POINT MODULES OVER REGULAR GRADED SKEW CLIFFORD ALGEBRAS

by

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ABSTRACT

POINT MODULES OVER REGULAR GRADED SKEW CLIFFORD ALGEBRAS

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In this thesis, I consider point modules over regular graded skew Clifford algebras.

First, I define a notion of rank (called μ-rank) on noncommutative quadratic forms. To every (commutative) quadratic form is associated a symmetric matrix, and one has the standard notions of rank and determinant function defined on the matrix, and, thus, on the quadratic form. In 2010, in [15], the notion of quadratic form was extended to the noncommutative setting and a one-to-one correspondence was established between these quadratic forms and certain matrices. Using this generalization, I define a notion of rank (called μ-rank) for such noncommutative quadratic forms, where $n = 2$ or 3. Since writing an arbitrary quadratic form as a sum of squares fails in this context, my methods entail rewriting an arbitrary quadratic form as a sum

of products. In so doing, I find analogs for 2×2 minors and determinant of a 3×3 matrix in this noncommutative setting.

Second, I use the μ-rank of a noncommutative quadratic form to determine the point modules over regular graded skew Clifford algebras. Results of Vancliff, Van Rompay and Willaert in 1998 ([16]) prove that point modules over a regular graded Clifford algebra (GCA) are determined by (commutative) quadrics of rank at most two that belong to the quadric system associated to the GCA. The results in this thesis show that the results of [16] may be extended, with suitable modification, to GSCAs. In particular, using the notion of μ-rank, the point modules over a regular GSCA are determined by (noncommutative) quadrics of μ-rank at most two that belong to the noncommutative quadric system associated to the GSCA.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

One of the main goals in research mathematics is to find methods that solve equations. Considering all the modules over an algebra is a commonly used technique, since it converts the problem of equation solving to a problem of classifying modules over some algebra. Suppose we have m polynomial-style equations $f_1 = 0, ..., f_m = 0$ in N (possibly noncommuting) variables $x_1, ..., x_N$ with coefficients in a field. If $A = \frac{\Bbbk\langle x_1,...,x_N\rangle}{\langle f_1,...,f_m\rangle}$, then there exists a bijection between the set of $n \times n$ matrix solutions to the equations $f_1 = 0, ..., f_m = 0$ and the set of isomorphism classes of n-dimensional left A-modules [6]. This result follows from the fact that any module action of an algebra A on a module M induces an algebra homomorphism that maps the elements of A onto linear transformations that act on M; and conversely. In the case where M is finite-dimensional of dimension n, the linear transformations on M can be identified with $n \times n$ matrices. The main objective in this thesis will be to consider certain modules over certain algebras.

In the 1980s, many noncommutative algebras appeared from quantum physics and many traditional algebraic techniques failed on these algebras. For instance, in the 1980s, Sklyanin was interested in finding the solutions to a certain equation related to the quantum Yang-Baxter equation and the quantum inverse scattering method. He found that certain algebras and their modules provided solutions to that equation. These algebras, later dubbed Sklyanin algebras, were subsequently proved to be *regular* algebras by Smith and Stafford in [14], as described below.

Indeed, the introduction of the notion of a noncommutative regular algebra by M. Artin and W. Schelter in [1] in the mid 1980's was motivated in part by the emergence of the above-mentioned 'new' noncommutative algebras. A desire to find a noncommutative algebraic geometry that would be as successful as commutative algebraic geometry had been for commutative algebra also motivated this development. We note that noncommutative regular algebras are viewed as noncommutative analogs of polynomial algebras. The classification of the generic classes of regular algebras was completed for algebras of global dimension three that are generated by degree one elements in three seminal papers in the late 1980's [1, 2, 3]. The main idea behind this classification was introduced by Artin, Tate and Van den Bergh in [2] and it involved using certain graded modules in place of geometric data, for example, "point modules" in place of certain points and "line modules" in place of certain lines [2, 3].

On employing their technique in the context of regular algebras A of *global dimension* three, Artin, Tate, and Van den Bergh showed that such algebras could be associated to certain subschemes E (typically of dimension one) of $\mathbb{P}^2$ where points in the scheme E parametrize certain A-modules called point modules. The technique involved the definition of a quantum analog of the projective plane $\mathbb{P}^2$. The classification of regular algebras of global dimension four is still an open problem. As it stands now, even *quadratic* regular algebras of global dimension four are still unclassified.

In 2010 [15], Cassidy and Vancliff introduced a quantized analog of a graded Clifford algebra (GCA) called a graded *skew* Clifford algebra (GSCA). Given $n \in \mathbb{N}$, GSCAs provide a relatively 'easy' way of producing examples of quadratic regular algebras of global dimension n. Moreover, in [15], several examples of GSCAs are discussed that are candidates for generic regular algebras of global dimension four, and in [12], the authors prove that almost all quadratic regular algebras of global dimension three can be classified using GSCAs.

In light of these recent results, the main objective of this thesis are the results of Chapter 5 where I generalize results in [16] for GCAs to GSCAs. To be able to do so, a new notion of rank on the noncommutative quadratic forms of [15] needs to be defined. Chapter 3 is devoted to the notion of rank for noncommutative quadratic forms on two generators. The main result of that section is Proposition 3.5.7, which relates the factoring of a quadratic form Q on two generators as a perfect square to a noncommutative analog of the determinant of a 2×2 matrix associated to Q. That result motivates the definition of rank, in Definition 3.5.9, of a quadratic form on two generators. Since our noncommutative setting depends on the entries in a certain scalar matrix μ, the generalization of rank and determinant are called *μ-rank* and *μ-determinant*, respectively.

The case of a notion of rank on quadratic forms on three generators is discussed in Chapter 4, with the main results relating the writing of an arbitrary quadratic form Q on three generators as a sum of products to analogs of the 2×2 minors, and determinant, of a 3×3 matrix associated to Q. In this chapter, the main result is Theorem 4.1.3, and the definition of μ-rank of a quadratic form on three generators is given in Definition 4.1.4.

To generalize results for GCAs to GSCAs in Chapter 5, I use the notion of μ-rank defined in Chapters 3 and 4. In particular, in [16] the point modules over a GCA are determined by (commutative) quadrics of rank at most two that belong to the quadric system associated to the GCA. Theorem 5.3.11 generalizes these results to GSCAs and noncommutative quadratic forms of μ-rank at most two. Moreover for GCAs, if the number of matrices of rank one is greater than one, then the number of left (respectively, right) point modules over A will be infinite [16]. In contrast, my work in Example 5.3.13 shows that, with regard to GSCAs, if the number of matrices of μ-rank one is greater than one, then the number of left (respectively, right) point modules over A may be finite.

CHAPTER 2

PRELIMINARY DEFINITIONS AND CONCEPTS

2.1 Introduction

In this chapter, we present basic definitions that will be used throughout this thesis. Moreover, throughout this thesis, $\Bbbk$ denotes an algebraically closed field such that $\mathrm{char}(\Bbbk) \neq 2$, and $M(n, \Bbbk)$ denotes the vector space of $n \times n$ matrices with entries in $\Bbbk$. The notation $T(V)$ will denote the tensor algebra on the vector space V, and, if C is any ring or vector space, then $C^\times$ will denote the nonzero elements in C.

2.1.1 *Graded Connected Algebras*

2.1.1.1 Graded Algebras (c.f. [10])

A $\Bbbk$-algebra A is said to be $\mathbb{Z}$-graded if:

(1) $A = \oplus_{i \in \mathbb{Z}} A_i$ where A_i are vector spaces over $\Bbbk$, and

(2) $A_i A_j \subset A_{i+j}$ for all i, j.

For each i, A_i denotes the span of the homogeneous elements in A of degree i.

2.1.1.2 Graded *Connected* Algebras [2]

A graded $\Bbbk$-algebra A is said to be *connected* if $A_0 = \Bbbk$.

2.1.1.3 Examples of Graded Connected Algebras

(1) The polynomial ring on n generators, $\mathbb{k}[x_1, ..., x_n]$, where $\deg(x_i) = 1$ for all i.

Here, $A_0 = \mathbb{k}$ and $A_1 = \mathbb{k}x_1 \oplus \mathbb{k}x_2 \oplus \cdots \oplus \mathbb{k}x_n$.

(2) The free algebra $\mathbb{k}\langle z_1, ..., z_n\rangle$ on n generators where $\deg(z_i) = 1$ for all i.

(3) The $\mathbb{k}$-algebra,

$$S = \frac{\mathbb{k}\langle z_1, ..., z_n\rangle}{\langle z_j z_i - \mu_{ij} z_i z_j : 1 \leq i, j \leq n\rangle},$$

where $\deg(z_i) = 1$ for all i and $\mu_{ij} \in \mathbb{k}^{\times}$, $\mu_{ii} = 1$, and $\mu_{ij}\mu_{ji} = 1$ for $1 \leq i, j \leq n$.

2.1.1.4 Example of an Algebra that is not graded

The $\mathbb{k}$-algebra,

$$A = \frac{\mathbb{k}\langle x, y\rangle}{\langle x^3 - y^2\rangle}.$$

where $\deg(x) = 1 = \deg(y)$. Since the relation $x^3 - y^2$ is not homogeneous, this means that $A_2 \cap A_3 \neq \{0\}$. This violates part (1) in Definition 2.1.1.1.

2.1.2 *Hilbert series* of a module (c.f. [3])

The Hilbert series of a graded $\mathbb{Z}$-module or a $\mathbb{Z}$-graded $\mathbb{k}$-vector space $M = \oplus M_n$ is the formal series

$$h_M(t) = \sum_n (\dim_{\mathbb{k}} M_n) t^n.$$

2.1.2.1 Example of Hilbert series

Suppose M is the commutative polynomial ring on n variables. Its Hilbert series is

$$\begin{aligned} h_M(t) &= 1 + nt + \frac{n(n+1)}{2}t^2 + \frac{n(n+1)(n+2)}{3!}t^3 + \cdots + \\ &\quad + \frac{n(n+1)(n+2)\cdots(2n-1)}{n!}t^n + \cdots \\ &= \frac{1}{(1-t)^n} \end{aligned}$$

2.1.3 Definition of a *Quadratic* Algebra (c.f. [13])

A $\Bbbk$-algebra A is quadratic if:

(1) A is $\mathbb{Z}$-graded,

(2) the generators of A have degree 1, and

(3) the relations of A are homogeneous of degree two.

2.1.3.1 Examples of Quadratic Algebras

(1) The algebra S in Example 2.1.1.3.3 is quadratic.

(2) The $\Bbbk$-algebra,

$$A = \frac{\Bbbk\langle x, y\rangle}{\langle x^2 - xy\rangle},$$

where $\deg(x) = 1 = \deg(y)$ is quadratic.

2.1.3.2 Examples of Algebras that are not Quadratic

The algebra,

$$A = \frac{\Bbbk\langle x, y\rangle}{\langle x^3 - y^2\rangle},$$

where $\deg(x) = 1 = \deg(y)$ is not quadratic since the relation $x^3 - y^2$ is not homogeneous of degree two.

2.1.4 Definition of *Global Dimension* of a Graded Connected Algebra (c.f. [2])

A graded connected algebra A has global dimension d if every left A-module and every right A-module has projective dimension at most d and at least one left module and at least one right module has projective dimension equal to d.

2.1.5 Definition of *Polynomial Growth* of a Graded Algebra A (c.f. [2])

Suppose A is a $\mathbb{Z}$-graded algebra such that $A = \oplus_{i\geq 0} A_i$. The algebra A has polynomial growth if $\dim_{\Bbbk} A_n \leq cn^{\delta}$ for some positive real numbers c and δ for all n.

2.1.6 Definition of *Gorenstein* [1]

A $\mathbb{Z}$-graded algebra A, of finite global dimension, is Gorenstein if:

(i) the projective modules appearing in a minimal resolution of the left trivial module ${}_A\Bbbk$ are finitely generated, and

(ii) the transposed complex (or the "dual sequence") obtained by applying the functor $M \rightsquigarrow M^* = \mathrm{Hom}_A(M, A)$ to a minimal resolution of ${}_A\Bbbk$ is a resolution of a graded right module isomorphic to the right trivial module $\Bbbk_A$.

2.1.7 Definition of *Regular* Algebras [2]

Suppose A is a $\mathbb{Z}$-graded algebra such that $A = \oplus_{n\geq 0} A_n$ and generated by A_1. The algebra A is regular of dimension d if it satisfies these conditions:

(i) has global dimension d,

(ii) has polynomial growth, and

(iii) is Gorenstein.

Sometimes a regular algebra is called Artin-Schelter regular or AS-regular.

2.1.7.1 Example of a Regular Algebra

The commutative polynomial algebra $\Bbbk[t_1, t_2]$ is a regular algebra. Applying the functor $\mathrm{Hom}(\_\_\_, \Bbbk[t_1, t_2])$ to the projective resolution

$$0 \longrightarrow \Bbbk[t_1, t_2] \xrightarrow{g} \Bbbk[t_1, t_2]^2 \xrightarrow{f} \Bbbk[t_1, t_2] \longrightarrow_{\Bbbk[t_1,t_2]} \Bbbk \longrightarrow 0$$

where f and g are right multiplication by appropriate matrices, gives dual maps that turn out to be left multiplication by the same matrices.

2.1.8 Definition of a *Normal Element* [10]

An element a of a ring R is a normal element if $aR = Ra$.

2.1.9 Definition of a *Normalizing Sequence* (Centralizing Sequence) [10]

A sequence $a_1, ..., a_n$ of elements of a ring R is called a normalizing sequence (respectively, centralizing sequence) if:

(1) a_1 is a normal (respectively, central) element of R,

(2) for each $j \in \{1, ..., n-1\}$ the image of a_{j+1} in $\frac{R}{\sum_{i=1}^{j} a_i R}$ is a normal (respectively, central) element, and

(3) $\sum_{i=1}^{n} a_i R \neq R$.

2.1.9.1 Example of a normalizing sequence

Suppose S is as in Example 2.1.1.3.3 with $n = 2$ and suppose $q_1 = z_1 z_2$ and $q_2 = z_1^2 + \lambda z_2^2$ where $\lambda \in \Bbbk$. The sequence $\{q_1, q_2\}$ is normalizing since q_1 is normal in S and q_2 is normal in $S/\langle q_1 \rangle$ since $z_1 q_2 = z_1^3 = q_2 z_1$ and $z_2 q_2 = \lambda z_2^3 = q_2 z_2$.

2.1.10 Definition of a *Graded Clifford Algebra* [4, 5]

Let $M_1, ..., M_n \in M(n, \Bbbk)$ denote symmetric matrices. A graded Clifford algebra is the $\Bbbk$-algebra C on degree-one generators $x_1, ..., x_n$ and on degree-two generators $y_1, ..., y_n$ with defining relations given by the following:

(a) $x_i x_j + x_j x_i = \sum_{k=1}^{n} (M_k)_{ij} y_k$ for all $i, j = 1, ..., n$, and

(b) y_k is central for all $k = 1, ..., n$.

2.1.11 Remark

There is a one-to-one correspondence between quadratic forms in n variables with coefficients in $\Bbbk$ and symmetric matrices in $M(n, \Bbbk)$. Lemma 3.2.1 expounds on this correspondence further.

2.1.12 Theorem [4, 5]

Let $M_1, ..., M_n$ be symmetric $n \times n$ matrices. The graded Clifford algebra C associated to $M_1, ..., M_n$ is quadratic, Auslander-regular of global dimension n and satisfies the Cohen-Macaulay property with Hilbert series $\frac{1}{(1-t)^n}$ if and only if the quadric system in $\mathbb{P}^{n-1}$ determined by the M_k's is base-point free. In this case, C is noetherian and has no zero divisors.

2.1.12.1 Example

Let $\lambda \in \Bbbk$, $M_1 = \left[\begin{smallmatrix} 2 & \lambda \\ \lambda & 0 \end{smallmatrix}\right]$, $M_2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, and let C be the graded Clifford algebra on generators x_1, x_2, y_1, y_2 where $\deg(x_i) = 1$, and $\deg(y_i) = 2$ for all i. Using the defining relations given in Definition 2.1.10(a), we have

$$2x_2^2 = y_2, \qquad x_1x_2 + x_2x_1 = \lambda y_1 = \lambda x_1^2, \qquad 2x_1^2 = 2y_1.$$

This implies that $C \twoheadleftarrow \frac{\Bbbk\langle x_1, x_2\rangle}{\langle x_1x_2 + x_2x_1 - \lambda x_1^2\rangle}$. By Remark 2.1.11, we have a one-to-one correspondence between M_1 and the quadratic form $q_1 = 2(t_1^2 + \lambda t_1 t_2)$ in commuting variables t_1, t_2, and similarly, between M_2 and $q_2 = t_2^2$. The quadrics corresponding to q_1 and q_2 are $\{(0,1),(\lambda,-1)\}$ and $\{(1,0)\}$, respectively, so they do not intersect. By the previous theorem, this implies that C is regular and quadratic and thus $C \cong \frac{\Bbbk\langle x_1,x_2\rangle}{\langle x_1x_2+x_2x_1-\lambda x_1^2\rangle}$.

CHAPTER 3

GENERALIZING THE NOTION OF RANK TO NONCOMMUTATIVE QUADRATIC FORMS ON TWO GENERATORS

3.1 Introduction

In this chapter, we introduce a notion of rank, called μ-rank, on noncommutative quadratic forms on two generators that generalizes the traditional notion of rank from the commutative setting. As in the commutative setting, this new notion of rank is based on the way a noncommutative quadratic form may be written as a sum of products.

3.2 Commutative Quadratic Forms

First, we present standard definitions and results for completeness.

3.2.1 Lemma (c.f. [8])

Associated to a quadratic form, q, is a symmetric matrix A as follows. If $x = [x_1 \quad x_2 \quad \dots \quad x_n]^T$, we have,

$$q(x_1, \dots, x_n) = \sum_{\substack{i,j=1 \\ i \leq j}}^{n} \alpha_{ij} x_i x_j = \begin{bmatrix} x_1 & \cdots & \cdots & \cdots & x_n \end{bmatrix} A \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{bmatrix}, \qquad \alpha_{ij} \in \Bbbk,$$

where

$$A = \begin{bmatrix} \alpha_{11} & \frac{1}{2}\alpha_{12} & \frac{1}{2}\alpha_{13} & \cdots & \cdots & \frac{1}{2}\alpha_{1n} \\ \frac{1}{2}\alpha_{12} & \alpha_{22} & \frac{1}{2}\alpha_{23} & \cdots & \cdots & \vdots \\ \frac{1}{2}\alpha_{13} & \frac{1}{2}\alpha_{23} & \alpha_{33} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{1}{2}\alpha_{1n} & \cdots & \cdots & \cdots & \cdots & \alpha_{nn} \end{bmatrix}.$$

Proof. The result follows from multiplying out the matrix product. ■

An example of this lemma is given below where q is a commutative quadratic form on three generators and A is a 3×3 symmetric matrix.

3.2.2 Example

Let

$$q = 3x^2 + 4xy + y^2 + 5yz + 8z^2 \in \Bbbk[x, y, z].$$

By Lemma 3.2.1,

$$q = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 1 & \frac{5}{2} \\ 0 & \frac{5}{2} & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Upon multiplication, this gives

$$3x^2 + 2yx + 2xy + y^2 + \frac{5}{2}zy + \frac{5}{2}yz + 8z^2 = 3x^2 + 4xy + y^2 + 5yz + 8z^2 = q.$$

Lemma 3.2.1 shows the one-to-one correspondence between quadratic forms in n variables with coefficients in $\Bbbk$ and symmetric matrices in $M(n, \Bbbk)$. Since $\Bbbk$ is an algebraically closed field, symmetric matrices can be diagonalized by a change of basis and the associated quadratic forms can be written in a particularly nice way as shown in the proposition that follows.

3.2.3 Proposition (c.f. [8])

Given a quadratic form $q \in \Bbbk[x_1, ..., x_n]$, there exists a basis $\{X_1, ..., X_n\}$ for $\Bbbk x_1 \oplus ... \oplus \Bbbk x_n$ such that $q = X_1^2 + X_2^2 + ... + X_m^2$, where $1 \leq m \leq n$. In particular, the symmetric matrix associated to q, with respect to $\{X_1, ..., X_n\}$, is the diagonal matrix

$$\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

where only the first m rows are nonzero.

Proof. This is a more detailed version of Harris' proof in [8].

If $q = 0$, then $m = 0$ and the result holds. So, suppose $q \neq 0$. We define a bilinear form $b : \Bbbk^n \times \Bbbk^n \to \Bbbk$ by

$$b(v, w) = \frac{q(v + w) - q(v) - q(w)}{2},$$

and choose a basis $e_1, ..., e_n$ for $\Bbbk^n$ as follows. Firstly, since $q \neq 0$, we may choose $e_1 \in \Bbbk^n$ such that $q(e_1) = 1$. We choose $e_2 \in \Bbbk^n$ such that $b(e_1, e_2) = 0$ and $q(e_2) = 1$, and $e_3 \in \Bbbk^n$ such that $b(e_1, e_3) = b(e_2, e_3) = 0$ and $q(e_3) = 1$ and so on. In this way, we obtain $\{e_1, ..., e_m\}$, where m is determined as follows:

$$b(e_i, e_m) = 0 \text{ for } 1 \leq i \leq m - 1, \quad q(e_m) = 1,$$

and

$$q(w) = 0 \text{ for all } w \in W = (\Bbbk e_1 + ... + \Bbbk e_m)^{\perp} \subset \Bbbk^n,$$

that is,

$$q(w) = 0 \text{ for all } w \in \Bbbk^n \text{ such that } b(e_i, w) = 0 \text{ for } 1 \leq i \leq m.$$

We will show that $\{e_1, ..., e_m\}$ is linearly independent. Suppose there exists $\alpha_1, ..., \alpha_m \in \Bbbk$ such that $\sum_{i=1}^{m} \alpha_i e_i = 0$. It follows that, for all $j = 1, ..., n$,

$$0 = b(e_j, \sum_{i=1}^{m} \alpha_i e_i) = \alpha_j b(e_j, e_j) = \alpha_j q(e_j) = \alpha_j.$$

Thus, $\{e_1, ..., e_m\}$ is linearly independent.

We can extend $\{e_1, ..., e_m\}$ to a basis $\mathcal{B} = \{e_1, ..., e_m, e_{m+1}, ..., e_n\}$ for $\Bbbk^n$ where $\{e_{m+1}, ..., e_n\}$ is an arbitrary basis for W. Let $\{X_1, ..., X_n\}$ be the basis of $\Bbbk x_1 \oplus ... \oplus \Bbbk x_n$ that is dual to $\mathcal{B}$. The above conditions on the elements of $\mathcal{B}$ imply that, with respect to $\{X_1, ..., X_n\}$, $q = X_1^2 + X_2^2 + ... + X_m^2$. ■

The rank of a matrix is one of the most fundamental features of a matrix, so one might ask how it influences properties of the quadratic form. This motivates the following definition and proposition.

3.3 Rank on Commutative Quadratic Forms

3.3.1 Definition (c.f. [9])

The rank of a quadratic form q is defined to be the rank of *any* symmetric matrix associated to q.

By Proposition 3.2.3, Definition 3.3.1 is well defined since the rank of a matrix is invariant under change of basis.

Remark. We can prove the following well-known result using Proposition 3.2.3, but we use a method that will be useful for noncommutative quadratic forms.

3.3.2 Proposition

If q is a quadratic form on two generators x_1, x_2, then $q = L_1L_2$ where $L_1, L_2 \in \Bbbk x_1 \oplus \Bbbk x_2$. Moreover, if A is any symmetric matrix associated to q, we have

$$0 \neq L_1 \neq L_2 \neq 0 \quad \Leftrightarrow \quad \mathrm{rk}(q) = 2 \quad \Leftrightarrow \quad \det A \neq 0 \text{ where } L_1, L_2 \text{ are linearly independent.}$$

$$0 \neq L_1 = L_2 \quad \Leftrightarrow \quad \mathrm{rk}(q) = 1 \quad \Leftrightarrow \quad \det A = 0 \text{ and } A \neq 0$$

$$0 = L_1 = L_2 \quad \Leftrightarrow \quad \mathrm{rk}(q) = 0 \quad \Leftrightarrow \quad A = 0,$$

where $\det A$ is the determinant of A.

Proof. Write $x_1 = x$ and $x_2 = y$ and suppose $q = ax^2 + 2bxy + cy^2 \in \Bbbk[x, y]$, where $a, b, c \in \Bbbk$. By Proposition 3.2.3, we may write the matrix A associated to q as $A = \left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right]$, where $\det A = ac - b^2$ is the determinant of A.

Case 1: If $a = 0$, then $q = (2bx + cy)y = L_1L_2$, where $L_2 \in \Bbbk^\times y$.

- If $b \neq 0$, then $\mathrm{rk}(q) = 2$ and $0 \neq L_1 \neq L_2 \neq 0$.
- If $b = 0$, then $\mathrm{rk}(q) \leq 1$ and $q = cy^2 = (\sqrt{c}y)^2 = L_1^2$, where $L_1 = (\sqrt{c}y) = L_2$.

Case 2: If $a \neq 0$, then the quadratic formula implies that

$$q = \frac{1}{\sqrt{a}}[ax + (b + \sqrt{b^2 - ac})y] \cdot \frac{1}{\sqrt{a}}[ax + (b - \sqrt{b^2 - ac})y].$$

Let $L_1 = \frac{1}{\sqrt{a}}[ax + (b + \sqrt{b^2 - ac})y]$ and let $L_2 = \frac{1}{\sqrt{a}}[ax + (b - \sqrt{b^2 - ac})y]$.

- $0 \neq L_1 \neq L_2 \neq 0$ if and only if $b^2 \neq ac$ and this holds if and only if $\text{rk}(q) = 2$.
- $L_1 = L_2 \neq 0$, then $b^2 = ac$ so $\text{rk}(q) = 1$ $(a \neq 0)$ and $L_1 = \frac{1}{\sqrt{a}}(ax + by) = L_2$. ■

By Proposition 3.3.2 and Definition 3.3.1, it follows that if $n \geq 3$, then there is a choice of variables x_1, x_2, x_3 such that

$$\text{rk}(q) = 3 \quad \Leftrightarrow \quad q = x_1^2 + x_2^2 + x_3^2 \quad = \quad XY + Z^2, \text{ where } X, Y, Z \in \Bbbk x_1 \oplus \Bbbk x_2 \oplus \Bbbk x_3$$

$$\text{where } X, Y, Z \text{ are linearly independent.}$$

$$\text{rk}(q) = 2 \quad \Leftrightarrow \quad q = x_1^2 + x_2^2 \quad = \quad L_1 L_2, \quad \text{(as in Proposition 3.3.2)}$$

$$\text{where } L_1, L_2 \text{ are linearly independent.}$$

$$\text{rk}(q) = 1 \quad \Leftrightarrow \quad q = x_1^2$$

Next, we explore quadratic forms in a noncommutative setting.

3.4 Noncommutative Quadratic Forms

In [15], the notion of quadratic form is extended to the noncommutative setting and a one-to-one correspondence is established between such noncommutative quadratic forms and certain matrices. In this section, we first describe the noncommutative quadratic forms in [15] and the matrices associated to them, and suggest a notion of rank for noncommutative quadratic forms defined on two generators. This new notion of rank, which we call μ-rank, is based on writing the quadratic form as a sum of products and generalizes the notion of rank on commutative quadratic forms.

The following definition sets the stage for the noncommutative setting where our noncommutative quadratic forms "live".

3.4.1 Definition [15]

Let $\mu_{ij} \in \Bbbk^\times$ where $\mu_{ii} = 1$, $\mu_{ij}\mu_{ji} = 1$ for $1 \leq i, j \leq n$. Let S denote the $\Bbbk$-algebra on generators $z_1, ..., z_n$ with defining relations

$$z_j z_i = \mu_{ij} z_i z_j, \text{ for } 1 \leq i, j \leq n;$$

that is,

$$S = \frac{\Bbbk\langle z_1, ..., z_n\rangle}{\langle z_j z_i - \mu_{ij} z_i z_j : 1 \leq i, j \leq n\rangle}.$$

The algebra S is a $\mathbb{Z}$-graded $\Bbbk$-algebra, and S may be constructed iteratively from $\Bbbk$ using $n-1$ Ore extensions and so S is an iterated Ore extension of $\Bbbk$. Such an algebra S is sometimes called a skew polynomial ring as in [15] and [7]. In this context, we use S_i to denote the span of the homogeneous elements of S of degree i.

We note that elements of degree two in S_2 are called quadratic forms [15].

Next, we present a lemma and a definition that generalize, for noncommutative quadratic forms, the correspondence between commutative quadratic forms and symmetric matrices and so allow us to work with noncommutative quadratic forms the same way we do with commutative ones.

3.4.2 Lemma (See discussion before Lemma 1.3 in [15])

$$\text{If } q(z_1, ..., z_n) = \sum_{\substack{i,j=1 \\ i \le j}}^{n} \alpha_{ij} z_i z_j \text{ in } S, \qquad \alpha_{ij} \in \mathbb{k},$$

then

$$q = \begin{bmatrix} z_1 & \cdots & \cdots & \cdots & z_n \end{bmatrix} M \begin{bmatrix} z_1 \\ \vdots \\ \vdots \\ \vdots \\ z_n \end{bmatrix}$$

where

$$M = \begin{bmatrix} \alpha_{11} & \frac{1}{2}\alpha_{12} & \frac{1}{2}\alpha_{13} & \cdots & \cdots & \frac{1}{2}\alpha_{1n} \\ \frac{1}{2}\mu_{21}\alpha_{12} & \alpha_{22} & \frac{1}{2}\alpha_{23} & \cdots & \cdots & \vdots \\ \frac{1}{2}\mu_{31}\alpha_{13} & \frac{1}{2}\mu_{32}\alpha_{23} & \alpha_{33} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{1}{2}\mu_{n1}\alpha_{1n} & \cdots & \cdots & \cdots & \cdots & \alpha_{nn} \end{bmatrix}.$$

3.4.3 Definition [15, Definition 1.2]

Let $\mu = (\mu_{ij}) \in M(n, \mathbb{k})$ where $\mu_{ij}\mu_{ji} = 1$ for all $i \neq j$. A matrix $M \in M(n, \mathbb{k})$ is called μ-symmetric if $M_{ij} = \mu_{ij} M_{ji}$ for $1 \le i, j \le n$.

We use $M^{\mu}(n, \mathbb{k})$ to denote the set of all μ-symmetric matrices in $M(n, \mathbb{k})$.

Remarks. If $\mu_{ij} = 1$ for all i, j, then a μ-symmetric matrix is a symmetric matrix, and if $\mu_{ij} = -1$ for all i, j, then a μ-symmetric matrix is a skew-symmetric matrix since char($\mathbb{k}$) $\neq 2$.

The following example is an illustration of the above lemma with q being a non-commutative quadratic form on three generators and M being a 3×3 μ-symmetric matrix.

3.4.4 Example

Let

$$q = 3z_1^2 + 4z_1z_2 + z_2^2 + 5z_2z_3 + 8z_3^2 \in S_2.$$

By Lemma 3.4.2,

$$q = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2\mu_{21} & 1 & \frac{5}{2} \\ 0 & \frac{5}{2}\mu_{32} & 8 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}.$$

Upon multiplication, we obtain

$$\begin{aligned} q &= 3z_1^2 + 2\mu_{21}z_2z_1 + 2z_1z_2 + z_2^2 + \frac{5}{2}\mu_{32}z_3z_2 + \frac{5}{2}z_2z_3 + 8z_3^2 \\ &= 3z_1^2 + 2z_1z_2 + 2z_1z_2 + z_2^2 + \frac{5}{2}z_2z_3 + \frac{5}{2}z_2z_3 + 8z_3^2, \end{aligned}$$

using the relations defining S; this yields

$$q = 3z_1^2 + 4z_1z_2 + z_2^2 + 5z_2z_3 + 8z_3^2.$$

We will now explore a new notion of rank, called μ-rank, on noncommutative quadratic forms. The notion of μ-rank takes into account some technical problems discussed below.

3.5 Rank on Noncommutative Quadratic Forms

If $q \in S_2$ is a noncommutative quadratic form, then a direct generalization using a sum of squares leads to problems (see Example 3.5.1 below) depending on the choice of the μ_{ij} and so, in general, such a generalization is inappropriate for noncommutative quadratic forms.

Below, we present two examples that illustrate two technical issues that arise for noncommutative quadratic forms.

3.5.1 Example

Suppose $n = 2$, $\mu_{12} = -1$ and $Q = z_1^2 + 2bz_1z_2 + cz_2^2$, where $b, c \in \Bbbk$. If $b \neq 0$, then $Q \neq \sum_{i=1}^{m} X_i^2$ for any $m \in \mathbb{N}$, where $X_i \in S_1$ for all i. Moreover, if $b = 0$, then $Q = z_1^2 + cz_2^2 = (z_1 + \alpha z_2)^2$, where $\alpha \in \Bbbk$, $\alpha^2 = c$. Hence, if $b \neq 0$, then a sum of squares is not possible; whereas if $b = 0$, then a sum of square terms is possible but the number of such terms is not unique if c is nonzero.

The next example highlights that an element of S_2 can factor as a perfect square and also as a product of linearly independent elements. We note that this is unique to noncommutative quadratic forms.

3.5.2 Example

Let $Q = z_1^2+6z_1z_2+4z_2^2 \in S_2$ and let $\mu_{12} = 2$. Using the relations for S as in Definition 3.4.1, $Q = z_1^2 + 2z_1z_2 + 2z_2z_1 + 4z_2^2 = (z_1 + 2z_2)^2$ and $Q = (z_1 + z_2)(z_1 + 4z_2)$.

We will, now, generalize Proposition 3.3.2 which concerns rank in the commutative case.

3.5.3 Remark

As was shown in [2], if the point modules of S are parametrized by $\mathbb{P}^{n-1}$, then S is a twist (in the sense of [3, §8]) of the polynomial ring by a graded degree-zero automorphism $\tau \in \mathrm{Aut}(R)$ (see Definition 3.5.4 below). This case occurs if and only if $\mu_{ik} = \mu_{ij}\mu_{jk}$ for all $i,\ j,\ k$. This is the situation throughout Chapter 3, since the assumption that $n = 2$ causes the point modules of S to be parametrized by $\mathbb{P}^1$.

3.5.4 Definition [3, §8]

Let $B = \bigoplus_{m\geq 0} B_m$ be a quadratic algebra and let ϕ be a graded degree-zero automorphism of B. The twist B^ϕ of B by ϕ is the vector space $\bigoplus_{m\geq 0} B_m$ with a new multiplication $*$ defined as follows: if $x, y \in B_1$, then $x * y = x\phi(y)$, where the right-hand side is computed using the original multiplication in B.

For the rest of this chapter, we denote multiplication in S by $*$ and the action of τ by $r^\tau = \tau(r)$ for all $r \in R$. By [11, Lemma 5.6], we may choose τ to be given by

$$\tau(z_1) = \mu_{12}z_1 \quad \text{and} \quad \tau(z_2) = z_2. \tag{$*$}$$

3.5.5 Lemma

If $Q \in S_2$ is a quadratic form on two variables, then Q factors in at most two distinct ways.

Proof. Suppose $Q = r_1 * r_2 = r_3 * r_4 = r_5 * r_6$ in S, where $r_i \in S_1$ for all i. Using τ given above in $(*)$, it follows that $Q = r_1 r_2^{\tau} = r_3 r_4^{\tau} = r_5 r_6^{\tau}$ in R. However, in R, the element Q factors in at most two distinct ways, so, without loss of generality, we may assume $r_5 \in \mathbb{k}^{\times} r_3$ and $r_6 \in \mathbb{k}^{\times} r_4$. Hence, in S, Q factors in at most two ways. ■

For the rest of this chapter, we will be concerned with a quadratic form $az_1 * z_1 + 2bz_1 * z_2 + cz_2 * z_2 \in S_2$, where $a,\ b,\ c \in \mathbb{k}$. As explained in Lemma 3.4.2, to such a quadratic form is associated a μ-symmetric matrix $M = \left[\begin{smallmatrix} a & b \\ \mu_{21} b & c \end{smallmatrix}\right]$. It will be useful to use an analog of the determinant function on M in the next result.

3.5.6 Definition

Let $D : M^{\mu}(2,\ \mathbb{k}) \to \mathbb{k}$ be given by

$$D(M) = 4b^2 - (1 + \mu_{12})^2 ac, \quad \text{where} \quad M = \left[\begin{smallmatrix} a & b \\ \mu_{21} b & c \end{smallmatrix}\right];$$

we call $D(M)$ the μ-determinant of M.

We remark that if $S = R$, that is, if $\mu_{12} = 1$, then $D(M) = -4\det(M)$.

3.5.7 Proposition

Let $Q = az_1 * z_1 + 2bz_1 * z_2 + cz_2 * z_2 \in S_2^{\times}$, where $a,\ b,\ c \in \Bbbk$, be a quadratic form with associated μ-symmetric matrix $M \in M^{\mu}(2,\ \Bbbk)$.

(a) There exists $L_1, L_2 \in S_1$ such that $Q = L_1 * L_2$ in S.

(b) There exists $L \in S_1$ such that $Q = L * L$ in S if and only if $D(M) = 0$.

(c) The element Q factors uniquely, up to a nonzero scalar multiple, in S if and only if $b^2 = \mu_{12}ac$.

Proof. Viewing $Q \in R$, we have $Q = a\mu_{12}z_1^2 + 2bz_1z_2 + cz_2^2$.

(a) Since Q factors in R, we have $Q = r_1r_2$, where $r_i \in R_1 = S_1$ for all i. Thus, in S, $Q = r_1 * \tau^{-1}(r_2)$, which proves (a).

(b) If $Q = r * r$ in S, for some $r \in S_1$, then

$$Q = rr^{\tau} = \mu_{12}\alpha_1^2z_1^2 + (1 + \mu_{12})\alpha_1\alpha_2z_1z_2 + \alpha_2^2z_2^2$$

in R, where $r = \alpha_1z_1 + \alpha_2z_2$ for some $\alpha_1,\ \alpha_2 \in \Bbbk$. Comparing coefficients, it follows that this situation occurs if and only if $2b = (1+\mu_{12})\alpha_1\alpha_2$, where $\alpha_1^2 = a$ and $\alpha_2^2 = c$. Hence, $Q = r * r$ for some $r \in S_1$ implies that $D(M) = 0$. Conversely, if $D(M) = 0$, then $2b = (1 + \mu_{12})\beta$, where $\beta \in \Bbbk$ and $\beta^2 = ac$. If also $ac = 0$, then (b) follows; whereas if $ac \neq 0$, then we may choose $\alpha_1,\ \alpha_2 \in \Bbbk$ such that $\alpha_1^2 = a$ and $\alpha_2 = \beta/\alpha_1$, which implies that $Q = r * r$ in S, where $r = \alpha_1z_1 + \alpha_2z_2$.

(c) A quadratic form factors uniquely in S if and only if it factors uniquely in R, and the latter occurs if and only if the discriminant is zero. Since the discriminant of $a\mu_{12}z_1^2 + 2bz_1z_2 + cz_2^2 \in R_2$ belongs to $\mathbb{k}^\times(b^2 - \mu_{12}ac)$, the result follows. ■

Below, we consider quadratic forms that factor uniquely. Such quadratic forms are especially useful when working on examples such as the ones in Theorem 5.3.11 in Chapter 5.

3.5.8 Corollary

Let Q be as in Proposition 3.5.7.

(a) Suppose Q does not factor uniquely. If $ac = 0$, then $Q \in \langle z_i \rangle$ for some $i \in \{1,\ 2\}$; whereas if $ac \neq 0$, then

$$Q = \left(z_1 + \frac{cz_2}{b+H} \right) * (a\, z_1 + [b + H]\, z_2\,),$$

where $H^2 = b^2 - \mu_{12}ac$.

(b) Suppose Q factors uniquely, up to a nonzero scalar multiple, in S. If $b = 0$, then $Q \in \mathbb{k}^\times z_i^2$ for some $i \in \{1,\ 2\}$; whereas if $b \neq 0$, then

$$Q = b^{-1}(bz_1 + cz_2) * (az_1 + bz_2).$$

Proof. (a) If $ac = 0$, the result in (a) clearly holds. If $ac \neq 0$, we may write $Q = a^{-1}(az_1+\alpha z_2)*(az_1+\beta z_2)$, where $\alpha,\ \beta \in \Bbbk^\times$. Comparing coefficients, we find $ac = \alpha\beta$ and $2b = \beta + \mu_{12}\alpha$. Solving for β yields $\beta = b + H$, where $H^2 = b^2 - \mu_{12}ac$. Since $\alpha = ac/(b+H)$, part (a) follows.

(b) By Proposition 3.5.7(c), $b^2 = \mu_{12}ac$. Thus, if $b = 0$, the result in (b) clearly holds. If $b \neq 0$, then $ac \neq 0$, so part (a) applies with $H = 0$. ■

Proposition 3.5.7 suggests the following generalization of the rank of a quadratic form on two generators.

3.5.9 Definition

Let $Q = az_1 * z_1 + 2bz_1 * z_2 + cz_2 * z_2 \in S_2$, where $a,\ b,\ c \in \Bbbk$, let $M \in M^\mu(2,\ \Bbbk)$ be the μ-symmetric matrix associated to Q and let $D : M^\mu(2,\ \Bbbk) \to \Bbbk$ be defined as in Definition 3.5.6. If $n = 2$, we define μ-rank : $S_2 \to \mathbb{N}$ as follows:

(a) if $Q = 0$, we define $\mu\text{-rank}(Q) = 0$;

(b) if $Q \neq 0$ and $D(M) = 0$, we define $\mu\text{-rank}(Q) = 1$;

(c) if $D(M) \neq 0$, we define $\mu\text{-rank}(Q) = 2$.

3.5.10 Example

If Q is the quadratic form in Example 3.5.2, then $\mu\text{-rank}(Q) = 1$.

CHAPTER 4

GENERALIZING THE NOTION OF RANK TO NONCOMMUTATIVE QUADRATIC FORMS ON THREE GENERATORS

4.1 Introduction

In this chapter, we explore further the notion of rank on noncommutative quadratic forms, and extend the results of the previous chapter concerning μ-rank of quadratic forms on two generators to quadratic forms on three generators. The main result of this chapter is Theorem 4.1.3, which uses analogs of the determinant and minors of a 3×3 matrix to describe factoring properties of a quadratic form. The definition of μ-rank of a noncommutative quadratic form on three generators is given in Definition 4.1.4.

Since $n = 3$ throughout this chapter, the methods of Chapter 3 cannot be employed directly since the algebra S, where $n \geq 3$, need not be a twist of a polynomial ring. In this chapter, we henceforth use juxtaposition to denote the multiplication in S.

The next result generalizes for noncommutative quadratic forms the fact that any commutative quadratic form on three generators can be written as the sum of a product of two linearly independent elements of S_1 and the square of a third linearly independent element of S_1.

4.1.1 Theorem

If $Q = az_1^2 + bz_2^2 + cz_3^2 + 2dz_1z_2 + 2ez_1z_3 + 2fz_2z_3 \in S_2$, where $a, \ldots, f \in \Bbbk$, is a quadratic form, then $Q = L_1L_2 + L_3^2$ for some $L_1,\ L_2,\ L_3 \in S_1$.

Proof. If $a = b = c = e = 0$, then the result clearly holds. Moreover, if $a = b = c = 0 \neq e$, then

$$Q = (z_1 + \alpha z_2)(dz_2 + ez_3) - \alpha dz_2^2,$$

where $\alpha \in \Bbbk$ and $\alpha e = f$. Hence, by symmetry, it suffices to prove the result in the case $a \neq 0$. Thus, we henceforth assume that $a = 1$.

If $\mu_{12} \neq -1 \neq \mu_{13}$, then

$$Q = Q' + \left(z_1 + \frac{2d}{1+\mu_{12}}z_2 + \frac{2e}{1+\mu_{13}}z_3\right)^2,$$

where $Q' \in \Bbbk z_2^2 + \Bbbk z_3^2 + \Bbbk z_2z_3$. Applying Theorem 3.5.7(a) to Q' implies the result in this case.

Suppose $\mu_{12} = -1 \neq \mu_{13}$. If $c \neq 0$ or $e \neq 0$, then there exists $\delta \in \Bbbk$ such that $\delta^2 = c$ and $2e \neq (1 + \mu_{13})\delta$. In this case,

$$Q = (z_1 + \gamma z_2 + \delta z_3)^2 + (z_1 + \alpha z_2)(2dz_2 + \beta z_3),$$

where $\alpha, \ldots, \delta \in \Bbbk$ satisfy

$$\delta^2 = c, \quad \gamma^2 = b - 2d\alpha, \quad \beta = 2e - (1+\mu_{13})\delta \neq 0 \quad \text{and} \quad (1+\mu_{23})\gamma\delta + \alpha\beta = 2f.$$

However, if $c = 0 = e$, then $Q = (z_1 + \epsilon z_2)^2 - 2z_2(dz_1 - fz_3)$, where $\epsilon \in \Bbbk$, $\epsilon^2 = b$. Similarly, if $\mu_{12} \neq -1 = \mu_{13}$.

It remains to consider $\mu_{12} = -1 = \mu_{13}$. If $e \neq 0$, then there exist solutions α, β, $\gamma \in \Bbbk$ to the equations

$$\alpha^2 + 2d\gamma = b, \quad \beta^2 = c \quad \text{and} \quad (1+\mu_{23})\alpha\beta + 2e\gamma = 2f,$$

so that

$$Q = (z_1 + \alpha z_2 + \beta z_3)^2 + 2(z_1 + \gamma z_2)(dz_2 + ez_3).$$

On the other hand, if $e = 0$, then $Q = (z_1 + \delta z_3)^2 + (2dz_1 + bz_2 + 2\mu_{32} f z_3)z_2$, where $\delta \in \Bbbk$, $\delta^2 = c$. $\blacksquare$

The next step is to generalize Theorem 3.5.7 and Definition 3.5.9 to the three-generator case. To do so, we now introduce analogs of the determinant and 2×2 minors of a 3×3 matrix.

4.1.2 Definition

Let $M = \left[\begin{smallmatrix} a & d & e \\ \mu_{21}d & b & f \\ \mu_{31}e & \mu_{32}f & c \end{smallmatrix}\right] \in M^\mu(3, \Bbbk)$ and, for $1 \leq i \leq 8$, define the functions D_i : $M^\mu(3, \Bbbk) \to \Bbbk$ by

$$D_1(M) = 4d^2 - (1+\mu_{12})^2 ab,$$

$$D_2(M) = 4e^2 - (1+\mu_{13})^2 ac,$$

$$D_3(M) = 4f^2 - (1+\mu_{23})^2 bc,$$

$$D_4(M) = 2(1+\mu_{23})de - (1+\mu_{12})(1+\mu_{13})af,$$

$$D_5(M) = 2(1+\mu_{12})ef - (1+\mu_{13})(1+\mu_{23})cd,$$

$$D_6(M) = 2(1+\mu_{13})df - (1+\mu_{12})(1+\mu_{23})be,$$

$$D_7(M) = (\mu_{23}cd^2 - 2def + be^2)(\mu_{13}\mu_{21}cd^2 - 2def + \mu_{12}\mu_{23}\mu_{31}be^2),$$

$$D_8(M) = \mu_{21}(d+X)(e-Y) + \mu_{23}\mu_{31}(d-X)(e+Y) - 2af,$$

where $X^2 = d^2 - \mu_{12}ab$ and $Y^2 = e^2 - \mu_{13}ac$. We call $D_1, \ldots, D_6$ the 2×2 μ-minors of M. The functions D_7 and D_8 will play a role analogous to that of the determinant of M and so could be called the μ-determinants of M, even though D_8 is not a polynomial in the entries of M. (Attempting to convert D_8 to a polynomial leads to unwieldy polynomials such as the one given after Theorem 4.1.3.)

Using our μ-determinants as defined in Definition 4.1.2, we will now prove results regarding the way a noncommutative quadratic form "factors".

4.1.3 Theorem

Let $Q = az_1^2 + bz_2^2 + cz_3^2 + 2dz_1z_2 + 2ez_1z_3 + 2fz_2z_3 \in S_2$, where $a, \ldots, f \in \Bbbk$, and let $M \in M^{\mu}(3, \Bbbk)$ be the μ-symmetric matrix associated to Q.

(a) There exists $L \in S_1$ such that $Q = L^2$ if and only if $D_i(M) = 0$ for all $i = 1, \ldots, 6$.

(b) (i) If $a = 0$, then there exists $L_1,\ L_2 \in S_1$ such that $Q = L_1L_2$ if and only if $D_7(M) = 0$;

(ii) if $a \neq 0$, then there exists $L_1,\ L_2 \in S_1$ such that $Q = L_1L_2$ if and only if $D_8(M) = 0$ for some X and Y satisfying $X^2 = d^2 - \mu_{12}ab$ and $Y^2 = e^2 - \mu_{13}ac$.

Proof. By Proposition 4.1.1, $Q = L_1L_2 + L_3^2$ for some $L_1,\ L_2,\ L_3 \in S_1$.

(a) Suppose there exist $\alpha_1,\ \alpha_2,\ \alpha_3 \in \Bbbk$ such that $Q = (\alpha_1z_1 + \alpha_2z_2 + \alpha_3z_3)^2$. Comparing coefficients, it follows that

$$\begin{array}{llll} \text{(i)} & 2d = (1+\mu_{12})\alpha_1\alpha_2, & \text{(iv)} & a = \alpha_1^2, \\ \text{(ii)} & 2e = (1+\mu_{13})\alpha_1\alpha_3, & \text{(v)} & b = \alpha_2^2, \\ \text{(iii)} & 2f = (1+\mu_{23})\alpha_2\alpha_3, & \text{(vi)} & c = \alpha_3^2, \end{array}$$

so $D_i(M) = 0$ for $i = 1,\ 2,\ 3$. Moreover, from equations (i)-(iv), we have

$$\begin{aligned} 4de(1+\mu_{23}) &= (2d)(2e)(1+\mu_{23}) \\ &= (1+\mu_{12})(1+\mu_{13})(1+\mu_{23})\alpha_1^2\alpha_2\alpha_3 \\ &= (1+\mu_{12})(1+\mu_{13})2af, \end{aligned}$$

so $D_4(M) = 0$. By symmetry, $D_i(M) = 0$ for $i = 5,\ 6$.

Conversely, suppose that $D_i(M) = 0$ for all $i = 1, \ldots, 6$. If $a = 0$, then $d = 0 = e$, since $D_1(M) = 0 = D_2(M)$. In this case, $Q \in \Bbbk z_2^2 + \Bbbk z_3^2 + \Bbbk z_2z_3$, so

Proposition 3.5.7(b) applies to Q (since $D_3(M) = 0$), and so $Q = L^2$, where $L \in S_1$. Thus, to complete the proof of (a), we may assume $a \neq 0$.

Since $D_i(M) = 0$ for $i = 1, 2, 3$, there exist $w_1, w_2, w_3 \in \Bbbk$ such that

$$2d = (1+\mu_{12})w_1, \qquad 2e = (1+\mu_{13})w_2, \qquad 2f = (1+\mu_{23})w_3, \tag{vii}$$

where $w_1^2 = ab$, $w_2^2 = ac$, $w_3^2 = bc$. Since $a \neq 0$, let $Q' = a^{-1}(az_1 + w_1z_2 + w_2z_3)^2 \in S_2$. By (vii), it follows that

$$Q' = az_1^2 + bz_2^2 + cz_3^2 + 2dz_1z_2 + 2ez_1z_3 + a^{-1}(1+\mu_{23})w_1w_2z_2z_3.$$

If $(1+\mu_{23})bc = 0$, then $Q' = Q$ and (a) follows. If $\mu_{12} = -1$, then w_1 may be chosen so that $Q' = Q$; similarly for w_2 if $\mu_{13} = -1$. Hence, we may assume

$$(1+\mu_{12})(1+\mu_{13})(1+\mu_{23})bc \neq 0. \tag{viii}$$

Moreover,

$$\begin{aligned}
(1+\mu_{12})(1+\mu_{13})(1+\mu_{23})w_1w_2 &= 4de(1+\mu_{23}), && \text{using (vii)}\\
&= 2(1+\mu_{12})(1+\mu_{13})af, && \text{as } D_4(M) = 0\\
&= (1+\mu_{12})(1+\mu_{13})(1+\mu_{23})aw_3, && \text{using (vii).}
\end{aligned}$$

Thus, since (viii) holds, $w_1w_2 = aw_3$, from which it follows that $Q' = Q$, which completes the proof of (a).

(b)(i) Suppose $a = 0$. If also $d = 0$, then, by Proposition 3.5.7(a), Q factors if and only if $be = 0$, and the latter holds if and only if $D_7(M) = 0$. Since a similar argument applies if instead $a = 0 = e$, we may assume $de \neq 0$. Let $Q_1,\ Q_2 \in S_2$ be given by

$$\begin{aligned}
Q_1 &= 2[z_1 + (2d)^{-1}bz_2 + (2e)^{-1}cz_3][dz_2 + ez_3] \\
&= bz_2^2 + cz_3^2 + 2dz_1z_2 + 2ez_1z_3 + (bed^{-1} + cd\mu_{23}e^{-1})z_2z_3, \\
Q_2 &= 2[d\mu_{21}z_2 + e\mu_{31}z_3][z_1 + b\mu_{12}(2d)^{-1}z_2 + c\mu_{13}(2e)^{-1}z_3] \\
&= bz_2^2 + cz_3^2 + 2dz_1z_2 + 2ez_1z_3 + [be\mu_{12}\mu_{23}(d\mu_{13})^{-1} + cd\mu_{13}(e\mu_{12})^{-1}]z_2z_3.
\end{aligned}$$

If Q factors, then the coefficients of z_2^2, z_3^2, z_1z_2 and z_1z_3 of Q imply that $Q = Q_1$ or $Q = Q_2$. By comparing the coefficients of z_2z_3 in each case, we find $D_7(M) = 0$. Conversely, if $D_7(M) = 0$, then $Q = Q_1$ or $Q = Q_2$, so Q factors.

(b)(ii) Suppose $a \neq 0$ and that Q factors. We may write

$$Q = a^{-1}(az_1 + \alpha_2z_2 + \alpha_3z_3)(az_1 + \beta_2z_2 + \beta_3z_3),$$

for some α_2, α_3, β_2, $\beta_3 \in \Bbbk$. Comparing coefficients, we have

$$ab = \alpha_2\beta_2, \qquad 2d = \beta_2 + \mu_{12}\alpha_2, \qquad 2e = \beta_3 + \mu_{13}\alpha_3, \tag{ix}$$

$$ac = \alpha_3\beta_3, \qquad 2af = \alpha_2\beta_3 + \mu_{23}\alpha_3\beta_2. \tag{x}$$

Equations (ix) imply that $ab = \alpha_2(2d - \mu_{12}\alpha_2)$, and so $\alpha_2 = \mu_{21}(d + X)$, where $X^2 = d^2 - \mu_{12}ab$. Similarly, $\alpha_3 = \mu_{31}(e + Y)$, where $Y^2 = e^2 - \mu_{13}ac$.

From the second equation in (x), it follows that

$$\begin{aligned} 2af &= \alpha_2(2e - \mu_{13}\alpha_3) + \mu_{23}\alpha_3(2d - \mu_{12}\alpha_2) \\ &= \mu_{21}(d + X)(e - Y) + \mu_{23}\mu_{31}(d - X)(e + Y), \end{aligned}$$

where X and Y are as above. Hence, $D_8(M) = 0$ for some X and Y such that $X^2 = d^2 - \mu_{12}ab$ and $Y^2 = e^2 - \mu_{13}ac$.

Conversely, suppose $a \neq 0$ and that $D_8(M) = 0$ for some X and Y satisfying $X^2 = d^2 - \mu_{12}ab$ and $Y^2 = e^2 - \mu_{13}ac$. Let $Q' \in S_2$, where

$$\begin{aligned} Q' &= a^{-1}[az_1 + \mu_{21}(d + X)z_2 + \mu_{31}(e + Y)z_3][az_1 + (d - X)z_2 + (e - Y)z_3] \\ &= az_1^2 + bz_2^2 + cz_3^2 + 2dz_1z_2 + 2ez_1z_3 + \\ &\qquad +a^{-1}[\mu_{21}(d + X)(e - Y) + \mu_{23}\mu_{31}(e + Y)(d - X)]z_2z_3. \end{aligned}$$

The last coefficient equals $2f$, since $D_8(M) = 0$, and so $Q' = Q$, which completes the proof of (b)(ii). ∎

We remark that, in Theorem 4.1.3(b)(ii), converting the equation $D_8(M) = 0$ to a polynomial equation yields, at best, a user-unfriendly polynomial equation of degree six:

$$
\begin{aligned}
0 = {} & (\mu_{13} + \mu_{12}\mu_{23})^4 a^2b^2c^2 + 64\mu_{12}\mu_{13}\mu_{23}d^2e^2f^2 + \\
& +16(\mu_{12}^2\mu_{13}^2a^2f^4 + \mu_{12}^2\mu_{23}^2b^2e^4 + \mu_{13}^2\mu_{23}^2c^2d^4) + \\
& +16(\mu_{13}^2 + \mu_{12}^2\mu_{23}^2)(\mu_{12}abe^2f^2 + \mu_{13}acd^2f^2 + \mu_{23}bcd^2e^2) + \\
& -32(\mu_{13} + \mu_{12}\mu_{23})(\mu_{12}\mu_{13}adef^3 + \mu_{12}\mu_{23}bde^3f + \mu_{13}\mu_{23}cd^3ef) + \\
& -8(\mu_{13} + \mu_{12}\mu_{23})^2(\mu_{12}\mu_{13}a^2bcf^2 + \mu_{12}\mu_{23}ab^2ce^2 + \mu_{13}\mu_{23}abc^2d^2) + \\
& -8(\mu_{13}^3 - 5\mu_{12}\mu_{13}^2\mu_{23} - 5\mu_{12}^2\mu_{13}\mu_{23}^2 + \mu_{12}^3\mu_{23}^3)abcdef.
\end{aligned}
$$

Theorem 4.1.3 suggests the following generalization of μ-rank in Definition 3.5.9 to the three-generator case.

4.1.4 Definition

Let $Q = az_1^2 + bz_2^2 + cz_3^2 + 2dz_1z_2 + 2ez_1z_3 + 2fz_2z_3 \in S_2$, where $a, \ldots, f \in \Bbbk$, with $a = 0$ or 1, let $M \in M^\mu(3,\ \Bbbk)$ be the μ-symmetric matrix associated to Q and let $D_i : M^\mu(3,\ \Bbbk) \to \Bbbk$, for $i = 1, \ldots, 8$, be defined as in Definition 4.1.2. If $n = 3$, we define the function μ-rank : $S_2 \to \mathbb{N}$ as follows:

(a) if $Q = 0$, we define $\mu\text{-rank}(Q) = 0$;

(b) if $Q \neq 0$ and if $D_i(M) = 0$ for all $i = 1, \dots, 6$, we define μ-rank$(Q) = 1$;

(c) if $D_i(M) \neq 0$ for some $i = 1, \dots, 6$ and if

$$(1-a)D_7(M) + aD_8(M) = 0,$$

we define μ-rank$(Q) = 2$;

(d) if $(1-a)D_7(M) + aD_8(M) \neq 0$, we define μ-rank$(Q) = 3$.

The following example uses Definition 4.1.4 above for the μ-rank of a noncommutative quadratic form on three generators. In this example, Q factors both as a perfect square and as a product of linearly independent 'factors'; nevertheless, according to our definition, the μ-rank of Q is one.

4.1.5 Example

If $Q = (2z_1 + z_2 + 8z_3)^2 = (2\mu_{12}z_1 + z_2 + 8z_3)(2\mu_{21}z_1 + z_2 + 8z_3)$, where $\mu_{12} = \mu_{13}$, then μ-rank$(Q) = 1$, by Definition 4.1.4 and Theorem 4.1.3(a).

4.1.6 Corollary

Let $n = 3$.

(a) If $Q \in S_2^\times$, then μ-rank$(Q) \leq 2$ if and only if $Q = L_1L_2$ for some $L_1,\ L_2 \in S_1^\times$.

(b) If $Q \in S_2^\times$, then μ-rank$(Q) = 1$ if and only if $Q = L^2$ for some $L \in S_1^\times$.

Proof. The result follows from Theorem 4.1.3. ■

The following result gives simplified versions of D_7 and D_8 in the special case where S is a twist of the polynomial ring (see Remark 3.5.3).

4.1.7 Corollary

Let $n = 3$. If S is a twist of the polynomial ring by an automorphism (see Remark 3.5.3), then

$$D_7(M) = (\mu_{23}cd^2 - 2def + be^2)^2 \qquad \text{and} \qquad D_8(M) = 2[\,\mu_{21}(de - XY) - af\,],$$

where $X^2 = d^2 - \mu_{12}ab$ and $Y^2 = e^2 - \mu_{13}ac$.

Proof. By Remark 3.5.3, $\mu_{13} = \mu_{12}\mu_{23}$, and so the above definition of $D_7(M)$ follows. For $D_8(M)$, since $\mu_{13} = \mu_{12}\mu_{23}$, using notation as in Definition 4.1.2, we have

$$\begin{aligned} D_8(M) &= \mu_{21}(d+X)(e-Y) + \mu_{21}(d-X)(e+Y) - 2af \\ &= \mu_{21}de - \mu_{21}dY + \mu_{21}Xe - \mu_{21}XY + \\ &\quad + \mu_{21}de + \mu_{21}dY - \mu_{21}Xe - \mu_{21}XY - 2af \\ &= 2[\,\mu_{21}(de - XY) - af\,]. \quad \blacksquare \end{aligned}$$

The results in this chapter suggest that generalizing the notion of rank to quadratic forms on four or more generators is likely to be very computation heavy. However, in the spirit of Corollary 4.1.6, one could define μ-rank one, respectively

μ-rank two, of a (noncommutative) quadratic form on n generators for any $n \in \mathbb{N}$ by using factoring as follows.

4.1.8 Definition

Let S be as in Definition 5.2.1, where n is an arbitrary positive integer, and let $Q \in S_2$.

(a) If $Q = 0$, we define μ-rank$(Q) = 0$.

(b) If $Q = L^2$ for some $L \in S_1^{\times}$, we define μ-rank$(Q) = 1$.

(c) If $Q \neq L^2$ for any $L \in S_1^{\times}$, but $Q = L_1 L_2$ where $L_1,\ L_2 \in S_1^{\times}$, we define μ-rank$(Q) = 2$.

CHAPTER 5

POINT MODULES OVER GRADED SKEW CLIFFORD ALGEBRAS

5.1 Introduction

In this chapter, the notion of μ-rank defined in the previous two chapters, Chapters 3 and 4, is used to show that point modules over certain AS-regular algebras are related to noncommutative quadrics in the sense of Definition 5.2.6. Moreover, the definition of a graded skew Clifford algebra (GSCA) from [15], and other relevant definitions and results are given. The goal in this chapter is to be able to count the number of point modules over a regular GSCA when the number of point modules is finite. Indeed, our main objective is to generalize to GSCAs results given in [16] that were applicable to graded Clifford algebras.

5.2 Graded Skew Clifford Algebras

For $\{i, j\} \subset \{1, ..., n\}$, let $\mu_{ij} \in \Bbbk^\times$ satisfy the property that $\mu_{ij}\mu_{ji} = 1$ for $i \neq j$. We write $\mu = (\mu_{ij}) \in M(n, \Bbbk)$. As in [15], we write S for the quadratic $\Bbbk$-algebra on generators $z_1, \ldots, z_n$ with defining relations $z_j z_i = \mu_{ij} z_i z_j$ for all $i, j = 1, 2, ..., n$, where $\mu_{ii} = 1$ for all i. We set $U \subset T(S_1)_2$ to be the span of the defining relations of S and write $V = S_1^*$ and $z = (z_1, \ldots, z_n)^T$.

5.2.1 Definition [15, §1.2]

(a) With μ and S as above, a quadratic form Q is any element of S_2.

(b) A matrix $M \in M(n, \Bbbk)$ is called μ-symmetric if $M_{ij} = \mu_{ij} M_{ji}$ for all $i, j = 1, \ldots, n$.

Henceforth, we assume $\mu_{ii} = 1$ for all i, and write $M^\mu(n, \Bbbk)$ for the vector space of μ-symmetric $n \times n$ matrices with entries in $\Bbbk$. By [15], there is a one-to-one correspondence between elements of $M^\mu(n, \Bbbk)$ and S_2 via $M \mapsto z^T M z \in S$.

5.2.2 Notation

Let $\tau : \mathbb{P}(M^\mu(n, \Bbbk)) \to \mathbb{P}(S_2)$ be defined by $\tau(M) = z^T M z$.

5.2.3 Remark

Henceforth, we fix $M_1, \ldots, M_n \in M^\mu(n, \Bbbk)$, and for each k we fix representatives $q_k = \tau(M_k)$. By [15, Lemma 1.3], $\{q_k\}_{k=1}^n$ is linearly independent in S if and only if $\{M_k\}_{k=1}^n$ is linearly independent. This correspondence mirrors the correspondence between symmetric matrices and commutative quadratic forms. The following definition generalizes that given in Definition 2.1.10.

5.2.4 Definition [15]

A *graded skew Clifford algebra* $A = A(\mu, M_1, \ldots, M_n)$ associated to μ and $M_1, \ldots, M_n$ is a graded $\Bbbk$-algebra on degree-one generators $x_1, \ldots, x_n$ and on degree-two generators $y_1, \ldots, y_n$ with defining relations given by:

(a) $x_ix_j + \mu_{ij}x_jx_i = \sum_{k=1}^{n}(M_k)_{ij}y_k$ for all $i, j = 1, \ldots, n$, and

(b) the existence of a normalizing sequence $\{y'_1, \ldots, y'_n\}$ that spans $\Bbbk y_1 + \cdots + \Bbbk y_n$.

5.2.5 Remark

If A is a graded skew Clifford algebra, then [15, Lemma 1.13] implies that $y_i \in (A_1)^2$ for all $i = 1, ..., n$ if and only if $M_1, ..., M_n$ are linearly independent. Thus, hereafter, we assume that $M_1, .., M_n$ are linearly independent. By [15], the degree of the defining relations of A and certain homological properties of A are tied to certain geometric data associated to A as follows.

5.2.6 Definition [15]

(a) Let $\mathcal{V}(U) \subset \mathbb{P}((S_1)^*) \times \mathbb{P}((S_1)^*)$ denote the zero locus of U. For any $q \in S_2^{\times}$, we call the zero locus of q in $\mathcal{V}(U)$ the *quadric associated to* q, and denote it by $\mathcal{V}_{_U}(q)$; in other words, $\mathcal{V}_{_U}(q) = \mathcal{V}(\Bbbk\hat{q} + U) = \mathcal{V}(\hat{q}) \cap \mathcal{V}(U)$, where $\hat{q}$ is any lift of q to $T(S_1)_2$. The span of elements $Q_1, \ldots, Q_m$ in S_2 will be called the *quadric system* associated to $Q_1, \ldots, Q_m$

(b) If a quadric system is given by a normalizing sequence in S, then it is called a *normalizing quadric system.*

(c) We call a point $(a,\ b) \in \mathcal{V}(U)$ a *base point* of the quadric system associated to $Q_1, \ldots, Q_m \in S_2$ if $(a,\ b) \in \mathcal{V}_{_U}(Q_k)$ for all $k = 1, \ldots, m$. We say such a quadric system is *base-point free* if $\bigcap_{k=1}^{m} \mathcal{V}_{_U}(Q_k)$ is empty.

5.2.6.1 Example

Let $q_1 = z_1z_2$, $q_2 = z_1^2 + \lambda z_2^2$, S be as above and $\lambda \in \Bbbk^\times$. From Example 2.1.9.1, we know that $\{q_1, q_2\}$ is a normalizing quadric system. We evaluate $\mathcal{V}(U)$ as follows: for $((a_1, a_2), (b_1, b_2)) \in \mathbb{P}^1 \times \mathbb{P}^1$, consider $(z_2z_1 - \mu_{12}z_1z_2)((a_1, a_2), (b_1, b_2)) = 0$ which implies $a_2b_1 - \mu_{12}a_1b_2 = 0$. Next, we evaluate $z_1z_2((a_1, a_2), (b_1, b_2)) = 0$ and this implies that $a_1b_2 = 0$. Thus, $\mathcal{V}_U(q_1) = \{((0, 1), (0, 1)), ((1, 0), (1, 0))\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$. Similarly, $\mathcal{V}_U(q_2) \subseteq \{((0, 1), (1, 0)), ((\lambda, a_2), (-a_2, 1)) : a_2 \in \Bbbk\}$. Hence, $\mathcal{V}_U(q_1) \cap \mathcal{V}_U(q_2) = \emptyset$, so $\{q_1, q_2\}$ is base-point free.

5.2.7 Theorem [15]

For all $k = 1, \dots, n$, let M_k and q_k be as in Remark 5.2.3. A graded skew Clifford algebra $A = A(\mu, M_1, \dots, M_n)$ is a quadratic, Auslander-regular algebra of global dimension n that satisfies the Cohen-Macaulay property with Hilbert series $1/(1-t)^n$ if and only if the quadric system associated to $\{q_1, \dots, q_n\}$ is normalizing and base-point free; in this case, A is a noetherian Artin-Schelter regular domain and is unique up to isomorphism.

5.2.8 Remark

(a) Henceforth, we assume that the quadric system associated to $\{q_1, \dots, q_n\}$ is normalizing and base-point free. By Theorem 5.2.7, this assumption allows us to write $A = T(V)/\langle W\rangle$ where $W \subseteq (T(V))_2$. Thus, $W^\perp = \{v \in T(V^*)_2 : v(w) = 0 \text{ for all } w \in W\}$, and so the Koszul dual of A equals $T(V^*)/\langle W^\perp\rangle =$

$S/\langle q_1, ..., q_n\rangle$. In this setting, $\{x_1, ..., x_n\}$ is the dual basis in V to $\{z_1, ..., z_n\}$ and we write $\sum_{i,j} \alpha_{ijm}(x_i x_j + \mu_{ij} x_j x_i)$ for the defining relations of A where $\alpha_{ijm} \in \Bbbk$, for all i, j, m, and $1 \leq m \leq \frac{n(n-1)}{2}$.

(b) By [15, Lemma 5.1] and its proof, the set of pure tensors in $\mathbb{P}(W^\perp)$, that is, $\{a \otimes b \in \mathbb{P}(W^\perp) : a, b \in T(V^*)_1\}$, is in one-to-one correspondence with the zero locus, in $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$, of W given by

$$\Gamma = \{(a, b) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) : w(a, b) = 0 \text{ for all } w \in W\}.$$

To see this in detail, let $w = \sum_{i,j}^n \beta_{ij} x_i \otimes x_j \in W$, where $\beta_{ij} \in \Bbbk$ for all i, j, and let $a = (a_1, ..., a_n) \in \mathbb{P}(V^*) = \mathbb{P}^{n-1}, b = (b_1, ..., b_n) \in \mathbb{P}(V^*) = \mathbb{P}^{n-1}$. So,

$$w\left(\sum_{k,l}^{n} a_k b_l z_k \otimes z_l\right) = \sum_{i,j}^{n} \beta_{ij} a_i b_j x_i(z_i) x_j(z_j) = \sum_{i,j}^{n} \beta_{ij} a_i b_j = w((a, b)).$$

We will now make more precise the connection between points in the zero locus of W and certain quadratic forms.

5.2.9 Lemma

If $a, b \in S_1^\times$, then the quadratic form $ab \in \mathbb{P}\left(\sum_{i=1}^n \Bbbk q_i\right)$ if and only if $(a, b) \in \Gamma$.

Proof. We note first that $ab \neq 0$ in S, since if ab were zero in S, then either a or b would be a zero divisor in S, which is a contradiction since S is a domain. Suppose $ab \in \mathbb{P}\left(\sum_{i=1}^n \Bbbk q_i\right)$. By Remark 5.2.8(a), this implies that $a \otimes b \in W^\perp$, that is,

$w(a \otimes b) = 0$ for all $w \in W$ and Remark 5.2.8 (b) implies that $w((a, b)) = 0$ for all $w \in W$. Thus, $(a, b) \in \Gamma$.

Suppose now that $w((a, b)) = 0$. By Remark 5.2.8(b), this implies that $w(a \otimes b) = 0$ for all $w \in W$ which implies that $a \otimes b \in W^{\perp}$. Since S is a domain, we have $ab \neq 0$ in S, so $ab \in \mathbb{P}\left(\sum_{i=1}^{n} \Bbbk q_i\right)$, as desired. ∎

5.3 Point Modules over Graded Skew Clifford Algebras

In this section, we prove results that relate point modules over graded skew Clifford algebras, as defined in Section 5.2.4, to noncommutative quadrics in the sense of Definition 5.2.6. In particular, we use our notion of μ-rank in Definition 4.1.8 of Chapter 4 on noncommutative quadratic forms to extend results in [16] about graded Clifford algebras (GCAs) to GSCAs, with our main result being Theorem 5.3.11. Although the overall approach and some of the proofs in this section are influenced by those in [16], many of the proofs involve new arguments.

In Chapters 3 and 4, a notion of μ-rank of a noncommutative quadratic form on n generators was defined, where $n = 2$ or 3. In Definition 4.1.8, the property of having μ-rank at most two is defined for a quadratic form on n generators for *any* arbitrary positive integer n. Moreover, if $M \in \mathbb{P}(M^{\mu}(n,\ \Bbbk))$ and if μ-rank$(\tau(M)) \leq 2$, where τ is given in Notation 5.2.2, then we define μ-rank(M) to be the μ-rank of $\tau(M)$.

5.3.1 Remark

In contrast to the commutative setting, there exist noncommutative quadratic forms q where $0 \neq q = L^2 = L_1 L_2$, with $L, L_1, L_2 \in S_1$ and $L_1,\ L_2$ linearly independent. For example, let $n = 2 = \mu_{12}$ and $q = (z_1 + 2z_2)^2 = (z_1 + z_2)(z_1 + 4z_2)$.

We now define a map Φ that will play a role similar to that played by the map ϕ in [16, §1].

5.3.2 Definition

Let $a,\ b \in \mathbb{P}^{n-1}$, with $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$, where $a_i,\ b_i \in \Bbbk$ for all i. We define $\Phi\ :\ \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}(M^{\mu}(n,\ \Bbbk))$ by

$$(a,\ b) \mapsto (a_i b_j + \mu_{ij} a_j b_i) \quad \text{for all } i,\ j = 1, \dots, n.$$

5.3.3 Remark

With $a,\ b$ as in Definition 5.3.2, let $q \in S_2$ be the quadratic form

$$q = \left(\sum_{i=1}^{n} a_i z_i\right)\left(\sum_{i=1}^{n} b_i z_i\right) \in \mathbb{P}(S_2),$$

so μ-rank$(q) \leq 2$. However, using the relations of S, we find

$$q = \sum_{i=1}^{n} a_i b_i z_i^2 + \sum_{\substack{i,j=1 \\ i<j}}^{n} (a_i b_j + \mu_{ij} a_j b_i)\, z_i z_j.$$

It follows that $q = \tau(M)$, where $M = (a_i b_j + \mu_{ij} a_j b_i)$, so $M \in \Bbbk^\times \Phi(a,\ b)$. Hence, μ-rank$(\Phi(a,\ b)) \le 2$ for all $a,\ b \in \mathbb{P}^{n-1}$.

5.3.4 Proposition

$\mathrm{Im}(\Phi) = \{X \in \mathbb{P}(M^\mu(n, \Bbbk)) : \mu\text{-rank}(X) \le 2\}$.

Proof. By the preceding discussion, $\mathrm{Im}(\Phi) \subseteq \{X \in \mathbb{P}(M^\mu(n, \Bbbk)) : \mu\text{-rank}(X) \le 2\}$. Conversely, let X be a nonzero μ-symmetric matrix of μ-rank at most two. Since X is μ-symmetric, $\tau(X) = q \in S_2^\times$, and, since μ-rank$(X) \le 2$, $q = ab$ for some $a = \sum_{i=1}^n a_i z_i$, $b = \sum_{i=1}^n b_i z_i \in S_1^\times$ where $a_i, b_j \in \Bbbk$ for all i, j. By Remark 5.3.3, this implies that $X = \Phi((a_i), (b_j)) \in \mathbb{P}(M^\mu(n, \Bbbk))$. ∎

5.3.5 Remark

Recall the notation in Remark 5.2.8, and suppose $(a,\ b) \in \mathbb{P}(S_1) \times \mathbb{P}(S_1)$. By our assumption in Remark 5.2.8(a), the point $(a,\ b) \in \Gamma$ if and only if $\sum_{i,j} \alpha_{ijm}(a_i b_j + \mu_{ij} a_j b_i) = 0$ for all m, where $a = (a_i)$, $b = (b_j)$; that is, if and only if the μ-symmetric matrix $\Phi(a,\ b)$ is a zero of $\sum_{i,j} \alpha_{ijm} X_{ij}$ for all m, where X_{ij} is the ij'th coordinate function on $M(n,\ \Bbbk)$.

5.3.6 Proposition

With the assumption in Remark 5.2.8(a),

$$\mathrm{Im}(\Phi|_\Gamma) = \left\{ M \in \mathbb{P}\big(\sum_{k=1}^n \Bbbk M_k\big) : \mu\text{-rank}(M) \le 2 \right\}.$$

Proof. Let $H = \{M \in \mathbb{P}\big(\sum_{k=1}^{n} \Bbbk M_k\big) : \mu\text{-rank}(M) \leq 2\}$ and let $M \in H$. Since M is μ-symmetric of μ-rank at most two, there exists $(a,\ b) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ such that $\Phi(a,\ b) = M$, by Proposition 5.3.4. Thus, by Lemma 5.2.9, $(a,\ b) \in \Gamma$, so $H \subseteq \mathrm{Im}(\Phi|_\Gamma)$.

For the converse, our argument follows that of [16, Proposition 1.5]. Let $M = (a_{ij}) \in \mathrm{Im}(\Phi|_\Gamma)$. So, by Proposition 5.3.4, μ-rank$(M) \leq 2$ and, by Remark 5.3.5, $\sum_{i,j=1}^{n} \alpha_{ijm} a_{ij} = 0$ for all m. We will prove $M = \sum_{k=1}^{n} \beta_k M_k$, where $\beta_1, \ldots, \beta_n \in \Bbbk$ are defined as follows. By Remark 5.2.5, for each $k \in \{1, \ldots, n\}$, $y_k \in (A_1)^2$, so $y_k = \sum_{i,j=1}^{n} \gamma_{ijk} Y_{ij}$, where $Y_{ij} = x_i x_j + \mu_{ij} x_j x_i$ and $\gamma_{ijk} \in \Bbbk$ for all i, j, k. For each $k = 1, \ldots, n$, we define $\beta_k \in \Bbbk$ by $\beta_k = \sum_{i,j=1}^{n} \gamma_{ijk} a_{ij}$. By Remark 5.2.8(a), $\sum_{i,j=1}^{n} \alpha_{ijm} Y_{ij} = 0$ in A for all m, and, by Definition 5.2.4(a), $(Y_{ij}) = \sum_{k=1}^{n} M_k y_k$. Since the behavior of the Y_{ij} is mirrored by the a_{ij}, it follows that $(Y_{ij})|_{(\beta_1,\ldots,\beta_n)} = (a_{ij}) = M$, since $(Y_{ij})|_{(y_1,\ldots,y_n)} = (Y_{ij})$. Hence,

$$\sum_{k=1}^{n} \beta_k M_k = \sum_{k=1}^{n} M_k y_k|_{(\beta_1,\ldots,\beta_n)} = (Y_{ij})|_{(\beta_1,\ldots,\beta_n)} = M,$$

as desired. It follows that $M \in H$ and so $\mathrm{Im}(\Phi|_\Gamma) \subseteq H$. ■

To use the map Φ to count the point modules over a regular GSCA, we need to determine which (noncommutative) quadratic forms factor uniquely. Theorem 5.3.7 shows that a quadratic form can be factored in at most two distinct ways.

5.3.7 Theorem

A quadratic form can be factored in at most two distinct ways up to a nonzero scalar multiple.

Proof. Let $q \in S_2^{\times}$. If q cannot be factored, then the result is trivially true. Hence, we may assume

$$q = \left(\sum_{i=1}^{n} \beta_i z_i\right)\left(\sum_{i=1}^{n} \beta_i' z_i\right),$$

where $\beta_i,\ \beta_i' \in \Bbbk$ for all i. If $n = 2$, then the result follows from Lemma 3.5.5. Hereafter, suppose that $n \geq 3$ and that the result holds for $n-1$ generators.

Case I. Suppose $\beta_i\beta_i' \neq 0$ for some i. Without loss of generality, we may assume that $i = n$ and that $\beta_n = 1 = \beta_n'$. Suppose q factors in the following three ways:

$$q = (a + z_n)(a' + z_n) = (b + z_n)(b' + z_n) = (c + z_n)(c' + z_n),$$

where $a, a', b, b', c, c' \in \sum_{k=1}^{n-1} \Bbbk z_k$. Let $\bar{q}$ denote the image of q in $S/\langle z_n \rangle$; clearly, $\bar{q} = aa' = bb' = cc'$. The induction hypothesis implies that $\bar{q}$ factors in at most two distinct ways up to a nonzero scalar multiple. Thus, without loss of generality, we may assume that $c = b$ and $c' = b'$. It follows that q factors in at most two distinct ways up to a nonzero scalar multiple.

Case II. Suppose $\beta_i\beta_i' = 0$ for all i, so $q = \sum_{i<j} \delta_{ij} z_i z_j$ where $\delta_{ij} \in \Bbbk$ for all i, j. We may assume, without loss of generality, that there exists $k \in \{1, \ldots, n\}$ such that

$\beta_i = 0$ for all $i > k$ and $\beta'_i = 0$ for all $i \le k$. By the induction hypothesis, we may also assume that $\beta_i \ne 0$ for all $i \le k$ and $\beta'_i \ne 0$ for all $i > k$.

If $q \in \langle z_i \rangle$ for some i, we may assume $i = n$ and so $k = n-1$. It follows that $q = az_n = z_n b$, where $a, b \in \sum_{i=1}^{n-1} \Bbbk z_i$. If $q = z_n b'$, where $b' \in S_1$, then $b = b'$ since S is a domain; similarly, if $q = a' z_n$. Moreover, the image of q in the domain $S/\langle z_n \rangle$ is zero, so if also $q = cd$, where $c,\ d \in S_1$, then $c \in \Bbbk z_n$ or $d \in \Bbbk z_n$, so q factors in at most two distinct ways up to a nonzero scalar multiple.

Suppose $q \notin \langle z_i \rangle$ for all $i = 1, \ldots, n$, and let $\bar{q}$ denote the image of q in $S/\langle z_n \rangle$. By the induction hypothesis, $\bar{q}$ factors in at most two distinct ways up to a nonzero scalar multiple, so we may assume $\bar{q} = ab = cd$, where $c, d \in \sum_{i=1}^{n-1} \Bbbk z_i$ and $a = \sum_{i=1}^{k} \beta_i z_i$ and $b = \sum_{i=k+1}^{n-1} \beta'_i z_i$. Lifting to S, we have

$$q = a(b + \beta'_n z_n) \quad \text{and} \quad q = c(d + \alpha z_n) \text{ or } (c + \gamma z_n)d,$$

where $\alpha, \gamma \in \Bbbk^\times$, and these are the only ways q can factor in S. Hence, if q factors in three distinct ways in S, then $\beta'_n a z_n = \alpha c z_n = \gamma z_n d$, since $ab = cd$. It follows that $c = \beta'_n \alpha^{-1} a$, since S is a domain, and $b = \beta'_n \alpha^{-1} d$, since $S/\langle z_n \rangle$ is a domain, and so $a(b + \beta'_n z_n)$ is a nonzero scalar multiple of $c(d + \alpha z_n)$ and γ has a unique solution. Thus, q factors in at most two distinct ways up to a nonzero scalar multiple. ■

We next need one last technical result before generalizing (most of) [16, Theorem 1.7] from the setting of GCAs to the setting of GSCAs.

5.3.8 Lemma

Let Δ_μ denote the points $(a,\ b) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ such that $(\tau \circ \Phi)(a,\ b)$ factors uniquely (up to nonzero scalar multiple). The restriction of $\tau \circ \Phi$ to $(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}) \setminus \Delta_\mu$ has degree two and is unramified, whereas $\tau \circ \Phi|_{\Delta_\mu}$ is one-to-one.

Proof. The result is an immediate consequence of Theorem 5.3.7 and the definition of Δ_μ. ■

The next result generalizes (most of) [16, Theorem 1.7], which we now state for comparison.

5.3.9 Theorem [16, Theorem 1.7]

Let C denote a GCA determined by symmetric matrices $N_1, \ldots, N_n \in M(n, \Bbbk)$ and let $\mathcal{Q}$ denote the corresponding quadric system in $\mathbb{P}^{n-1}$. If $\mathcal{Q}$ has no base points, then the number of isomorphism classes of left (respectively, right) point modules over C is equal to $2r_2 + r_1 \in \mathbb{N} \cup \{0, \infty\}$, where r_j denotes the number of matrices in $\mathbb{P}\big(\sum_{k=1}^{n} \Bbbk N_k\big)$ that have rank j. If the number of left (respectively, right) point modules is finite, then $r_1 \in \{0, 1\}$.

5.3.10 Remark

In the setting of GCAs, if M is a symmetric matrix, then $\tau(M)$ is a commutative quadratic form where S, in this case, is commutative; thus, if $a,\ b \in S_1^\times$ are linearly independent, then we view $q = ab = ba$ as two different ways to factor q in S. It follows that a symmetric matrix M has rank j, where $j = 1$ or 2, if and only if $\tau(M)$ factors in j distinct ways, up to a nonzero scalar multiple. In light of this, the next result is clearly a generalization of the first part of Theorem 5.3.9.

5.3.11 Theorem

If the quadric system $\{q_1, \ldots, q_n\}$ associated to the GSCA, A, is normalizing and base-point free, then the number of isomorphism classes of left (respectively, right) point modules over A is equal to $2f_2 + f_1 \in \mathbb{N} \cup \{0, \infty\}$, where f_j denotes the number of matrices M in $\mathbb{P}\big(\sum_{k=1}^{n} \Bbbk M_k\big)$ such that μ-rank$(M) \leq 2$ and such that $\tau(M)$ factors in j distinct ways (up to a nonzero scalar multiple).

Proof. Using the notation from Remark 5.2.8, by [2], the hypotheses on A imply that the set of isomorphism classes of left (respectively, right) point modules over A is in bijection with Γ. Hence, the result follows from Lemma 5.2.9, Proposition 5.3.6 and Lemma 5.3.8. ■

The last part of Theorem 5.3.9 appears not to extend to the setting of GSCAs. More precisely, the proof of the last part of Theorem 5.3.9 uses the correspondence between rank and factoring described in Remark 5.3.10. Given Remark 5.3.1, the obvious counterpart in the setting of GSCAs is either $f_1 \in \{0, 1\}$ or the number of elements of μ-rank one being at most one. However, the following two examples demonstrate that both these properties are unsuitable for generalizing the last part of Theorem 5.3.9 to the setting of GSCAs.

5.3.12 Example

Take $n = 4$ and let

$$\mu_{12} = \mu_{13} = \mu_{14} = -\mu_{23} = \mu_{24} = \mu_{34} = 1,$$

$$q_1 = z_4^2, \qquad q_2 = z_2 z_3, \qquad q_3 = (z_1 + z_2)(z_1 + z_4),$$

$$q_4 = b^2 z_1^2 - a^2 z_2^2 + z_3^2 + 2b z_1 z_3,$$

where $a,\ b \in \Bbbk^{\times}$ and $a^2 \neq b^2$. Since the quadric system is normalizing and base-point free, the corresponding GSCA, A, is quadratic and regular of global dimension four (by Theorem 5.2.7), and is the $\Bbbk$-algebra on generators $x_1, \ldots, x_4$ with defining relations:

$$\begin{aligned} x_1x_2 + x_2x_1 &= x_1^2 - b^2x_3^2, & x_1x_3 + x_3x_1 &= 2bx_3^2, \\ x_1x_4 + x_4x_1 &= x_1^2 - b^2x_3^2, & x_3x_4 + x_4x_3 &= 0, \\ x_2x_4 + x_4x_2 &= x_1^2 - b^2x_3^2, & x_2^2 + a^2x_3^2 &= 0, \end{aligned}$$

and has exactly eleven point modules. In this example, A is a GCA, but the algebra S has been chosen to be noncommutative (via the choice of μ_{23}). Here, $\mathbb{P}\left(\sum_{k=1}^{4} \Bbbk q_k\right)$ contains three elements that factor uniquely, namely

$$q_1, \qquad q_4 + 2aq_2 \qquad \text{and} \qquad q_4 - 2aq_2.$$

(To see that $q_4 + 2aq_2$ factors uniquely, we note that the only way it can factor is as $q_4 + 2aq_2 = (bz_1 + \alpha z_2 + z_3)(bz_1 + \beta z_2 + z_3)$, for some $\alpha,\ \beta \in \Bbbk$, since its image factors uniquely in $S/\langle z_2\rangle$; solving for $\alpha,\ \beta$ yields only one solution: $\alpha = a$, $\beta = -a$. Similarly, for $q_4 - 2aq_2$.) Hence, A has a finite number of point modules, yet $f_1 = 3 > 1$.

In the previous example, if, instead, one takes $\mu_{23} = 1$, so that S is now commutative (as in [16]), then the quadric system contains only one element of rank one (up to nonzero scalar multiple), which agrees with Theorem 5.3.9.

5.3.13 Example

For our second example, we consider a GSCA in [15, §5.3] with $n = 4$, where

$$q_1 = z_1z_2, \qquad q_2 = z_3^2, \qquad q_3 = z_1^2 - z_2z_4, \qquad q_4 = z_2^2 + z_4^2 - z_2z_3,$$
$$\mu_{23} = 1 = -\mu_{34}, \qquad (\mu_{14})^2 = \mu_{24} = -1, \qquad \mu_{13} = -\mu_{14},$$

so the quadric system is normalizing and base-point free. By Theorem 5.2.7, the corresponding GSCA, A, is quadratic and regular of global dimension four, and is the $\Bbbk$-algebra on generators $x_1, \ldots, x_4$ with defining relations:

$$x_1x_3 = \mu_{14}x_3x_1, \qquad x_3x_4 = x_4x_3, \qquad x_2x_3 + x_3x_2 = -x_4^2,$$
$$x_1x_4 = -\mu_{14}x_4x_1, \qquad x_4^2 = x_2^2, \qquad x_2x_4 - x_4x_2 = -x_1^2,$$

and has exactly five nonisomorphic point modules, two of which correspond to $q_1 = z_1 z_2 = z_2 z_1$. The other three point modules correspond to two quadratic forms in $\mathbb{P}\big(\sum_{k=1}^{4} \Bbbk q_k\big)$ that have μ-rank one, namely

$$q_2 = z_3^2 \qquad \text{and} \qquad q_2 + 4q_4 = (z_2 - \frac{z_3}{2} + z_4)^2 = (-z_2 + \frac{z_3}{2} + z_4)^2,$$

where the latter quadratic form clearly factors in two distinct ways. Hence, A has a finite number of point modules even though two distinct elements of $\mathbb{P}\big(\sum_{k=1}^{4} \Bbbk q_k\big)$ have μ-rank one.

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BIOGRAPHICAL STATEMENT

Padmini P. Veerapen was born in Curepipe, Mauritius, Indian Ocean in 1978. She completed her primary school education at Hugh Otter Barry Government School in 1989 and her secondary education at the Queen Elizabeth College in 1998. In 1999, she began her college education in Computer Systems Engineering at Carleton University in Ottawa, Canada, and very quickly realized that applied mathematics was not her 'cup of tea'. She received her Honors B.S. degree in Psychology, Honors B.A. degree in Philosophy in 2005 and M.S. degree in Mathematics in 2008 from the University of Texas at Arlington.

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