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Research Article

On the Order Statistics of Standard Normal-Based Power Method Distributions

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This paper derives a procedure for determining the expectations of order statistics associated with the standard normal distribution (Z) and its powers of order three and five (Z^3 and Z^5). The procedure is demonstrated for sample sizes of $n \leq 9$. It is shown that Z^3 and Z^5 have expectations of order statistics that are functions of the expectations for Z and can be expressed in terms of explicit elementary functions for sample sizes of $n \leq 5$. For sample sizes of $n = 6, 7$ the expectations of the order statistics for Z, Z^3, and Z^5 only require a single remainder term.

1. Introduction

Order statistics have played an important role in the development of techniques associated with estimation [1, 2], hypothesis testing [3, 4], and describing data in the context of L-moments [5, 6]. In terms of the latter, L-moments are based on the expectations of linear combinations of order statistics associated with a random variable X. Specifically, the first four L-moments are expressed as

$$\begin{aligned}
\lambda_1 &= E[X_{1:1}], \\
\lambda_2 &= \frac{1}{2}E[X_{2:2} - X_{1:2}], \\
\lambda_3 &= \frac{1}{3}E[X_{3:3} - 2X_{2:3} + X_{1:3}], \\
\lambda_4 &= \frac{1}{4}E[X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}]
\end{aligned} \tag{1.1}$$

or more generally as

$$\lambda_r = \frac{1}{r}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}E\left[X_{r-j:r}\right], \tag{1.2}$$

where the order statistics $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ are drawn from the random variable X. The values of λ_1 and λ_2 are measures of location and scale and are the arithmetic mean and one-half the coefficient of mean difference (or Gini's index of spread), respectively. Higher-order L-moments are transformed to dimensionless quantities referred to as L-moment ratios defined as $\tau_r = \lambda_r/\lambda_2$ for $r \geq 3$, and where τ_3 and τ_4 are the analogs to the conventional measures of skew and kurtosis. In general, L-moment ratios are bounded in the interval $-1 < \tau_r < 1$ as is the index of L-skew (τ_3) where a symmetric distribution implies that all L-moment ratios with odd subscripts are zero. Other smaller boundaries can be found for more specific cases. For example, the index of L-kurtosis (τ_4) has the boundary condition for continuous distributions of [7]

$$\frac{5\tau_3^2 - 1}{4} < \tau_4 < 1. \tag{1.3}$$

Headrick [8] derived classes of standard normal-L-moment-based power method distributions using the polynomial transformation

$$p(Z) = \sum_{i=1}^{m} c_i Z^{i-1}, \tag{1.4}$$

where $Z \sim i.i.d.\ N(0,1)$. Setting $m = 4$ ($m = 6$) gives the third- (fifth-) order class of power method distributions. The shape of $p(Z)$ in (1.4) is contingent on the values of the constant coefficients c_i. For the larger class of nonnormal distributions associated with $m = 6$, the coefficients are computed from the system of equations given in Headrick ([8, Equations (2.8)–(2.13)] for specified values of L-moment ratios ($\tau_{3,\ldots,6}$). In general, λ_1 and λ_2 are standardized to the unit normal distribution as

$$\begin{aligned} \lambda_1 &= c_1 + c_3 + 3c_5 = 0, \\ \lambda_2 &= \frac{(4c_2 + 10c_4 + 43c_6)}{4\sqrt{\pi}} = \frac{1}{\sqrt{\pi}}. \end{aligned} \tag{1.5}$$

The pdf and cdf associated with (1.4) are given in parametric form as in [8, Equations (1.3) and (1.4)]

$$\begin{aligned} f_{p(z)}(p(z)) &= \overline{f}(z) = \left(p(z), \frac{\phi(z)}{p'(z)}\right), \\ F_{p(z)}(p(z)) &= \overline{F}(z) = (p(z), \Phi(z)), \end{aligned} \tag{1.6}$$

where $\overline{f} : \Re \mapsto \Re^2$ and $\overline{F} : \Re \mapsto \Re^2$ are the parametric forms of the pdf and cdf with the mappings $z \mapsto (x, y)$ and $z \mapsto (x, v)$ with $x = p(z)$, $y = \phi(z)/p'(z)$, $v = \Phi(z)$, and where $\phi(z)$ and $\Phi(z)$ are the standard normal pdf and cdf, respectively. For further details on the distributional properties associated with power method transformations see [9, pages 9–30] and [8] in terms of conventional moment and L-moment theory, respectively.

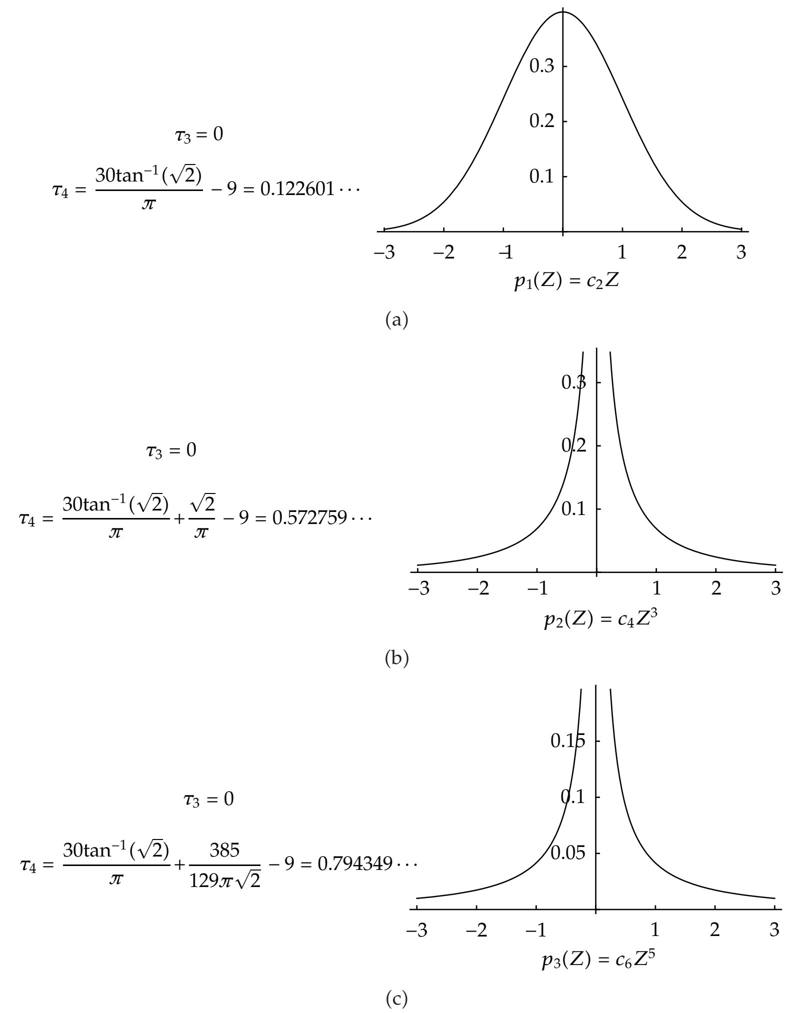

Figure 1: Graphs of the three standard normal-based power method distributions $p_t(Z)$ in (1.7) and their values of L-skew (τ_3) and L-kurtosis (τ_4).

Of concern in this study are three power method distributions related to (1.4) and (1.5) as

$$p_t(Z) = c_{2t} Z^{2t-1}, \quad \text{where if} \begin{cases} t = 1, & c_2 = 1,\ c_4 = 0,\ c_6 = 0, \\ t = 2, \quad \text{then} & c_2 = 0,\ c_4 = 2/5,\ c_6 = 0, \\ t = 3, & c_2 = 0,\ c_4 = 0,\ c_6 = 4/43, \end{cases} \tag{1.7}$$

and thus $p_1(Z) = Z$, $p_2(Z) = (2/5)Z^3$ and $p_3(Z) = (4/43)Z^5$. Note that these power method distributions are symmetric and imply that $c_{1,3,5} = 0$ in (1.4). The graphs of the pdfs associated with the distributions in (1.7) are given in Figure 1 along with their values of L-skew and L-kurtosis. We would point out that the importance of these distributions was noted by Stoyanov [10, page 281], "…power transformations [such as $p_2(Z)$ and $p_3(Z)$] can be considered as functional transformations on random data, usually called Box-Cox transformations. Their importance in the area of statistics and its applications is well known."

The standard normal distribution $p_1(Z)$ in (1.7) is the only case of the three distributions considered that is moment determinant. That is, $p_2(Z)$ and $p_3(Z)$ have the so-called classical problem of moments insofar as their respective cdfs have nonunique solutions (i.e., they are moment indeterminant, see [10–12]). However, as pointed out by Huang [12], $p_2(Z)$ and $p_3(Z)$ are determinant in the context of order statistics moments.

The derivation of the expected values of single order statistics associated with $p_1(Z)$ in terms of explicit elementary functions has been attempted by numerous authors (see [13–17]). As indicated by Johnson et al. [18, pages 93-94] these attempts fail to give explicit expressions in terms of elementary functions for the expected values of order statistics with sample sizes of $n > 5$. However, Renner [19] provides a technique for expressing the expected values of order statistics associated with $p_1(Z)$ for $n = 6,\ 7$ based on a single power series.

There is a paucity of research on the expectations of order statistics associated with $p_2(Z)$ and $p_3(Z)$ in the context of explicit elementary functions. Thus, what follows in Section 2 is the development of an approach for determining the expected values of the order statistics for $p_2(Z)$ and $p_3(Z)$, which is based on a generalization of Renner's [19] discussion in the context of $p_1(Z)$. In Section 3, some specific evaluations of the generalization are provided to demonstrate the methodology.

2. Methodology

The expected values of the order statistics associated with (1.7) can be determined based on the following expression [20, page 34]:

$$E\left[p(Z)_{j:n}\right] = n2^{-n}\binom{n-1}{j-1}\int_0^{\infty} p_t(z)\varphi(z)\left([1+\Psi(z)]^{j-1}[1-\Psi(z)]^{n-j} - [1-\Psi(z)]^{j-1}[1+\Psi(z)]^{n-j}\right)dz, \tag{2.1}$$

where $p_t(z)$ is defined as in (1.7) and $\varphi(z) = 2\phi(z)$ and $\Psi(z) = 2\Phi(z) - 1$ are the pdf and cdf of the folded unit normal distribution at $z = 0$. Table 1 gives a summary of some specific expansions of the polynomial in (2.1) for sample sizes of $n = 1, \ldots, 9$, which are applicable to all three distributions related to $p_t(z)$. Inspection of Table 1 indicates that we have in general (a) $E[p(Z)_{j:n}] = -E[p(Z)_{n+1-j:n}]$, (b) the median $E[p(Z)_{j:n}] = -E[p(Z)_{j:n}] = 0$, and (c) the $E[p(Z)_{j:n}]$ are linear combinations of the integrals I_{2r-1} for $r = 1, 2, \ldots$, with only odd subscripts appearing as only odd powers of $\Psi(z)$ appear in the polynomial expansions associated with (2.1). As such, I_{2r-1} in (2.1) can be expressed as

$$I_{2r-1} = \int_0^{\infty} p_t(z)\varphi(z)[\Psi(z)]^{2r-1}dz. \tag{2.2}$$

Equation (2.2) may be integrated by parts as

$$I_{2r-1} = (2r-1)\int_0^{\infty} q_t(z)\varphi(z)^2[\Psi(z)]^{2r-2}dz, \tag{2.3}$$

Table 1: General expressions for the expected values of the order statistics for $p_{t=1,2,3}(Z)$ in (1.7) and sample sizes of $n = 1, \dots, 9$. I_{2r-1} denotes an integral in (2.1) where $r = 1, \dots, 4$.

Sample size (n)	Expected value
1	$E[p_t(Z)_{1:1}] = -E[p_t(Z)_{1:1}] = 0$
2	$E[p_t(Z)_{2:2}] = -E[p_t(Z)_{1:2}] = I_1 = 1/\sqrt{\pi}$
3	$E[p_t(Z)_{2:3}] = -E[p_t(Z)_{2:3}] = 0$ $E[p_t(Z)_{3:3}] = -E[p_t(Z)_{1:3}] = (3/2)I_1 = 3/(2\sqrt{\pi})$
4	$E[p_t(Z)_{3:4}] = -E[p_t(Z)_{2:4}] = (3/2)(I_1 - I_3)$ $E[p_t(Z)_{4:4}] = -E[p_t(Z)_{1:4}] = (1/2)(3I_1 + I_3)$
5	$E[p_t(Z)_{3:5}] = -E[p_t(Z)_{3:5}] = 0$ $E[p_t(Z)_{4:5}] = -E[p_t(Z)_{2:5}] = (5/2)(I_1 - I_3)$ $E[p_t(Z)_{5:5}] = -E[p_t(Z)_{1:5}] = (5/4)(I_1 + I_3)$
6	$E[p_t(Z)_{4:6}] = -E[p_t(Z)_{3:6}] = (15/8)(I_1 - 2I_3 + I_5)$ $E[p_t(Z)_{5:6}] = -E[p_t(Z)_{2:6}] = (15/16)(3I_1 - 2I_3 - I_5)$ $E[p_t(Z)_{6:6}] = -E[p_t(Z)_{1:6}] = (3/16)(5I_1 + 10I_3 + I_5)$
7	$E[p_t(Z)_{4:7}] = -E[p_t(Z)_{4:7}] = 0$ $E[p_t(Z)_{5:7}] = -E[p_t(Z)_{3:7}] = (105/32)(I_1 - 2I_3 + I_5)$ $E[p_t(Z)_{6:7}] = -E[p_t(Z)_{2:7}] = (21/8)(I_1 - I_5)$ $E[p_t(Z)_{7:7}] = -E[p_t(Z)_{1:7}] = (7/32)(3I_1 + 10I_3 + 3I_5)$
8	$E[p_t(Z)_{5:8}] = -E[p_t(Z)_{4:8}] = (35/16)(I_1 - 3I_3 + 3I_5 - I_7)$ $E[p_t(Z)_{6:8}] = -E[p_t(Z)_{3:8}] = (21/16)(3I_1 - 5I_3 + I_5 + I_7)$ $E[p_t(Z)_{7:8}] = -E[p_t(Z)_{2:8}] = (7/16)(5I_1 + 5I_3 - 9I_5 - I_7)$ $E[p_t(Z)_{8:8}] = -E[p_t(Z)_{1:8}] = (1/16)(7I_1 + 35I_3 + 21I_5 + I_7)$
9	$E[p_t(Z)_{5:9}] = -E[p_t(Z)_{5:9}] = 0$ $E[p_t(Z)_{6:9}] = -E[p_t(Z)_{4:9}] = (63/16)(I_1 - 3I_3 + 3I_5 - I_7)$ $E[p_t(Z)_{7:9}] = -E[p_t(Z)_{3:9}] = (63/16)(I_1 - I_3 - I_5 + I_7)$ $E[p_t(Z)_{8:9}] = -E[p_t(Z)_{2:9}] = (9/16)(3I_1 + 7I_3 - 7I_5 + 3I_7)$ $E[p_t(Z)_{9:9}] = -E[p_t(Z)_{1:9}] = (9/32)(I_1 + 7I_3 + 7I_5 + I_7)$

where $q_1(z) = 1$, $q_2(z) = (2/5)(z^2 + 2)$ and $q_3(z) = (4/43)(z^4 + 4z^2 + 8)$, for $p_1(z)$, $p_2(z)$, and $p_3(z)$, respectively. Note that $\Psi(0) = 0$ and $\lim_{z\to+\infty}\varphi(z) = 0$. Evaluating (2.3) for $r = 1$ gives a coefficient of mean difference of

$$I_1 = \int_0^{\infty} q_t(z)\varphi(z)^2 dz = \frac{1}{\sqrt{\pi}} \tag{2.4}$$

for all $p_t(z)$ in (1.7), which is consistent with the specification in (1.5) and given in Table 1.

The expression $[\Psi(z)]^{2r-2}$ in (2.3) can be expressed as

$$[\Psi(z)]^{2r-2} = \left(\frac{2}{\pi}\right)^{r-1}\left[\iint_0^z \exp\left\{-\frac{1}{2}u^2\right\}du\right]^{2r-2} \tag{2.5}$$

or analogously as a double integral over $\Re^2$ as

$$[\Psi(z)]^{2r-2} = \left(\frac{2}{\pi}\right)^{r-1}\left[\iiint_0^z \exp\left\{-\frac{1}{2}\left(z_1^2 + z_2^2\right)\right\}dz_1 dz_2\right]^{r-1}. \tag{2.6}$$

Using (2.6), let $z_2 = z_1 \tan\theta_1$ and thus $dz_2 = z_1 \sec^2\theta_1 d\theta_1$. Further, let $z_1^2 + z_2^2 = z_1^2 \sec^2\theta_1$. As such, the region of integration will be reduced to one-half of the area of the original rectangle associated with (2.6). Thus, we have

$$\begin{aligned}
[\Psi(z)]^{2r-2} &= \left(\frac{2}{\pi}\right)^{r-1} \left[2\int_0^{\pi/4}\int_0^{z} \exp\left\{-\frac{1}{2}\left(z_1^2\sec^2\theta_1\right)\right\} dz_1\left(z_1\sec^2\theta_1 d\theta_1\right)\right]^{r-1} \\
&= \left(\frac{4}{\pi}\right)^{r-1} \left[\int_0^{\pi/4}\left\{\int_0^{z} \exp\left\{-\frac{1}{2}\left(z_1^2\sec^2\theta_1\right)\right\} z_1 dz_1\right\}\sec^2\theta_1 d\theta_1\right]^{r-1}.
\end{aligned} \tag{2.7}$$

Subsequently, setting $z_1^2 = w$ in (2.7), where $z_1 dz_1 = dw/2$, gives

$$\begin{aligned}
[\Psi(z)]^{2r-2} &= \left(\frac{4}{\pi}\right)^{r-1} \left[\int_0^{\pi/4}\left\{\int_0^{z^2} \exp\left(-\frac{1}{2}w\sec^2\theta_1\right)\frac{dw}{2}\right\}\sec^2\theta_1 d\theta_1\right]^{r-1} \\
&= \left(\frac{4}{\pi}\right)^{r-1} \left[\int_0^{\pi/4}\left\{\frac{1}{2}\cdot\frac{\exp\left(-(1/2)w\sec^2\theta_1\right)}{-(1/2)\sec^2\theta_1}\right\}_0^{z^2}\sec^2\theta_1 d\theta_1\right]^{r-1},
\end{aligned} \tag{2.8}$$

and hence

$$[\Psi(z)]^{2r-2} = \left(\frac{4}{\pi}\right)^{r-1} \left[\int_0^{\pi/4}\left(1-\exp\left\{-\frac{1}{2}\left(z^2\sec^2\theta_1\right)\right\}\right)d\theta_1\right]^{r-1}. \tag{2.9}$$

Expanding (2.9) yields

$$[\Psi(z)]^{2r-2} = 1 + \left\{\sum_{k=1}^{r-1}(-1)^k\binom{r-1}{k}\left(\frac{4}{\pi}\right)^k\int_0^{\pi/4}\cdots\int_0^{\pi/4}\exp\left\{-\frac{1}{2}z^2\sum_{i=1}^{k}\sec^2\theta_i\right\}d\theta_1\cdots d\theta_k\right\}, \tag{2.10}$$

where the subscript i runs faster than k. For example, if $r = 4$, then (2.10) would appear more specifically as

$$\begin{aligned}
[\Psi(z)]^{2r-2} = 1 &- \binom{r-1}{1}\left(\frac{4}{\pi}\right)\int_0^{\pi/4}\exp\left\{-\frac{1}{2}z^2\sec^2\theta_1\right\}d\theta_1 \\
&+ \binom{r-1}{2}\left(\frac{4}{\pi}\right)^2\iint_0^{\pi/4}\exp\left\{-\frac{1}{2}z^2\left(\sec^2\theta_1+\sec^2\theta_2\right)\right\}d\theta_1 d\theta_2 \\
&- \binom{r-1}{3}\left(\frac{4}{\pi}\right)^3\iiint_0^{\pi/4}\exp\left\{-\frac{1}{2}z^2\left(\sec^2\theta_1+\sec^2\theta_2+\sec^2\theta_3\right)\right\}d\theta_1 d\theta_2 d\theta_3.
\end{aligned} \tag{2.11}$$

Substituting (2.10) into (2.3) and initially integrating with respect to z (Lichtenstein, [21]) yields

$$\sqrt{\pi}\int_0^{\infty} q_t(z)\varphi(z)^2 \exp\left\{-\frac{1}{2}z^2\sum_{i=1}^{k}\sec^2\theta_i\right\}dz = g_t\left(\sec^2\theta_i\right), \tag{2.12}$$

where the specific forms of $g_t(\sec^2\theta_i)$, which are associated with $p_t(z)$, are

$$\begin{aligned}
g_1\left(\sec^2\theta_i\right) &= \frac{\sqrt{2}}{\left(2+\sum_{i=1}^{k}\sec^2\theta_i\right)^{1/2}},\\
g_2\left(\sec^2\theta_i\right) &= \frac{2\sqrt{2}\left(5+2\sum_{i=1}^{k}\sec^2\theta_i\right)}{5\left(2+\sum_{i=1}^{k}\sec^2\theta_i\right)^{3/2}},\\
g_3\left(\sec^2\theta_i\right) &= \frac{4\sqrt{2}\left(3+4\left(2+\sum_{i=1}^{k}\sec^2\theta_i\right)+8\left(2+\sum_{i=1}^{k}\sec^2\theta_i\right)^2\right)}{43\left(2+\sum_{i=1}^{k}\sec^2\theta_i\right)^{5/2}}.
\end{aligned} \tag{2.13}$$

Equations (2.13) can be more conveniently expressed as

$$g_t\left(\sec^2\theta_i\right) = g_1\left(\sec^2\theta_i\right) - h_t\left(\sec^2\theta_i\right), \tag{2.14}$$

where the specific forms of $h_t(\sec^2\theta_i)$ are

$$h_1\left(\sec^2\theta_i\right) = 0, \tag{2.15}$$

$$h_2\left(\sec^2\theta_i\right) = \frac{\sqrt{2}\left(\sum_{i=1}^{k}\sec^2\theta_i\right)}{5\left(2+\sum_{i=1}^{k}\sec^2\theta_i\right)^{3/2}}, \tag{2.16}$$

$$h_3\left(\sec^2\theta_i\right) = \frac{\sqrt{2}\left(11\sum_{i=1}^{k}\sec^4\theta_i + 28\sum_{i=1}^{k}\sec^2\theta_i + 22\sum_{i<j}\sec^2\theta_i\sec^2\theta_j\right)}{43\left(2+\sum_{i=1}^{k}\sec^2\theta_i\right)^{5/2}} \tag{2.17}$$

and where $\sum_{i<j}$ in (2.17) indicates summing over all $k(k-1)/2$ pairwise combinations. Hence, the integral in (2.3) can be expressed as

$$I_{2r-1} = \frac{2r-1}{\sqrt{\pi}}\left(1+\left\{\sum_{k=1}^{r-1}(-1)^k\binom{r-1}{k}\left(\frac{4}{\pi}\right)^k\int_0^{\pi/4}\cdots\int_0^{\pi/4} g_t\left(\sec^2\theta_i\right)d\theta_1\cdots d\theta_k\right\}\right), \tag{2.18}$$

and subsequently substituting (2.14) into (2.18) gives

$$\begin{aligned} &I_{2r-1} \\ &= \frac{2r-1}{\sqrt{\pi}}\left(1+\left\{\sum_{k=1}^{r-1}(-1)^k\binom{r-1}{k}\left(\frac{4}{\pi}\right)^k\int_0^{\pi/4}\cdots\int_0^{\pi/4}\left(g_1\left(\sec^2\theta_i\right)-h_t\left(\sec^2\theta_i\right)\right)d\theta_1\cdots d\theta_k\right\}\right). \end{aligned} \tag{2.19}$$

The integral associated with $g_1(\sec^2\theta_i)$ in (2.19) cannot be expressed in terms of explicit elementary functions for $k > 1$, which also implies $r > 2$ and sample sizes of $n > 5$ in Table 1. As such, we will consider the approximating function $g_1^*(\sec^2\theta_i)$ as

$$g_1^*\left(\sec^2\theta_i\right) = \left(2^{k/2}\right)\prod_{i=1}^{k}\frac{1}{\left(2+\sec^2\theta_i\right)^{1/2}}, \tag{2.20}$$

where

$$\begin{aligned} \int_0^{\pi/4}\cdots\int_0^{\pi/4} g_1\left(\sec^2\theta_i\right)d\theta_1\cdots d\theta_k &= \int_0^{\pi/4}\cdots\int_0^{\pi/4} g_1^*\left(\sec^2\theta_i\right)d\theta_1\cdots d\theta_k \\ &= \begin{cases} \tan^{-1}\left(1/\sqrt{2}\right), & k = 1, \\ 0, & k \longrightarrow \infty. \end{cases} \end{aligned} \tag{2.21}$$

Thus, for finite $k > 1$ we have

$$\begin{aligned} \int_0^{\pi/4}\cdots\int_0^{\pi/4} g_1\left(\sec^2\theta_i\right)d\theta_1\cdots d\theta_k &= \int_0^{\pi/4}\cdots\int_0^{\pi/4} g_1^*\left(\sec^2\theta_i\right)d\theta_1\cdots d\theta_k + \varepsilon_k \\ &= \left(\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)^k + \varepsilon_k, \end{aligned} \tag{2.22}$$

where ε_k is the remainder term required for $k > 1$ and where $\varepsilon_1 = 0$ for $r = 1,\ 2$ and $n \le 5$. Thus, using (2.22), (2.19) can be expressed as

$$\begin{aligned} &I_{2r-1} \\ &= \frac{2r-1}{\sqrt{\pi}}\left(\left\{1+\sum_{k=1}^{r-1}(-1)^k\binom{r-1}{k}\left(\frac{4}{\pi}\right)^k \right.\right. \\ &\qquad \left.\left.\times\left(\left(\left(\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)^k+\varepsilon_k\right)-\int_0^{\pi/4}\cdots\int_0^{\pi/4} h_t\left(\sec^2\theta_i\right)d\theta_1\cdots d\theta_k\right)\right\}\right). \end{aligned} \tag{2.23}$$

The remainder terms $\varepsilon_{k>1}$ in (2.23) can be solved by using (2.3), (2.15), (2.23), and the error function Erf [22], where Erf would replace $\Phi(z)$ in (2.3) where $\Psi(z) = 2\Phi(z) - 1$. More specifically, Table 2 gives the values of ε_k for $k = 1,\ldots 12,\ 25$, and 50 with 40-digit precision.

Table 2: Computed values of the remainder term ε_k associated with (2.23). The values were computed with 40-digit precision.

Sample size (n)	Integral	Remainder term
$1,\ldots,5$	I_1, I_3	$\varepsilon_1 = 0.0$
$6,7$	I_5	$\varepsilon_2 = 0.03140698829552010270731937950881276500595$
$8,9$	I_7	$\varepsilon_3 = 0.05156068650031409787170392919312656858246$
$10,11$	I_9	$\varepsilon_4 = 0.05900198710355817149868423817928465212298$
$12,13$	I_{11}	$\varepsilon_5 = 0.05808975458203638968882522593413660371348$
$14,15$	I_{13}	$\varepsilon_6 = 0.05274763616761422221709626523935998463539$
$16,17$	I_{15}	$\varepsilon_7 = 0.04559236574104643530748593758544745949676$
$18,19$	I_{17}	$\varepsilon_8 = 0.03815223895234453779274127861572423887877$
$20,21$	I_{19}	$\varepsilon_9 = 0.03122205691467168489718556870682270636055$
$22,23$	I_{21}	$\varepsilon_{10} = 0.02514855254614865670209122288596241803047$
$24,25$	I_{23}	$\varepsilon_{11} = 0.02002429921405354560405588075438666460570$
$26,27$	I_{25}	$\varepsilon_{12} = 0.01580928681263632398753707685232879723154$
$\vdots$	$\vdots$	$\vdots$
$52,53$	I_{51}	$\varepsilon_{25} = 0.00057455597453332805073409074487236584232$
$\vdots$	$\vdots$	$\vdots$
$102,103$	I_{101}	$\varepsilon_{50} = 0.00000099193614769461065745252616987082859$

Table 3: Expected values of order statistics for $p_1(Z) = Z$ for $n = 4,5$.

$$E[p_1(Z)_{3:4}] = -\frac{3}{\sqrt{\pi}} + \frac{18\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} = 0.29701138\ldots$$

$$E[p_1(Z)_{4:4}] = \frac{3}{\sqrt{\pi}} - \frac{6\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} = 1.02937537\ldots$$

$$E[p_1(Z)_{3:5}] = 0$$

$$E[p_1(Z)_{4:5}] = -\frac{5}{\sqrt{\pi}} + \frac{30\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} = 0.49501897\ldots$$

$$E[p_1(Z)_{5:5}] = \frac{5}{\sqrt{\pi}} - \frac{15\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} = 1.16296447\ldots$$

Table 4: Expected values of order statistics for $p_2(Z) = (2/5)Z^3$ for $n = 4,5$.

$$E[p_1(Z)_{3:4}] = -\frac{3\sqrt{2}}{5\pi^{3/2}} + E[p_1(Z)_{3:4}] = 0.14462665\ldots$$

$$E[p_2(Z)_{4:4}] = \frac{\sqrt{2}}{5\pi^{3/2}} + E[p_1(Z)_{4:4}] = 1.08017028\ldots$$

$$E[p_2(Z)_{3:5}] = 0$$

$$E[p_2(Z)_{4:5}] = -\frac{\sqrt{2}}{\pi^{3/2}} + E[p_1(Z)_{4:5}] = 0.24104442\ldots$$

$$E[p_2(Z)_{5:5}] = \frac{1}{\sqrt{2}\pi^{3/2}} + E[p_1(Z)_{5:5}] = 1.28995174\ldots$$

Inspection of Table 2 indicates that the (positive) remainder term achieves a maximum at ε_4 and thereafter tends to zero as k increases (i.e., $\varepsilon_k \to 0$ for $k > 4$).

Table 5: Expected values of order statistics for $p_3(Z) = (4/43)Z^5$ for $n = 4, 5$.

$E[p_3(Z)_{3:4}] = -\dfrac{77}{43\sqrt{2}\pi^{3/2}} + E[p_1(Z)_{3:4}] = 0.069615569\ldots$
$E[p_3(Z)_{4:4}] = \dfrac{77}{129\sqrt{2}\pi^{3/2}} + E[p_1(Z)_{4:4}] = 1.10517397\ldots$
$E[p_3(Z)_{3:5}] = 0$
$E[p_3(Z)_{4:5}] = -\dfrac{385}{129\sqrt{2}\pi^{3/2}} + E[p_1(Z)_{4:5}] = 0.11602594\ldots$
$E[p_3(Z)_{5:5}] = \dfrac{385}{258\sqrt{2}\pi^{3/2}} + E[p_1(Z)_{5:5}] = 1.35246098\ldots$

Table 6: Expected values of order statistics for $p_1(Z) = Z$ for $n = 6, 7$.

$E[p_1(Z)_{4:6}] = \dfrac{150\varepsilon_2}{\pi^{5/2}} - \dfrac{30\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} + \dfrac{150\tan^{-1}(1/\sqrt{2})^2}{\pi^{5/2}} = 0.20154683\ldots$
$E[p_1(Z)_{5:6}] = -\dfrac{15}{2\sqrt{\pi}} - \dfrac{75\varepsilon_2}{\pi^{5/2}} + \dfrac{60\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} - \dfrac{75\tan^{-1}(1/\sqrt{2})^2}{\pi^{5/2}} = 0.64177503\ldots$
$E[p_1(Z)_{6:6}] = \dfrac{15}{2\sqrt{\pi}} + \dfrac{15\varepsilon_2}{\pi^{5/2}} - \dfrac{30\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} + \dfrac{15\tan^{-1}(1/\sqrt{2})^2}{\pi^{5/2}} = 1.26720636\ldots$
$E[p_1(Z)_{4:7}] = 0$
$E[p_1(Z)_{5:7}] = \dfrac{525\varepsilon_2}{2\pi^{5/2}} - \dfrac{105\tan^{-1}(1/\sqrt{2})}{2\pi^{3/2}} + \dfrac{525\tan^{-1}(1/\sqrt{2})^2}{2\pi^{5/2}} = 0.35270695\ldots$
$E[p_1(Z)_{6:7}] = -\dfrac{21}{2\sqrt{\pi}} - \dfrac{210\varepsilon_2}{\pi^{5/2}} + \dfrac{105\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} - \dfrac{210\tan^{-1}(1/\sqrt{2})^2}{\pi^{5/2}} = 0.75737427\ldots$
$E[p_1(Z)_{7:7}] = \dfrac{21}{2\sqrt{\pi}} + \dfrac{105\varepsilon_2}{2\pi^{5/2}} - \dfrac{105\tan^{-1}(1/\sqrt{2})}{2\pi^{3/2}} + \dfrac{105\tan^{-1}(1/\sqrt{2})^2}{2\pi^{5/2}} = 1.35217837\ldots$

We would note that the approach taken here to determine ε_2 is analogous to Renner's [19] approach of developing a power series for this value. That is, the remainder term ε_2 in Table 2 is also the value approximated in [19] for $p_1(Z)$. Further, we would note that extending the approach in [19] for computing the remainder terms for $k > 2$ would become computationally burdensome.

To demonstrate (2.23) more specifically, if $r = 4$ and $t = 2$ in (1.7), then the integral I_7 associated with $p_2(Z)$ would appear as

$$\begin{aligned} I_7 = \frac{2r-1}{\sqrt{\pi}} \Bigg\{ 1 &- \binom{r-1}{1}\left(\frac{4}{\pi}\right)\left(\left(\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right) - \int_0^{\pi/4} h_2\left(\sec^2\theta_i\right) d\theta_1\right) \\ &+ \binom{r-1}{2}\left(\frac{4}{\pi}\right)^2\left(\left(\left(\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)^2 + \varepsilon_2\right) - \iint_0^{\pi/4} h_2\left(\sec^2\theta_i\right) d\theta_1 d\theta_2\right) \\ &- \binom{r-1}{3}\left(\frac{4}{\pi}\right)^3\left(\left(\left(\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)^3 + \varepsilon_3\right) - \iiint_0^{\pi/4} h_2\left(\sec^2\theta_i\right) d\theta_1 d\theta_2 d\theta_3\right)\Bigg\}. \end{aligned} \tag{2.24}$$

Table 7: Expected values of order statistics for $p_2(Z) = (2/5)Z^3$ for $n = 6,7$.

$E[p_2(Z)_{4:6}] = \frac{\sqrt{2}}{\pi^{3/2}} - \frac{10\sqrt{2}\tan^{-1}(3\sqrt{3/2}/7)}{\pi^{5/2}} + E[p_1(Z)_{4:6}] = 0.06475951\ldots$
$E[p_2(Z)_{5:6}] = -\frac{2\sqrt{2}}{\pi^{3/2}} + \frac{5\sqrt{2}\tan^{-1}(3\sqrt{3/2}/7)}{\pi^{5/2}} + E[p_1(Z)_{5:6}] = 0.32918688\ldots$
$E[p_2(Z)_{6:6}] = \frac{2\sqrt{2}}{\pi^{3/2}} - \frac{\sqrt{2}\tan^{-1}(3\sqrt{3/2}/7)}{\pi^{5/2}} + E[p_1(Z)_{6:6}] = 1.48210471\ldots$
$E[p_2(Z)_{4:7}] = 0$
$E[p_2(Z)_{5:7}] = \frac{7}{2\sqrt{2}\pi^{3/2}} - \frac{35\tan^{-1}(3\sqrt{3/2}/7)}{\sqrt{2}\pi^{5/2}} + E[p_1(Z)_{5:7}] = 0.11332914\ldots$
$E[p_2(Z)_{6:7}] = -\frac{7}{\sqrt{2}\pi^{3/2}} + \frac{14\sqrt{2}\tan^{-1}(3\sqrt{3/2}/7)}{\pi^{5/2}} + E[p_1(Z)_{6:7}] = 0.41552998\ldots$
$E[p_2(Z)_{7:7}] = \frac{7}{2\sqrt{2}\pi^{3/2}} - \frac{7\tan^{-1}(3\sqrt{3/2}/7)}{\sqrt{2}\pi^{5/2}} + E[p_1(Z)_{7:7}] = 1.65986717\ldots$

Table 8: Expected values of order statistics for $p_3(Z) = (4/43)Z^5$ for $n = 6,7$.

$E[p_3(Z)_{4:6}] = \frac{10\sqrt{3}}{43\pi^{5/2}} + \frac{385}{129\sqrt{2}\pi^{3/2}} - \frac{1925\sqrt{2}\tan^{-1}(3\sqrt{3/2}/7)}{129\pi^{5/2}} + E[p_1(Z)_{4:6}] = 0.02045216\ldots$
$E[p_3(Z)_{5:6}] = -\frac{5\sqrt{3}}{43\pi^{5/2}} - \frac{385}{129\sqrt{2}\pi^{3/2}} + \frac{1925\sqrt{2}\tan^{-1}(3\sqrt{3/2}/7)}{129\sqrt{2}\pi^{5/2}} + E[p_1(Z)_{5:6}] = 0.16381284\ldots$
$E[p_3(Z)_{6:6}] = \frac{\sqrt{3}}{43\pi^{5/2}} + \frac{385}{129\sqrt{2}\pi^{3/2}} - \frac{385\tan^{-1}(3\sqrt{3/2}/7)}{129\sqrt{2}\pi^{5/2}} + E[p_1(Z)_{6:6}] = 1.59019061\ldots$
$E[p_3(Z)_{4:7}] = 0$
$E[p_3(Z)_{5:7}] = \frac{35\sqrt{3}}{86\pi^{5/2}} + \frac{2695}{516\sqrt{2}\pi^{3/2}} - \frac{13475\tan^{-1}(3\sqrt{3/2}/7)}{258\sqrt{2}\pi^{5/2}} + E[p_1(Z)_{5:7}] = 0.03579128\ldots$
$E[p_3(Z)_{6:7}] = -\frac{14\sqrt{3}}{43\pi^{5/2}} - \frac{2695}{258\sqrt{2}\pi^{3/2}} + \frac{2695\sqrt{2}\tan^{-1}(3\sqrt{3/2}/7)}{129\pi^{5/2}} + E[p_1(Z)_{6:7}] = 0.21502146\ldots$
$E[p_3(Z)_{7:7}] = \frac{7\sqrt{3}}{86\pi^{5/2}} + \frac{2695}{516\sqrt{2}\pi^{3/2}} - \frac{2695\tan^{-1}(3\sqrt{3/2}/7)}{258\sqrt{2}\pi^{5/2}} + E[p_1(Z)_{7:7}] = 1.81938546\ldots$

3. Evaluations

Tables 3–5 give evaluations for the expected values of the order statistics for $p_1(Z)$, $p_2(Z)$, and $p_3(Z)$ in (1.7), which are based on (2.23) and the general formulae given in Table 1 for sample sizes of $n = 4, 5$. Inspection of Tables 4 and 5 indicates that the expected values for $p_2(Z)$ and $p_3(Z)$ are all expressed in terms of elementary functions and are also functions of the expectations associated with $p_1(Z)$ in Table 3.

Presented in Tables 6, 7, and 8 are the evaluations for all three distributions in (1.7) for samples of sizes $n = 6, 7$ where the expectations of the order statistics for $p_1(Z)$, $p_2(Z)$, and $p_3(Z)$ are all expressed in terms of explicit elementary functions and a single remainder term. Tables 9 and 10 give the expected values of the order statistics associated with the standard

Table 9: Expected values of order statistics for $p_1(Z) = Z$ for $n = 8$.

$$E[p_1(Z)_{5:8}] = -\frac{210\varepsilon_2}{\pi^{5/2}} + \frac{980\varepsilon_3}{\pi^{7/2}} - \frac{210\tan^{-1}(1/\sqrt{2})^2}{\pi^{5/2}} + \frac{980\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} = 0.15251439\ldots$$

$$E[p_1(Z)_{6:8}] = \frac{546\varepsilon_2}{\pi^{5/2}} - \frac{588\varepsilon_3}{\pi^{7/2}} - \frac{84\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} + \frac{546\tan^{-1}(1/\sqrt{2})^2}{\pi^{5/2}} - \frac{588\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} = 0.47282249\ldots$$

$$\begin{aligned} E[p_1(Z)_{7:8}] &= -\frac{14}{\sqrt{\pi}} - \frac{462\varepsilon_2}{\pi^{5/2}} + \frac{196\varepsilon_3}{\pi^{7/2}} + \frac{168\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} - \frac{462\tan^{-1}(1/\sqrt{2})^2}{\pi^{5/2}} + \frac{196\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} \\ &= 0.85222486\ldots \end{aligned}$$

$$E[p_1(Z)_{8:8}] = \frac{14}{\sqrt{\pi}} + \frac{126\varepsilon_2}{\pi^{5/2}} - \frac{28\varepsilon_3}{\pi^{7/2}} - \frac{84\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} + \frac{126\tan^{-1}(1/\sqrt{2})^2}{\pi^{5/2}} - \frac{28\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} = 1.42360030\ldots$$

Table 10: Expected values of order statistics for $p_1(Z) = Z$ for $n = 9$.

$$E[p_1(Z)_{5:9}] = 0$$

$$E[p_1(Z)_{6:9}] = -\frac{378\varepsilon_2}{\pi^{5/2}} + \frac{1764\varepsilon_3}{\pi^{7/2}} - \frac{378\tan^{-1}(1/\sqrt{2})^2}{\pi^{5/2}} + \frac{1764\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} = 0.27452591\ldots$$

$$E[p_1(Z)_{7:9}] = \frac{1008\varepsilon_2}{\pi^{5/2}} - \frac{1764\varepsilon_3}{\pi^{7/2}} - \frac{126\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} + \frac{1008\tan^{-1}(1/\sqrt{2})^2}{\pi^{5/2}} - \frac{1764\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} = 0.57197078\ldots$$

$$\begin{aligned} E[p_1(Z)_{8:9}] &= -\frac{18}{\sqrt{\pi}} - \frac{882\varepsilon_2}{\pi^{5/2}} + \frac{756\varepsilon_3}{\pi^{7/2}} + \frac{252\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} - \frac{882\tan^{-1}(1/\sqrt{2})^2}{\pi^{5/2}} + \frac{756\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} \\ &= 0.93229745\ldots \end{aligned}$$

$$\begin{aligned} E[p_1(Z)_{9:9}] &= \frac{18}{\sqrt{\pi}} + \frac{252\varepsilon_2}{\pi^{5/2}} - \frac{126\varepsilon_3}{\pi^{7/2}} - \frac{126\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} + \frac{252\tan^{-1}(1/\sqrt{2})^2}{\pi^{5/2}} - \frac{126\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} \\ &= 1.48501316\ldots \end{aligned}$$

normal distribution $p_1(Z)$ for sample sizes of $n = 8$ and $n = 9$, respectively. We would also note that Mathematica [22] software is available from the authors for implementing the methodology.

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