-module M is a complex of free R-modules,

$$\mathbb{F} : \cdots \longrightarrow F_n \xrightarrow{f_n} \cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0$$

such that $\operatorname{coker}(f_1) = M$ and $\mathbb{F}$ is exact. We will abuse notation slightly and say that,

$$\mathbb{F} : \cdots \longrightarrow F_n \xrightarrow{f_n} \cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \xrightarrow{f_{-1}} 0$$

is a free resolution M and $\ker(f_i) = \operatorname{im}(f_{i+1})$ for $i \geq -1$. The image of the map f_i is called the ith syzygy module of M.

It is not difficult to see that every R-module has a free resolution. To construct a free resolution of an R-module M, we begin by finding a set of generators for M. If M has n_0 generators, we define the map $f_0 : F_0 \longrightarrow M$, where F_0 is a free R-module of rank n_0, such that f_0 sends the generators of R^{n_0} to the generators of M. Next we find a generating set for $M_1 = \ker(f_0)$. If M_1 is generated by n_1 elements in R^{n_0}, we define the map $\phi_1 : F_1 \longrightarrow M_1$, where F_1 is a free R-module of rank n_1, so that ϕ_1 sends the generators of F_1 to the generators of M_1. We then define the map $\pi_1 : M_1 \longrightarrow R^{n_0}$ to be the natural projection of M_1 into R^{n_0}, and set $f_1 = \pi_1\phi_1 : R^{n_1} \longrightarrow R^{n_0}$. Since $\operatorname{im}(\pi_1) = M_1$ and ϕ_1 is a surjection we have that $\operatorname{im}(f_1) = M_1$ also, which ensures that we are building an exact sequence. Repeating this procedure we obtain a free resolution for M over R. If for some $r < \infty$ we have that $F_{r+1} = 0$, but $F_i \neq 0$ for $0 \leq i \leq r$, then we say $\mathbb{F}$ is a finite free resolution of length r.

One of the most famous results in commutative algebra is the Hilbert Syzygy Theorem, which states that if $R = \Bbbk[x_1, \ldots, x_r]$, then every finitely generated R-module has a finite free resolution of length $\leq r$. When R is a polynomial ring all projective modules are also free by Serre's Theorem. Thus we define the *projective dimension* of an R-module M, denoted $\mathrm{pd}(M)$, to be the minimum of the lengths of all free resolutions for M over R. When M is a finitely generated R-module, we have that $\mathrm{pd}(M) \leq r$ by the Hilbert Syzygy Theorem. As previously stated, we will think of the maps f_i as matrices with respect to the standard bases of the free modules R^{n_i}.

Example 2.2.2. Let $R = \Bbbk[x, y]$ and $I = \langle x^2, xy, y^2 \rangle$. We will compute a free resolution for $S = R/I$, adopting the notation that the ith syzygy module of S is Z_i, with minimal generating set S_i. Since $\ker(S \longrightarrow 0) = S$, we need $\mathrm{im}(f_0) = S$, which is generated by $\bar{1}$. So we choose the free module $F_0 = R$ and let f_0 be the natural projection of R onto S such that $1 \mapsto \bar{1}$. Now set $Z_1 = \ker(f_0) = I$, which is generated by $\langle x^2, xy, y^2 \rangle$. Since I is generated by three monomials we choose $F_1 = R^3$ and the map $\phi_1 : R^3 \longrightarrow Z_1$ by sending $e_1 \mapsto x^2$, $e_2 \mapsto xy$, and $e_3 \mapsto y^2$. If we let $f_1 = \pi_1 \phi_1$, where π_1 is the natural projection from $Z_1 = I$ into R, we get that $f_1 = \begin{bmatrix} x^2 & xy & y^2 \end{bmatrix}$. This gives us the following diagram,

$$\begin{tikzcd}
R^3 \arrow[rr, "f_1 = \begin{bmatrix} x^2 & xy & y^2 \end{bmatrix}"] \arrow[dr, "\phi_1"'] & & R \arrow[r, "f_0"] & S \arrow[r] & 0 \\
& Z_1 \arrow[ur, "\pi_1"'] & & &
\end{tikzcd}$$

The $\ker(f_1)$ is generated by elements of R^3. We can find generators for $\ker(f_1)$ by finding the syzygy-pairs between the generators of $Z_1 = I$, see [20]. This gives,

$$Z_2 = \ker(f_1) \text{ is generated by } \{-ye_1 + xe_2, -y^2 e_1 + x^2 e_3, -ye_2 + xe_3\},$$

where e_1, e_2, and e_3 are the standard basis elements of R^3. This generates Z_2, but is not a minimal generating set. The second generator in the list can be obtained by taking a linear combination of the first and the third with the computation,

$$-y^2e_1 + x^2e_3 = y(-ye_1 + xe_2) + x(-ye_2 + xe_3).$$

Since both $-ye_1+xe_2$ and $-ye_2+xe_3$ are clearly algebraically independent the minimal generating set for Z_2 is $S_2 = \{-ye_1+xe_2, -ye_2+xe_3\}$. Since Z_2 is minimally generated by 2 elements so we choose $F_2 = R^2$ and define the map $\phi_2 : R^2 \longrightarrow Z_2$, by mapping $e_1 \mapsto -ye_1 + xe_2$ and $e_2 \mapsto -ye_2 + xe_3$. If we let $f_2 = \pi_2\phi_2$, where π_2 is the natural projection from Z_2 to R^3, then we have that $\text{im}(f_2) = Z_2$. This gives the following diagram,

$$\begin{tikzcd}
R^2 \arrow[rr, "f_2=\begin{bmatrix} -y & 0 \\ x & -y \\ 0 & x \end{bmatrix}"] \arrow[dr, "\phi_2"'] & & R^3 \arrow[rr, "f_1=\begin{bmatrix} x^2 & xy & y^2 \end{bmatrix}"] \arrow[dr, "\phi_1"'] & & R \arrow[r, "f_0"] & S \arrow[r] & 0 \\
& Z_2 \arrow[ur, "\pi_2"'] & & Z_1 \arrow[ur, "\pi_1"'] & & &
\end{tikzcd}$$

Notice that $Z_3 = \ker(f_2) = 0$ which implies that Z_3 is free and our resolution terminates leaving us with,

$$0 \longrightarrow R^2 \xrightarrow{\begin{bmatrix} -y & 0 \\ x & -y \\ 0 & x \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} x^2 & xy & y^2 \end{bmatrix}} R \longrightarrow S \longrightarrow 0$$

In fact the resolution we have constructed is a minimal free resolution of S.

Definition 2.2.3. Using the notation adopted in the previous example a free resolution of S is said to be *minimal* if each of the generating sets S_i for the ith syzygy module Z_i is a minimal generating set. This implies that each generator of Z_i must be contained in $Z_i - \mathfrak{m}Z_i$, by Nakayama's lemma [15, 4.8].

Remark 2.2.4. In Example 2.2.2 R was a polynomial ring in two variables and we expected $\mathrm{pd}(S) \leq 2$. In general an ideal I in $R = \Bbbk[x_1, \ldots, x_r]$ is $\mathfrak{m}$-primary if $x_1^{\alpha_1}, \ldots, x_r^{\alpha_r}$ are all minimal generators of I with $\alpha_i > 0$ for all i. In Example 2.2.2 we can see that I is $\mathfrak{m}$-primary, and in fact we also have that $\mathrm{pd}(S) = 2$. It turns out that if I is $\mathfrak{m}$-primary then $\mathrm{pd}(S)$ will be equal to the number of variables in the polynomial ring, by the Auslander-Buchsbaum formula, see [10].

2.2.2 Monomial Resolutions

Another observation that we can make from Example 2.2.2 is that the module of second syzygies was generated by the syzygy-pairs between the minimal generators of I. Buchberger's Criterion, [20, 3.3], shows that this is true in general for monomial ideals and allows us to give a nice description of the generators for the second syzygy module.

Proposition 2.2.5. *If I is a monomial ideal with minimal generating set $\{m_1, \ldots, m_n\}$, then the second syzygies of $S = R/I$ are generated by the syzygy-pairs between the minimal generators of I:*

$$\sigma_{ij} = \frac{m_{ij}}{m_j} e_j - \frac{m_{ij}}{m_i} e_i, \text{ for } 1 \leq i < j \leq n.$$

The set $\{\sigma_{ij}\}_{i<j}$ is rarely a minimal generating set for the second syzygy module of S. We denote the set of all second syzygies by $Z_2 = \sum_{i<j} R\sigma_{ij} \subseteq R^n$. We will define a unique minimal generating set for Z_2 denoted S_2, which we will call the *ordered minimal second syzygies.* This minimal generating set will be defined from the following orderings on the generators of I and the σ_{ij}'s. For the duration of this paper we will use the graded reverse lexicographic ordering (GRevLex) on elements

of R with $x < y < z$, and define a standard dictionary order on the indices of σ_{ij}. That is, $\sigma_{ij} < \sigma_{kl}$ if and only if either $i < k$, or $j < l$ when $i = k$. It should be noted that when we write m_{ij} it is not implied that $i < j$, since $m_{ij} = m_{ji}$. However when we write σ_{ij}, it is always assumed that $i < j$. We also note that in some situations we may choose to use a different ordering on the minimal generators of I, to make calculations easier. We can now define a specific minimal generating set S_2 of Z_2, which we will use throughout this paper.

Definition 2.2.6. If I is a monomial ideal, then $\sigma_{ij} \in S_2$ if and only if the following conditions are satisfied,

1. $\sigma_{ij} \in Z_2 - \mathfrak{m}Z_2$ and
2. $\sigma_{ij} \neq \sum_{k<l} a_{kl}\sigma_{kl}$, $a_{kl} \in R$, in which $\sigma_{ij} < \sigma_{kl}$ for all k, l such that a_{kl} is a unit.

We refer to the set S_2 as the *ordered minimal second syzygies* of S.

Condition (2) in this definition is needed because in some situations the problem arises where we may have a choice of which second syzygy we remove to construct a minimal generating set for Z_2. If this occurs, then by Definition 2.2.6 we will always remove the smallest second syzygy, in the sense of (2), with respect to GRevLex and the chosen dictionary ordering on $\{\sigma_{ij}\}_{i<j}$, of those from which we have a choice. In terms of the resolution of S, the columns of the matrix of f_2 will consist of the σ_{ij} which are in S_2.

Since we are choosing the maximal second syzygies based on the chosen ordering of $\{\sigma_{ij}\}_{i<j}$ we will refer to the resolution obtained from Definition 2.2.6 as the *maximal ordered resolution.* The resolutions we are considering here all have projective dimension 3, thus the column ordering of f_3 will have no affect on the remaining resolution. To be consistent, we will choose the same ordering in f_3 that is being used in f_2. The following lemma gives conditions on the minimal generators of I so that a syzygy-pair $\sigma_{ij} \notin S_2$ and in turn gives us conditions for σ_{ij} to be minimal.

Lemma 2.2.7 (Second Syzygy Lemma)**.** *Let I be a monomial ideal with minimal generating set $\{m_1, \ldots, m_n\}$.*

1. *If $\sigma_{ij} \in \mathfrak{m}Z_2$ then there exists a minimal generator m_k such that $m_{ik}|_< m_{ij}$.*
2. *If there exists a minimal generator m_k such that $m_{ik}|_< m_{ij}$ and $m_{jk}|_< m_{ij}$ then $\sigma_{ij} \in \mathfrak{m}Z_2$.*
3. *If $\sigma_{ij} \notin S_2$ then there exists a minimal generator m_k such that $m_k|_< m_{ij}$.*

Proof: (1): If $\sigma_{ij} \in \mathfrak{m}Z_2$, then

$$\sigma_{ij} = \sum_{1 \leq k < l \leq n} a_{kl}\sigma_{kl}, \text{ with } a_{kl} \in \mathfrak{m}.$$

It is enough to look at what we would need to get the i^{th} row entry in σ_{ij}. We have,

$$\frac{m_{ij}}{m_i} = \sum_{k=1}^{i-1} a_{ki}\frac{m_{ki}}{m_i} - \sum_{l=i+1}^{n} a_{il}\frac{m_{il}}{m_i} \implies m_{ij} = \sum_{k=1}^{i-1} a_{ki}m_{ki} - \sum_{l=i+1}^{n} a_{il}m_{il}.$$

Since m_{ij} is a monomial we know that at least one of the terms in the above sum must equal m_{ij} up to multiplication by a unit. That is, there exists a minimal generator m_k such that $m_{ij} = ua_{ik}m_{ik}$ where u is a unit in R. Thus we have that $m_{ik}|_< m_{ij}$ since $a_{ik} \in \mathfrak{m}$.

(2): Since $m_{ik}|_< m_{ij}$ and $m_{jk}|_< m_{ij}$ we know that both $a = \dfrac{m_{ij}}{m_{ik}}, b = \dfrac{m_{ij}}{m_{jk}} \in \mathfrak{m}$. Thus we have that,

$$\begin{aligned} a\sigma_{ik} - b\sigma_{jk} &= a\left(\frac{m_{ik}}{m_k}e_k - \frac{m_{ik}}{m_i}e_i\right) - b\left(\frac{m_{jk}}{m_k}e_k - \frac{m_{jk}}{m_j}e_j\right) \\ &= \frac{m_{ij}}{m_k}e_k - \frac{m_{ij}}{m_i}e_i - \frac{m_{ij}}{m_k}e_k + \frac{m_{ij}}{m_j}e_j \\ &= \frac{m_{ij}}{m_j}e_j - \frac{m_{ij}}{m_i}e_i \;=\; \sigma_{ij}. \end{aligned}$$

This implies that $\sigma_{ij} \in \mathfrak{m}Z_2$.

(3): If the hypothesis of (1) from Definition 2.2.6 is not satisfied, then from the proof of (1) we have that there is a minimal generator m_k such that $m_k|_< m_{ij}$. The other possibility is that

$$\sigma_{ij} = \sum_{k<l} a_{kl}\sigma_{kl}, a_{kl} \in R, \text{ in which } \sigma_{ij} < \sigma_{kl} \text{ for all } k, l \text{ such that } a_{kl} \text{ is a unit.}$$

Using a similar argument as in the proof of (1), we have that there exists a minimal generator m_k such that $m_{ij} = ua_{ik}m_{ik}$, a_{ik} is not necessarily in $\mathfrak{m}$. Hence $m_{ik}|m_{ij}$ which implies that $m_k|m_{ij}$ by Lemma 2.1.1. Thus we must have that $m_k|_< m_{ij}$ since we are assuming that m_k is a minimal generator of I. □

Remark 2.2.8. There are two important facts that we obtain from the previous lemma. First, if $\sigma_{ij} \notin S_2$ then we know that there must be a minimal generator m_k of I such that $m_k|_< m_{ij}$, which implies that both m_{ik} and m_{jk} divide m_{ij} by Lemma 2.1.1. If we let $a = \dfrac{m_{ij}}{m_{ik}}$, and $b = \dfrac{m_{ij}}{m_{jk}}$ as we did in the proof of (2), we have that $\sigma_{ij} = a\sigma_{ik} + b\sigma_{jk}$. Thus every $\sigma_{ij} \notin S_2$ can be obtained from a linear combination of exactly two second syzygies. Secondly, the contrapositive of (3) says that if there is no minimal generator m_k of I such that $m_k|_< m_{ij}$ then $\sigma_{ij} \in S_2$.

Since we would like to know how to find all of the minimal second syzygies we need to know what to do if there is a minimal generator m_k of I such that $m_k|_< m_{ij}$. When this happens it does not necessarily mean that $\sigma_{ij} \notin S_2$. If we assume that m_k is the only minimal generator strictly dividing m_{ij} then we can say the following.

1. If $m_{ik}|_< m_{ij}$ and $m_{jk}|_< m_{ij}$ then $\sigma_{ij} \notin S_2$ by Lemma 2.2.7.
2. If m_k has no positive degree on some variable equal to the degree on the same variable in m_i and m_j, then $m_k \| m_{ij}$ which implies that $\sigma_{ij} \notin S_2$ by Lemma 2.1.1 and Lemma 2.2.7.
3. If either $m_{ik} = m_{ij}$ or $m_{jk} = m_{ij}$ and $\sigma_{ij} < \sigma_{ik}$ or $\sigma_{ij} < \sigma_{jk}$ respectively, then $\sigma_{ij} \notin S_2$, by Definition 2.2.6.

We will consider these properties to determine if σ_{ij} is in S_2 or not.

Example 2.2.9. Let $I = \langle yz, x^3, y^3, x^2z, z^3, x^2y^2\rangle$. We will find all of the $\sigma_{ij} \in S_2$. In general we will always have $\binom{n}{2}$ distinct σ_{ij} when I is minimally generated by n elements. Here we have $\binom{6}{2} = 15$. Looking at each m_{ij}, for distinct minimal generators m_i and m_j we see that for m_{13}, m_{14}, m_{15}, m_{24}, m_{26}, m_{36}, and m_{45} there will not be another minimal generator of I dividing any one of these. We notice that $m_{16} = m_{46}$ and m_{14} strictly divides both of these. So we can choose either σ_{16} or σ_{46} to be minimal since $\sigma_{16} = y\cdot\sigma_{14} + \sigma_{46}$ and $\sigma_{46} = \sigma_{16} - y\cdot\sigma_{14}$. By our definition of S_2 we will remove the smallest of the σ_{ij} from which we have a choice. Since $\sigma_{16} < \sigma_{46}$ from our chosen ordering we have that $\sigma_{16} \notin S_2$ and $\sigma_{46} \in S_2$. We then find that for all other m_{ij} there is a minimal generator m_k such that m_{ik} and m_{jk} both strictly divide m_{ij}. Thus we have found $S_2 = \{\sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{24}, \sigma_{26}, \sigma_{36}, \sigma_{45}, \sigma_{46}\}$, which implies that,

$$f_2 = \begin{bmatrix} -y^2 & -x^2 & -z^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z & -y^2 & 0 & 0 & 0 \\ z & 0 & 0 & 0 & 0 & -x^2 & 0 & 0 \\ 0 & y & 0 & x & 0 & 0 & -z^2 & -y^2 \\ 0 & 0 & y & 0 & 0 & 0 & x^2 & 0 \\ 0 & 0 & 0 & 0 & x & y & 0 & z \end{bmatrix}.$$

In Chapter 3 we will discuss how we can use similar methods to compute a minimal generating set for the third syzygies. Since we will be working with trivariate monomial ideals we will find that we can represent the minimal free resolution by some planar graph. Also if I is generic as defined in Definition 3.3.1 the third syzygies will have a description similar to that of the second syzygies.

2.3 Tor as an Algebra

In this section we will discuss how we can represent $\mathrm{Tor}^R(\Bbbk, S)$ as a graded $\Bbbk$-algebra, where $\Bbbk$ is the residue class field of the local ring S. We do this using the induced algebra structure from the Koszul complex. In addition we will outline classes for this algebra structure and properties of these classes when S has codepth 3, see [4]. We can give specific descriptions of the Poincaré series of $\Bbbk$, $P_{\Bbbk}^{S}(t) = \sum \left(\mathrm{rank}_{\Bbbk}\mathrm{Tor}_i^S(\Bbbk,\Bbbk)\right) t^i$ for each class, which will ultimately provide us with information about the Bass series $I_S^S(t) = \sum \left(\mathrm{rank}_{\Bbbk}\mathrm{Ext}_S^i(\Bbbk, S)\right) t^i$ for S.

We should note that we will give many of the definitions in this section in a general sense, and will later reduce to the specific case that we are interested in. One thing in particular that should be observed is that the images of the generators for the homogeneous maximal ideal $\mathfrak{m} = \langle x, y, z\rangle$ of $R = \Bbbk[x, y, z]$ also generate the maximal ideal of $S = R/I$. We may abuse notation and refer to the maximal ideal of S as $\mathfrak{m}$ also.

2.3.1 Koszul Algebra

The Koszul complex is studied frequently in commutative and homological algebra, as it is a useful tool. Since the Koszul complex is defined in terms of the exterior algebra it inherits a graded algebra structure. The Kozsul complex is in fact a *differential graded* (DG) R-algebra, since it is a graded algebra together with a differential. Here we will review the exterior algebra and define the Koszul complex in general. We will then focus on some specific properties of the Koszul complex and explore the Koszul complex of $\mathfrak{m}$ over S. For more information on the exterior algebra or the Koszul complex we refer the reader to [10, 15].

The *exterior algebra* $\bigwedge M$ of an R-module M is the residue class algebra

$$\bigwedge M = (\bigotimes M)/\mathfrak{F}$$

where $\mathfrak{F}$ is the two sided ideal generated by elements of the form $x \otimes x$, $x \in M$. We refer the reader to [10, 1.6] for more on the tensor algebra $\bigotimes M$. Since $\mathfrak{F}$ is generated by homogeneous elements of $\bigotimes M$, $\bigwedge M$ inherits a graded R-algebra structure from $\bigotimes M$. Products between elements $x, y \in \bigwedge M$ are denoted by "little wedges" $x \wedge y$, and $\bigwedge M$ is generally not commutative, but is anti-commutative. Specifically if $x, y \in \bigwedge M$ are both homogeneous elements of degree 1 then,

$$x \wedge y = -y \wedge x \text{ and } x \wedge x = 0.$$

We denote the ith graded component of $\bigwedge M$ by $\bigwedge^i M$ and is called the ith *exterior power* of M. In general we have that $\bigwedge^0 M \cong R$ and $\bigwedge^1 \cong M$.

If M is a free R-module such that $M \cong R^j$ then $\bigwedge^1 M \cong R^j$. We can choose the standard basis elements $\{e_1, \ldots, e_j\}$ to generate M and hence also $\bigwedge^1 M$. Each of these basis elements are homogeneous elements of degree 1 in $\bigwedge^1 M$. If $x, y \in \bigwedge^1 M$, then $x \wedge y \in \bigwedge^2 M$, which consists of all the homogeneous degree 2 elements of $\bigwedge M$. Thus it is not difficult to see that elements of the form $e_i \wedge e_k$, $i < k$ will minimally generate $\bigwedge^2 M$. If we impose a specific ordering on these generators, for instance the standard dictionary ordering on the indices for $e_i \wedge e_k$, we get that $\bigwedge^2 M$ is a free R-module and is isomorphic to $R^{\binom{j}{2}}$. This only occurs if we specify how each of the generators $e_i \wedge e_k$ relates to each of the standard basis elements of $R^{\binom{j}{2}}$. Specifically we would have that

$$e_i \wedge e_k = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \longleftarrow \left[j(i-1) + k - i - \binom{i}{2} \right]^{\text{th}} \text{ row .}$$

Continuing this method we have that each $\bigwedge^i M \cong R^{\binom{j}{i}}$. Moreover we have that $\bigwedge^{j-1} M \cong R^j$, $\bigwedge^j M \cong R$, and $\bigwedge^i M = 0$ for all $i > j$.

We will now define the Koszul complex on a sequence of elements $\mathbf{x} = (x_1, \ldots, x_j)$ in a free R-module $N \cong R^j$.

Definition 2.3.1. The *Koszul complex* on $\mathbf{x} \in N \cong R^j$ is the complex,

$$K(\mathbf{x}):\ 0 \longrightarrow \bigwedge^j N \xrightarrow{\varphi_j} \bigwedge^{j-1} N \longrightarrow \cdots \longrightarrow \bigwedge^2 N \xrightarrow{\varphi_2} N \xrightarrow{\mathbf{x}} R \longrightarrow 0$$

where $\varphi_i(e_{n_1} \wedge \cdots \wedge e_{n_i}) = \sum_{k=1}^{i} (-1)^{k+1} x_k (e_{n_1} \wedge \cdots \wedge \hat{e_{n_k}} \wedge \cdots \wedge e_{n_i})$ for $2 \leq i \leq j$, with $\hat{e_{n_k}}$ meaning we omit e_{n_k} from the product.

To be precise if $\mathbf{x} = (x_1, \ldots, x_j)$ then we will write $K(x_1, \ldots, x_j)$ instead of $K(\mathbf{x})$. It is not difficult to see that the zeroth homology for $K(x_1, \ldots, x_j)$, $H_0(K) = R/\langle x_1, \ldots, x_j \rangle$. Moreover it is known that $H_i(K(x_1, \ldots, x_j)) = 0$ for $i > 0$ when $x_1, \ldots, x_j$ is an R-sequence or more generally $H_i(M \otimes_R K(x_1, \ldots, x_j)) = 0$ for $i > 0$ when $x_1, \ldots, x_j$ is an M-sequence, and $H_0(M \otimes_R K(x_1, \ldots, x_j)) = M/\langle x_1, \ldots, x_j \rangle M$, [15, 17.4].

Example 2.3.2. We are interested in the homology structure when $H^i(K) \neq 0$ for $i > 0$. To explore this we will first look at the Koszul complex on a minimal set of generators of the maximal ideal $\mathfrak{m}$ of $R = \Bbbk[x, y, z]$. Since x, y, z is a regular sequence in R it is not difficult to see that $K(x, y, z)$ is just the deleted resolution of $R/\mathfrak{m}$,

$$0 \longrightarrow R \xrightarrow{\varphi_3 = \begin{bmatrix} z \\ -y \\ x \end{bmatrix}} R^3 \xrightarrow{\varphi_2 = \begin{bmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{bmatrix}} R^3 \xrightarrow{\varphi_1 = \begin{bmatrix} x & y & z \end{bmatrix}} R \longrightarrow 0$$

We will choose the standard basis elements e_1, e_2, e_3 of R^3 for the graded degree 1 part of $K(x, y, z)$, and let $e_1 \wedge e_2$, $e_1 \wedge e_3$ and $e_2 \wedge e_3$ be the ordered generators of the graded degree 2 part. We then have that $e_1 \wedge e_2 \wedge e_3$ generates the degree 3 part of $K(x, y, z)$. Moreover from the differentials in $K(x, y, z)$ we see that,

$e_1 \mapsto x \quad e_2 \mapsto y \quad e_3 \mapsto z$,

$e_1 \wedge e_2 \mapsto -ye_1 + xe_2 \quad e_1 \wedge e_3 \mapsto -ze_1 + xe_3 \quad e_2 \wedge e_3 \mapsto -ze_2 + ye_3$, and

$e_1 \wedge e_2 \wedge e_3 \mapsto z(e_1 \wedge e_2) - y(e_1 \wedge e_3) + x(e_2 \wedge e_3).$

These mappings will be important when we begin finding generators for the graded pieces of the Koszul homology.

Our primary interest in the Koszul complex is in it's homology. In the previous example we already know everything there is to know about the homology of this complex. Specifically $H_0(K(x,y,z)) = R/\langle x,y,z\rangle \cong \Bbbk$, and $H_i(K(x,y,z)) = 0$ for $i > 0$. Since the zeroth homology is isomorphic to the residue class field we have that the homology complex is a graded $\Bbbk$-algebra. Although in this instance it is not very interesting. However if we tensor $K(x,y,z)$ with an R-module M, in which $\{x,y,z\}$ is not an M-sequence, then we may have non-vanishing homology. Specifically if $M = R/I$ where I is $\mathfrak{m}$-primary, then $H_i(M \otimes_R K(x,y,z)) \neq 0$ for $0 \leq i \leq 3$ and $H_i(M \otimes_R K(x,y,z)) = 0$ for $i > 3$. This is the situation we are interested in, but we will be even more specific by restricting to quotients of monomial ideals. We give a precise definition of this graded $\Bbbk$-algebra as follows.

Definition 2.3.3. Let $R = \Bbbk[x,y,z]$, let I be an $\mathfrak{m}$-primary monomial ideal of R, and set $S = R/I$. We define $A = H(S \otimes_S K(x,y,z))$ which is a graded $\Bbbk$-algebra.

Since S is a quotient ring of R we have that $S \otimes_S K(x,y,z) \cong S \otimes_R K(x,y,z)$, both of which represent the Koszul complex on the set of minimal generators for the maximal ideal of S. To simplify notation we will let $K = S \otimes_R K(x,y,z)$, thus $A = H(K)$. Our primary goal will be to classify the algebra structure on A for different I. In general we will find that we can break up A into six categories, which we will outline in Section 2.3.3, and provide additional details in Chapter 4. Finding the Bass series for these categories requires the use of the Poincaré series of $\Bbbk$. Since this is defined using $\mathrm{Tor}^S(\Bbbk,\Bbbk)$ we will first explore Tor in more detail and find that it also has a graded $\Bbbk$-algebra structure which is induced from the algebra structure on the Koszul complex. In fact we will show that $\mathrm{Tor}^R(\Bbbk, S) \cong A$.

2.3.2 Tor

The homology of a complex of R-modules gives us a measurement as to the lack of exactness of the complex. We are interested in how the homology of a complex C is affected when we build new complexes from C using certain operations. With this perspective in mind Tor measures the lack of exactness of the complex obtained by tensoring a free resolution an R-module M with another R-module N. We will provide the general definition of $\mathrm{Tor}^R(M,N)$ where M and N are both R-modules, then consider $\mathrm{Tor}^R(\Bbbk, S)$ as a special case.

Let M and N be R-modules and let M have free resolution given by,

$$\mathbb{F}: \ \cdots \longrightarrow F_{i+i} \xrightarrow{f_{i+1}} F_i \xrightarrow{f_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

If we tensor the deleted resolution of $\mathbb{F}$ with N over R we have the complex,

$$\mathbb{F} \otimes_R N: \ \cdots \longrightarrow F_{i+i} \otimes_R N \xrightarrow{f_{i+1} \otimes_R N} F_i \otimes_R N \xrightarrow{f_i \otimes_R N} F_{i-1} \otimes_R N \longrightarrow \cdots$$

which, in general is no longer exact everywhere.

Definition 2.3.4. Let M and N be R-modules and let $\mathbb{F}$ be a free resolution of M over R with ith free module F_i and ith differential f_i. We define $\mathrm{Tor}^R(M,N)$ to be the homology on the complex $\mathbb{F} \otimes_R N$. More precisely,

$$\mathrm{Tor}_i^R(M,N) = \ker(f_i \otimes_R N)/\mathrm{im}(f_{i+1} \otimes_R N).$$

Note we could have also computed Tor from $N \otimes_R \mathbb{F}$.

Fact 2.3.5 (Facts about Tor)**.** We have the following facts about $\mathrm{Tor}^R(M,N)$.

1. $\mathrm{Tor}_0^R(M,N) = M \otimes_R N$
2. If M or N are free then $\mathrm{Tor}_i^R(M,N) = 0$ for $i > 0$
3. If R is Noetherian and M and N are finitely generated R modules, then $\mathrm{Tor}_i^R(M,N)$ is a finitely generated module.

We will now switch our focus back to R being a trivariate polynomial ring over a field $\Bbbk$. Generally we have that $\mathrm{Tor}_0^R(\Bbbk, S) = \Bbbk \otimes_R S \cong \Bbbk$. Moreover since $\Bbbk$ and S are Noetherian and finitely generated we know that $\mathrm{Tor}_i^R(\Bbbk, S)$ is finitely generated. Recall that $K(x, y, z)$ is isomorphic to a minimal free resolution of $\Bbbk$ over R, thus $\mathrm{Tor}_i^R(\Bbbk, S)$ is the ith homology of K. This also implies that $\mathrm{Tor}^R(\Bbbk, S)$ is a *graded* $\Bbbk$-*algebra* and is equal to A. We will show that we can compute $\mathrm{Tor}^R(\Bbbk, S)$ in two ways with the following fact from [4, 1.2.1].

Fact 2.3.6. The homology of K is isomorphic to the homology of $\mathbb{F} \otimes_R \Bbbk$, where $\mathbb{F}$ is a minimal free resolution of S over R. Thus we can compute $\mathrm{Tor}^R(\Bbbk, S)$ in the following two ways;

1. we can compute $\mathrm{Tor}^R(\Bbbk, S)$ from a minimal free resolution of S, or
2. we can compute $\mathrm{Tor}^R(\Bbbk, S)$ from the Koszul complex K.

In the next section we will introduce some of the specific properties for the algebra A when S has codepth 3, see Definition 2.3.8 for the definition of codepth. In this case $A = A_1 \oplus A_2 \oplus A_3$ and $A_i \neq 0$ for $1 \leq i \leq 3$. We will introduce certain invariants involving these algebras and discuss basic methods of computing these invariants. We will also give the general classification table for A from [4, 1.3].

2.3.3 Codepth 3 Algebra Structure

From now on we will refer to the algebra A as the Tor-algebra for $S = R/I$ or the Tor-algebra for I. Here we will explain the Tor-algebra structure when S has codepth 3, and provide a classification as given by Avramov in [4]. We will first need to introduce some invariants related to these structures as well as set notation for some general characteristics of S. Note that each A_i is a $\Bbbk$-vector space. We also have the following fact relating ranks of free modules in a free resolution of S over R to the ranks of each A_i over $\Bbbk$, from [4, 1.2.2].

Fact 2.3.7. If $S = R/I$ and I is minimally generated by n elements, then S has minimal free resolution over R given by,

$$\mathbb{F} := \ 0 \longrightarrow R^m \xrightarrow{f_3} R^{m+n-1} \xrightarrow{f_2} R^n \xrightarrow{f_1} R \longrightarrow S \longrightarrow 0,$$

and the equality $\operatorname{rank}_{\Bbbk}(A_i) = \operatorname{rank}_R(F_i)$ holds for all i. Specifically $\operatorname{rank}_{\Bbbk}(A_1) = n$, $\operatorname{rank}_{\Bbbk}(A_2) = m + n - 1$, and $\operatorname{rank}_{\Bbbk}(A_3) = m$.

Thus we obtain information about each A_i by knowing what the minimal free resolution of S is. Knowing that I is minimally generated by n elements and $\operatorname{rank}_R(F_3) = m$ we have that A_1 is minimally generated by n degree 1 elements of $S \otimes_R \bigwedge^1 R^3$, A_2 is minimally generated by $m+n-1$ degree 2 elements of $S \otimes_R \bigwedge^2 R^3$, and A_3 is minimally generated by m degree 3 elements of $S \otimes_R \bigwedge^3 R^3$. To determine the structure of A we will need to consider the ranks of some $\Bbbk$-vectors spaces associated with A.

Definition 2.3.8. Let $R = \Bbbk[x, y, z]$ for a field $\Bbbk$, let I be an ideal of R, and let $S = R/I$. We define the following characteristics of S and invariants for $A = H(K)$.

1. $e = \operatorname{rank}_{\Bbbk}(\mathfrak{m}/\mathfrak{m}^2)$ is the embedding dimension of S
2. $d = \inf\{i \in \mathbb{Z} | \operatorname{rank}_{\Bbbk}\operatorname{Ext}^i_S(\Bbbk, S) \neq 0\}$ is the depth of S
3. $c = e - d$ is the codepth of S
4. $p = \operatorname{rank}_{\Bbbk}(A_1^2)$
5. $q = \operatorname{rank}_{\Bbbk}(A_1 \cdot A_2)$
6. $r = \operatorname{rank}_{\Bbbk}(\delta_2)$, where $\delta_2 : A_2 \longrightarrow \operatorname{Hom}_{\Bbbk}(A_1, A_3)$ where $\delta_2(x)(y) = xy$ for $x \in A_2$ and $y \in A_1$

Since we will be restricting to $\mathfrak{m}$-primary monomial ideals we will have that $c = e - d = 3 - 0 = 3$. We will however give the general list of classes for A provided by Avramov. Before we give the general classification we need to describe some components of this classification. Let E be a graded algebra over a commutative ring, where $E_i = 0$ for $i < 0$. We will also assume that E is *graded commutative*, meaning that $xy = (-1)^{ij}yx$ for all $x \in E_i$ and $y \in E_j$, and $x^2 = 0$ when i is odd. Notice these are the same properties that we have for $\bigwedge M$. Let M be a left E-module, then M is a graded module and we define a *degree shift* on M by saying that $(\Sigma^j M)_i = M_{i-j}$ for every $i, j \in \mathbb{Z}$. It is not difficult to see that $\Sigma^j M$ is also a graded E-module.

The *trivial extension* $E \ltimes M$ is the graded algebra with underlying complex $E \oplus M$ and the product $(x, m)(x', m') = (xx', xm' + (-1)^{ij'}x'm)$ for $x' \in E_{j'}$ and $m \in M_i$. The following table will describe the various structure classes for A, see [4, 1.3].

Fact 2.3.9. Let B, C, and D be graded $\Bbbk$-algebras, and W a graded B-module with $(B_+)W = 0$. We can describe A, up to isomorphism by the following table:

Table 2.1: Tor-algebra Structure

Class	[range]	c	A	B	C	D
$\mathbf{C}(c)$	$[c \geq 0]$	c	B	$\bigwedge_{\Bbbk} \Sigma\Bbbk^c$		
$\mathbf{S}$		2	$B \ltimes W$	$\Bbbk$		
$\mathbf{T}$		3	$B \ltimes W$	$C \ltimes \Sigma(C/C_{\geq 2})$	$\bigwedge_{\Bbbk} \Sigma\Bbbk^2$	
$\mathbf{B}$		3	$B \ltimes W$	$C \ltimes \Sigma C_+$	$\bigwedge_{\Bbbk} \Sigma\Bbbk^2$	
$\mathbf{G}(r)$	$[r \geq 2]$	3	$B \ltimes W$	$C \ltimes \mathrm{Hom}_{\Bbbk}(C, \Sigma^3\Bbbk)$	$\Bbbk \ltimes \Sigma\Bbbk^r$	
$\mathbf{H}(p, q)$	$[p, q \geq 0]$	3	$B \ltimes W$	$C \otimes_{\Bbbk} D$	$\Bbbk \ltimes (\Sigma\Bbbk^p \oplus \Sigma^2\Bbbk^q)$	$\Bbbk \ltimes \Sigma\Bbbk$

No two algebras A in the table are isomorphic and neither are any two algebras B.

Our investigation will restrict to $c = 3$, so we can remove $\mathbf{S}$ from the list. It turns out that $\mathbf{G}(r) \cong \mathbf{H}(0, r)$ for $r = 0, 1$, which is why we make the restriction that $r \geq 2$. When I is an $\mathfrak{m}$-primary monomial ideal it is known that S is a complete intersection if and only if S is Gorenstein. Also S is Gorenstein but *not* a complete intersection if and only if it is in $\mathbf{G}(r)$. This allows us to remove $\mathbf{G}(r)$ from the list for the rings that we are concerned with.

Remark 2.3.10. It is also known that if $c = 3$ then S is Golod if and only if S is in $\mathbf{H}(0,0)$. Here, S is Golod if an only if $A_+^2 = 0$, which will only occur when S is in $\mathbf{H}(0,0)$. In Chapter 4 we will use this to find a class of examples of rings given by monomial ideals which are Golod.

2.3.4 Bass Numbers

In this section we will define the Bass numbers for commutative local rings. There are many longstanding questions involving Bass numbers. Avramov's primary goal in [4] was to show that the sequence of Bass numbers is increasing. He gives a complete answer to this question for rings of codepth 3. He does this by finding nice closed form expressions for the Bass series. Since our rings are also codepth 3, we learn how to relate the Bass numbers to the Betti numbers of the canonical module of our ring. We begin by giving a general definition for the Bass numbers and Bass series for a commutative local ring S.

Definition 2.3.11. The ith Bass number of a commutative local ring S is the number of copies of the injective envelope of the residue field $\Bbbk$ in the ith injective module of a minimal injective resolution of S. More precisely the ith Bass number for S is $\mu_S^i = \mathrm{rank}_{\Bbbk}\mathrm{Ext}_S^i(\Bbbk, S)$ and the Bass series for S is given by $I_S^S(t) = \sum\limits_{i \geq 0} \mu_S^i t^i$.

Fact 2.3.12. Since S is local and Artinian then the Betti numbers of $\omega_S = \mathrm{Ext}_R^3(S, R)$ (the canonical module of S) are equal to the Bass numbers of S, see [18].

Using this Fact we are able to obtain information about μ_S^0 and μ_S^1 from the minimal free resolution of S. Recall the the minimal free resolution of S will have the following form,

$$\mathbb{F} := \ 0 \longrightarrow F_3 \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \longrightarrow S \longrightarrow 0.$$

In [15, 20.9] it is shown that a complex of free R modules is exact if and only if $\operatorname{rank}_R(F_i) = \operatorname{rank}_R(f_i) + \operatorname{rank}_R(f_{i+1})$. Using this formula along with the assumption that I is minimally generated by n elements and $\operatorname{rank}_R(F_3) = m$ we obtain a more precise construction of the free resolution,

$$\mathbb{F} := \ 0 \longrightarrow R^m \xrightarrow{f_3} R^{m+n-1} \xrightarrow{f_2} R^n \xrightarrow{f_1} R \longrightarrow S \longrightarrow 0 \ \text{ with } n > 3, m > 1.$$

By applying $\operatorname{Hom}(\_, R)$ to the deleted resolution we get,

$$0 \to \operatorname{Hom}(R, R) \xrightarrow{f_1^*} \operatorname{Hom}(R^n, R) \xrightarrow{f_2^*} \operatorname{Hom}(R^{m+n-1}, R) \xrightarrow{f_3^*} \operatorname{Hom}(R^m, R) \to \omega_S \to 0,$$

is a minimal free resolution of ω_S over R, see [10, 3.3.9].

Now if we tensor $\operatorname{Hom}(\mathbb{F}, R)$ with S we will get a free presentation of ω_S as an S-module,

$$\operatorname{Hom}(R^{m+n-1}, R) \otimes_R S \xrightarrow{f_3^* \otimes_R S} \operatorname{Hom}(R^m, R) \otimes_R S \to \omega_S \to 0.$$

Though this is a free presentation of ω_S as an S-module, it may not be minimal. We may think of each map f_i^* as the transpose of the matrix for f_i, and thus the map $f_3^* \otimes_R S$ is the transpose of the matrix of f_3 with entries in S. This presentation for ω_S tells us that $\mu_S^0 = \operatorname{rank}_S(\operatorname{Hom}(R^m, R) \otimes_R S) = m$. The map $f_3^* \otimes_R S$ will be minimal if the rows of f_3 are algebraically independent mod I. To clarify, we say that the k^{th} row of f_3, denoted r_k, is algebraically *dependent* mod I if there exists $a_i \in R$ such that $r_k - (a_1 r_1 + \cdots + a_{k-1} r_{k-1} + a_{k+1} r_{k+1} + \cdots + a_{m+n-1} r_{m+n-1}) \in IR^m$. This allows us to find an expression for μ_S^1 as the number of algebraically independent rows of f_3 mod I. Let $\hat{r}$ be the number of rows in f_3 that are dependent mod I, then $\mu_S^1 = m + n - 1 - \hat{r}$.

Using this formula for μ_S^1 we could show that $\mu_S^1 > \mu_S^0$, if we could show that $m + n - 1 - \hat{r} > m \iff n - 2 \geq \hat{r}$. This allows us to state this question on lower Bass numbers in the following way.

Question 2.3.13. Is the number of rows in the matrix of f_3 which are dependent mod I less than or equal to $n - 2$?

In Chapter 3 we will find a positive answer to this question when I is a generic monomial ideal. Using this method to obtain information about the Bass numbers is inefficient in general. This is because the permissable row operations can become quite complicated even for general monomial ideals.

Remark 2.3.14. This question acts as one of the primary motivations for the work done in Chapter 3, and ultimately Chapter 4 as well. We also use the fact that $\mu_S^1 = m + n - 1 - \hat{r}$ heavily in Chapter 4, as we will prove that the invariant $r = \operatorname{rank}_{\Bbbk}(\delta_2) = \hat{r}$ in Proposition 4.1.8.

2.3.5 $P_{\Bbbk}^S(t)$ and $I_S^S(t)$

One of the most important tools we will use to determine when a ring S has a specific Tor-algebra structure is the Bass series $I_S^S(t)$ for S. We will list the expressions for the Bass series of S outlined in [4, 2.1] based upon their respective Tor-algebra structure. From Section 2.3.3 we know that if I is an $\mathfrak{m}$-primary monomial ideal we will only need to consider the Bass series for the structures $\mathbf{C}(c)$, $\mathbf{T}$, $\mathbf{B}$, and $\mathbf{H}(p, q)$. In Chapter 4 we will also remove $\mathbf{C}(c)$ from our list, as this case is easy to classify. We can use the expressions for Bass series of our respective rings to determine μ_S^1

from μ_S^0, given a minimal free resolution of S. In Chapter 4 we will relate results obtained in Chapter 3 to the computation of μ_S^0 and μ_S^1.

The Poincaré series of $\Bbbk$, $P_{\Bbbk}^S(t) = \sum \left(\operatorname{rank}_{\Bbbk} \operatorname{Tor}_i^S(\Bbbk, \Bbbk)\right) t^i$ is commonly studied. For the Koszul complex K we have a natural map $K \longrightarrow \Bbbk$ that makes $\Bbbk$ a DG K-module. From [2, 3.2] and [5, 4.1], respectively, we get

$$P_{\Bbbk}^S(t) = (1+t)^e \cdot P_{\Bbbk}^K(t) \quad \text{and} \quad I_S^S(t) = t^e \cdot I_K^K(t).$$

We have already seen how $H(\mathbb{F} \otimes_R \Bbbk) \cong A$, thus from [4, 1.6] we get that

$$P_{\Bbbk}^K(t) = P_{\Bbbk}^A \quad \text{and} \quad I_K^K(t) = I_A^A(t).$$

For our specific case we have that $e = 3$, and we get

$$P_{\Bbbk}^S(t) = (1+t)^3 \cdot P_{\Bbbk}^A(t) \quad \text{and} \quad I_S^S(t) = t^3 \cdot I_A^A(t).$$

We will now provide the expressions for the Poincaré series and Bass series given by Avramov in [4, 2.1]. We do note that the expressions given by Avramov do hold in more generality than what we state here, but we are only concerned with the case when $e = 3$ and $d = 0$ for S.

Theorem 2.3.15 (Avramov)**.** *Let $R = \Bbbk[x, y, z]$ for a field $\Bbbk$, let I be an $\mathfrak{m}$-primary monomial ideal minimally generated by n monomials, and let $S = R/I$. We have that $e = \operatorname{edim}(S) = 3$ and $d = \operatorname{depth}(S) = 0$. Also let $l = n - 1$, $m = \operatorname{rank}_R(F_3)$ and let the numbers p, q, and r be the numbers defined in Definition 2.3.8. There are equalities*

$$P_{\Bbbk}^S(t) = \frac{(1+t)^2}{g(t)} \quad \textit{and} \quad I_S^S(t) = \frac{f(t)}{g(t)}$$

where $f(t)$ and $g(t)$ are polynomials in $\mathbb{Z}[t]$, listed in the following table:

Table 2.2: Poincaré and Bass Series for S

Class	$g(t)$	$f(t)$
$\mathbf{C}(c)$	$(1-t)^3(1+t)^2$	$(1-t)^3(1+t)^2$
$\mathbf{T}$	$1-t-lt^2-(m-3)t^3-t^5$	$m+lt-2t^2-t^3+t^4$
$\mathbf{B}$	$1-t-lt^2-(m-1)t^3+t^4$	$m+(l-2)t-t^2+t^4$
$\mathbf{H}(0,0)$	$1-t-lt^2-mt^3$	$m+lt+t^2-t^3$
$\mathbf{H}(p,q),\ p+q\geq 1$	$1-t-lt^2-(m-p)t^3+qt^4$	$m+(l-q)t-pt^2-t^3+t^4$

We notice that since the rings in $\mathbf{C}(c)$ are complete intersections they are also Gorenstein and $I_S^S(t)=1$. This is not surprising since all of the positive Bass numbers vanish for S in this case and it is commonly known that $\mu_S^0=1$.

CHAPTER 3

TRIVARIATE MONOMIAL RESOLUTIONS

3.1 Motivation

In this chapter we are interested in when we get nonzero elements from I as entries in f_i. We will also be interested in the maximum number of such entries we can have in f_i when I is a trivariate monomial ideal. Our primary focus will be on determining when we get elements from I in f_3. We will use our special form of the minimal free resolution, given in Definition 2.2.6 to find the maximum number of rows of f_3 that contain only elements from I. We are primarily interested in generic monomial ideals. When I is generic the free resolution for $S = R/I$ will have a nice construction. More precisely, if I is generic then each column of the matrix of f_3 in the minimal free resolution of S will have exactly three nonzero *pure power* entries. In Theorem 3.3.11 we will show that for generic monomial ideals the maximum number of nonzero elements from I that we get in f_3 is $n-2$, when I is minimally generated by n elements. We will conclude by considering some examples of resolutions for non-generic monomial ideals and contrast the differences of f_3 when I is generic. We will also consider examples where we relax the condition that I is $\mathfrak{m}$-primary. Many of our results will not hold without this condition.

As discussed in Section 2.3.4 one of the motivations for this work relates to a question on the Bass numbers of S. Specifically, is the first Bass number of S always larger than the zeroth Bass number of S? In Question 2.3.13 we found that we can answer this question if we can show that the number rows in f_3, which are algebraically dependent mod I, is less than or equal to $n-2$. Moreover if we can show that row operations do not affect whether or not a row in f_3 is dependent mod I, we only need to show that the number of rows in f_3 which are contained in I is less than or equal to $n-2$. We discuss this in more detail in Remark 3.4.7 and find that we get a positive answer to this question when I is generic. In Chapter 4 we will also find that the number $p = \mathrm{rank}_{\Bbbk}(A_1 \cdot A_2)$ is related to the number of minimal second syzygies of S which are contained in I.

3.2 Trivariate Resolutions

To answer our original question of when we get nonzero elements from I as entries in f_i we will need to describe the conditions needed for such entries to appear in each f_i. If I is minimally generated by monomials $m_1, \ldots, m_n$ then the matrix f_1 is $[m_1 \ldots m_n]$. Since f_2 is generated by the syzygy-pairs between the minimal generators, defined in 2.2.5, we will only get nonzero elements from I in f_2 when $(m_i, m_j) = 1$ and the syzygy-pair between m_i and m_j is a minimal second syzygy. From this we will find that if I is minimally generated by n monomials, then $2n-2$ is the maximum number of nonzero entries we can get in f_2 which are also elements of I.

We can now address the question of when we get elements from I as entries in f_2 and what the maximum number of such entries will be. Since we are only dealing with trivariate monomial ideals, if two minimal generators m_i and m_j have $(m_i, m_j) = 1$ then one of the generators must be a pure power and the other generator must only have nonzero degrees on the other two variables. Using this in combination with Remark 2.2.8 we can classify the maximum number of nonzero entries of f_2 which are also in I.

Theorem 3.2.1. *Let I be a monomial ideal of R minimally generated by n elements.*

1. *If m_i and m_j are minimal generators of I then the following are equivalent.*

(i) $(m_i, m_j) = 1$ *and there is no other minimal generator* m_k *of* I *such that* $m_k|_< m_{ij}$

(i) $\sigma_{ij} \in S_2$ *and* $\sigma_{ij} = m_i e_j - m_j e_i$

2. *The matrix of* f_2 *from Definition 2.2.6 contains at most* $2n - 2$ *nonzero entries from* I*.*

Proof: (1): This follows immediately from Definition 2.2.5 and Remark 2.2.8, which is a consequence of Lemma 2.2.7.

(2): Since we are assuming that $n > 3$ we must have at least one generator with nonzero degrees on at least two variables. For simplicity order the generators so that $m_1 = x^a, m_2 = y^b, m_3 = z^c$. Note we are not assuming that these are the only generators of I. There are two cases that need to be considered. First, if there are no minimal second syzygies between a pure power generator and a mixed double generator, then we can only get entries from I in f_2 from the second syzygies σ_{12}, σ_{13}, and σ_{23}. This gives a maximum of 3 columns in f_2 with entries from I, which means the number such columns is less than or equal to $n - 1$ since $n \geq 4$, which satisfies our hypothesis using part (1) of the Proposition.

For the second case we will show that if we have a minimal second syzygy between a pure power generator and a mixed double generator involving the other two variables, then this is the only pure power generator that can appear more than once in f_2. Without loss of generality assume that x^a is the pure power generator that appears, and that $m_4 = y^\beta z^\gamma$, $\beta, \gamma > 0$ is the mixed double generator with $\sigma_{14} \in S_2$. We will first show that the only second syzygies that could yield a y^b or z^c in f_2 must be σ_{12} and σ_{13} respectively. This will imply that all of the second syzygies in S_2 that have nonzero entries from I are of the form σ_{1i} where $m_i = y^{b'} z^{c'}$. Secondly we will show that the maximum number of these minimal second syzygies that we may have is $n-1$. This implies that the maximum number of nonzero entries from I that we can have in f_2 is $2n-2$ by part (1) of the Proposition.

It is implied that $\sigma_{23} \notin S_2$ since m_4 is a minimal generator of I. Suppose that $\sigma_{25} \in S_2$ such that $m_5 = x^{a'}z^{c'}, a', c' > 0$. Then either $c' \geq \gamma$ or $\gamma \geq c'$. Assume that $c' \geq \gamma$, then both $m_{24} = y^b z^\gamma$ and $m_{45} = x^{a'}y^\beta z^{c'}$ strictly divide $m_{25} = x^{a'}y^b z^{c'}$. This implies that $\sigma_{25} \notin S_2$ by Lemma 2.2.7. Similarly if we assume that $\gamma \geq c'$ we have that $\sigma_{14} \notin S_2$ which is a contradiction. Thus if $\sigma_{14} \in S_2$, the only minimal second syzygies that can give us a y^b or z^c in f_2 are σ_{12} and σ_{13}.

Now since the only possible minimal second syzygies that give the desired entries in f_2 are $\sigma_{1i}, 2 \geq i \geq n$, then we have at most $n-1$ of these second syzygies. Part (1) of the proposition says that each of these has exactly two nonzero entries from I thus we can have at most $2(n-1)$ nonzero elements from I as entries in f_2. $\square$

We will see later that if I is generic and $n \geq 5$ we achieve the maximum number of nonzero entries in f_2 from I if and only if we achieve the maximum number of nonzero entries in f_3 from I. This is a direct consequence of Proposition 3.2.1 and Theorem 3.3.11. Refer to Example 3.3.12 to illustrate Proposition 3.2.1.

3.3 Generic Monomial Resolutions

The focus of this section will be to discuss free resolutions of S when I is a generic monomial ideal. Free resolutions of generic monomial ideals have been studied extensively in [8, 19–21]. These resolutions have a specific structure. In particular, f_3 will contain exactly three nonzero entries in each column. Here we will discuss how we can use Buchberger graphs to represent these resolutions when I is generic as shown in [8, 20]. We will also discuss more specific properties of f_3, namely, the types of nonzero elements of I we can get in f_3 and the maximum number of such entries.

In [19, 20] we learn that the matrix for f_3 is *completely determined* by the minimal second syzygies which are used in f_2. This is because we can represent the minimal resolution of S by some labeled planar graph. The minimal second syzygies are represented by the edges of the planar graph and the minimal third syzygies are represented by the faces of the planar graph. The labeling is given by the ordering chosen on the minimum generators of I which are the vertices of the graph. The edges that are chosen will determine the faces of the graph. These are the minimal cycles formed by the minimal second syzygies. These cycles are the faces of the planar graph which represents the minimal third syzygies. For more on planar graphs we refer the reader to [17, 19].

Definition 3.3.1. A monomial ideal $I = \langle m_1, \ldots, m_n \rangle$ is *generic* if whenever two distinct minimal generators m_i and m_j have the same positive degree in some variable, there is another minimal generator m_k such that $m_k \| m_{ij}$.

The way generic monomial ideals are defined we have that $\sigma_{ij} \notin S_2$ whenever m_i and m_j have the same positive degree in some variable, see [20, 6.26]. This is also a consequence of the second syzygy lemma. By the definition of generic, if m_i and m_j have the same positive degree in some variable, then there will always be a minimal generator m_k such that $m_k \| m_{ij}$. This implies that $m_{ik}|_< m_{ij}$ and $m_{jk}|_< m_{ij}$ by Lemma 2.1.1. We will now give a stronger version of Lemma 2.2.7 for generic monomial ideals.

Lemma 3.3.2. *Let I be a generic monomial ideal with minimal generating set $\{m_1, \ldots, m_n\}$, then $\sigma_{ij} \notin S_2$ if and only if there exists a minimal generator m_k such that $m_{ik}|_< m_{ij}$ and $m_{jk}|_< m_{ij}$.*

Proof: We have already proven the reverse direction in Lemma 2.2.7. For the forward direction we need to show that if either condition (1) or (2) in Definition 2.2.6 is not satisfied then we will get the desired result. In fact we will show that if (2) is not satisfied that this implies that (1) is also not satisfied when I is generic. First assume that (1) is not satisfied, then there exists a minimal generator m_k such that $m_{ik}|_< m_{ij}$. We only need to show that $m_{jk}|_< m_{ij}$. It can also be assumed that m_i and m_j do not share any positive degrees in some variable since I is generic. If m_i and m_k do not have the same positive degree in some variable then we are done because $m_{jk} \neq m_{ij}$. If m_j and m_k do have the same positive degree in some variable, then there exists a minimal generator m_l such that $m_l \| m_{jk}$. Thus $m_l \| m_{ij}$.

We will now show that if (2) is not satisfied in Definition 2.2.6 then (1) is not satisfied either. If (2) is not satisfied then Lemma 2.2.7 says that there exists a minimal generator m_k such that $m_k|_< m_{ij}$. This implies that both m_{ik} and m_{jk} divide m_{ij}. Notice that both $m_{ik}|_< m_{ij}$ and $m_{jk}|_< m_{ij}$ would imply that (1) is not satisfied and we are done. Without loss of generality say $m_{ik} = m_{ij}$, then at least two of these three minimal generators must have the same positive degree on the same variable. Since I is generic there must exist a minimal generator m_l such that m_l strongly divides either m_{ij}, m_{ik} or m_{jk}. This implies that $m_l \| m_{ij}$ which gives us that (1) is not satisfied. $\square$

Using Lemma 3.3.2 or [20, 6.26] we have that all of the minimal second syzygies in f_2 must correspond to minimal generators m_i and m_j such that these generators do not have the same positive degree in some variable. This also tells us that we will not have a choice of which second syzygies are in S_2. Thus condition (2) in Definition 2.2.6 is not needed to define S_2 for generic monomial ideals.

One of the nice properties about trivariate monomial ideals is that we can also represent these ideals with three dimensional staircase diagrams. Since we are assuming that S artinian, this also implies that these diagrams will be bounded on all axes. In general staircase diagrams provide a template for which we can draw a graphical representation of the free resolution of S as shown in [19] and [20]. For a generic monomial ideal one way this can be done is by constructing the Buchberger graph for I.

Definition 3.3.3. The *Buchberger graph* $\mathrm{Buch}(I)$ of a monomial ideal $I = \langle m_1, \ldots, m_n \rangle$ has vertices $1, \ldots, n$ and an edge (i, j) whenever there is no monomial m_k such that $m_k \| m_{ij}$.

From this definition it is clear that if I is generic, then (i, j) will not be an edge on $\mathrm{Buch}(I)$ if m_i and m_j have the same positive degree in some variable. In [20, 6.10] it is shown that $\mathrm{Buch}(I)$ is equal to the edges of the Scarf complex $\triangle_I$, which uniquely generates Z_2 when I is generic. Using this along with a result from [8] we are able to give a detailed description of the free resolution of R/I when I is generic.

Proposition 3.3.4. *Let $I = \langle m_1, \ldots, m_n \rangle$ be generic, then the following hold.*

1. *$(i, j) \in \mathrm{Buch}(I)$ if and only if $\sigma_{ij} \in S_2$.*
2. *$\mathrm{Buch}(I)$ is a planar triangulation.*

Part (1) follows from [20, 6.10] and (2) is obtained from a more general result [8, 5.5], which says that if I is a generic monomial ideal in r variables then the Scarf complex $\triangle_I$ is a regular triangulation. Since we are only dealing with trivariate monomial ideals we can say that $\mathrm{Buch}(I)$ is a planar triangulation, we refer the reader to [17] for more on planar triangulations. In general, S_2 is contained in the edges of $\mathrm{Buch}(I)$, but $\mathrm{Buch}(I)$ will not give us a minimal representation of a free resolution unless I is generic. Part (2) of Proposition 3.3.4 tells us that every column in f_3 contains exactly three nonzero entries. This is not surprising since it is also known that $\triangle_I$ is a simplicial complex, [20, 6.8]. In Lemma 3.3.8 we will give a precise description of these nonzero entries. First, we will motivate this with an example.

We will not display the first matrix, f_1 for some of our examples. Instead we will assume that given an ideal $I = \langle m_1, \dots, m_n \rangle$ that $f_1 = \begin{bmatrix} m_1 & \dots & m_n \end{bmatrix}$.

Example 3.3.5. Let $I = \langle yz^2, x^5, x^3y^2, y^5, z^5, x^3z^3 \rangle$, which is generic, and the minimal free resolution for S obtained from Definition 2.2.6 is,

$$0 \longrightarrow R^4 \xrightarrow{\begin{bmatrix} y & z & 0 & 0 \\ -x^2 & 0 & y^3 & 0 \\ 0 & 0 & -x^3 & 0 \\ 0 & 0 & 0 & x^3 \\ 0 & -x^2 & 0 & -z^2 \\ z^2 & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z^2 & 0 \\ 0 & 0 & 0 & y \end{bmatrix}} R^9 \xrightarrow{\begin{bmatrix} -x^5 & -x^3y & -y^4 & -z^3 & -x^3z & 0 & 0 & 0 & 0 \\ yz^2 & 0 & 0 & 0 & 0 & -y^2 & -z^3 & 0 & 0 \\ 0 & z^2 & 0 & 0 & 0 & x^2 & 0 & -y^3 & 0 \\ 0 & 0 & z^2 & 0 & 0 & 0 & 0 & x^3 & 0 \\ 0 & 0 & 0 & y & 0 & 0 & 0 & 0 & -x^3 \\ 0 & 0 & 0 & 0 & y & 0 & x^2 & 0 & z^2 \end{bmatrix}} R^6 \longrightarrow \cdots$$

The main observation here is that each column in f_3 contains exactly three nonzero *pure power* entries. We will show that this is true in general for minimal resolutions given by generic monomial ideals. We first give a precise description of the third syzygies for S when I is generic.

Proposition 3.3.6. *Let $I = \langle m_1, \dots, m_n \rangle$ be generic, then every column in f_3 is given by,*

$$\tau_{ijk} = \frac{m_{ijk}}{m_{ij}} e_{|\sigma_{ij}|} - \frac{m_{ijk}}{m_{ik}} e_{|\sigma_{ik}|} + \frac{m_{ijk}}{m_{jk}} e_{|\sigma_{jk}|}, \text{ such that } 1 \le i < j < k \le n,$$

and $|\sigma_{ij}|$ denotes the column number in f_2, in which σ_{ij} lies.

The reason that this is true is due to the fact that the algebraic Scarf complex gives a minimal free resolution of S, [20, 6.26], and that the following identity holds for all $1 \le i < j < k \le n$,

$$\tau_{ijk} = \frac{m_{ijk}}{m_{ij}} \sigma_{ij} - \frac{m_{ijk}}{m_{ik}} \sigma_{ik} + \frac{m_{ijk}}{m_{jk}} \sigma_{jk} = 0.$$

In general the τ_{ijk} generate the third syzygy module, as the set of all of these represent the third syzygies given by the Taylor complex, [20, 6.1]. However, for our special case the set $\{\tau_{ijk}\}_{i<j<k}$ does not minimally generate the third syzygy module unless S is Gorenstein. We will use this to define the set of all third syzygies for S by $Z_3 = \sum_{i<j<k} R\tau_{ijk} \subseteq R^{m+n-1}$, with minimal generating set S_3. We make note that to find the minimal third syzygies we only need to find all combinations of three columns in f_2 so that these three columns only have nonzero entries in three distinct rows. The $|\sigma_{ij}|$, $|\sigma_{ik}|$, and $|\sigma_{jk}|$ will be the actual column numbers of σ_{ij}, σ_{ik} and σ_{jk} in f_2. The column numbers will also tell us which rows in τ_{ijk} have nonzero entries. It should be noted that τ_{ijk} is unique up to the ordering chosen on the second syzygies. Also since the minimal third syzygies are determined by the chosen minimal second syzygies we do not have a choice of which τ_{ijk} minimally generate Z_3 when I is generic.

Lemma 3.3.7. *Let I be a monomial ideal of R, such that m_i, m_j, m_k and m_l are distinct minimal generators of I with $i < j < k < l$. If $\sigma_{ij}, \sigma_{ik}, \sigma_{jk}, \sigma_{il}, \sigma_{jl}, \sigma_{kl} \in S_2$ and m_{ijk}, m_{ijl} and m_{ikl} all strictly divide m_{jkl} then $\tau_{jkl} \in \mathfrak{m}Z_3$.*

Proof: First since m_{ijk}, m_{ijl} and m_{ikl} all strictly divide m_{jkl} then,

$$\frac{m_{jkl}}{m_{ijk}}, \frac{m_{jkl}}{m_{ijl}}, \frac{m_{jkl}}{m_{ikl}} \in \mathfrak{m}.$$

Thus we have that,

$$\begin{aligned}
&\frac{m_{jkl}}{m_{ijk}}\tau_{ijk} - \frac{m_{jkl}}{m_{ijl}}\tau_{ijl} + \frac{m_{jkl}}{m_{ikl}}\tau_{ikl} \\
&\quad = \frac{m_{jkl}}{m_{ijk}}\left(\frac{m_{ijk}}{m_{ij}}e_{|\sigma_{ij}|} - \frac{m_{ijk}}{m_{ik}}e_{|\sigma_{ik}|} + \frac{m_{ijk}}{m_{jk}}e_{|\sigma_{jk}|}\right) - \\
&\qquad \frac{m_{jkl}}{m_{ijl}}\left(\frac{m_{ijl}}{m_{ij}}e_{|\sigma_{ij}|} - \frac{m_{ijl}}{m_{il}}e_{|\sigma_{il}|} + \frac{m_{ijl}}{m_{jl}}e_{|\sigma_{jl}|}\right) + \\
&\qquad\quad \frac{m_{jkl}}{m_{ikl}}\left(\frac{m_{ikl}}{m_{ik}}e_{|\sigma_{ik}|} - \frac{m_{ikl}}{m_{il}}e_{|\sigma_{il}|} + \frac{m_{ikl}}{m_{kl}}e_{|\sigma_{kl}|}\right)
\end{aligned}$$

$$= \frac{m_{jkl}}{m_{jk}}e_{|\sigma_{jk}|} - \frac{m_{jkl}}{m_{jl}}e_{|\sigma_{jl}|} + \frac{m_{jkl}}{m_{kl}}e_{|\sigma_{kl}|} = \tau_{jkl}. \quad \square$$

In general it is not difficult to see that $\{\tau_{ijk}|\sigma_{ij}, \sigma_{ik}, \sigma_{jk} \in S_2 \text{ for all } i < j < k\}$ generates Z_3 for any monomial ideal I. However, all of the second syzygies involved with an arbitrary τ_{ijk} may not be minimal. If this occurs we must simply replace the non-minimal second syzygy with a linear combination of two minimal second syzygies that generate it, as described in Remark 2.2.8. Since by definition $Z_3 \subseteq R^{m+n-1}$ the second syzygies corresponding to τ_{jkl} must be minimal to be able to say that $\tau_{jkl} \in \mathfrak{m}Z_3$. Because of this we could just assume that σ_{jk}, σ_{jl} and σ_{kl} are minimal in Lemma 3.3.7 and get the same result.

Lemma 3.3.8. *Let I be a generic monomial ideal of R, then the matrix of f_3 in Definition 2.2.6 contains only pure powers of x, y, and z.*

The proof of this lemma is a consequence of Proposition 3.4.1 (which we will prove in the next section) and Definition 3.3.6.

This lemma tells us that each column in f_3 will have exactly three nonzero entries of the form $(\pm)x^{a'}$, $(\pm)y^{b'}$ and $(\pm)z^{c'}$ where $a', b', c' > 0$. From this we can also determine exactly when we get rows in f_3 which contain only elements from the ideal I. First we will see what conditions must be satisfied for us to get nonzero entries in f_3 from I. By applying the previous lemma it is clear that if x^a, y^b, and z^c are the minimal pure power generators of I, that these are the only possible nonzero entries that we can have in f_3 from I. So, if we wanted to have x^a as an entry in f_3, we would need a set of three minimal generators, $\{x^a, y^{b_1}z^{c_1}, y^{b_2}z^{c_2}\}$, $b_i, c_i \geq 0$, which corresponds to a minimal third syzygy. Notice that $[x^a, y^{b_1}z^{c_1}, y^{b_2}z^{c_2}] = x^a y^{b'} z^{c'}$ where $b' = \max\{b_1, b_2\}$ and $c' = \max\{c_1, c_2\}$. Here we get x^a as an entry in this third syzygy from the following computation,

$$\frac{[x^a, y^{b_1}z^{c_1}, y^{b_2}z^{c_2}]}{[y^{b_1}z^{c_1}, y^{b_2}z^{c_2}]} = \frac{x^a y^{b'} z^{c'}}{y^{b'} z^{c'}} = x^a.$$

Now we know what is required to obtain a nonzero entry from I in f_3. The following lemma will show that when we get nonzero entries, from I in f_3, then we will not get any other nonzero entries from I in f_3, when I is generic.

Lemma 3.3.9. *Let I be a generic monomial ideal of R, then if the matrix of f_3 in Definition 2.2.6 contains nonzero entries from I, each such entry must be the same.*

Proof: To show this we will consider different possibilities for nonzero pure power generators to appear in f_3 and eliminate all these possibilities except for what is stated in the lemma. First, if one column contains all three pure power generators we will see that this implies this is the only column in f_3, which implies that S is Gorenstein. For sake of contradiction suppose that there is more than one column in f_3. This means we must have that I is minimally generated by at least four monomials. To simplify calculation we will order our generators so that $m_1 = x^a$, $m_2 = y^b$, $m_3 = z^c$ and $m_4 = x^\alpha y^\beta z^\gamma$ such that $\alpha < a$, $\beta < b$ and $\gamma < c$, where at least two of these degrees are positive. Since I is generic we know that τ_{123}, τ_{124}, τ_{134} and τ_{234} are all in Z_3 but not necessarily in S_3. It is clear that m_{124}, m_{134} and m_{234} all strictly divide m_{123}, which implies that $\tau_{123} \in \mathfrak{m}Z_3$ by Lemma 3.3.7. Thus $\tau_{123} \notin S_3$, which implies that if f_3 contains a column with all three pure power generators this must be the only column in f_3.

Next we will show that a single column cannot contain two different pure power generators because this would require that either more than three minimal generators had to correspond to this minimal third syzygy (which cannot happen since I is generic) or S is Gorenstein. Without loss of generality suppose that we have x^a and y^b as pure power entries in the same column of f_3. Then we know that we must have another generator with nonzero degrees only on y and z to get x^a as an entry, and a generator with nonzero degrees only on x and z to get y^b as an entry. Since I is generic we can only have one other generator corresponding to this third syzygy by

Proposition 3.3.4 and Definition 3.3.6. Thus this third generator would just be z^c, which would imply that S is Gorenstein by the previous argument.

We will now show that it is not possible to get two different pure power generators in two different columns of f_3. Suppose that x^a is an entry in one column of f_3 and y^b is an entry in another column of f_3. Then we must have two sets of minimal generators $\{x^a, y^{b_1}z^{c_1}, y^{b_2}z^{c_2}\}$, $b \geq b_1 > b_2 \geq 0$, $c_2 > c_1 \geq 0$ and $\{y^b, x^{a_1}z^{c_3}, x^{a_2}z^{c_4}\}$, $a \geq a_1 > a_2 \geq 0$, $c_4 > c_3 \geq 0$ that correspond respectively to these minimal third syzygies. Here we must have that either $c_1 \geq c_3$ or $c_3 \geq c_1$, choosing either case will yield a similar contradiction. Assume that $c_1 \geq c_3$, then

$$[x^a, x^{a_1}z^{c_3}]|_{<}[x^a, y^{b_1}z^{c_1}] \text{ and } [x^{a_1}z^{c_3}, y^{b_1}z^{c_1}]|_{<}[x^a, y^{b_1}z^{c_1}].$$

Thus there is no minimal second syzygy between x^a and $y^{b_1}z^{c_1}$ by Lemma 2.2.7, which implies the minimal generators $\{x^a, y^{b_1}z^{c_1}, y^{b_2}z^{c_2}\}$ cannot correspond to a minimal third syzygy. Therefore we cannot get two different pure power generators as entries in two different columns of f_3. Thus our only other option is that we may have entries from one of the pure power generators of I in f_3. □

We may want to note that we could have also looked at the proof of Lemma 3.3.9 by analyzing the planar graph representation of the resolution. First to get all three pure power entries we would only be able to have one face on the graph in which the vertices would be the pure power generators, which implies S is Gorenstein. To get two different pure power entries in the same column, we would need at least four edges which cannot happen since the graphs associated with generic resolutions are planar triangulations. In Section 3.4 we will see that this is possible when I is not generic. To get two different entries in different columns we would describe two different faces such that both have a different pure power vertex and the other two vertices in the face involve only the other two variables. But this would contradict the planarity of the graph since we would have at least two of the edges crossing at a location where there is not a vertex on the graph. The following example illustrates the previous two lemmas.

Example 3.3.10. Let $I = \langle x^4, x^2y^2, xy^3, y^4, x^3z, z^5 \rangle$, which is generic. Then the free resolution for S obtained from Definition 2.2.6 is,

$$0 \longrightarrow R^4 \xrightarrow{\begin{bmatrix} z & 0 & 0 & 0 \\ -y^2 & 0 & 0 & 0 \\ 0 & z^5 & 0 & 0 \\ x & 0 & z^4 & 0 \\ 0 & -y & -x & 0 \\ 0 & 0 & 0 & z^5 \\ 0 & x & 0 & -y \\ 0 & 0 & 0 & x \\ 0 & 0 & y^2 & 0 \end{bmatrix}} R^9 \xrightarrow{\begin{bmatrix} -y^2 & -z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x^2 & 0 & -y & -xz & -z^5 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & -y & -z^5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & -z^5 & 0 \\ 0 & x & 0 & y^2 & 0 & 0 & 0 & 0 & -z^4 \\ 0 & 0 & 0 & 0 & x^2y^2 & 0 & xy^3 & y^4 & x^3 \end{bmatrix}} R^6 \longrightarrow \cdots$$

In this example we have two occurrences of z^5 in f_3, which is also a minimal generator of I. We obtained these two entries from the sets $\{x^2y^2, xy^3, z^5\}$ and $\{xy^3, y^4, z^5\}$ which correspond to τ_{236} and τ_{346} respectively. Using the previous two lemmas we can now find a bound on the maximum number of nonzero entries allowed in f_3 which are also in I.

Theorem 3.3.11. *Let I be a generic monomial ideal of R, minimally generated by n elements, then the matrix of f_3 in Definition 2.2.6 will contain at most $n-2$ nonzero entries from I.*

Proof: Using Lemmas 3.3.8 and 3.3.9 we need only show that the maximum number of entries in f_3 for one of the pure power generators of I is $n-2$. To do this we will show that there is no possible construction of I where we can get more than $n-2$ entries in f_3. We will first show exactly which combination of generators will give us entries from I in f_3. Without loss of generality assume x^a is the entry we get in f_3. Let I be generic monomial ideal such that $\{x^a, y^{b_1}z^{c_1}, y^{b_2}z^{c_2}, \ldots, y^{b_r}z^{c_r}\}$ are minimal generators of I with $b_r = c_1 = 0$, $b_{i-1} > b_i > 0$, and $0 < c_i < c_{i+1}$ for all $2 \leq i \leq r-1$. Here we are not necessarily assuming that these are the only minimal generators of I, we are just picking out x^a and all generators with nonzero degrees only on y and z. Notice that this still satisfies the blanket condition that I is $\mathfrak{m}$-primary, since $y^{b_1}z^{c_1} = y^{b_1} = y^b$ and $y^{b_r}z^{c_r} = z^{c_r} = z^c$ by assumption. Now we will show that the only possible third syzygies that can give us an x^a in f_3 must involve three of these generators in the form, $\{x^a, y^{b_i}z^{c_i}, y^{b_{i+1}}z^{c_{i+1}}\}$ such that $1 \leq i \leq r-1$. Suppose the contrary, that the set $\{x^a, y^{b_j}z^{c_j}, y^{b_i}z^{c_i}\}$ with $i, j \in \{1, \ldots, r\}$ and $|i-j| \geq 2$, corresponds to a minimal third syzygy. Then there are two cases, that $i < j$ or $i > j$. First suppose that $i < j$, then since $b_i > b_{i+1} > b_j$ and $c_i < c_{i+1} < c_j$ we have that both

$$[y^{b_i}z^{c_i}, y^{b_{i+1}}z^{c_{i+1}}] = y^{b_i}z^{c_{i+1}} \text{ and } [y^{b_{i+1}}z^{c_{i+1}}, y^{b_j}z^{c_j}] = y^{b_{i+1}}z^{c_j},$$

strictly divide $[y^{b_i}z^{c_i}, y^{b_j}z^{c_j}] = y^{b_i}z^{c_j}$. Thus $\{y^{b_i}z^{c_i}, y^{b_j}z^{c_j}\}$ cannot correspond to a minimal second syzygy and hence $\{x^a, y^{b_j}z^{c_j}, y^{b_i}z^{c_i}\}$ cannot correspond to a minimal third syzygy. A similar argument may be used when $i > j$. Taking all possible sets satisfying our conditions we get,

$$\{x^a, y^{b_1}z^{c_1}, y^{b_2}z^{c_2}\}, \{x^a, y^{b_2}z^{c_2}, y^{b_3}z^{c_3}\}, \ldots, \{x^a, y^{b_{r-1}}z^{c_{r-1}}, y^{b_r}z^{c_r}\}$$

are the only possible sets that can correspond to minimal third syzygies with x^a as an entry, and there are exactly $r-1$ of these sets. Now it is easy to see that we cannot get more than $n-2$ of these sets for an ideal generated by n elements. This is because we have r minimal generators with nonzero degrees only on y and z, which gives at most $r-1$ entries with x^a and we must have a minimum of $r+1$ minimal generators for I. If we wanted greater than or equal to $n-1$ copies of x^a in f_3, then $r-1 \geq n-1 \Longrightarrow r \geq n$, but this implies that I must have at least $n+1$ generators, which is a contradiction. Thus we can only have at most $n-2$ copies of x^a in f_3, which completes our proof. $\square$

Example 3.3.12. Let $I = \langle x^5, x^4y, x^2y^3, xy^4, y^5, z^5 \rangle$, which is generic. Then the free resolution for S obtained from Definition 2.2.6 is,

$$0 \longrightarrow R^4 \xrightarrow{\begin{bmatrix} z^5 & 0 & 0 & 0 \\ -y & 0 & 0 & 0 \\ 0 & z^5 & 0 & 0 \\ x & -y^2 & 0 & 0 \\ 0 & 0 & z^5 & 0 \\ 0 & x^2 & -y & 0 \\ 0 & 0 & 0 & z^5 \\ 0 & 0 & x & -y \\ 0 & 0 & 0 & x \end{bmatrix}} R^9 \xrightarrow{\begin{bmatrix} -y & -z^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x & 0 & -y^2 & -z^5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x^2 & 0 & -y & -z^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & -y & -z^5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & -z^5 \\ 0 & x^5 & 0 & x^4y & 0 & x^2y^3 & 0 & xy^4 & y^5 \end{bmatrix}} R^6 \longrightarrow \cdots$$

Here we see that we can get $n-2$ nonzero entries from I in f_3 and that we also have exactly $n-2$ rows in f_3 with nonzero entries from I. Theorem 3.3.11 shows that $n-2$ is an upper bound on the number of entries we can get from I in f_3 when I is generic. In Example 3.3.12 we also achieve $2n-2$ nonzero entries in f_2 which are also elements of I. In Proposition 3.2.1 we proved that this is an upper bound of such entries. If we achieve $n-2$ nonzero entries in f_3 from I, then we can see from the proof of Theorem 3.3.11 that we have $n-1$ minimal second syzygies with nonzero entries from I. Thus if I is generic with $n \geq 5$ we get $n-2$ nonzero entries in f_3 from I if and only if we get $2n-2$ nonzero entries in f_2 from I. The reason we must have that $n \geq 5$ is because if $I = \langle x^a, y^b, z^c, x^\alpha y^\beta z^\gamma \rangle$ with positive degrees on all variables then we get exactly $2n-2=6$ nonzero entries from I in f_2 but no entries from I in f_3. This is the only case where we can achieve the upper bound on the number of nonzero entries from I and f_2 and not achieve the upper bound in f_3.

Remark 3.3.13. Something that is also of interest is what are the various numbers of nonzero entries from I can we have in f_3? Though we don't have a complete answer to this question yet, we found something surprising when looking at this for generic monomial ideals. If I is generic, generated by n elements, it seems that we are unable to have $n-3$ nonzero entries from I in f_3. However, it does seem possible to get any other number of such entries less than or equal to $n-2$. The easiest way to look at this is by trying to construct a planar graph with n vertices, which has $n-3$ faces corresponding to a single pure power generator and two generators involving only the other two variables. When we attempt to do this it seems that it can only be done if our planar graph is not a planar triangulation, which would imply that I is not generic. Even for non-generic monomial ideals it is unclear as to whether or not we can get $n-3$ nonzero entries from I in f_3. To illustrate this further we will provide some examples of generic ideals generated by 6 elements where we get 4,2,1, and 0 nonzero entries from I in f_3.

Table 3.1: Possible number of nonzero entries from I in f_3

I	$\langle x^4, y^4, y^3z, y^2z^2, yz^3, z^4\rangle$	$\langle x^4, y^4, y^3z, y^2z^2, xz^3, z^4\rangle$	$\langle x^4, y^4, y^3z, x^2z^2, yz^3, z^4\rangle$	$\langle x^4, x^3y, y^4, y^3z, xz^3, z^4\rangle$
f_3	$\begin{bmatrix} z & 0 & 0 & 0 \\ -y & z & 0 & 0 \\ 0 & -y & z & 0 \\ 0 & 0 & -y & z \\ 0 & 0 & 0 & -y \\ x^4 & 0 & 0 & 0 \\ 0 & x^4 & 0 & 0 \\ 0 & 0 & x^4 & 0 \\ 0 & 0 & 0 & x^4 \end{bmatrix}$	$\begin{bmatrix} z & 0 & 0 & 0 \\ -y & z & 0 & 0 \\ 0 & -y & z & 0 \\ 0 & 0 & -y^2 & 0 \\ x^4 & 0 & 0 & 0 \\ 0 & x^4 & 0 & 0 \\ 0 & 0 & x^3 & z \\ 0 & 0 & 0 & -x \\ 0 & 0 & 0 & y^2 \end{bmatrix}$	$\begin{bmatrix} z & 0 & 0 & 0 \\ -y & z & 0 & 0 \\ 0 & -y^3 & 0 & 0 \\ x^4 & 0 & 0 & 0 \\ 0 & x^2 & z & 0 \\ 0 & 0 & -x^2 & 0 \\ 0 & 0 & y^2 & z \\ 0 & 0 & 0 & -y \\ 0 & 0 & 0 & x^2 \end{bmatrix}$	$\begin{bmatrix} z^3 & 0 & 0 & 0 \\ -y & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & -y & z & 0 \\ x & 0 & -y^2 & 0 \\ 0 & x^3 & 0 & 0 \\ 0 & 0 & x^2 & z \\ 0 & 0 & 0 & -x \\ 0 & 0 & 0 & y^3 \end{bmatrix}$

It seems fairly clear that we can have any number of nonzero entries from I in f_3 less than or equal to $n-2$ except for $n-3$, when I is generic. However, it is unclear as to how we would prove this, so we offer it as a conjecture.

Conjecture 3.3.14. *Let $\hat{r}$ be the number of nonzero entries from I in the matrix of f_3 from Definition 2.2.6. Then there exists a generic monomial ideal I, such that $\hat{r}$ can be any nonnegative integer $\leq n-2$, with the single exception that $\hat{r} \neq n-3$.*

3.4 Non-Generic Resolutions

In this section we will consider some of the differences between the resolutions of S when I is *not* generic in comparison to resolutions when I is generic. The main theorem of the section will show that when I is not generic we will never have the same structure on the matrix of f_3 as described in the previous section for a generic ideal. Specifically, if I is not generic then there will be at least one column in f_3 which contains more than three nonzero entries. We will also see the interesting nature of these results by considering examples where we relax the conditions that I be a *trivariate* monomial ideal that is $\mathfrak{m}$-*primary*. When we remove these restrictions on I we will find that many of our results from this section do not hold.

Proposition 3.4.1. *Let $I = \langle m_1, \ldots, m_n \rangle$ be a monomial ideal. If a column in f_3 from Definition 2.2.6 has exactly three nonzero entries then these entries must all be pure power entries.*

Proof: Suppose τ_{ijk} is a minimal third syzygy. Then by definition,

$$\tau_{ijk} = \frac{m_{ijk}}{m_{ij}} e_{|\sigma_{ij}|} - \frac{m_{ijk}}{m_{ik}} e_{|\sigma_{ik}|} + \frac{m_{ijk}}{m_{jk}} e_{|\sigma_{jk}|}, \text{ such that } 1 \leq i < j < k \leq n.$$

Each of the nonzero entries can be described using the following form,

$$\frac{m_{ijk}}{m_{ij}} = x^{\alpha_1} y^{\beta_1} z^{\gamma_1}, \ \frac{m_{ijk}}{m_{ik}} = x^{\alpha_2} y^{\beta_2} z^{\gamma_2}, \text{ and } \frac{m_{ijk}}{m_{jk}} = x^{\alpha_3} y^{\beta_3} z^{\gamma_3}.$$

Notice that it is not possible for $\alpha_l, \beta_l, \gamma_l > 0$ for $l \in \{1, 2, 3\}$, because this would imply that at least one of our generators was not minimal. Thus we will assume that two of the powers are nonzero for one of the entries in τ_{ijk}. Without loss of generality suppose that $\dfrac{m_{ijk}}{m_{ij}} = x^{\alpha_1}y^{\beta_1}$ with $\alpha_1, \beta_1 > 0$. This implies that $a_k > a_i, a_j$ and $b_k > b_i, b_j$. From this we can compute the other two nonzero entries in τ_{ijk} to be

$$\frac{m_{ijk}}{m_{ik}} = z^{\gamma_2} \text{ and } \frac{m_{ijk}}{m_{jk}} = z^{\gamma_3}.$$

Now we have that $z^{\gamma_2}m_{ik} = z^{\gamma_3}m_{jk}$ which means that either $m_{ik}|m_{jk}$ or $m_{jk}|m_{ik}$. Suppose that $m_{ik}|m_{jk}$, then $m_{ij}|_{<}m_{jk}$ since $a_k > a_i, a_j$ and $b_k > b_i, b_j$. We now have two possibilities, either, $m_{ik} = m_{jk}$ or m_{ik} strictly divides m_{jk}. If $m_{ik} = m_{jk}$, then $\sigma_{ik} \notin S_2$ by Definition 2.2.6 and Lemma 2.2.7. If m_{ik} strictly divides m_{jk} then $\sigma_{jk} \notin S_2$ by Lemma 2.2.7. Both cases contradict our assumption that τ_{ijk} is a minimal third syzygy. Thus we can have no mixed entries in τ_{ijk} and in turn all nonzero entries in τ_{ijk} are pure powers. $\square$

This proposition tells us that to get nonzero mixed entries in a column of f_3 that we must have at least four nonzero entries in the column. Another interesting fact that we obtain in the proof of this proposition is if we have a column in f_3 with exactly three nonzero entries, then we will have a pure power entry for each variable. It is not true that having four or more nonzero entries in a column of f_3 implies that we get a mixed entry. For this we can just look at resolutions where I is a power of the homogeneous maximal ideal of R.

Example 3.4.2. Let $I = \mathfrak{m}^2$, and $J = \langle x^3, x^2y, y^3, z^3, x^2z^2 \rangle$, neither of which are generic. Then a minimal free resolution for R/I obtained from Definition 2.2.6 is,

$$0 \to R^3 \xrightarrow{\begin{bmatrix} z & 0 & 0 \\ -y & 0 & 0 \\ 0 & z & 0 \\ x & -y & 0 \\ 0 & x & 0 \\ -x & 0 & z \\ 0 & 0 & -y \\ 0 & 0 & x \end{bmatrix}} R^8 \xrightarrow{\begin{bmatrix} -y & -z & 0 & 0 & 0 & 0 & 0 & 0 \\ x & 0 & -y & -z & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & -z & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 & -y & -z & 0 \\ 0 & 0 & 0 & x & y & x & 0 & -z \\ 0 & 0 & 0 & 0 & 0 & 0 & x & y \end{bmatrix}} R^6 \xrightarrow{\begin{bmatrix} x^2 & xy & y^2 & xz & yz & z^2 \end{bmatrix}} R \to \cdots$$

and a minimal resolution for R/J obtained from Definition 2.2.6 is,

$$0 \to R^2 \xrightarrow{\begin{bmatrix} z^2 & 0 \\ -y & 0 \\ 0 & z^3 \\ x & -y^2z \\ 0 & x^2 \\ 0 & y^3 \end{bmatrix}} R^6 \xrightarrow{\begin{bmatrix} -y & -z^2 & 0 & 0 & 0 & 0 \\ x & 0 & -y^2 & -z^2 & 0 & 0 \\ 0 & 0 & x^2 & 0 & -z^3 & 0 \\ 0 & 0 & 0 & 0 & y^3 & -x^2 \\ 0 & x & 0 & y & 0 & z \end{bmatrix}} R^5 \xrightarrow{\begin{bmatrix} x^3 & x^2y & y^3 & z^3 & x^2z^2 \end{bmatrix}} R \to \cdots$$

As we can see in the resolution of R/I all the nonzero entries in f_3 are pure powers but we have a column with four such entries. In the resolution of R/J we observe that there is a column in f_3 with four nonzero entries and that one of these entries is the mixed double, $-y^2z$. Something else we can observe is that this entry is in row four of f_3 which corresponds to σ_{26} in f_2. This minimal second syzygy is obtained from the two minimal generators x^2y and x^2z^2, which have the same nonzero degree on x. In a generic ideal we would be guaranteed the existence of another minimal generator that would strongly divide the $[x^2y, x^2z^2]$, so that σ_{26} would not be minimal. The fact that this does not happen here is what leads to the presence of the entry $-y^2z$ in row four of f_3. A consequence we can observe from this is that any column in f_3 that contains four or more nonzero entries must involve a minimal second syzygy, σ_{ij} such that m_i and m_j have the same nonzero degree on some variable. However the converse of this is not true. It is possible to have a resolution so that a column in f_3 has exactly three nonzero entries and there is a corresponding minimal second syzygy, σ_{ij} where m_i and m_j have the same nonzero degree on some variable. This can be seen in the first column of f_3 for the resolution of R/J in Example 3.4.2. We can say that if I is not generic, there will be at least two minimal generators m_i and m_j with the same nonzero degree on some variable such that $\sigma_{ij} \in S_2$.

Lemma 3.4.3. *If $I = \langle m_1, \ldots, m_n \rangle$ is not generic then there is at least two minimal generators m_i and m_j with the same nonzero degree in some variable such that $\sigma_{ij} \in S_2$.*

We make note that Lemma 3.4.3 follows from [20, 6.26] where it lists equivalent definitions for I to be generic. Specifically, I is generic if and only if the algebraic Scarf complex is a free resolution of S, and every edge (i, j) of $\triangle_I$ is such that m_i and m_j do not have the same nonzero degree on the same variable. We will however provide a proof of Lemma 3.4.3 using the construction given in Definition 2.2.6 and Lemma 2.2.7.

Proof: To prove this we will show that either σ_{ij} is indeed in S_2 or we may choose two other minimal generators satisfying our hypothesis, which will also satisfy the desired result. Without loss of generality assume that $a_i = a_j = a'$, $b_i > b_j$ and $c_j > c_i$. Also since I is not generic we may assume that there is no minimal generator m_k such that $m_k \| m_{ij}$. If σ_{ij} does not satisfy condition (1) in Definition 2.2.6 then it does not satisfy condition (2). We will show that if σ_{ij} does not satisfy each of the conditions in Definition 2.2.6 that we will be able to find another pair of minimal generators which satisfy the desired result.

First suppose that $\sigma_{ij} \in \mathfrak{m}Z_2$. By Lemma 2.2.7 there exists a minimal generator m_k such that $m_{ik}|_< m_{ij}$. This implies that either $\max\{b_i, b_k\} < \max\{b_i, b_j\}$ or $\max\{c_i, c_k\} < \max\{c_i, c_j\}$ since $a_i = a_j$. But since $b_i > b_j$ by assumption we must have that $\max\{c_i, c_k\} < \max\{c_i, c_j\}$, which implies that $c_k < c_j$. We now have two possibilities either $b_i = b_k$ or $b_i > b_k$. If $b_i = b_k$ then $m_{ij} = m_{jk}$. We could now choose m_i and m_k to satisfy the conditions of our hypothesis. If $b_i > b_k$ we may assume that $a_k = a'$, since $m_k \| m_{ij}$ if $a_k < a'$. This implies that $b_i > b_k > b_j$ and $c_j > c_k > c_i$ which in turn implies that $m_{ik}|_< m_{ij}$ and $m_{jk}|_< m_{ij}$. We can now see that either m_i

and m_k or m_j and m_k would satisfy the conditions of our hypothesis and we may change our original choice of generators.

If condition (1) of Definition 2.2.6 is satisfied and (2) is not then we would have that there exists a minimal generator m_k such that $m_k|_< m_{ij}$ which implies that $m_{ik}|m_{ij}$. If $m_{ik}|_< m_{ij}$ then our argument is the same as it was above. If $m_{ik} = m_{ij}$, then $c_k = c_j = c'$. This implies that $a_k < a'$ and $b_i > b_k > b_j$ and hence $m_{jk} = x^{a'}y^{b_k}z^{c'}$ which strictly divides m_{ik} and m_{ij}. Thus we may change our choice of minimal generators to m_j and m_k to satisfy the conditions of our hypothesis.

In summary the previous arguments show that if $\sigma_{ij} \notin S_2$ we can always find two other minimal generators which satisfy the conditions of our hypothesis. Since I is finitely generated we may apply all of the above arguments above inductively on the two minimal generators chosen to satisfy the conditions of our hypothesis, so that eventually we will be able to find two minimal generators $m_{i'}$ and $m_{j'}$, with the same positive degree in some variable such that there will not exist a minimal generator $m_{k'}$ such that $m_{k'}|_< m_{i'j'}$. Thus $\sigma_{i'j'} \in S_2$ by Lemma 2.2.7 . $\square$

We can now give a description of the structure of f_3 from Definition 2.2.6 when I is not generic.

Theorem 3.4.4. *If $I = \langle m_1, \ldots, m_n \rangle$ is not generic then there is at least one column in the matrix of f_3 from Definition 2.2.6, which contains more than three nonzero entries.*

Proof: Since I is not generic there are at least two minimal generators m_i and m_j with the same nonzero degree on the same variable such that $\sigma_{ij} \in S_2$ by Lemma 3.4.3. Suppose that $a_i = a_j = a'$, then $b_i > b_j$ and $c_i < c_j$ which gives that $m_{ij} = x^{a'}y^{b_i}z^{c_j}$. We notice that σ_{ij} will correspond to two different minimal third syzygies. This is due to the fact that σ_{ij} cannot lie on the outer boundary of the planar graph associated with the resolution of S. If this were to occur it would mean that m_{ij} would only have nonzero powers on exactly two variables, which implies that m_i and m_j cannot have the same positive degree on some variable. We will assume that both of the minimal third syzygies associated with σ_{ij} contain exactly three entries and get a contradiction. First let τ_{ijk} and τ_{ijl} be minimal third syzygies. These two third syzygies are constructed from five minimal second syzygies, $\sigma_{ij}, \sigma_{ik}, \sigma_{jk}, \sigma_{il},$ and σ_{jl}. We would only need to show that one of these second syzygies is not in S_2 or that either τ_{ijk} or τ_{ijl} is not in S_3 to get a contradiction. It is clear that $a_k, a_l \neq a'$ since this would contradict the minimality of one of our second syzygies. We have three cases that we must consider, either $a_k > a'$ and $a_l < a'$, $a_k, a_l > a'$ or $a_k, a_l < a'$. The case of $a_k < a'$ and $a_l > a'$ would be exactly the same proof as $a_k > a'$ and $a_l < a'$.

(1): Let $a_k > a'$ and $a_l < a'$. Then we have the following nonzero entries for τ_{ijk},

$$\frac{m_{ijk}}{m_{ij}} = x^{\alpha_1}, \quad \frac{m_{ijk}}{m_{ik}} = z^{\gamma_1}, \quad \frac{m_{ijk}}{m_{jk}} = y^{\beta_1}.$$

Before we write the nonzero entries for τ_{ijl} we observe that since $a_l < a'$ we must have that either $b_l > b_i$ or $c_l > c_j$. Without loss of generality say $b_l > b_i$, then the nonzero entries for τ_{ijl} will be

$$\frac{m_{ijl}}{m_{ij}} = y^{\beta_2}, \quad \frac{m_{ijl}}{m_{il}} = z^{\gamma_2}, \quad \frac{m_{ijl}}{m_{jl}} = z^{\hat{\gamma_2}}.$$

This implies that $m_{il}z^{\gamma_2} = m_{jl}z^{\hat{\gamma_2}}$ and using the same technique in the proof of Proposition 3.4.1 we find that either σ_{il} or σ_{jl} would not be minimal. Thus τ_{ijl} is not a minimal third syzygy.

(2): Let $a_k, a_l > a'$. Recall from Lemmas 2.1.1 and 2.2.7 if there is a minimal generator $m_k|_< m_{ij}$ such that $m_{ik}|_< m_{ij}$ and $m_{jk}|_< m_{ij}$ then $\sigma_{ij} \notin S_2$. To maintain that both τ_{ijk} and τ_{ijl} are minimal we may assume that no two of these four minimal generators divides the least common multiple of the other two. We will now construct the least common multiples for all five of the second syzygies we need here:

$$m_{ij} = x^{a'} y^{b_i} z^{c_j},$$
$$m_{ik} = x^{a_k} y^{\max\{b_i,b_k\}} z^{\max\{c_i,c_k\}},$$
$$m_{jk} = x^{a_k} y^{\max\{b_j,b_k\}} z^{\max\{c_j,c_k\}},$$
$$m_{il} = x^{a_l} y^{\max\{b_i,b_l\}} z^{\max\{c_i,c_l\}},$$
$$m_{jl} = x^{a_l} y^{\max\{b_j,b_l\}} z^{\max\{c_j,c_l\}}.$$

Note that it is implied that m_k and m_l do not divide m_{ij} since $a_k, a_l > a'$. We will now simplify the above least common multiples with the following:

(i) m_j does not divide m_{ik} implies that $c_k < c_j$ since $b_j < \max\{b_i, b_k\}$,

(ii) m_i does not divide m_{jk} implies that $b_k < b_i$ since $c_i < \max\{c_j, c_k\}$,

(iii) m_j does not divide m_{il} implies that $c_l < c_j$ since $b_j < \max\{b_i, b_l\}$,

(iv) m_i does not divide m_{jl} implies that $b_l < b_i$ since $c_i < \max\{c_j, c_l\}$,

(v) m_l does not divide m_{ik} implies that either $a_l > a_k$ or $c_l > \max\{c_i, c_k\}$,

(vi) m_l does not divide m_{jk} implies that either $a_l > a_k$ or $b_l > \max\{b_j, b_k\}$,

(vii) m_k does not divide m_{il} implies that either $a_k > a_l$ or $c_k > \max\{c_i, c_l\}$,

(viii) m_k does not divide m_{jl} implies that either $a_k > a_l$ or $b_k > \max\{b_j, b_l\}$.

From this we have that:

$$m_{ik} = x^{a_k} y^{b_i} z^{\max\{c_i,c_k\}},$$
$$m_{jk} = x^{a_k} y^{\max\{b_j,b_k\}} z^{c_j},$$
$$m_{il} = x^{a_l} y^{b_i} z^{\max\{c_i,c_l\}},$$
$$m_{jl} = x^{a_l} y^{\max\{b_j,b_l\}} z^{c_j}.$$

If $a_l > a_k$ then we have that $m_{ijk} = x^{a_k} y^{b_i} z^{c_j}, m_{ikl} = x^{a_l} y^{b_i} z^{\max\{c_i,c_k,c_l\}}$ and $m_{jkl} = x^{a_l} y^{\max\{b_j,b_k,b_l\}} z^{c_j}$ all strictly divide $m_{ijl} = x^{a_l} y^{b_i} z^{c_j}$, which implies that $\tau_{ijl} \in \mathfrak{m}Z_3$ by Lemma 3.3.7. Similarly if $a_k > a_l$ we will have that m_{ijl}, m_{ikl} and m_{jkl} all strictly divide m_{ijk} implying that τ_{ijk} is not minimal. We can see that $a_k \neq a_l$ because this would imply that $c_l > \max\{c_i, c_k\}, b_l > \max\{b_j, b_k\}, c_k > \max\{c_i, c_l\}$ and $b_k >$

$\max\{b_j, b_l\}$ from conditions (v) - (viii), which cannot happen. In any of these cases we have that either τ_{ijk} or τ_{ikl} is not minimal which contradicts our assumption.

(3): Let $a' > a_k, a_l$. Without loss of generality we must have that some of the exponents differ on m_k and m_l, say $b_k > b_l$. Then we have that $m_{jl} = x^{a'}y^{b_l}z^{c_j}$ and $m_{kl} = x^{\max\{a_k,a_l\}}y^{b_k}z^{\max\{c_k,c_l\}}$ both strictly divide $m_{jk} = x^{a'}y^{b_k}z^{c_j}$. Thus by lemma 2.2.7 $\sigma_{jk} \in \mathfrak{m}Z_2$ and is not minimal which implies that τ_{ijk} is not minimal which is a contradiction. We would see a similar result for any other choice of exponents on m_k and m_l.

Thus we have shown for all cases that when I is not generic there are at least two minimal generators m_i and m_j of I, with the same nonzero degree in some variable, so that we cannot have two minimal third syzygies τ_{ijk} and τ_{ijl}. Since these two generators must correspond with two minimal third syzygies one of these third syzygies must have more than three nonzero entries. $\square$

The assumption that I is a trivariate monomial ideal that is $\mathfrak{m}$-primary is crucial for this theorem. The $\mathfrak{m}$-primary condition is what forces the minimal second syzygy σ_{ij} to correspond to exactly two minimal third syzygies, when m_i and m_j have the same positive degree on the same variable. If we remove the $\mathfrak{m}$-primary condition on I then Theorem 3.4.4 does not hold.

Example 3.4.5. Let $I = \langle x^4, x^3yz, x^3y^3, x^3z^3, y^3z^3\rangle$, which is not $\mathfrak{m}$-primary. Then the minimal free resolution for S obtained from Definition 2.2.6 is,

$$0 \longrightarrow R^2 \xrightarrow{\begin{bmatrix} y^2 & z^2 \\ -z & 0 \\ 0 & -y \\ x & 0 \\ 0 & x \\ 0 & 0 \end{bmatrix}} R^6 \xrightarrow{\begin{bmatrix} -yz & -y^3 & -z^3 & 0 & 0 & 0 \\ x & 0 & 0 & -y^2 & -z^2 & 0 \\ 0 & x & 0 & z & 0 & 0 \\ 0 & 0 & x & 0 & y & -y^3 \\ 0 & 0 & 0 & 0 & 0 & x^3 \end{bmatrix}} R^5 \longrightarrow \cdots$$

We observe that each column in f_3 contains exactly three nonzero pure power entries, but I is clearly not generic. The reason for this is that the minimal free resolution for S here is supported on a simplicial complex.

This theorem gives us the converse to what we already knew about generic monomial ideals. Specifically, that if I is generic then every column in f_3 has exactly three nonzero pure power entries. The following corollary gives us an alternate definition for an $\mathfrak{m}$-primary generic monomial ideal in three variables.

Corollary 3.4.6. *A monomial ideal $I = \langle m_1, \dots, m_n \rangle$, which is $\mathfrak{m}$-primary in R, is generic if and only if each column in the matrix of f_3 from Definition 2.2.6 contains exactly three nonzero entries.*

We also make an observation that these results only hold in general over *trivariate* monomial ideals. If $R = \Bbbk[x_1, \dots, x_r]$ with $r > 3$ and I was a monomial ideal which is $\mathfrak{m}$-primary, then S will have projective dimension r. Our original question of when we get entries in the matrices of f_i which are also elements of I becomes significantly more complicated when $r \geq 4$, even if I is generic. Although resolutions for generic monomial ideals do maintain some structure in general. Namely, the resolutions are simplicial, see [20].

Remark 3.4.7. To answer the question of whether the first Bass number of S is always larger than the zeroth Bass number of S, we refer back to Question 2.3.13. We recall that this question states, after permissible row operations, is the number of rows in f_3 which contain only entries from I, less than or equal to $n-2$? First we describe what we mean by permissable row operations. In general, we want to ensure that none of the rows in f_3 are dependant mod I. We say that the k^{th} row of f_3, denoted r_k, is dependent mod I if, $r_k - (a_1 r_1 + \cdots + a_{k-1} r_{k-1} + a_{k+1} r_{k+1} + \cdots + a_{m+n-1} r_{m+n-1}) \in IR^m$, for $a_i \in R$. When I is generic the row operations do not change the number of rows in f_3 from Definition 2.2.6, which are contained in I. This is due to the fact that each column of f_3 has exactly three nonzero pure power entries in each variable. Thus if a row in f_3 contains an element that is not already in I, say an $x^{a'}$, we would be unable to get this row to be contained in I by applying our row operations. If the other two nonzero entries in this column are $y^{b'}$ and $z^{c'}$ then the only option is to take $x^{a'} - (a_1 y^{b'} + a_2 z^{c'})$, which cannot be in I because $x^{a'}$ is not in I. So for resolutions given by generic monomial ideals we only need know that the number of rows in f_3 from Definition 2.2.6 which contain only entries from I is less than or equal to $n-2$. In Theorem 3.3.11 we showed that the maximum number of nonzero entries that we can get from I in f_3 will be exactly $n-2$. Thus we can conclude that the first Bass number of S is always larger than the zeroth Bass number of S, for rings defined by trivariate generic monomial ideals.

CHAPTER 4

TOR ALGEBRA STRUCTURE

In this Chapter we will discuss methods for computing the invariants defined in Definition 2.3.8. We will see how to compute these directly, and then find results relating these invariants to properties of the minimal free resolution of S from Definition 2.2.6. We will also state some more specific properties for A, which will aid in our computations. In Section 4.2 we will compute multiple examples of the Tor-algebra for a variety of monomial ideals. We will conclude by defining classes of monomial ideals which have the desired Tor-algebra structure, and provide very specific information about this structure for these ideals.

4.1 Computational Methods

In Section 2.3.5 we claimed that we would remove $\mathbf{C}(c)$ from our classes we are concerned with because this case was easy. Since S is in $\mathbf{C}(c)$ if and only if it is a complete intersection, proved by Assmus in [1], and for our ideals S is a complete intersection if and only if it is Gorenstein we can provide the complete classification for $\mathbf{C}(c)$ for trivariate monomial ideals. Specifically we know that $S = R/I$ is Gorenstein if and only if $I = \langle x^a, y^b, z^c \rangle$ with $a, b, c > 0$.

4.1.1 Direct Computation

Before we begin providing the computational methods we will use to compute the invariants of the Tor-algebra, we will provide an example of a direct computation for an ideal. We will compute $p = \operatorname{rank}_{\Bbbk}(A_1^2)$ and $q = \operatorname{rank}_{\Bbbk}(A_1 \cdot A_2)$ using multiplication tables after we have found a set of minimal generators for A_1 and A_2.

Example 4.1.1. Let $I = \langle x^3, x^2y, y^3, z^3, x^2z^2 \rangle$ and let $S = R/I$. We know that we can determine the number of minimal generators we will need for A_1, A_2, and A_3 by finding the ranks of the respective free modules in a minimal free resolution of S. Computing the minimal resolution from Definition 2.2.6 we have,

$$0 \longrightarrow R^2 \xrightarrow{\begin{bmatrix} z^2 & 0 \\ -y & 0 \\ 0 & z^3 \\ x & -y^2z \\ 0 & x^2 \\ 0 & y^3 \end{bmatrix}} R^6 \xrightarrow{\begin{bmatrix} -y & -z^2 & 0 & 0 & 0 & 0 \\ x & 0 & -y^2 & -z^2 & 0 & 0 \\ 0 & 0 & x^2 & 0 & -z^3 & 0 \\ 0 & 0 & 0 & 0 & y^3 & -x^2 \\ 0 & x & 0 & y & 0 & z \end{bmatrix}} R^5 \longrightarrow \cdots$$

We know that A_1 will always have the same number of minimal generators as I does, and in this case $\operatorname{rank}_{\Bbbk}(A_2) = 6$ and $\operatorname{rank}_{\Bbbk}(A_3) = 2$. It is possible to compute the generators for A_1, A_2, and A_3 in Macaulay 2 by computing the homology of the Koszul complex over S. Here we will explain how to do this by hand. As usual we must ensure that the set of generators we compute are independent, but we must also ensure that they in the kernel of the differentials on the Koszul complex K, which is the Koszul complex $K(x, y, z)$ given in Example 2.3.2 tensored with S. Also since each A_i represents the homology of K we must ensure that the generators we choose for each A_i are not dependent mod $\operatorname{im}(\varphi_{i+1})$ from (2.3.2). We will abuse notation slightly here saying that the differentials φ_i in (2.3.2) are the maps in K.

The generators of A_1 will be comprised of degree 1 elements in K which have the form m_je_i where m_j is a nonzero monomial in S and $i = 1, 2, 3$. To ensure that this element is in $\ker(\varphi_1)$ we must have that $\varphi_1(m_je_i) = 0$ for each generator we choose. Since $\varphi_1(m_je_i) = m_j\varphi_1(e_i) = m_j \cdot x_i$ this makes it fairly simple to find a minimal generating set for A_1. Specifically we find that,

$$A_1 = \langle x^2e_1, xye_1, xze_1, y^2e_2, z^2e_3\rangle.$$

It is easy to see that these are all independent with respect to each other but we must also make sure that they are independent with respect to $\mathrm{im}(\varphi_2)$. The easiest way to do this is to view the generators of A_1 along with the generators for $\mathrm{im}(\varphi_2)$ as $\Bbbk$-vectors. This gives,

A_1					$\mathrm{im}(\varphi_2)$		
x^2	xy	xz	0	0	$-y$	$-z$	0
0	0	0	y^2	0	x	0	$-z$
0	0	0	0	z^2	0	x	y

We can see from this that the chosen minimal generators for A_1 do indeed represent a minimal generating set.

Finding a minimal generating set for A_2 is generally not as simple. It requires more effort to verify that an element is in $\ker(\varphi_2)$ for these generators. We do know that each of these generators will be made up of degree 2 elements from K which have the form $m_j(e_i \wedge e_j)$ with $1 \leq i < j \leq 3$. We want $m_j(e_i \wedge e_j) \in \ker(\varphi_2)$, so we need $\varphi_2[m_j(e_i \wedge e_j)] = m_j\varphi_2(e_i \wedge e_j) = m_j(-x_ie_i + x_je_j)$ to be zero. Once we have a set of minimal generators we must also make sure that it is independent mod $\mathrm{im}(\varphi_3)$. We choose the following generators for A_2,

$$A_2 = \langle x^2(e_1 \wedge e_2), xy^2(e_1 \wedge e_2), x^2z(e_1 \wedge e_3), xz^2(e_1 \wedge e_3), x^2z(e_2 \wedge e_3), y^2z^2(e_2 \wedge e_3)\rangle.$$

Again to verify the independence of these generators we will view them as $\Bbbk$-vectors along with the generator for $\mathrm{im}(\varphi_3)$.

A_2						$\mathrm{im}(\varphi_3)$
x^2	xy^2	0	0	0	0	z
0	0	x^2z	xz^2	0	0	$-y$
0	0	0	0	x^2z	y^2z^2	x

We are less concerned with finding a minimal generating set for A_3 at the moment, because it is not needed to compute the numbers p and q. We can now compute these numbers using the multiplication tables for $A_1 \cdot A_1$ and $A_1 \cdot A_2$.

Table 4.1: Example 4.1.1 – A_1^2

A_1^2	x^2e_1	xye_1	xze_1	y^2e_2	z^2e_3
x^2e_1	0	0	0	0	0
xye_1	0	0	0	0	$xyz^2(e_1 \wedge e_3)$
xze_1	0	0	0	$xy^2z(e_1 \wedge e_2)$	0
y^2e_2	0	0	$-xy^2z(e_1 \wedge e_2)$	0	$y^2z^2(e_2 \wedge e_3)$
z^2e_3	0	$-xyz^2(e_1 \wedge e_3)$	0	$-y^2z^2(e_2 \wedge e_3)$	0

Table 4.2: Example 4.1.1 – $A_1 \cdot A_2$

$A_1 \cdot A_2$	x^2e_1	xye_1	xze_1	y^2e_2	z^2e_3
$x^2(e_1 \wedge e_2)$	0	0	0	0	0
$xy^2(e_1 \wedge e_2)$	0	0	0	0	$xy^2z^2(e_1 \wedge e_2 \wedge e_3)$
$x^2z(e_1 \wedge e_3)$	0	0	0	0	0
$xz^2(e_1 \wedge e_3)$	0	0	0	$-xy^2z^2(e_1 \wedge e_2 \wedge e_3)$	0
$x^2z(e_2 \wedge e_3)$	0	0	0	0	0
$y^2z^2(e_2 \wedge e_3)$	0	0	0	0	0

The $\Bbbk$-vector space rank of these will be the number of distinct nonzero independent products that we get up to multiplication by a unit. From this we may find that we are tempted to say that $p = 3$ and $q = 1$, but this is not the case. We have yet to check if the degree 2 elements in $A_1 \cdot A_1$ and the degree 3 elements in $A_1 \cdot A_2$ are nonzero modulo the incoming maps. Since the incoming map for A_3 is just zero we can conclude that $q = 1$. We must however look at the three nonzero products from $A_1 \cdot A_1$ along with $\mathrm{im}(\varphi_3)$. This gives the following,

$$\begin{array}{ccc|c} & & & \mathrm{im}(\varphi_3) \\ \hline xy^2z & 0 & 0 & z \\ 0 & xyz^2 & 0 & -y \\ 0 & 0 & y^2z^2 & x \end{array}$$

It is clear that we cannot obtain $y^2z^2(e_2 \wedge e_3)$ from $\mathrm{im}(\varphi_3)$ or any of the other two nonzero elements. We do find that both $xy^2z(e_1 \wedge e_2)$ and $xyz^2(e_1 \wedge e_3)$ are zero mod $\mathrm{im}(\varphi_3)$. To see this, we obtain $xy^2z(e_1 \wedge e_2)$ from $\mathrm{im}(\varphi_3)$ by,

$$xy^2 \cdot \begin{pmatrix} z \\ -y \\ x \end{pmatrix} = \begin{pmatrix} xy^2z \\ -xy^3 \\ x^2y^2 \end{pmatrix} = \begin{pmatrix} xy^2z \\ 0 \\ 0 \end{pmatrix}$$

since both $-xy^3$ and x^2y^2 are in I. We could preform a similar computation on $xyz^2(e_1 \wedge e_3)$. Thus we have that $p = 1$.

Taking this example to the next stage we would want to know what the Tor-algebra class for $I = \langle x^3, x^2y, y^3, z^3, x^2z^2 \rangle$ is. We know that this particular ideal can only fall in $\mathbf{T}$, $\mathbf{B}$, or $\mathbf{H}(1, 1)$. The following Theorem is a collection of information from [4, 2.1 and 3.1]. We have already stated [4, 2.1] in Theorem 2.3.15, but we will restate the Bass series for the three classes we are interested in to help us with our computations.

Theorem 4.1.2. *Let I be an $\mathfrak{m}$-primary monomial ideal minimally generated by n monomials, such that S is not Gorenstein. Let p, q, r and l be the numbers defined in Definition 2.3.8 and Theorem 2.3.15. The following table lists the Bass series $I_S^S(t)$ along with values for the previously defined invariants for various classes of the Tor-algebra for S.*

Table 4.3: Bass series and invariants for Tor-algebra classes

Class	$I_S^S(t)$	p	q	r
T	$\dfrac{m + lt - 2t^2 - t^3 + t^4}{1 - t - lt^2 - (m-3)t^3 - t^5}$	3	0	0
B	$\dfrac{m + (l-2)t - t^2 + t^4}{1 - t - lt^2 - (m-1)t^3 + t^4}$	1	1	2
$\mathbf{H}(0,0)$	$\dfrac{m + lt + t^2 - t^3}{1 - t - lt^2 - mt^3}$	0	0	0
$\mathbf{H}(p,q)$ $p+q \geq 1$	$\dfrac{m + (l-q)t - pt^2 - t^3 + t^4}{1 - t - lt^2 - (m-p)t^3 + qt^4}$	p	q	q

If we were to revisit Example 4.1.1 with this Theorem in mind we would have that I was either in **B** or $\mathbf{H}(1,1)$. One approach to determining this would be to compute r since it differs for both of these classes. Since we have expressions for the Bass series of the respective classes it may be easier to determine what μ_S^1 is for this ideal using results from Chapter 3 and compare this to our expressions for the Bass series from the previous Theorem.

We know that $\mu_S^0 = m = 2$, and looking at the free resolution for S given in Example 4.1.1 we can see that there are exactly two rows in f_3 which have only nonzero elements from I. It is not difficult to see that none of the other rows will be dependent mod I. From Section 2.3.4 we have that $\mu_S^1 = m + n - 1 - \hat{r} = 6 - 2 = 4$ where $\hat{r}$ is the number of rows in f_3 which are dependent mod I. We will now compare this to the expressions for $\mathbf{B}$ and $\mathbf{H}(1,1)$ in Theorem 4.1.2. For $\mathbf{H}(1,1)$,

$$I_S^S(t) \;=\; \sum_{i\geq 0} \mu_S^i t^i \;=\; \mu_S^0 + \mu_S^1 t + \mu_S^2 t^2 + \cdots \;=\; \frac{2+3t-t^2-t^3+t^4}{1-t-4t^2-t^3+t^4}$$

$$\implies (\mu_S^0 + \mu_S^1 t + \mu_S^2 t^2 + \cdots)(1-t-4t^2-t^3+t^4) = 2+3t-t^2-t^3+t^4$$

$$\implies \mu_S^1 - \mu_S^0 = 3 \implies \mu_S^1 = 5.$$

Since we know that $\mu_S^1 = 4$ this formula does not work and we can conclude that I is in $\mathbf{B}$. However we will verify that the Bass series formula will work for $\mathbf{B}$, which gives,

$$I_S^S(t) = \sum_{i\geq 0} \mu_S^i t^i = \mu_S^0 + \mu_S^1 t + \mu_S^2 t^2 + \cdots = \frac{2+2t-t^2+t^4}{1-t-4t^2-t^3+t^4}$$

$$\implies (\mu_S^0 + \mu_S^1 t + \mu_S^2 t^2 + \cdots)(1-t-4t^2-t^3+t^4) = m+2t-t^2+t^4$$

$$\implies \mu_S^1 - \mu_S^0 = 2 \implies \mu_S^1 = 4, \text{ which is what we wanted.}$$

With Theorem 4.1.2 in mind there are some cases when n is small in which we can easily determine what the Tor-algebra class is for I. These are outlined in [4, 3.4.2 and 3.4.2].

Fact 4.1.3. 1. If $n = 4$ then S is in one of the following classes:

(i) $\mathbf{H}(3,2)$ with $m = 2$.

(ii) $\mathbf{T}$ with $m \geq 3$.

(iii) $\mathbf{H}(3,0)$ with even $m \geq 4$.

2. If $n \geq 5$, $m = 2$, and $p > 0$, then S is in one of the following classes:

(i) $\mathbf{B}$ with odd n.

(ii) $\mathbf{H}(1,2)$ with even n.

It is not difficult to see that for an $\mathfrak{m}$-primary monomial ideal minimally generated by 4 monomials we must have that either $m = 2$ or $m = 3$. It can also be shown that if $m = 2$ then the largest number of minimal generators that we may have for I is $n = 5$, illustrated in Example 4.1.1. We will see that we can only satisfy 1.(i), 1.(ii), and 2.(i) from Fact 4.1.3. We will provide examples to all of these in Section 4.2. Before we begin computing our examples we will find some more efficient way to compute the invariants p, q and r.

4.1.2 Computation from Free Resolutions

In this section we will learn how to relate the invariants for the Tor-algebra to information that we can obtain from the minimal free resolution of S from Definition 2.2.6. This will allow us to give more precise descriptions of the Tor-algebra for monomial ideals. We begin by showing how we can always find a specific minimal generating set for A_1 from the minimal generators of I.

Proposition 4.1.4. *Let I be a monomial ideal with minimal generating set*

$$\{x^a, y^b, z^c, x^{a_1}y^{b_1}z^{c_1}, \ldots, x^{a_\rho}y^{b_\rho}z^{c_\rho}\}, \text{ where } a_j = 0 \text{ for } i \leq j \leq \rho.$$

Then the set

$$\{x^{a-1}e_1, y^{b-1}e_2, z^{c-1}e_3, x^{a_1-1}y^{b_1}z^{c_1}e_1, \ldots, x^{c_{i-1}-1}y^{b_{i-1}}z^{c_{i-1}}e_1, y^{b_i-1}z^{c_i}e_2, \ldots, y^{b_\rho-1}z^{c_\rho}e_2\}$$

is a minimal generating set for A_1.

Proof: We already know that if I is minimally generated by n monomials then A_1 is minimally generated by n degree 1 elements from K. The minimal generating set we specified for I is general. We have only specified which minimal mixed generators have only positive degrees on y and z, and which ones have positive degrees on x with the condition that $a_j = 0$ for $i \leq j \leq \rho$. To show that this is a minimal generating set for A_1 we will first show that each of the generators are in the kernel of the differential φ_1 from K, then we will show that this set is independent module $\mathrm{im}(\varphi_2)$.

It is clear that all of the chosen generators are in $\ker(\varphi_1)$, since for any minimal generator of I, m_j with positive degree on x_i we have that $\varphi_1(\frac{m_j}{x_i}e_i) = 0$. We can represent all of the mixed generators as $x^{a_j-1}y^{b_j}z^{c_j}e_1$ with $1 \leq j < i$ and $y^{b_j-1}z^{c_j}e_2$ with $i \leq j \leq \rho$. To check for independence we will view the generators as $\Bbbk$-vectors,

$$\begin{array}{cccccc|ccc}
A_1 & & & & & & \mathrm{im}(\varphi_2) & & \\
\hline
x^{a-1} & 0 & 0 & \{x^{a_j-1}y^{b_j}z^{c_j}\}_{1\leq j<i} & 0 & & -y & -z & 0 \\
0 & y^{b-1} & 0 & 0 & \{y^{b_j-1}z^{c_j}\}_{i\leq j\leq \rho} & & x & 0 & -z \\
0 & 0 & z^{c-1} & 0 & 0 & & 0 & x & y
\end{array}$$

It is clear that $x^{a-1}e_1, y^{b-1}e_2$, and $z^{c-1}e_3$ are independent. For the other generators it is not possible to get any $x^{a_j-1}y^{b_j}z^{c_j}e_1$ from another $x^{a_k-1}y^{b_k}z^{c_k}e_1$ with $j \neq k$, because this would imply that $x^{a_j}y^{b_j}z^{c_j}$ was a multiple of $x^{a_k}y^{b_k}z^{c_k}$ which cannot happen because they are both minimal generators. A similar argument can be made for $y^{b_j-1}z^{c_j}e_2$. We can also see that the only way we could generate $x^{a_j-1}y^{b_j}z^{c_j}e_1$ from $\mathrm{im}(\varphi_2)$ is with the following multiplications,

$$-x^{a_j-1}y^{b_j-1}z^{c_j}\cdot\begin{pmatrix}-y\\ x\\ 0\end{pmatrix} = \begin{pmatrix}x^{a_j-1}y^{b_j}z^{c_j}\\ -x^{a_j}y^{b_j-1}z^{c_j}\\ 0\end{pmatrix} \text{ or } -x^{a_j-1}y^{b_j}z^{c_j-1}\cdot\begin{pmatrix}-z\\ 0\\ x\end{pmatrix} = \begin{pmatrix}x^{a_j-1}y^{b_j}z^{c_j}\\ 0\\ -x^{a_j}y^{b_j}z^{c_j-1}\end{pmatrix}$$

which could only equal $x^{a_j-1}y^{b_j}z^{c_j}e_1$ if both $x^{a_j}y^{b_j-1}z^{c_j}$ and $x^{a_j}y^{b_j}z^{c_j-1}$ were in I. But this contradicts the assumption that $x^{a_j}y^{b_j}z^{c_j}$ is a minimal generator of I. Similarly we can only generate $y^{b_j-1}z^{c_j}e_2$ from $\mathrm{im}(\varphi_2)$ with the following,

$$-y^{b_j-1}z^{c_j-1}\cdot\begin{pmatrix}0\\ -z\\ y\end{pmatrix} = \begin{pmatrix}0\\ -y^{b_j-1}z^{c_j}\\ y^{b_j}z^{c_j-1}\end{pmatrix}$$

which again cannot be equal to $y^{b_j-1}z^{c_j}e_2$ because $y^{b_j}z^{c_j-1} \notin I$. Thus we have shown that this is indeed a minimal generating set for A_1. $\square$

Using Proposition 4.1.4 were are able to easily find a minimal generating set for A_1 given a minimal generating set for I. While this is nice, it still doesn't tell us anything about our invariants for the Tor-algebra. We will use this proposition to prove our next result, which will directly relate the computation of $p = \operatorname{rank}_{\Bbbk}(A_1^2)$ to the number of distinct $\sigma_{ij} \in S_2$ with only nonzero entries from I. This was described in Theorem 3.2.1, which will also be used for the next Theorem.

Theorem 4.1.5. *If $I = \langle m_1, \ldots, m_n \rangle$ is an $\mathfrak{m}$-primary monomial ideal such that $m'_k e_i = \dfrac{m_k}{x_i} e_i$ and $m'_l e_j = \dfrac{m_l}{x_j} e_j$, $1 \le i < j \le 3$, $1 \le k < l \le n$ are minimal generators of A_1, then $m'_k \cdot m'_l (e_i \wedge e_j) \in A_1^2 \subseteq A_2$ is not zero if and only if $\sigma_{kl} \in S_2$ and σ_{kl} only has nonzero entries from I.*

Proof: ($\Longrightarrow$) Without loss of generality we may assume that $i = 1$ and $j = 2$. Then $m'_k e_1 = x^{a_k - 1} y^{b_k} z^{c_k}$, $m'_l e_2 = x^{a_l} y^{b_l - 1} z^{c_l}$, and by assumption,

$$m'_k \cdot m'_l (e_1 \wedge e_2) \;=\; x^{a_k + a_l - 1} y^{b_k + b_l - 1} z^{c_k + c_l} (e_1 \wedge e_2) \neq 0.$$

This means that $m'_k \cdot m'_l \notin I$ and that $m'_k \cdot m'_l (e_1 \wedge e_2)$ is not dependent mod $\operatorname{im}(\varphi_3)$. To proceed we will construct the generators m_k and m_l so that the previous statement holds. To construct m_k and m_l so that $m'_k \cdot m'_l \notin I$ we only need to ensure that there is no minimal generator of I which divides $m'_k \cdot m'_l$. First we notice that neither m_k or m_l may have positive degrees on all variables. Suppose that m_k does have positive degrees on all variables, that is $a_k, b_k, c_k > 0$. Then we have that $a_l \le a_k + a_l - 1$, $b_l \le b_k + b_l - 1$, and $c_l \le c_k + c_l$, which implies that $m_l | (m'_k \cdot m'_l)$. It is also easy to see that we get a similar contradiction when m_k and m_l only have positive degrees on the same two variables. By assumption m_k must have positive degree on x and m_l must have positive degree on y. From this there are only two options, which we will describe by (m_k, m_l). Either (m_k, m_l) has positive degree on only x or only y, or $(m_k, m_l) = 1$. We will show that the second option is the only possible option. Suppose that

$(m_k, m_l) = x^\alpha$ with $\alpha = \min\{a_k, a_l\} > 0$. Then without loss of generality we must have that $m_k = x^{a_k} z^{c_k}$ and $m_l = x^{a_l} y^{b_l}$. From this we see that

$$m_k | m_k' \cdot m_l' = x^{a_k + a_l - 1} y^{b_l - 1} z^{c_k} \text{ since } a_k \leq a_k + a_l - 1.$$

This is a contradiction which means that $(m_k, m_l) = 1$. Since there is no minimal generator of I that divides $m_k' \cdot m_l'$. This implies that $\sigma_{kl} \in S_2$ by Lemma 2.2.7, and since $(m_k, m_l) = 1$ we have that σ_{kl} only has nonzero entries from I by Theorem 3.2.1.

$(\Longleftarrow)$ Suppose $\sigma_{kl} \in S_2$ and σ_{kl} only has nonzero entries from I. Then by Theorem 3.2.1 $(m_k, m_l) = 1$ and there is no minimal generator m_i of I such that $m_i |_< m_{kl}$. We know from construction that one of these minimal generators is a pure power and the other generator only has positive degrees in the other two variables. Without loss of generality let $m_k = x^a$ and $m_l = y^{b_l} z^{c_l}$ with either $b_l > 0$ or $c_l > 0$. By Proposition 4.1.4 we have that $m_k' e_1 = x^{a-1} e_1$ and $m_l' e_2 = y^{b_l - 1} z^{c_l} e_2$ are both minimal generators of A_1. We only need to show that $m_k' \cdot m_l' (e_1 \wedge e_2) \neq 0$ in A_1^2. Computing this we have,

$$m_k' \cdot m_l' (e_1 \wedge e_2) = x^{a-1} y^{b_l - 1} z^{c_l} (e_1 \wedge e_2) = \begin{pmatrix} x^{a-1} y^{b_l - 1} z^{c_l} \\ 0 \\ 0 \end{pmatrix}.$$

This element of A_1^2 is zero if $x^{a-1} y^{b_l - 1} z^{c_l} \in I$ or $m_k' \cdot m_l' (e_1 \wedge e_2)$ is dependent modulo $\operatorname{im}(\varphi_3)$. Since there is no minimal generator m_i of I such that $m_i |_< m_{kl} = x^a y^{b_l} z^{c_l}$ then $m_k' \cdot m_l' \notin I$ since $m_k' \cdot m_l' |_< m_{kl}$. For sake of contradiction if $m_k' \cdot m_l' (e_1 \wedge e_2)$ was dependent mod $\mathrm{im}(\varphi_3)$ then the only possible way we could obtain $m'_k \cdot m'_l(e_1 \wedge e_2)$ from φ_3 would be,

$$\begin{pmatrix} x^{a-1}y^{b_l-1}z^{c_l} \\ 0 \\ 0 \end{pmatrix} = x^{a-1}y^{b_l-1}z^{c_l-1} \cdot \begin{pmatrix} z \\ -y \\ x \end{pmatrix} = \begin{pmatrix} x^{a-1}y^{b_l-1}z^{c_l} \\ -x^{a-1}y^{b_l}z^{c_l-1} \\ x^{a}y^{b_l-1}z^{c_l-1} \end{pmatrix}$$

$$= \begin{pmatrix} x^{a-1}y^{b_l-1}z^{c_l} \\ -x^{a-1}y^{b_l}z^{c_l-1} \\ 0 \end{pmatrix}.$$

For this to be true we must have that $-x^{a-1}y^{b_l}z^{c_l-1} \in I$, which means there is a minimal generator m_i of I such that $m_i | x^{a-1}y^{b_l}z^{c_l-1}$. But this would also imply that $m_i | (m'_k \cdot m'_l)$ which is a contradiction. Thus we have shown that $m'_k \cdot m'_l(e_1 \wedge e_2) \neq 0$. $\square$

This Theorem provides some informative corollaries.

Corollary 4.1.6. *Let $I = \langle m_1, \ldots, m_n \rangle$ be an $\mathfrak{m}$-primary monomial ideal, then $p = \mathrm{rank}_{\Bbbk}(A_1^2)$ is precisely then number of distinct $\sigma_{ij} \in S_2$ which have only nonzero entries from I.*

The proof of this follows immediately from Theorem 4.1.5. This gives us an easy way to find p from the free resolution of S. We only need to determine how many columns of f_2 have only entries from I. Another way we could see this is by examining the planar graph representing the minimal free resolution of S. We would find p by counting the number of edges between the pure power generators along with the edges between a pure power generator and any mixed double generator involving only the other two variables.

Corollary 4.1.7. *Let $I = \langle m_1, \ldots, m_n \rangle$ be an $\mathfrak{m}$-primary monomial ideal, then $p =$* $\mathrm{rank}_{\Bbbk}(A_1^2) \leq n - 1$.

This follows immediately from Theorem 3.2.1 and Theorem 4.1.5.

It would be nice if we also had way to describe q from the minimal free resolution of S. In general this description is not as clear. But we can observe that if S is in either $\mathbf{H}(p,q)$ or $\mathbf{T}$ we have that $q=r$. Moreover when S is in $\mathbf{T}$ or $\mathbf{B}$ we know the values of p,q, and r from Theorem 4.1.2. So we provide the following proposition which will follow from Theorem 4.1.2.

Proposition 4.1.8. *Let I be an $\mathfrak{m}$-primary monomial ideal minimally generated by n monomials, such that S is not Gorenstein. Then $r = \operatorname{rank}_{\Bbbk}(\delta_2)$ from Definition 2.3.8 is precisely the number of rows in the matrix of f_3 which are dependent* mod I.

Proof: Recall from Section 2.3.4 we showed that $\mu_S^1 = m+n-1-\hat{r}$ where $\hat{r}$ was the number of rows in f_3 which were dependent mod I. Since we are assuming that S is not Gorenstein then S is either in $\mathbf{T}$, $\mathbf{B}$, or $\mathbf{H}(p,q)$. Thus to prove this we only need to show that $r=\hat{r}$ from the expressions for the Bass Series in each respective class.

Class $\mathbf{T}$: From the expression for the Bass series for $\mathbf{T}$ we have that,

$$I_S^S(t) \;=\; \sum_{i\geq 0}\mu_S^i t^i \;=\; \mu_S^0+\mu_S^1 t+\cdots \;=\; \frac{m+lt-2t^2-t^3+t^4}{1-t-lt^2-(m-3)t^3-t^5}$$

$$\implies (\mu_S^0+\mu_S^1 t+\cdots)(1-t-lt^2-(m-3)t^3-t^5) = m+lt-2t^2-t^3+t^4$$

$$\implies \mu_S^1-\mu_S^0 = l \implies \mu_S^1 = m+n-1.$$

So there are now rows in f_3 which are dependent mod I and consequently for $\mathbf{T}$, $r=0$.

Class $\mathbf{B}$: From the expression for the Bass series for $\mathbf{B}$ we have that,

$$I_S^S(t) \;=\; \sum_{i\geq 0}\mu_S^i t^i \;=\; \mu_S^0+\mu_S^1 t+\cdots \;=\; \frac{m+(l-2)t-t^2+t^4}{1-t-lt^2-(m-1)t^3+t^4}$$

$$\implies (\mu_S^0+\mu_S^1 t+\cdots)(1-t-lt^2-(m-1)t^3+t^4) = m+(l-2)t-t^2+t^4$$

$$\implies \mu_S^1 - \mu_S^0 = l - 2 \implies \mu_S^1 = m + n - 1 - 2.$$

This says that there are exactly two rows in f_3 that are dependent mod I and for **B**, $r = 2$.

Class $\mathbf{H}(p, q)$: For $p = q = 0$ we observe that the terms in the numerator and the denominator of the expression for the Bass series with degrees ≤ 1 are the same as they were in **T**. Thus we can conclude for $\mathbf{H}(0, 0)$ that there are no rows in f_3 that are dependent mod I. We also know that $r = 0$ in this case. If $p+q \geq 1$ we find that the expression for the Bass series for $\mathbf{H}(p, q)$ gives,

$$I_S^S(t) = \sum_{i\geq 0} \mu_S^i t^i = \mu_S^0 + \mu_S^1 t + \cdots = \frac{m + (l-q)t - pt^2 - t^3 + t^4}{1 - t - lt^2 - (m-p)t^3 + qt^4}$$

$$\implies (\mu_S^0 + \mu_S^1 t + \cdots)(1 - t - lt^2 - (m-p)t^3 + qt^4) = m + (l-q)t - pt^2 - t^3 + t^4$$

$$\implies \mu_S^1 - \mu_S^0 = l - q \implies \mu_S^1 = m + n - 1 - q.$$

Thus for $\mathbf{H}(p, q)$ we have that there are q rows in f_3 which are dependent mod I and we also know that $r = q$. □

4.2 Examples

In this section we will list several examples of ideals along with their Tor-algebra classification. We will rely on the computational methods used from Section 4.1.2. This will allow us to determine the Tor-algebra classification for various ideals by looking at a minimal free resolution. We will conclude this section by giving a small class of examples with a very specific structure. These will relate to Theorems 3.2.1 and 3.3.11 from Chapter 3.

Example 4.2.1. For this example we will satisfy the conditions 1.(i),1.(ii), and 2.(i) from Fact 4.1.3. We have already seen that $I = \langle x^3, x^2y, y^3, z^3, x^2z^2\rangle$ from Example 4.1.1 satisfies 2.(i) and is in class **B**. So we will consider two general ideals.

1.(i): Let $I = \langle x^a, y^b, z^c, x^\alpha y^\beta z^\gamma \rangle$ with $a, b, c > 0$ and exactly one of the degrees α, β, or γ is zero. Since the computation of the free resolution for S here requires that we choose which α, β, or γ is zero, we will assume that $\alpha = 0$. This gives the following minimal free resolution from Definition 2.2.6

$$0 \longrightarrow R^2 \xrightarrow{\begin{bmatrix} z^\gamma & 0 \\ 0 & y^\beta \\ -y^{b-\beta} & -z^{c-\gamma} \\ x^a & 0 \\ 0 & x^a \end{bmatrix}} R^5 \xrightarrow{\begin{bmatrix} -y^b & -z^c & -y^\beta z^\gamma & 0 & 0 \\ x^a & 0 & 0 & -z^\gamma & 0 \\ 0 & x^a & 0 & 0 & -y^\beta \\ 0 & 0 & x^a & y^{b-\beta} & z^{c-\gamma} \end{bmatrix}} R^4 \longrightarrow \cdots$$

We quickly see that f_2 has exactly 3 columns with only nonzero entries from I thus $p = 3$ by Corollary 4.1.6. We also se that there are exactly 2 rows in f_3 which contain only nonzero entries from I so $r = 2$ by Proposition 4.1.8, since I is generic. Thus we conclude that I is in $\mathbf{H}(3, 2)$.

1.(ii): Now let $I = \langle x^a, y^b, z^c, x^\alpha y^\beta z^\gamma \rangle$ with $a, b, c, \alpha, \beta, \gamma > 0$. Then the minimal free resolution for S from Definition 2.2.6 is,

$$0 \longrightarrow R^3 \xrightarrow{\begin{bmatrix} z^\gamma & 0 & 0 \\ 0 & y^\beta & 0 \\ -y^{b-\beta} & -z^{c-\gamma} & 0 \\ 0 & 0 & x^\alpha \\ x^{a-\alpha} & 0 & -z^{c-\gamma} \\ 0 & x^{a-\alpha} & y^{b-\beta} \end{bmatrix}} R^6 \xrightarrow{\begin{bmatrix} -y^b & -z^c & -y^\beta z^\gamma & 0 & 0 & 0 \\ x^a & 0 & 0 & -z^c & -x^\alpha z^\gamma & 0 \\ 0 & x^a & 0 & y^b & 0 & -x^\alpha y^\beta \\ 0 & 0 & x^{a-\alpha} & 0 & y^{b-\beta} & z^{c-\gamma} \end{bmatrix}} R^4 \longrightarrow \cdots$$

Since there are three columns in f_2 with only nonzero entries from I we have that $p = 3$ by Corollary 4.1.6. This time we find that there are no rows in f_3 that are dependent mod I which implies that $r = 0$ and in turn implies $q = 0$. Thus I is either in $\mathbf{T}$ or $\mathbf{H}(3, 0)$. From Fact 4.1.3 we know that I must be in $\mathbf{T}$.

Remark 4.2.2. We may want to note in general how we would determine if I is in $\mathbf{T}$ or in $\mathbf{H}(3, 0)$, when we are faced with this question. The expressions for the Bass series gives that the zeroth and first Bass numbers yield the same result in both cases. Since it is difficult to compute the higher Bass numbers in general we need a better method to differentiate the two classes. The class **T** is studied in [3]. We refer to **T** as the truncated exterior algebra. In [3, 3.5] we learn that one of the properties of this algebra is that the graded $\Bbbk$-algebra B from Fact 2.3.9 is generated by three distinct elements in degree 1, and generated by the products of the degree 1 generators in degree 2. For us this translates into having three distinct minimal generators of A_1 so that the products of these degree 1 generators are all nonzero and minimally generate A_1^2. We will also have that these products are minimal generators of A_2. In the case of 1.(ii) in Example 4.2.1 we have that the minimal generators x^a, y^b, and z^c of I give us three minimal second syzygies, σ_{12}, σ_{13}, and σ_{23}, which have only nonzero entries from I. Connecting this with the proof of Theorem 4.1.5 we have that the minimal generators $x^{a-1}e_1, y^{b-1}e_2$, and $z^{c-1}e_3$ of A_1 give us the minimal generators of A_1^2, which are $x^{a-1}y^{b-1}(e_1 \wedge e_2), x^{a-1}z^{c-1}(e_1 \wedge e_3)$, and $y^{b-1}z^{c-1}(e_2 \wedge e_3)$. This description for **T** will be important when we give our general classification of **T** for generic monomial ideals.

Example 4.2.3. Let $I = \langle x^5, y^5, z^5, y^3z^3, xy^4z^2, xy^2z^4 \rangle$ which is generic. Then the maps for f_2 and f_3 from the minimal free resolution of S from Definition 2.2.6 is,

$$R^6 \xrightarrow{\begin{bmatrix} z^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & y^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & z & 0 & 0 \\ -y & 0 & -z & 0 & 0 & 0 \\ 0 & -z & 0 & -y & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 \\ -x^4 & 0 & 0 & 0 & -z & 0 \\ 0 & 0 & 0 & 0 & 0 & x \\ 0 & x^4 & 0 & 0 & 0 & -y \\ 0 & 0 & x^4 & 0 & y & 0 \\ 0 & 0 & 0 & x^4 & 0 & z \end{bmatrix}} R^{11} \xrightarrow{\begin{bmatrix} -y^5 & -z^5 & -y^3z^3 & -y^4z^2 & -y^2z^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ x^5 & 0 & 0 & 0 & 0 & -z^3 & -xz^2 & 0 & 0 & 0 & 0 \\ 0 & x^5 & 0 & 0 & 0 & 0 & 0 & -y^3 & -xy^2 & 0 & 0 \\ 0 & 0 & x^5 & 0 & 0 & y^2 & 0 & z^2 & 0 & -xy & -xz \\ 0 & 0 & 0 & x^4 & 0 & 0 & y & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & x^4 & 0 & 0 & 0 & z & 0 & y \end{bmatrix}} R^6$$

We notice that there are exactly three columns in f_2 with only nonzero entries from I, so $p = 3$ by Corollary 4.1.6. We also notice that there are no rows in f_3 with entries from I so $r = 0$ since I is generic, which implies that $q = 0$. Thus we are faced with the same dilemma as in the previous example, either S is in $\mathbf{T}$ or in $\mathbf{H}(3,0)$. However the minimal generators of A_1 which contribute to $\operatorname{rank}_{\Bbbk}(A_1^2)$ are $x^4e_1, y^4e_2, y^2z^2e_2$, and z^4e_3. This does not agree with the description for $\mathbf{T}$ in Remark 4.2.2 so we can conclude that S is in $\mathbf{H}(3,0)$.

Example 4.2.4. For this example we will list several non-generic ideals along with their respective Tor-algebra structure. The reader is encouraged to check this using the methods we have introduced in this section.

Table 4.4: Example 4.2.4 – Some Ideals and their Tor-algebras

	Ideal	p	q	r	Class
1.	$I = \langle x^a, y^b, z^c, x^3y^2z, x^2y^3z, xyz^3\rangle$	3	0	0	$\mathbf{T}$
2.	$I = \langle x^a, y^b, z^c, x^3z, y^3z, xyz^3\rangle$	1	1	2	$\mathbf{B}$
3.	$I = \langle x^a, y^b, z^c, x^3y^3, x^3y^2z\rangle$	2	1	1	$\mathbf{H}$(2,1)
4.	$I = \langle x^a, y^b, z^c, x^3y^3, x^3z^3, y^3z^3, xyz^4\rangle$	0	0	0	$\mathbf{H}$(0,0)
5.	$I = \mathfrak{m}^i,\ i \geq 1$	0	0	0	$\mathbf{H}$(0,0)

We will be able to see later from some of these examples how it may be difficult to find general classifications for non-generic monomial ideals, but they do seem to fit a pattern, which we will discuss prior to Conjecture 4.3.6.

The next Proposition gives us a specific class of examples where we know exactly what the zeroth and first Bass numbers are. These are exactly the generic ideals which give us the maximum number of nonzero entries from I in both f_2 and f_3 from Theorems 3.2.1 and 3.3.11.

Proposition 4.2.5. *If I is minimally generated by $n = \rho + 3 \geq 4$ monomials, $\{x^a, y^b, z^c, y^{b_1}z^{c_1}, \dots, y^{b_\rho}z^{c_\rho}\}$ with $0 < b_i < b_{i+1} < b$, and $0 < c_{i+1} < c_i < c$ for $1 \leq i < \rho$,, then R is in $\mathbf{H}(p,q)$ with $p = \mu_R^1 = n-1$ and $q = \mu_R^0 = n-2$.*

Proof: Notice that I is minimally generated by $n \geq 4$ and is generic based upon our assumptions. The free resolution of S from Definition 2.2.6 is the same as the resolution described in Theorem 3.3.11 so that we get exactly $n-2$ pure power entries from I in f_3. Using this along with Theorem 3.2.1 we can see that we have exactly $n-1$ minimal second syzygies with only nonzero entries from I. Thus buy Theorem 4.1.5 and Proposition 4.1.8 we have that $p = n-1 \geq 3$ and $r = n-2 \geq 2$. Thus we have that S must be in $\mathbf{H}(p,q)$ with $p = n-1$ and $q = r = n-2$. It is also easy to see that $\mu_S^1 = p$ and $\mu_S^0 = q$ in this case. □

4.3 Classes for Trivariate Monomial Ideals

For all of the examples we have seen so far, $p \geq q$. In general this is not true, see Fact 4.1.3. Even in the case of general monomial ideals we do not have an answer to this question. However we can give a positive answer to this for generic monomial ideals.

Proposition 4.3.1. *If I is $\mathfrak{m}$-primary and generic then $p \geq q$.*

Proof: In $\mathbf{T}$, $\mathbf{B}$, and $\mathbf{H}(0,0)$ this is clear, so we only need to show this for $\mathbf{H}(p,q)$ with $p+q \geq 1$. For this case $q = r$ thus by Proposition 4.1.8 q is precisely the number of rows in f_3 which are dependent mod I. For sake of contradiction, suppose that $q > p$. Then there are q rows in f_3 which are dependent mod I. Moreover, we know that since I is generic that there are exactly q entries of the same nonzero pure power generator of I in q rows of f_3, by Lemma 3.3.9. Suppose that this entry is x^a, then we have q sets of generators of the form $\{x^a, y^{b_1}z^{c_1}, y^{b_2}z^{c_2}\}, \dots, \{x^a, y^{b_q}z^{c_q}, y^{b_{q+1}}z^{c_{q+1}}\}$ which all represent minimal third syzygies for S. However this would imply that we have $q+1$ minimal second syzygies generated by the sets of generators $\{x^a, y^{b_1}z^{c_1}\}, \ldots, \{x^a, y^{b_{q+1}}z^{c_{q+1}}\}$. By Theorem 3.2.1 each of these second syzygies would have only nonzero entries from I and thus $p \geq q+1$ by Corollary 4.1.6, which is a contradiction. $\square$

4.3.1 Generic Classification

Theorem 4.3.2. *Let I be an $\mathfrak{m}$-primary generic monomial ideal minimally generated by n monomials, then we have the following Tor-algebra classifications for S:*

1. *If $I = \langle x^a, y^b, z^c, x^{a_1}y^{b_1}z^{c_1}, \ldots x^{a_\rho}y^{b_\rho}z^{c_\rho}\rangle$ with $a_i, b_i, c_i > 0$ for all $1 \leq i \leq \rho$ and $\rho > 0$, then S is in $\mathbf{T}$.*
2. *If $I = \langle x^a, y^b, z^c, x^{\alpha_1}y^{\beta_1}, x^{\alpha_2}z^{\gamma_1}, y^{\beta_2}z^{\gamma_2}, x^{a_1}y^{b_1}z^{c_1}, \ldots x^{a_\rho}y^{b_\rho}z^{c_\rho}\rangle$, $\alpha_i, \beta_i, \gamma_i > 0$, then S is in $\mathbf{H}(0,0)$ and is Golod if and only if there are no $\sigma_{ij} \in S_2$ with only nonzero entries from I.*
3. *Otherwise S is in $\mathbf{H}(p,q)$ with $p+q \geq 1$.*

Proof: **Case 1:** Let $m_1 = x^a$, $m_2 = y^b$, and $m_3 = z^c$, then it is clear that $\sigma_{12}, \sigma_{13}, \sigma_{23} \in S_2$ and have only nonzero entries from I. Applying Theorem 3.2.1 we have that these are the only $\sigma_{ij} \in S_2$ that satisfy this. Thus $p = 3$ by Corollary 4.1.6. Also from our construction of I and our results from Section 3.3 we have that there are no rows in f_3 that are dependent mod I. By Proposition 4.1.8 this implies that $r = 0$ which implies $q = 0$ by Theorem 4.1.2. Using Theorem 4.1.2 we have that either S is in $\mathbf{T}$ or $\mathbf{H}(3,0)$. However since the products $x^{a-1}y^{b-1}(e_1 \wedge e_2)$, $x^{a-1}z^{c-1}(e_1 \wedge e_3)$, and $y^{b-1}z^{c-1}(e_2 \wedge e_3)$ from $A_1 \cdot A_1$ generate A_1^2 we must have that S is in $\mathbf{T}$ by Remark 4.2.2.

Case 2: We know that S is in $\mathbf{H}(0,0)$ if and only if S is Golod from Remark 2.3.10. Both directions of this proof are immediate consequences of Corollary 4.1.6 and Proposition 4.3.1. We note here that the only reason that we state that $x^{\alpha_1}y^{\beta_1}, x^{\alpha_2}z^{\gamma_1}, y^{\beta_2}z^{\gamma_2}$, $\alpha_i, \beta_i, \gamma_i > 0$ are minimal generators of I is because this is required for p to equal zero but does not necessarily imply p is zero. The condition that there are no $\sigma_{ij} \in S_2$ with only nonzero entries from I is what implies $p = 0$.

Case 3: To prove this case we need to show that there are no other constructions for I in $\mathbf{T}$, and then we must show that when I is generic we cannot get anything in $\mathbf{B}$. From Example 4.2.3 we saw that we can have an example in $\mathbf{H}(3,0)$ having a minimal second syzygies involving the sets of generators $\{x^5, y^5\}$, $\{x^5, z^5\}$, and $\{x^5, y^3z^3\}$. We know in general that if we have a minimal second syzygy between generators of the form $\{x^a, y^\beta z^\gamma\}$, $\beta, \gamma > 0$, then we cannot have any minimal second syzygies between minimal generators $\{y^b, x^\alpha z^{\gamma'}\}$ or $\{z^c, x^{\alpha'} y^{\beta'}\}$. To have $p = 3$ and $q = 0$ we must have two other minimal second syzygies from generators of the form, $\{x^a, y^{\beta_1} z^{\gamma_1}\}$ and $\{x^a, y^{\beta_2} z^{\gamma_2}\}$ with $0 \leq \beta_1 < \beta < \beta_2 \leq b$ and $0 \leq \gamma_2 < \gamma < \gamma_1 \leq c$. It is clear that this is not in $\mathbf{T}$ from Remark 4.2.2 because this implies that four minimal generators of A_1 would give us three distinct nonzero elements from A_1^2. Thus the only constructions we have from $\mathbf{T}$ are in Case 1.

If S was in $\mathbf{B}$ then we would have that $p = 1$ and $r = 2$. If $r = 2$ then since I is generic we must have exactly two of the same nonzero pure power entries from I in f_3. If we assume these two entries are x^a, then we must have sets of minimal generators of the form, $\{x^a, y^{b_1} z^{c_1}, y^{b_2} z^{c_2}\}$ and $\{x^a, y^{b_3} z^{c_3}, y^{b_4} z^{c_4}\}$ which correspond to minimal third syzygies in f_3. We make note here that b_2 and c_2 may or may not be equal to b_3 and c_3 respectively. This means that we would have at least three minimal second syzygies from generators of the form $\{x^a, y^{b_1} z^{c_1}\}$, $\{x^a, y^{b_2} z^{c_2}\}$, and $\{x^a, y^{b_4} z^{c_4}\}$. Thus $p \geq 3$ by Corollary 4.1.6. This implies that S cannot be in $\mathbf{B}$ when I is generic and the only possibility that we may have is that S is in $\mathbf{H}(p,q)$ with $p + q \geq 1$. □

We should note that it is possible to find monomial ideals that are in $\mathbf{B}$ as we have already done so in Examples 4.1.1 and 4.2.4. The class $\mathbf{B}$ comes from special ideals constructed by A. Brown in [9]. All of the ideals given in [9] have $m = 2$ and it has previously been unknown if any examples with $m > 2$ exist. In the next section we will show that the answer to this question is yes by providing a class of monomial ideals in $\mathbf{B}$ where m can be larger than 2.

Remark 4.3.3. Recall that in Remark 3.3.13 we conjectured that for a generic monomial ideal generated by n elements that we could not have $n-3$ nonzero entries from I in f_3. If this is true then it would mean that we will never have $q = n-3$, more specifically we would not be able to find generic monomial ideals in the class $\mathbf{H}(p, n-3)$.

4.3.2 Other Classes

The following theorem will give us a class of monomial ideals in $\mathbf{B}$. The construction of this class is just an extension of the ideal from Example 4.1.1.

Theorem 4.3.4. *Let I be minimally generated by $n = \rho + 3$ monomials, $\{x^a, y^b, z^c, x^{a_1}z^{c'}, x^{a_2}y^{b_2}z^{c'}, \ldots, x^{a_{\rho-1}}y^{b_{\rho-1}}z^{c'}, y^{b_\rho}z^{c'}\}$ with $\rho \geq 2$ and $0 < c' < c$. Then R is in $\mathbf{B}$ with $m = \rho \geq 2$.*

Proof: To prove this we will first construct the second and third syzygies in the free resolution of S so that we may compute p and r. We will then only need to show that $q \neq r$ which will prove our result.

We begin by identifying the minimal second syzygies of S. Notice that $[x^a, x^{a_1}z^{c'}] = x^a z^{c'}$ and $[x^{a_1}z^{c'}, x^{a_i}y^{b_i}z^{c'}] = x^{a_1}y^{b_i}z^{c'}$ both strictly divide $[x^a, x^{a_i}y^{b_i}z^{c'}] = x^a y^{b_i}z^{c'}$, so there are no minimal second syzygies between the minimal generators x^a and $x^{a_i}y^{b_i}z^{c'}$ for $2 \leq i \leq \rho$ by Lemma 2.2.7. Similarly we can see that there are no minimal second syzygies between y^b and $x^{a_i}y^{b_i}z^{c'}$ for $1 \leq i \leq \rho - 1$. On the other hand we have that $[z^c, x^{a_i}y^{b_i}z^{c'}] = x^{a_i}y^{b_i}z^c$, and the only minimal generators that divide this are z^c and $x^{a_i}y^{b_i}z^{c'}$, which implies that each second syzygy between z^c and $x^{a_i}y^{b_i}z^{c'}$ is minimal for $1 \leq i \leq \rho$ by Remark 2.2.8. In addition we have that each second syzygy between

$x^{a_i}y^{b_i}z^{c'}$ and $x^{a_j}y^{b_j}z^{c'}$ is minimal if and only if $|i-j|=1$. We also notice that the second syzygy between $\{x^a, y^b\}$ is minimal. This gives us all of the minimal second syzygies for S which will come from the following sets of minimal generators in order, assuming that f_1 is ordered in the manner we wrote the generators for I from above,

$$\{x^a, y^b\}, \{x^a, x^{a_1}z^{c'}\}, \{y^b, y^{b_\rho}z^{c'}\}, \{z^c, x^{a_i}y^{b_i}z^{c'}\}_{1\leq i\leq \rho}, \{x^{a_i}y^{b_i}z^{c'}, x^{a_{i+1}}y^{b_{i+1}}z^{c'}\}_{1\leq i<\rho}.$$

This implies that we have exactly $2\rho+2$ minimal second syzygies.

For the minimal third syzygies we need to find the minimal cycles between the sets of minimal second syzygies. These cycles come from the following sets of generators of I in the given order,

$$\{x^a, y^b, \{x^{a_i}y^{b_i}z^{c'}\}_{1\leq i\leq \rho}\}, \{z^c, x^{a_i}y^{b_i}z^{c'}, x^{a_{i+1}}y^{b_{i+1}}z^{c'}\}_{1\leq i<\rho}.$$

Counting these we have exactly ρ minimal third syzygies.

It would be difficult to write down a general form of the free resolution for S here so instead we will analyze what kind of entries we would have from the minimal second and third syzygies we have listed above. From the list for the second syzygies, $\{x^a, y^b\}$ is the only pair in which $(x^a, y^b)=1$. This implies we have only one minimal second syzygy in S_2 with only nonzero entries from I, which implies that $p=1$ by Corollary 4.1.6. For the third syzygies we have that all of the entries from the syzygies given by $\{z^c, x^{a_i}y^{b_i}z^{c'}, x^{a_{i+1}}y^{b_{i+1}}z^{c'}\}$ will only have nonzero pure power entries of each of the variables by Proposition 3.4.1, but none of these entries will also be in I. The minimal third syzygy given by $\{x^a, y^b, \{x^{a_i}y^{b_i}z^{c'}\}_{1\leq i\leq\rho}\}$ will have precisely the following form up to the sign on the entries,

$$\begin{pmatrix} z^{c'} & y^b & x^a & 0 & \cdots & 0 & x^{a-a_1}y^{b-b_2} & x^{a-a_2}y^{b-b_3} & \cdots & x^{a-a_{\rho-2}}y^{b-b_{\rho-1}} & x^{a-a_{\rho-1}}y^{b-b_\rho} \end{pmatrix}$$

where $a-a_i<a-a_{i+1}$ and $b-b_i>b-b_{i+1}$ for $1\leq i\leq \rho-1$. We note that this is actually a column in f_3 we are writing it as a row vector for clarity. We can already see that we have the entries y^b and x^a in this third syzygy. In f_3 these will entries will be in rows 2 and 3 from our ordering, which correspond with minimal second syzygies between generators $\{x^a, x^{a_1}z^{c'}\}$ and $\{y^b, y^{b_\rho}z^{c'}\}$. Since these are not involved with any of the other cycles we know that all the other values in rows 2 and 3 of f_3 will be zeros. Also since $a - a_i < a - a_{i+1}$ and $b - b_i > b - b_{i+1}$ for $1 \leq i \leq \rho - 1$ none of the other rows from this third syzygy will be dependent mod I. Thus we can conclude that $r = 2$ from Proposition 4.1.8.

We will now compute minimal generating sets for A_1 and A_2 and show that $q = \operatorname{rank}_{\Bbbk}(A_1 \cdot A_2) = 1$. From Proposition 4.1.4 we have that,

$$A_1 = \langle x^{a-1}e_1, y^{b-1}e_2, z^{c-1}e_3, \{x^{a_i-1}y^{b_i}z^{c'}e_1\}_{i=1}^{\rho-1}, y^{b_\rho-1}z^{c'}e_2\rangle.$$

We will show that,

$$A_2 = \langle x^{a-1}y^{b-1}(e_1 \wedge e_2), \{x^{a_i-1}y^{b_{i+1}-1}z^{c'}(e_1 \wedge e_2)\}_{i=1}^{\rho-1}, x^{a-1}z^{c'-1}(e_1 \wedge e_3),$$
$$\{x^{a_i-1}y^{b_i}z^{c-1}(e_1 \wedge e_3)\}_{i=1}^{\rho-1}, y^{b-1}z^{c'-1}(e_2 \wedge e_3), y^{b_\rho-1}z^{c-1}(e_2 \wedge e_3)\rangle.$$

We notice that we have precisely $2\rho + 2$ generators here. It is relatively simple to verify that each of these generators satisfies the differential φ_2, and that all of these generators are linearly independent with each other. We need to ensure that each of these generators is independent mod $\operatorname{im}(\varphi_3)$. We first consider the following,

						$\operatorname{im}(\varphi_3)$
$x^{a-1}y^{b-1}$	0	0	0	$x^{a_i-1}y^{b_{i+1}-1}z^{c'}$	0	z
0	$x^{a-1}z^{c'-1}$	0	0	0	$x^{a_i-1}y^{b_i}z^{c-1}$	$-y$
0	0	$y^{b-1}z^{c'-1}$	$y^{b_\rho-1}z^{c-1}$	0	0	x

It is clear that the first four columns are independent so we only need to check the last two. To obtain $x^{a_i-1}y^{b_{i+1}-1}z^{c'}(e_1 \wedge e_2)$ from $\mathrm{im}(\varphi_3)$ we would have to have

$$x^{a_i-1}y^{b_{i+1}-1}z^{c'-1} \cdot \begin{pmatrix} z \\ -y \\ x \end{pmatrix} = \begin{pmatrix} x^{a_i-1}y^{b_{i+1}-1}z^{c'} \\ -x^{a_i-1}y^{b_{i+1}}z^{c'-1} \\ x^{a_i}y^{b_{i+1}-1}z^{c'-1} \end{pmatrix}.$$

But this cannot equal $x^{a_i-1}y^{b_{i+1}-1}z^{c'}(e_1 \wedge e_2)$ because neither $x^{a_i-1}y^{b_{i+1}}z^{c'-1}$ nor $x^{a_i}y^{b_{i+1}-1}z^{c'-1}$ are in I.

To obtain $x^{a_i-1}y^{b_i}z^{c-1}(e_1 \wedge e_3)$ from $\mathrm{im}(\varphi_3)$ we would have to have,

$$-x^{a_i-1}y^{b_i-1}z^{c-1} \cdot \begin{pmatrix} z \\ -y \\ x \end{pmatrix} = \begin{pmatrix} -x^{a_i-1}y^{b_i-1}z^{c} \\ x^{a_i-1}y^{b_i}z^{c-1} \\ -x^{a_i}y^{b_i-1}z^{c-1} \end{pmatrix} = \begin{pmatrix} 0 \\ x^{a_i-1}y^{b_i}z^{c-1} \\ -x^{a_i}y^{b_i-1}z^{c-1} \end{pmatrix}.$$

But $x^{a_i}y^{b_i-1}z^{c-1} \notin I$ so this cannot be equal to $x^{a_i-1}y^{b_i}z^{c-1}(e_1 \wedge e_3)$. Since we have show that each arbitrary element is not dependent mod I we can conclude that this is a minimal generating set for A_2.

We will now compute q using the multiplication table for $A_1 \cdot A_2$. In this table we will write only the monomial for the multiplication since every element in $A_1 \cdot A_2 \subseteq A_3 \cong S$.

Table 4.5: Proof of Theorem 4.3.4 – $A_1 \cdot A_2$

	$x^{a-1}e_1$	$x^{a_i-1}y^{b_i}z^{c'}e_1$	$y^{b-1}e_2$	$y^{b_\rho-1}z^{c'}e_2$	$z^{c-1}e_3$
$x^{a-1}y^{b-1}(e_1 \wedge e_2)$	0	0	0	0	0
$x^{a_i-1}y^{b_{i+1}-1}z^{c'}(e_1 \wedge e_2)$	0	0	0	0	0
$x^{a-1}z^{c'-1}(e_1 \wedge e_3)$	0	0	$x^{a-1}y^{b-1}z^{c'-1}$	0	0
$x^{a_i-1}y^{b_i}z^{c-1}(e_1 \wedge e_3)$	0	0	0	0	0
$y^{b-1}z^{c'-1}(e_2 \wedge e_3)$	$x^{a-1}y^{b-1}z^{c'-1}$	0	0	0	0
$y^{b_\rho-1}z^{c-1}(e_2 \wedge e_3)$	0	0	0	0	0

Here we have that $A_1 \cdot A_2$ is generated by $x^{a-1}y^{b-1}z^{c'-1}(e_1 \wedge e_2 \wedge e_3)$, which implies that $q = 1$. Thus we have shown that $p = q = 1$ and $r = 2$, therefore S is in $\mathbf{B}$. $\square$

In Theorem 4.3.2 we saw what kind of generic monomial ideals would be in $\mathbf{T}$. We can also find a class of non-generic monomial ideals in $\mathbf{T}$ by removing the two mixed double generators of I that we have in Theorem 4.3.4. This gives us the following theorem.

Theorem 4.3.5. *Let I be minimally generated by $n = \rho + 3 \geq 4$ monomials, $\{x^a, y^b, z^c, x^{a_1}y^{b_1}z^{c'}, \ldots, x^{a_\rho}y^{b_\rho}z^{c'}\}$ with $a_i, b_i, c' > 0$ for $1 \leq i \leq \rho$. Then R is in $\mathbf{T}$ with $m = \rho + 2 \geq 3$.*

Proof: The proof here will be similar to what we did in the previous theorem except we will not have to compute the minimal generating sets for A_1 and A_2. Without loss of generality we will assume that

$$I = \langle x^a, y^b, z^c, x^{a_1}y^{b_1}z^{c'}, \ldots, x^{a_\rho}y^{b_\rho}z^{c'}\rangle$$

where $c' > 0$, $a_i > a_{i+1} > 0$ and $0 < b_i < b_{i+1}$ for all $1 \leq i < \rho$. Using similar arguments as in the previous theorem we have that the following pairs of minimal generators represent all of the minimal second syzygies for S,

$$\{x^a, y^b\}, \{x^a, z^c\}, \{x^a, x^{a_1}y^{b_1}z^{c'}\}, \{y^b, z^c\}, \{y^b, x^{a_\rho}y^{b_\rho}z^{c'}\},$$
$$\{z^c, x^{a_i}y^{b_i}z^{c'}\}_{i=1}^{\rho}, \{x^{a_i}y^{b_i}z^{c'}, x^{a_{i+1}}y^{b_{i+1}}z^{c'}\}_{i=1}^{\rho-1}.$$

We notice that the pairs $\{x^a, y^b\}$, $\{x^a, z^c\}$, and $\{y^b, z^c\}$ represent the only minimal second syzygies which have only nonzero entries from I. Thus $p = 3$ by Corollary 4.1.6. We will also note that we only need to show that $q = 0$ because this would imply that S satisfies the structure for $\mathbf{T}$ by Remark 4.2.2.

Using the minimal second syzygies from above we find that the minimal cycles for the minimal third syzygies are given by the sets of generators,

$$\{x^a, y^b, \{x^{a_i}y^{b_i}z^{c'}\}_{i=1}^{\rho}\}, \{x^a, z^c, x^{a_1}y^{b_1}z^{c'}\}, \{y^b, z^c, x^{a_\rho}y^{b_\rho}z^{c'}\}, \{z^c, x^{a_i}y^{b_i}z^{c'}, x^{a_{i+1}}y^{b_{i+1}}z^{c'}\}_{i=1}^{\rho-1}.$$

All but the first cycle only involve three minimal generators none of which satisfy the criterion needed to admit a nonzero entry from I in f_3. The cycle given by $\{x^a, y^b, \{x^{a_i}y^{b_i}z^{c'}\}_{i=1}^{\rho}\}$ will be given up to a sign on the nonzero entries by,

$$\begin{pmatrix} z^{c'} & 0 & y^{b-b_1} & 0 & x^{a-a_\rho} & 0 & \cdots & 0 & x^{a-a_1}y^{b-b_2} & x^{a-a_2}y^{b-b_3} & \cdots & x^{a-a_{\rho-1}}y^{b-b_\rho} \end{pmatrix}$$

where $a - a_i < a - a_{i+1}$ and $b - b_i > b - b_{i+1}$ for $1 \leq i \leq \rho - 1$. It is clear that none of these nonzero elements are in I, and none of them are dependent mod I after row operations. Thus we have shown that there are no rows in f_3 which are dependent mod I which implies that $q = 0$. $\square$

In both of the previous theorems the classes are less than robust because generally we would not have ideals generated like this by randomly picking minimal generators. We would eventually like to have a complete classification of the Tor-algebra for trivariate monomial ideals. The issue this that in general we do not know how the permissable row operations will affect r for a general monomial ideal. It is also unclear as to whether or not we can get other nonzero entries from I in f_3 for these ideals in general. These ideas seem to be the key to showing the Tor-Algebra classification in general.

Another interesting observation we note from our class in **B** is that in f_3 we will have exactly two different nonzero pure power entries in the same column, which are also in I. In some sense this explains the reason that $q \neq r$ in **B**. One could go back to Example 4.1.1 and see that we have both y^3 and z^3 as entries in the same column of f_3, and these are both minimal generators of I. Looking at the multiplication between $A_1 \cdot A_2$ we can see that both $y^2e_2 \cdot xz^2(e_1 \wedge e_3)$ and $z^2e_3 \cdot xy^2(e_1 \wedge e_2)$ yield a nonzero value in $A_1 \cdot A_2$. However since we get that both of these multiplications are the same $q = 1$. In any case it seems that this situation where $q = 1$ and $r = 2$ is unique to the ideals having exactly two different pure power entries from I in the same column of f_3.

We can also see from Example 4.2.4 that we can find instances of non-generic monomial ideals in each class. The pattern does not seem that far off from our generic classification, the only difference being that we have to take the class **B** into account. In Example 4.2.4 we even provide an example in **B** that is outside the scope of our classification from Theorem 4.3.4. The common trait here is that in both cases we have exactly two different nonzero pure power entries from I in the same column in f_3. So we will leave the reader with our conjecture for the complete classification of trivariate monomial ideals.

Conjecture 4.3.6. *Let I be an $\mathfrak{m}$-primary monomial ideal minimally generated by n monomials, then we have the following Tor-algebra classifications for S:*

1. *If $I = \langle x^a, y^b, z^c, x^{a_1}y^{b_1}z^{c_1}, \dots x^{a_\rho}y^{b_\rho}z^{c_\rho}\rangle$ with $a, b, c, a_i, b_i, c_i > 0$ for all $1 \leq i \leq \rho$ and $\rho > 0$, then S is in* **T**.
2. *If $I = \langle x^a, y^b, z^c, x^{\alpha_{11}}y^{\alpha_{21}}z^{\alpha_{31}}, \dots, x^{\alpha_{1\rho}}y^{\alpha_{2\rho}}z^{\alpha_{3\rho}}, x^{a_{11}}y^{a_{21}}z^{a_{31}}, \dots, x^{a_{1\beta}}y^{a_{2\beta}}z^{a_{3\beta}}\rangle$ with $\rho \geq 2$, satisfying the following,*
   1. *(a) for exactly one $i \in \{1, 2, 3\}$ we have that $\alpha_{ij} = \alpha_{ik} > 0$ for all $k, j \in \{1, \dots, \rho\}$, and $a_{il} > \alpha_{ij}$ for all $1 \leq l \leq \beta$,*
   2. *(b) for $s, j \in \{1, 2, 3\}$, $s \neq t$ and $s, t \neq i$ we have that $\alpha_{s1}, \alpha_{t\rho} = 0$, $\alpha_{sj} < \alpha_{sj+1}$ and $\alpha_{tk} > \alpha_{tk+1}$ for every $j, k \in \{1, \dots, \rho - 1\}$.*

   Then S is in **B**.

3. *If* $I = \langle x^a, y^b, z^c, x^{\alpha_1}y^{\beta_1}, x^{\alpha_2}z^{\gamma_1}, y^{\beta_2}z^{\gamma_2}, x^{a_1}y^{b_1}z^{c_1}, \ldots x^{a_\rho}y^{b_\rho}z^{c_\rho}\rangle$, $a, b, c, \alpha_i, \beta_i, \gamma_i > 0$, *then* R *is in* $\mathbf{H}(0,0)$ *and is Golod if and only if there are no* $\sigma_{ij} \in S_2$ *with only nonzero entries from* I.
4. *Otherwise* S *is in* $\mathbf{H}(p,q)$ *with* $p + q \geq 1$.

REFERENCES

[1] E. F. Assmus, *On the homology of local rings*, J. Math **3** (1959), 187–199.

[2] L. L. Avramov, *Small homomorphisms of local rings*, J. Algebra **50** (1978), 400–453.

[3] ———, *Homological asymptotics of modules over local rings* (1987). Math. Sci. Res. Inst. Publ. 15.

[4] ———, *A cohomological study of local rings of embedding codepth 3* (2012). arXiv:1105.3991v2.

[5] L. L. Avramov, H.-B. Foxby, and J. Lescot, *Bass series of local ring homomorphisms of finite flat dimension*, Trans. Amer. Math Soc. **335** (1993), 497–523.

[6] L. L. Avramov and J. Lescot, *Bass numbers and golod rings*, Math. Scand. **51** (1982), 199–211.

[7] H. Bass, *On the ubiquity of gorenstein rings*, Math. Z. **82** (1963), 8–28.

[8] D. Bayer, I. Peeva, and B. Sturmfels, *Monomial resolutions*, Mathematical Research Letters **5** (1998), 31–46.

[9] A. E. Brown, *A structure theorem for a class of grade three perfect ideals*, J. Algebra **105** (1987), 308–327.

[10] W. Bruns and J. Herzog, *Cohen-macaulay rings*, Cambridge University Press, Cambridge, 1993. volume 39 or Cambridge Studies in Advanced Mathematics.

[11] D. Buchsbaum and D. Eisenbud, *Some structure theorems for finite free resolutions*, Adv. Math. **12** (1974), 84–139.

[12] ———, *Algebra structures on minimal free resolutions and gorenstein ideals of codimension 3*, Amer. J. Math. **99** (1977), 447–485.

[13] L. W. Christensen, J. Striuli, and O. Veleche, *Growth in the minimal injective resolution of a local ring*, J. Lond. Math. Soc. (2) **81** (2010), 24–44.

[14] D. Cox, J. Little, and D. O'Shea, *Ideals, varieties, and algorithms*, Second, Springer-Verlag, New York, 1997.

[15] D. Eisenbud, *Introduction to commutative algebra with a view towards algebraic geometry*, Springer-Verlag, New York, 1995.

[16] ———, *The geometry of syzygies*, Springer, New York, 2005. Graduate Texts in Mathematics.

[17] C. Godsil and G. Royle, *Algebraic graph theory*, Springer-Verlag, New York, 2001. volume 207 of Graduate Texts in Mathematics.

[18] D. Jorgensen and G. Leuschke, *On the growth of the betti sequence of the canonical module*, Math. Z. **256** (2006), 647–659.

[19] E. Miller, *Planar graphs as minimal resolutions of trivariate monomial ideals*, Doc. Math. **7** (2002), 43–90.

[20] E. Miller and B. Sturmfels, *Combinatorial commutative algebra*, Springer, New York, 2000. Graduate Texts in Mathematics.

[21] E. Miller, B. Sturmfels, and K. Yanagawa, *Generic and cogeneric monomial ideals*, J. Symbolic Computation **29** (2000), 691–708.

BIOGRAPHICAL STATEMENT

Jared Lafayette Painter was born in Tyler, Texas on July 25, 1981 to Jerry and Debra Painter. He was raised the eldest boy out of five children, and grew up in Tyler, Texas. Jared attended a private school from kindergarten through seventh grade, and was home-schooled through the rest of his junior high and high school education.

Jared first enrolled in Tyler Junior College in August 1999 with the intention of studying Mechanical Engineering. After changing his major several times Jared worked for several years in Information Technology for the City of Tyler and for Wood Networks in Tyler, while working toward a Computer Science degree at the University of Texas at Tyler. In the Summer of 2004 he finally decided that he wanted to study Mathematics and received a Bachelor of Science degree in Mathematics in May of 2005 and a Master of Science degree in Mathematics in May of 2007 from the University of Texas at Tyler. He began the Ph.D. program at the University of Texas at Arlington in August of 2007. In May 2012, he was awarded a Ph.D. in Mathematics under the direction of David Jorgensen.

Jared's research interests lie in commutative algebra, with focus in free resolutions and homological properties over commutative rings.