A NOVEL APPROACH FOR TWO-DIMENSIONAL INCLUSION/HOLE PROBLEMS USING THE AIRY STRESS FUNCTION METHOD

by

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I would like to dedicate this thesis to my parents, Sathiabhama and Madhavan, and my sister, Vijayashri, for their love and affection, without whom I would be unable to complete my work.

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ABSTRACT

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This thesis demonstrates the application of the Airy stress function in solving plane elasticity problems. An Airy stress function is determined for two-dimensional infinite plates containing a circular inclusion embedded with a circular hole subjected to tensile loading and the traction and displacement functions for various boundary conditions satisfy the equilibrium and continuity conditions. Mathematica was useful for solving the complex expressions and to calculate the elastic fields for a variety of elastic problems in this study.

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CHAPTER 1

INTRODUCTION

1.1 Introduction to Elasticity

A body is called elastic if it possesses the property of recovering its original shape when the forces causing deformations are removed. All structural materials exhibit the property of elasticity to a certain extent. The mathematical analysis of elastic behavior of structural materials is called the theory of elasticity. Elasticity of materials describes the relation between stress and strain, giving rise to Hooke's law. According to Hooke's law, within the elastic limit, the stress applied to a body is directly proportional to the strain produced. The elasticity theory establishes a mathematical formulation of the deformation problem which requires mathematical knowledge to understand the basic theory and formulation procedures [1]. Governing partial differential field equations are developed using the basic principles of continuum mechanics commonly formulated using vectors and tensor language, which is used to solve the mathematical problems associated with elasticity. Various techniques are used to solve the field equations such as the Fourier methods, integral transforms, complex variable, finite element, finite difference, potential theory, variational calculus, etc.

The most generalized analytical technique is the complex variable theory, which is a very powerful tool used for the solution of many applications including torsion problems, anisotropic, thermo-elastic materials in elasticity and most importantly, the plane stress problem. This method was introduced in plane elastic problems based on the reduction of the elasticity boundary value problems to the formulation in the complex domain. This formulation then allows many powerful mathematical techniques available from the complex theory to be applied to the elasticity problem. In complex variable notation, a representation is derived for the elastic displacement in terms of two arbitrary analytic functions of a complex variable known as the complex potential [2].

As the solutions of analytical closed form to fully three dimensional problems are extremely difficult to execute, only simple regions will be considered. Most solutions have been developed for reduced problems in elasticity including axis-symmetry or two-dimensionality to simplify particular aspects of the formulation and solution. In reality, all realistic structures are three dimensional and approximate models are initiated for these theories.

The states of plane stress and plane strain are the two basic problems developed to represent the fundamental plane problem in elasticity. These two formulations yield very similar field equations which can be applied to various two-dimensional problems in elasticity. A distinct stress function technique is implemented for the solutions to plane stress and plane strain problems. The Airy stress function method is a promising approach to reduce the general formulation to a single unknown for a scalar valued function which is then used to deduce stress and strain components. The idea of developing the stress field formulation is to form a single governing equation that satisfies equilibrium and compatibility equations [3].

Earlier, many approaches have been proposed for two-dimensional infinite plates with circular holes, inclusions and concentric circular inclusions using the Airy stress function method. However, due to the complexity of algebraic expressions involved, there has been no work about a two-dimensional infinite plate with a circular inclusion and a circular hole using the Airy stress function. This thesis demonstrates the derivation of Airy stress function for two-dimensional infinite plates having a circular inclusion embedded with a circular hole subjected to external loading to find traction and displacements for various boundary conditions which satisfies the equilibrium and continuity conditions. Symbolically capable software, Mathematica, was useful for solving such complex expressions in this study.

1.2 Introduction to Mathematica

A computer algebra system is a software program which is used in manipulation of mathematical expressions in symbolic form. A computer algebra system appears in the late 1960s which is the base for developing various symbolic software packages such as muMATH, Reduce, Derive and Macsyma [5]. Macsyma was the first popular computer algebra system using LISP as the programming basis and other commercial symbolic software packages such as MATLAB, MAPLE, MATHEMATICA were written in the C language. Symbolic algebra systems facilitate mathematical expressions analytically without any approximation [6].

Mathematica has been widely used by researchers, mathematicians and theoretical physicists. Mathematica fully integrates technical computational software using symbolic expressions to provide a mathematical representation and other structures. Mathematical computations can be classified into three main groups: numerical, symbolic and graphical. All these three classes are handled in a unified way by Mathematica. Mathematica is used as a calculator. However, the range of calculations which can be done with this software is greater than the traditional electronic calculator or with a traditional programming language such as Fortran or Basic. It is capable of handling numbers of any precision later than a traditional calculator and can do numerical computations as well numerical operations in functions, such as numerical integration, numerical minimization and linear programming.

Mathematica can derive and manipulate power series approximations and also includes a full range of higher mathematical functions, from elliptic integrals and complex Bessel functions to hypergeometric functions. Manipulation of algebraic formulas is made possible by Mathematica's symbolic computation capabilities. Mathematica incorporates a graphic language, in which symbolic representation of geometry is given as input, then render the objects graphically. In addition, Mathematica can communicate at a high level with other programs using the Math-Link communication standard. The interface used in Mathematica is divided into two parts, the kernel, which performs computations, and the front end, which does interact with a user [6].

1.2.1 Features of Mathematica

1) High level programming knowledge which can create programs, modeling and data analysis.
2) An embedded system where Mathematica runs automatically multiple parts of computation concurrently making parallel computing easy in everyday use.
3) A software platform on which different packages can be executed for specific applications and the most important subsystem for system reliability with built-in important measures.
4) Mathematica provides a comprehensive set of tools for understanding basic descriptive statistics to developing and visualizing multidimensional non linear models.
5) A powerful computational system that can create design and develop control solutions for analog and digital systems using classical and state spacing techniques.
6) Mathematica is widely used in many technical areas such as computational biology, wavelet analysis, financial engineering and geographical information systems.

CHAPTER 2

AIRY STRESS FUNCTION

2.1 Complex Variable Method

A complex variable z is defined by real numbers x and y in the Cartesian form as

$$z = x + iy, \qquad (2.1.1)$$

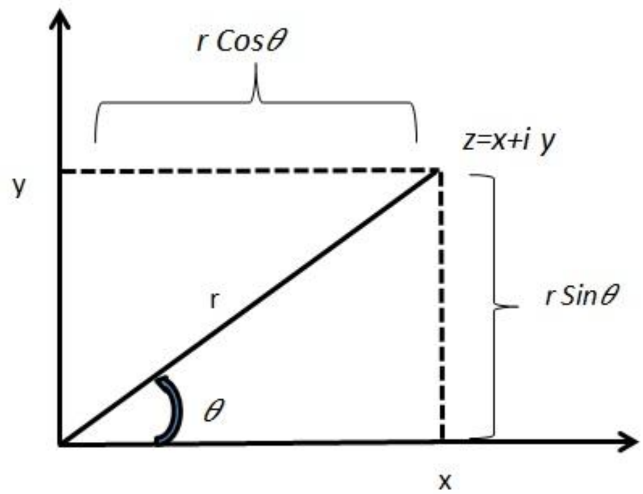

Figure 2.1 Complex Plane

where $i = \sqrt{-1}$ is called the imaginary unit, x is known as the real part of z, that is, x=Re(z) and y is the imaginary part of z, y=Im(z). This definition can also be expressed in polar form as,

$$z = r(\cos\theta + i\,\sin\theta) = re^{i\theta}, \qquad (2.1.2)$$

where $r = \sqrt(x\text{^}2 + y\text{^}2)$ is the modulus of z and $\theta = \tan^{-1}(y/x)$ is the argument. A complex function includes independent and dependent variables which is separated into real and imaginary parts. It can be used as a two-dimensional vector components in x and y. This type of representation is used in plane elasticity problems. The complex conjugate $\bar{z}$ of the variable z is defined by

$$\bar{z} = x - i\, y = \mathrm{re}^{-\mathrm{i}\theta}. \qquad (2.1.3)$$

Complex conjugate $\bar{z}$ is a reflection of z about the real axis. Note that $r = \sqrt(z\bar{z})$

Using definitions (2.1.1) and (2.1.3) , the differential operators can be developed as follows:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \\ \frac{\partial}{\partial y} &= i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right), \qquad (2.1.4) \\ \frac{\partial}{\partial z} &= \frac{1}{2}(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}), \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right). \end{aligned}$$

Addition, subtraction, division and multiplication of complex numbers are defined by

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2), \\ z_1 - z_2 &= (x_1 - x_2) + i(y_1 - y_2), \qquad (2.1.5) \\ z_1 z_2 &= (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2), \\ \frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 x_2 + y_1 y_2}{{x^2}_2 + {y^2}_2} + i\,\frac{y_1 x_2 - x_1 y_2}{{x^2}_2 + {y^2}_2}. \end{aligned}$$

2.2 Formulation of Airy Stress Function

The Airy stress function, a distinct stress function method, is implemented for the solutions to plane stress and plane strain problems which can reduce the general formulation to a single governing equation in terms of a single unknown. The stress function formulation is based on the general idea of developing a representation for the stress field that yields a single governing equation and satisfies the equilibrium equation from the compatibility condition [3].

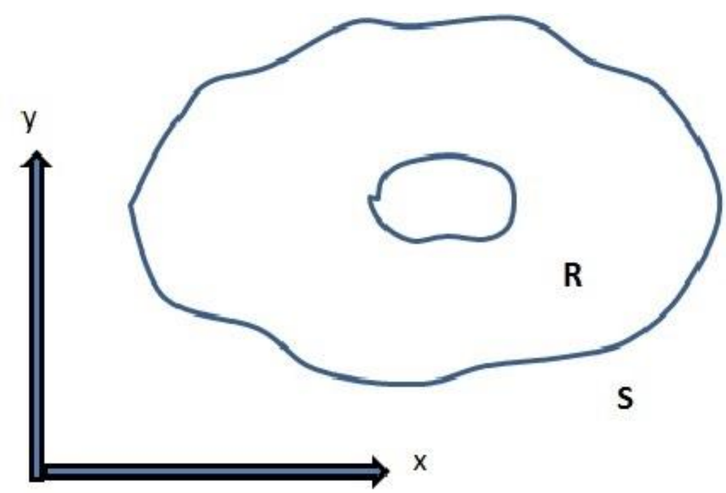

Figure 2.2 Typical Complex Domain

The equilibrium equations for a plane problem are written in the form

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 ,$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 . \qquad (2.2.1)$$

This equilibrium equations (2.2.1), neglecting the body forces, will be identically satisfied by choosing the representation in the form

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2},$$
$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2},\qquad (2.2.2)$$
$$\tau_{\mathrm{xy}} = -\frac{\partial^2 \phi}{\partial x \partial y},$$

where $\phi = \phi(x, y)$ is an arbitrary form called the Airy stress function.

The compatibility condition which can be written in terms of stress is expressed as

$$\nabla^2(\sigma_x + \sigma_y) = 0\,, \qquad (2.2.3)$$

where ∇ is the Laplace operator.

The field equations for compatibility relations in terms of stress for both plane strain and plane stress are given by

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = -\frac{1-2\nu}{1-\nu}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \ldots\text{. plane strain} \qquad (2.2.4)$$

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = -(1-\nu)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \ldots\text{. plane stress} \qquad (2.2.5)$$

These equations can be further reduced to

$$\nabla^4 \phi = -\frac{1-2\nu}{1-\nu}\nabla^2 V \ldots\text{plane strain} \qquad (2.2.6)$$

$$\nabla^4 \phi = -(1-\nu)\nabla^2 V \ldots\text{plane stress} \qquad (2.2.7)$$

where $\nabla^4 = \nabla^2\nabla^2$ is called the biharmonic operators. Neglecting the body forces, the plane strain and plane stress compatibility equations are reduce to a single governing equations as

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = \nabla^4 \phi = 0 \qquad (2.2.8)$$

This relation is called the biharmonic equation and its solution ϕ is known as the biharmonic function. This function is determined in the two-dimensional region R in the figure enclosed by the boundary condition S where appropriate boundary conditions are applied. Thus the plane problem of elasticity has been reduced to a single governing equation using the Airy stress function such that the stress components satisfy the boundary condition [3].

2.2.1 Airy Stress Function for Polar Coordinates

It is advantageous to use the polar coordinate in plane elasticity problems such as the stresses in circular rings and disks, curved bars of narrow rectangular cross-section with a circular axis, etc. For a two-dimensional polar coordinate system, the solution to plane strain and plane stress problems involves the determination of in plane displacement, stresses and strains $(U_r, U_\theta, \sigma_r, \sigma_\theta, \sigma_{\mathrm{r\theta}}, e_r, e_\theta, e_{\mathrm{r\theta}})$ in the region R from Figure 2.2 subjected to the prescribed boundary condition on S.

The stress transformation from the Cartesian coordinate to the polar coordinate is as follows:

$$\begin{aligned} \sigma_r &= \sigma_x \mathrm{Cos}^2\theta + \sigma_y \mathrm{Sin}^2\theta + 2\tau_{\mathrm{xy}} \mathrm{Sin}\theta\mathrm{Cos}\theta \,, \\ \sigma_\theta &= \sigma_x \mathrm{Sin}^2\theta + \sigma_y \mathrm{Cos}^2\theta - 2\tau_{\mathrm{xy}} \mathrm{Sin}\theta\mathrm{Cos}\theta \,, \\ \tau_{\mathrm{r\theta}} &= -\sigma_x \mathrm{Sin\theta Cos\theta} + \sigma_y \mathrm{Sin}\theta\mathrm{Cos}\theta + \tau_{\mathrm{xy}}(\mathrm{Cos}^2\theta - \mathrm{Sin}^2\theta). \end{aligned} \tag{2.2.9}$$

Equation (2.2.4) shows the relation between the stress components and the Airy stress function which can be transformed into polar form as

$$\begin{aligned} \sigma_r &= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial^2\theta} \,, \\ \sigma_\theta &= \frac{\partial^2\phi}{\partial r^2} \,, \\ \tau_{\mathrm{r\theta}} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right). \end{aligned} \tag{2.2.10}$$

This form will satisfy the equilibrium equation and the compatibility relation is reduced to the biharmonic equation in the absence of body forces as

$$\nabla^4\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\phi = 0 \qquad (2.2.11)$$

The plane problem of elasticity in terms of the Airy stress function for the curvilinear system is reduced to a single governing equation such that the stress components satisfy the appropriate boundary condition.

2.3 Complex Formulation of the Plane Elasticity Problem

Plane strain and plane stress are the two basic theories developed to represent the fundamental plane problem in elasticity. The formulation of equations for both plane stress and plane strain remains the same by simple changes in elastic constants. With appropriate boundary conditions, the complex variable approach in plane problems of elasticity can reduce the problem to the solution of the Navier's displacement equations of equilibrium [2].

The basic equation for plane strain includes expressions for the stresses in terms of displacement as

$$\sigma_x = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x},$$

$$\sigma_y = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y}, \qquad (2.3.1)$$

$$\tau_{\mathrm{xy}} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

where λ and μ are the Lame constants.

The Navier equation without body force is reduced to

$$\mu\nabla^2 u + (\lambda + \mu)\nabla(\nabla . u) = 0 \ , \qquad (2.3.2)$$

where the Laplacian operator is defined as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \ . \qquad (2.3.3)$$

Using the relations (2.1.1) and (2.1.3), the variables x and y can be expressed in complex terms z and $\bar{z}$ and applying this to the Airy stress function, we can write $\phi = \phi(z, \bar{z})$**.** Repeated use of differential operator from the equation allows the following representation of harmonic and biharmonic operators

$$\nabla^2() = 4\frac{\partial^2()}{\partial z \partial \bar{z}}, \qquad (2.3.4)$$

$$\nabla^4() = 16\frac{\partial^4()}{\partial z^2 \partial \bar{z}^2}. \qquad (2.3.5)$$

The governing biharmonic elasticity equation can be expressed as

$$\frac{\partial^4 \phi}{\partial z^2 \partial \bar{z}^2} = 0 \ , \qquad (2.3.6)$$

Integrating the above equation (2.3.6) yields

$$\phi(z, \bar{z}) = \frac{1}{2}(z\bar{\gamma}(\bar{z}) + \bar{z}\gamma(z) + \chi(z) + \bar{\chi}(\bar{z}))$$

$$= \mathrm{Re}(\bar{z}\gamma(z) + \chi(z)). \qquad (2.3.7)$$

This result demonstrates that the Airy stress function can be formulated in terms of two functions of a complex variable where γ and χ are arbitrary analytic functions of the indicated variables.

Introducing the complex displacement $U = u + \mathrm{iv}$ into the governing Navier equation (2.3.2), we get

$$(\lambda + \mu)\frac{\partial}{\partial \bar{z}}\left(\frac{\partial U}{\partial z} + \frac{\partial \bar{U}}{\partial z}\right) + 2\mu\frac{\partial^2 U}{\partial \bar{z} \partial z} = 0 \ . \qquad (2.3.8)$$

This expression is integrated which yields a solution for the complex displacement as

$$2\mu U = \kappa\gamma(z) - z\bar{\gamma}'(\bar{z}) - \bar{\psi}(\bar{z}), \qquad (2.3.9)$$

where $\gamma(z)$ and $\psi(z) = \chi'(z)$ are arbitrary analytic functions of a complex variable and the parameter κ depends only on the Poisson ratio ν

$$\kappa = 3 - 4\nu \quad \text{for plane strain} \tag{2.3.10}$$

$$\kappa = \frac{3-\nu}{1+\nu} \quad \text{for plane stress} \tag{2.3.11}$$

Equation (2.3.9) is the complex variable formulation for the displacement field equation in terms of two arbitrary functions of complex variable [1].

Using these relations (2.2.2) and (2.3.7) yields the fundamental stress combinations

$$\begin{aligned} \sigma_x + \sigma_y &= 2(\gamma'(z) + \overline{\gamma'(z)}), \\ \sigma_y - \sigma_x + 2i\tau_{\mathrm{xy}} &= (\bar{z}\gamma''(z) + \psi'(z)). \end{aligned} \tag{2.3.12}$$

Individual stress components can be solved from the fundament stress combination equations (2.3.12). Using standard transformation laws, the individual stresses and displacement in the polar coordinate can be written as

$$\begin{aligned} \sigma_r + \sigma_\theta &= \sigma_x + \sigma_y, \\ \sigma_\theta - \sigma_r + 2i\tau_{\mathrm{r\theta}} &= \left(\sigma_y - \sigma_x + 2i\tau_{\mathrm{xy}}\right)e^{2i\theta}, \\ u_r + iu_\theta &= (u + iv)e^{-i\theta}. \end{aligned} \tag{2.3.13}$$

2.4 Structure of the Complex Potential

Investigation of complex potential in plane elasticity problems involves the determination of the two complex potential functions, $\gamma(z)$ and $\psi(z)$. These potential functions have some similar properties and structures which are determined by applying appropriate boundary conditions for the stress and displacement. Most problems of interest in plane elasticity involve finite simply connected, finite multiple connected and infinite multiple connected domain [1].

2.4.1 Finite Simply Connected Domains

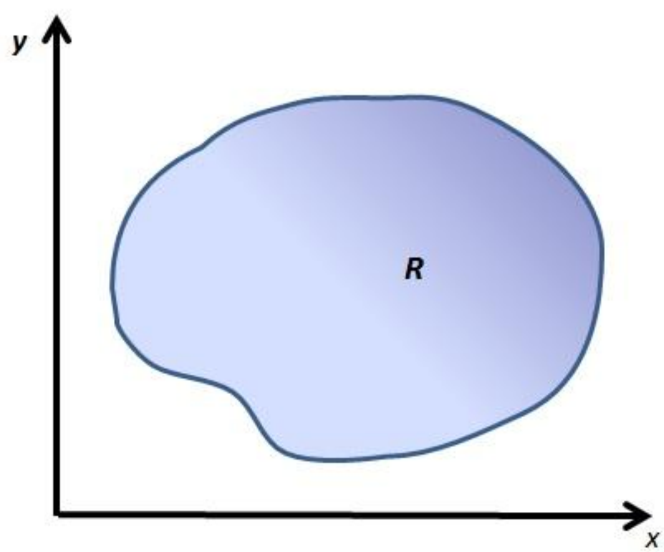

Figure 2.3 Finite Simply Connected Domain

Assume a finite simply connected domain shown in Fig 2.3. In this case, the potential functions, $\gamma(z)$ and $\psi(z)$ are single valued analytic functions in the domain R. The functions represented in power series are as follows:

$$\gamma(z) = \sum_{n=0}^{\infty} a_n z^n,$$

$$(2.4.1)$$

$$\psi(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where a_n and b_n are complex constants to be determined by the specific boundary conditions.

2.4.2 Finite Multiply Connected Domains

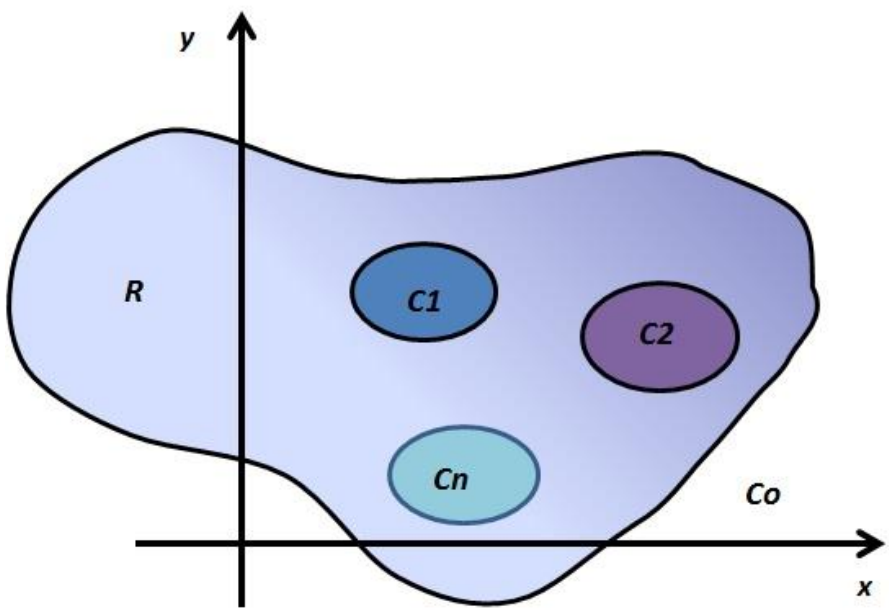

Figure 2.4 Finite Multiply Connected Domain

Assume that the domain R being considered is multiply connected, being bounded internally by contours C_1, C_2…. C_n and externally bounded by a contour C_0 from the figure. The stress and displacement fields are assumed to be single-valued throughout the region R that the complex potentials are multiple valued functions in domain R of the form,

$$\gamma(z) = \sum_{k=0}^{\infty} \frac{-F_k}{2\pi(1+\kappa)} \log(z - z_k) + \gamma^*(z),$$
$$\psi(z) = \sum_{k=0}^{\infty} \frac{-\kappa \bar{F}_k}{2\pi(1+\kappa)} \log(z - z_k) + \psi^*(z), \qquad (2.4.2)$$

where $\gamma^*(z)$ and $\psi^*(z)$ are analytic functions in R, F_k is the resultant vector applied to the contour C_k, z_k is a arbitrary point within the contour and κ is the material constant.

2.4.3 Infinite Multiply Connected Domains

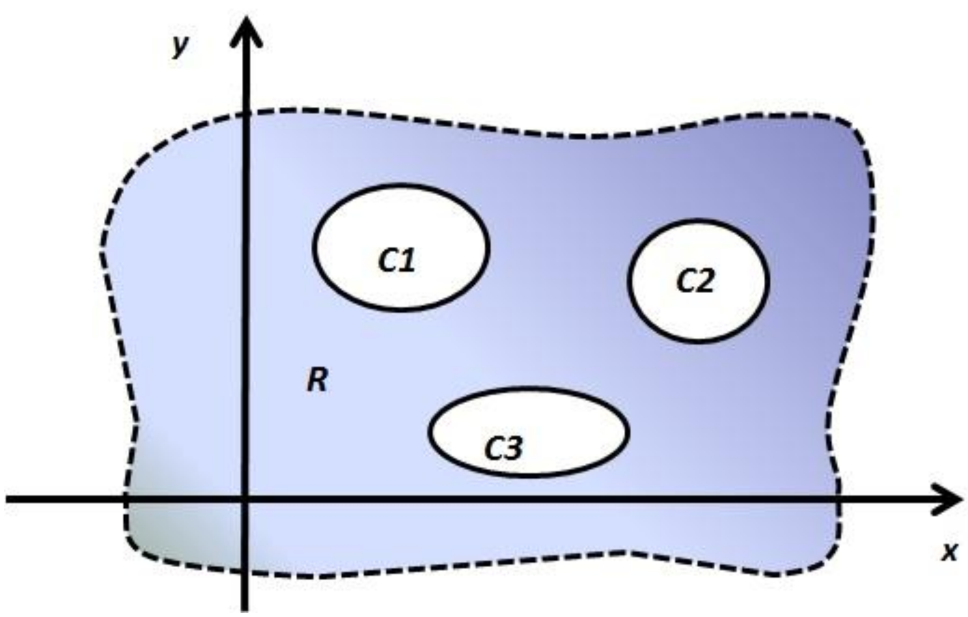

Figure 2.5 Infinite Multiply Connected Domain

To examine the behavior of the stress and displacement fields at infinity, the general form of the potential is determined in an analogous method. The logarithmic terms must be added written in the form

$$\log(z - z_k) = \log z + \log(1 - \frac{z_k}{z}) = \log z - (\frac{z_k}{z} + \frac{1}{2}(\frac{z_k}{z})^2 + \ldots)$$
$$= \log z + \text{(arbitrary analytic function).} \qquad (2.4.3)$$

Using this result, the stresses which remain bounded over the contours C_k at infinity give the general-form

$$\gamma(z) = -\frac{\sum_{k=1}^{m} F_k}{2\pi(1+\kappa)} \log z + \frac{\sigma^{\infty}{}_x + \sigma^{\infty}{}_y}{4} z + \gamma^{**}(z),$$

$$(2.4.4)$$

$$\psi(z) = \frac{\kappa \sum_{k=1}^{m} \bar{F}_k}{2\pi(1+\kappa)} \log z + \frac{\sigma^{\infty}{}_y - \sigma^{\infty}{}_x + 2i\tau^{\infty}{}_{xy}}{4} z + \psi^{**}(z),$$

where $\sigma^{\infty}{}_x, \sigma^{\infty}{}_y$ and $\tau^{\infty}{}_{xy}$ are the stresses at infinity and $\gamma^{**}(z)$ and $\psi^{**}(z)$ are arbitrary analytic functions outside the region enclosing all m contours. These analytic functions can be expressed as Laurent series as

$$\gamma^{**}(z) = \sum_{n=1}^{\infty} a_n z^{-n},$$

$$(2.4.5)$$

$$\psi^{**}(z) = \sum_{n=1}^{\infty} b_n z^{-n}.$$

At Infinity, a bounded strain over an infinite length will produce infinite displacement. In a similar way, the displacement would indicate unbounded behavior unless all stresses vanish at infinity and the summation of forces equals to zero.

CHAPTER 3

APPLICATIONS OF THE AIRY STRESS FUNCTION

3.1 Stressed Infinite Plate with a Circular Hole

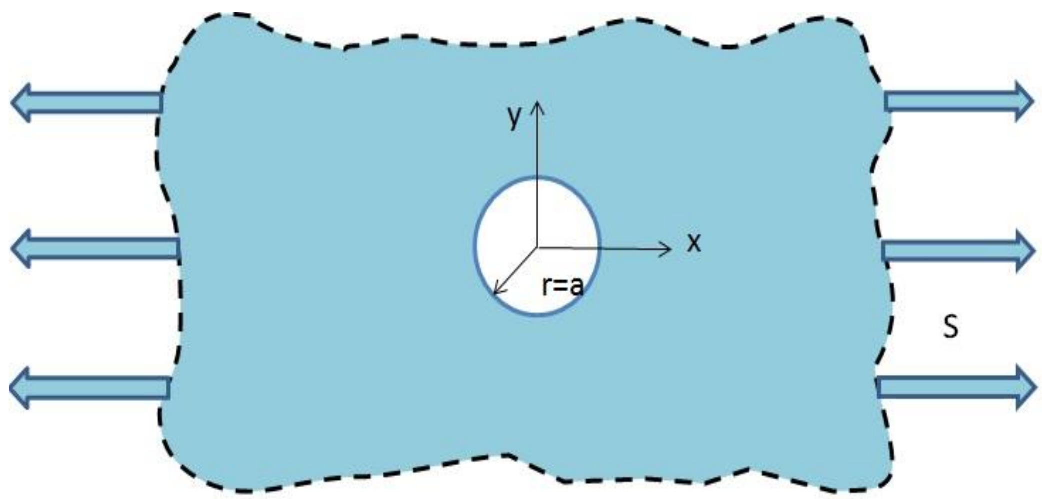

Figure 3.1 Infinite Plate with a Circular Hole

An infinitely stressed plate containing a stress-free circular hole with a radius r=a is subjected to the general system of uniform tensile stress $\sigma_x^{\infty} = S$ at infinity [7]. From equation (2.4.4), the logarithmic terms are negligible because the circular hole is stress free. Therefore, the complex potential is written in the form:

$$\gamma(z) = \frac{\sigma^{\infty}{}_x + \sigma^{\infty}{}_y}{4} z + \gamma^{**}(z),$$

$$(3.1.1)$$

$$\psi(z) = \frac{\sigma^{\infty}{}_y - \sigma^{\infty}{}_x + 2\mathrm{i}\tau^{\infty}{}_{\mathrm{xy}}}{4} z + \psi^{**}(z).$$

Substituting the summation for three terms n=3 to the above equation, we get

$$\gamma(z) = \frac{Sz}{4} + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3},$$

$$(3.1.2)$$

$$\psi(z) = -\frac{Sz}{2} + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3}.$$

By integration, $\psi(z) = \chi'(z)$, we get the complex potential function as

$$\chi(z) = -\frac{Sz^2}{4} + \mathrm{Log}[z] b_1 - \frac{b_2}{z} - \frac{b_3}{2z^2}. \qquad (3.1.3)$$

Substituting $z = \mathrm{re}^{\mathrm{i}\theta}$ in the equation (2.3.7) yields the Airy stress function expressed in polar form as

$$\phi(r,\theta) = \frac{r^4 S\mathrm{Sin}[\theta]^2 + 2\mathrm{Cos}[4\theta]a_3 + r(2r\mathrm{Cos}[2\theta]a_1 + 2\mathrm{Cos}[3\theta]a_2 + r\mathrm{Log}[r^2]b_1 - 2\mathrm{Cos}[\theta]b_2) - \mathrm{Cos}[2\theta]b_3}{2r^2}. \quad (3.1.4)$$

Applying the stress free condition ($\sigma_r - I\tau_{\mathrm{r\theta}} = 0$) at r=a, the unknown constants are solved in the complex potential functions and the final form of the Airy stress function may be written as

$$\phi(r,\theta) = -\frac{1}{4} a^2 S(-1 + \mathrm{Log}[a^2]) . \qquad (3.1.5)$$

The stress components can be derived from the Airy stress function. They are

$$\sigma_r = \frac{(-a^2 + r^2) S (r^2 + (-3a^2 + r^2)\mathrm{Cos}[2\theta])}{2r^4},$$

$$\sigma_\theta = \frac{S(r^2(a^2 + r^2) - (3a^4 + r^4)\mathrm{Cos}[2\theta])}{2r^4}, \qquad (3.1.6)$$

$$\tau_{\mathrm{r\theta}} = -\frac{(-3a^4 + 2a^2 r^2 + r^4) S\mathrm{Cos}[\theta]\mathrm{Sin}[\theta]}{r^4}.$$

Hence the above equations satisfy the equilibrium equations.

3.2 Two-Dimensional Infinite Plate with a Circular Inclusion

Consider a two-dimensional circular inclusion enclosed in an infinite plate having a radius "a" as the material properties, κ and κ_1 depends only on Poisson's ratio, μ and $\mu_1$are the shear moduli for the circular inclusion and infinite plate grouped with finite simply connected domain, respectively. This type of geometry is similar to the infinite plate with a hole so the stress and displacement can be validated by maintaining the equilibrium and continuity at the boundary of the interface.

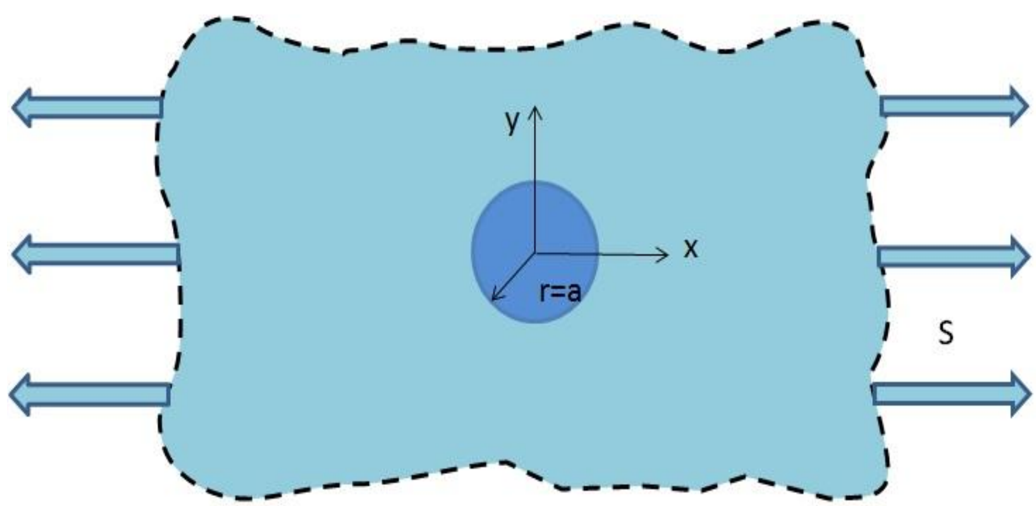

Figure 3.2 Two-Dimensional Circular Inclusion

3.2.1 Stress Field inside the Circular Inclusion

The complex potentials for the two-dimensional circular inclusion are similar to a finite simply connected domain by

$$\gamma(z) = \sum_{n=0}^{\infty} a_n z^n,$$

$$(3.2.1)$$

$$\psi(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Applying the similar boundary condition for the two-dimensional circular inclusion, we get

$$\gamma(z) = a_0 + a_1 z + a_2 z^2,$$

$$(3.2.2)$$

$$\psi(z) = b_0 + b_1 + b_2 z^2.$$

The individual stresses can be derived from the stress combination equation as follows:

$$\sigma_x + \sigma_y = 2(\gamma'(z) + \bar{\gamma'(z)}),$$

$$(3.2.3)$$

$$\sigma_y - \sigma_x + 2i\tau_{\mathrm{xy}} = (\bar{z}\gamma''(z) + \psi'(z)).$$

Substituting the complex potentials into equation (3.2.3) to evaluate stress components, we have

$$\sigma_{\mathrm{rin}} = 2a_1 + 2r\mathrm{Cos}[\theta]a_2 - \mathrm{Cos}[2\theta]b_1 - 2r\mathrm{Cos}[3\theta]b_2,$$

$$\sigma_{\theta\mathrm{in}} = 2a_1 + 6r\mathrm{Cos}[\theta]a_2 + \mathrm{Cos}[2\theta]b_1 + 2r\mathrm{Cos}[3\theta]b_2, \qquad (3.2.4)$$

$$\tau_{\mathrm{r}\theta\mathrm{in}} = 2\mathrm{Sin}[\theta](ra_2 + \mathrm{Cos}[\theta]b_1 + r(1 + 2\mathrm{Cos}[2\theta])b_2).$$

The displacements from the equation (2.3.8) are given as

$$U_{\mathrm{rin}} = \frac{k\mathrm{Cos}[\theta]a_0 + (-1+k)ra_1 - 2r^2\mathrm{Cos}[\theta]a_2 + kr^2\mathrm{Cos}[\theta]a_2 - \mathrm{Cos}[\theta]b_0 - r\mathrm{Cos}[2\theta]b_1 - r^2\mathrm{Cos}[3\theta]b_2}{2\mu},$$

$$(3.2.5)$$

$$U_{\theta \text{in}} = \frac{-k\text{Sin}[\theta]a_0 + (2+k)r^2\text{Sin}[\theta]a_2 + \text{Sin}[\theta]b_0 + r\text{Sin}[2\theta]b_1 + r^2\text{Sin}[3\theta]b_2}{2\mu}.$$

3.2.2 Stress Field in Infinite Matrix Surrounding the Disc

The complex potential for the infinite matrix surrounding the circular inclusion is similar to an infinite multiply connected domain as

$$\gamma^{**}(z) = \sum_{n=1}^{\infty} a_n z^{-n},$$

$$(3.2.6)$$

$$\psi^{**}(z) = \sum_{n=1}^{\infty} b_n z^{-n},$$

which are further expressed as power series as

$$\gamma(z) = \frac{Sz}{4} + \frac{a_3}{z} + \frac{a_4}{z^2},$$

$$(3.2.7)$$

$$\psi(z) = -\frac{\text{Sz}}{2} + \frac{b_3}{z} + \frac{b_4}{z^2} + \frac{b_5}{z^3}.$$

Similarly, the stresses and displacements can be evaluated from the general stress combination equation as

$$\sigma_{\text{rout}} = \frac{r^4S + r^4S\text{Cos}[2\theta] - 8r^2\text{Cos}[2\theta]a_3 - 20r\text{Cos}[3\theta]a_4 + 2r^2b_3 + 4r\text{Cos}[\theta]b_4 + 6\text{Cos}[2\theta]b_5}{2r^4},$$

$$\sigma_{\theta\text{out}}$$

$$= -\frac{-r^4S + r^4S\text{Cos}[2\theta] - 4r\text{Cos}[3\theta]a_4 + 2r^2b_3 + 4r\text{Cos}[\theta]b_4 + 6\text{Cos}[2\theta]b_5}{2r^4}, \quad (3.2.8)$$

$$\tau_{r\theta\text{out}} = -\frac{\text{Sin}[\theta](r^4S\text{Cos}[\theta] + 4r^2\text{Cos}[\theta]a_3 + 6r(1 + 2\text{Cos}[2\theta])a_4 - 2rb_4 - 6\text{Cos}[\theta]b_5)}{r^4},$$

$$U_{\text{rout}} = \frac{-r^4 S + 2r^4 S\text{Cos}[2\theta] - 4r^2 b_3 - 4r\text{Cos}[\theta] b_4 - 4\text{Cos}[2\theta] b_5 + r^4 S k_1 + 4r^2 \text{Cos}[2\theta] a_3 (1 + k_1) + 4r\text{Cos}[3\theta] a_4 (2 + k_1)}{8r^3 \mu_1},$$

(3.2.9)

$$U_{\theta\text{out}} = -\frac{\text{Sin}[\theta](r^4 S\text{Cos}[\theta] + r b_4 + 2\text{Cos}[\theta] b_5 + r(1 + 2\text{Cos}[2\theta]) a_4 (-2 + k_1) + 2r^2 \text{Cos}[\theta] a_3 (-1 + k_1))}{2r^3 \mu_1}.$$

The continuity condition can be satisfied when the traction force and displacements are equal at the boundary of the interface and the constants are solved by equating the equations as

$$b_0 \to k a_0, \qquad a_1 \to \frac{(S + kS)\mu}{4(2\mu - \mu_1 + k\mu_1)}, \qquad b_1 \to -\frac{S(\mu + \mu k_1)}{2(\mu k_1 + \mu_1)}$$

$$a_2 \to 0, \quad b_2 \to 0, \qquad a_3 \to -\frac{a^2 S(\mu - \mu_1)}{2(\mu k_1 + \mu_1)}, \qquad b_3 \to -\frac{a^2 S}{2} + \frac{a^2 (S + kS)\mu}{2(2\mu - \mu_1 + k\mu_1)}$$

$$a_4 \to 0, \quad b_4 \to 0, \qquad b_5 \to -\frac{a^4 S(\mu - \mu_1)}{2(\mu k_1 + \mu_1)} \qquad (3.2.10)$$

We get the final stress and displacement equations for the two-dimensional circular inclusion and the infinite plate by substituting the above constants (3.2.10) into equations (3.2.4), (3.2.5), (3.2.8) and (3.2.9).

The final stress and displacement expressions for the two-dimensional circular inclusion are as follows:

$$\sigma_{\mathrm{rin}} = \frac{S\mathrm{Cos}[2\theta](\mu + \mu k_1)}{2(\mu k_1 + \mu_1)} + \frac{(S + kS)\mu}{2(2\mu - \mu_1 + k\mu_1)},$$

$$\sigma_{\theta\mathrm{in}} = -\frac{S\mathrm{Cos}[2\theta](\mu + \mu k_1)}{2(\mu k_1 + \mu_1)} + \frac{(S + kS)\mu}{2(2\mu - \mu_1 + k\mu_1)},$$

$$\tau_{\mathrm{r}\theta\mathrm{in}} = -\frac{S\mathrm{Cos}[\theta]\mathrm{Sin}[\theta](\mu + \mu k_1)}{\mu k_1 + \mu_1}\ , \tag{3.2.11}$$

$$U_{\mathrm{rin}} = \frac{1}{8} rS\left(\frac{2\mathrm{Cos}[2\theta](1 + k_1)}{\mu k_1 + \mu_1} + \frac{-1 + k^2}{2\mu + (-1 + k)\mu_1}\right),$$

$$U_{\theta\mathrm{in}} = -\frac{rS\mathrm{Sin}[2\theta](\mu + \mu k_1)}{4\mu(\mu k_1 + \mu_1)}.$$

The final stress and displacement expressions for the infinite matrix surrounding the inclusion are

$$\sigma_{\mathrm{rout}} = \frac{S}{2r^4}\left(r^4 + r^4\mathrm{Cos}[2\theta] + \frac{4a^2r^2\mathrm{Cos}[2\theta](\mu - \mu_1)}{\mu k_1 + \mu_1} + \frac{3a^4\mathrm{Cos}[2\theta](-\mu + \mu_1)}{\mu k_1 + \mu_1} + \frac{a^2(-1 + k)r^2(\mu - \mu_1)}{2\mu + (-1 + k)\mu_1}\right),$$

$$\sigma_{\theta\mathrm{out}} = -\frac{S}{2r^4}\left(-r^4 + r^4\mathrm{Cos}[2\theta] + \frac{3a^4\mathrm{Cos}[2\theta](-\mu + \mu_1)}{\mu k_1 + \mu_1} + \frac{a^2(-1 + k)r^2(\mu - \mu_1)}{2\mu + (-1 + k)\mu_1}\right),$$

$$\tau_{\mathrm{r}\theta\mathrm{out}} = -\frac{\mathrm{Sin}[\theta]}{r^4}\left(r^4S\mathrm{Cos}[\theta] + \frac{3a^4S\mathrm{Cos}[\theta](\mu - \mu_1)}{\mu k_1 + \mu_1} - \frac{2a^2r^2S\mathrm{Cos}[\theta](\mu - \mu_1)}{\mu k_1 + \mu_1}\right), \tag{3.2.12}$$

$$U_{\mathrm{rout}} = \frac{S}{8r^3\mu_1}\left(-r^4 + 2r^4\mathrm{Cos}[2\theta] + r^4k_1 + \frac{2a^4\mathrm{Cos}[2\theta](\mu - \mu_1)}{\mu k_1 + \mu_1} - \frac{2a^2r^2\mathrm{Cos}[2\theta](1 + k_1)(\mu - \mu_1)}{\mu k_1 + \mu_1} - \frac{2a^2(-1 + k)r^2(\mu - \mu_1)}{2\mu + (-1 + k)\mu_1}\right)\ ,$$

$$U_{\theta\mathrm{out}} = \frac{1}{4r^3\mu_1}\left(-r^4 S\mathrm{Sin}[2\theta] + \frac{a^4 S\mathrm{Sin}[2\theta](\mu-\mu_1)}{\mu k_1+\mu_1} - \frac{a^2 r^2 S\mathrm{Sin}[2\theta](\mu-\mu_1)}{\mu k_1+\mu_1} + \frac{a^2 r^2 S\mathrm{Sin}[2\theta]k_1(\mu-\mu_1)}{\mu k_1+\mu_1}\right).$$

Thus Equations (3.2.11) and (3.2.12) can be validated by substituting them into the equilibrium equations.

3.3 Stress Analysis for a Two-Dimensional Circular Inclusion Embedded with a Circular Hole

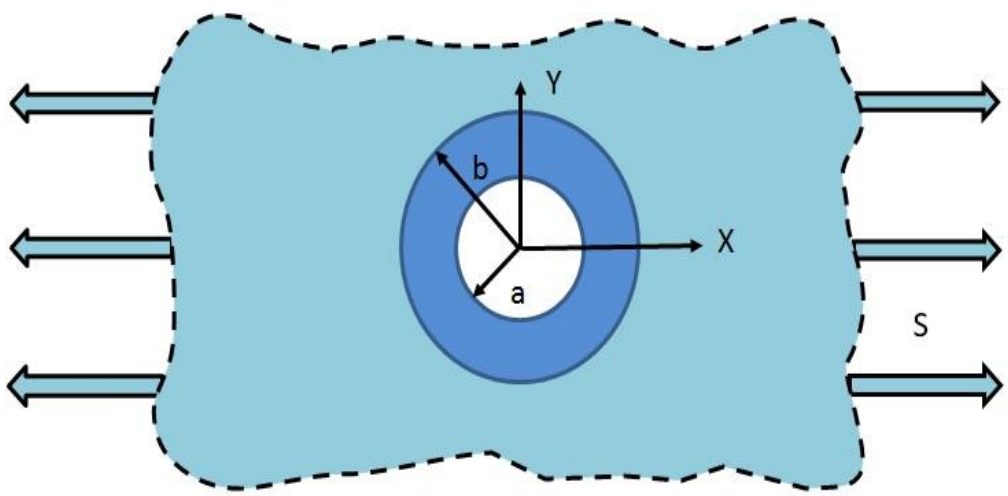

Figure 3.3 Two-Dimensional Circular Inclusion Plate Embedded with a Circular Hole

Consider a two-dimensional infinite plate containing a circular inclusion of a radius “a” embedded with a circular hole of a radius “b” which is subjected to general system of uniform tensile stress $\sigma_x^{\infty} = S$ at infinity. The material properties are assumed to be μ , μ_1 as the shear moduli and the parameter κ , κ_1 depends only on the Poisson’s ratio for the circular inclusion embedded with a hole and for the infinite plate outside the inclusion. In this problem, the circular inclusion enclosed with a circular hole is similar to the finite simply connected domain and the infinite matrix outside the inclusion region is similar to the infinite multiply connected domain.

3.3.1 Fundamental Stress and Displacement Equations

The individual stresses and displacements can be derived from the fundamental stress and displacement equations as

$$\sigma_x + \sigma_y = 2(\gamma'(z) + \overline{\gamma'(z)}),$$

$$\sigma_y - \sigma_x + 2i\tau_{\mathrm{xy}} = (\bar{z}\gamma''(z) + \psi'(z)), \qquad (3.3.1)$$

$$2\mu\mathrm{U} = \kappa\gamma(z) - z\overline{\gamma'(z)} - \overline{\psi(z)}\ .$$

where U = u + i v.

The stress and displacement in the polar coordinates can be written as

$$\sigma_r + \sigma_\theta = \sigma_x + \sigma_y,$$

$$\sigma_\theta - \sigma_r + 2i\tau_{\mathrm{r\theta}} = (\sigma_y - \sigma_x + 2i\tau_{\mathrm{xy}})e^{2i\theta}, \qquad (3.3.2)$$

$$u_r + iu_\theta = (u + iv)e^{-i\theta}.$$

3.3.2 *Stress Field inside the Circular Inclusion Embedded with a Circular Hole*

The expressions for the complex potentials may then be written as

$$\gamma_{\mathrm{in}}(z) = \frac{a_1 + ib_1}{z} + z(a_2 + ib_2) + z^2(a_3 + ib_3) + z^3(a_4 + ib_4),$$

$$(3.3.3)$$

$$\psi_{\mathrm{in}}(z) = \frac{a_5 + ib_5}{z^3} + \frac{a_6 + ib_6}{z^2} + \frac{a_7 + ib_7}{z} + z(a_8 + ib_8)\ .$$

The stress and displacement are determined from the general relations (3.3.1) and (3.3.2) as

$\sigma_{\mathrm{rin}} = \frac{1}{r^4}(-4r^2\mathrm{Cos}[2\theta]a_1 + 2r^4a_2 + 2r^5\mathrm{Cos}[\theta]a_3 + 3\mathrm{Cos}[2\theta]a_5 + 2r\mathrm{Cos}[\theta]a_6 + r^2a_7 -$

$r^4\mathrm{Cos}[2\theta]a_8 - 4r^2\mathrm{Sin}[2\theta]b_1 - 2r^5\mathrm{Sin}[\theta]b_3 + 3\mathrm{Sin}[2\theta]b_5 + 2r\mathrm{Sin}[\theta]b_6 + r^4\mathrm{Sin}[2\theta]b_8),$

$$\sigma_{\theta\mathrm{in}} = \frac{1}{r^4}(2r^4a_2 + 6r^5\mathrm{Cos}[\theta]a_3 + 12r^6\mathrm{Cos}[2\theta]a_4 - 3\mathrm{Cos}[2\theta]a_5 - 2r\mathrm{Cos}[\theta]a_6 - r^2a_7$$
$$+ r^4\mathrm{Cos}[2\theta]a_8 - 6r^5\mathrm{Sin}[\theta]b_3 - 12r^6\mathrm{Sin}[2\theta]b_4 - 3\mathrm{Sin}[2\theta]b_5 - 2r\mathrm{Sin}[\theta]b_6$$
$$- r^4\mathrm{Sin}[2\theta]b_8)\,, \qquad (3.3.4)$$

$$\tau_{\mathrm{r}\theta\mathrm{in}} = \frac{1}{r^4}(-2r^2\mathrm{Sin}[2\theta]a_1 + 2r^5\mathrm{Sin}[\theta]a_3 + 6r^6\mathrm{Sin}[2\theta]a_4 + 3\mathrm{Sin}[2\theta]a_5 + 2r\mathrm{Sin}[\theta]a_6$$
$$+ r^4\mathrm{Sin}[2\theta]a_8 + 2r^2\mathrm{Cos}[2\theta]b_1 + 2r^5\mathrm{Cos}[\theta]b_3 + 6r^6\mathrm{Cos}[2\theta]b_4 - 3\mathrm{Cos}[2\theta]b_5$$
$$- 2r\mathrm{Cos}[\theta]b_6 - r^2b_7 + r^4\mathrm{Cos}[2\theta]b_8),$$

$$U_{\mathrm{rin}} = \frac{1}{2r^3\mu}(r^2\mathrm{Cos}[2\theta]a_1 + kr^2\mathrm{Cos}[2\theta]a_1 - r^4a_2 + kr^4a_2 - 2r^5\mathrm{Cos}[\theta]a_3 + kr^5\mathrm{Cos}[\theta]a_3$$
$$- 3r^6\mathrm{Cos}[2\theta]a_4 + kr^6\mathrm{Cos}[2\theta]a_4 - \mathrm{Cos}[2\theta]a_5 - r\mathrm{Cos}[\theta]a_6 - r^2a_7 - r^4\mathrm{Cos}[2\theta]a_8$$
$$- r^2\mathrm{Sin}[2\theta]b_1 + kr^2\mathrm{Sin}[2\theta]b_1 - 2r^5\mathrm{Sin}[\theta]b_3 - kr^5\mathrm{Sin}[\theta]b_3 - 3r^6\mathrm{Sin}[2\theta]b_4$$
$$- kr^6\mathrm{Sin}[2\theta]b_4 - \mathrm{Sin}[2\theta]b_5 - r\mathrm{Sin}[\theta]b_6 + r^4\mathrm{Sin}[2\theta]b_8),$$

(3.3.5)

$U_{\theta\mathrm{in}} = \frac{1}{2r^3\mu}(-r^2\mathrm{Sin}[2\theta]a_1 - kr^2\mathrm{Sin}[2\theta]a_1 - 2r^5\mathrm{Sin}[\theta]a_3 + kr^5\mathrm{Sin}[\theta]a_3 - 3r^6\mathrm{Sin}[2\theta]a_4 +$

$kr^6\mathrm{Sin}[2\theta]a_4 - \mathrm{Sin}[2\theta]a_5 - r\mathrm{Sin}[\theta]a_6 + r^4\mathrm{Sin}[2\theta]a_8 - r^2\mathrm{Cos}[2\theta]b_1 + kr^2\mathrm{Cos}[2\theta]b_1 + r^4b_2 +$

$kr^4b_2 + 2r^5\mathrm{Cos}[\theta]b_3 + kr^5\mathrm{Cos}[\theta]b_3 + 3r^6\mathrm{Cos}[2\theta]b_4 + kr^6\mathrm{Cos}[2\theta]b_4 + \mathrm{Cos}[2\theta]b_5 + r\mathrm{Cos}[\theta]b_6 +$

$r^2b_7 + r^4\mathrm{Cos}[2\theta]b_8).$

3.3.3 Stress Field for the Infinite Matrix

The stresses at infinity are assumed to be $\sigma_x^\infty = S$, $\sigma_y^\infty = 0$ and $\tau_{xy}^\infty = 0$. Then, the complex potential functions for the infinite plate outside the circular inclusion can be written as

$$\gamma_{\text{out}}(z) = \frac{Sz}{4} + \frac{a_9 + ib_9}{z},$$

$$\psi_{\text{out}}(z) = \frac{Sz}{2} + \frac{a_{10} + ib_{10}}{z} + \frac{a_{11} + ib_{11}}{z^2} + \frac{a_{12} + ib_{12}}{z^3}. \quad (3.3.6)$$

Substituting the complex potentials to the general stress combination and displacement equations, we get the individual stress and displacement as

$$\sigma_{\text{rout}} = \frac{1}{2r^4}(r^4S - r^4S\text{Cos}[2\theta] - 8r^2\text{Cos}[2\theta]a_9 + 2r^2a_{10} + 4r\text{Cos}[\theta]a_{11} + 6\text{Cos}[2\theta]a_{12} - 8r^2\text{Sin}[2\theta]b_9 + 4r\text{Sin}[\theta]b_{11} + 6\text{Sin}[2\theta]b_{12}),$$

$$\sigma_{\theta\text{out}} = \frac{1}{2r^4}(r^4S + r^4S\text{Cos}[2\theta] - 2r^2a_{10} - 4r\text{Cos}[\theta]a_{11} - 6\text{Cos}[2\theta]a_{12} - 4r\text{Sin}[\theta]b_{11} - 6\text{Sin}[2\theta]b_{12}), \quad (3.3.7)$$

$$\tau_{r\theta\text{out}} = \frac{1}{2r^4}(r^4S\text{Sin}[2\theta] - 4r^2\text{Sin}[2\theta]a_9 + 4r\text{Sin}[\theta]a_{11} + 6\text{Sin}[2\theta]a_{12} + 4r^2\text{Cos}[2\theta]b_9 - 2r^2b_{10} - 4r\text{Cos}[\theta]b_{11} - 6\text{Cos}[2\theta]b_{12}),$$

$$U_{\text{rout}} = \frac{1}{8r^3\mu_1}(-r^4S - 2r^4S\text{Cos}[2\theta] + 4r^2\text{Cos}[2\theta]a_9 - 4r^2a_{10} - 4r\text{Cos}[\theta]a_{11} - 4\text{Cos}[2\theta]a_{12} - 4r^2\text{Sin}[2\theta]b_9 - 4r\text{Sin}[\theta]b_{11} - 4\text{Sin}[2\theta]b_{12} + r^4Sk_1 + 4r^2\text{Cos}[2\theta]a_9k_1 + 4r^2\text{Sin}[2\theta]b_9k_1),$$

$$(3.3.8)$$

$$U_{\theta\text{out}} = \frac{1}{4r^3\mu_1}(r^4S\text{Sin}[2\theta] - 2r^2\text{Sin}[2\theta]a_9 - 2r\text{Sin}[\theta]a_{11} - 2\text{Sin}[2\theta]a_{12} - 2r^2\text{Cos}[2\theta]b_9 + 2r^2b_{10} + 2r\text{Cos}[\theta]b_{11} + 2\text{Cos}[2\theta]b_{12} - 2r^2\text{Sin}[2\theta]a_9k_1 + 2r^2\text{Cos}[2\theta]b_9k_1).$$

3.3.4 Continuity Equations

The unknown constants can be found by satisfying the stress free condition and continuity condition at the interface of the circular inclusion and the infinite matrix which can be expressed by equating Equations (3.3.4) to (3.3.7) and (3.3.5) to (3.3.8).

At r=a, $\sigma_{\mathrm{rin}} - I\tau_{\mathrm{r\theta in}} = 0,$ (3.3.9)

At r=b, $\sigma_{\mathrm{rin}} = \sigma_{\mathrm{rout}}$

$\tau_{\mathrm{r\theta in}} = \tau_{\mathrm{r\theta out}},$

(3.3.10)

$U_{\mathrm{rin}} = U_{\mathrm{rout}},$

$U_{\mathrm{\theta in}} = U_{\mathrm{\theta out.}}$

By equating the equations, we get

$$[\sigma_{\mathrm{rin}} - I\tau_{\mathrm{r\theta in}}]$$

$$= \frac{1}{a^4}\big(-4a^2\mathrm{Cos}[2\theta]a_1 + 2a^4a_2 + 2a^5\mathrm{Cos}[\theta]a_3 + 3\mathrm{Cos}[2\theta]a_5 + 2a\mathrm{Cos}[\theta]a_6 + a^2a_7 - a^4\mathrm{Cos}[2\theta]a_8$$

$$- 4a^2\mathrm{Sin}[2\theta]b_1 - 2a^5\mathrm{Sin}[\theta]b_3 + 3\mathrm{Sin}[2\theta]b_5 + 2a\mathrm{Sin}[\theta]b_6 + a^4\mathrm{Sin}[2\theta]b_8$$

$$- i(-2a^2\mathrm{Sin}[2\theta]a_1 + 2a^5\mathrm{Sin}[\theta]a_3 + 6a^6\mathrm{Sin}[2\theta]a_4 + 3\mathrm{Sin}[2\theta]a_5 + 2a\mathrm{Sin}[\theta]a_6 + a^4\mathrm{Sin}[2\theta]a_8$$

$$+ 2a^2\mathrm{Cos}[2\theta]b_1 + 2a^5\mathrm{Cos}[\theta]b_3 + 6a^6\mathrm{Cos}[2\theta]b_4 - 3\mathrm{Cos}[2\theta]b_5 - 2a\mathrm{Cos}[\theta]b_6 - a^2b_7$$

$$+ a^4\mathrm{Cos}[2\theta]b_8)\big) \text{ ,} \qquad (3.3.11)$$

$$[\sigma_{\mathrm{rin}} - \sigma_{\mathrm{rout}}] = \frac{1}{2b^4}(-b^4S + b^4S\mathrm{Cos}[2\theta] - 8b^2\mathrm{Cos}[2\theta]a_1 + 4b^4a_2 + 4b^5\mathrm{Cos}[\theta]a_3 + 6\mathrm{Cos}[2\theta]a_5$$

$$+ 4b\mathrm{Cos}[\theta]a_6 + 2b^2a_7 - 2b^4\mathrm{Cos}[2\theta]a_8 + 8b^2\mathrm{Cos}[2\theta]a_9 - 2b^2a_{10} - 4b\mathrm{Cos}[\theta]a_{11}$$

$$- 6\mathrm{Cos}[2\theta]a_{12} - 8b^2\mathrm{Sin}[2\theta]b_1 - 4b^5\mathrm{Sin}[\theta]b_3 + 6\mathrm{Sin}[2\theta]b_5 + 4b\mathrm{Sin}[\theta]b_6$$

$$+ 2b^4\mathrm{Sin}[2\theta]b_8 + 8b^2\mathrm{Sin}[2\theta]b_9 - 4b\mathrm{Sin}[\theta]b_{11}$$

$$- 6\mathrm{Sin}[2\theta]b_{12}) \text{ ,} \qquad (3.3.12)$$

$$[\tau_{\text{r}\theta\text{in}} - \tau_{\text{r}\theta\text{out}}] = \frac{1}{2b^4}(-b^4 S\text{Sin}[2\theta] - 4b^2\text{Sin}[2\theta]a_1 + 4b^5\text{Sin}[\theta]a_3 + 12b^6\text{Sin}[2\theta]a_4 + 6\text{Sin}[2\theta]a_5$$

$$+ 4b\text{Sin}[\theta]a_6 + 2b^4\text{Sin}[2\theta]a_8 + 4b^2\text{Sin}[2\theta]a_9 - 4b\text{Sin}[\theta]a_{11} - 6\text{Sin}[2\theta]a_{12}$$

$$+ 4b^2\text{Cos}[2\theta]b_1 + 4b^5\text{Cos}[\theta]b_3 + 12b^6\text{Cos}[2\theta]b_4 - 6\text{Cos}[2\theta]b_5 - 4b\text{Cos}[\theta]b_6$$

$$- 2b^2 b_7 + 2b^4\text{Cos}[2\theta]b_8 - 4b^2\text{Cos}[2\theta]b_9 + 2b^2 b_{10} + 4b\text{Cos}[\theta]b_{11}$$

$$+ 6\text{Cos}[2\theta]b_{12}) \quad , \tag{3.3.13}$$

$$[U_{\text{rin}} - U_{\text{rout}}] = \frac{1}{8b^3\mu\mu_1}(b^4 S\mu + 2b^4 S\mu\text{Cos}[2\theta] - 4b^2\mu\text{Cos}[2\theta]a_9 + 4b^2\mu a_{10} + 4b\mu\text{Cos}[\theta]a_{11}$$

$$+ 4\mu\text{Cos}[2\theta]a_{12} + 4b^2\mu\text{Sin}[2\theta]b_9 + 4b\mu\text{Sin}[\theta]b_{11} + 4\mu\text{Sin}[2\theta]b_{12} - b^4 S\mu k_1$$

$$- 4b^2\mu\text{Cos}[2\theta]a_9 k_1 - 4b^2\mu\text{Sin}[2\theta]b_9 k_1 + 4b^2\text{Cos}[2\theta]a_1\mu_1 + 4b^2 k\text{Cos}[2\theta]a_1\mu_1$$

$$- 4b^4 a_2\mu_1 + 4b^4 k a_2\mu_1 - 8b^5\text{Cos}[\theta]a_3\mu_1 + 4b^5 k\text{Cos}[\theta]a_3\mu_1 - 12b^6\text{Cos}[2\theta]a_4\mu_1$$

$$+ 4b^6 k\text{Cos}[2\theta]a_4\mu_1 - 4\text{Cos}[2\theta]a_5\mu_1 - 4b\text{Cos}[\theta]a_6\mu_1 - 4b^2 a_7\mu_1$$

$$- 4b^4\text{Cos}[2\theta]a_8\mu_1 - 4b^2\text{Sin}[2\theta]b_1\mu_1 + 4b^2 k\text{Sin}[2\theta]b_1\mu_1 - 8b^5\text{Sin}[\theta]b_3\mu_1$$

$$- 4b^5 k\text{Sin}[\theta]b_3\mu_1 - 12b^6\text{Sin}[2\theta]b_4\mu_1 - 4b^6 k\text{Sin}[2\theta]b_4\mu_1 - 4\text{Sin}[2\theta]b_5\mu_1$$

$$- 4b\text{Sin}[\theta]b_6\mu_1 + 4b^4\text{Sin}[2\theta]b_8\mu_1) \, . \tag{3.3.14}$$

$$[U_{\theta\text{in}} - U_{\theta\text{out}}]$$

$$= \frac{1}{4b^3\mu\mu_1}(-b^4 S\mu\text{Sin}[2\theta] + 2b^2\mu\text{Sin}[2\theta]a_9 + 2b\mu\text{Sin}[\theta]a_{11} + 2\mu\text{Sin}[2\theta]a_{12} + 2b^2\mu\text{Cos}[2\theta]b_9$$

$$- 2b^2\mu b_{10} - 2b\mu\text{Cos}[\theta]b_{11} - 2\mu\text{Cos}[2\theta]b_{12} + 2b^2\mu\text{Sin}[2\theta]a_9 k_1 - 2b^2\mu\text{Cos}[2\theta]b_9 k_1$$

$$- 2b^2\text{Sin}[2\theta]a_1\mu_1 - 2b^2 k\text{Sin}[2\theta]a_1\mu_1 - 4b^5\text{Sin}[\theta]a_3\mu_1 + 2b^5 k\text{Sin}[\theta]a_3\mu_1 - 6b^6\text{Sin}[2\theta]a_4\mu_1$$

$$+ 2b^6 k\text{Sin}[2\theta]a_4\mu_1 - 2\text{Sin}[2\theta]a_5\mu_1 - 2b\text{Sin}[\theta]a_6\mu_1 + 2b^4\text{Sin}[2\theta]a_8\mu_1 - 2b^2\text{Cos}[2\theta]b_1\mu_1$$

$$+ 2b^2 k\text{Cos}[2\theta]b_1\mu_1 + 2b^4 b_2\mu_1 + 2b^4 k b_2\mu_1 + 4b^5\text{Cos}[\theta]b_3\mu_1 + 2b^5 k\text{Cos}[\theta]b_3\mu_1 + 6b^6\text{Cos}[2\theta]b_4\mu_1$$

$$+ 2b^6 k\text{Cos}[2\theta]b_4\mu_1 + 2\text{Cos}[2\theta]b_5\mu_1 + 2b\text{Cos}[\theta]b_6\mu_1 + 2b^2 b_7\mu_1$$

$$+ 2b^4\text{Cos}[2\theta]b_8\mu_1) \, . \tag{3.3.15}$$

The constants are found by equating the coefficients as

$$\text{Eqn1} = \frac{1}{a^4}(2a^4a_2 + a^2a_7) = 0\,, \tag{3.3.16}$$

$$\text{Eqn2} = \frac{1}{a^4}(-a^2b_7) = 0 \quad , \tag{3.3.17}$$

$$\text{Eqn3} = \frac{1}{a^4}(2a^5a_3 + 2aa_6) = 0 \quad , \tag{3.3.18}$$

$$\text{Eqn4} = \frac{1}{a^4}(-4a^2a_1 + 3a_5 - a^4a_8) = 0 \quad , \tag{3.3.19}$$

$$\text{Eqn5} = \frac{1}{a^4}(-2a^5b_3 + 2ab_6) = 0 \quad , \tag{3.3.20}$$

$$\text{Eqn6} = \frac{1}{a^4}(-4a^2b_1 + 3b_5 + a^4b_8) = 0 \ , \tag{3.3.21}$$

$$\text{Eqn7} = -\frac{2(a^4b_3 - b_6)}{a^3} = 0 \quad , \tag{3.3.22}$$

$$\text{Eqn8} = \frac{-2a^2b_1 - 6a^6b_4 + 3b_5 - a^4b_8}{a^4} = 0 \quad , \tag{3.3.23}$$

$$\text{Eqn9} = -\frac{2(a^4a_3 + a_6)}{a^3} = 0 \quad , \tag{3.3.24}$$

$$\text{Eqn10} = \frac{2a^2a_1 - 6a^6a_4 - 3a_5 - a^4a_8}{a^4} = 0 \quad , \tag{3.3.25}$$

$$\text{Eqn11} = \tfrac{1}{2b^4}(-b^4S + 4b^4a_2 + 2b^2a_7 - 2b^2a_{10}) = 0 \quad , \tag{3.3.26}$$

$$\text{Eqn12} = \frac{4b^5a_3 + 4ba_6 - 4ba_{11}}{2b^4} = 0 \quad , \tag{3.3.27}$$

$$\text{Eqn13} = \frac{b^4S - 8b^2a_1 + 6a_5 - 2b^4a_8 + 8b^2a_9 - 6a_{12}}{2b^4} = 0 \quad , \tag{3.3.28}$$

$$\text{Eqn14} = \frac{-4b^5b_3 + 4bb_6 - 4bb_{11}}{2b^4} = 0 \quad , \tag{3.3.29}$$

$$\text{Eqn15} = \frac{-8b^2b_1 + 6b_5 + 2b^4b_8 + 8b^2b_9 - 6b_{12}}{2b^4} = 0 \quad , \tag{3.3.30}$$

$$\text{Eqn16} = \frac{1}{2b^4}(-2b^2b_7 + 2b^2b_{10}) = 0 \quad , \tag{3.3.31}$$

$$\text{Eqn17} = \frac{4b^5b_3 - 4bb_6 + 4bb_{11}}{2b^4} = 0 \quad , \tag{3.3.32}$$

$$\text{Eqn18} = \frac{4b^2 b_1 + 12b^6 b_4 - 6b_5 + 2b^4 b_8 - 4b^2 b_9 + 6b_{12}}{2b^4} = 0 \quad , \tag{3.3.33}$$

$$\text{Eqn19} = \frac{4b^5 a_3 + 4ba_6 - 4ba_{11}}{2b^4} = 0 \quad , \tag{3.3.34}$$

$$\text{Eqn20} = \frac{-b^4 S - 4b^2 a_1 + 12b^6 a_4 + 6a_5 + 2b^4 a_8 + 4b^2 a_9 - 6a_{12}}{2b^4} = 0 \quad , \tag{3.3.35}$$

$$\text{Eqn21} = \frac{1}{8b^3 \mu\mu_1}(b^4 S\mu + 4b^2 \mu a_{10} - b^4 S\mu k_1 - 4b^4 a_2 \mu_1 + 4b^4 k a_2 \mu_1 - 4b^2 a_7 \mu_1) = 0 \ , \tag{3.3.36}$$

$$\text{Eqn22} = \frac{4b\mu a_{11} - 8b^5 a_3 \mu_1 + 4b^5 k a_3 \mu_1 - 4ba_6 \mu_1}{8b^3 \mu\mu_1} = 0 \quad , \tag{3.3.37}$$

Eqn23

$$= \frac{2b^4 S\mu - 4b^2 \mu a_9 + 4\mu a_{12} - 4b^2 \mu a_9 k_1 + 4b^2 a_1 \mu_1 + 4b^2 k a_1 \mu_1 - 12b^6 a_4 \mu_1 + 4b^6 k a_4 \mu_1 - 4a_5 \mu_1 - 4b^4 a_8 \mu_1}{8b^3 \mu\mu_1}$$

$$= 0 \quad , \tag{3.3.38}$$

$$\text{Eqn24} = \frac{4b\mu b_{11} - 8b^5 b_3 \mu_1 - 4b^5 k b_3 \mu_1 - 4bb_6 \mu_1}{8b^3 \mu\mu_1} = 0 \quad , \tag{3.3.39}$$

Eqn25

$$= \frac{4b^2 \mu b_9 + 4\mu b_{12} - 4b^2 \mu b_9 k_1 - 4b^2 b_1 \mu_1 + 4b^2 k b_1 \mu_1 - 12b^6 b_4 \mu_1 - 4b^6 k b_4 \mu_1 - 4b_5 \mu_1 + 4b^4 b_8 \mu_1}{8b^3 \mu\mu_1}$$

$$= 0 \quad , \tag{3.3.40}$$

$$\text{Eqn26} = \frac{1}{4b^3 \mu\mu_1}(-2b^2 \mu b_{10} + 2b^4 b_2 \mu_1 + 2b^4 k b_2 \mu_1 + 2b^2 b_7 \mu_1) = 0 \quad , \tag{3.3.41}$$

$$\text{Eqn27} = \frac{-2b\mu b_{11} + 4b^5 b_3 \mu_1 + 2b^5 k b_3 \mu_1 + 2bb_6 \mu_1}{4b^3 \mu\mu_1} = 0 \quad , \tag{3.3.42}$$

Eqn28

$$= \frac{2b^2 \mu b_9 - 2\mu b_{12} - 2b^2 \mu b_9 k_1 - 2b^2 b_1 \mu_1 + 2b^2 k b_1 \mu_1 + 6b^6 b_4 \mu_1 + 2b^6 k b_4 \mu_1 + 2b_5 \mu_1 + 2b^4 b_8 \mu_1}{4b^3 \mu\mu_1}$$

$$= 0 \quad , \tag{3.3.43}$$

$$\text{Eqn29} = \frac{2b\mu a_{11} - 4b^5 a_3 \mu_1 + 2b^5 k a_3 \mu_1 - 2ba_6 \mu_1}{4b^3 \mu\mu_1} = 0 \quad , \tag{3.3.44}$$

Eqn30

$$= \frac{-b^4 S\mu + 2b^2\mu a_9 + 2\mu a_{12} + 2b^2\mu a_9 k_1 - 2b^2 a_1\mu_1 - 2b^2 k a_1\mu_1 - 6b^6 a_4\mu_1 + 2b^6 k a_4\mu_1 - 2a_5\mu_1 + 2b^4 a_8\mu_1}{4b^3\mu\mu_1}$$

$$= 0 \quad . \tag{3.3.45}$$

Solving the unknowns using Mathematica, we have

$a_1 \to (a^2b^2S\mu(\mu(2a^6 + 3a^2b^4 - 5b^6 + (a^6 - b^6)k_1) - (2a^6 + 3a^2b^4 + 2b^6(-3 + k) + (a^6 + b^6(-3$

$+ k))k_1)\mu_1))\ \ /\ \ (2(a^2 - b^2)^4\mu^2(1 + k_1) - 2(a - b)(a + b)\mu(b^6(1 + k) + a^6(2$

$+ k) + a^4b^2(2 + k) + 4a^2b^4(2 + k) + (a^6 - 3a^4b^2 + b^6(-3 + k))k_1)\mu_1$

$+ 2(4a^6b^2 + b^8(-3 + k) + a^8(1 + k) + a^2b^6(-3 + k)(1 + k))\mu_1^2),$

$a_2 \to -\left(b^2S\mu(1 + k_1)\right)\ /\ \ (8(a - b)(a + b)\mu - 4\left(2a^2 + b^2(-1 + k)\right)\mu_1),$

$a_4 \to \left(b^2S\mu\left((a - b)(a + b)\mu(2a^2 + b^2 + a^2k_1) + (-2a^4 + b^4 + a^2b^2(1 + k) - a^4k_1)\mu_1\right)\right)$

$/\ \ (2(a^2 - b^2)^4\mu^2(1 + k_1) - 2(a - b)(a + b)\mu(b^6(1 + k) + a^6(2 + k) + a^4b^2(2$

$+ k) + 4a^2b^4(2 + k) + (a^6 - 3a^4b^2 + b^6(-3 + k))k_1)\mu_1 + 2(4a^6b^2 + b^8(-3 + k)$

$+ a^8(1 + k) + a^2b^6(-3 + k)(1 + k))\mu_1^2)\ \ ,$

$a_5 \to \left(a^4b^4S\mu\left((a^4 + 4a^2b^2 - 5b^4)\mu + (a^4 - b^4)\mu k_1\right.\right.$

$\left.\left.- (4a^2b^2 + 2b^4(-3 + k) + a^4(1 + k) + b^4(-3 + k)k_1)\mu_1\right)\right)\ \ /\ \ (2((a^2 - b^2)^4\mu^2$

$+ (a^2 - b^2)^4\mu^2k_1 - (a - b)(a + b)\mu(b^6(1 + k) + a^6(2 + k) + a^4b^2(2 + k)$

$+ 4a^2b^4(2 + k) + (a^6 - 3a^4b^2 + b^6(-3 + k))k_1)\mu_1 + 4a^6b^2 + b^8(-3 + k) + a^8(1$

$+ k) + a^2b^6(-3 + k)(1 + k))\mu_1^2))\ ,$

$a_7 \to \left(a^2b^2S\mu(1 + k_1)\right)\ \ /\ \ (4(a - b)(a + b)\mu - 2(2a^2 + b^2(-1 + k))\mu_1),$

$$a_8 \to \Big(b^2 S\mu\big(\mu(-8a^6+3a^4b^2+5b^6+(-4a^6+3a^4b^2+b^6)k_1)$$

$$+\big(8a^6+2b^6(-3+k)-3a^4b^2(1+k)+\big(4a^6+b^6(-3+k)\big)k_1\big)\mu_1\big)\Big)$$

$$/\quad (2(a^2-b^2)^4\mu^2(1+k_1)-2(a-b)(a+b)\mu(b^6(1+k)+a^6(2+k)+a^4b^2(2$$

$$+k)+4a^2b^4(2+k)+(a^6-3a^4b^2+b^6(-3+k))k_1)\mu_1+2(4a^6b^2+b^8(-3+k)$$

$$+a^8(1+k)+a^2b^6(-3+k)(1+k))\mu_1^2)\,,$$

$$a_9 \to -\Big(b^2 S\big(-(a^2-b^2)^4\mu^2-(8a^6b^2-6a^4b^4+b^8(-4+k)-2a^2b^6(-1+k)+a^8k)\mu\mu_1$$

$$+\big(4a^6b^2+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1+k)\big)\mu_1^2\big)\Big)$$

$$/\;\; (2((a^2-b^2)^4\mu^2+(a^2-b^2)^4\mu^2k_1-(a-b)(a+b)\mu(b^6(1+k)+a^6(2+k)$$

$$+a^4b^2(2+k)+4a^2b^4(2+k)+(a^6-3a^4b^2+b^6(-3+k))k_1)\mu_1+(4a^6b^2$$

$$+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1+k))\mu_1^2))\,,$$

$$a_{10} \to \frac{1}{2}b^2 S\left(-1+\frac{(a-b)(a+b)\mu(1+k_1)}{2(a-b)(a+b)\mu+\big(-2a^2-b^2(-1+k)\big)\mu_1}\right),$$

$$a_{12} \to -\Big(b^4 S\mu_1\big(-(a-b)(a+b)\big(a^6+5a^4b^2-2a^2b^4(-3+k)-b^6(-3+k)\big)\mu$$

$$-a^2\big(a^6+b^6(-3+k)-a^2b^4(-2+k)\big)\mu k_1$$

$$+\big(4a^6b^2+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1+k)\big)\mu_1\big)\Big)$$

$$/\quad (2((a^2-b^2)^4\mu^2+(a^2-b^2)^4\mu^2k_1-(a-b)(a+b)\mu(b^6(1+k)+a^6(2+k)$$

$$+a^4b^2(2+k)+4a^2b^4(2+k)+(a^6-3a^4b^2+b^6(-3+k))k_1)\mu_1+(4a^6b^2$$

$$+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1+k))\mu_1^2))\,,$$

$$a_3 \to 0, a_6 \to 0, a_{11} \to 0, b_1 \to 0, b_2 \to 0, b_3 \to 0, b_4 \to 0, b_5 \to 0,$$

$$b_6 \to 0, b_7 \to 0, b_8 \to 0, b_9 \to 0, b_{10} \to 0, b_{11} \to 0, b_{12} \to 0 \quad . \qquad (3.3.46)$$

Substituting the above solutions into Equations (3.3.4) and (3.3.5), we get the final stress and displacement expressions for the circular inclusion embedded with the circular hole as

$$\sigma_{\mathrm{rin}} = \{\frac{1}{2r^4} b^2(a-r)(a+r)S\mu(\frac{r^2(1+k_1)}{2(a-b)(a+b)\mu + \left(-2a^2 - b^2(-1+k)\right)\mu_1}$$

$$+ \Big(\mathrm{Cos}[2\theta]\Big(\mu\big(3a^2b^2(a^4 + 4a^2b^2 - 5b^4) + (-8a^6 + 3a^4b^2 + 5b^6)r^2$$

$$+ \left(-3a^2b^6 + 3a^4b^2r^2 + b^6r^2 + a^6(3b^2 - 4r^2)\right)k_1\big)$$

$$- \big(6a^2b^6(-3+k) - 2b^6(-3+k)r^2 + a^6(3b^2(1+k) - 8r^2)$$

$$+ 3a^4b^2(4b^2 + (1+k)r^2) - \left(-3a^2b^6(-3+k) + \left(4a^6 + b^6(-3+k)\right)r^2\right)k_1\big)\mu_1\Big)\Big)$$

$$/ ((a^2 - b^2)^4\mu^2(1+k_1) - (a-b)(a+b)\mu(b^6(1+k) + a^6(2+k) + a^4b^2(2+k)$$

$$+ 4a^2b^4(2+k) + (a^6 - 3a^4b^2 + b^6(-3+k))k_1)\mu_1 + (4a^6b^2 + b^8(-3+k)$$

$$+ a^8(1+k) + a^2b^6(-3+k)(1$$

$$+ k))\mu_1^2))\} \quad , \qquad (3.3.47)$$

$$\sigma_{\theta\mathrm{in}} = \{\frac{1}{2r^4} b^2 S\mu(-\frac{4r^4}{8(a-b)(a+b)\mu - 4\left(2a^2 + b^2(-1+k)\right)\mu_1} - (\mathrm{Cos}[2\theta](2(a-b)(a+b)\mu$$

$$+ (-2a^2 - b^2(-1+k))\mu_1)((a-b)(a+b)\mu(3a^4b^2(a^2 + 5b^2) + (8a^4 + 5a^2b^2$$

$$+ 5b^4)r^4 - 12(2a^2 + b^2)r^6 + (3a^4b^2(a^2 + b^2) + (4a^4 + a^2b^2 + b^4)r^4$$

$$- 12a^2r^6)k_1) - (3a^4b^2(4a^2b^2 + 2b^4(-3+k) + a^4(1+k)) + (8a^6 + 2b^6(-3$$

$$+ k) - 3a^4b^2(1+k))r^4 + 12(-2a^4 + b^4 + a^2b^2(1+k))r^6 + (4a^6r^4 + b^6(-3$$

$$+ k)r^4 + 3a^4(b^6(-3+k) - 4r^6))k_1)\mu_1) + r^2(a^2 + (a^2 + r^2)k_1)((a^2 - b^2)^4\mu^2(1$$

$$+ k_1) - (a-b)(a+b)\mu(b^6(1+k) + a^6(2+k) + a^4b^2(2+k) + 4a^2b^4(2+k)$$

$$+ (a^6 - 3a^4b^2 + b^6(-3+k))k_1)\mu_1 + (4a^6b^2 + b^8(-3+k) + a^8(1+k)$$

$$+ a^2b^6(-3+k)(1+k))\mu_1^2)) /((2(a-b)(a+b)\mu + (-2a^2 - b^2(-1$$

$$+ k))\mu_1)((a^2 - b^2)^4\mu^2(1+k_1) - (a-b)(a+b)\mu(b^6(1+k) + a^6(2+k)$$

$$+ a^4b^2(2+k) + 4a^2b^4(2+k) + (a^6 - 3a^4b^2 + b^6(-3+k))k_1)\mu_1 + (4a^6b^2$$

$$+ b^8(-3+k) + a^8(1+k) + a^2b^6(-3+k)(1$$

$$+ k))\mu_1^2)))\} \quad , \qquad (3.3.48)$$

$$\tau_{\mathrm{r\theta in}} = \{(b^2(-a^2+r^2)S\mu \mathrm{Sin}[2\theta](-(a^2-b^2)(5b^4r^2-6b^2r^4+a^4(3b^2-4r^2)+a^2(15b^4-b^2r^2$$

$$-12r^4))\mu+(a^6(3b^2(1+k)-4r^2)+2b^4r^2(b^2(-3+k)+3r^2)+3a^4(4b^4$$

$$+b^2(1+k)r^2-4r^4)+6a^2b^2(b^4(-3+k)+b^2r^2+(1+k)r^4))\mu_1+k_1(-(a^2$$

$$-b^2)(b^4r^2+a^4(3b^2-2r^2)+a^2(3b^4+b^2r^2-6r^4))\mu+(3a^2b^6(-3+k)$$

$$-2a^6r^2+b^6(-3+k)r^2-6a^4r^4)\mu_1)))\ /\ (2r^4((a^2-b^2)^4\mu^2+(b^8(1+k)-a^8(2$$

$$+k)-3a^4b^4(2+k)+a^2b^6(7+3k))\mu\mu_1+(4a^6b^2+b^8(-3+k)+a^8(1+k)$$

$$+a^2b^6(-3-2k+k^2))\mu_1^2+(a^2-b^2)\mu k_1((a^2-b^2)^3\mu-(a^6-3a^4b^2+b^6(-3$$

$$+k))\mu_1))))\}\ , \qquad (3.3.49)$$

$$U_{\mathrm{rin}} = \{\frac{1}{8r^3}b^2S(-\frac{r^2(2a^2+(-1+k)r^2)(1+k_1)}{2(a-b)(a+b)\mu+\left(-2a^2-b^2(-1+k)\right)\mu_1}-(2\mathrm{Cos}[2\theta]((a-b)(a$$

$$+b)(a^4b^2(a^2+5b^2)-a^2(2a^4+2a^2b^2+5b^4)(1+k)r^2-(8a^4+5a^2b^2$$

$$+5b^4)r^4-(2a^2+b^2)(-3+k)r^6)\mu+(a^4b^2(a^4-b^4)-a^2(a^6-b^6)(1+k)r^2$$

$$+(-4a^6+3a^4b^2+b^6)r^4-a^2(a-b)(a+b)(-3+k)r^6)\mu k_1+(-a^4b^2(4a^2b^2$$

$$+2b^4(-3+k)+a^4(1+k))+a^2(2a^6+3a^2b^4+2b^6(-3+k))(1+k)r^2+(8a^6$$

$$+2b^6(-3+k)-3a^4b^2(1+k))r^4-(-3+k)(-2a^4+b^4+a^2b^2(1+k))r^6$$

$$+(-a^4b^6(-3+k)+a^2(a^6+b^6(-3+k))(1+k)r^2+(4a^6+b^6(-3+k))r^4$$

$$+a^4(-3+k)r^6)k_1)\mu_1))/((a^2-b^2)^4\mu^2(1+k_1)-(a-b)(a+b)\mu(b^6(1+k)$$

$$+a^6(2+k)+a^4b^2(2+k)+4a^2b^4(2+k)+(a^6-3a^4b^2+b^6(-3+k))k_1)\mu_1$$

$$+(4a^6b^2+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1$$

$$+k))\mu_1^2))\}\ , \qquad (3.3.50)$$

$$U_{\theta\mathrm{in}} = \{(b^2 S\mathrm{Sin}[2\theta](-(a-b)(a+b)(a^4b^2(a^2+5b^2)+a^2(2a^4+2a^2b^2+5b^4)(1+k)r^2+(8a^4 + 5a^2b^2+5b^4)r^4-(2a^2+b^2)(-3+k)r^6)\mu+(b^4(-3+k)r^4(2b^2+r^2)+a^8(1 + k)(b^2+2r^2)+a^2b^2(-3+k)(1+k)r^2(2b^4+r^4)+4a^6(b^4+2r^4)+a^4(b - r)(b+r)(2b^4(-3+k)+b^2(-3+5k)r^2+2(-3+k)r^4))\mu_1+k_1((-a^8b^2 + a^4b^6-a^2(a^6-b^6)(1+k)r^2+(-4a^6+3a^4b^2+b^6)r^4+a^2(a-b)(a+b)(-3 + k)r^6)\mu+(a^8(1+k)r^2+a^2b^6(-3+k)(1+k)r^2+4a^6r^4+b^6(-3+k)r^4 + a^4(-3+k)(b^6-r^6))\mu_1)))/(4r^3((a^2-b^2)^4\mu^2+(a^2-b^2)^4\mu^2k_1-(a-b)(a + b)\mu(b^6(1+k)+a^6(2+k)+a^4b^2(2+k)+4a^2b^4(2+k)+(a^6-3a^4b^2 + b^6(-3+k))k_1)\mu_1+(4a^6b^2+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1 + k))\mu_1^2))\}. \qquad (3.3.51)$$

Similarly by substituting the constants into Equations (3.3.7) and (3.3.8), we get the final stress and displacement equations for the infinite matrix surrounding the disc as

$$\sigma_{\mathrm{rout}} = \{\frac{1}{2r^4}S(r^4-r^4\mathrm{Cos}[2\theta]-(3b^4\mathrm{Cos}[2\theta]\mu_1(-(a-b)(a+b)(a^6+5a^4b^2-2a^2b^4(-3+k) - b^6(-3+k))\mu-a^2(a^6+b^6(-3+k)-a^2b^4(-2+k))\mu k_1+(4a^6b^2+b^8(-3 + k)+a^8(1+k)+a^2b^6(-3+k)(1+k))\mu_1))/((a^2-b^2)^4\mu^2+(a^2-b^2)^4\mu^2k_1 - (a-b)(a+b)\mu(b^6(1+k)+a^6(2+k)+a^4b^2(2+k)+4a^2b^4(2+k)+(a^6 - 3a^4b^2+b^6(-3+k))k_1)\mu_1+(4a^6b^2+b^8(-3+k)+a^8(1+k)+a^2b^6(-3 + k)(1+k))\mu_1^2)+(4b^2r^2\mathrm{Cos}[2\theta](-(a^2-b^2)^4\mu^2-(8a^6b^2-6a^4b^4+b^8(-4 + k)-2a^2b^6(-1+k)+a^8k)\mu\mu_1+(4a^6b^2+b^8(-3+k)+a^8(1+k)+a^2b^6(-3 + k)(1+k))\mu_1^2))/((a^2-b^2)^4\mu^2+(a^2-b^2)^4\mu^2k_1-(a-b)(a+b)\mu(b^6(1+k) + a^6(2+k)+a^4b^2(2+k)+4a^2b^4(2+k)+(a^6-3a^4b^2+b^6(-3+k))k_1)\mu_1 + (4a^6b^2+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1+k))\mu_1^2)+b^2r^2(-1 + \frac{(a-b)(a+b)\mu(1+k_1)}{2(a-b)(a+b)\mu+\left(-2a^2-b^2(-1+k)\right)\mu_1})))\}\ , \qquad (3.3.52)$$

$$\sigma_{\theta\mathrm{out}} = \{\frac{1}{2r^4}S(r^4 + r^4\mathrm{Cos}[2\theta] + (3b^4\mathrm{Cos}[2\theta]\mu_1(-(a-b)(a+b)(a^6 + 5a^4b^2 - 2a^2b^4(-3+k) - b^6(-3+k))\mu - a^2(a^6 + b^6(-3+k) - a^2b^4(-2+k))\mu k_1 + (4a^6b^2 + b^8(-3+k) + a^8(1+k) + a^2b^6(-3+k)(1+k))\mu_1))/((a^2-b^2)^4\mu^2 + (a^2-b^2)^4\mu^2k_1 - (a-b)(a+b)\mu(b^6(1+k) + a^6(2+k) + a^4b^2(2+k) + 4a^2b^4(2+k) + (a^6 - 3a^4b^2 + b^6(-3+k))k_1)\mu_1 + (4a^6b^2 + b^8(-3+k) + a^8(1+k) + a^2b^6(-3+k)(1+k))\mu_1^2) - b^2r^2(-1 + \frac{(a-b)(a+b)\mu(1+k_1)}{2(a-b)(a+b)\mu + \left(-2a^2 - b^2(-1+k)\right)\mu_1}))\} \quad , \qquad (3.3.53)$$

$$\tau_{r\theta\mathrm{out}} = \{\frac{1}{2r^4}S\mathrm{Sin}[2\theta](r^4 - (3b^4\mu_1(-(a-b)(a+b)(a^6 + 5a^4b^2 - 2a^2b^4(-3+k) - b^6(-3+k))\mu - a^2(a^6 + b^6(-3+k) - a^2b^4(-2+k))\mu k_1 + (4a^6b^2 + b^8(-3+k) + a^8(1+k) + a^2b^6(-3+k)(1+k))\mu_1))/((a^2-b^2)^4\mu^2 + (a^2-b^2)^4\mu^2k_1 - (a-b)(a+b)\mu(b^6(1+k) + a^6(2+k) + a^4b^2(2+k) + 4a^2b^4(2+k) + (a^6 - 3a^4b^2 + b^6(-3+k))k_1)\mu_1 + (4a^6b^2 + b^8(-3+k) + a^8(1+k) + a^2b^6(-3+k)(1+k))\mu_1^2) + (2b^2r^2(-(a^2-b^2)^4\mu^2 - (8a^6b^2 - 6a^4b^4 + b^8(-4+k) - 2a^2b^6(-1+k) + a^8k)\mu\mu_1 + (4a^6b^2 + b^8(-3+k) + a^8(1+k) + a^2b^6(-3+k)(1+k))\mu_1^2))/((a^2-b^2)^4\mu^2 + (a^2-b^2)^4\mu^2k_1 - (a-b)(a+b)\mu(b^6(1+k) + a^6(2+k) + a^4b^2(2+k) + 4a^2b^4(2+k) + (a^6 - 3a^4b^2 + b^6(-3+k))k_1)\mu_1 + (4a^6b^2 + b^8(-3+k) + a^8(1+k) + a^2b^6(-3+k)(1+k))\mu_1^2))\} \;, \qquad (3.3.54)$$

$$U_{\text{rout}} = \{\frac{1}{8r^3\mu_1}S(2b^2r^2 - r^4 - 2r^4\text{Cos}[2\theta] + (r^2(-2(a-b)(a+b)\mu(b^2 + (b-r)(b+r)k_1)$$

$$+ (-2a^2 - b^2(-1+k))r^2k_1\mu_1)((a^2-b^2)^4\mu^2(1+k_1) - (a-b)(a+b)\mu(b^6(1$$

$$+ k) + a^6(2+k) + a^4b^2(2+k) + 4a^2b^4(2+k) + (a^6 - 3a^4b^2 + b^6(-3$$

$$+ k))k_1)\mu_1 + (4a^6b^2 + b^8(-3+k) + a^8(1+k) + a^2b^6(-3+k)(1+k))\mu_1^2)$$

$$+ 2b^2\text{Cos}[2\theta](2(-a^2+b^2)\mu + (2a^2 + b^2(-1+k))\mu_1)(-(a^2-b^2)^4r^2\mu^2(1+k_1)$$

$$+ (a-b)(a+b)\mu(b^2(a^6 + 5a^4b^2 - 2a^2b^4(-3+k) - b^6(-3+k)) - (-b^6(-4$$

$$+ k) + a^6k + a^2b^4(2+k) + a^4b^2(8+k))r^2 + (a^2b^2(a^4 + a^2b^2 - b^4(-3+k))$$

$$- (-b^6(-4+k) + a^6k + a^2b^4(2+k) + a^4b^2(8+k))r^2)k_1)\mu_1 - (4a^6b^2 + b^8(-3$$

$$+ k) + a^8(1+k) + a^2b^6(-3+k)(1+k))(b^2 - r^2 - r^2k_1)\mu_1^2))/((2(a-b)(a$$

$$+ b)\mu + (-2a^2 - b^2(-1+k))\mu_1)((a^2-b^2)^4\mu^2(1+k_1) - (a-b)(a+b)\mu(b^6(1$$

$$+ k) + a^6(2+k) + a^4b^2(2+k) + 4a^2b^4(2+k) + (a^6 - 3a^4b^2 + b^6(-3$$

$$+ k))k_1)\mu_1 + (4a^6b^2 + b^8(-3+k) + a^8(1+k) + a^2b^6(-3+k)(1$$

$$+ k))\mu_1^2))))\} \quad , \tag{3.3.55}$$

$$U_{\theta\text{out}} = \{(S\text{Sin}[2\theta]((a-b)^4(a+b)^4(b-r)r^2(b+r)\mu^2 + (a-b)^4(a+b)^4(b-r)r^2(b+r)\mu^2k_1$$

$$+ (a-b)(a+b)\mu(b^4(a^6 + 5a^4b^2 - 2a^2b^4(-3+k) - b^6(-3+k)) + b^2(-b^6(-4$$

$$+ k) + a^6k + a^2b^4(2+k) + a^4b^2(8+k))r^2 + (b^6(1+k) + a^6(2+k) + a^4b^2(2$$

$$+ k) + 4a^2b^4(2+k))r^4 + (-b^8(-4+k)r^2 + b^6(-3+k)r^4 + a^2b^6(-b^2(-3+k)$$

$$+ (2+k)r^2) + a^6(b^4 + b^2kr^2 + r^4) + a^4(b^6 + b^4(8+k)r^2 - 3b^2r^4))k_1)\mu_1$$

$$- (4a^6b^2 + b^8(-3+k) + a^8(1+k) + a^2b^6(-3+k)(1+k))(b^4 + b^2r^2 + r^4$$

$$+ b^2r^2k_1)\mu_1^2))/(4r^3\mu_1(-(a^2-b^2)^4\mu^2(1+k_1) + (a-b)(a+b)\mu(b^6(1+k)$$

$$+ a^6(2+k) + a^4b^2(2+k) + 4a^2b^4(2+k) + (a^6 - 3a^4b^2 + b^6(-3+k))k_1)\mu_1$$

$$- (4a^6b^2 + b^8(-3+k) + a^8(1+k) + a^2b^6(-3+k)(1+k))\mu_1^2))\}. \tag{3.3.56}$$

The final stress and displacement expression are validated by substituting them into the equilibrium equations (2.2.1).

3.3.5 Examples:

Consider a problem to find the stress and displacement for an infinite plate made of material A, containing a circular inclusion made of material B embedded with a circular hole of radius a=1 mm and the circular inclusion of radius b=2 mm, subjected to far field tensile loading of S= 100 N/mm^2.

For material A, the shear modulus $\mu = 10$ GPa and $\kappa = 1.8$ (Poisson's ratio $\boldsymbol{\upsilon} = 0.3$).

For material B, the shear modulus $\mu 1 = 70$ GPa and $\kappa 1 = 1.4$ (Poisson's ration $\boldsymbol{\upsilon} = 0.4$).

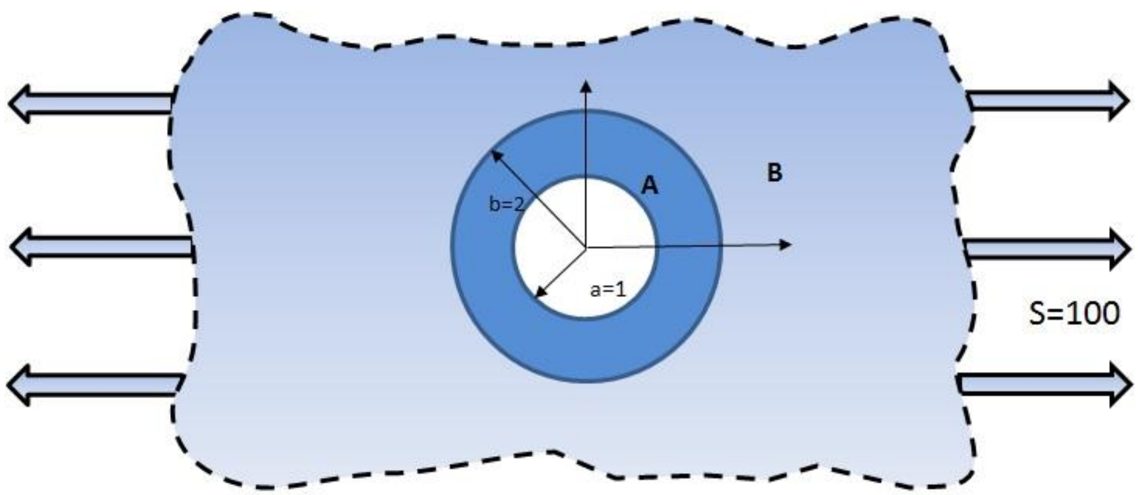

Figure 3.4 Two-Dimensional Circular Inclusion / Hole

Substituting the above properties into the derived stress and displacement equations and converting from polar to Cartesian form, we get

$$\sigma_{\text{rin}} = 11.320754716981131 - \frac{11.320754716981131}{x^2} \text{ ,}$$

$$\sigma_{\theta\text{in}} = 11.32075471698113 + \frac{11.32075471698113}{x^2},$$

$$\tau_{\mathrm{r\theta in}} = 17.351097037683406 - \frac{32.56839056160773}{x^4} + \frac{24.959743799645565}{x^2}$$
$$- 9.742450275721241x^2 \ ,$$

$$U_{\mathrm{rin}} = \frac{0. + 0.5660377358490566x^2 + 0.22641509433962265x^4}{x^3} \ ,$$

$$U_{\theta \mathrm{in}}$$
$$= \frac{0.5428065093601289 + 1.7471820659751893x^2 + 0.8675548518841703x^4 + 0.09742450275721239x^6}{x^3} \ ,$$

$$\sigma_{\mathrm{rout}} = \frac{50(0. - 3.3207547169811322x^2 + x^4)}{x^4},$$

$$\sigma_{\theta \mathrm{out}} = \frac{50(0. + 3.3207547169811322x^2 + x^4)}{x^4},$$

$$\tau_{\mathrm{r\theta out}} = \frac{50(-56.93498736090246 + 8.840603438194362x^2 + x^4)}{x^4},$$

$$U_{\mathrm{rout}} = \frac{5(0. + 6.6415094339622645x^2 + 0.3999999999999999x^4)}{28x^3} \ ,$$

$$U_{\theta \mathrm{out}} = \frac{5(18.97832912030082 + 10.608724125833234x^2 + x^4)}{14x^3}.$$

3.3.6 Graph Plot for Various Stress Components

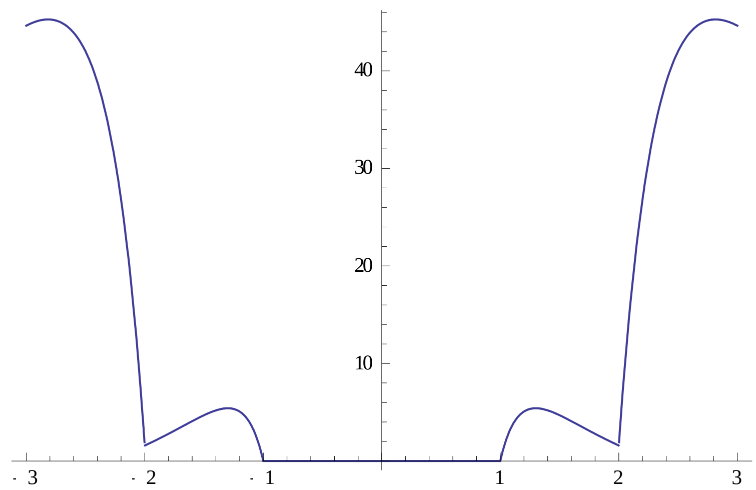

Figure 3.5 Variation of Stress σ_{rr} along the Plate

The radial stress function is symmetrical and continuous along the plate.

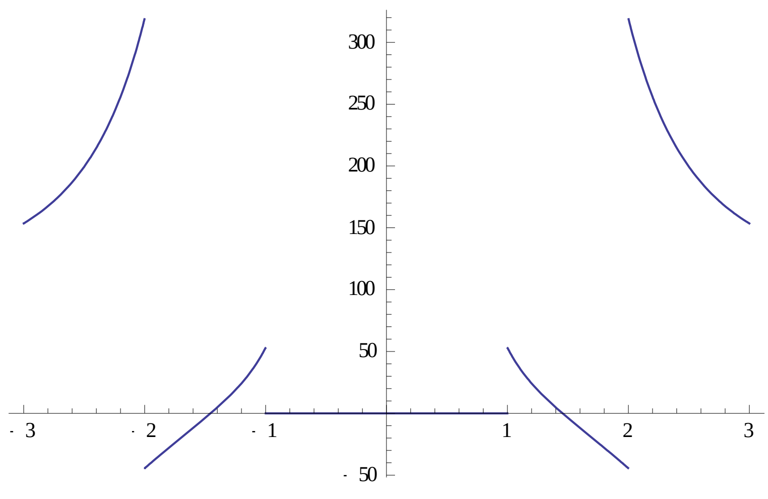

Figure 3.6 Variation of Stress $\boldsymbol{\sigma_{\theta\theta}}$ along the Plate

The circumferential stress function is symmetrical and discontinuous along the plate.

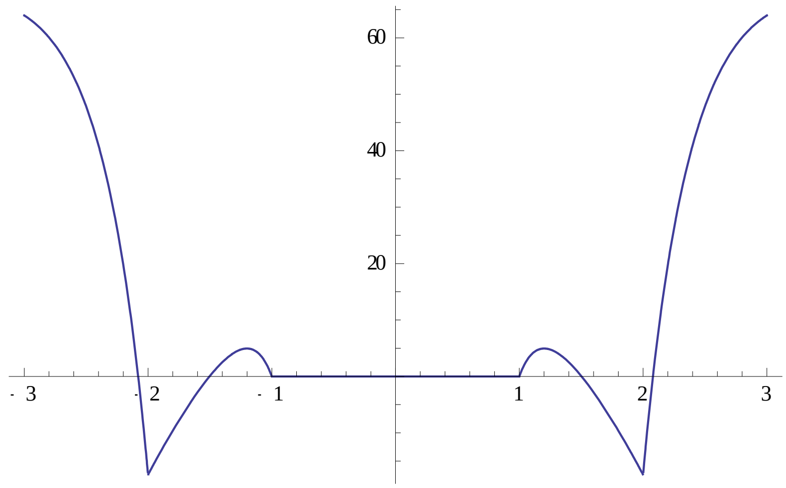

Figure 3.7 Variation of Shear Stress $\tau_{r\theta}$ along the Plate

The shear stress vanishes inside the circular hole region and the function is symmetrical and continuous throughout the plate.

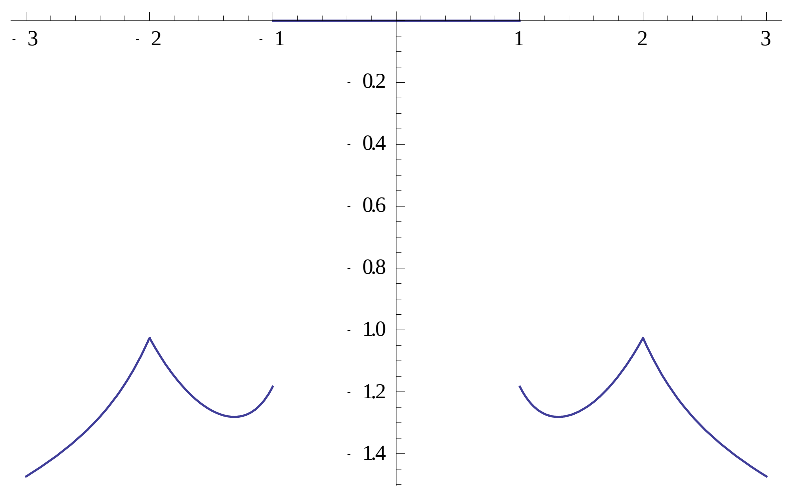

Figure 3.8 Variation of Displacement U_r along the Plate

The radial displacement function is symmetrical and continuous throughout the plate.

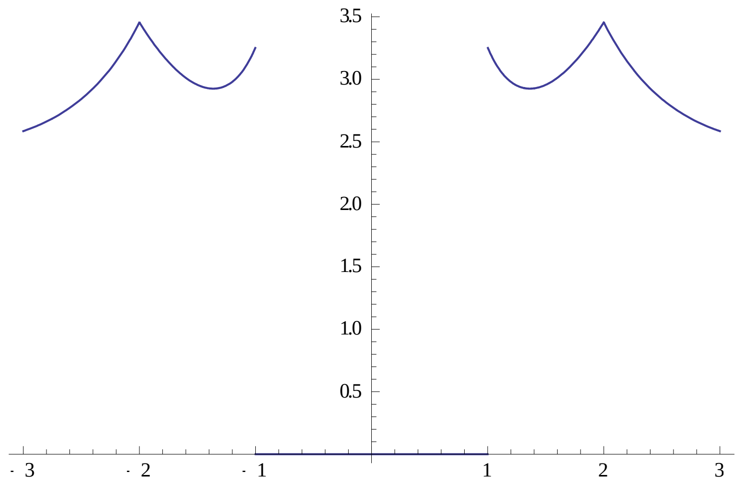

Figure 3.9 Variation of Displacement U_θ along the Plate

The circumferential displacement function is symmetrical and continuous throughout the plate.

3.3.7 3D Plot

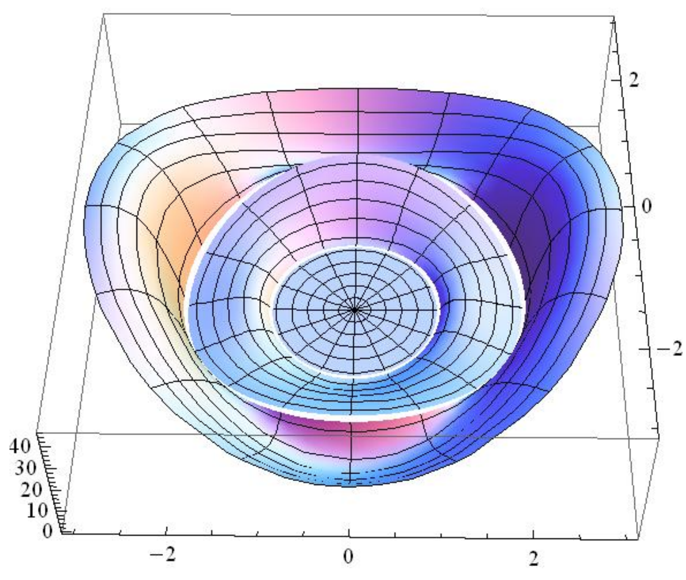

Figure 3.10 Variation of Radial Stress over the Plate

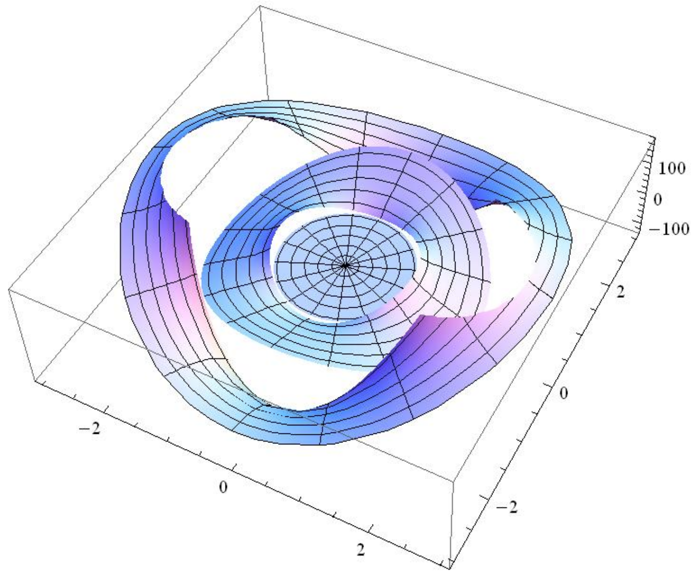

Figure 3.11 Variation of Angular Stress over the Plate

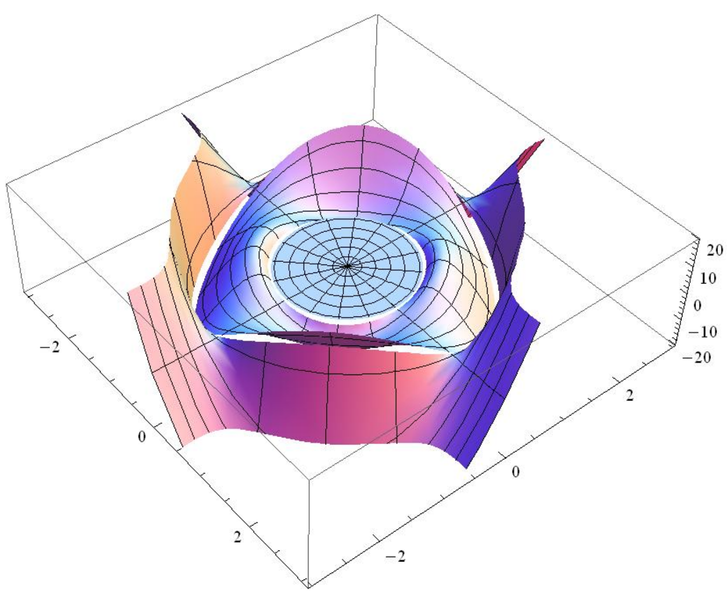

Figure 3.12 Variation of Shear Stress over the Plate

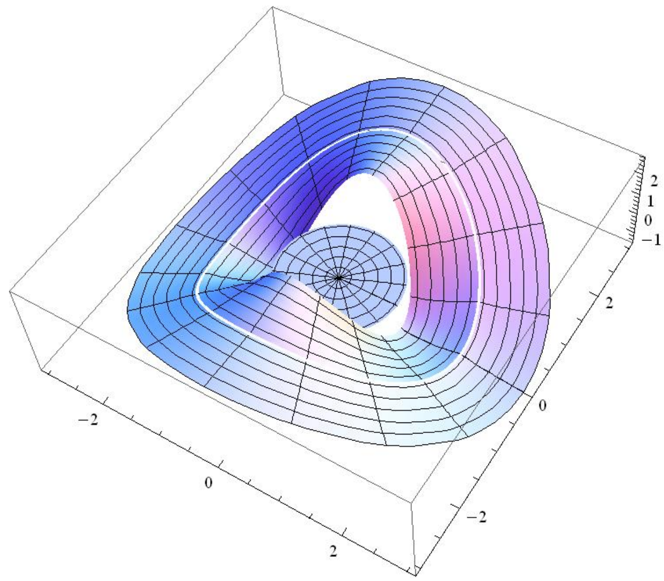

Figure 3.13 Variation of Radial Displacement over the Plate

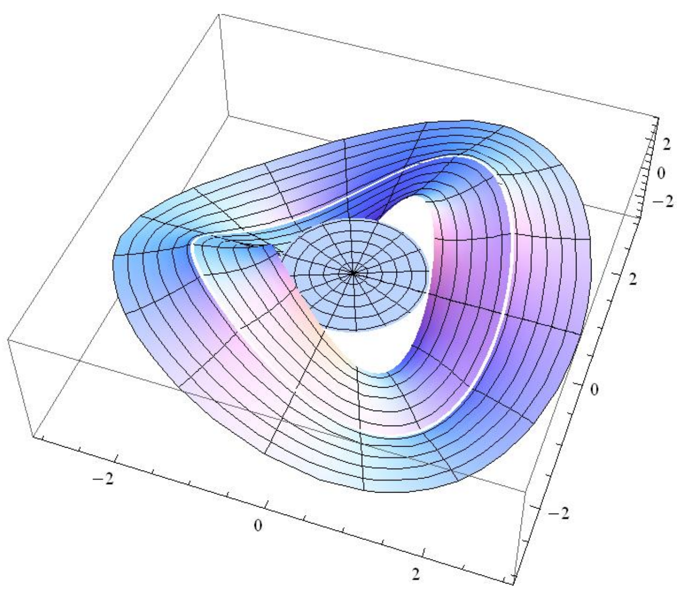

Figure 3.14 Variation of Circumferential Displacement over the Plate

CHAPTER 4

CONCLUSIONS AND FUTURE WORK

Application of the Airy stress function in solving plane elasticity problems has been demonstrated in this research which would have been difficult without the usage of symbolic software. The use of symbolic software is essential which enables to calculate the exact elastic fields and plot the solutions for a variety of elasticity problems in this study. This particular software, Mathematica, has all the necessary computational and plotting tools which provide efficient and accurate ways of analyzing the elasticity problems. In the previous works, the Airy stress function for a two-dimensional infinite plate with a circular inclusion and three phase plate with concentric circular inclusion were derived [7],[8].

In this research, it was shown that the Airy stress function was applied to a two-dimensional infinite plate containing a circular inclusion embedded with a circular hole subjected to tensile loading and the stress components and displacement were successfully obtained at any point subjected to various boundary conditions. Mathematica was very useful for solving complex simultaneous equations with unknown variables and the results generated by the symbolic software were exact. By analyzing the graphical results, it shows that the various stress components and displacements vanished inside the circular hole region and the traction and displacement functions were continuous throughout the plate. The final stress and displacement equations derived from the Airy stress function satisfy the equilibrium and continuity equations.

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Suggestions for Future Work

1. Multiple circular inclusions in finite circular matrix.
2. Concentric circular inclusion/elliptic inclusion embedded with a circular hole.
3. Anisotropic medium can be considered.
4. Analysis of more complex geometries using Airy stress function method.

APPENDIX A

MATHEMATICA CODE

The stress field inside the circular inclusion embedded with a circular hole:

$$\gamma = (a_1 + Ib_1)\, 1/z + (a_2 + Ib_2)z + (a_3 + Ib_3)z^2 + (a_4 + Ib_4)z^3$$

$$\frac{a_1 + ib_1}{z} + z(a_2 + ib_2) + z^2(a_3 + ib_3) + z^3(a_4 + ib_4)$$

$$\varphi = (a_5 + Ib_5)\, 1/z^3 + (a_6 + Ib_6)\, 1/z^2 + (a_7 + Ib_7)\, 1/z + (a_8 + Ib_8)z$$

$$\frac{a_5 + ib_5}{z^3} + \frac{a_6 + ib_6}{z^2} + \frac{a_7 + ib_7}{z} + z(a_8 + ib_8)$$

$$\gamma 1 = D[\gamma, z]$$

$$a_2 - \frac{a_1 + ib_1}{z^2} + ib_2 + 2z(a_3 + ib_3) + 3z^2(a_4 + ib_4)$$

$$\gamma 2 = D[\gamma 1, z]$$

$$\frac{2(a_1 + ib_1)}{z^3} + 2(a_3 + ib_3) + 6z(a_4 + ib_4)$$

$$\gamma 3 = \text{ComplexExpand}[\text{Conjugate}[\gamma 1]]$$

$$-\frac{a_1}{z^2} + a_2 + 2za_3 + 3z^2a_4 + i(\frac{b_1}{z^2} - b_2 - 2zb_3 - 3z^2b_4)$$

$$\varphi 1 = D[\varphi, z]$$

$$a_8 - \frac{3(a_5 + ib_5)}{z^4} - \frac{2(a_6 + ib_6)}{z^3} - \frac{a_7 + ib_7}{z^2} + ib_8$$

$$\varphi 2 = \text{ComplexExpand}[\text{Conjugate}[\varphi]]$$

$$\frac{a_5}{z^3} + \frac{a_6}{z^2} + \frac{a_7}{z} + za_8 - i(\frac{b_5}{z^3} + \frac{b_6}{z^2} + \frac{b_7}{z} + zb_8)$$

$$z = \text{ComplexExpand}[r * \text{Exp}[I * \theta]]$$

$$r\text{Cos}[\theta] + ir\text{Sin}[\theta]$$

$$\text{z1} = \text{ComplexExpand}[\text{Conjugate}[z]]$$

$$r\text{Cos}[\theta] - ir\text{Sin}[\theta]$$

$$\sigma_r + \sigma_\theta == \text{FullSimplify}[4 * \text{ComplexExpand}[\text{Re}[\gamma 1]]]$$

$$\sigma_r + \sigma_\theta == 4(a_2 + 2r\text{Cos}[\theta]a_3 + \frac{\text{Cos}[2\theta](-a_1 + 3r^4a_4)}{r^2} - 2r\text{Sin}[\theta]b_3 - \frac{\text{Sin}[2\theta](b_1 + 3r^4b_4)}{r^2})$$

$$\sigma_\theta - \sigma_r == \text{FullSimplify}[2 * \text{ComplexExpand}[\text{Re}[(\text{Cos}[2\theta] + I\text{Sin}[2\theta]) * ((\text{z1} * \gamma 2) + \varphi 1)]]]$$

$$-\boldsymbol{\sigma_r} + \boldsymbol{\sigma_\theta} == \frac{\mathbf{1}}{\boldsymbol{r^4}}(\mathbf{4}\boldsymbol{r}\mathbf{Cos}[\boldsymbol{\theta}](\boldsymbol{r^4 a_3} - \boldsymbol{a_6}) - \mathbf{2}\boldsymbol{r^2 a_7} + \mathbf{2Cos}[\mathbf{2}\boldsymbol{\theta}](\mathbf{2}\boldsymbol{r^2 a_1} + \mathbf{6}\boldsymbol{r^6 a_4} - \mathbf{3}\boldsymbol{a_5} + \boldsymbol{r^4 a_8}) - \mathbf{4}\boldsymbol{r}\mathbf{Sin}[\boldsymbol{\theta}](\boldsymbol{r^4 b_3} + \boldsymbol{b_6}) - \mathbf{2Sin}[\mathbf{2}\boldsymbol{\theta}](-\mathbf{2}\boldsymbol{r^2 b_1} + \mathbf{6}\boldsymbol{r^6 b_4} + \mathbf{3}\boldsymbol{b_5} + \boldsymbol{r^4 b_8}))$$

$\mathbf{2}\boldsymbol{\tau_{r\theta}}$ **== FullSimplify[2 * ComplexExpand[Im[(Cos[2**$\boldsymbol{\theta}$**] + *I*Sin[2**$\boldsymbol{\theta}$**])((z1 * γ2) + φ1)]]]**

$$\mathbf{2}\boldsymbol{\tau_{r\theta}} == \frac{\mathbf{1}}{\boldsymbol{r^4}}(\mathbf{4}\boldsymbol{r}\mathbf{Sin}[\boldsymbol{\theta}](\boldsymbol{r^4 a_3} + \boldsymbol{a_6}) + \mathbf{2Sin}[\mathbf{2}\boldsymbol{\theta}](-\mathbf{2}\boldsymbol{r^2 a_1} + \mathbf{6}\boldsymbol{r^6 a_4} + \mathbf{3}\boldsymbol{a_5} + \boldsymbol{r^4 a_8}) + \mathbf{4}\boldsymbol{r}\mathbf{Cos}[\boldsymbol{\theta}](\boldsymbol{r^4 b_3} - \boldsymbol{b_6}) - \mathbf{2}\boldsymbol{r^2 b_7} + \mathbf{2Cos}[\mathbf{2}\boldsymbol{\theta}](\mathbf{2}\boldsymbol{r^2 b_1} + \mathbf{6}\boldsymbol{r^6 b_4} - \mathbf{3}\boldsymbol{b_5} + \boldsymbol{r^4 b_8}))$$

Solve[{$\boldsymbol{\sigma_r} + \boldsymbol{\sigma_\theta}$ **== FullSimplify[4 * ComplexExpand[Re[γ1]]],** $\boldsymbol{\sigma_\theta} - \boldsymbol{\sigma_r}$ **=**

= FullSimplify[2 * ComplexExpand[Re[(Cos[2$\boldsymbol{\theta}$**] + *I*Sin[2**$\boldsymbol{\theta}$**]) * ((z1 * γ2)**

+ φ1)]]], $\mathbf{2}\boldsymbol{\tau_{r\theta}}$ **=**

= FullSimplify[2 * ComplexExpand[Im[(Cos[2$\boldsymbol{\theta}$**] + *I*Sin[2**$\boldsymbol{\theta}$**]) * ((z1 * γ2)**

+ φ1)]]]},{$\boldsymbol{\sigma_r}, \boldsymbol{\sigma_\theta}, \boldsymbol{\tau_{r\theta}}$**}]//TrigReduce**

$$\sigma_{\mathrm{rin}} = \frac{1}{r^4}(-4r^2\mathrm{Cos}[2\theta]a_1 + 2r^4 a_2 + 2r^5\mathrm{Cos}[\theta]a_3 + 3\mathrm{Cos}[2\theta]a_5 + 2r\mathrm{Cos}[\theta]a_6 + r^2 a_7 - r^4\mathrm{Cos}[2\theta]a_8 - 4r^2\mathrm{Sin}[2\theta]b_1 - 2r^5\mathrm{Sin}[\theta]b_3 + 3\mathrm{Sin}[2\theta]b_5 + 2r\mathrm{Sin}[\theta]b_6 + r^4\mathrm{Sin}[2\theta]b_8)$$

$$\sigma_{\theta\mathrm{in}} = \frac{1}{r^4}(2r^4 a_2 + 6r^5\mathrm{Cos}[\theta]a_3 + 12r^6\mathrm{Cos}[2\theta]a_4 - 3\mathrm{Cos}[2\theta]a_5 - 2r\mathrm{Cos}[\theta]a_6 - r^2 a_7 + r^4\mathrm{Cos}[2\theta]a_8 - 6r^5\mathrm{Sin}[\theta]b_3 - 12r^6\mathrm{Sin}[2\theta]b_4 - 3\mathrm{Sin}[2\theta]b_5 - 2r\mathrm{Sin}[\theta]b_6 - r^4\mathrm{Sin}[2\theta]b_8)$$

$$\tau_{r\theta\mathrm{in}} = \frac{1}{r^4}(-2r^2\mathrm{Sin}[2\theta]a_1 + 2r^5\mathrm{Sin}[\theta]a_3 + 6r^6\mathrm{Sin}[2\theta]a_4 + 3\mathrm{Sin}[2\theta]a_5 + 2r\mathrm{Sin}[\theta]a_6 + r^4\mathrm{Sin}[2\theta]a_8 + 2r^2\mathrm{Cos}[2\theta]b_1 + 2r^5\mathrm{Cos}[\theta]b_3 + 6r^6\mathrm{Cos}[2\theta]b_4 - 3\mathrm{Cos}[2\theta]b_5 - 2r\mathrm{Cos}[\theta]b_6 - r^2 b_7 + r^4\mathrm{Cos}[2\theta]b_8)$$

$\boldsymbol{U_{rin}}$ **= 1/(2**$\boldsymbol{\mu}$**) * FullSimplify[ComplexExpand[Re[(Cos[**$\boldsymbol{\theta}$**] − *I*Sin[**$\boldsymbol{\theta}$**])(**$\boldsymbol{k} * \boldsymbol{\gamma} - \boldsymbol{z}$ *** γ3**

− φ2)]]]//TrigReduce

$$\frac{1}{2r^3\mu}(r^2\text{Cos}[2\theta]a_1 + kr^2\text{Cos}[2\theta]a_1 - r^4a_2 + kr^4a_2 - 2r^5\text{Cos}[\theta]a_3 + kr^5\text{Cos}[\theta]a_3 - 3r^6\text{Cos}[2\theta]a_4$$
$$+ kr^6\text{Cos}[2\theta]a_4 - \text{Cos}[2\theta]a_5 - r\text{Cos}[\theta]a_6 - r^2a_7 - r^4\text{Cos}[2\theta]a_8 - r^2\text{Sin}[2\theta]b_1$$
$$+ kr^2\text{Sin}[2\theta]b_1 - 2r^5\text{Sin}[\theta]b_3 - kr^5\text{Sin}[\theta]b_3 - 3r^6\text{Sin}[2\theta]b_4 - kr^6\text{Sin}[2\theta]b_4$$
$$- \text{Sin}[2\theta]b_5 - r\text{Sin}[\theta]b_6 + r^4\text{Sin}[2\theta]b_8)$$

$$\boldsymbol{U_{\theta\text{in}} = 1/(2\mu) * \textbf{FullSimplify}[\textbf{ComplexExpand}[\textbf{Im}[(\textbf{Cos}[\theta] - I\textbf{Sin}[\theta])(k * \gamma - z * \gamma 3}$$
$$\boldsymbol{- \varphi 2)]]]//\textbf{TrigReduce}}$$

$$\frac{1}{2r^3\mu}(-r^2\text{Sin}[2\theta]a_1 - kr^2\text{Sin}[2\theta]a_1 - 2r^5\text{Sin}[\theta]a_3 + kr^5\text{Sin}[\theta]a_3 - 3r^6\text{Sin}[2\theta]a_4 + kr^6\text{Sin}[2\theta]a_4$$
$$- \text{Sin}[2\theta]a_5 - r\text{Sin}[\theta]a_6 + r^4\text{Sin}[2\theta]a_8 - r^2\text{Cos}[2\theta]b_1 + kr^2\text{Cos}[2\theta]b_1 + r^4b_2$$
$$+ kr^4b_2 + 2r^5\text{Cos}[\theta]b_3 + kr^5\text{Cos}[\theta]b_3 + 3r^6\text{Cos}[2\theta]b_4 + kr^6\text{Cos}[2\theta]b_4$$
$$+ \text{Cos}[2\theta]b_5 + r\text{Cos}[\theta]b_6 + r^2b_7 + r^4\text{Cos}[2\theta]b_8)$$

The stress field and displacement for the infinite matrix:

$$\boldsymbol{\gamma = (S)\, z/4 + (a_9 + Ib_9)\, 1/z}$$

$$\boldsymbol{\frac{Sz}{4} + \frac{a_9 + ib_9}{z}}$$

$$\boldsymbol{\varphi = (S)\, z/2 + (a_{10} + Ib_{10})\, 1/z + (a_{11} + Ib_{11})\, 1/z^2 + (a_{12} + Ib_{12})\, 1/z^3}$$

$$\boldsymbol{\frac{Sz}{2} + \frac{a_{10} + ib_{10}}{z} + \frac{a_{11} + ib_{11}}{z^2} + \frac{a_{12} + ib_{12}}{z^3}}$$

$$\boldsymbol{\gamma 1 = D[\gamma, z]}$$

$$\boldsymbol{\frac{S}{4} - \frac{a_9 + ib_9}{z^2}}$$

$$\boldsymbol{\gamma 2 = D[\gamma 1, z]}$$

$$\boldsymbol{\frac{2(a_9 + ib_9)}{z^3}}$$

$$\boldsymbol{\gamma 3 = \textbf{ComplexExpand}[\textbf{Conjugate}[\gamma 1]]}$$

$$\frac{S}{4} - \frac{a_9}{z^2} + \frac{ib_9}{z^2}$$

$$\varphi 1 = D[\varphi, z]$$

$$\frac{S}{2} - \frac{a_{10} + ib_{10}}{z^2} - \frac{2(a_{11} + ib_{11})}{z^3} - \frac{3(a_{12} + ib_{12})}{z^4}$$

$$\varphi 2 = \text{ComplexExpand}[\text{Conjugate}[\varphi]]$$

$$\frac{Sz}{2} + \frac{a_{10}}{z} + \frac{a_{11}}{z^2} + \frac{a_{12}}{z^3} - i(\frac{b_{10}}{z} + \frac{b_{11}}{z^2} + \frac{b_{12}}{z^3})$$

$$z = \text{ComplexExpand}[r * \text{Exp}[I * \theta]]$$

$$r\text{Cos}[\theta] + ir\text{Sin}[\theta]$$

$$\text{z1} = \text{ComplexExpand}[\text{Conjugate}[z]]$$

$$r\text{Cos}[\theta] - ir\text{Sin}[\theta]$$

$$\sigma_r + \sigma_\theta == \text{FullSimplify}[4 * \text{ComplexExpand}[\text{Re}[\gamma 1]]]$$

$$\sigma_r + \sigma_\theta == S - \frac{4(\text{Cos}[2\theta]a_9 + \text{Sin}[2\theta]b_9)}{r^2}$$

$$\sigma_\theta - \sigma_r == \text{FullSimplify}[2 * \text{ComplexExpand}[\text{Re}[(\text{Cos}[2\theta] + I\text{Sin}[2\theta])((\text{z1} * \gamma 2) + \varphi 1)]]]$$

$$-\sigma_r + \sigma_\theta == \frac{1}{r^4}(4r^2\text{Cos}[2\theta]a_9 + \text{Cos}[2\theta](r^4 S - 6a_{12}) - 2r(ra_{10} + 2\text{Cos}[\theta](a_{11} - 2r\text{Sin}[\theta]b_9)$$

$$+ 2\text{Sin}[\theta]b_{11}) - 6\text{Sin}[2\theta]b_{12})$$

$$2\tau_{r\theta} == \text{FullSimplify}[2 * \text{ComplexExpand}[\text{Im}[(\text{Cos}[2\theta] + I\text{Sin}[2\theta])((\text{z1} * \gamma 2) + \varphi 1)]]]$$

$$2\tau_{r\theta} == \frac{1}{r^4}(4r\text{Sin}[\theta]a_{11} + \text{Sin}[2\theta](r^4 S - 4r^2 a_9 + 6a_{12}) - 2r^2 b_{10} - 4r\text{Cos}[\theta]b_{11}$$

$$+ 2\text{Cos}[2\theta](2r^2 b_9 - 3b_{12}))$$

$$\text{Solve}[\{\sigma_r + \sigma_\theta == \text{FullSimplify}[4 * \text{ComplexExpand}[\text{Re}[\gamma 1]]], \sigma_\theta - \sigma_r =$$

$$= \text{FullSimplify}[2 * \text{ComplexExpand}[\text{Re}[(\text{Cos}[2\theta] + I\text{Sin}[2\theta])((\text{z1} * \gamma 2)$$

$$+ \varphi 1)]]], 2\tau_{r\theta} =$$

$$= \text{FullSimplify}[2 * \text{ComplexExpand}[\text{Im}[(\text{Cos}[2\theta] + I\text{Sin}[2\theta])((\text{z1} * \gamma 2)$$

$$+ \varphi 1)]]]\}, \{\sigma_r, \sigma_\theta, \tau_{r\theta}\}]//\text{TrigReduce}$$

$$\sigma_{\text{rout}} = \frac{1}{2r^4}(r^4S - r^4S\text{Cos}[2\theta] - 8r^2\text{Cos}[2\theta]a_9 + 2r^2a_{10} + 4r\text{Cos}[\theta]a_{11} + 6\text{Cos}[2\theta]a_{12} - 8r^2\text{Sin}[2\theta]b_9 + 4r\text{Sin}[\theta]b_{11} + 6\text{Sin}[2\theta]b_{12})$$

$$\sigma_{\theta\text{out}} = \frac{1}{2r^4}(r^4S + r^4S\text{Cos}[2\theta] - 2r^2a_{10} - 4r\text{Cos}[\theta]a_{11} - 6\text{Cos}[2\theta]a_{12} - 4r\text{Sin}[\theta]b_{11} - 6\text{Sin}[2\theta]b_{12})$$

$$\tau_{\text{r}\theta\text{out}} = \frac{1}{2r^4}(r^4S\text{Sin}[2\theta] - 4r^2\text{Sin}[2\theta]a_9 + 4r\text{Sin}[\theta]a_{11} + 6\text{Sin}[2\theta]a_{12} + 4r^2\text{Cos}[2\theta]b_9 - 2r^2b_{10} - 4r\text{Cos}[\theta]b_{11} - 6\text{Cos}[2\theta]b_{12})$$

$$\boldsymbol{U_{\text{rout}} = 1/(2\mu_1)\,\text{FullSimplify}[\text{ComplexExpand}[\text{Re}[(\text{Cos}[\theta] - I\text{Sin}[\theta])(k_1 * \gamma - z * \gamma 3 - \varphi 2)]]]//\text{TrigReduce}}$$

$$\frac{1}{8r^3\mu_1}(-r^4S - 2r^4S\text{Cos}[2\theta] + 4r^2\text{Cos}[2\theta]a_9 - 4r^2a_{10} - 4r\text{Cos}[\theta]a_{11} - 4\text{Cos}[2\theta]a_{12} - 4r^2\text{Sin}[2\theta]b_9 - 4r\text{Sin}[\theta]b_{11} - 4\text{Sin}[2\theta]b_{12} + r^4Sk_1 + 4r^2\text{Cos}[2\theta]a_9k_1 + 4r^2\text{Sin}[2\theta]b_9k_1)$$

$$\boldsymbol{U_{\theta\text{out}} = 1/(2\mu_1)\,\text{FullSimplify}[\text{ComplexExpand}[\text{Im}[(\text{Cos}[\theta] - I\text{Sin}[\theta])(k_1 * \gamma - z * \gamma 3 - \varphi 2)]]]//\text{TrigReduce}}$$

$$\boldsymbol{\frac{1}{4r^3\mu_1}(r^4S\text{Sin}[2\theta] - 2r^2\text{Sin}[2\theta]a_9 - 2r\text{Sin}[\theta]a_{11} - 2\text{Sin}[2\theta]a_{12} - 2r^2\text{Cos}[2\theta]b_9 + 2r^2b_{10} + 2r\text{Cos}[\theta]b_{11} + 2\text{Cos}[2\theta]b_{12} - 2r^2\text{Sin}[2\theta]a_9k_1 + 2r^2\text{Cos}[2\theta]b_9k_1)}$$

The continuity conditions:

$$\boldsymbol{\text{EqnA} = \text{Simplify}[\sigma_{\text{rin}} - I\tau_{\text{r}\theta\text{in}}]/.\,r \to a}$$

$$\frac{1}{a^4}(-4a^2\text{Cos}[2\theta]a_1 + 2a^4a_2 + 2a^5\text{Cos}[\theta]a_3 + 3\text{Cos}[2\theta]a_5 + 2a\text{Cos}[\theta]a_6 + a^2a_7$$

$$- a^4\text{Cos}[2\theta]a_8 - 4a^2\text{Sin}[2\theta]b_1 - 2a^5\text{Sin}[\theta]b_3 + 3\text{Sin}[2\theta]b_5 + 2a\text{Sin}[\theta]b_6$$

$$+ a^4\text{Sin}[2\theta]b_8 - i(-2a^2\text{Sin}[2\theta]a_1 + 2a^5\text{Sin}[\theta]a_3 + 6a^6\text{Sin}[2\theta]a_4$$

$$+ 3\text{Sin}[2\theta]a_5 + 2a\text{Sin}[\theta]a_6 + a^4\text{Sin}[2\theta]a_8 + 2a^2\text{Cos}[2\theta]b_1 + 2a^5\text{Cos}[\theta]b_3$$

$$+ 6a^6\text{Cos}[2\theta]b_4 - 3\text{Cos}[2\theta]b_5 - 2a\text{Cos}[\theta]b_6 - a^2b_7 + a^4\text{Cos}[2\theta]b_8))$$

$$\text{EqnB} = \text{Simplify}[\sigma_{\text{rin}} - \sigma_{\text{rout}}]/.\, r \to b$$

$$\frac{1}{2b^4}(-b^4S + b^4S\text{Cos}[2\theta] - 8b^2\text{Cos}[2\theta]a_1 + 4b^4a_2 + 4b^5\text{Cos}[\theta]a_3 + 6\text{Cos}[2\theta]a_5 + 4b\text{Cos}[\theta]a_6$$

$$+ 2b^2a_7 - 2b^4\text{Cos}[2\theta]a_8 + 8b^2\text{Cos}[2\theta]a_9 - 2b^2a_{10} - 4b\text{Cos}[\theta]a_{11}$$

$$- 6\text{Cos}[2\theta]a_{12} - 8b^2\text{Sin}[2\theta]b_1 - 4b^5\text{Sin}[\theta]b_3 + 6\text{Sin}[2\theta]b_5 + 4b\text{Sin}[\theta]b_6$$

$$+ 2b^4\text{Sin}[2\theta]b_8 + 8b^2\text{Sin}[2\theta]b_9 - 4b\text{Sin}[\theta]b_{11} - 6\text{Sin}[2\theta]b_{12})$$

$$\text{EqnC} = \text{Simplify}[\tau_{\text{r}\theta\text{in}} - \tau_{\text{r}\theta\text{out}}]/.\, r \to b$$

$$\frac{1}{2b^4}(-b^4S\text{Sin}[2\theta] - 4b^2\text{Sin}[2\theta]a_1 + 4b^5\text{Sin}[\theta]a_3 + 12b^6\text{Sin}[2\theta]a_4 + 6\text{Sin}[2\theta]a_5$$

$$+ 4b\text{Sin}[\theta]a_6 + 2b^4\text{Sin}[2\theta]a_8 + 4b^2\text{Sin}[2\theta]a_9 - 4b\text{Sin}[\theta]a_{11} - 6\text{Sin}[2\theta]a_{12}$$

$$+ 4b^2\text{Cos}[2\theta]b_1 + 4b^5\text{Cos}[\theta]b_3 + 12b^6\text{Cos}[2\theta]b_4 - 6\text{Cos}[2\theta]b_5 - 4b\text{Cos}[\theta]b_6$$

$$- 2b^2b_7 + 2b^4\text{Cos}[2\theta]b_8 - 4b^2\text{Cos}[2\theta]b_9 + 2b^2b_{10} + 4b\text{Cos}[\theta]b_{11}$$

$$+ 6\text{Cos}[2\theta]b_{12})$$

$\mathbf{EqnD = FullSimplify}[U_{\mathrm{rin}} - U_{\mathrm{rout}}]/.r \to b//\mathbf{TrigReduce}$

$$\frac{1}{8b^3\mu\mu_1}(b^4S\mu + 2b^4S\mu\mathrm{Cos}[2\theta] - 4b^2\mu\mathrm{Cos}[2\theta]a_9 + 4b^2\mu a_{10} + 4b\mu\mathrm{Cos}[\theta]a_{11} + 4\mu\mathrm{Cos}[2\theta]a_{12}$$

$$+ 4b^2\mu\mathrm{Sin}[2\theta]b_9 + 4b\mu\mathrm{Sin}[\theta]b_{11} + 4\mu\mathrm{Sin}[2\theta]b_{12} - b^4S\mu k_1$$

$$- 4b^2\mu\mathrm{Cos}[2\theta]a_9k_1 - 4b^2\mu\mathrm{Sin}[2\theta]b_9k_1 + 4b^2\mathrm{Cos}[2\theta]a_1\mu_1$$

$$+ 4b^2k\mathrm{Cos}[2\theta]a_1\mu_1 - 4b^4a_2\mu_1 + 4b^4ka_2\mu_1 - 8b^5\mathrm{Cos}[\theta]a_3\mu_1$$

$$+ 4b^5k\mathrm{Cos}[\theta]a_3\mu_1 - 12b^6\mathrm{Cos}[2\theta]a_4\mu_1 + 4b^6k\mathrm{Cos}[2\theta]a_4\mu_1 - 4\mathrm{Cos}[2\theta]a_5\mu_1$$

$$- 4b\mathrm{Cos}[\theta]a_6\mu_1 - 4b^2a_7\mu_1 - 4b^4\mathrm{Cos}[2\theta]a_8\mu_1 - 4b^2\mathrm{Sin}[2\theta]b_1\mu_1$$

$$+ 4b^2k\mathrm{Sin}[2\theta]b_1\mu_1 - 8b^5\mathrm{Sin}[\theta]b_3\mu_1 - 4b^5k\mathrm{Sin}[\theta]b_3\mu_1 - 12b^6\mathrm{Sin}[2\theta]b_4\mu_1$$

$$- 4b^6k\mathrm{Sin}[2\theta]b_4\mu_1 - 4\mathrm{Sin}[2\theta]b_5\mu_1 - 4b\mathrm{Sin}[\theta]b_6\mu_1 + 4b^4\mathrm{Sin}[2\theta]b_8\mu_1)$$

$\mathbf{EqnE = FullSimplify}[U_{\theta\mathrm{in}} - U_{\theta\mathrm{out}}]/.r \to b//\mathbf{TrigReduce}$

$$\frac{1}{4b^3\mu\mu_1}(-b^4S\mu\mathrm{Sin}[2\theta] + 2b^2\mu\mathrm{Sin}[2\theta]a_9 + 2b\mu\mathrm{Sin}[\theta]a_{11} + 2\mu\mathrm{Sin}[2\theta]a_{12} + 2b^2\mu\mathrm{Cos}[2\theta]b_9$$

$$- 2b^2\mu b_{10} - 2b\mu\mathrm{Cos}[\theta]b_{11} - 2\mu\mathrm{Cos}[2\theta]b_{12} + 2b^2\mu\mathrm{Sin}[2\theta]a_9k_1$$

$$- 2b^2\mu\mathrm{Cos}[2\theta]b_9k_1 - 2b^2\mathrm{Sin}[2\theta]a_1\mu_1 - 2b^2k\mathrm{Sin}[2\theta]a_1\mu_1 - 4b^5\mathrm{Sin}[\theta]a_3\mu_1$$

$$+ 2b^5k\mathrm{Sin}[\theta]a_3\mu_1 - 6b^6\mathrm{Sin}[2\theta]a_4\mu_1 + 2b^6k\mathrm{Sin}[2\theta]a_4\mu_1 - 2\mathrm{Sin}[2\theta]a_5\mu_1$$

$$- 2b\mathrm{Sin}[\theta]a_6\mu_1 + 2b^4\mathrm{Sin}[2\theta]a_8\mu_1 - 2b^2\mathrm{Cos}[2\theta]b_1\mu_1 + 2b^2k\mathrm{Cos}[2\theta]b_1\mu_1$$

$$+ 2b^4b_2\mu_1 + 2b^4kb_2\mu_1 + 4b^5\mathrm{Cos}[\theta]b_3\mu_1 + 2b^5k\mathrm{Cos}[\theta]b_3\mu_1$$

$$+ 6b^6\mathrm{Cos}[2\theta]b_4\mu_1 + 2b^6k\mathrm{Cos}[2\theta]b_4\mu_1 + 2\mathrm{Cos}[2\theta]b_5\mu_1 + 2b\mathrm{Cos}[\theta]b_6\mu_1$$

$$+ 2b^2b_7\mu_1 + 2b^4\mathrm{Cos}[2\theta]b_8\mu_1)$$

For solving the unknowns by equating the coefficients:

$$\text{Eqn1} = \frac{1}{a^4}(2a^4a_2 + a^2a_7) = 0$$

$$\text{Eqn2} = \frac{1}{a^4}(-a^2b_7) = 0$$

$$\text{Eqn3} = \frac{1}{a^4}(2a^5a_3 + 2aa_6) = 0$$

$$\text{Eqn4} = \frac{1}{a^4}(-4a^2a_1 + 3a_5 - a^4a_8) = 0$$

$$\text{Eqn5} = \frac{1}{a^4}(-2a^5b_3 + 2ab_6) = 0$$

$$\text{Eqn6} = \frac{1}{a^4}(-4a^2b_1 + 3b_5 + a^4b_8) = 0$$

$$\text{Eqn7} = -\frac{2(a^4b_3 - b_6)}{a^3} = 0$$

$$\text{Eqn8} = \frac{-2a^2b_1 - 6a^6b_4 + 3b_5 - a^4b_8}{a^4} = 0$$

$$\text{Eqn9} = -\frac{2(a^4a_3 + a_6)}{a^3} = 0$$

$$\text{Eqn10} = \frac{2a^2a_1 - 6a^6a_4 - 3a_5 - a^4a_8}{a^4} = 0$$

$$\text{Eqn11} = \frac{1}{2b^4}(-b^4S + 4b^4a_2 + 2b^2a_7 - 2b^2a_{10}) = 0$$

$$\text{Eqn12} = \frac{4b^5a_3 + 4ba_6 - 4ba_{11}}{2b^4} = 0$$

$$\text{Eqn13} = \frac{b^4S - 8b^2a_1 + 6a_5 - 2b^4a_8 + 8b^2a_9 - 6a_{12}}{2b^4} = 0$$

$$\text{Eqn14} = \frac{-4b^5b_3 + 4bb_6 - 4bb_{11}}{2b^4} = 0$$

$$\text{Eqn15} = \frac{-8b^2b_1 + 6b_5 + 2b^4b_8 + 8b^2b_9 - 6b_{12}}{2b^4} = 0$$

$$\text{Eqn16} = \frac{1}{2b^4}(-2b^2b_7 + 2b^2b_{10}) = 0$$

$$\text{Eqn17} = \frac{4b^5b_3 - 4bb_6 + 4bb_{11}}{2b^4} = 0$$

$$\text{Eqn18} = \frac{4b^2b_1 + 12b^6b_4 - 6b_5 + 2b^4b_8 - 4b^2b_9 + 6b_{12}}{2b^4} = 0$$

$$\text{Eqn19} = \frac{4b^5a_3 + 4ba_6 - 4ba_{11}}{2b^4} = 0$$

$$\text{Eqn20} = \frac{-b^4S - 4b^2a_1 + 12b^6a_4 + 6a_5 + 2b^4a_8 + 4b^2a_9 - 6a_{12}}{2b^4} = 0$$

$$\text{Eqn21} = \frac{1}{8b^3\mu\mu_1}(b^4S\mu + 4b^2\mu a_{10} - b^4S\mu k_1 - 4b^4a_2\mu_1 + 4b^4ka_2\mu_1 - 4b^2a_7\mu_1) = 0$$

$$\text{Eqn22} = \frac{4b\mu a_{11} - 8b^5a_3\mu_1 + 4b^5ka_3\mu_1 - 4ba_6\mu_1}{8b^3\mu\mu_1} = 0$$

Eqn23

$$= \frac{2b^4S\mu - 4b^2\mu a_9 + 4\mu a_{12} - 4b^2\mu a_9k_1 + 4b^2a_1\mu_1 + 4b^2ka_1\mu_1 - 12b^6a_4\mu_1 + 4b^6ka_4\mu_1 - 4a_5\mu_1 - 4b^4a_8\mu_1}{8b^3\mu\mu_1}$$

$= 0$

$$\text{Eqn24} = \frac{4b\mu b_{11} - 8b^5b_3\mu_1 - 4b^5kb_3\mu_1 - 4bb_6\mu_1}{8b^3\mu\mu_1} = 0$$

Eqn25

$$= \frac{4b^2\mu b_9 + 4\mu b_{12} - 4b^2\mu b_9k_1 - 4b^2b_1\mu_1 + 4b^2kb_1\mu_1 - 12b^6b_4\mu_1 - 4b^6kb_4\mu_1 - 4b_5\mu_1 + 4b^4b_8\mu_1}{8b^3\mu\mu_1}$$

$= 0$

$$\text{Eqn26} = \frac{1}{4b^3\mu\mu_1}(-2b^2\mu b_{10} + 2b^4b_2\mu_1 + 2b^4kb_2\mu_1 + 2b^2b_7\mu_1) = 0$$

$$\text{Eqn27} = \frac{-2b\mu b_{11} + 4b^5b_3\mu_1 + 2b^5kb_3\mu_1 + 2bb_6\mu_1}{4b^3\mu\mu_1} = 0$$

Eqn28

$$= \frac{2b^2\mu b_9 - 2\mu b_{12} - 2b^2\mu b_9k_1 - 2b^2b_1\mu_1 + 2b^2kb_1\mu_1 + 6b^6b_4\mu_1 + 2b^6kb_4\mu_1 + 2b_5\mu_1 + 2b^4b_8\mu_1}{4b^3\mu\mu_1}$$

$= 0$

$$\text{Eqn29} = \frac{2b\mu a_{11} - 4b^5 a_3\mu_1 + 2b^5 k a_3\mu_1 - 2ba_6\mu_1}{4b^3\mu\mu_1} = 0$$

Eqn30

$$= \frac{-b^4S\mu + 2b^2\mu a_9 + 2\mu a_{12} + 2b^2\mu a_9 k_1 - 2b^2 a_1\mu_1 - 2b^2 k a_1\mu_1 - 6b^6 a_4\mu_1 + 2b^6 k a_4\mu_1 - 2a_5\mu_1 + 2b^4 a_8\mu_1}{4b^3\mu\mu_1}$$

$$= \mathbf{0}$$

Solution = Solve[{Eqn1 == 0, Eqn2 == 0, Eqn3 == 0, Eqn4 == 0, Eqn5 == 0, Eqn6 == 0, Eqn7 =

= 0, Eqn8 == 0, Eqn9 == 0, Eqn10 == 0, Eqn11 == 0, Eqn12 == 0, Eqn13 == 0, Eqn14 =

= 0, Eqn15 == 0, Eqn16 == 0, Eqn17 == 0, Eqn18 == 0, Eqn19 == 0, Eqn20 == 0, Eqn21 =

= 0, Eqn22 == 0, Eqn23 == 0, Eqn24 == 0, Eqn25 == 0, Eqn26 == 0, Eqn27 == 0, Eqn28 =

= 0, Eqn29 == 0, Eqn30 =

= 0}, $\{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5, a_6, b_6, a_7, b_7, a_8, b_8, a_9, b_9, a_{10}, b_{10}, a_{11}, b_{11}, a_{12}, b_{12}\}$]/

/FullSimplify

$$a_1 \to (a^2b^2S\mu(\mu(2a^6 + 3a^2b^4 - 5b^6 + (a^6 - b^6)k_1) - (2a^6 + 3a^2b^4 + 2b^6(-3 + k) + (a^6 + b^6(-3 + k))k_1)\mu_1)) \;/\; (2(a^2 - b^2)^4\mu^2(1 + k_1) - 2(a - b)(a + b)\mu(b^6(1 + k) + a^6(2 + k) + a^4b^2(2 + k) + 4a^2b^4(2 + k) + (a^6 - 3a^4b^2 + b^6(-3 + k))k_1)\mu_1 + 2(4a^6b^2 + b^8(-3 + k) + a^8(1 + k) + a^2b^6(-3 + k)(1 + k))\mu_1^2)$$

$$a_2 \to -\left(b^2S\mu(1 + k_1)\right) \;/\; (8(a - b)(a + b)\mu - 4\left(2a^2 + b^2(-1 + k)\right)\mu_1)$$

$$a_4 \to \left(b^2S\mu\left((a - b)(a + b)\mu(2a^2 + b^2 + a^2k_1) + (-2a^4 + b^4 + a^2b^2(1 + k) - a^4k_1)\mu_1\right)\right) \;/\; (2(a^2 - b^2)^4\mu^2(1 + k_1) - 2(a - b)(a + b)\mu(b^6(1 + k) + a^6(2 + k) + a^4b^2(2 + k) + 4a^2b^4(2 + k) + (a^6 - 3a^4b^2 + b^6(-3 + k))k_1)\mu_1 + 2(4a^6b^2 + b^8(-3 + k) + a^8(1 + k) + a^2b^6(-3 + k)(1 + k))\mu_1^2)$$

$$a_5 \to \Big(a^4b^4S\mu\big((a^4+4a^2b^2-5b^4)\mu+(a^4-b^4)\mu k_1$$

$$-(4a^2b^2+2b^4(-3+k)+a^4(1+k)+b^4(-3+k)k_1)\mu_1\big)\Big) \;\; / \;\; (2((a^2-b^2)^4\mu^2$$

$$+(a^2-b^2)^4\mu^2k_1-(a-b)(a+b)\mu(b^6(1+k)+a^6(2+k)+a^4b^2(2+k)$$

$$+4a^2b^4(2+k)+(a^6-3a^4b^2+b^6(-3+k))k_1)\mu_1+4a^6b^2+b^8(-3+k)+a^8(1$$

$$+k)+a^2b^6(-3+k)(1+k))\mu_1^2))$$

$$a_7 \to \big(a^2b^2S\mu(1+k_1)\big) \;\; / \;\; (4(a-b)(a+b)\mu-2(2a^2+b^2(-1+k))\mu_1)$$

$$a_8 \to \Big(b^2S\mu\big(\mu(-8a^6+3a^4b^2+5b^6+(-4a^6+3a^4b^2+b^6)k_1)$$

$$+\big(8a^6+2b^6(-3+k)-3a^4b^2(1+k)+\big(4a^6+b^6(-3+k)\big)k_1\big)\mu_1\big)\Big)$$

$$/ \quad (2(a^2-b^2)^4\mu^2(1+k_1)-2(a-b)(a+b)\mu(b^6(1+k)+a^6(2+k)+a^4b^2(2$$

$$+k)+4a^2b^4(2+k)+(a^6-3a^4b^2+b^6(-3+k))k_1)\mu_1+2(4a^6b^2+b^8(-3+k)$$

$$+a^8(1+k)+a^2b^6(-3+k)(1+k))\mu_1^2)$$

$$a_9 \to -\Big(b^2S\big(-(a^2-b^2)^4\mu^2-(8a^6b^2-6a^4b^4+b^8(-4+k)-2a^2b^6(-1+k)+a^8k)\mu\mu_1$$

$$+\big(4a^6b^2+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1+k)\big)\mu_1^2\big)\Big)$$

$$/ \;\; (2((a^2-b^2)^4\mu^2+(a^2-b^2)^4\mu^2k_1-(a-b)(a+b)\mu(b^6(1+k)+a^6(2+k)$$

$$+a^4b^2(2+k)+4a^2b^4(2+k)+(a^6-3a^4b^2+b^6(-3+k))k_1)\mu_1+(4a^6b^2$$

$$+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1+k))\mu_1^2))$$

$$a_{10} \to \frac{1}{2}b^2S(-1+\frac{(a-b)(a+b)\mu(1+k_1)}{2(a-b)(a+b)\mu+(-2a^2-b^2(-1+k))\mu_1})$$

$$a_{12} \to -\Big(b^4S\mu_1\big(-(a-b)(a+b)\big(a^6+5a^4b^2-2a^2b^4(-3+k)-b^6(-3+k)\big)\mu$$

$$-a^2\big(a^6+b^6(-3+k)-a^2b^4(-2+k)\big)\mu k_1$$

$$+\big(4a^6b^2+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1+k)\big)\mu_1\big)\Big)$$

$$/ \quad (2((a^2-b^2)^4\mu^2+(a^2-b^2)^4\mu^2k_1-(a-b)(a+b)\mu(b^6(1+k)+a^6(2+k)$$

$$+a^4b^2(2+k)+4a^2b^4(2+k)+(a^6-3a^4b^2+b^6(-3+k))k_1)\mu_1+(4a^6b^2$$

$$+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1+k))\mu_1^2))$$

$a_3 \to 0, a_6 \to 0, a_{11} \to 0, b_1 \to 0, b_2 \to 0, b_3 \to 0, b_4 \to 0, b_5 \to 0$

$b_6 \to 0, b_7 \to 0, b_8 \to 0, b_9 \to 0, b_{10} \to 0, b_{11} \to 0, b_{12} \to 0$

The final radial stress for inner and outer region:

$\boldsymbol{\sigma_{\mathbf{rin}}}$**/. Solution/Simplify**

$$\sigma_{\mathrm{rin}} = \{\frac{1}{2r^4} b^2(a-r)(a+r)S\mu(\frac{r^2(1+k_1)}{2(a-b)(a+b)\mu + \left(-2a^2 - b^2(-1+k)\right)\mu_1}$$

$$+ \left(\mathrm{Cos}[2\theta]\left(\mu\left(3a^2b^2(a^4 + 4a^2b^2 - 5b^4) + (-8a^6 + 3a^4b^2 + 5b^6)r^2\right.\right.\right.$$

$$\left.+ \left(-3a^2b^6 + 3a^4b^2r^2 + b^6r^2 + a^6(3b^2 - 4r^2)\right)k_1\right)$$

$$- \left(6a^2b^6(-3+k) - 2b^6(-3+k)r^2 + a^6(3b^2(1+k) - 8r^2)\right)$$

$$\left.\left. + 3a^4b^2(4b^2 + (1+k)r^2) - \left(-3a^2b^6(-3+k) + \left(4a^6 + b^6(-3+k)\right)r^2\right)k_1\right)\mu_1\right)\right)$$

$$/ ((a^2 - b^2)^4\mu^2(1+k_1) - (a-b)(a+b)\mu(b^6(1+k) + a^6(2+k) + a^4b^2(2+k)$$

$$+ 4a^2b^4(2+k) + (a^6 - 3a^4b^2 + b^6(-3+k))k_1)\mu_1 + (4a^6b^2 + b^8(-3+k)$$

$$+ a^8(1+k) + a^2b^6(-3+k)(1$$

$$+ k))\mu_1^2))\}$$

$\boldsymbol{\sigma_{\mathrm{rout}}}$**/. Solution/Simplify**

$$\sigma_{\mathrm{rout}} = \{\frac{1}{2r^4}S(r^4 - r^4\mathrm{Cos}[2\theta] - (3b^4\mathrm{Cos}[2\theta]\mu_1(-(a-b)(a+b)(a^6 + 5a^4b^2 - 2a^2b^4(-3+k) - b^6(-3+k))\mu - a^2(a^6 + b^6(-3+k) - a^2b^4(-2+k))\mu k_1 + (4a^6b^2 + b^8(-3+k) + a^8(1+k) + a^2b^6(-3+k)(1+k))\mu_1))/((a^2-b^2)^4\mu^2 + (a^2-b^2)^4\mu^2k_1 - (a-b)(a+b)\mu(b^6(1+k) + a^6(2+k) + a^4b^2(2+k) + 4a^2b^4(2+k) + (a^6 - 3a^4b^2 + b^6(-3+k))k_1)\mu_1 + (4a^6b^2 + b^8(-3+k) + a^8(1+k) + a^2b^6(-3+k)(1+k))\mu_1^2) + (4b^2r^2\mathrm{Cos}[2\theta](-(a^2-b^2)^4\mu^2 - (8a^6b^2 - 6a^4b^4 + b^8(-4+k) - 2a^2b^6(-1+k) + a^8k)\mu\mu_1 + (4a^6b^2 + b^8(-3+k) + a^8(1+k) + a^2b^6(-3+k)(1+k))\mu_1^2))/((a^2-b^2)^4\mu^2 + (a^2-b^2)^4\mu^2k_1 - (a-b)(a+b)\mu(b^6(1+k) + a^6(2+k) + a^4b^2(2+k) + 4a^2b^4(2+k) + (a^6 - 3a^4b^2 + b^6(-3+k))k_1)\mu_1 + (4a^6b^2 + b^8(-3+k) + a^8(1+k) + a^2b^6(-3+k)(1+k))\mu_1^2) + b^2r^2(-1 + \frac{(a-b)(a+b)\mu(1+k_1)}{2(a-b)(a+b)\mu + \left(-2a^2 - b^2(-1+k)\right)\mu_1}))\}$$

The circumferential stresses for the inner and outer region:

$\sigma_{\theta in}$/. Solution/Simplify

$$\sigma_{\theta\text{in}} = \{\frac{1}{2r^4} b^2 S\mu(-\frac{4r^4}{8(a-b)(a+b)\mu - 4\left(2a^2 + b^2(-1+k)\right)\mu_1} - (\text{Cos}[2\theta](2(a-b)(a+b)\mu$$

$$+ (-2a^2 - b^2(-1+k))\mu_1)((a-b)(a+b)\mu(3a^4b^2(a^2+5b^2) + (8a^4 + 5a^2b^2$$

$$+ 5b^4)r^4 - 12(2a^2+b^2)r^6 + (3a^4b^2(a^2+b^2) + (4a^4 + a^2b^2 + b^4)r^4$$

$$- 12a^2r^6)k_1) - (3a^4b^2(4a^2b^2 + 2b^4(-3+k) + a^4(1+k)) + (8a^6 + 2b^6(-3$$

$$+ k) - 3a^4b^2(1+k))r^4 + 12(-2a^4 + b^4 + a^2b^2(1+k))r^6 + (4a^6r^4 + b^6(-3$$

$$+ k)r^4 + 3a^4(b^6(-3+k) - 4r^6))k_1)\mu_1) + r^2(a^2 + (a^2+r^2)k_1)((a^2-b^2)^4\mu^2(1$$

$$+ k_1) - (a-b)(a+b)\mu(b^6(1+k) + a^6(2+k) + a^4b^2(2+k) + 4a^2b^4(2+k)$$

$$+ (a^6 - 3a^4b^2 + b^6(-3+k))k_1)\mu_1 + (4a^6b^2 + b^8(-3+k) + a^8(1+k)$$

$$+ a^2b^6(-3+k)(1+k))\mu_1^2)) /((2(a-b)(a+b)\mu + (-2a^2 - b^2(-1$$

$$+ k))\mu_1)((a^2-b^2)^4\mu^2(1+k_1) - (a-b)(a+b)\mu(b^6(1+k) + a^6(2+k)$$

$$+ a^4b^2(2+k) + 4a^2b^4(2+k) + (a^6 - 3a^4b^2 + b^6(-3+k))k_1)\mu_1 + (4a^6b^2$$

$$+ b^8(-3+k) + a^8(1+k) + a^2b^6(-3+k)(1+k))\mu_1^2)))\}$$

$\sigma_{\theta out}$/. Solution/Simplify

$$\sigma_{\theta\text{out}} = \{\frac{1}{2r^4} S(r^4 + r^4\text{Cos}[2\theta] + (3b^4\text{Cos}[2\theta]\mu_1(-(a-b)(a+b)(a^6 + 5a^4b^2 - 2a^2b^4(-3+k)$$

$$- b^6(-3+k))\mu - a^2(a^6 + b^6(-3+k) - a^2b^4(-2+k))\mu k_1 + (4a^6b^2 + b^8(-3$$

$$+ k) + a^8(1+k) + a^2b^6(-3+k)(1+k))\mu_1))/((a^2-b^2)^4\mu^2 + (a^2-b^2)^4\mu^2k_1$$

$$- (a-b)(a+b)\mu(b^6(1+k) + a^6(2+k) + a^4b^2(2+k) + 4a^2b^4(2+k) + (a^6$$

$$- 3a^4b^2 + b^6(-3+k))k_1)\mu_1 + (4a^6b^2 + b^8(-3+k) + a^8(1+k) + a^2b^6(-3$$

$$+ k)(1+k))\mu_1^2) - b^2r^2(-1 + \frac{(a-b)(a+b)\mu(1+k_1)}{2(a-b)(a+b)\mu + \left(-2a^2 - b^2(-1+k)\right)\mu_1}))\}$$

The final shear stresses for the inner and outer region:

$\boldsymbol{\tau_{r\theta in}}$/. **Solution/Simplify**

$$\tau_{r\theta in} = \{(b^2(-a^2+r^2)S\mu \text{Sin}[2\theta](-(a^2-b^2)(5b^4r^2-6b^2r^4+a^4(3b^2-4r^2)+a^2(15b^4-b^2r^2-12r^4))\mu+(a^6(3b^2(1+k)-4r^2)+2b^4r^2(b^2(-3+k)+3r^2)+3a^4(4b^4+b^2(1+k)r^2-4r^4)+6a^2b^2(b^4(-3+k)+b^2r^2+(1+k)r^4))\mu_1+k_1(-(a^2-b^2)(b^4r^2+a^4(3b^2-2r^2)+a^2(3b^4+b^2r^2-6r^4))\mu+(3a^2b^6(-3+k)-2a^6r^2+b^6(-3+k)r^2-6a^4r^4)\mu_1)))\ /\ (2r^4((a^2-b^2)^4\mu^2+(b^8(1+k)-a^8(2+k)-3a^4b^4(2+k)+a^2b^6(7+3k))\mu\mu_1+(4a^6b^2+b^8(-3+k)+a^8(1+k)+a^2b^6(-3-2k+k^2))\mu_1^2+(a^2-b^2)\mu k_1((a^2-b^2)^3\mu-(a^6-3a^4b^2+b^6(-3+k))\mu_1))))\}$$

$\boldsymbol{\tau_{r\theta out}}$/. **Solution/Simplify**

$$\tau_{r\theta out} = \{\frac{1}{2r^4}S\text{Sin}[2\theta](r^4-(3b^4\mu_1(-(a-b)(a+b)(a^6+5a^4b^2-2a^2b^4(-3+k)-b^6(-3+k))\mu-a^2(a^6+b^6(-3+k)-a^2b^4(-2+k))\mu k_1+(4a^6b^2+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1+k))\mu_1))/((a^2-b^2)^4\mu^2+(a^2-b^2)^4\mu^2k_1-(a-b)(a+b)\mu(b^6(1+k)+a^6(2+k)+a^4b^2(2+k)+4a^2b^4(2+k)+(a^6-3a^4b^2+b^6(-3+k))k_1)\mu_1+(4a^6b^2+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1+k))\mu_1^2)+(2b^2r^2(-(a^2-b^2)^4\mu^2-(8a^6b^2-6a^4b^4+b^8(-4+k)-2a^2b^6(-1+k)+a^8k)\mu\mu_1+(4a^6b^2+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1+k))\mu_1^2))/((a^2-b^2)^4\mu^2+(a^2-b^2)^4\mu^2k_1-(a-b)(a+b)\mu(b^6(1+k)+a^6(2+k)+a^4b^2(2+k)+4a^2b^4(2+k)+(a^6-3a^4b^2+b^6(-3+k))k_1)\mu_1+(4a^6b^2+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1+k))\mu_1^2))\}$$

The final radial and circumferential displacements for the inner region:

$\boldsymbol{U}_{\mathbf{rin}}$/. **Solution/Simplify**

$$U_{\text{rin}} = \{\frac{1}{8r^3}b^2S(-\frac{r^2(2a^2+(-1+k)r^2)(1+k_1)}{2(a-b)(a+b)\mu+\left(-2a^2-b^2(-1+k)\right)\mu_1} - (2\text{Cos}[2\theta]((a-b)(a+b)(a^4b^2(a^2+5b^2)-a^2(2a^4+2a^2b^2+5b^4)(1+k)r^2-(8a^4+5a^2b^2+5b^4)r^4-(2a^2+b^2)(-3+k)r^6)\mu+(a^4b^2(a^4-b^4)-a^2(a^6-b^6)(1+k)r^2+(-4a^6+3a^4b^2+b^6)r^4-a^2(a-b)(a+b)(-3+k)r^6)\mu k_1+(-a^4b^2(4a^2b^2+2b^4(-3+k)+a^4(1+k))+a^2(2a^6+3a^2b^4+2b^6(-3+k))(1+k)r^2+(8a^6+2b^6(-3+k)-3a^4b^2(1+k))r^4-(-3+k)(-2a^4+b^4+a^2b^2(1+k))r^6+(-a^4b^6(-3+k)+a^2(a^6+b^6(-3+k))(1+k)r^2+(4a^6+b^6(-3+k))r^4+a^4(-3+k)r^6)k_1)\mu_1))/((a^2-b^2)^4\mu^2(1+k_1)-(a-b)(a+b)\mu(b^6(1+k)+a^6(2+k)+a^4b^2(2+k)+4a^2b^4(2+k)+(a^6-3a^4b^2+b^6(-3+k))k_1)\mu_1+(4a^6b^2+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1+k))\mu_1^2))\}$$

$\boldsymbol{U}_{\boldsymbol{\theta}\mathbf{in}}$/. **Solution/Simplify**

$$U_{\theta\text{in}} = \{(b^2S\text{Sin}[2\theta](-(a-b)(a+b)(a^4b^2(a^2+5b^2)+a^2(2a^4+2a^2b^2+5b^4)(1+k)r^2+(8a^4+5a^2b^2+5b^4)r^4-(2a^2+b^2)(-3+k)r^6)\mu+(b^4(-3+k)r^4(2b^2+r^2)+a^8(1+k)(b^2+2r^2)+a^2b^2(-3+k)(1+k)r^2(2b^4+r^4)+4a^6(b^4+2r^4)+a^4(b-r)(b+r)(2b^4(-3+k)+b^2(-3+5k)r^2+2(-3+k)r^4))\mu_1+k_1((-a^8b^2+a^4b^6-a^2(a^6-b^6)(1+k)r^2+(-4a^6+3a^4b^2+b^6)r^4+a^2(a-b)(a+b)(-3+k)r^6)\mu+(a^8(1+k)r^2+a^2b^6(-3+k)(1+k)r^2+4a^6r^4+b^6(-3+k)r^4+a^4(-3+k)(b^6-r^6))\mu_1)))/(4r^3((a^2-b^2)^4\mu^2+(a^2-b^2)^4\mu^2k_1-(a-b)(a+b)\mu(b^6(1+k)+a^6(2+k)+a^4b^2(2+k)+4a^2b^4(2+k)+(a^6-3a^4b^2+b^6(-3+k))k_1)\mu_1+(4a^6b^2+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1+k))\mu_1^2))\}$$

The final radial and circumferential displacements for the outer region:

$\boldsymbol{U}_{\mathbf{rout}}$/. **Solution/Simplify**

$$U_{\text{rout}} = \{\frac{1}{8r^3\mu_1}S(2b^2r^2 - r^4 - 2r^4\text{Cos}[2\theta] + (r^2(-2(a-b)(a+b)\mu(b^2+(b-r)(b+r)k_1)$$

$$+ (-2a^2 - b^2(-1+k))r^2k_1\mu_1)((a^2-b^2)^4\mu^2(1+k_1) - (a-b)(a+b)\mu(b^6(1$$

$$+ k) + a^6(2+k) + a^4b^2(2+k) + 4a^2b^4(2+k) + (a^6 - 3a^4b^2 + b^6(-3$$

$$+ k))k_1)\mu_1 + (4a^6b^2 + b^8(-3+k) + a^8(1+k) + a^2b^6(-3+k)(1+k))\mu_1^2)$$

$$+ 2b^2\text{Cos}[2\theta](2(-a^2+b^2)\mu + (2a^2+b^2(-1+k))\mu_1)(-(a^2-b^2)^4r^2\mu^2(1+k_1)$$

$$+ (a-b)(a+b)\mu(b^2(a^6 + 5a^4b^2 - 2a^2b^4(-3+k) - b^6(-3+k)) - (-b^6(-4$$

$$+ k) + a^6k + a^2b^4(2+k) + a^4b^2(8+k))r^2 + (a^2b^2(a^4 + a^2b^2 - b^4(-3+k))$$

$$- (-b^6(-4+k) + a^6k + a^2b^4(2+k) + a^4b^2(8+k))r^2)k_1)\mu_1 - (4a^6b^2 + b^8(-3$$

$$+ k) + a^8(1+k) + a^2b^6(-3+k)(1+k))(b^2 - r^2 - r^2k_1)\mu_1^2))/((2(a-b)(a$$

$$+ b)\mu + (-2a^2 - b^2(-1+k))\mu_1)((a^2-b^2)^4\mu^2(1+k_1) - (a-b)(a+b)\mu(b^6(1$$

$$+ k) + a^6(2+k) + a^4b^2(2+k) + 4a^2b^4(2+k) + (a^6 - 3a^4b^2 + b^6(-3$$

$$+ k))k_1)\mu_1 + (4a^6b^2 + b^8(-3+k) + a^8(1+k) + a^2b^6(-3+k)(1+k))\mu_1^2)))\}$$

$\boldsymbol{U_{\theta\text{out}}}$/. **Solution/Simplify**

$$U_{\theta\text{out}} = \{(S\text{Sin}[2\theta]((a-b)^4(a+b)^4(b-r)r^2(b+r)\mu^2 + (a-b)^4(a+b)^4(b-r)r^2(b+r)\mu^2 k_1 + (a-b)(a+b)\mu(b^4(a^6+5a^4b^2-2a^2b^4(-3+k)-b^6(-3+k)) + b^2(-b^6(-4+k)+a^6k+a^2b^4(2+k)+a^4b^2(8+k))r^2 + (b^6(1+k)+a^6(2+k)+a^4b^2(2+k)+4a^2b^4(2+k))r^4 + (-b^8(-4+k)r^2+b^6(-3+k)r^4+a^2b^6(-b^2(-3+k)+(2+k)r^2)+a^6(b^4+b^2kr^2+r^4)+a^4(b^6+b^4(8+k)r^2-3b^2r^4))k_1)\mu_1 - (4a^6b^2+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1+k))(b^4+b^2r^2+r^4+b^2r^2k_1)\mu_1^2))/(4r^3\mu_1(-(a^2-b^2)^4\mu^2(1+k_1)+(a-b)(a+b)\mu(b^6(1+k)+a^6(2+k)+a^4b^2(2+k)+4a^2b^4(2+k)+(a^6-3a^4b^2+b^6(-3+k))k_1)\mu_1 - (4a^6b^2+b^8(-3+k)+a^8(1+k)+a^2b^6(-3+k)(1+k))\mu_1^2))\}$$

The equilibrium equations:

$\mathbf{Simplify}[\boldsymbol{D}[\boldsymbol{\sigma}_{\mathbf{rin}}, \boldsymbol{r}] + \boldsymbol{D}[\boldsymbol{\tau}_{\mathbf{r\theta in}}, \boldsymbol{\theta}] * \mathbf{1}/\boldsymbol{r} + (\boldsymbol{\sigma}_{\mathbf{rin}} - \boldsymbol{\sigma}_{\mathbf{\theta in}})/\boldsymbol{r}]$

0

$\mathbf{Simplify}[\boldsymbol{D}[\boldsymbol{\sigma}_{\mathbf{rout}}, \boldsymbol{r}] + \boldsymbol{D}[\boldsymbol{\tau}_{\mathbf{r\theta out}}, \boldsymbol{\theta}] * \mathbf{1}/\boldsymbol{r} + (\boldsymbol{\sigma}_{\mathbf{rout}} - \boldsymbol{\sigma}_{\mathbf{\theta out}})/\boldsymbol{r}]$

0

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BIOGRAPHICAL INFORMATION

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