AIRY STRESS FUNCTION FOR TWO DIMENSIONAL INCLUSION PROBLEMS

by

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Presented to the Faculty of the Graduate School of

The University of Texas at Arlington in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE IN MECHANICAL ENGINEERING

THE UNIVERSITY OF TEXAS AT ARLINGTON

December 2005

ACKNOWLEDGEMENTS

I would like to express my gratitude to all those who gave me the possibility to complete this thesis. I am deeply indebted to my supervising professor, Dr. Seiichi Nomura for his guidance, patience and motivation. I sincerely thank him for his help not only in completing my thesis but also for his invaluable suggestions and encouragement.

I sincerely thank committee members Dr. Wen Chan and Dr. Dereje Agonafer for serving on my committee. They have been a continuous source of inspiration throughout my study at The University of Texas at Arlington.

I would like to end by saying that this thesis came to a successful end due to the support and blessings from my family. Last but not the least I thank Sameer Chandragiri and all my other friends for their help and encouragement.

November 18th, 2005

ABSTRACT

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Publication No. ______

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The University of Texas at Arlington, 2005

Supervising Professor: Dr. Seiichi Nomura

This thesis addresses a problem of finding the elastic fields in a two-dimensional body containing an inhomogeneous inclusion using the Airy stress function. The Airy stress function is determined so that the prescribed boundary condition at a far field and the continuity condition of the traction force and the displacement field at the interface are satisfied exactly. All the derivations and solving simultaneous equations are carried out using symbolic software.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

CHAPTER 1

INTRODUCTION

1.1 Overview

Elasticity is an elegant and fascinating subject that deals with the determination of the stress, strain and distribution in an elastic solid under the influence of external forces. A particular form of elasticity which applies to a large range of engineering materials, at least over part of their load range produces deformations which are proportional to the loads producing them, giving rise to the Hooke's Law. The theory establishes mathematical models of a deformation problem, and this requires mathematical knowledge to understand the formulation and solution procedures. The variable theory provides a very powerful tool for the solution of many problems in elasticity. Employing complex variable methods enables many problems to be solved that would be intractable by other schemes. The method is based on the reduction of the elasticity boundary value problem to a formulation in the complex domain. This formulation then allows many powerful mathematical techniques available from the complex variable theory to be applied to the elasticity problem.

Another problem faced is the complexity of the elastic field equations as analytical closed-form solutions to fully three-dimensional problems are very difficult to accomplish. Thus, most solutions are developed for reduced problems that typically include axisymmetry or two-dimensionality to simplify particular aspects of the formulation and solution. Because all real elastic structures are three-dimensional, the theories set forth here will be approximate models. The nature and accuracy of the

approximation depend on the problem and loading geometry. Two basic theories, plane stress and plane strain represent the fundamental plane problem in elasticity. These two theories apply to significantly different types of two-dimensional bodies although their formulations yield very similar field equations.

Numerous solutions to plane stress and plane strain problems can be obtained through the use of a particular stress functions technique. The method employs the *Airy stress function* [1] and will reduce the general formulation to a single governing equation in terms of a single unknown. The resulting governing equation is then solvable by several methods of applied mathematics, and thus many analytical solutions to problems of interest can be generated. The stress function formulation is based on the general idea of developing a representation for the stress field that satisfies equilibrium and yields a single governing equation from the compatibility statement. This thesis is a successful attempt to apply the above-mentioned method to a plate of infinite length and width with a central hole and a disc separately. The problem of a circular hole in an infinite plate has been studied for many years with various approaches [1] including the Airy stress function approach. This problem has many applications in engineering as it can reveal stress singularity around the hole. However, due to the complexity of algebra involved, there has been no work about a two-dimensional plate with a circular inclusion (disc) using the Airy stress function to our best knowledge. This thesis addresses such a problem using symbolic algebra software

1.2 Use of Symbolic Software

The development of hardware and software of computers has made available symbolic algebra software packages such as MATLAB, MAPLE, *MATHEMATICA* [2]. Older packages such as Macsyma- one of the very first general-purpose symbolic computations systems were written in LISP whereas new ones such as *Mathematica* are written in the C language and its variations and is one of the most widely available symbolic systems.

Using symbolic algebra systems, one can evaluate mathematical expressions analytically without any approximation. Differentiations, integrations, expansions and solving equations exactly are the major features of symbolic algebra systems. Most of the symbolic algebra systems have been used by mathematicians and theoretical physicists [8]. The ability to deal with symbolic formulae, as well as with numbers, is one of its most powerful features. This is what makes it possible to do algebra and calculus.

It has been demonstrated that in certain circumstances the widely held view that one can always dramatically improve on the CPU time required for lengthy computations by using compiled C or Fortran code instead of advanced quantitative programming environments such as *Mathematica*, MATLAB etc… is wrong. A well written C program can be expected to outperform *Mathematica*, R, S-Plus or MATLAB [7] but, if the C program is not efficiently programmed using the best possible algorithm then in fact it may take longer than using a symbolic software byte-code compiler.

At a technical level, *Mathematica* performs both symbolic and numeric calculations of cross-sectional properties such as areas, centroids, and moments of inertia. Symbolic softwares such as Mathematica can derive closed-form solutions for beams with circular, elliptical, equilateral-triangular, and rectangular cross sections [2]. Symbolic software also addresses the finite element method and is useful in finding shape functions, creating different types of meshes and can solve problems for both isotropic and anisotropic materials. They are also useful in the kinematic modeling of fully constrained systems[2].

This thesis will formulate a boundary value problem cast within a two-dimensional domain (subjected to far field loading) in the x-y plane using the Cartesian coordinates and then reformulating in the polar coordinates to allow further development and study in that coordinate system. The Airy stress function for specific two-dimensional plane conditions is computed and the stresses and displacements at a given point can be found using *Mathematica*.

The thesis is divided into 4 chapters.

Chapter 2 discusses the theory of two-dimensional elasticity behind the Airy stress function and the foundation of its formulation. It establishes a single governing equation for the plane stress and plane strain conditions by reducing the Navier equation to a form from which the Airy stress function can be derived.

Chapter 3 contains the application of the Airy stress function to:

- A finite plate with a hole subjected to tensile loading
- An infinite plate with a hole subjected to far field loading
- A plate with a circular inclusion

The formulations for all of the above problems were carried out using *Mathematica* and as a result the stresses and displacement given a certain point on the plate can be found.

Graphs will be shown to describe the variation of stresses at the various points on the plate.

Chapter 4 contains the conclusion and recommendation

The Appendix will contain the description of all the formatted codes used to carry out the above.

CHAPTER 2

FORMULATION OF THE AIRY STRESS FUNCTION

2.1 Airy Stress Function

Solutions to plane strain and plane stress problems can be obtained by using various stress function techniques which employ the Airy stress function to reduce the generalized formulation to the governing equations with solvable unknowns. The stress function formulation is based on the idea representing the stress fields that satisfy the equilibrium equations.

The method is started by reviewing the equilibrium equations for the plane problem without a body force as follows:

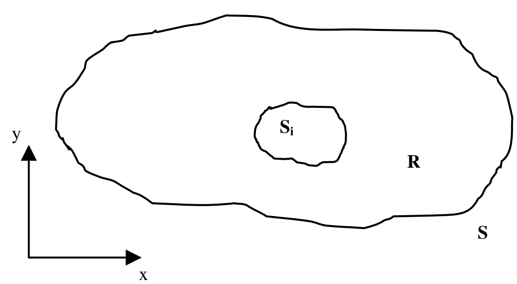

Figure 2-1 *Typical Domain for the Plane Elasticity Problem.*

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \tag{2.1.1}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0. \tag{2.1.2}$$

It is observed that these equations will be identically satisfied by choosing a representation

$$\sigma_x = \frac{\partial^2 \phi}{\partial x^2},$$
$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \qquad (2.1.3)$$
$$\tau_{xy} = \frac{-\partial^2 \phi}{\partial x \partial y}.$$

where $\phi = \phi(x, y)$ is called the Airy stress function.

The compatibility relationship, assuming no body forces, in terms of stress can be written as

$$\nabla^2(\sigma_x + \sigma_y) = 0, \qquad (2.1.4)$$

where ∇ is the Laplace operator.

Now representing the relation in terms of the Airy stress function using relations (2.1.3), we get

$$\frac{\partial^4 \phi}{\partial x^4} + \frac{2\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = \nabla^4 \phi = 0. \qquad (2.1.5)$$

This relation is called the biharmonic equation, and its solutions are known as biharmonic functions[1]. Now that we have the problem of elasticity reduced to a single equation in terms of the Airy stress function, ϕ, it has to be determined in the two-dimensional region *R* bounded by the boundary *S* as shown in Figure 2-1. Appropriate boundary conditions over *S* are necessary to complete the solution.

2.1.1 Polar Coordinate Formulation

The polar coordinates play an important part in solving many plane problems in elasticity and the above developed equation can be worked out by converting it into the polar coordinates thus giving us the governing equations in this curvilinear system. For the polar coordinate system, the solution to plane stress and plane strain basic vector differential problems involves the determination of the in-plane displacement and stresses $\{u_r, u_\theta, \varepsilon_r, \varepsilon_\theta, \gamma_{r\theta}, \sigma_r, \sigma_\theta, \tau_{r\theta}\}$ in region *R* subjected to prescribed boundary condition on *S*.

The two-dimensional in-plane stresses from the Cartesian to the polar coordinates will transform as follows:

$$\begin{aligned}
\sigma_r &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta, \\
\sigma_\theta &= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta, \qquad (2.1.6) \\
\tau_{r\theta} &= -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta).
\end{aligned}$$

The relations (2.1.3) between the stress components and the Airy stress function can be transformed to the polar form to yield

$$\begin{aligned}
\sigma_r &= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial^2\theta}, \\
\sigma_\theta &= \frac{\partial^2\phi}{\partial r^2}, \qquad (2.1.7) \\
\tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right).
\end{aligned}$$

In the absence of a body force, the biharmonic equation (2.1.5) changes to the polar coordinates as

$$\nabla^4\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi = 0. \qquad (2.1.8)$$

The plane problem again is formulated in term of the Airy stress function, $\phi(r,\theta)$, with a single governing biharmonic equation as required.

2.2 Complex Variable Methods

A complex variable z is defined by two real variables x and y

$$z = x + iy. \qquad (2.2.1)$$

This definition can also be expressed in polar form by

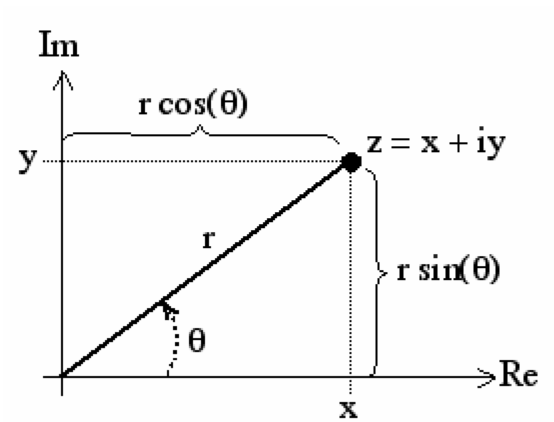

Figure 2-2 *Complex Plane.*

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta} \qquad (2.2.2)$$

where $r = \sqrt{x^2 + y^2}$ known as the modulus of z and

$\theta = \tan^{-1}(y/x)$ the argument

$$\bar{z} = x - iy = re^{-i\theta}. \qquad (2.2.3)$$

$\bar{z}$ is the complex conjugate of the variable z

Using definitions (2.2.1) and (2.2.3), the following differential operators can be developed as follows:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}},$$
$$\frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right),$$
$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \tag{2.2.4}$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

2.2.1 Plane Elasticity Problem using Complex Variables

Complex Variable theory [1] provides a very powerful tool for the solution of many problems in elasticity. Such applications include solutions of torsion problems and the plane stress and plane strain problems. Although each case is related to a completely different two-dimensional model, the basic formulations are quiet similar, and by simple changes in elastic constants, solutions are interchangeable.

For a linear elastic 2-D body, the relations between the stress components and the displacements are expressed as:

$$\sigma_x = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x},$$
$$\sigma_y = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y}, \tag{2.2.5}$$
$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

where λ is called the *Lame's constant*, and μ is referred to as the *shear modulus or modulus of rigidity*.

We now wish to represent the Airy stress function in terms of functions of a complex variable and transform the plane problem onto one involving complex variable theory. Using relations (2.2.1) and (2.2.3), the variables *x* and *y* can be expressed in terms of z and $\bar{z}$. Applying this concept to the Airy stress function, we can write $\phi = \phi(z, \bar{z})$. Repeated use of the differential operators defined in equations (2.2.4) allows the following representation of the harmonic and biharmonic [1] operators:

$$\nabla^2() = 4\frac{\partial^2()}{\partial z \partial \bar{z}}, \nabla^4() = 16\frac{\partial^2()}{\partial z^2 \partial \bar{z}^2}. \tag{2.2.6}$$

Therefore, the governing biharmonic elasticity equation (2.1.5) can be expressed as:

$$\frac{\partial^4 \phi}{\partial z^2 \partial \bar{z}^2} = 0. \tag{2.2.7}$$

Integrating the above result yields

$$\phi(z, \bar{z}) = \frac{1}{2}(\overline{z\gamma(z)} + \bar{z}\gamma(z) + \chi(z) + \overline{\chi(z)})$$

$$= \mathrm{Re}(\bar{z}\gamma(z) + \chi(z)), \tag{2.2.8}$$

where $\gamma(z)$ and $\chi(\ z)$ are arbitrary functions of the indicated variables, and we conclude that ϕ must be real. This result demonstrates that the Airy stress function can be formulated in terms of two functions of a complex variable.

Following along another path, consider the governing Navier equation[1]

$$\mu\nabla^2 u + (\lambda + \mu)\nabla(\nabla . u) = 0. \quad (2.2.9)$$

Introduce the complex variable $U=u + iv$ into the above equation to get

$$(\lambda + \mu)\frac{\partial}{\partial \bar{z}}\left(\frac{\partial U}{\partial z} + \frac{\overline{\partial U}}{\partial z}\right) + 2\mu\frac{\partial^2 U}{\partial \bar{z}\partial z} = 0. \quad (2.2.10)$$

Integrating the above expression yields a solution form for the complex displacement

$$2\mu U = \kappa\gamma(z) - \overline{z\gamma'(z)} - \overline{\psi(z)}. \quad (2.2.11)$$

where again γ (z) and $\psi(z) = \chi'(z)$ are arbitrary functions of a complex variable and the parameter κ depends on the Poisson's ratio ν

$$\kappa = 3 - 4v, \text{ plane strain}$$

$$\kappa = \frac{3 - v}{1 + v}, \text{ plane stress} \quad (2.2.12)$$

Equation (2.2.11) is the complex variable formulation for the displacement field and is written in terms of two arbitrary functions of a complex variable.

The relations (2.1.3) and (2.2.11) yields the fundamental stress combinations.

$$\begin{aligned} \sigma_x + \sigma_y &= 2(\gamma'(z) + \overline{\gamma'(z)}), \\ \sigma_y - \sigma_x + 2i\tau_{xy} &= 2(\bar{z}\gamma''(z) + \psi'(z)). \end{aligned} \quad (2.2.13)$$

By adding and subtracting and equating the real and imaginary parts, relations (2.2.12) can be easily solved for the individual stresses. Using standard transformation laws [Appendix B [1]], the stresses and displacements in the polar coordinates can be written as

$$\sigma_r + \sigma_\theta = \sigma_x + \sigma_y,$$

$$\sigma_\theta - \sigma_r + 2i\tau_{r\theta} = (\sigma_y - \sigma_x + 2i\tau_{xy})e^{2i\theta}, \tag{2.2.14}$$

$$u_r + iu_\theta = (u + iv)e^{2i\theta}.$$

2.3 Investigation of Complex Potentials

The solution to plane elasticity problems involves determination of the two potential functions, γ(z) and ψ(z), which have certain properties that can be determined by applying the appropriate stress and displacement conditions. Particular general forms of these potentials exist for regions of different topology. Most problems of interest involve finite simply connected, finite multiply connected and infinite multiply connected domains as shown in the Figure 2-3.

2.3.1 Finite Simply Connected Domain

Consider a finite simply connected region shown in Figure 2-3(a). For this case, the potential functions, γ(z) and ψ(z), are analytical and single-valued in the region *R*.

$$\gamma(z) = \sum_{n=0}^{\infty} a_n z^n,$$

$$\psi(z) = \sum_{n=0}^{\infty} b_n z^n, \tag{2.3.1}$$

where a_n and b_n are constants to be determined by the boundary conditions of the problem under study.

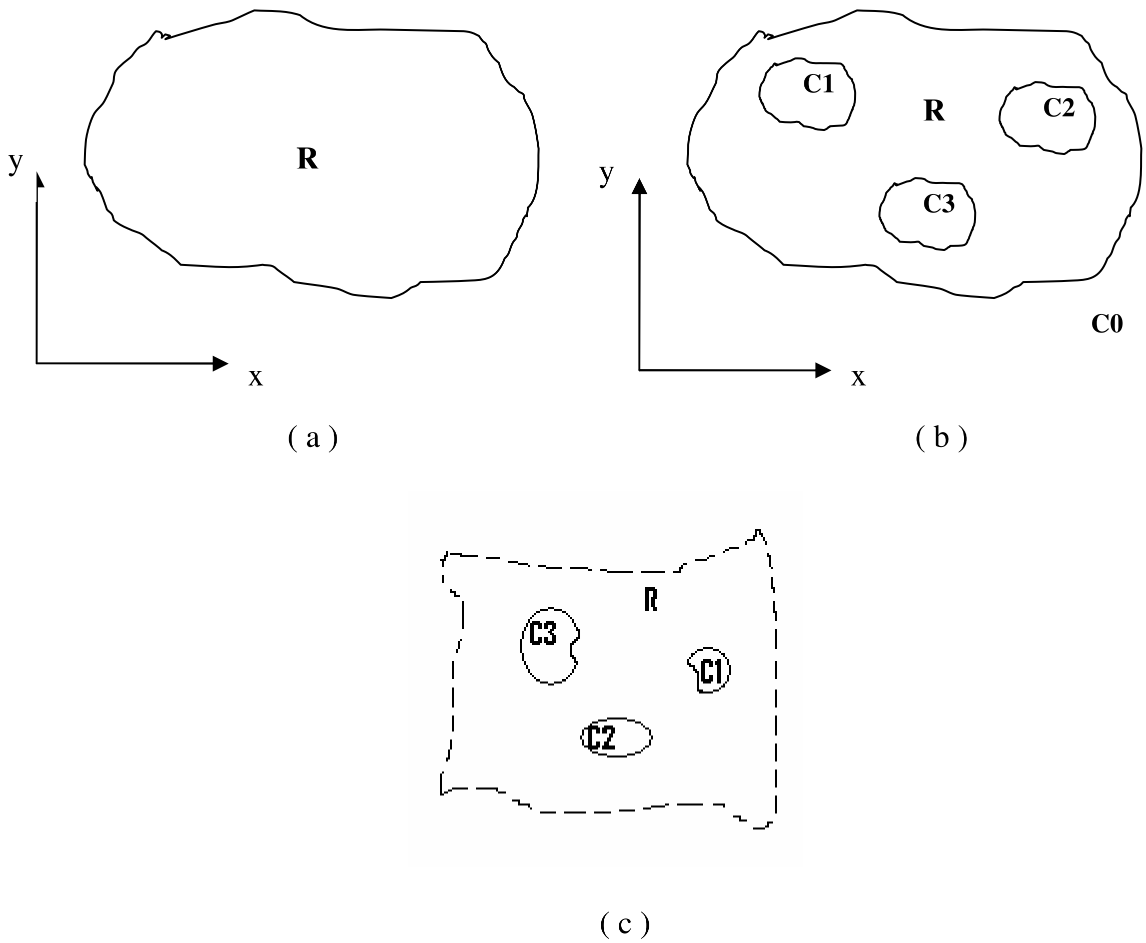

Figure 2-3 *Typical Domains of Interest* : *(a) Finite Simply Connected, (b) Finite Multiply Connected, (c) Infinite Multiply Connected.*

2.3.2 Finite Multiply Connected Domain

For a general region surrounded with a defined external boundary assumed to have n internal boundaries as shown in Figure 2-3(b), the potential need not be single valued. This can be demonstrated by considering the behavior of the stresses and displacements around each of the n contours C_k in region R. Assuming that the displacement and stresses are single valued and continuous everywhere, we get

$$\gamma(z) = -\sum_{k=1}^{n} \frac{F_k}{2\pi(1+\kappa)} \log(z - z_k) + \gamma^*(z),$$
$$\psi(z) = \sum_{k=1}^{n} \frac{\kappa \overline{F}_k}{2\pi(1+\kappa)} \log(z - z_k) + \psi^*(z), \tag{2.3.2}$$

where F_k is the resultant force on each contour C_k, z_k is a point within the contour C_k, $\gamma^*(z)$ and $\psi^*(z)$ are arbitrary analytic functions in R, and κ is the material constant

2.3.3 Infinite Domain

For an externally unbounded region having m internal boundaries as shown in Figure 2-3©, the potentials can be determined by taking into consideration that the stresses remain bounded at infinity. Taking the requirement into consideration we get

$$\gamma(z) = -\frac{\sum_{k=1}^{m} F_k}{2\pi(1+\kappa)} \log z + \frac{\sigma_x^\infty + \sigma_y^\infty}{4} z + \gamma^{**}(z),$$
$$\psi(z) = \frac{\kappa \sum_{k=1}^{m} \overline{F}_k}{2\pi(1+k)} \log z + \frac{\sigma_y^\infty - \sigma_x^\infty + 2i\tau_{xy}^\infty}{2} z + \psi^{**}(z). \tag{2.3.4}$$

where $\sigma_x^\infty, \sigma_y^\infty, \tau_{xy}^\infty$ are the stresses at infinity and $\gamma^{**}(z)$ and $\psi^{**}(z)$ are arbitrary analytic functions outside the region enclosing all n contours. Using power series theory these analytic functions can be expressed as:

$$\gamma^{**}(z) = \sum_{n=1}^{\infty} a_n z^{-n},$$
$$\psi^{**}(z) = \sum_{n=1}^{\infty} b_n z^{-n}. \tag{2.3.5}$$

The displacements at infinity would indicate unbounded behavior as even a bounded strain over an infinite length will produce infinite displacements. Therefore the case of the above region is obtained by dropping the summation terms in (2.3.4).

CHAPTER 3

ADVANCED APPLICATIONS OF THE AIRY STRESS FUNCTION

3.1 Finite Plate with a Hole Subjected to Tensile Loading

Applying the same approach as the finite multiply connected domains in Chapter 2 for the plate shown below, the respective potential functions would be the same as discussed in (2.3.2).

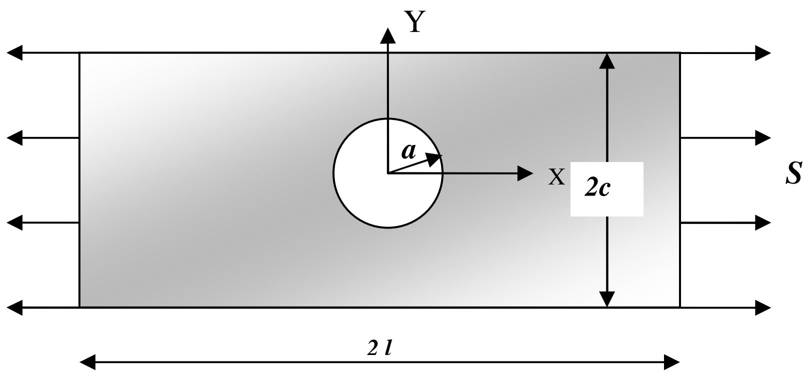

Figure 3-1 *Finite Plate with a Hole Subjected to Tensile Loading*.

Assume a finite plate of length *2l* and width *2c* with a central hole of radius *a* subjected to uniform tension *S* along the x-axis. This is the case of a finite multiply connected region, whose complex potential functions can be expressed as (2.3.2). Since $\psi(z) = \chi'(z)$, integrating the second complex function in (2.3.2) yields

$$\chi(z) = \int \left(\sum_{k=1}^{n} \frac{\kappa \overline{F}_k}{2\pi(1+\kappa)} \log(z - z_k) + \psi^*(z) \right) dz, \qquad (3.1.1)$$

where $k = 1$ since there is only one internal boundary and z_k is 0 since the center of the circle is taken as the origin (0,0) and $\gamma^*(z)$ and $\psi^*(z)$ are arbitrary analytic functions.

Since they are single-valued within the region *R*, they can be expressed as the following series.

$$\gamma^*(z) = \sum_{n=0}^{\infty} a_n z^n,$$
$$\psi^*(z) = \sum_{n=0}^{\infty} b_n z^n. \tag{3.1.2}$$

Substituting the above in the complex potential functions we get

$$\gamma(z) = -\sum_{k=1}^{n} \frac{F_k}{2\pi(1+\kappa)} \log(z - z_k) + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots,$$
$$\psi(z) = \sum_{k=1}^{n} \frac{\kappa \overline{F}_k}{2\pi(1+\kappa)} \log(z - z_k) + b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \ldots, \tag{3.1.3}$$
$$\chi(z) = \int \left(\sum_{k=1}^{n} \frac{\kappa \overline{F}_k}{2\pi(1+\kappa)} \log(z - z_k) + b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \ldots \right) dz,$$

where k=1 as there is only one internal boundary, $z_k = 0$ as the center of the internal boundary which is a circle is the origin (0, 0). In fact, the logarithmic part of the equations is omitted as it corresponds to discontinuity in the displacements or dislocation which does not exist in this case as it is an elastic plate. Therefore, the Airy stress function of (2.2.8) is

$$\phi(z) = \mathrm{Re}\left[\overline{z} \sum_{n=0}^{\infty} a_n z^n + \int \sum_{n=0}^{\infty} b_n z^n dz \right].$$

For the given tensile loading the boundary conditions for the above plate will be

$$
\begin{aligned}
&\sigma_x(\pm l, y) = S, \\
&\sigma_y(x, \pm c) = 0, \\
&\tau_{xy}(\pm l, y) = \tau_{xy}(x, \pm c) = 0, \\
&\sigma_r(\pm a, \pm a) - \tau_{r\theta}(\pm a, \pm a) = 0.
\end{aligned} \tag{3.1.4}
$$

Solving for the Airy stress function, we get

$$
\phi(x, y) = \frac{S(x^2 - y^2)(-6a^2 + x^2 + y^2) + 12b_0 x(a^2 - y^2) + 12a_0 x(a^2 - y^2)}{12(a^2 - y^2)}. \tag{3.1.5}
$$

From the above result we can see that the Airy stress function is an indefinite value. Therefore, we can conclude that for a finite plate with a central hole infinite series having appropriate boundary conditions has to be taken into consideration to get a valid solution.

3.2 Infinite Plate with a Hole Subjected to Tensile Loading

Consider an infinite plate with a central hole subjected to uniform tensile far field loading $\sigma_x^\infty = S$ in the x direction. From (2.3.4), we get the complex potentials to be

$$
\gamma(z) = -\frac{\sum_{k=1}^{m} F_k}{2\pi(1+\kappa)} \log z + \frac{\sigma_x^\infty + \sigma_y^\infty}{4} z + \gamma^{**}(z),
$$

$$
\psi(z) = \frac{\kappa \sum_{k=1}^{m} \overline{F}_k}{2\pi(1+k)} \log z + \frac{\sigma_y^\infty - \sigma_x^\infty + 2i\tau_{xy}^\infty}{2} z + \psi^{**}(z),
$$

where F_k is the resultant force on the central hole, but the logarithmic part of the equations is omitted as it corresponds to discontinuity in the displacements or dislocation which does not exist in this case as it is an elastic plate and $\sigma_x^\infty = S, \sigma_y^\infty = \sigma_z^\infty = \tau_{xy}^\infty = 0$.

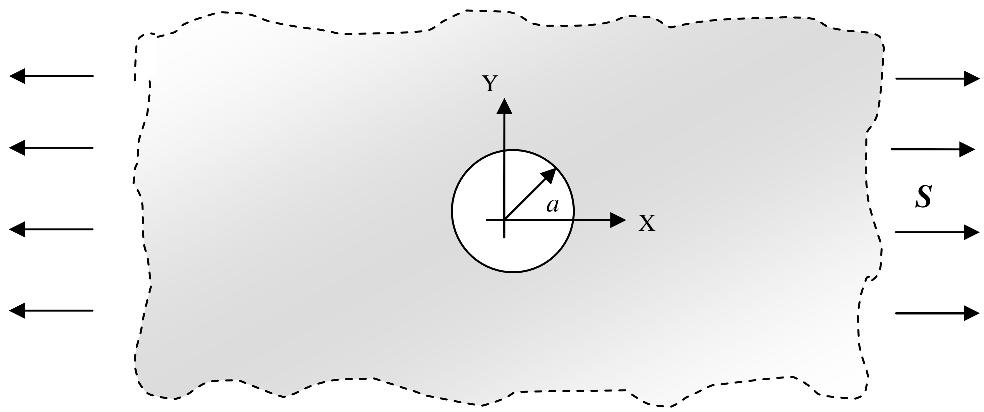

Figure 3-2 *Infinite Plate with a Hole Subjected to Tensile Loading.*

Substituting the power series for the arbitrary analytic functions $\gamma^{**}(z)$ and $\psi^{**}(z)$, we get

$$\gamma(z) = \frac{S}{4}z + \sum_{n=1}^{\infty} a_n z^{-n},$$

$$\psi(z) = -\frac{S}{2}z + \sum_{n=1}^{\infty} b_n z^{-n}.$$

The number of terms to be taken into consideration for the summation is determined by the boundary condition. Therefore, taking the first three terms for the above condition, we get

$$\gamma(z) = \frac{Sz}{4} + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3}, \qquad \psi(z) = \frac{-Sz}{2} + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3}. \tag{3.2.1}$$

Integrating the second potential function, we can obtain $\chi(z)$ as:

$$\chi(z) = \frac{-Sz^2}{4} + (\log z)b_1 - \frac{b_2}{z} - \frac{b_3}{2z^2}. \tag{3.2.2}$$

A plate with a hole is better manipulated using the polar coordinates.

Substituting $z = re^{i\theta}$ in (3.2.1) and (3.2.2) gives the Airy stress function in terms of the polar coordinates as

$$\begin{aligned}\phi(r,\theta) = \frac{1}{2r^2}(&Sr^4 \sin^2\theta - b_3 \cos 2\theta + b_1 r^2 \log r^2 \\ &- 2b_2 \cos\theta + 2a_1 r \cos 2\theta + 2a_2 \cos 3\theta + 2a_3 \cos 4\theta).\end{aligned} \tag{3.2.3}$$

Applying (2.1.7) to the above Airy stress function, we get the stresses in polar form as follows:

$$\begin{aligned}\sigma_{rr} = \frac{1}{2r^4}(&Sr^4 + Sr^4 \cos 2\theta + 2b_1 r^2 + 4b_2 r \cos\theta \\ &+ 6b_3 \cos 2\theta - 8a_1 r^2 \cos 2\theta - 20a_2 r \cos 3\theta),\end{aligned} \tag{3.2.4}$$

$$\begin{aligned}\sigma_{\theta\theta} = \frac{1}{2r^4}(&Sr^4 - Sr^4 \cos 2\theta - 2b_1 r^2 - 4b_2 r \cos\theta \\ &- 6b_3 \cos 2\theta + 4a_2 r \cos 3\theta),\end{aligned} \tag{3.2.5}$$

$$\begin{aligned}\tau_{r\theta} = \frac{1}{2r^4}(&-Sr^4 \sin 2\theta + 4b_2 r \sin\theta + 6b_3 \sin 2\theta \\ &- 4a_1 r^2 \sin 2\theta - 12a_2 r \sin 3\theta).\end{aligned} \tag{3.2.6}$$

The displacements can be determined using (2.2.11) as

$$\begin{aligned}u_r = \frac{1}{8\mu_1 r^3}(&-Sr^4 + S\kappa_1 r^4 - 2Sr^4 \cos 2\theta - 4b_1 r^2 \\ &- 4b_2 r \cos\theta - 4b_3 \cos 2\theta + 4a_1 r^2 \cos 2\theta \\ &+ 4\kappa_1 a_1 r^2 \cos 2\theta + 8a_2 r \cos 3\theta + 4\kappa_1 a_2 r \cos 3\theta),\end{aligned} \tag{3.2.7}$$

$$u_\theta = \frac{1}{4r^3\mu_1}(-Sr^4\sin 2\theta - 2b_2 r\sin\theta - 2b_3\sin 2\theta + 2a_1 r^2\sin 2\theta$$
$$- 2\kappa_1 a_1 r^2 \sin 2\theta + 4a_2\sin 3\theta - 2\kappa_1 a_2 r\sin 3\theta). \tag{3.2.8}$$

where the constants can be found using the boundary conditions. The stress free condition on the hole is denoted as

$$\begin{aligned} \sigma_{rr} &= 0, \\ \tau_{r\theta} &= 0, \end{aligned} \quad \text{at } r = a$$

which can be expressed as

$$\sigma_{rr} - i\tau_{r\theta} = 0 \quad \text{at } r = a. \tag{3.2.9}$$

Therefore,

$$\frac{S}{2} + \frac{1}{2}Se^{2i\theta} - \frac{3a_1e^{-2i\theta}}{a^2} - \frac{a_1e^{2i\theta}}{a^2} - \frac{8a_2e^{-3i\theta}}{a^3} - \frac{2a_2e^{3i\theta}}{a^3} -$$
$$\frac{15a_3e^{3i\theta}}{a^4} - \frac{3a_3e^{4i\theta}}{a^4} + \frac{b_1}{a^2} + \frac{2b_2e^{-i\theta}}{a^3} + \frac{3b_3e^{-2i\theta}}{a^4} = 0. \tag{3.2.10}$$

The constants can be determined from (3.2.10) by equating like powers of $e^{in\theta}$ as

$$a_1 = \frac{a^2S}{2}, \quad a_2 = 0, \quad a_3 = 0,$$
$$b_1 = -\frac{-a^2S}{2}, \quad b_2 = 0, \quad b_3 = \frac{a^4S}{2}. \tag{3.2.11}$$

Substituting the above constants in (3.2.3), the Airy stress function is determined as:

$$\phi(r,\theta) = \frac{-2a^2(a^2-3r^2)S\cos 2\theta - 3a^2Sr^2\log r^2 + 6r^4S\sin^2\theta}{12r^2}, \tag{3.2.12}$$

from which the stress components can be obtained using equations (3.2.4) to (3.2.6) as:

$$\sigma_{rr} = \frac{r^4 S - a^2 r^2 S + 2a^4 S\cos 2\theta - 4a^2 r^2 S\cos 2\theta + r^4 S\cos 2\theta}{2r^4},$$

$$\sigma_{\theta\theta} = \frac{r^4 S + a^2 r^2 S - 2a^4 S\cos 2\theta - r^4 S\cos 2\theta}{2r^4},$$

$$\tau_{r\theta} = \frac{2a^4 S\sin 2\theta - 2a^2 r^2 S\sin 2\theta - r^4 S\sin 2\theta}{2r^4}. \qquad (3.2.13)$$

3.3 Two Dimensional Circular Inclusion

Consider a circular inclusion with radius a and material constants, κ and μ, where κ is a parameter depending only on the Poisson's Ratio and μ is the shear modulus, embedded in an infinite plate with material constants, κ_1 and μ_1. If the plate is taken as a two-dimensional object with a disc inserted instead of the central hole, the above observations of an infinite plate with a hole

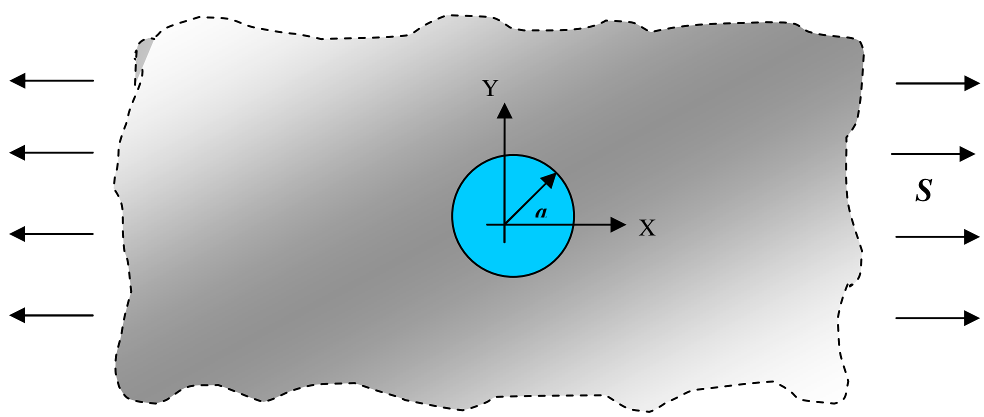

Figure 3-3 *Infinite Plate with a Circular Inclusion Subjected to Tensile Loading.*

can be clubbed with the finite simply connected domain, by maintaining the equilibrium and continuity of stresses and displacements at the boundary of the two phases.

3.3.1 Stress Field Inside the Two Dimensional Circular Inclusion

The two-dimensional circular disc is similar to a finite simply bounded region. Complex potentials are assumed to be

$$\gamma(z) = \sum_{n=0}^{2} c_n z^n, \qquad \psi(z) = \sum_{n=0}^{2} d_n z^n. \tag{3.3.1}$$

Since the disc is going to be clubbed with the infinite plate, the same boundary conditions apply to it. Therefore we take the first three terms.

$$\gamma(z) = c_0 + c_1 z + c_2 z^2,$$

$$\psi(z) = d_0 + d_1 z + d_2 z^2,$$

$$\chi(z) = d_0 z + \frac{d_1 z^2}{2} + \frac{d_2 z^3}{3}. \tag{3.3.2}$$

The Airy stress function in polar form, $z = re^{i\theta}$, is expressed as

$$\begin{aligned}\phi(r,\theta) = {} & c_0 r\cos\theta + c_1 r^2\cos^2\theta + c_1 r^2 \sin^2\theta + c_2 r^3 \cos\theta\cos 2\theta \\ & + c_2 r^3 \sin\theta \sin 2\theta + d_0 r\cos\theta\cos 2\theta + d_0 r\sin\theta\sin 2\theta \\ & + \frac{1}{2} d_1 r^2 \cos\theta\cos 3\theta + \frac{1}{2} d_1 r^2 \sin\theta \sin 3\theta \\ & + \frac{1}{3} d_2 r^3 \cos\theta\cos 4\theta + \frac{1}{3} d_2 r^3 \sin\theta\sin 4\theta.\end{aligned} \tag{3.3.3}$$

The stresses in polar form using (2.1.6) are expressed as

$$\sigma_{rr}^{in} = 2c_1 + 2c_2 r\cos\theta - d_1\cos 2\theta - 2d_2 r\cos 3\theta,$$
$$\sigma_{\theta\theta}^{in} = 2c_1 + 6c_2 r\cos\theta + d_1\cos 2\theta + 2d_2 r\cos 3\theta, \qquad (3.3.4)$$
$$\tau_{r\theta}^{in} = 2c_1 + 6c_2 r\cos\theta + d_1\cos 2\theta + 2d_2 r\cos 3\theta.$$

Determining the displacements using (2.2.11), we get

$$u_r^{in} = \frac{1}{2\mu}(\kappa c_0\cos\theta - c_1 r + \kappa c_1 r - 2c_2 r^2\cos\theta + \kappa c_2 r^2\cos\theta$$
$$- d_0\cos\theta - d_1 r\cos 2\theta - d_2 r^2\cos 3\theta), \qquad (3.3.5)$$
$$u_\theta^{in} = \frac{1}{2\mu}(-\kappa c_0\sin\theta + 2c_2 r^2\sin\theta + \kappa c_2 r^2\sin\theta + d_0\sin\theta$$
$$+ d_1 r\sin 2\theta + d_2 r^2\sin 3\theta).$$

The stresses and displacements in a circular disc of radius a and material constants of κ and μ at $r = a$ are expressed as

$$\sigma_{rr}^{in} = 2c_1 + 2ac_2\cos\theta - d_1\cos 2\theta - 2ad_2\cos 3\theta,$$
$$\sigma_{\theta\theta}^{in} = 2c_1 + 6ac_2\cos\theta + d_1\cos 2\theta + 2ad_2\cos 3\theta, \qquad (3.3.6)$$
$$\tau_{r\theta}^{in} = 2ac_2\sin\theta - d_1\sin 2\theta - 2ad_2\sin 3\theta.$$

$$u_r^{in} = \frac{1}{2\mu}(a(-1+\kappa)c_1 + (\kappa c_0 + a^2(-2+\kappa)c_2 - d_0)\cos\theta$$
$$- ad_1\cos 2\theta - a^2 d_2\cos 3\theta), \qquad (3.3.7)$$
$$u_\theta^{in} = \frac{1}{2\mu}(-\kappa c_0\sin\theta + 2a^2 c_2\sin\theta + a^2\kappa c_2\sin\theta + d_0\sin\theta$$
$$+ ad_1\sin 2\theta + a^2 d_2\sin 3\theta).$$

3.3.2 Infinite Plate Surrounding the Disc

Deriving the stress and displacement components for the part outside the inclusion which material constant κ_1 which depends only on the Poisson's Ratio and a shear modulus of μ_1 .From (2.3.4) and (2.3.5) at $r = a$ yields

$$\sigma_{rr}^{out} = \frac{1}{2a^4}(a^4 S + a^4 S\cos 2\theta + 2a^2 b_1 + 4ab_2 \cos\theta + 6b_3 \cos 2\theta - 8a^2 a_1 \cos 2\theta - 20aa_2 \cos 3\theta),$$

$$\sigma_{\theta\theta}^{out} = \frac{1}{2a^4}(a^4 S - a^4 S\cos 2\theta - 2a^2 b_1 - 4ab_2 \cos\theta - 6b_3 \cos 2\theta + 4aa_2 \cos 3\theta),$$

$$\tau_{r\theta}^{out} = \frac{1}{2a^4}(-a^4 S\sin 2\theta + 4ab_2 \sin\theta + 6b_3 \sin 2\theta - 4a^2 a_1 \sin 2\theta - 12aa_2 \sin 3\theta), \tag{3.3.8}$$

$$u_r^{out} = \frac{1}{8a^3\mu_1}(-a^4 S + a^4 S\kappa_1 - 2a^4 S\cos 2\theta - 4a^2 b_1 - 4ab_2 \cos\theta - 4b_3 \cos 2\theta + 4a^2 a_1 \cos 2\theta + 4a^2\kappa_1 a_1 \cos 2\theta + 8aa_2 \cos 3\theta + 4a\kappa_1 a_2 \cos 3\theta),$$

$$u_\theta^{out} = \frac{1}{4a^3\mu_1}(-a^4 S\sin 2\theta - 2ab_2 \sin\theta - 2b_3 \sin 2\theta + 2a^2 a_1 \sin 2\theta - 2a^2\kappa_1 a_1 \sin 2\theta + 4a_2 \sin 3\theta - 2a\kappa_1 a_2 \sin 3\theta). \tag{3.3.9}$$

The continuity condition can be satisfied if the traction force and displacements are the same at the interface.

Equating (3.3.6) to (3.3.8) and (3.3.7) to (3.3.9) we get the following equations:

$$\frac{1}{2a^4}(a^4 S + a^4 S\cos 2\theta + 2a^2 b_1 + 4ab_2\cos\theta +$$
$$6b_3\cos 2\theta - 8a^2 a_1\cos 2\theta - 20aa_2\cos 3\theta) - 2c_1 - \quad (3.3.10)$$
$$2ac_2\cos\theta + d_1\cos 2\theta + 2ad_2\cos 3\theta = 0,$$

$$-2c_1 - 6ac_2\cos\theta - d_1\cos 2\theta - 2ad_2\cos 3\theta +$$
$$\frac{1}{2a^4}(a^4 S - a^4 S\cos 2\theta - 2a^2 b_1 - 4ab_2\cos\theta \quad (3.3.11)$$
$$-6b_3\cos 2\theta + 4aa_2\cos 3\theta) = 0,$$

$$-2ac_2\sin\theta + d_1\sin 2\theta + 2ad_2\sin 3\theta +$$
$$\frac{1}{2a^4}(-a^4 S\sin 2\theta + 4ab_2\sin\theta + 6b_3\sin 2\theta \quad (3.3.12)$$
$$-4a^2 a_1\sin 2\theta - 12aa_2\sin 3\theta) = 0,$$

$$-\frac{1}{2\mu}(a(-1+\kappa)c_1 + \cos\theta(\kappa c_0 + a^2(-2+\kappa)c_2 - d_0)$$
$$-ad_1\cos 2\theta - a^2 d_2\cos 3\theta) +$$
$$\frac{1}{8a^3\mu_1}(-a^4 S + a^4 S\kappa_1 - 2a^4 S\cos 2\theta - 4a^2 b_1 \quad (3.3.13)$$
$$-4ab_2\cos\theta - 4b_3\cos 2\theta + 4a^2 a_1\cos 2\theta$$
$$+4a^2\kappa_1 a_1\cos 2\theta + 8aa_2\cos 3\theta + 4a\kappa_1 a_2\cos 3\theta) = 0,$$

$$-\frac{1}{2\mu}(-\kappa c_0\sin\theta + 2a^2 c_2\sin\theta + a^2\kappa c_2\sin\theta + d_0\sin\theta$$
$$+ad_1\sin 2\theta + a^2 d_2\sin 3\theta) + \frac{1}{4a^3\mu_1}(-a^4 S\sin 2\theta - \quad (3.3.14)$$
$$2ab_2\sin\theta - 2b_3\sin 2\theta + 2a^2 a_1\sin 2\theta - 2a^2\kappa_1 a_1\sin 2\theta +$$
$$4a_2\sin 3\theta - 2a\kappa_1 a_2\sin 3\theta) = 0.$$

Equating the coefficients of $\cos\theta, \cos 2\theta, \cos 3\theta, \sin\theta, \sin 2\theta, \sin 3\theta$ and the constants in the above equations to zero, the following equations are obtained.

$$2ac_2 - \frac{2b_2}{a^3} = 0, \tag{3.3.15}$$

$$-\frac{S}{2} - d_1 - \frac{3b_3}{a^4} + \frac{4a_1}{a^3} = 0, \tag{3.3.16}$$

$$-2ad_2 + \frac{10a_2}{a^3} = 0, \tag{3.3.17}$$

$$2ac_2 - \frac{2b_2}{a^3} = 0, \tag{3.3.18}$$

$$\frac{S}{2} + d_1 - \frac{3b_3}{a^4} + \frac{2a_1}{a^2} = 0, \tag{3.3.19}$$

$$2ad_2 + \frac{6a_2}{a^3} = 0, \tag{3.3.20}$$

$$\frac{\kappa c_0}{2\mu} - \frac{a^2 c_2}{\mu} + \frac{a^2 \kappa c_2}{2\mu} - \frac{d_0}{2\mu} + \frac{b_2}{2a^2 \mu_1} = 0, \tag{3.3.21}$$

$$-\frac{aS}{4\mu_1} - \frac{ad_1}{2\mu} + \frac{b_3}{2a^3 \mu_1} - \frac{a_1}{2a\mu_1} - \frac{\kappa_1 a_1}{2a\mu_1} = 0, \tag{3.3.22}$$

$$-\frac{a^2 d_2}{2\mu} - \frac{a_2}{a^2 \mu_1} - \frac{\kappa_1 a_2}{2a^2 \mu_1} = 0, \tag{3.3.23}$$

$$-\frac{\kappa c_0}{2\mu} + \frac{a^2 c_2}{\mu} + \frac{a^2 \kappa c_2}{2\mu} + \frac{d_0}{2\mu} + \frac{b_2}{2a^2 \mu_1} = 0, \tag{3.3.24}$$

$$\frac{aS}{4\mu_1} + \frac{ad_1}{2\mu} + \frac{b_3}{2a^3 \mu_1} - \frac{a_1}{2a\mu_1} + \frac{\kappa_1 a_1}{2a\mu_1} = 0, \tag{3.3.25}$$

$$\frac{a^2 d_2}{2\mu} - \frac{a_2}{a^2\mu_1} + \frac{\kappa_1 a_2}{2a^2\mu_1} = 0, \tag{3.3.26}$$

$$\frac{aS}{8\mu_1} - \frac{aS\kappa_1}{8\mu_1} - \frac{ac_1}{2\mu} + \frac{a\kappa c_1}{2\mu} + \frac{b_1}{2a\mu_1} = 0, \tag{3.3.27}$$

$$-\frac{S}{2} + 2c_1 - \frac{b_1}{a^2} = 0. \tag{3.3.28}$$

Solving the above equations simultaneously to find the constants, we get

$$a_1 = -\frac{a^2(S\mu - S\mu_1)}{2(\kappa_1\mu + \mu_1)}, \quad a_2 = 0, \quad a_3 = 0,$$

$$b_1 = \frac{a(-aS\mu + aS\kappa_1\mu + aS\mu_1 - aS\kappa\mu_1)}{2(2\mu - \mu_1 + \kappa\mu_1)}, \quad b_2 = 0, \quad b_3 = -\frac{a^4 S\mu - a^4 S\mu_1}{2(\kappa_1\mu + \mu_1)},$$

$$c_0 = \frac{d_0}{\kappa}, \quad c_1 = \frac{(S + S\kappa_1)\mu}{4(2\mu - \mu_1 + \kappa\mu_1)}, \quad c_2 = 0,$$

$$d_0 = \kappa c_0, \quad d_1 = -\frac{S\mu + S\kappa_1\mu}{2(\kappa_1\mu + \mu_1)}, \quad d_2 = 0.$$

Substituting the above solutions into (3.3.6) and (3.3.7), we get the final stress and displacement equations for the disc as

$$\sigma_{rr}^{in} = \frac{S\mu}{2(2\mu - \mu_1 + \kappa\mu_1)} + \frac{S\kappa_1\mu}{2(2\mu - \mu_1 + \kappa\mu_1)} + \frac{S\mu\cos 2\theta}{2(\kappa_1\mu + \mu_1)} + \frac{S\kappa_1\mu\cos 2\theta}{2(\kappa_1\mu + \mu_1)},$$

$$\sigma_{\theta\theta}^{in} = \frac{1}{2}S(1+\kappa_1)\mu\left(\frac{1}{2\mu + (-1+\kappa)\mu_1} - \frac{\cos 2\theta}{\kappa_1\mu + \mu_1}\right),$$

$$\tau_{r\theta}^{in} = -\frac{(S\mu + S\kappa_1\mu)\sin 2\theta}{2(\kappa_1\mu + \mu_1)}, \tag{3.3.29}$$

$$u_r^{in} = \frac{rS(-1+\kappa)(1+\kappa_1)}{16\mu + 8(-1+\kappa)\mu_1},$$

$$u_{\theta}^{in} = -\frac{rS(1+\kappa_1)\sin 2\theta}{4(\kappa_1\mu + \mu_1)}. \tag{3.3.30}$$

The above equations can be verified by substituting them into the equilibrium equations i.e. (2.1.7).

Substitute the constants into equations (3.3.8) and (3.3.9) to obtain the final stress and displacement equations for the part surrounding the circular inclusion i.e.

$$\begin{aligned}
\sigma_{rr}^{out} &= \frac{S}{2} - \frac{a^2 S\mu}{2r^2(2\mu - \mu_1 + \kappa\mu_1)} + \frac{a^2 S\kappa_1\mu}{2r^2(2\mu - \mu_1 + \kappa\mu_1)} + \\
&\frac{a^2 S\mu_1}{2r^2(2\mu - \mu_1 + \kappa\mu_1)} - \frac{a^2 S\kappa\mu_1}{2r^2(2\mu - \mu_1 + \kappa\mu_1)} + \frac{1}{2}S\cos 2\theta - \\
&\frac{3a^4 S\mu\cos 2\theta}{2r^4(\kappa_1\mu + \mu_1)} + \frac{2a^2 S\mu\cos 2\theta}{r^2(\kappa_1\mu + \mu_1)} + \frac{3a^4 S\mu_1\cos 2\theta}{2r^4(\kappa_1\mu + \mu_1)} - \frac{2a^2 S\mu_1\cos 2\theta}{r^2(\kappa_1\mu + \mu_1)}, \\
\sigma_{\theta\theta}^{out} &= \frac{S}{2} + \frac{a^2 S}{2r^2} - \frac{a^2 S\mu}{2r^2(2\mu - \mu_1 + \kappa\mu_1)} - \frac{a^2 S\kappa_1\mu}{2r^2(2\mu - \mu_1 + \kappa\mu_1)} \\
&- \frac{1}{2}S\cos 2\theta + \frac{3a^4 S\mu\cos 2\theta}{2r^4(\kappa_1\mu + \mu_1)} - \frac{3a^4 S\mu_1\cos 2\theta}{2r^4(\kappa_1\mu + \mu_1)}, \\
\tau_{r\theta}^{out} &= -\frac{1}{2}S\sin 2\theta - \frac{3a^4 S\mu\sin 2\theta}{2r^4(\kappa_1\mu + \mu_1)} - \frac{a^2 S\mu\sin 2\theta}{r^2(\kappa_1\mu + \mu_1)} + \\
&\frac{3a^4 S\mu_1\sin 2\theta}{2r^4(\kappa_1\mu + \mu_1)} - \frac{a^2 S\mu_1\sin 2\theta}{r^2(\kappa_1\mu + \mu_1)},
\end{aligned} \tag{3.3.31}$$

$$\begin{aligned}
u_r^{out} &= \frac{a^2 S}{4r\mu_1} - \frac{rS}{8\mu_1} + \frac{rS\kappa_1}{8\mu_1} - \frac{a^2 S\mu}{4r\mu_1(2\mu - \mu_1 + \kappa\mu_1)} - \frac{a^2 S\kappa_1\mu}{4r\mu_1(2\mu - \mu_1 + \kappa\mu_1)} + \\
&\frac{rS\cos 2\theta}{4\mu_1} - \frac{a^4 S\cos 2\theta}{4r^3(\kappa_1\mu + \mu_1)} + \frac{a^2 S\cos 2\theta}{4r(\kappa_1\mu + \mu_1)} + \frac{a^2 S\kappa_1\cos 2\theta}{4r(\kappa_1\mu + \mu_1)} + \\
&\frac{a^4 S\mu\cos 2\theta}{4r^3\mu_1(\kappa_1\mu + \mu_1)} - \frac{a^2 S\mu\cos 2\theta}{4r\mu_1(\kappa_1\mu + \mu_1)} - \frac{a^2 S\kappa_1\mu\cos 2\theta}{4r\mu_1(\kappa_1\mu + \mu_1)},
\end{aligned}$$

$$u_\theta^{out} = -\frac{S(a^4(-\mu+\mu_1)+a^2r^2(-1+\kappa_1)(-\mu+\mu_1)+r^4(\kappa_1\mu+\mu_1))\sin 2\theta}{4r^3\mu_1(\kappa_1\mu+\mu_1)}. \quad (3.3.32)$$

The above equations can be verified by substituting them into the equilibrium equations (2.1.7).

Example Problem: Circular Inclusion Problem

Consider finding the stresses for a thin Infinite plate with a hole made of Iron, having a two-dimensional circular inclusion made of carbon in the center, subjected to far field tensile loading of 1000N/mm^2 problem.

we know that,

for Carbon,

Poisson's Ratio, $\nu = 0.24$,

Shear Modulus μ= 12.4GPa,

Parameter κ for the plane strain from (2.2.12)= $3 - 4\nu = 2.04$.

for Iron

Poissons Ratio, $\nu 1 = 0.29$,

Shear Modulus $\mu 1$= 77.5GPa,

Parameter $\kappa 1$ for the plane strain from (2.2.12)= = $3 - 4\nu = 1.836$.

Let,

the radius of the central disc be, a= 100mm,

Substituting the above values in the stress and displacement components derived above, we get

$$\sigma_{rr}^{in} = 166.993 + 175.481\cos 2\theta,$$

$$\sigma_{\theta\theta}^{in} = 166.93 - 175.481\cos 2\theta,$$

$$\tau_{r\theta}^{in} = -175.481\sin 2\theta,$$

$$\sigma_{rr}^{out} = 500 - \frac{3.33\times 10^{6}}{r^{2}} + 500\cos 2\theta - \frac{9.736\times 10^{6}\cos 2\theta}{r^{4}} - \frac{1.298\times \cos 2\theta}{r^{2}}, \qquad (3.3.33)$$

$$\sigma_{\theta\theta}^{out} = 500 + \frac{3.33\times 10^{6}}{r^{2}} - 500\cos 2\theta - \frac{9.736\times 10^{6}\cos 2\theta}{r^{4}},$$

$$\tau_{r\theta}^{out} = -500\sin 2\theta + \frac{9.74\times 10^{10}\sin 2\theta}{r^{4}} - \frac{6.49\times 10^{6}\sin 2\theta}{r^{2}}.$$

Converting the above to the Cartesian coordinate system, we get

$$\sigma_{xx}^{in} = 342.413,$$

$$\sigma_{xx}^{out} = 500(2 + \frac{1}{(x^{2}+y^{2})^{2}}(9.46\times 10^{-7}(-2.076\times 10^{9}(x-y)(x+y) - \qquad (3.3.34)$$

$$6.86\times 10^{9}(-30000 + 2(x^{2}+y^{2}))\cos(4\tan^{-1}\frac{y}{x}))),$$

from which we can see that the stresses in x-direction at any point on the inner disc are constant but changes out side the disc with respect to the location.

$$\sigma_{yy}^{in} = -8.54812,$$

$$\sigma_{yy}^{out} = \frac{1}{(x^{2}+y^{2})^{2}}(0.000473(-6.68\times 10^{9}(x-y)(x+y) - \qquad (3.3.35)$$

$$6.86\times 10^{9}(30000 - 2(x^{2}+y^{2}))\cos(4\tan^{-1}\frac{y}{x}))),$$

and same observation can be made for the stresses in the y-direction also.

$$\tau_{xy}^{in} = 0,$$

$$\tau_{xy}^{out} = \frac{-6.66\times 10^{6}xy + (9.74\times 10^{10} - 6.49\times 10^{6}x^{2} - 6.49\times 10^{6}y^{2})\sin(4\tan^{-1}\frac{y}{x})}{(x^{2}+y^{2})^{2}}. \tag{3.3.36}$$

The stresses along the x and y-axes can be shown using the following graphs:

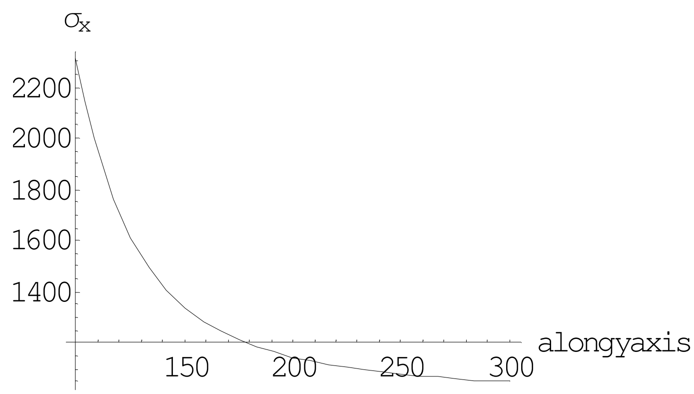

Figure 3-4 *Variation of the tensile stress along the y- axis.*

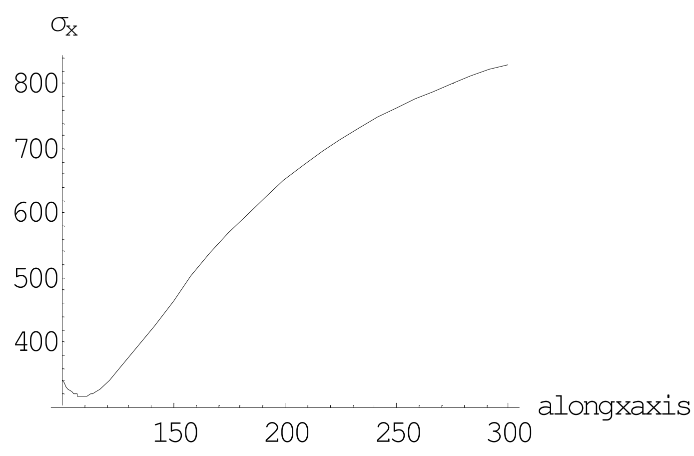

Figure 3-5 *Variation of the tensile stress along the x- axis.*

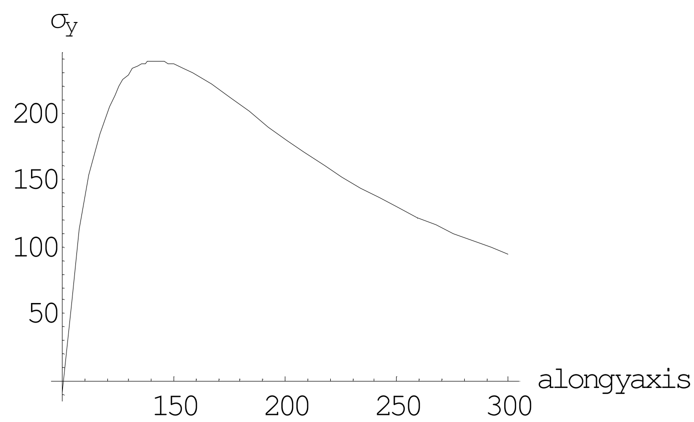

Figure 3-6 *Variation of the compressive stress along the y- axis.*

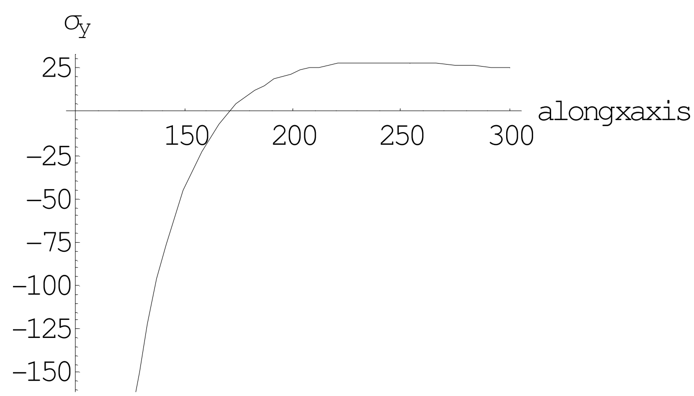

Figure 3-7 *Variation of the compressive stress along the x- axis.*

The shear stresses turn out to be zero along the x and y axes therefore, the 3-D graph for the shear stresses can be demonstrated as follows:

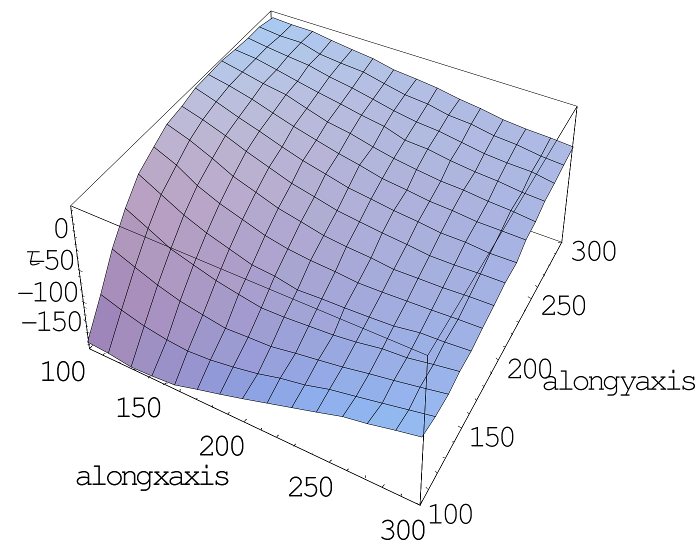

Figure 3-8 *Variation of the shear stress over the plate.*

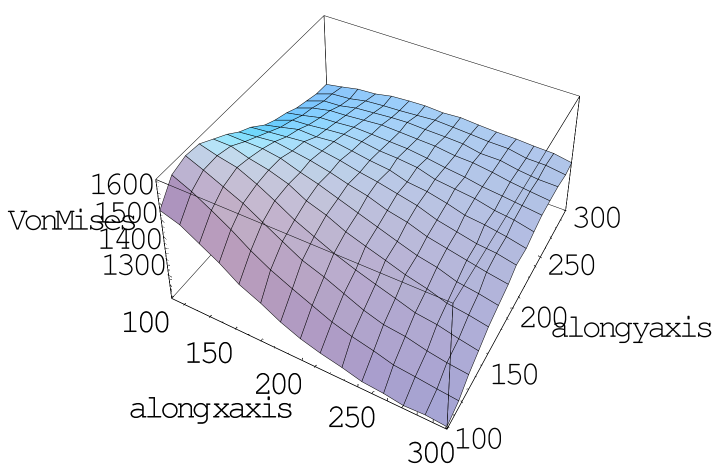

Figure 3-9 Variation of the Von Mises stress over the plate.

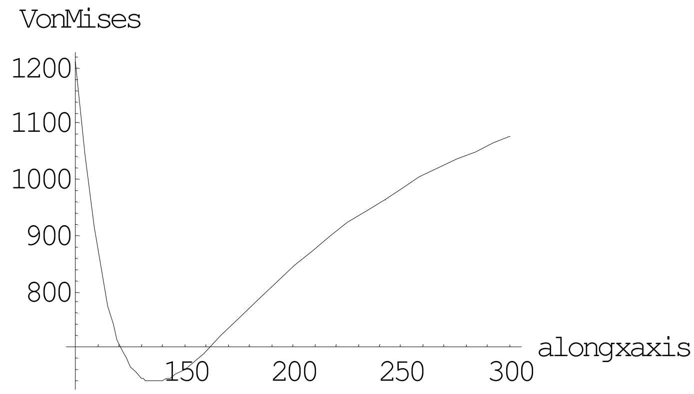

Figure 3-10 Variation of the Von Mises stress along the x-axis.

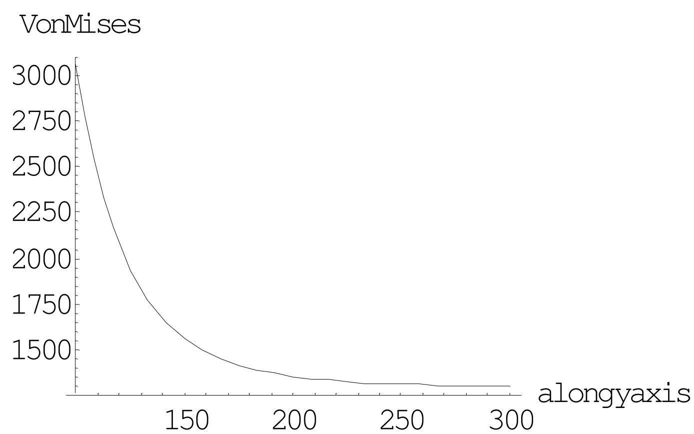

Figure 3-11 Variation of the Von Mises stress along the y-axis.

CHAPTER 4

CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

This thesis demonstrated how to solve an elasticity problem using the Airy stress function. It showed how the method can be applied to find the stresses and displacements at any point on a two-dimensional plate subjected to different boundary conditions. This eventually led to how the Airy stress function can be applied to a two-phase plate with a circular inclusion in finding the stresses and displacements at any point.

The problem studied in Chapter 3 further demonstrated how the Airy stress function is applied to an infinite plate with a circular inclusion. On studying the graphical representation of the result, it can be seen that all stresses within the inclusion are constant and the shear stress is zero when subjected to a far-field stress. The maximum tensile stress occurs at the boundary of the disc intersecting the y-axis and is decreased along the boundary of the disc as it nears the x-axis. The maximum compressive stress occurs at the boundary intersecting with the x-axis and decreases as it nears the y-axis along the interfacing boundary.

The following extension is suggested to further exploit the content of the present thesis:

a. The boundary conditions can be generalized to include all the stress components at far fields.

b. Multi-layer circular inclusions can be considered.

c. The shape of the inclusion can be extended to an elliptic shape.

APPENDIX A

MATHEMATICA CODE

Mathematica code for finding the stresses and displacements for a circular inclusion

$$\gamma = \sum_{n=0}^{2} a_n * z^n$$

$$\varphi = \sum_{n=0}^{2} b_n * z^n$$

$$a_0 + z\,a_1 + z^2\,a_2$$

$$b_0 + z\,b_1 + z^2\,b_2$$

$$\gamma 1 = (S * z / 4) + \sum_{n=1}^{2} m_n * z^{-n}$$

$$\frac{S\,z}{4} + \frac{m_1}{z} + \frac{m_2}{z^2}$$

$$\varphi 1 = (G * z / 2) + \sum_{n=1}^{3} f_n * z^{-n}$$

$$\frac{G\,z}{2} + \frac{f_1}{z} + \frac{f_2}{z^2} + \frac{f_3}{z^3}$$

```
d = D[γ, z]
d1 = D[γ1, z]
```

$$a_1 + 2\,z\,a_2$$

$$\frac{S}{4} - \frac{m_1}{z^2} - \frac{2\,m_2}{z^3}$$

```
χ = Integrate[φ,z]

χ1 = Integrate[φ1,z]
```

$$z\,b_0 + \frac{z^2 b_1}{2} + \frac{z^3 b_2}{3}$$

$$\frac{G z^2}{4} + \mathrm{Log}[z]\, f_1 - \frac{f_2}{z} - \frac{f_3}{2 z^2}$$

```
z = ComplexExpand[r * Exp[I * θ]]
```

$$r\,\mathrm{Cos}[\theta] + i\, r\, \mathrm{Sin}[\theta]$$

```
e = ComplexExpand[Conjugate[z]]
```

$$r\,\mathrm{Cos}[\theta] - i\, r\, \mathrm{Sin}[\theta]$$

```
h = FullSimplify[ComplexExpand[(e * γ) + χ]] // TrigReduce
h1 = FullSimplify[ComplexExpand[χ1 + (γ1 * e)]] // TrigReduce
```

$$\frac{1}{6} e^{-i\theta} r \left(6 a_0 + 6 e^{i\theta} r a_1 + 6 e^{2i\theta} r^2 a_2 + 6 e^{2i\theta} b_0 + 3 e^{3i\theta} r b_1 + 2 e^{4i\theta} r^2 b_2\right)$$

$$\frac{1}{4\,\mathrm{Abs}[r]^4}\left(e^{4\,\mathrm{Im}[\theta]} r^2 \left(-e^{4i(\theta-\mathrm{Re}[\theta])}\, \mathrm{Conjugate}[r]^2 \left(\left(-1+e^{2i\theta}\right) r^2 S + 4\left(\mathrm{Im}[\theta] - \mathrm{Log}\left[e^{i\,\mathrm{Re}[\theta]} r\right]\right) f_1\right) - 2 e^{-2i\theta} f_3 + 4 e^{-2i\,\mathrm{Re}[\theta]}\, \mathrm{Conjugate}[r] \left(-e^{i\theta} f_2 + r m_1\right) + 4 e^{-3i\theta} r m_2\right)\right)$$

```
ϕ = FullSimplify[ComplexExpand[Re[h]]]

ϕ1= FullSimplify[ComplexExpand[Re[h1]]]
```

$$\frac{1}{6} r \left(6 r a_1 + 6\,\mathrm{Cos}[\theta] \left(a_0 + r^2 a_2 + b_0\right) + 3 r\,\mathrm{Cos}[2\theta]\, b_1 + 2 r^2 \mathrm{Cos}[3\theta]\, b_2\right)$$

$$\frac{r^4 S\,\mathrm{Sin}[\theta]^2 - \mathrm{Cos}[2\theta]\, f_3 + r\left(r\,\mathrm{Log}[r^2]\, f_1 - 2\,\mathrm{Cos}[\theta]\, f_2 + 2 r\,\mathrm{Cos}[2\theta]\, m_1 + 2\,\mathrm{Cos}[3\theta]\, m_2\right)}{2 r^2}$$

```
σrrIni =
 ((D[ϕ, r] * 1/r) + (D[ϕ, {θ, 2}] * 1/r^2)) /. Cos'[θ] → -Sin[θ] /. Cos''[θ] → -Cos[θ] //
  TrigReduce
```

$$2\,a_1 + 2\,r\,\mathrm{Cos}[\theta]\,a_2 - \mathrm{Cos}[2\,\theta]\,b_1 - 2\,r\,\mathrm{Cos}[3\,\theta]\,b_2$$

```
σrr1Ini = ( (D[ϕ1, r] * 1/r) + (D[ϕ1, {θ, 2}] * 1/r^2)) /. Cos'[θ] → -Sin[θ] /.
   Cos''[θ] → -Cos[θ] // TrigReduce
```

$$\frac{1}{2\,r^4}\left(r^4\,S + r^4\,S\,\mathrm{Cos}[2\,\theta] + 2\,r^2\,f_1 + 4\,r\,\mathrm{Cos}[\theta]\,f_2 + 6\,\mathrm{Cos}[2\,\theta]\,f_3 - 8\,r^2\,\mathrm{Cos}[2\,\theta]\,m_1 - 20\,r\,\mathrm{Cos}[3\,\theta]\,m_2\right)$$

```
σrr =
 (((FullSimplify[(D[ϕ, r] * 1/r) + (D[ϕ, {θ, 2}] * 1/r^2)]) /. Cos'[θ] → -Sin[θ]) /.
    Cos''[θ] → -Cos[θ]) /. r → a // TrigReduce
σrr1 =
 (FullSimplify[ (D[ϕ1, r] * 1/r) + (D[ϕ1, {θ, 2}] * 1/r^2)]) /. Cos'[θ] → -Sin[θ] /.
   Cos''[θ] → -Cos[θ] /. r → a // TrigReduce
```

$$2\,a_1 + 2\,a\,\mathrm{Cos}[\theta]\,a_2 - \mathrm{Cos}[2\,\theta]\,b_1 - 2\,a\,\mathrm{Cos}[3\,\theta]\,b_2$$

$$\frac{1}{2\,a^4}\left(a^4\,S + a^4\,S\,\mathrm{Cos}[2\,\theta] + 2\,a^2\,f_1 + 4\,a\,\mathrm{Cos}[\theta]\,f_2 + 6\,\mathrm{Cos}[2\,\theta]\,f_3 - 8\,a^2\,\mathrm{Cos}[2\,\theta]\,m_1 - 20\,a\,\mathrm{Cos}[3\,\theta]\,m_2\right)$$

```
σθθIni = D[ϕ, {r, 2}] // TrigReduce
```

$$2\,a_1 + 6\,r\,\mathrm{Cos}[\theta]\,a_2 + \mathrm{Cos}[2\,\theta]\,b_1 + 2\,r\,\mathrm{Cos}[3\,\theta]\,b_2$$

```
σθθ1Ini = D[ϕ1, {r, 2}] // TrigReduce
```

$$\frac{r^4\,S - r^4\,S\,\mathrm{Cos}[2\,\theta] - 2\,r^2\,f_1 - 4\,r\,\mathrm{Cos}[\theta]\,f_2 - 6\,\mathrm{Cos}[2\,\theta]\,f_3 + 4\,r\,\mathrm{Cos}[3\,\theta]\,m_2}{2\,r^4}$$

```
σθθ = Simplify[(D[ϕ, {r, 2}]) /. r → a // TrigReduce]
```

$$2\,a_1 + 6\,a\,\mathrm{Cos}[\theta]\,a_2 + \mathrm{Cos}[2\,\theta]\,b_1 + 2\,a\,\mathrm{Cos}[3\,\theta]\,b_2$$

```
σθθ1 = FullSimplify[ (D[ϕ1, {r, 2}]) /. r → a // TrigReduce]
```

$$\frac{a^4\, S\, \text{Sin}[\theta]^2 - 3\, \text{Cos}[2\,\theta]\, f_3 - a\,(a\, f_1 + 2\, \text{Cos}[\theta]\, f_2 - 2\, \text{Cos}[3\,\theta]\, m_2)}{a^4}$$

```
τrθIni =
 (FullSimplify[-D[D[ϕ, θ]*1/r, r]]) /. Cos′[θ] → -Sin[θ] /. Cos″[θ] → -Cos[θ] //
  TrigReduce
```

$$2\, r\, \text{Sin}[\theta]\, a_2 + \text{Sin}[2\,\theta]\, b_1 + 2\, r\, \text{Sin}[3\,\theta]\, b_2$$

```
τrθ1Ini = (FullSimplify[- D[D[ϕ1, θ] * 1 / r, r]]) // TrigReduce
```

$$\frac{-r^4\, S\, \text{Sin}[2\theta] + 4\, r\, \text{Sin}[\theta]\, f_2 + 6\, \text{Sin}[2\theta]\, f_3 - 4\, r^2\, \text{Sin}[2\theta]\, m_1 - 12\, r\, \text{Sin}[3\theta]\, m_2}{2\, r^4}$$

```
τrθ =
 (((FullSimplify[- D[D[ϕ, θ] * 1 / r, r]] // TrigReduce) /. r → a) /.
   Cos′[θ] → -Sin[θ]) /. Cos″[θ] → -Cos[θ]
τrθ1 =
 (((FullSimplify[- D[D[ϕ1, θ] * 1 / r, r]] // TrigReduce) /. r → a) /.
   Cos′[θ] → -Sin[θ]) /. Cos″[θ] → -Cos[θ]
```

$$2\, a\, \text{Sin}[\theta]\, a_2 + \text{Sin}[2\,\theta]\, b_1 + 2\, a\, \text{Sin}[3\,\theta]\, b_2$$

$$\frac{-a^4\, S\, \text{Sin}[2\theta] + 4\, a\, \text{Sin}[\theta]\, f_2 + 6\, \text{Sin}[2\theta]\, f_3 - 4\, a^2\, \text{Sin}[2\theta]\, m_1 - 12\, a\, \text{Sin}[3\theta]\, m_2}{2\, a^4}$$

```
urIni = FullSimplify[ur] // TrigReduce
```

$$\frac{1}{2\,\mu}\,(\kappa\, \text{Cos}[\theta]\, a_0 - r\, a_1 + r\,\kappa\, a_1 - 2\, r^2\, \text{Cos}[\theta]\, a_2 + r^2\,\kappa\, \text{Cos}[\theta]\, a_2 - \text{Cos}[\theta]\, b_0 - r\, \text{Cos}[2\,\theta]\, b_1 - r^2\, \text{Cos}[3\,\theta]\, b_2)$$

u_{r1Ini} **= FullSimplify[ur1] // TrigReduce**

$$\frac{1}{8 r^3 \mu 1}(-r^4 S + r^4 S\kappa 1 + 2 r^4 S \mathrm{Cos}[2\theta] - 4 r^2 f_1 - 4 r \mathrm{Cos}[\theta]\, f_2 - 4 \mathrm{Cos}[2\theta]\, f_3 + 4 r^2 \mathrm{Cos}[2\theta]\, m_1 + 4 r^2 \kappa 1 \mathrm{Cos}[2\theta]\, m_1 + 8 r \mathrm{Cos}[3\theta]\, m_2 + 4 r \kappa 1 \mathrm{Cos}[3\theta]\, m_2)$$

u_r **= (FullSimplify[**u_{rIni} **// TrigReduce]) /. r → a**

u_{r1} **= (FullSimplify[** u_{r1Ini} **// TrigReduce]) /. r → a**

$$\frac{a(-1+\kappa)\, a_1 + \mathrm{Cos}[\theta]\,(\kappa a_0 + a^2(-2+\kappa)\, a_2 - b_0) - a \mathrm{Cos}[2\theta]\, b_1 - a^2 \mathrm{Cos}[3\theta]\, b_2}{2\mu}$$

$$\frac{1}{8 a^3 \mu 1}(a^4 S(-1+\kappa 1 + 2 \mathrm{Cos}[2\theta]) - 4 \mathrm{Cos}[2\theta]\, f_3 + 4 a(-a f_1 - \mathrm{Cos}[\theta]\, f_2 + a(1+\kappa 1)\mathrm{Cos}[2\theta]\, m_1 + (2+\kappa 1)\mathrm{Cos}[3\theta]\, m_2))$$

$u_{\theta 1Ini}$ **= (ComplexExpand[ui1] // TrigReduce) /. Cos[θ]^2 + Sin[θ]^2 → 1**

$$\frac{1}{4 r^3 \mu 1}(-r^4 S \mathrm{Sin}[2\theta] - 2 r \mathrm{Sin}[\theta]\, f_2 - 2 \mathrm{Sin}[2\theta]\, f_3 + 2 r^2 \mathrm{Sin}[2\theta]\, m_1 - 2 r^2 \kappa 1 \mathrm{Sin}[2\theta]\, m_1 + 4 r \mathrm{Sin}[3\theta]\, m_2 - 2 r \kappa 1 \mathrm{Sin}[3\theta]\, m_2)$$

u_θ **= (Simplify[**$u_{\theta Ini}$**] // TrigReduce) /. r → a**

$u_{\theta 1}$ **= (Simplify[** $u_{\theta 1Ini}$**] // TrigReduce) /. r → a**

$$\frac{-\kappa \mathrm{Sin}[\theta]\, a_0 + 2 a^2 \mathrm{Sin}[\theta]\, a_2 + a^2 \kappa \mathrm{Sin}[\theta]\, a_2 + \mathrm{Sin}[\theta]\, b_0 + a \mathrm{Sin}[2\theta]\, b_1 + a^2 \mathrm{Sin}[3\theta]\, b_2}{2\mu}$$

$$\frac{1}{4 a^3 \mu 1}(-a^4 S \mathrm{Sin}[2\theta] - 2 a \mathrm{Sin}[\theta]\, f_2 - 2 \mathrm{Sin}[2\theta]\, f_3 + 2 a^2 \mathrm{Sin}[2\theta]\, m_1 - 2 a^2 \kappa 1 \mathrm{Sin}[2\theta]\, m_1 + 4 a \mathrm{Sin}[3\theta]\, m_2 - 2 a \kappa 1 \mathrm{Sin}[3\theta]\, m_2)$$

THE SIMULTANEOUS EQUATIONS THAT HAVE TO BE SOLVED TO FIND THE CONSTANTS

```
Eq1 = FullSimplify[ Coefficient[σrr // Expand, Cos[θ]] -
   Coefficient[σrr1 // Expand, Cos[θ]]]
```

$$2\,a\,a_2 - \frac{2\,f_2}{a^3}$$

```
Eq2 = FullSimplify[Coefficient[σrr // Expand, Cos[2 θ]] -
   Coefficient[ σrr1 // Expand, Cos[2 θ]]]
```

$$-\frac{S}{2} - b_1 - \frac{3\,f_3}{a^4} + \frac{4\,m_1}{a^2}$$

```
Eq3 = FullSimplify[Coefficient[σrr // Expand, Cos[3 θ]] -
   Coefficient[ σrr1 // Expand, Cos[3 θ]]]
```

$$-2\,a\,b_2 + \frac{10\,m_2}{a^3}$$

```
Eq4 = FullSimplify[Coefficient[ τrθ // Expand, Sin[θ]] -
   Coefficient[ τrθ1 // Expand, Sin[θ]]]
```

$$2\,a\,a_2 - \frac{2\,f_2}{a^3}$$

```
Eq5 =
 FullSimplify[Coefficient[τrθ // Expand, Sin[2 θ]] -
   Coefficient [ τrθ1 // Expand, Sin[2 θ]]]
```

$$\frac{S}{2} + b_1 - \frac{3\,f_3}{a^4} + \frac{2\,m_1}{a^2}$$

```
Eq6 = FullSimplify[ Coefficient[τrθ // Expand, Sin[3 θ]] -
   Coefficient[ τrθ1 // Expand, Sin[3 θ]]]
```

$$2\,a\,b_2 + \frac{6\,m_2}{a^3}$$

```
Eq7 =
 Expand[Coefficient[u_r //Expand, Cos[θ]] - Coefficient[ u_r1 //Expand, Cos[θ]]]
```

$$\frac{\kappa\, a_0}{2\,\mu} - \frac{a^2\, a_2}{\mu} + \frac{a^2\, \kappa\, a_2}{2\,\mu} - \frac{b_0}{2\,\mu} + \frac{f_2}{2\, a^2\, \mu 1}$$

```
Eq8 =
 Expand[Coefficient[u_r //Expand, Cos[2θ]] - Coefficient[u_r1 //Expand, Cos[2θ]]]
```

$$-\frac{a\, S}{4\,\mu 1} - \frac{a\, b_1}{2\,\mu} + \frac{f_3}{2\, a^3\, \mu 1} - \frac{m_1}{2\, a\, \mu 1} - \frac{\kappa 1\, m_1}{2\, a\, \mu 1}$$

```
Eq9 = Coefficient[u_r //Expand, Cos[3θ]] - Coefficient[u_r1 //Expand, Cos[3θ]]
```

$$-\frac{a^2\, b_2}{2\,\mu} - \frac{m_2}{a^2\, \mu 1} - \frac{\kappa 1\, m_2}{2\, a^2\, \mu 1}$$

```
Eq10 = Coefficient[u_θ //Expand, Sin[θ]] - Coefficient[ u_θ1 //Expand, Sin[θ]]
```

$$-\frac{\kappa\, a_0}{2\,\mu} + \frac{a^2\, a_2}{\mu} + \frac{a^2\, \kappa\, a_2}{2\,\mu} + \frac{b_0}{2\,\mu} + \frac{f_2}{2\, a^2\, \mu 1}$$

```
Eq11 = Coefficient[u_θ //Expand, Sin[2θ]] - Coefficient[u_θ1 //Expand, Sin[2θ]]
```

$$\frac{a\, S}{4\,\mu 1} + \frac{a\, b_1}{2\,\mu} + \frac{f_3}{2\, a^3\, \mu 1} - \frac{m_1}{2\, a\, \mu 1} + \frac{\kappa 1\, m_1}{2\, a\, \mu 1}$$

```
Eq12 = Coefficient[u_θ //Expand, Sin[3θ]] - Coefficient[u_θ1 //Expand, Sin[3θ]]
```

$$\frac{a^2\, b_2}{2\,\mu} - \frac{m_2}{a^2\, \mu 1} + \frac{\kappa 1\, m_2}{2\, a^2\, \mu 1}$$

```
Eq13 =
 Expand[u_r - Coefficient [u_r, Cos[θ]] * Cos[θ] -
   Coefficient[u_r, Cos[2θ]] * Cos[2θ] - Coefficient[u_r, Cos[3θ]] * Cos[3θ] -
   (u_r1 - Coefficient[u_r1, Cos[θ]] * Cos[θ] - Coefficient[u_r1, Cos[2θ]] * Cos[2θ] -
     Coefficient[u_r1, Cos[3θ]] * Cos[3θ])]
```

$$\frac{a\,S}{8\,\mu 1}-\frac{a\,S\,\kappa 1}{8\,\mu 1}-\frac{a\,a_1}{2\,\mu}+\frac{a\,\kappa\,a_1}{2\,\mu}+\frac{f_1}{2\,a\,\mu 1}$$

**Eq14 = FullSimplify[u_θ – Coefficient [u_θ, Sin[θ]] * Sin[θ] –
Coefficient[u_θ, Sin[2θ]] * Sin[2θ] – Coefficient[u_θ, Sin[3θ]] * Sin[3θ] –
($u_{\theta 1}$ – Coefficient[$u_{\theta 1}$, Sin[θ]] * Sin[θ] – Coefficient[$u_{\theta 1}$, Sin[2θ]] * Sin[2θ] –
Coefficient[$u_{\theta 1}$, Sin[3θ]] * Sin[3θ])]**

0

**Eq15 = FullSimplify[σ_{rr} – Coefficient [σ_{rr}, Cos[θ]] * Cos[θ] –
Coefficient[σ_{rr}, Cos[2θ]] * Cos[2θ] – Coefficient[σ_{rr}, Cos[3θ]] * Cos[3θ] –
(σ_{rr1} – Coefficient[σ_{rr1}, Cos[θ]] * Cos[θ] –
Coefficient[σ_{rr1}, Cos[2θ]] * Cos[2θ] – Coefficient[σ_{rr1}, Cos[3θ]] * Cos[3θ])]**

$$-\frac{S}{2}+2\,a_1-\frac{f_1}{a^2}$$

**Eq16 = FullSimplify[$\tau_{r\theta}$ – Coefficient [$\tau_{r\theta}$, Sin[θ]] * Sin[θ] –
Coefficient[$\tau_{r\theta}$, Sin[2θ]] * Sin[2θ] – Coefficient[$\tau_{r\theta}$, Sin[3θ]] * Sin[3θ] –
($\tau_{r\theta 1}$ – Coefficient[$\tau_{r\theta 1}$, Sin[θ]] * Sin[θ] –
Coefficient[$\tau_{r\theta 1}$, Sin[2θ]] * Sin[2θ] – Coefficient[$\tau_{r\theta 1}$, Sin[3θ]] * Sin[3θ])]**

0

Solve[{ Eq13 == 0, Eq15 == 0}, {a_1, f_1}]

$$\left\{\left\{a_1\to-\frac{-S\,\mu-S\,\kappa 1\,\mu}{4\,(2\,\mu-\mu 1+\kappa\,\mu 1)},\;f_1\to\frac{a\,(-a\,S\,\mu+a\,S\,\kappa 1\,\mu+a\,S\,\mu 1-a\,S\,\kappa\,\mu 1)}{2\,(2\,\mu-\mu 1+\kappa\,\mu 1)}\right\}\right\}$$

**Solve[{ Eq1 == 0 , Eq2 == 0, Eq3 == 0, Eq4 == 0, Eq6 == 0, Eq9 == 0, Eq11 == 0,
Eq8 == 0, Eq12 == 0, Eq7 == 0, Eq10 == 0, Eq13 == 0, Eq15 == 0},
{a_0, a_2, b_0, f_2, b_1, b_2, m_1, m_2, f_3, f_1, a_1}]**

$$\left\{\left\{b_0\to\kappa a_0,\ f_1\to-\frac{a^2 S}{2}+\frac{a^2 (S+S\kappa 1)\,\mu}{2\,(2\mu-\mu 1+\kappa\mu 1)},\ a_1\to\frac{(S+S\kappa 1)\,\mu}{4\,(2\mu-\mu 1+\kappa\mu 1)},\ b_1\to-\frac{S\mu+S\kappa 1\mu}{2\,(\kappa 1\mu+\mu 1)},\right.\right.$$

$$\left.\left.f_3\to-\frac{a^4 S\mu-a^4 S\mu 1}{2\,(\kappa 1\mu+\mu 1)},\ m_1\to-\frac{a^2\,(S\mu-S\mu 1)}{2\,(\kappa 1\mu+\mu 1)},\ f_2\to 0,\ a_2\to 0,\ b_2\to 0,\ m_2\to 0\right\}\right\}$$

RADIAL STRESSES ON THE INNER AND OUTER PARTS

$$\sigma_{\text{rrFinal}} = (\sigma_{\text{rrIni}})\ /.\ a_1\to\frac{(S+S\kappa 1)\,\mu}{4\,(2\mu-\mu 1+\kappa\mu 1)}\ /.\ b_1\to-\frac{S\mu+S\kappa 1\mu}{2\,(\kappa 1\mu+\mu 1)}\ /.\ a_2\to 0\ /.\ b_2\to 0$$

$$\frac{(S+S\kappa 1)\ \mu}{2\ (2\mu-\mu 1+\kappa\mu 1)}+\frac{(S\mu+S\kappa 1\mu)\ \mathrm{Cos}[2\theta]}{2\ (\kappa 1\mu+\mu 1)}$$

$$\sigma_{\text{rr1Final}} =$$

$$\mathrm{Expand}\left[(\sigma_{\text{rr1Ini}})\ /.\ f_3\to-\frac{a^4 S\mu-a^4 S\mu 1}{2\,(\kappa 1\mu+\mu 1)}\ /.\ m_1\to-\frac{a^2\,(S\mu-S\mu 1)}{2\,(\kappa 1\mu+\mu 1)}\ /.\ f_2\to 0\ /.\ m_2\to 0\ /.\ f_1\to\frac{a\,(-aS\mu+aS\kappa 1\mu+aS\mu 1-aS\kappa\mu 1)}{2\,(2\mu-\mu 1+\kappa\mu 1)}\right]$$

$$\frac{S}{2}-\frac{a^2 S\mu}{2r^2\,(2\mu-\mu 1+\kappa\mu 1)}+\frac{a^2 S\kappa 1\mu}{2r^2\,(2\mu-\mu 1+\kappa\mu 1)}+\frac{a^2 S\mu 1}{2r^2\,(2\mu-\mu 1+\kappa\mu 1)}-\frac{a^2 S\kappa\mu 1}{2r^2\,(2\mu-\mu 1+\kappa\mu 1)}+\frac{1}{2}\,S\,\mathrm{Cos}[2\theta]-\frac{3a^4 S\mu\,\mathrm{Cos}[2\theta]}{2r^4\,(\kappa 1\mu+\mu 1)}+\frac{2a^2 S\mu\,\mathrm{Cos}[2\theta]}{r^2\,(\kappa 1\mu+\mu 1)}+\frac{3a^4 S\mu 1\,\mathrm{Cos}[2\theta]}{2r^4\,(\kappa 1\mu+\mu 1)}-\frac{2a^2 S\mu 1\,\mathrm{Cos}[2\theta]}{r^2\,(\kappa 1\mu+\mu 1)}$$

SHEAR STRESSES ON THE INNER AND OUTER PART

$$\tau_{\text{r}\theta\text{Final}} = (\tau_{\text{r}\theta\text{Ini}})\ /.\ b_1\to-\frac{S\mu+S\kappa 1\mu}{2\,(\kappa 1\mu+\mu 1)}\ /.\ a_2\to 0\ /.\ b_2\to 0$$

$$-\frac{(S\mu+S\kappa 1\mu)\ \mathrm{Sin}[2\theta]}{2\ (\kappa 1\mu+\mu 1)}$$

$\tau_{r\theta 1Final}$ =

$$\text{Expand}\left[\tau_{r\theta 1Ini} /.\ f_3 \to -\frac{a^4 S\mu - a^4 S\mu 1}{2(\kappa 1\mu + \mu 1)} /.\ m_1 \to -\frac{a^2 (S\mu - S\mu 1)}{2(\kappa 1\mu + \mu 1)} /.\ f_2 \to 0 /.\ m_2 \to 0\right]$$

$$-\frac{1}{2} S \operatorname{Sin}[2\theta] - \frac{3a^4 S\mu \operatorname{Sin}[2\theta]}{2r^4 (\kappa 1\mu + \mu 1)} + \frac{a^2 S\mu \operatorname{Sin}[2\theta]}{r^2 (\kappa 1\mu + \mu 1)} + \frac{3a^4 S\mu 1 \operatorname{Sin}[2\theta]}{2r^4 (\kappa 1\mu + \mu 1)} - \frac{a^2 S\mu 1 \operatorname{Sin}[2\theta]}{r^2 (\kappa 1\mu + \mu 1)}$$

ANGULAR STRESSES ON THE INNER AND OUTER PARTS

$\sigma_{\theta\theta Final}$ =

$$\text{FullSimplify}\left[\sigma_{\theta\theta Ini} /.\ a_1 \to \frac{(S + S\kappa 1)\mu}{4(2\mu - \mu 1 + \kappa\mu 1)} /.\ b_1 \to -\frac{S\mu + S\kappa 1\mu}{2(\kappa 1\mu + \mu 1)} /.\ a_2 \to 0 /.\ b_2 \to 0\right]$$

$$\frac{1}{2} S (1 + \kappa 1) \mu \left(\frac{1}{2\mu + (-1 + \kappa)\mu 1} - \frac{\operatorname{Cos}[2\theta]}{\kappa 1\mu + \mu 1}\right)$$

$\sigma_{\theta\theta 1Final}$ =

$$\text{Expand}\left[\sigma_{\theta\theta 1Ini} /.\ f_1 \to -\frac{a^2 S}{2} + \frac{a^2 (S + S\kappa 1)\mu}{2(2\mu - \mu 1 + \kappa\mu 1)} /.\ f_3 \to -\frac{a^4 S\mu - a^4 S\mu 1}{2(\kappa 1\mu + \mu 1)} /.\ m_2 \to 0 /.\ f_2 \to 0\right]$$

$$\frac{S}{2} + \frac{a^2 S}{2r^2} - \frac{a^2 S\mu}{2r^2 (2\mu - \mu 1 + \kappa\mu 1)} - \frac{a^2 S\kappa 1\mu}{2r^2 (2\mu - \mu 1 + \kappa\mu 1)} - \frac{1}{2} S \operatorname{Cos}[2\theta] + \frac{3a^4 S\mu \operatorname{Cos}[2\theta]}{2r^4 (\kappa 1\mu + \mu 1)} - \frac{3a^4 S\mu 1 \operatorname{Cos}[2\theta]}{2r^4 (\kappa 1\mu + \mu 1)}$$

RADIAL DISPLACEMENTS FOR THE INNER DISC AND THE OUTER PLATE

$$u_{rFinal} = \text{FullSimplify}\left[u_{rIni} /.\ a_0 \to \frac{b_0}{\kappa} /.\ b_1 \to 0 /.\ a_1 \to \frac{(S + S\kappa 1)\mu}{4(2\mu - \mu 1 + \kappa\mu 1)},\ b_1 \to -\frac{S\mu + S\kappa 1\mu}{2(\kappa 1\mu + \mu 1)} /.\ a_2 \to 0 /.\ b_2 \to 0\right]$$

$$\frac{r\,(S\,(-1+\kappa)\,(1+\kappa 1)\,\mu + 4\,r\,(2\mu + (-1+\kappa)\,\mu 1)\,((-2+\kappa)\,\mathrm{Cos}[\theta]\,a_2 - \mathrm{Cos}[3\theta]\,b_2))}{8\,\mu\,(2\mu + (-1+\kappa)\,\mu 1)}$$

$$u_{r1Final} = \mathbf{FullSimplify}\Big[u_{r1Ini}\ /.\ f_1 \to -\frac{a^2\,S}{2} + \frac{a^2\,(S + S\,\kappa 1)\,\mu}{2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)}\ /.\ f_3 \to -\frac{a^4\,S\,\mu - a^4\,S\,\mu 1}{2\,(\kappa 1\,\mu + \mu 1)}\ /.\ m_1 \to -\frac{a^2\,(S\,\mu - S\,\mu 1)}{2\,(\kappa 1\,\mu + \mu 1)}\ /.\ f_2 \to 0\ /.\ m_2 \to 0\Big]$$

$$\frac{1}{8\,r^3\,\mu 1}\left(S\left(\frac{r^2\,(r^2\,(-1+\kappa 1)\,(2\,\mu + (-1+\kappa)\,\mu 1) - 2\,a^2\,((-1+\kappa 1)\,\mu + \mu 1 - \kappa\,\mu 1))}{2\,\mu + (-1+\kappa)\,\mu 1} + \frac{2\,(a^4\,(\mu - \mu 1) + a^2\,r^2\,(1+\kappa 1)\,(-\mu + \mu 1) + r^4\,(\kappa 1\,\mu + \mu 1))\,\mathrm{Cos}[2\,\theta]}{\kappa 1\,\mu + \mu 1}\right)\right)$$

ANGULAR DISPLACEMENTS FOR THE INNER DISC AND THE OUTER PLATE

$$u_{\theta Final} = \mathbf{FullSimplify}\Big[u_{\theta Ini}\ /.\ b_0 \to \kappa\,a_0\ /.\ b_1 \to -\frac{S\,\mu + S\,\kappa 1\,\mu}{2\,(\kappa 1\,\mu + \mu 1)}\ /.\ a_2 \to 0\ /.\ b_2 \to 0\Big]$$

$$-\frac{r\,S\,(1+\kappa 1)\,\mathrm{Sin}[2\,\theta]}{4\,(\kappa 1\,\mu + \mu 1)}$$

$$u_{\theta 1Final} = \mathbf{FullSimplify}\Big[u_{\theta 1Ini}\ /.\ f_3 \to -\frac{a^4\,S\,\mu - a^4\,S\,\mu 1}{2\,(\kappa 1\,\mu + \mu 1)}\ /.\ m_1 \to -\frac{a^2\,(S\,\mu - S\,\mu 1)}{2\,(\kappa 1\,\mu + \mu 1)}\ /.\ f_2 \to 0\ /.\ m_2 \to 0\Big]$$

$$-\frac{S\,(a^4\,(-\mu + \mu 1) + a^2\,r^2\,(-1+\kappa 1)\,(-\mu + \mu 1) + r^4\,(\kappa 1\,\mu + \mu 1))\,\mathrm{Sin}[2\,\theta]}{4\,r^3\,\mu 1\,(\kappa 1\,\mu + \mu 1)}$$

VERIFICATION OF THE EQUILIBRIUM EQUATIONS

```
Simplify[D[σrrFinal, r] +D[τrθFinal, θ]*1/r + (σrrFinal - σθθFinal)/r ]
```

0

```
Simplify[D[σrr1Final, r] + D[τrθ1Final, θ]*1/r + (σrr1Final - σθθ1Final)/r]
```

0

```
Simplify[D[τrθFinal, r] + D[σθθFinal, θ]*1/r + (2*τrθFinal/r) ]
```

0

```
Simplify[D[τrθ1Final, r] + D[σθθ1Final, θ]*1/r + 2 *τrθ1Final/r]
```

0

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BIOGRAPHICAL INFORMATION

Dharshini, finished her Bachelor's in Mechanical Engineering from Vijaynagar Engineering college, Bellary, Karnataka, India. She worked on a project in Hindustan Aeronautics Limited for three months and realized her interest in design and analyses. In January 2004 she left her homeland to pursue her dream. She graduated as a Master of Science in Mechanical Engineering in December 2005 from the University of Texas at Arlington, with a great deal of confidence and knowledge and the satisfaction of being a step closer to her goal.