A STUDY ON THE B FAMILY OF SHALLOW WATER WAVE EQUATIONS

by

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ABSTRACT

A STUDY ON THE B FAMILY OF SHALLOW WATER WAVE EQUATIONS

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In this dissertation we study b family of shallow water wave equations which include classical Korteweg-de Vries, Camassa-Holm and Degasperis-Procesi equations. We first establish the models of the Camassa-Holm and Degasperis-Procesi equations, deriving them from the shallow water wave argument and then compare a large class of properties relating to the two equations. Then we consider the b family equation as the parent equation and derive the above mentioned two equations as special cases of the b-family as well as the classical KdV equation. Next we establish results of local well-posedness using Kato's semigroup theory,global existence and blow up solutions under certain special initial profiles(periodic) and relate those to periodic b-family equations. **Keywords:**Camassa-Holm(CH) and Degasperis-Procesi(DP) equations ; periodic b-family ;blow-up;local existence;global existence.

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CHAPTER 1

INTRODUCTION

1.1 Motivation for this Study

The main topic of this project is the study of nonlinear differential equations in the family

$$u_t + cu_x + \gamma u_{xxx} - \alpha^2 u_{xxt} = \left(c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx}\right)_x, \tag{1.1.1}$$

where constants $c, \gamma, \alpha, c_1, c_2, c_3 \in \mathbb{R}$. The primary equations of interest here are $\alpha = c_2 = c_3 = 0$, the Korteweg-de Vries equation; $c_1 = -\dfrac{3c_3}{2\alpha^2}, c_2 = \dfrac{c_3}{2}$, the Camassa–Holm equation; and $c_1 = -\dfrac{2c_3}{\alpha^2}, c_2 = c_3$ the Degasperis–Procesi equation [5]. As these completely integrable equations admit different types of solutions, have differing conservation laws, etc., we thus have the motivation to study and classify their differences as well as their similarities. To do this, we first derive a form of equation (1.1.1) using the variational principle applied to a shallow water wave environment. Specifically, we will start with Bernoulli's equation

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} + gz = 0, \tag{1.1.2}$$

where ϕ is the velocity potential, p, ρ the pressure acting on the fluid and density respectively, g the acceleration due to gravity, and show that by minimizing the pressure density, i.e. the integral of (1.1.2), we obtain the following shallow water wave equations:

$$\phi_{xx} + \phi_{zz} = 0, \ z \in (-h, \eta), \qquad x \in \mathbb{R}, \tag{1.1.3}$$

$$\eta_t + \eta_x \phi_x - \phi_z = 0 \quad \text{on } z = \eta, \tag{1.1.4}$$

$$\phi_z = 0 \quad \text{on } z = -h, \tag{1.1.5}$$

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = 0 \quad \text{on } z = \eta. \tag{1.1.6}$$

After obtaining these equations, we demonstrate how they lead to the equation of elevation for a free surface in a shallow water environment, i.e. the Korteweg-de Vries equation

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{xxx} = 0. \tag{1.1.7}$$

Of course, what is of interest is that this equation is derived by a linear perturbation of the Boussinesq system of equations. Hence, it follows that a quadratic perturbation of the Boussinesq system of equations would lead to

$$\underbrace{[\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{xxx}]}_{\text{KdV}} - \frac{3}{8}\alpha^2\eta^2\eta_x + \alpha\beta\left(\frac{23}{24}\eta_x\eta_{xx} + \frac{5}{12}\eta\eta_{xxx}\right) +$$

$$\beta^2\frac{19}{360}\eta_{xxxxx} = 0. \tag{1.1.8}$$

This equation of perturbation then becomes the starting point for the derivation of the so-called b-family of equations

$$\underbrace{m_t}_{\text{Evolution}} + c\underbrace{\left(m_x + \frac{1}{6}h^2 u_{xxx}\right)}_{\text{Dispersion}} + \underbrace{(um_x + bu_x m)}_{\text{Nonlinearity}} = 0, \tag{1.1.9}$$

which yields for $b = 2$ the canonical Camassa–Holm equation

$$\underbrace{m_t}_{\text{Evolution}} + c\underbrace{\left(m_x + \frac{1}{6}h^2 u_{xxx}\right)}_{\text{Dispersion}} + \underbrace{(um_x + 2u_x m)}_{\text{Nonlinearity}} = 0, \tag{1.1.10}$$

and for $b = 3$ the canonical Desgaperis-Procesi equation

$$\underbrace{m_t}_{\text{Evolution}} + \underbrace{c\left(m_x + \frac{1}{6}h^2 u_{xxx}\right)}_{\text{Dispersion}} + \underbrace{(um_x + 3u_x m)}_{\text{Nonlinearity}} = 0. \qquad (1.1.11)$$

Henceforth, we are equipped to study the unique and common attributes of each of these equations. These include, but are not limited to, the soliton solutions which all three equations admit but which nevertheless present differences, i.e. the Korteweg-de Vries possessing smooth soliton solutions and the Camassa–Holm and Degasperis–Procesi possessing so-called *weak* soliton solutions, the phenomenon of wave-breaking of each, noting here that the classical Korteweg-de Vries equation possesses no wave-breaking solutions [11] and the comparison of initial data. We show that the Korteweg-de Vries equation depends on both the size and smoothness of the initial data while the Camassa–Holm and Degasperis–Procesi equation depends only on the shape of the initial data. We then venture into the more subtle similarities and differences that exist between the Camassa–Holm and Degasperis–Procesi equations. These include the soliton solutions, the isospectral eigenvalue problem, and a comparison of the conservation laws of each equation. With a general outline of the study explicitly set forth, we now turn to some of the early developments on the study of nonlinear differential equations, particularly the Korteweg-de Vries equation, and then provide recent developments to further motivate the study of the Cammasa-Holm and Degasperis–Procesi equations.

1.2 Early Developments

In this section we note the historical development of nonlinear shallow water wave theory following Ablowitz and Clarkson [18].

"Solitons", or solitary waves, were first observed by J. Scott Russell in 1834 while he was riding on horseback along a narrow canal near Edinburgh, Scotland. This "...

solitary elevation ..." or "... Wave of Translation ..." as he called it, would become an extensive project for Russell as he conducted experiments and careful study on the phenomenon. Specifically, two results of Russell's are of importance to motivate the development of the nonlinear partial differential equations for modeling fluids, etc., i.e.

1. That he observed solitary waves and hence deduced their existence.

2. That he found the speed of propagation c of the solitary wave in a channel of depth h to be $c = \sqrt{g(h + \alpha)}$, where α is the amplitude of the wave and g the force due to gravity.

1.2.1 Explanation

Dispersion and non-linearity can interact to produce permanent and localized wave forms. Consider a pulse of light traveling in glass. This pulse can be thought of as consisting of light of several different frequencies. Since glass shows dispersion, these different frequencies will travel at different speeds and the shape of the pulse will therefore change over time. However, there is also the non-linear Kerr effect: the speed of light of a given frequency depends on the light's amplitude or strength. If the pulse has just the right shape, the Kerr effect will exactly cancel the dispersion effect, and the pulse's shape won't change over time: a soliton.

Some types of tidal bore, a wave phenomenon of a few rivers including the River Severn, are 'undular': a wavefront followed by a train of solitons. Other solitons occur as the undersea internal waves, initiated by seabed topography, that propagate on the oceanic pycnocline. Atmospheric solitons also exist, such as the Morning Glory Cloud of the Gulf of Carpentaria, where pressure solitons travelling in a temperature inversion layer produce vast linear roll clouds. The recent and not widely accepted soliton model in neuroscience proposes to explain the signal conduction within neurons as pressure solitons.

A topological soliton, or topological defect, is any solution of a set of partial differential equations that is stable against decay to the "trivial solution." Soliton stability is due to topological constraints, rather than integrability of the field equations. The constraints arise almost always because the differential equations must obey a set of boundary conditions, and the boundary has a non-trivial homotopy group, preserved by the differential equations. Thus, the differential equation solutions can be classified into homotopy classes. There is no continuous transformation that will map a solution in one homotopy class to another. The solutions are truly distinct, and maintain their integrity, even in the face of extremely powerful forces. Examples of topological solitons include the screw dislocation in a crystalline lattice, the Dirac string and the magnetic monopole in electromagnetism, the Skyrmion and the Wess-Zumino-Witten model in quantum field theory, and cosmic strings and domain walls in cosmology.

1.2.1.1 History

In 1834, John Scott Russell described his wave of translation. The discovery was described here in Russell's own words:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the

month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".

Russell's experimental work seemed at odds with the Isaac Newton and Daniel Bernoulli's theories of hydrodynamics. George Biddell Airy and George Gabriel Stokes had difficulty accepting Russell's experimental observations because they could not be explained by linear water wave theory. His contemporaries spent some time attempting to extend the theory but it would take until 1895 before Diederik Korteweg and Gustav de Vries provided the theoretical explanation.

However, the existence of this type of wave and other results were hotly debated. Airy came to the conclusion that Russell's wave could not exist. Meanwhile, Stokes used the right equation, but drew the wrong conclusions. However, it was Boussinesq and Rayleigh who independently obtained approximate descriptions, with Boussinesq *deriving* a one-dimensional nonlinear evolution equation to obtain his results. Their investigations provoked a controversy as to whether the inviscid equations of water waves would possess such solitary solutions. Finally, the problem was laid to rest by Korteweg and de Vries in 1895. They derived a nonlinear evolution equation governing long one dimensional, small amplitude, surface gravity waves propagating in shallow water. This now famous Korteweg-de Vries equation can be written from equation (1.1.1) in the normalized form

$$u_t + uu_x - u_{xxx} = 0. \tag{1.2.1}$$

Despite the early development of this equation it was not until 1960 that any new application was found. Study of the Korteweg-de Vries equation then proliferated and arose in a number of different physical contexts, from stratified internal waves to plasma physics. Moreover, there were numerous modifications to the original Korteweg-de Vries equation, e.g. KdV5, etc. which continue to be studied extensively. However, our story in essence begins with Korteweg-de Vries and ends with the so-called b-Family of equations.

1.3 Recent Developments

1.3.1 The Camassa–Holm Equation

The Camassa–Holm equation can be written from equation (1.1.1) in the normalized form

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0. \qquad (1.3.1)$$

The origin of equation (1.3.1) can by traced back to an article written in 1981 by Fuchssteiner and Fokas [8] where it appears as one member of a whole family of bi-Hamiltonian equations generated by the method of recursion operator. However, in 1993 Camassa and Holm derived equation (1.3.1) in the context of shallow water waves [19]. Specifically, they derived equation (1.3.1) as a model for unidirectional water wave propagation in shallow water with the solution u representing the height of the water's free surface above a flat bottom. Moreover, the *relevance* of this equation as a model for shallow water waves was investigated by Johnson [21].

1.3.1.1 The Mathematical Structure of the Camassa–Holm Equation

The Camassa–Holm equation has a host of interesting mathematical properties. It is bi-Hamiltonian, that is, it possesses two unique and yet compatible Hamiltonian structures [19]. Thus it possesses an infinite number of conservation laws. Equation (1.3.1) also admits a Lax-pair and is formally integrable by means of scattering and inverse scattering techniques. It turns out that as with the Korteweg-de Vries equation, the Camassa–Holm equation admits soliton solutions. Moreover, the Camassa–Holm equation possesses a type of soliton solution which due to their shape are known as peaked solitons or peakons. A single peakon solution is given by

$$u(t, x) = ce^{|x-ct|}, \qquad (1.3.2)$$

the traveling speed thus being equal to the height at its peak. However, of special interest in this project are the analyses of local existence, global existence and blow-up of solutions.

1.3.1.2   Local Existence and Blow-up of Solutions

Local existence of solutions has been studied in [20] with the help of Kato's theory and in [25] using regularization techniques. Utilizing Kato's theory, it was demonstrated that for $u_0 \in H^s(\mathbb{R}), s > \dfrac{3}{2}$, there is a unique solution u with

$$u \in C([0, T), H^s(\mathbb{R})) \cap C^1([0, T), H^{s-1}(\mathbb{R})), \tag{1.3.3}$$

where $T > 0$ depends only on the size of the initial data u_0, i.e., $\|u_0\|_{H^s(\mathbb{R})}$. As well as local existence of solutions there is the phenomenon of blow-up, e.g. the Camassa–Holm equation has solutions that exist for only a finite time. As an example, consider the following. Let T be the maximal time of existence, i.e. the time where the solution eventually looses its regularity. Then

$$\liminf_{t \to T} u_x = -\infty. \tag{1.3.4}$$

In others words, there exists a time when the profile of the solution steepens gradually, ultimately leading to a vertical slope, that is, *wave breaking* for a bounded solution. This fact was proved by Constantin and Escher [3].

1.3.1.3   Global Existence of Solutions

A particularly major advance in this direction was accomplished by Constantin and Escher [4]. They proved for initial data in the energy space $H^1(\mathbb{R})$ and the potential being a positive regular Borel measure on $\mathbb{R}$ with bounded total variation, equation (1.3.1) has a unique global weak solution.

1.3.2 The Degasperis–Procesi Equation

The Degasperis–Procesi equation can be written from equation (1.1.1) in the normalized form

$$u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0. \tag{1.3.5}$$

This equation was originally *isolated* by Degasperis and Procesi [6] as one equation in a family of three of the form (1.1.1), i.e. equation (1.3.5). It, like the Camassa–Holm equation, can be regarded as a model for nonlinear shallow water dynamics with asymptotic accuracy equal to the Camassa–Holm equation. Dullin, Gottwald and Holm [13] demonstrated that equation (1.3.5) can be obtained as a model for unidirectional water wave propagation in shallow water by an appropriate Kodama transformation with the solution u representing the height of the water's free surface above a flat bottom.

1.3.2.1 The Mathematical Structure of the Degasperis–Procesi Equation

The Degasperis–Procesi equation has a host of interesting mathematical properties. It is bi-Hamiltonian, that is, it possesses two unique and yet compatible Hamiltonian structures [5]. Thus it possesses a infinite number of conservation laws. Equation (1.3.5) admits a Lax-pair and is also formally integrable by means of scattering and inverse scattering techniques. It turns out that as with the Korteweg-de Vries equation, the Degasperis–Procesi equation admits of soliton solutions. Moreover, the Degasperis–Procesi equation possesses a type of soliton solution which due to their shape are known as peaked solitons or peakons. A single peakon solution is given by

$$u(t, x) = ce^{|x-ct|}, \tag{1.3.6}$$

the traveling speed thus being equal to the height at its peak. However, unlike the Camassa–Holm equation, the Degasperis–Procesi equation admits a type of solution known as a *shockpeakon*. A shockpeakon solution is given by

$$u(t, x) = -\frac{1}{t + k} \, \mathrm{sgn}(x) e^{-|x|}, \quad k > 0, \ x \in \mathbb{R}. \tag{1.3.7}$$

This is a major difference between the Camassa–Holm and Degasperis–Procesi equations and will be exploited later in greater detail.

1.3.2.2 Local Existence and Blow-up of Solutions

Local existence of the Degasperis–Procesi equation was first demonstrated by Yin [28]. As with the Camassa–Holm equation we have the following, given $u_0 \in H^s(\mathbb{R}), \forall s > \frac{3}{2}$ there exists a maximal $T = T(u_0) > 0$ and an unique solution u to equation (1.3.5), such that

$$u \in C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R})). \tag{1.3.8}$$

Liu et al. [26] showed using the work of Constantin [1] that both the Camassa–Holm and Degasperis–Procesi equations have a first derivative that approaches minus infinity in a finite time, i.e. given $u_0 \in H^s(\mathbb{R}), \forall s > \frac{3}{2}$, blow-up of the solution u in finite time $T < +\infty$ occurs if and only if

$$\lim_{t \to T} \inf[\inf[u_x(t, x)]] = -\infty. \tag{1.3.9}$$

In others words, there exists a time when the profile of the solution steepens gradually, ultimately leading to a vertical slope or gradient catastrophe.

1.3.2.3 Global Existence of Solutions

The first major step in this direction was accomplished by Liu et al. [10], [26]. They presented two major results, namely the existence of unique global strong solutions and unique global weak solutions. They proved that for $u_0 \in H^s(\mathbb{R}), s > \dfrac{3}{2}$,

$$\begin{cases} m_0 \leq 0 \text{ if } x \leq x_0, \\ \\ m_0 \geq 0 \text{ if } x \geq x_0, \end{cases} \tag{1.3.10}$$

equation (1.3.5) has a unique *global strong solution*

$$u \in C([0, \infty); H^s(\mathbb{R})) \cap C^1([0, \infty); H^{s-1}(\mathbb{R})), \tag{1.3.11}$$

and that for $u_0 \in H^1(\mathbb{R})$,

$$\begin{cases} m_0 \leq 0 \text{ if } x \in (-\infty, x_0), \\ \\ m_0 \geq 0 \text{ if } x \in (x_0, \infty), \end{cases} \tag{1.3.12}$$

equation (1.3.5) has a unique *global weak solution*

$$u \in W_{\text{loc}}^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L_{\text{loc}}^\infty(\mathbb{R}_+; H^1(\mathbb{R})). \tag{1.3.13}$$

CHAPTER 2

THE B-FAMILY OF SHALLOW WATER WAVE EQUATIONS

2.1 Preliminaries

In this section we discuss the preliminaries following Debnath [17] necessary to derive the b-Family equations from the water wave equations.

2.1.1 The Inviscid Fluid Equations of Motion

It is well known that if a fluid is incompressible and has constant density then the fundamental Euler equations for water waves are

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p + g\mathbf{k} = 0, \tag{2.1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2.1.2}$$

where $\mathbf{u}$ is the velocity field, p, ρ the pressure acting on the fluid and density respectively, g the acceleration due to gravity and $\mathbf{k}$ the unit vector in the positive z direction. Moreover, if $\nabla \times \mathbf{u} = 0$, i.e. the fluid is irrotational, there exists a single valued velocity potential ϕ such that

$$\nabla\phi = \mathbf{u}. \tag{2.1.3}$$

Hence, $\nabla \cdot \mathbf{u} = 0$ reduces to the Laplace equation $\nabla^2\phi = 0$. Now if we apply the formula $(\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{2}\nabla u^2 - \mathbf{u} \times (\nabla \times \mathbf{u})$ to (2.1.1) along with the fact that $\nabla \times \mathbf{u} = 0$ and $\nabla\phi = \mathbf{u}$ we have after integration with respect to the spatial variables and without loss of generality the Bernoulli equation

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} + gz = 0. \tag{2.1.4}$$

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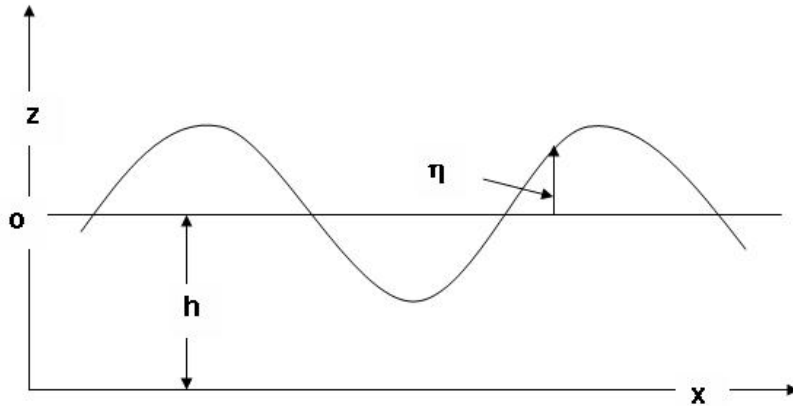
Figure 2.1. A fundamental surface gravity wave model with flat bottom.

2.1.2 The Variational Derivation of the Classical Water Wave Equations

In this section the equations of irrotational motion of an inviscid, incompressible, homogeneous fluid with a free surface are shown to arise naturally from the variational principle following Debnath [17] and Luke [16]. Let $\Omega \subseteq (t, x)$ be an arbitrary domain of the (t, x) plane. Also, let h be the constant undisturbed fluid depth and $\eta = \eta(t, x)$ the free surface height (Figure 2.1).

We now formulate the variational principle for two-dimensional water waves as

$$\delta \int_\Omega \mathfrak{L}\, d\Omega = 0, \tag{2.1.5}$$

where the Lagrangian $\mathfrak{L} = \mathfrak{L}(\eta, \phi_x, \phi_z, \phi_t)$ is assumed to be equal to the pressure density and hence defined by

$$\mathfrak{L} = -\rho \int_{-h}^{\eta} \left[\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + gz \right] dz, \tag{2.1.6}$$

and $\phi(t, x, z)$ is the velocity potential of an unbounded fluid lying between a rigid bottom $z = -h$, and the free surface $z = \eta$ (Figure 2.1). Note that the functions $\eta(t, x)$ and

$\phi(t, x, z)$ are allowed to vary provided $\delta\eta = 0$ and $\delta\phi = 0$ on $\partial\Omega$. Then using the standard calculus of variations we have from (2.1.5)

$$\int_\Omega \left[\frac{\delta\mathcal{L}}{\delta\eta}\delta\eta + \frac{\delta\mathcal{L}}{\delta\phi_x}\delta\phi_x + \frac{\delta\mathcal{L}}{\delta\phi_z}\delta\phi_z + \frac{\delta\mathcal{L}}{\delta\phi_t}\delta\phi_t \right] d\Omega = 0, \tag{2.1.7}$$

and hence,

$$\begin{aligned}
-\delta \int_\Omega \frac{\mathcal{L}}{\rho} d\Omega &= \delta \int_\Omega \left(\int_{-h}^{\eta} (\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + gz) \, dz \right) d\Omega \\
&= \int_\Omega \left(\delta \int_{-h}^{\eta} (\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + gz) \, dz \right) d\Omega \\
&= \int_\Omega \left[[\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + gz)]_{z=\eta} \, \delta\eta + \int_{-h}^{\eta} (\phi_x\delta\phi_x + \phi_z\delta\phi_z + \delta\phi_t) \, dz \right] d\Omega = 0,
\end{aligned} \tag{2.1.8}$$

which by Green's formula yields

$$\begin{aligned}
\int_\Omega \left[\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + gz) \right]_{z=\eta} \delta\eta \, d\Omega \\
- \int_\Omega [(\eta_t + \eta_x\phi_x - \phi_z)\delta\phi]_{z=\eta} \, d\Omega \\
- \int_\Omega [\phi_z\delta\phi]_{z=-h} \, d\Omega \\
- \int_\Omega \left[\int_{-h}^{\eta} (\phi_{xx} + \phi_{zz})\delta\phi dz \right] d\Omega = 0.
\end{aligned} \tag{2.1.9}$$

As the variation in ϕ and η are arbitrarily chosen, it follows that for (2.1.9) to vanish identically we must have the following:

$$\phi_{xx} + \phi_{zz} = 0, \; z \in (-h, \eta), \qquad x \in \mathbb{R}, \tag{2.1.10}$$

$$\eta_t + \eta_x\phi_x - \phi_z = 0 \quad \text{on } z = \eta, \tag{2.1.11}$$

$$\phi_z = 0 \quad \text{on } z = -h, \tag{2.1.12}$$

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = 0 \quad \text{on } z = \eta. \tag{2.1.13}$$

We obtain these equations by applying the following argument to equation (2.1.9). Equation (2.1.10) is obtained by choosing $\delta\eta = \delta\phi = 0$ on $z = \eta$ and $z = -h$ and applying

the variational argument for $z \in (-h, \eta)$, e.g. $\delta\phi$ is arbitrary in the space $(-h, \eta)$ which implies (2.1.10). Equations (2.1.11), (2.1.12) arise by setting $\delta\eta = 0$ and considering appropriate choices of positive $\delta\phi > 0$ on $z = \eta$ and $\delta\phi = 0$ on $z = -h$ for (2.1.11) and again $\delta\phi > 0$ on $z = -h$ and $\delta\phi = 0$ on $z = \eta$ for (2.1.12). Once these equations are eliminated we consider $\delta\eta$ as arbitrary and apply the variational argument to obtain (2.1.13) on $z = \eta$. Hence, we conclude (2.1.10)–(2.1.13) represent the well known nonlinear system of equations for classical water waves. Note that in terms of continuum mechanics which has the basic assumption that the motion of a fluid can be described in terms of a topological deformation that depends on time, there exists a surface function $S(t, x, z)$ such that

$$S(t, x, z) = 0. \tag{2.1.14}$$

Utilizing this formulation forces us to impose a kinematic and dynamic boundary condition at the free surface of elevation . However, for the variational problem these arise naturally, i.e., (2.1.13) the kinematic boundary condition and (2.1.12) the dynamic boundary condition. Thus the variational principle generates the governing equations of free surface flow by simply minimizing the pressure density.

2.2 The Equation of Elevation η

In this section we derive the equation of elevation η to linear order following Debnath [17] from the nondimensional equations of classical water waves and then write down the equation of elevation to quadratic order that will be of primary importance in the derivation of the b-Family equations.
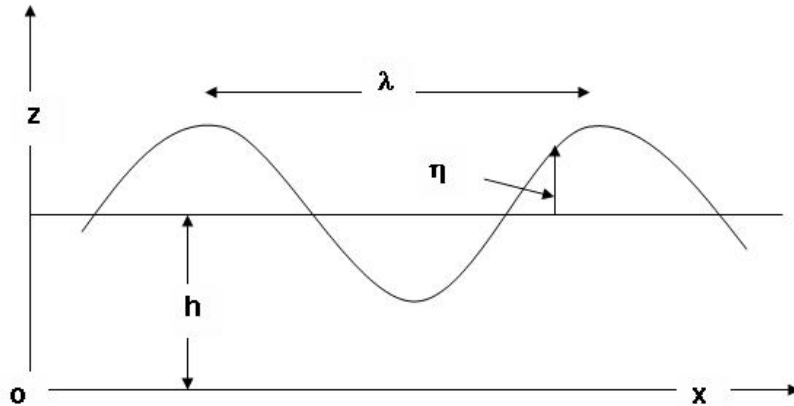
Figure 2.2. A fundamental shallow water wave model.

2.2.1   The Nondimensional Equations of Shallow Water Waves

We consider here the model of wave propagation on the surface of the water. The simplest possible scenario is considered, i.e. the fluid is inviscid and of constant depth h, stationary in its undisturbed state, of constant density ρ and without surface tension. The free surface elevation equation above the undisturbed depth h is taken to be $z = \eta(t, x)$ such that the free surface is at $z = h + \eta$ and $z = 0$ the rigid horizontal bottom (Figure 2.2).

To obtain the dimensionless forms of (2.1.10)–(2.1.13) we introduce the following variables. Let λ be the wavelength of the wave under consideration and a its amplitude. Then the dimensionless variables are

$$\hat{x} = \frac{x}{\lambda}, \hat{z} = \frac{z}{h}, \hat{t} = \frac{ct}{\lambda}, \hat{\eta} = \frac{\eta}{a}, \hat{\phi} = \frac{\phi h}{ac\lambda}, \tag{2.2.1}$$

where $c = \sqrt{gh}$ is the shallow water wave speed. We now introduce two variables that will be used from this point forward in there current form. These are the fundamen-

tal parameters of nonlinear shallow water waves and are characterized in the following manner:

$$\alpha = \frac{a}{h}, \quad \beta = \left(\frac{h}{\lambda}\right)^2, \tag{2.2.2}$$

where we assume $\alpha, \beta \ll 1$ and $\alpha \geq \beta > \alpha^2 \geq \alpha\beta \geq \beta^2$. Utilizing the nondimensional variables and the previous parameters we can write the classical equations for water waves (2.1.10)–(2.1.13) as

$$\beta\phi_{xx} + \phi_{zz} = 0, \tag{2.2.3}$$

$$\eta_t + \alpha\eta_x\phi_x - \frac{1}{\beta}\phi_z = 0 \quad \text{on } z = 1 + \alpha\eta, \tag{2.2.4}$$

$$\eta + \phi_t + \frac{1}{2}(\alpha\phi_x^2 + \frac{\alpha}{\beta}\phi_z^2) = 0 \quad \text{on } z = 1 + \alpha\eta, \tag{2.2.5}$$

$$\phi_z = 0 \quad \text{on } z = 0. \tag{2.2.6}$$

Now, take α arbitrary and asymptotically expand ϕ in terms of β, i.e.

$$\phi = \sum_{n=0}^{\infty} \beta^n \phi_n = \phi_0 + \beta\phi_1 + \beta^2\phi_2 + \cdots \tag{2.2.7}$$

substituting into equations (2.2.3)–(2.2.6). The $O(1)$ term in (2.2.3) is

$$\phi_{0zz} = 0, \tag{2.2.8}$$

which when combined with equation (2.2.6) yields $\phi_{0z} = 0, \forall z$. This implies $\phi_0 = \phi_0(t, x)$ and that the horizontal velocity component is independent of z in the lowest order. As a consequence we let $u(t, x) = \phi_{0x}$. Next, consider the first and second order terms of (2.2.3), i.e.

$$\phi_{0xx} + \phi_{1zz} = 0, \tag{2.2.9}$$

$$\phi_{1xx} + \phi_{2zz} = 0. \tag{2.2.10}$$

Thus integrating (2.2.9) with respect to z, i.e.

$$\int [\phi_{0xx} + \phi_{1zz}]\, dz, \tag{2.2.11}$$

and utilizing the fact $u(t, x) = \phi_{0x}$ we obtain the following:

$$\phi_{1z} = -zu_x + C(t, x), \tag{2.2.12}$$

where the arbitrary function $C(t, x) = 0$ by (2.2.6). Hence, integrating (2.2.12) with respect to z and utilizing the fact $C(t, x) = 0$, i.e.

$$\int \phi_{1z} \, dz, \tag{2.2.13}$$

we have

$$\phi_1 = \frac{-z^2}{2} u_x, \tag{2.2.14}$$

such that $\phi_1 = 0$ at $z = 0$, and u is then the horizontal velocity component at the bottom boundary. We then substitute (2.2.14) into (2.2.9)–(2.2.10) and perform the same integration as in the previous step (keeping in mind that $\phi_z = 0$ at $z = 0$ to determine the arbitrary function) to obtain the following equations:

$$\phi_{2z} = \frac{1}{6} z^3 u_{xxx}, \tag{2.2.15}$$

$$\phi_2 = \frac{1}{24} z^4 u_{xxx}. \tag{2.2.16}$$

Consequently, considering the free surface boundary conditions retaining all terms of $O(\alpha), O(\beta)$ for (2.2.4) and $O(\alpha^2), O(\beta^2), O(\alpha\beta)$ for (2.2.5) we obtain

$$\phi_{0t} - \frac{1}{2}\beta u_{tx} + \eta + \frac{1}{2}\alpha u^2 = 0, \tag{2.2.17}$$

$$\beta\left[\eta_t + \alpha u\eta_x + (1 + \alpha\eta)u_x\right] = \frac{\beta^2}{6} u_{xxx}. \tag{2.2.18}$$

If we differentiate (2.2.17) with respect to x and simplify (2.2.18) we then have the following equations:

$$u_t + \eta_x + \alpha u u_x - \frac{1}{2}\beta u_{txx} = 0, \tag{2.2.19}$$

$$\eta_t + [(1 + \alpha\eta)u]_x - \frac{1}{6}\beta u_{xxx} = 0. \tag{2.2.20}$$

Equations (2.2.19)–(2.2.20) are known as the Boussinesq system of equations or more commonly the nondimensional shallow water wave equations.

2.2.2 Derivation of the Equation of Elevation

Starting with equations (2.2.19), (2.2.20) we will now demonstrate how to obtain the equation of elevation to $O(\alpha, \beta)$ inclusive and $\beta < 1$. We seek only solutions traveling to the right, i.e. in the positive x direction so that $u = u(x - ct)$ and $\eta = \eta(x - ct)$ for some $c > 0$. With the zero order terms of α and β, and $c = 1$, we consider the solution

$$u = \eta + \alpha F + \beta G, \tag{2.2.21}$$

where F, G are functions to be determined. If we now substitute (2.2.21) into (2.2.19) and (2.2.20) for the zero order terms of α, β and $u = \eta$ for the higher order terms we have

$$(\eta + \alpha F + \beta G)_t + \eta_x + \alpha \eta \eta_x - \frac{1}{2}\beta \eta_{txx} = 0, \tag{2.2.22}$$

$$\eta_t + [(1 + \alpha \eta)((\eta + \alpha F + \beta G))]_x - \frac{1}{6}\beta \eta_{xxx} = 0. \tag{2.2.23}$$

Expanding equations (2.2.22), (2.2.23) and ignoring quadratic order terms we have

$$\eta_t + \eta_x + \alpha F_t + \beta G_t + \alpha \eta \eta_x - \frac{1}{2}\beta \eta_{txx} = 0, \tag{2.2.24}$$

$$\eta_t + \eta_x + \alpha F_x + \beta G_x + 2\alpha \eta \eta_x - \frac{1}{6}\beta \eta_{xxx} = 0. \tag{2.2.25}$$

This requires for the nontrivial terms that

$$F_t - F_x = \eta \eta_x \Leftrightarrow F_t = F_x + \eta \eta_x, \tag{2.2.26}$$

$$G_t - G_x = \frac{1}{2}\eta_{txx} - \frac{1}{6}\eta_{xxx} \Leftrightarrow G_t = G_x + \frac{1}{2}\eta_{txx} - \frac{1}{6}\eta_{xxx}. \tag{2.2.27}$$

Thus, for zero order we require

$$\eta_t = -\eta_x, \quad F = -\frac{1}{4}\eta^2, \quad G = \frac{1}{3}\eta_{xx} = -\frac{1}{3}\eta_{xt}. \tag{2.2.28}$$

Using these results with the stipulation that the ratio $\dfrac{\alpha}{\beta} = \dfrac{a\lambda^2}{h^3} \ll 1$, we obtain from the two equations (2.2.22) and (2.2.23) a single equation of elevation in η of the form

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{xxx} = 0. \tag{2.2.29}$$

This is the famous Korteweg-de Vries equation as described in the introduction. It should be noted here that the ratio $\dfrac{\alpha}{\beta} = \dfrac{a\lambda^2}{h^3}$ is one of the most *rudimentary* parameters in the theory of nonlinear shallow water waves. It is known as the Ursell parameter and describes the linearity and dispersiveness of waves, e.g. large Ursell parameters correspond to nonlinear/nondispersive waves and small Ursell parameters to linear/dispersive waves. Lastly, if we were to continue the above outlined procedure up to $O(\alpha^2, \beta^2)$, i.e. start with the asymptotic expansion

$$u = \eta + \alpha F_1 + \beta F_2 + \alpha^2 F_3 + \alpha\beta F_4 + \beta^2 F_5, \tag{2.2.30}$$

we would end up with the following equation of elevation:

$$\underbrace{[\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{xxx}]}_{\text{KdV}} - \frac{3}{8}\alpha^2\eta^2\eta_x + \alpha\beta\left(\frac{23}{24}\eta_x\eta_{xx} + \frac{5}{12}\eta\eta_{xxx}\right) +$$

$$\beta^2\frac{19}{360}\eta_{xxxxx} = 0. \tag{2.2.31}$$

which along with the preceding is the starting point for the derivation of the b-Family of equations.

2.3 The Derivation of the b-Family of Equations

In this section we will describe from Dullin, Gottwald and Holm [14] the derivation of the b-Family of equations by first considering the derivation of the integrable Camassa–Holm equation from (2.2.31) and then by demonstrating that for arbitrary parameters of the Kodama transformation the procedure would lead to the b-Family of equations.

2.3.1 The Derivation of the Camassa–Holm Equation

We will describe the derivation of the Camassa–Holm equation

$$m_t + m_x + \frac{\alpha}{2}(um_x + 2u_x m) + \frac{1}{6}\beta u_{xxx} = 0, \tag{2.3.1}$$

where $u(t, x)$ is the fluid velocity, $m = u - \mu u_{xx}$ and the parameter μ is to be determined. It should be noted here that as $\mu \to 0$ we recover the Korteweg-de Vries equation as derived in the previous section. The result (2.3.1) will be asymptotically equal to (2.2.31), e.g. (2.3.1) $\sim$ (2.2.31) after we perform two steps. The first step, following Kodama [24], is to apply the near-identity transformation

$$\eta = \eta(u) = u + \alpha F[u] + \beta G[u], \tag{2.3.2}$$

to (2.2.31) and seek functionals $F[u], G[u]$ such that the transformed equation is the integrable equation (2.3.1) at order $O(\beta^2)$. The functionals $F[u], G[u]$ are chosen as in Kodama [24] such that $G[u] \propto u_{xx}$, e.g. $G[u]$ is proportional to u_{xx} and $F[u] \propto u^2 + u_x \partial_x^{-1}$, e.g. $F[u]$ is proportional to the linear combination of u^2 and $u_x \partial_x^{-1}$ where $u_x \partial_x^{-1}$ is a non-local term and the operator ∂_x^{-1} denotes integration. Thus with constants of proportionality a_1, a_2, a_3 and $F[u], G[u]$ we have the Kodama transformation

$$\eta = \eta(u) = u + \alpha(a_1 u^2 + a_2 u_x \partial_x^{-1} u) + \beta(a_3 u_{xx}). \tag{2.3.3}$$

We note here that terms of degree n start contributing at degree $n+1$ in the transformed equation which implies that only terms of order $O(\alpha, \beta)$ are needed in the transformation. Next, we substitute the Kodama transformation (2.3.3) into (2.2.31). This in turn leads to a series of terms in asymptotic order which when we expand the time derivatives to linear order generates a series of higher order terms leading to a series of equations of order $O(1), O(\alpha), O(\beta), O(\alpha^2), O(\alpha\beta), O(\beta^2)$ where we note here the $O(\alpha\beta)$ equation

$$A u_x u_{xx} + B u u_{xxx}, \tag{2.3.4}$$

with

$$A = \frac{23}{24} + \frac{1}{2}(2a_1 + a_2) - 3a_3, \quad B = \frac{5}{12} + \frac{1}{2}a_2. \tag{2.3.5}$$

This completes the first step of the derivation. In the next step of the derivation we apply the Helmholtz operator $\mathfrak{H} = 1 - \nu\beta\partial_x^2$ which introduces a free parameter ν and creates two more derivatives in x. However, terms of $O(\alpha^2)$ are not affected, i.e. these terms are proportional to $u^2 u_x$ and must be identically zero in order for (2.3.1) to appear. Secondly, as a result of the application of $\mathfrak{H}$ we recover the u_{xxt} term which was eliminated previously and is also of particular importance for the emergence of (2.3.1). To eliminate the equation associated with the order $O(\alpha^2)$ term we must have the coefficient of

$$\left(\frac{3}{2}a_1 + \frac{3}{4}a_2 - \frac{3}{8}\right) u^2 u_x, \tag{2.3.6}$$

equals zero, e.g.

$$\frac{3}{2}a_1 + \frac{3}{4}a_2 - \frac{3}{8} = 0, \tag{2.3.7}$$

which occurs only if a_1, a_2 satisfy

$$4a_1 + 2a_2 = 1. \tag{2.3.8}$$

Eliminating the fifth order derivative at order $O(\beta^2)$ determines the value of the free parameter $\nu = \frac{19}{60}$. It should be noted that the removal of the fifth order derivative was only made possible by introducing the parameter ν into the Helmholtz operator. Lastly, to insure equivalence to (2.3.1) we need the following relative coefficients to appear in the following ratios:

$$\left(A - \frac{9}{2}\nu\right) : \left(B - \frac{3}{2}\nu\right) = 2 : 1,$$
$$\left(\frac{3}{2}\nu\right) : \left(B - \frac{3}{2}\nu\right) = 3 : 1, \tag{2.3.9}$$

which implies $B = 2\nu$ and $A = \dfrac{11}{2}\nu$. With these conditions we finally obtain the result

$$u_t - \nu\beta u_{xxt} + u_x + \frac{3}{2}\alpha u u_x - \frac{1}{2}\alpha\beta\nu(uu_{xxx} + 2u_x u_{xx}) + \beta\left(\frac{1}{6} - \nu\right)u_{xxx} = 0. \quad (2.3.10)$$

With $a_1 = \dfrac{7}{20}, a_2 = -\dfrac{1}{5}$ and $a_3 = \dfrac{1}{30}$, if we let $m = u - \nu\beta u_{xx}$, e.g. $\mathfrak{H}u$, and perform some algebraic simplification we have

$$m_t + m_x + \frac{\alpha}{2}(um_x + 2u_x m) + \frac{1}{6}\beta u_{xxx} = 0. \quad (2.3.11)$$

In order to insure the correctness of our derivation we must transform the solutions u back to η using the inverse transform

$$u = u(\eta) = \eta + \alpha(b_1\eta^2 + b_2\eta_x\partial_x^{-1}\eta) + \beta(b_3\eta_{xx}), \quad (2.3.12)$$

noting the new coefficients b_1, b_2, b_3 are not necessarily those coefficients used in the original transformation. However, the transformation of (2.2.31) also involved the Helmholtz operator $\mathfrak{H}$. Therefore, it is not clear that it is sufficient to use (2.3.12) to recover (2.2.31). However, after application of (2.3.12) we find that (2.2.31) is recovered with a sign-reversal. Thus we conclude that (2.3.1) $\sim$ (2.2.31) to order $O(\beta^2)$. If we now return to dimensional units recalling the transforms (2.2.1), we have after simplification

$$m_t + cm_x + \frac{1}{2}\left(\frac{ac}{h}\right)(um_x + 2u_x m) + \frac{1}{6}ch^2 u_{xxx} = 0, \quad (2.3.13)$$

where m is now $m = u - \nu h^2 u_{xx}$. In addition, u and u_x have units $\dfrac{ac}{h}$ which when combined with a scaling of $u \mapsto 2u$ gives (2.3.1), i.e. the canonical Camassa–Holm equation

$$m_t + cm_x + (um_x + 2u_x m) + \frac{1}{6}ch^2 u_{xxx} = 0, \quad (2.3.14)$$

and after a rearrangement yields the finalized canonical Camassa–Holm equation

$$\underbrace{m_t}_{\text{Evolution}} + \underbrace{c\left(m_x + \frac{1}{6}h^2 u_{xxx}\right)}_{\text{Dispersion}} + \underbrace{(um_x + 2u_x m)}_{\text{Nonlinearity}} = 0. \quad (2.3.15)$$

Remark 2.3.1. Equation (2.3.15) was first derived by Camassa and Holm [19] using an asymptotic expansion directly in the Hamiltonian for Euler's equation in the shallow water wave regime. It was thus shown to be completely integrable using Painlevé analysis, bi-Hamiltonian and hence possess an infinite number of conservation laws. However, families of integrable equations similar to (2.3.15) were known to be derivable in the general context of hereditary symmetries by Fokas and Fuchstiener [8]. However, the explicit formulation of (2.3.15) was not derived physically as a shallow water wave equation nor its solutions studied before Camassa and Holm's paper [19]. However, recently equation (2.3.15) was derived as a shallow water wave equation by using asymptotic methods, i.e. Fokas and Liu [7], Dullin et al. [12] and Johnson [21]. Each of these methods used asymptotic expansion in the absence of surface tension.

Let us now examine some of the individual components of (2.3.15). We will not consider the evolution term here. However, we will consider the dispersion term. Since the dispersion relation for equation (2.3.15), e.g.

$$\frac{\omega}{\kappa} = c - \frac{1}{6}\left[\frac{c(h\kappa)^2}{1 + \nu(h\kappa)^2}\right], \tag{2.3.16}$$

is real and has second derivative nonzero we conclude that equation (2.3.15) is dispersive. However, if $c \to 0$ then the dispersive term is eliminated and we obtain the so-called Camassa–Holm peaked soliton or peakon equation

$$\underbrace{m_t}_{\text{Evolution}} + \underbrace{um_x}_{\text{Convection}} + \underbrace{2u_x m}_{\text{Stretching}} = 0, \tag{2.3.17}$$

where we note the convection and stretching terms of the nonlinear part. We also note here that the dispersion relation for equation (2.3.15) remains unaffected by the Kodama transformation, e.g. is invariant under the Kodama transformation. To see this, one may observe that linear terms in η map to linear terms in $u \in C^\infty(\mathbb{R}^n)$ and nonlinear terms to nonlinear terms. Therefore we need only consider the linear terms in proving the

invariance of the linear dispersion relation under the Kodama transformation. Consider the transformation $\eta = u + \varepsilon L(u)$ where L is a linear differential operator with constant coefficients, $\eta_t = M(\eta)$ the linear equation to be transformed and M a second linear differential operator. To first order, we then have $u_t = M(u)$ and the full transformation gives

$$u_t + \varepsilon L(u_t) = M(u + \varepsilon L(u)). \tag{2.3.18}$$

Now, eliminate u_t in L by substituting $M(u)$ which yields

$$u_t + \varepsilon L(M(u)) = M(u + \varepsilon L(u)) = M(u) + \varepsilon M(L(u)). \tag{2.3.19}$$

As L, M are linear commutative differential operators with constant coefficients we conclude

$$u_t = M(u), \tag{2.3.20}$$

and as a consequence, the linear dispersion relation is invariant under the Kodama transformation. A similar result holds for the Helmholtz operator with the exception that we must determine an order of error to truncate from (2.3.15) so that the dispersion relations for (2.3.15) and (2.2.31) agree up to this desired order.

2.3.2   The Kodama Transform and the b-Family Equations

In fact, we can generalize the results of section 2.3.1 by constructing a Kodama transformation that when applied to (2.2.31) will yield the appropriate coefficient in the finalized equation, e.g. Camassa–Holm $b = 2$ and Degasperis–Procesi $b = 3$. We start with the previous derivation in 2.3.1 where (2.3.9) now reads

$$\left(A - \frac{9}{2}\nu\right) : \left(B - \frac{3}{2}\nu\right) = b : 1,$$
$$\left(\frac{3}{2}\nu\right) : \left(B - \frac{3}{2}\nu\right) = (b+1) : 1, \tag{2.3.21}$$

which implies

$$A = \left(\frac{3}{2}\nu\right)\left(\frac{4b+3}{b+1}\right), \quad B = \left(\frac{3}{2}\nu\right)\left(\frac{b+2}{b+1}\right). \tag{2.3.22}$$

The resulting Kodama transformation of the form (2.3.3) with coefficients

$$\gamma_1 = a_1 + \left(\frac{b-2}{b+1}\right),$$

$$\gamma_2 = a_2 - \left(\frac{b-2}{b+1}\right),$$

$$\gamma_3 = a_3 - \left(\frac{b-2}{b+1}\right), \tag{2.3.23}$$

is

$$\eta = \eta(u) = u + \alpha(\gamma_1 u^2 + \gamma_2 u_x \partial_x^{-1} u) + \beta(\gamma_3 u_{xx}). \tag{2.3.24}$$

Using this transformation and applying the necessary scaling and rearrangements yields the b-Family of shallow water waves equations at quadratic order accuracy

$$\underbrace{m_t}_{\text{Evolution}} + c\underbrace{\left(m_x + \frac{1}{6}h^2 u_{xxx}\right)}_{\text{Dispersion}} + \underbrace{(um_x + bu_x m)}_{\text{Nonlinearity}} = 0, \tag{2.3.25}$$

where $b = 2$ is the canonical Camassa–Holm equation

$$\underbrace{m_t}_{\text{Evolution}} + c\underbrace{\left(m_x + \frac{1}{6}h^2 u_{xxx}\right)}_{\text{Dispersion}} + \underbrace{(um_x + 2u_x m)}_{\text{Nonlinearity}} = 0, \tag{2.3.26}$$

and $b = 3$ is the canonical Desgaperis-Procesi equation

$$\underbrace{m_t}_{\text{Evolution}} + c\underbrace{\left(m_x + \frac{1}{6}h^2 u_{xxx}\right)}_{\text{Dispersion}} + \underbrace{(um_x + 3u_x m)}_{\text{Nonlinearity}} = 0. \tag{2.3.27}$$

Remark 2.3.2. We have now derived a more general family of equations known as the b-Family of equations. We point out here that the remarks made in relation to the

Camassa–Holm equation continue to hold for the b-Family equations. However, now the nonlinear term is parameterized by a constant $b \in \mathbb{R}$ which is dependent on the coefficients of the Kodama transformation, i.e. $\gamma_1, \gamma_2, \gamma_3$. An asymptotically equivalent shallow water wave equation for any $b \neq -1$ may be achieved by a Kodama transformation. The value $b = -1$ is of course excluded as the Kodama transformation will then admit unbounded coefficients $\gamma_1, \gamma_2, \gamma_3$. As has been shown in the previous sections, the cases $b = 2$ and $b = 3$ are the integrable Camassa–Holm and Degasperis—Procesi equations respectively. What is interesting to note here is that these particular cases exhaust the integrable candidates for (2.3.25) as was demonstrated using Painlevé analysis [5]. The Degasperis–Procesi equation was originally *isolated* by Degasperis and Procesi [6] as one equation in a family of three in the form

$$u_t + cu_x + \gamma u_{xxx} - \alpha^2 u_{xxt} = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x, \qquad (2.3.28)$$

which can be rewritten using the definition of the momentum density m in the dispersionless canonical form

$$m_t + um_x + 3u_x m = 0. \qquad (2.3.29)$$

As mentioned, this was one of the equations said to satisfy "asymptotic integrability of third order" which is a *necessary* condition for complete integrability. However, it was latter demonstrated by Degasperis, Holm, and Hone [5] that the Degasperis–Procesi equation is completely integrable as well by deriving a *Lax pair and a bi-Hamiltonian structure* for the equation. The Degasperis–Procesi equation, as derived here from shallow water waves, is asymptotically equivalent with (2.2.31). Although the Camassa–Holm equation and Degasperis–Procesi equation are similar in many respects, there exist several important differences as we shall point out in the following chapter.

The hydrodynamical relevance of the b-family of equations

Due to their integrable structure the nonlinear dispersive partial differential equations namely the Camassa Holm(CH) equation and the Degasperis Processi(DP) equation have attracted much attention.Recently Constantin et al. proved thsat both equations arise in the modeling of the propagation of shallow water waves over a flat bed.The equations capture stronger nonlinear effects than the classical nonlinear dispersive Benjamin-Bauna-Mahoney(BBM) and Korteweg De Vries(KdV) equation.Noteworthy fact is both CH and DP equations accomodate wave breaking.Constantin et al. put formal asymtotic procedures on a firm and mathematically rigorous basis.They have also explained in clear terms in what sense the two models reflect onto the wave breaking phenomenon by some numerical computations.

2.4 Unidirectional Asymptotics for water waves

Degasperis and Procesi[6] studied a family of third order dispersive nonlinear equations

$$u_t - \alpha^2 u_{xxt} + \gamma u_{xxx} + c_0 u_x = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x. \qquad (2.4.1)$$

with six real constants $c_0, c_1, c_2, c_3, \gamma, \alpha \in \mathbb{R}$. They found that there are only three equations from this family were asymptotically integrable up to third order, that is, the Korteweg-de Vries (KdV) equation ($\alpha = c_2 = c_3 = 0$), the Camassa-Holm (CH) equation ($c_1 = -\frac{3c_3}{2\alpha^2}, c_2 = \frac{c_3}{2}$), and one new equation ($c_1 = -\frac{2c_3}{\alpha^2}, c_2 = c_3$), which is called the Degasperis-Procesi equation. By rescaling, shifting the dependent variable, and finally

applying a Galilean transformation, those three completely integrable[1] equations can be transformed into the following forms, the Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + uu_x = 0,$$

the Camassa-Holm (CH) shallow water equation [13],

$$y_t + y_x u + 2yu_x = 0, \quad y = u - u_{xx}, \tag{2.4.2}$$

and the Depasperis-Procesi equation of the form (1.1). These three cases are all the completely integrable candidates for (1.2) [13]. Applying a reciprocal transformation to the Degasperis-Procesi equation, Degasperis, Holm and Hone [5] used the Painlevé analysis to show the formal integrability of the DP equation as Hamiltonian systems by constructing a Lax pair and a bi-Hamiltonian structure. Equation (1.1) was also derived as, in dimensionless space-time variables (x, t), an approximation to the incompressible Euler equations for shallow water under the Kodama transformation [24] and its asymptotic accuracy is the same as that of the Camassa-Holm (CH) shallow water equation, where $u(t, x)$ is considered as the fluid velocity at time t in the spatial x-direction with momentum density y. More interestingly, the DP equation is recently observed as a model supporting shock waves [6]. More recently, Constantin and Lannes give a rigorous proof of both the CH equation and the DP equation are valid approximation to the governing equations for water waves and also show the relevance of these two equations as models for the propagation of shallow water waves. To see this rigorous justification of the

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[1]Integrability is meant in the sense of the infinite-dimensional extension of a classical completely integrable Hamiltonian system: there is a transformation which converts the equation into an infinite sequence of linear ordinary differential equations which can be trivially integrated .

derivation, one can consider the water wave equations for one-dimensional surfaces in nondimensionalized form

$$
\begin{cases}
\mu \partial_x^2 \psi + \partial_z^2 \Psi = 0, & \text{in } \Omega_t, \\[2mm]
\partial_z \Psi = 0, & \text{at } z = -1, \\[2mm]
\partial_t \xi - \frac{1}{\mu}(-\mu \partial_x \xi \partial_x \Phi + \partial_z \Psi) = 0, & \text{at } z = \epsilon \xi, \\[2mm]
\partial_t \Psi + \frac{\epsilon}{2}(\partial_x \Psi)^2 + \frac{\epsilon}{2\mu}(\partial_z \Psi)^2 = 0, & \text{at } z = \epsilon \xi,
\end{cases}
$$

where $x \to \epsilon \xi(t,x)$ parameterizes the elevation of the free surface at time t, $\Omega_t = \{(x,z); \; -1 < z < \epsilon \xi(t,x)\}$ is the fluid domain delimited by the free surface and the flat bottom $\{z = -1\}$, $\Psi(t, \cdot)$ is the velocity potential associated to the flow, and ϵ and μ are two dimensionless parameters defined by

$$
\epsilon = \frac{a}{h}, \quad \mu = \frac{h^2}{\lambda^2},
$$

where h is the mean depth, a is the typical amplitude, and λ is the typical wavelength of the waves. In the shallow water scaling that is when $\mu \ll 1$ the so called Green-Naghdi equations can be derived,without any assumption on ε ($\varepsilon = \bigcirc(1)$). For on-dimensional surfaces and flat bottoms, these equations couple the the free elevation ζ to the vertically averaged horizontal component of the velocity,

Define the vertically averaged horizontal component of the velocity by

$$
u(t,x) = \frac{1}{1 + \epsilon \xi} \int_{-1}^{\epsilon \xi} \partial_x \Psi(t,x,z) dz.
$$

In the shallow-water scaling ($\mu \ll 1$), one can derive the Green-Naghdi equations for one-dimensional surfaces and flat bottoms without any assumption on $\epsilon(\epsilon = O(1))$. These equations couple the free surface elevation ξ to the vertically averaged horizontal component of the velocity u and can be written as

$$
\begin{cases}
\xi_t + \big((1 + \epsilon \xi)u\big)_x = 0 \\[2mm]
u_t + \xi_x + \epsilon u u_x = \frac{\mu}{3} \frac{1}{1 + \epsilon \xi} \big((1 + \epsilon \xi)^3 (u_{xt} + \epsilon u u_{xx} - \epsilon u_x^2)\big)_x,
\end{cases}
$$

where $O(\mu^2)$ terms have been neglected. In the so-called long-wave regime

$$\mu \ll 1, \qquad \epsilon = O(\mu),$$

the right-going wave should satisfy the KdV equation

$$u_t + u_x + \epsilon\frac{3}{2}uu_x + \mu\frac{1}{6}u_{xxx} = 0$$

with $\xi = u + O(\epsilon, \mu)$, or a wider class of equations, referred as the BBM equations (sometimes also called the regularized long-wave equations), which provide an approximation of the exact water wave equations of the same accuracy as the KdV equation.

$$u_t + u_x + \frac{3}{2}\epsilon uu_x + \mu(\alpha u_{xxx} + \beta u_{xxt}) = 0, \quad \text{with } \alpha - \beta = \frac{1}{6}.$$

Consider now the so-called Camassa-Holm scaling, that is

$$\mu \ll 1, \quad \epsilon = O(\sqrt{\mu}).$$

With this scaling, one still has $\epsilon \ll 1$, the dimensionless parameter is, however, larger here than in the long wave scaling, and the nonlinear effects are therefore stronger and it is possible that a stronger nonlinearity could allow the appearance of breaking waves, which is a fundamental phenomenon in the theory of water waves that is not captured by the BBM equations. Define the horizontal velocity u^θ ($\theta \in [0, 1]$) at the level line θ of the fluid domain by

$$v \equiv u^\theta(x) = \partial_x \Psi \big|_{z=(1+\epsilon\xi)\theta-1}.$$

Let $p \in \mathbb{R}$ and $\lambda = \frac{1}{2}(\theta^2 - \frac{1}{3})$, with $\theta \in [0, 1]$. Assume

$$\alpha = p + \lambda, \ \beta = p - \frac{1}{6} + \lambda, \ \gamma = -\frac{2}{3}p - \frac{1}{6} - \frac{3}{2}\lambda, \ \delta = -\frac{9}{2}p - \frac{23}{24} - \frac{3}{2}\lambda.$$

Under the Camassa-Holm scaling, one should have the following class of equations for $v \equiv u^\theta$ ($\theta \in [0, 1]$), namely

$$(\star) \qquad v_t + v_x + \frac{3}{2}\epsilon vv_x + \mu(\alpha v_{xxx} + \beta v_{xxt}) = \epsilon\mu(\gamma vv_{xxx} + \delta v_x v_{xx}),$$

where $O(\epsilon^4, \eta^2)$ terms have been discarded. The vertically averaged horizontal velocity u and the free surface ξ satisfy

$$u = u^\theta + \mu\lambda u^\theta_{xx} + 2\mu\epsilon\lambda u^\theta u^\theta_{xx},$$

$$\xi = u + \frac{\epsilon}{4}u^2 + \mu\frac{1}{6}u_{xt} - \epsilon\mu\left(\frac{1}{6}uu_{xx} + \frac{5}{48}u_x^2\right).$$

By rescaling, shifting the dependent variable, and applying a Galilean transformation, the Camassa-Holm equation

$$U_t + \kappa U_x + 3UU_x - U_{txx} = 2U_xU_{xx} + UU_{xxx}$$

can be obtained from $(\star)$ if the following conditions hold

$$\beta < 0, \quad \alpha \neq \beta, \quad \beta = -2\gamma, \quad \delta = 2\gamma,$$

where $p = -\frac{1}{3}$, $\theta^2 = \frac{1}{2}$. The solution u^θ of $(\star)$ is transformed to the solution U of the CH equation by

$$U(t,x) = \frac{1}{a}u^\theta\left(\frac{x}{b} + \frac{\nu}{c}t, \frac{t}{c}\right),$$

with $a = \frac{2}{\epsilon\kappa}(1 - \nu)$, $b^2 = -\frac{1}{\beta}\mu$, $\nu = \frac{\alpha}{\beta}$, and $c = \frac{b}{\kappa}(1 - \nu)$. On the other hand, the DP equation

$$U_t + \kappa U_x + 4UU_x - U_{txx} = 3U_xU_{xx} + UU_{xxx}$$

can also be derived if the following conditions hold

$$\beta < 0, \quad \alpha \neq \beta, \quad \beta = -\frac{8}{3}\gamma, \quad \delta = 3\gamma,$$

where $p = -\frac{77}{216}$, $\theta^2 = \frac{23}{36}$. The solution u^θ of $(\star)$ is also transformed to the solution U of the DP equation by

$$U(t,x) = \frac{1}{a}u^\theta\left(\frac{x}{b} + \frac{\nu}{c}t, \frac{t}{c}\right),$$

with $a = \frac{8}{3\epsilon\kappa}(1 - \nu)$, $b^2 = -\frac{1}{\beta}\mu$, $\nu = \frac{\alpha}{\beta}$, and $c = \frac{b}{\kappa}(1 - \nu)$. A detailed derivation of the CH and DP equations can be found in Constantin and Lanne's work. It is well known that

the KdV equation is an integrable Hamiltonian equation that possesses smooth solitons as traveling waves. In the KdV equation, the leading order asymptotic balance that confines the traveling wave solitons occurs between nonlinear steepening linear dispersion. However, the nonlinear dispersion and nonlocal balance in the CH equation and the DP equation, even in the absence of linear dispersion, can still produce a confined solitary traveling waves

$$u(t, x) = c\varphi(x - ct) = ce^{-|x-ct|},$$

traveling at constant speed $c > 0$, which are called the peakons. Peakons of both equations are true solitons that interact via elastic collisions under the CH dynamics, or the DP dynamics, respectively. The peakons of the CH equation and the DP equation are orbitally stable . The result of the stability of the DP equation will be discussed in the last section. The DP equation can be rewritten as the form

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad t > 0, \ x \in \mathbb{R}. \tag{2.4.3}$$

It is noted that the peaked solitons are not classical solutions of (1.4). They satisfy the Degasperis-Procesi equation in conservation law form

$$u_t + \partial_x \left(\frac{1}{2}u^2 + p * \left(\frac{3}{2}u^2 \right) \right) = 0, \quad t > 0, \ x \in \mathbb{R}, \tag{2.4.4}$$

where $p(x) = \frac{1}{2}e^{-|x|}$, $*$ stands for convolution with respect to the spatial variable $x \in \mathbb{R}$, and $p * f = (1 - \partial_x^2)^{-1}f$. Since $p(x) = \varphi(x)$, in view of the structure of Eq.(1.5), it is quite clear why the peakons can be understood as solutions.

CHAPTER 3

A COMPARATIVE ANALYSIS

The DP equation is presently of great interest due to its structure (integrability, special solutions presenting interesting features). While Eq.(1.1) has an apparent similarity to Eq.(1.3), which both are important model equations for shallow water waves with the breaking phenomena, there are major structural differences and it is not much to know about its qualitative aspects. One of the novel features of the DP equation is that it has not only peakon solitons[6], $u(t,x) = ce^{-|x-ct|}$, $c > 0$ but also shock peakons[15] of the form

$$u(t,x) = -\frac{1}{t+k}\text{sgn}(x)e^{-|x|}, \ k > 0.$$

It is easy to see from[15] that the above shock-peakon solutions can be observed by substituting $(x,t) \longmapsto (\epsilon x, \epsilon t)$ to Eq.(1.4) and letting $\epsilon \to 0$ so that it yields the "derivative Burgers equation" $(u_t + uu_x)_{xx} = 0$, from which shock waves form.

In the periodic case of the spatial variable, both the CH equation and DP equation have periodic peakons [28] of the form

$$u_c(t,x) = c\frac{\cosh(x - ct - [x - ct] - \frac{1}{2})}{\sinh(\frac{1}{2})}, \quad x \in \mathbb{R}, \ t \geq 0, \ c > 0.$$

However, it is recently shown by Escher, Liu and Yin that the the periodic DP equation possesses the periodic shock waves given by

$$u_c(t,x) = \begin{cases} \left(\frac{\cosh(\frac{1}{2})}{\sinh(\frac{1}{2})}t + c\right)^{-1}\frac{\sinh(x-[x]-\frac{1}{2})}{\sinh(\frac{1}{2})}, & x \in \mathbb{R} \setminus \mathbb{Z}, \ c > 0, \\ 0, & x \in \mathbb{Z}. \end{cases}$$

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On the other hand, the isospectral problem in the Lax pair for the DP equation is of third-order instead of second [5], and consequently is not self-adjoint,

$$\psi_x - \psi_{xxx} - \lambda y \psi = 0,$$

and

$$\psi_t + \frac{1}{\lambda}\psi_{xx} + u\psi_x - \left(u_x + \frac{2}{3\lambda}\right)\psi = 0,$$

while the isospectral problem for the CH equation is of second order[13],

$$\psi_{xx} - \frac{1}{4}\psi - \lambda y \psi = 0$$

and

$$\psi_t - \left(\frac{1}{2\lambda} - u\right)\psi_x - \frac{1}{2}u_x\psi = 0$$

(in both cases $y = u - u_{xx}$). The spectral analysis and the inverse spectral theory for the CH equation are presented by Constantin and McKean [1] and Johnson[21]. Lundmark and Szmigielski[15] presented an inverse scattering transform (IST) method for computing $n-$peakon solutions of the DP equation. The approach is similar to that used by Beals, Sattinger and Szmigielski to obtain $n-$peakon solutions of the CH equation, but the present case does involve substantially new features as mentioned above. It is also noted that the CH equation is a re-expression of geodesic flow on the diffeomorphism group[9] or on the Bott-Virasoro group , while no such geometric derivation of the DP equation is available. Another indication of the fact that there is no simple transformation of the DP equation into the CH equation is the entirely different form of conservation laws for these two equations[5]. The following are three useful conservation laws of the DP equation.

$$E_1(u) = \int_{\mathbb{R}} y\,dx, \quad E_2(u) = \int_{\mathbb{R}} yv\,dx, \quad E_3(u) = \int_{\mathbb{R}} u^3\,dx,$$

where $y = (1-\partial_x^2)u$ and $v = (4-\partial_x^2)^{-1}u$, while the corresponding three useful conservation laws of the CH equation are the following.

$$F_1(u) = \int_{\mathbb{R}} y \, dx, \quad F_2(u) = \int_{\mathbb{R}} (u^2 + u_x^2) \, dx, \quad F_3(u) = \int_{\mathbb{R}} (u^3 + uu_x^2) \, dx.$$

It is observed that the corresponding conservation laws of the DP equation are much weaker than those of the CH equation. Therefore, the issue of if and how particular initial data generate a blow-up in finite time is more subtle.

It is worth noticing the following result obtained by Henry and Mustafa which implies that, analogous to the case of the CH equation, smooth solutions of the DP equation have infinite propagation speed.

Proposition 3.0.1. *Assume u_0 is a smooth function with compact support. If the solution u with initial data u_0 of (1.4) exists on some time interval $[0, \epsilon)$ with $\epsilon > 0$ and, at any time instant $t \in [0, \epsilon)$, the solution $u(t, \cdot)$ has compact support, then u is identically zero.*

3.1 The KdV and CH/DP Equations

In this section we present a comparative analysis of certain salient features of the Korteweg-de Vries, Camassa–Holm and Degasperis–Procesi equations. Most notably, we compare the soliton solutions in each case, the phenomenon of wave breaking and how certain characteristics of initial data in each of the Korteweg-de Vries and Camassa–Holm/Degasperis–Procesi cases affect solutions.

3.1.1 A Comparison of Soliton Solutions

We begin with the definition of a soliton.

Definition 3.1.1. A *soliton* is a solitary wave which asymptotically preserves its shape and velocity upon nonlinear interaction with other solitary waves, or more generally, with another (arbitrary) localized disturbance [18], or alternatively a soliton is a self-reinforcing solitary wave (a wave packet or pulse) that maintains its shape while it travels at constant speed; solitons are caused by a delicate balance between nonlinear and dispersive effects in the medium.

This is in important contradistinction to solitary waves which even though being an isolated wavefront, may not preserve shape, etc. upon interaction with other solitary waves. The Korteweg-de Vries, Camassa–Holm and Degasperis–Procesi equations are three completely integrable equations that possess solitons as traveling wave solutions. To start our analysis, we rewrite equation (2.2.29) in terms of dimensional variables, i.e.

$$\eta_t + c\eta_x + \frac{3c}{2h}\eta\eta_x + \frac{ch^2}{6}\eta_{xxx} = 0, \tag{3.1.1}$$

where, as previously, $c = \sqrt{gh}$ and the total water depth is $h + \eta$. In the Korteweg-de Vries case, the balance that confines the traveling wave soliton occurs between non-linear steepening and linear dispersion. Physically, nonlinear steepening is the tendency of the wave to steepen, i.e. for its slope to increase. In contradistinction, linear dispersion is the tendency of the wave to *spread out* or disperse over time. Mathematically, the term $\frac{3c}{2h}\eta\eta_x$ represents the nonlinear steepening while $c\left(\eta_x + \frac{h^2}{6}\eta_{xxx}\right)$ describes the linear dispersion. To insure that solutions to the Korteweg-de Vries equation are indeed dispersive we prove the following. Consider the normalized linear Korteweg-de Vries equation

$$u_t - u_{xxx} = 0. \tag{3.1.2}$$

Moreover, consider a solution to (3.1.2)

$$u(t, x) = A e^{i(\kappa x - wt)}, \tag{3.1.3}$$

where $w(\kappa)$ is the dispersion relation. Then inserting (3.1.3) into (3.1.2) we have

$$iw A e^{i(\kappa x - wt)} = i\kappa^3 A e^{i(\kappa x - wt)} \Leftrightarrow w = \kappa^3. \tag{3.1.4}$$

Since $w''(\kappa) \neq 0$ we conclude that solutions to the Korteweg-de Vries equation are dispersive. A soliton solution moving to the right (Figure 3.1) for equation (3.1.1) is given now without derivation, i.e.

$$\eta(t, x) = a \operatorname{sech}^2 \left[\sqrt{\left(\frac{3a}{4h^3} \right)} (x - ct) \right]. \tag{3.1.5}$$

It should be noted here that although solution (3.1.5) is exact for all $\frac{a}{h}$, keep in mind the derivation of (3.1.1) was made with the assumption of shallow water, e.g. $\frac{a}{h} \ll 1$. It can also be observed that the solution is smooth over the domain, e.g. $\eta \in C^\infty([0, \infty) \cup \mathbb{R})$. However, even in the absence of the linear dispersion term, the parameter b in the Camassa–Holm equation (2.3.25) introduces additional possibilities for balance, including the nonlinear/nonlocal balance in the dispersionless Camassa–Holm case for $b = 2$, i.e.

$$m_t + u m_x + 2u_x m = 0. \tag{3.1.6}$$
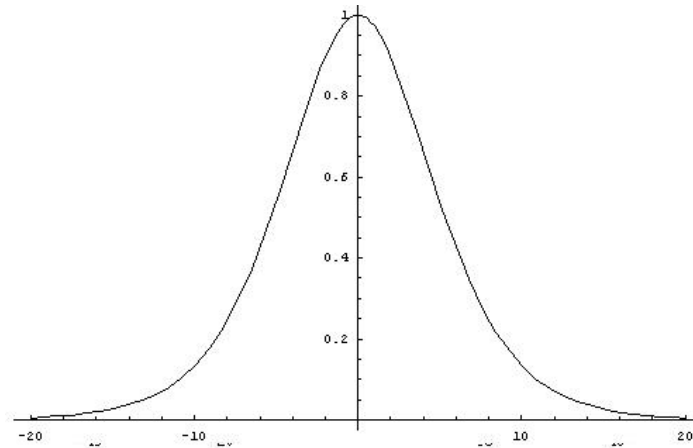
Figure 3.1. Mathematica profile of the soliton with $h = 3, c = 1, a = 1$ at $t = 0$.

The nonlinear/nonlocal balance in this equation can, even in the absence of linear dispersion, still produce a soliton solution moving to the right (Figure 3.2), i.e.

$$u(t, x) = ce^{-|x-ct|}, \ c > 0. \tag{3.1.7}$$

It is immediate that the derivative of (3.1.7) at $t = 0$

$$\varphi_x = -\,\text{sgn}(x)e^{-|x|}, \tag{3.1.8}$$

with limits

$$\lim_{x \to 0^+} \varphi_x = -1 \text{ and } \lim_{x \to 0^-} \varphi_x = 1, \tag{3.1.9}$$

has a jump discontinuity at its peak. Hence, we have $u \notin C^1([0, \infty) \cup \mathbb{R})$. With this information we can conclude that the Korteweg-de Vries and Camassa–Holm soliton solutions are quite different in structure, the one being smooth and hence a strong solution
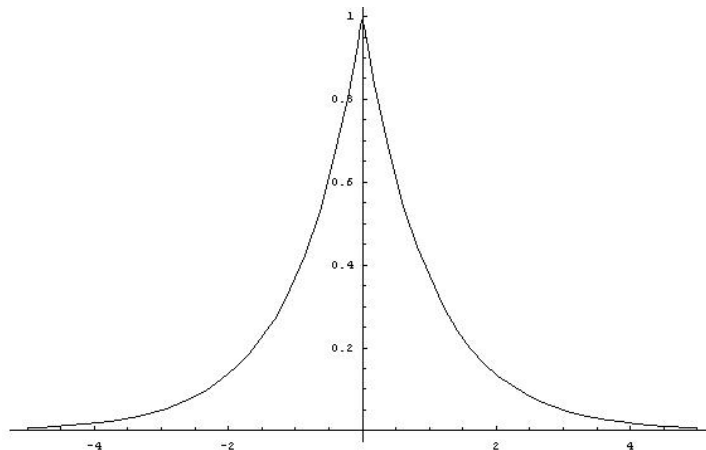
Figure 3.2. Mathematica profile of the peakon.
with $c = 1$ at $t = 0$.

and the other possessing a discontinuous jump in its derivative and hence representing a *weak solution*. One manner of specifying the weak solutions of the Camassa–Holm equation is simply to enforce a convention that $\varphi_x(x = 0) = 0$. However, a more satisfying approach is the following [15]. Let $G(x) = \varphi(x)$ be the peakon solution to equation (3.1.6). Then, write the dispersionless b family equation as

$$
\begin{aligned}
0 &= m_t + u m_x + b u_x m \\
&= (u - u_{xx})_t + (b+1) u u_x - b u_x u_{xx} - u u_{xxx} \\
&= (1 - \partial_x^2)\left[u_t + \left(\frac{1}{2}u^2\right)_x\right] + b \left(\frac{1}{2}u^2\right)_x + (3 - b)\left(\frac{1}{2}u_x^2\right)_x, \quad (3.1.10)
\end{aligned}
$$

and apply $(1 - \partial_x^2)^{-1}$, which turns out to be convolution with $\frac{1}{2}G(x)$. This gives

$$
u_t + \partial_x\left[\frac{1}{2}u^2 + \frac{1}{2}G(x) * (\frac{b}{2}u^2 + \frac{3 - b}{2}u_x^2)\right] = 0, \quad (3.1.11)
$$

so that the Camassa–Holm ($b = 2$) and Degasperis–Procesi ($b = 3$) equations can be written respectively as

$$u_t + \partial_x \left[\frac{1}{2}u^2 + \frac{1}{2}G(x) * (u^2 + \frac{1}{2}u_x^2) \right] = 0, \qquad (3.1.12)$$

and

$$u_t + \partial_x \left[\frac{1}{2}u^2 + \frac{1}{2}G(x) * \frac{3}{2}u^2 \right] = 0. \qquad (3.1.13)$$

This is the exact meaning in which peakons are solutions. For the Camassa–Holm equation (3.1.12) it is natural to impose the $H^1(\mathbb{R})$[1] regularity (and hence continuity) with respect to x in the definition of the weak solution because of the u_x^2 term in the equation [15]. The Degasperis–Procesi equation (3.1.13) was also studied for weak solutions in the $H^1(\mathbb{R})$ class by Yin [28]. However, it was demonstrated by Coclite and Karlsen [9] that since (3.1.13) does not involve a u_x term the $H^1(\mathbb{R})$ restriction could be weakened. They defined a *weak solution* of the Degasperis–Procesi equation as a function $u \in L^\infty(\mathbb{R}^+; L_2(\mathbb{R}))$ which satisfies (3.1.13) in the sense of *distributions*[2].

3.1.2 A Comparison of the Wave Breaking Phenomenon

In this section we discuss the differences in wave breaking between solutions to the Korteweg-de Vries and Camassa–Holm/Degasperis–Procesi equations. However, we first need a definition of wave breaking. We can describe wave breaking with the following definition:

Definition 3.1.2. Wave breaking (Figure 3.3) occurs in a solution(wave) if the solution remains bounded but the slope becomes infinite in finite time. This happens as the wave

---

[1]$H^m(\mathbb{R}) = \{u \in L^2(\mathbb{R}) | \partial^\alpha u \in L^2(\mathbb{R}) \, \forall \alpha, |\alpha| \leq m\}, m \in \mathbb{N}$, stands for the Sobolev space of functions with derivatives up to order m having finite $L^2(\mathbb{R})$ norm.

[2]A distribution T is a linear functional on $F(\Omega)$, where $F(\Omega)$ is the space of functions of class C_c^∞, such that $\lim_k T(\varphi_k) = T(\varphi)$ for any sequence φ_k converging to $\varphi \in F(\Omega)$
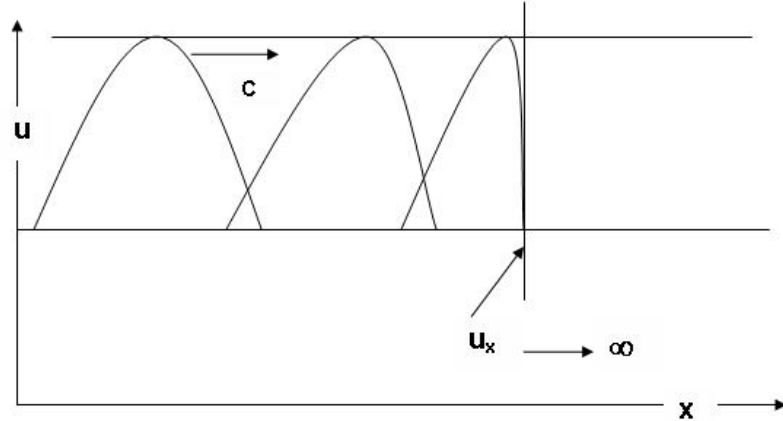
Figure 3.3. An example of wave breaking with velocity $c > 0$.

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velocity of the top of the wave is greater than the velocity at the bottom and hence the top of the wave over takes the bottom thus leading to a collapse or break in the wave.

As we have noted, the Korteweg-de Vries equation admits traveling wave solutions, i.e.

$$u(t, x) = \varphi(x - ct), \tag{3.1.14}$$

which travel with a fixed speed $c > 0$ and vanish at infinity. Moreover, the Korteweg-de Vries equation admits soliton solutions as described in section 3.1.1. However, and most importantly, solutions to the Korteweg-de Vries equation do not model the occurrence of the wave breaking phenomenon for shallow water waves [11]. Consider the following Cauchy problem for the family of third order partial differential equations, i.e.

$$\begin{cases} u_t + cu_x + \gamma u_{xxx} - \alpha^2 u_{xxt} = \left(c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx}\right)_x, & t \geq 0, \ x \in \mathbb{R} \\ u(0, x) = u_0(x), & u_0 \in H^1(\mathbb{R}), \ t > 0. \end{cases} \tag{3.1.15}$$

With $\alpha = c_2 = c_3 = 0$, (3.1.15) reduces to the Korteweg-de Vries equation. As the solution of equation (3.1.15) in the Korteweg-de Vries case is global as soon as $u_0 \in H^1(\mathbb{R})$ [1], i.e. exists for all $t \in [0, \infty)$, we conclude from the definition of wave breaking, i.e. the solution is bounded and has infinite slope in finite time, that the Korteweg-de Vries equation admits no wave breaking solution. Moreover, this equation has a nonlinear term $\frac{3}{2}\alpha\eta\eta_x$ accounting for the breaking of the wave – the higher a wave particle, the larger its velocity – and a dispersion term $\frac{1}{6}\beta\eta_{xxx}$ accounting for the broadening of the wave profile. These two effects balance each other and give rise to the stable stationary behavior.

As we will now show, solutions to the Camassa–Holm and Degasperis–Procesi equations describe the wave breaking phenomenon explicitly and without modification. To satisfy the definition of wave breaking we must show that a solution to the Camassa–Holm and Degasperis–Procesi equation is bounded and obtains infinite slope in finite time. It was recently demonstrated by (Liu and Yin) [26] that the Camassa–Holm equation, e.g. $c_1 = -\frac{3c_3}{2\alpha^2}, c_2 = \frac{c_3}{2}$ with the indicated initial data in the solution to equation (3.1.15) is bounded, i.e.

$$\|u(t,x)\|_{L^\infty(\mathbb{R})} \leq \sqrt{2}\|u(t,x)\|_{H^1(\mathbb{R})} \leq \sqrt{2}\|u_0(x)\|_{H^1(\mathbb{R})}, \tag{3.1.16}$$

and the Degasperis–Procesi equation, e.g. $c_1 = -\frac{2c_3}{\alpha^2}, c_2 = c_3$ with the indicated initial data in (3.1.15) is bounded, i.e.

$$\|u(t,x)\|_{L^\infty(\mathbb{R})} \leq 3t\|u_0(x)\|^2_{L^2(\mathbb{R})} + \|u_0(x)\|_{L^\infty(\mathbb{R})}. \tag{3.1.17}$$

They went on to show using the work of Constantin [1] that solutions to the Camassa–Holm and Degasperis–Procesi equations have a first derivative that approaches infinity in a finite time in the following lemma:

Lemma 3.1.1. *Given $u_0 \in H^s(\mathbb{R}), \forall s > \dfrac{3}{2}$, blow-up of the solution $u = u(\cdot, u_0)$ in finite time $T < +\infty$ occurs $\Leftrightarrow$*

$$\liminf_{t \to T} [\inf[u_x(t, x)]] = -\infty. \tag{3.1.18}$$

We give a proof of this lemma for the Degasperis–Procesi equation by arguing the contrapositive of the above. A similar proof holds for the Camassa–Holm equation.

Proof. Let $m = u - u_{xx}$. Then, by a direct computation we have

$$\|m\|_{L^2}^2 = \int_{\mathbb{R}} (u - u_{xx})^2 \, dx = \int_{\mathbb{R}} (u^2 + 2u_x^2 + u_{xx}^2) \, dx. \tag{3.1.19}$$

Hence,

$$\|u\|_{H^2}^2 \leq \|m\|_{L^2}^2 \leq 2\|u\|_{H^2}^2. \tag{3.1.20}$$

Now let $m = Mu$ where $M = (1 - \partial_x^2)^{-1}$. Then the equation for m reads

$$\begin{cases} m_t + m_x u + 3mu_x = 0, \\ m(0, x) = m_0(x) = Mu_0(x). \end{cases} \tag{3.1.21}$$

Applying the operator m to (3.1.21) and using integration by parts we have

$$\frac{d}{dt} \int_{\mathbb{R}} m^2 dx = -5 \int_{\mathbb{R}} u_x m^2 dx. \tag{3.1.22}$$

Then by Gronwall's Inequality[3] we have from (3.1.22) that *if u_x is bounded from below on $[0, T)$ then the H^2 norm of the solution is also bounded on $[0, T)$.* Hence, it

---

[3]If $u : [t_0, t_1] \to R$ is continuous and nonnegative then for $K \geq 0$ such that

follows from (3.1.20) that for some constant $C(t)$, $\|u\|_{H^2}^2 \leq C(t), \forall t \in [0, T)$. Since this is true for any $T > 0$, we conclude the solution has global existence. On the other hand, if

$$u = \int_{\mathbb{R}} G(x - \xi)m(\xi)\,d\xi, \tag{3.1.25}$$

for some Green's function $G(x; \xi)$ then

$$
\begin{aligned}
\|u_x\|_{L^\infty} &\leq \left| \int_{\mathbb{R}} G(x - \xi)m(\xi)\,d\xi \right| \\
&\leq \|G_x\|_{L^2} \|m\|_{L^2} = \frac{1}{2}\|m\|_{L^2} \leq \|u\|_{H^2}.
\end{aligned}
\tag{3.1.26}
$$

Hence, (3.1.26) informs us that *if the H^2 norm of the solution is bounded then the L^∞ norm of the first derivative of the solution is also bounded.* This completes the proof. □

Hence, we can conclude both the Camassa–Holm and Degasperis–Procesi equations have solutions that blow-up and thus by (3.1.16) and (3.1.17) have wave breaking.

$$u(t) \leq K + \int_{t_0}^{t_1} \Phi(s)u(s)\,ds, \; t \in [t_0, t_1], \tag{3.1.23}$$

we have

$$u(t) \leq K \exp\left(\int_{t_0}^{t_1} \Phi(s)\,ds \right), \; t \in [t_0, t_1]. \tag{3.1.24}$$

3.1.3 A Comparison of Dependence on Initial Data

In this section we describe the differences between solutions to the Korteweg-de Vries and Camassa–Holm/Degasperis–Procesi equations with respect to their initial data. In particular, we look at how size and smoothness are factors in determining solutions to the Korteweg-de Vries equation and shape the factor in determining solutions to the Camassa–Holm/Degasperis–Procesi equations.

3.1.3.1 Korteweg-de Vries: Size and Smoothness of Initial Data

To show solutions to the Korteweg-de Vries equation are dependent upon the size and smoothness of the initial data we will use a theorem due to Kato [22]. Given the normalized form of the Korteweg-de Vries equation and its subsequent Cauchy problem

$$
\begin{cases}
u_t + uu_x - u_{xxx} = 0, & t \geq 0, \ x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R},
\end{cases}
\tag{3.1.27}
$$

and for any $u_0 \in H^s, s \geq 1$, there is a unique global solution u to (3.1.27) obtained in the class

$$
u \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1}),
\tag{3.1.28}
$$

so that T is dependent upon $\|u_0\|_{H^s}$ and $u_0 \mapsto u(t)$ is a continuous map in the H^s norm [23]. Since the maximal time T is dependent upon the H^s norm or magnitude of u_0 in a general Sobolev space with the prescribed condition $s \geq 1$ and $u_0 \mapsto u(t)$ is a continuous map in the H^s norm, then we can conclude that the solutions to the Korteweg-de Vries equation are dependent upon both the size and smoothness of the initial data.

3.1.3.2 Camassa–Holm and Degasperis–Procesi: Shape of Initial Data

To show the solutions to the Camassa–Holm and Degasperis–Procesi equations are dependent on the shape of their initial data we consider the following Cauchy problem:

$$\begin{cases} q_t = u(t, q), & t \in [0, T), \\ q(0, x) = x, & x \in \mathbb{R}, \end{cases} \tag{3.1.29}$$

where u is a solution of the Camassa–Holm/Degasperis–Procesi equations. Then, applying the classical results in the theory of ordinary differential equations, one can obtain the following result on q:

Lemma 3.1.2. *Let $u_0 \in H^s(\mathbb{R}), \forall s > \dfrac{3}{2}$ and let $T > 0$ be the maximal existence time of the corresponding solution u to Eq. (2.3.25). Then Eq. (3.1.29) has a unique solution $q \in C^1([0, T) \times \mathbb{R}, \mathbb{R})$ and*

$$q_x(t, x) = exp\left(\int_0^t u_x(s, q(s, x)) \, ds\right) > 0, \ (t, x) \in [0, T) \times \mathbb{R}. \tag{3.1.30}$$

Moreover, if $m = u - u_{xx}$ and $b = 2$ or $b = 3$ we have

$$m(t, q(t, x))q_x^b(t, x) = m_0(x). \tag{3.1.31}$$

A key point from lemma 3.1.2 consists in the observation that all solutions of the Camassa–Holm/Degasperis–Procesi equations with initial data $u_0 \in H^s(\mathbb{R}), \forall s > \dfrac{3}{2}$, are uniformly bounded $\forall t \in (0, T)$ and the only way that a solution fails to exist for all time is if wave breaking occurs. Therefore the maximal existence time is not dependent upon the size of $u_0 \in H^s(\mathbb{R}), \forall s > \dfrac{3}{2}$, i.e. $T \neq T(\|u_0\|_s)$ nor is the solution u dependent upon $u_0 \mapsto u(t)$ being a continuous map in the H^s norm [2], but instead the solution depends only on the sign of m_0 or the shape of the initial data.

3.2 The Camassa–Holm and Degasperis–Procesi Equations

3.2.1 A Comparison of Soliton Solutions

We recall from section 3.1.1 that both the Camassa–Holm and Degasperis–Procesi equations admit soliton solutions, i.e.

$$u(t, x) = ce^{-|x-ct|}, \ c > 0. \tag{3.2.1}$$

However, recently Lundmark [15] demonstrated that concrete examples of the *entropy weak solutions*[4] of Coclite and Karlsen [9] to the Degasperis–Procesi equation may develop discontinuities in finite time. He studied the interactions of peakons and antipeakons and showed that a jump discontinuity formed for a particular solution to the Degasperis–Procesi equation when these two collided thus leading to a new phenomenon described by a *"shockpeakon"*, i.e.

$$u(t, x) = -\frac{1}{t + k} \ \mathrm{sgn}(x)e^{-|x|}, \quad k > 0, \ x \in \mathbb{R}. \tag{3.2.2}$$

It turns out that if we allow the peakon and antipeakon solutions to collide with initial data

$$u_0(t, x) = c_1 e^{-|x-x_1|} + c_2 e^{-|x-x_2|}, \tag{3.2.3}$$

$c_1 > 0, c_2 < 0$ and $x_1 + x_2 = 0, x_2 > 0$, then the collision occurs at $x = 0$ (Figures 3.4, 3.5, 3.6) and the solution (the sum of a peakon and antipeakon, the antipeakon moving off to the right and the peakon to the left),

---

[4]An entropy weak solution is a weak solution in the sense of distributions that satisfies the entropy condition $\dfrac{u(t, x + h) - u(t, x)}{h} \leq \dfrac{C}{t}$ for some $C > 0$.
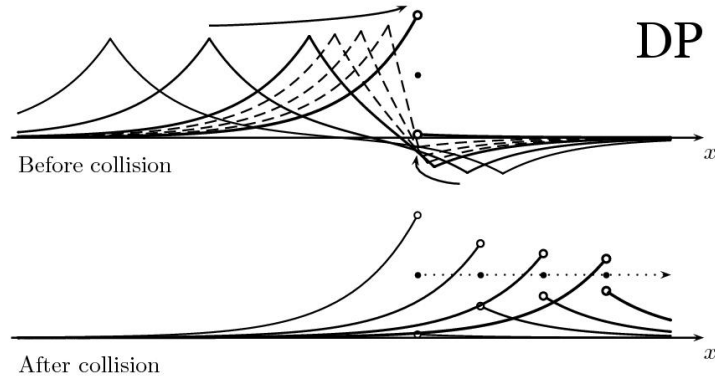
Figure 3.4. Degasperis–Procesi peakon-antipeakon collision.

$$u(t,x) = p_1 e^{-|x-q_1(x)|} + p_2 e^{-|x-q_2(x)|}, \ (t,x) \in \mathbb{R}_+ \times \mathbb{R} \tag{3.2.4}$$

only satisfies the Degasperis–Procesi equation for $t < T$. The unique continuation of the solution into the entropy weak solution is then given by the stationary decaying shockpeakon

$$u(t,x) = -\frac{1}{k+(t-T)} \, \text{sgn}(x) e^{-|x|}, \ t \geq T. \tag{3.2.5}$$

$\boxed{\text{THREE DIFFERENT PROFILES OF SOLUTION 3.2.2.}}$

This is important as no such phenomenon is known to arise from the Camassa–Holm equation peakon-antipeakon collision (Figure 3.7) and hence we have a difference in soliton solutions.

Remark 3.2.1. It turns out there is a simplified approach to the problem of existence of shocks in solutions to the Degasperis–Procesi equation. It was recently demonstrated by Coclite and Karlsen [9] that equation (3.1.13) can be viewed as Burgers' or the Riemann shock equation perturbed by a source term, albeit a nonlocal one. Thus, let us consider the normalized Degasperis–Procesi equation
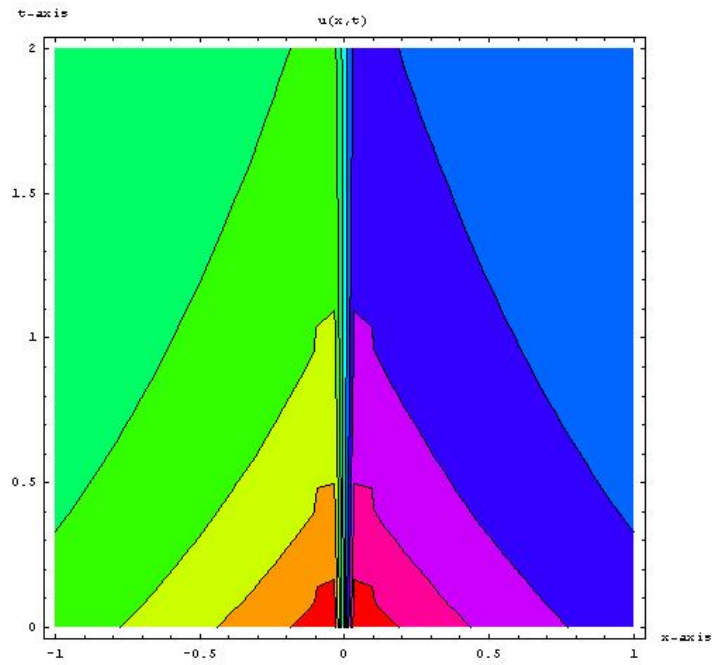
Figure 3.5. Mathematica contour profile of a stationary decaying shockpeakon for $0 \leq t \leq 2$.
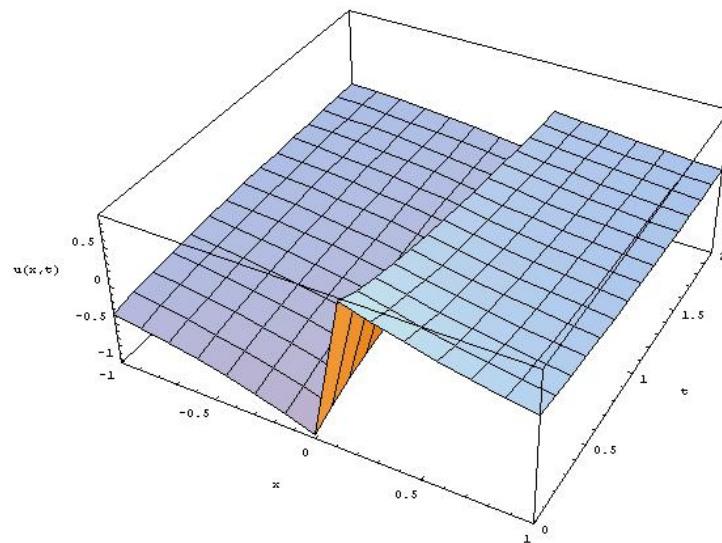
Figure 3.6. Mathematica 3D profile of a stationary decaying shockpeakon for $0 \leq t \leq 2$.
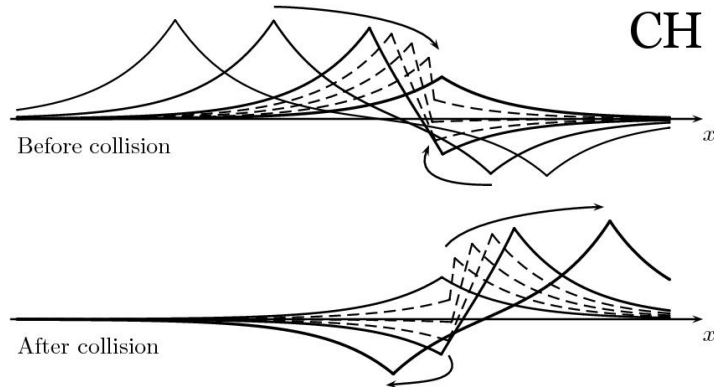
Figure 3.7. Camassa–Holm peakon-antipeakon collision.

$$(u - u_{xx})_t + 4uu_x = uu_{xxx} + 3u_xu_{xx}. \tag{3.2.6}$$

Moreover, consider the high-frequency limit obtained by changing variables $(t, x) \mapsto (\varepsilon x, \varepsilon t)$ for $\varepsilon > 0$ in (3.2.6), and then letting $\varepsilon \to 0$. Then the Degasperis–Procesi equation (3.2.6) reduces after scaling to

$$(u_t + uu_x)_{xx} = 0. \tag{3.2.7}$$

It turns out that this is the *derivative* Burgers' equation. Since the inviscid Burgers' equation $u_t + uu_x = 0$ admits shock wave solutions, we can conclude that the Degasperis–Procesi equation admits shock wave solutions, which is precisely our findings with in solution (3.2.2).

3.2.2 A Comparison of Eigenvalue problems

In this section we return to the peakon solution of both the Camassa–Holm and Degasperis–Procesi equations. Recall the b-family equation (2.3.25), i.e.

$$m_t + c \left(m_x + \frac{1}{6} h^2 u_{xxx} \right) + (u m_x + b u_x m) = 0. \tag{3.2.8}$$

If we take $b = 2$ we have the Camassa–Holm equation. It was demonstrated by Camassa and Holm [19] that this equation has the following *Lax pair*[5], i.e.

$$\psi_{xx} = \frac{1}{4} \psi + \lambda m \psi, \tag{3.2.11}$$

$$\psi_t = - (u + \lambda) \psi_x + \frac{1}{2} u_x \psi. \tag{3.2.12}$$

The Lax pair for the Camassa–Holm equation thus consists of a second order eigenvalue problem and a first order evolutionary equation for the eigenfunction ψ with the compatibility condition $\psi_{xxt} = \psi_{txx}$. However, if we take $b = 3$ we have the Degasperis–Procesi equation. It was demonstrated by Degasperis, Holm and Hone [5] that this equation has the following Lax pair:

$$\psi_{xxx} = \psi_x - \lambda m \psi, \tag{3.2.13}$$

$$\psi_t = -\frac{1}{\lambda} \psi_{xx} - u \psi_x + \left(u_x + \frac{2}{3\lambda} \right) \psi. \tag{3.2.14}$$

The Lax pair for the Degasperis–Procesi equation thus consists of a third order eigenvalue problem and a second order evolutionary equation for the eigenfunction ψ

---

[5]Given operators L, M and the following system,

$$L v = \lambda v, \tag{3.2.9}$$

$$v_t = M v. \tag{3.2.10}$$

we obtain the equation $L_t + [L, M] = 0$ to solve for nontrivial eigenfunctions, where the commutator operator $[L, M] = LM - ML$, iff $\lambda_t = 0$. In other words, our operators must satisfy $L_t + [L, M] = 0$ and the equations can be thought of as the compatibility condition between these two linear operators.

with the compatibility condition $\psi_{xxxt} = \psi_{txxx}$. Our interest here lies in the main with equations (3.2.11) and (3.2.13). We observe the eigenvalue problem for equation (3.2.11) consists of a second order differential operator while the eigenvalue problem for equation (3.2.13) consists of a third order differential operator. This implies the integrable structures for these two equations are considerably different, the Degasperis–Procesi equation of course being more complex than the Camassa–Holm equation [27].

3.2.3 A Comparison of Conservation Laws

We begin with the definition of a conservation law.

Definition 3.2.1. A function ρ is called a conserved *density* of a local conservation law for an equation of evolution $u_t = F(u, u_x, u_{xx}, ...)$ if there exists a function ϕ such that

$$\partial_t(\rho) = \partial_x(\phi), \tag{3.2.15}$$

where we refer to ϕ as the *flux* function and (3.2.15) as a conservation law. [6]

In this section we note some of the conservation laws associated with the Camassa–Holm and Degasperis–Procesi equations and make a generic comparison of each. For this exposition, the following three local conservation laws of the Degasperis–Procesi equation are useful:

$$E_1(u) = \int_{\mathbb{R}} m \, dx, \tag{3.2.16}$$

$$E_2(u) = \int_{\mathbb{R}} mv \, dx, \tag{3.2.17}$$

$$E_3(u) = \int_{\mathbb{R}} u^3 \, dx, \tag{3.2.18}$$

---

[6] Consider Burgers' equation $u_t + uu_x = 0$. If $\rho = u$ and $\phi = \frac{1}{2}u^2$ then we have $\rho_t = \phi_x$ which is an example of a conservation law for Burgers' equation.

where we recall the operator $m = u - u_{xx}$ and note $v = (4 - \partial_x^2)^{-1}u$. We will now show m is indeed a conserved density of the Degasperis–Procesi equation. Hence, we need only apply the definition of a conservation law and conclude that

$$\partial_t(m) = \partial_t(u - u_{xx}) = \partial_x(2u^2 - \frac{3}{2}u_x^2 - \frac{1}{2}u_x^2 + uu_{xx}). \qquad (3.2.19)$$

Similar results hold for the densities mv, u^3. Again, we consider the following three local conservation laws of the Camassa–Holm equation:

$$F_1(u) = \int_{\mathbb{R}} m \, dx, \qquad (3.2.20)$$

$$F_2(u) = \int_{\mathbb{R}} (u^2 + u_x^2) \, dx, \qquad (3.2.21)$$

$$F_3(u) = \int_{\mathbb{R}} (u^3 + uu_x^2) \, dx, \qquad (3.2.22)$$

where F_1 is the total mass as in (3.2.16) and F_2 the total energy, etc. Now we wish to note some similarities of the Camassa–Holm and Degasperis–Procesi equations in terms of the conservation laws. However, "can we locate more conservation laws for the Camassa–Holm and Degasperis–Procesi equations?". Since each equation is completely integrable, the answer is yes. For the Degasperis–Procesi equation it was demonstrated by Degasperis et al. [5], that the Degasperis–Procesi equation has infinitely many conservation laws. This was accomplished by first introducing a density function $\rho = (\ln\psi)_x \Leftrightarrow \psi = e^{\int \rho dx}$ and then calculating the necessary derivatives ψ_x, ψ_{xxx} which yield after substitution into the eigenvalue problem of the Lax pair (3.2.13),

$$\mathfrak{H}\rho = (1 - \partial_x^2)\rho = 3\rho\rho_x + \rho^3 + \lambda m, \qquad (3.2.23)$$

and hence, from (3.2.14), ignoring the $\dfrac{2}{3\lambda}$ term, the conservation law for ρ,

$$\partial_t(\rho) = \partial_x(u_x - u\rho - \lambda^{-1}(\rho_x + \rho^2)). \tag{3.2.24}$$

The density ρ could then be written as a formal power series in λ with the corresponding coefficients determined recursively from (3.2.23). Substituting a corresponding expansion of the flux $u_x - u\rho - \lambda^{-1}(\rho_x + \rho^2)$ into (3.2.24) and comparing the powers of λ would yield an infinite sequence of conservation laws. However, this method admits of two different expansions,

$$\rho = (m\lambda)^{\frac{1}{3}} \sum_{n=0}^{\infty} \rho^{(n)} \lambda^{-\frac{n}{3}}, \tag{3.2.25}$$

and a sequence in positive powers of λ,

$$\rho = \sum_{n=0}^{\infty} r^{(n)} \lambda^{n+1}, \tag{3.2.26}$$

that lead to two infinite sequences of conserved quantities[7] and hence infinitely many conserved quantities. For the Camassa–Holm equation it was demonstrated by Camassa et al. [19] and Fokas et al. [8] that there exists a *bi-Hamiltonian structure*. Moreover, as it possesses this bi-Hamiltonian structure then the equation can be written in two different ways, i.e.

$$m_t = -(m\partial_x + \partial_x m)\,\delta\left(H_1 = \frac{1}{2}\int_{\mathbb{R}} mu\,dx\right), \tag{3.2.27}$$

$$m_t = -(\partial_x - \partial_x^3)\,\delta\left(H_2 = \frac{1}{2}\int_{\mathbb{R}} (u^3 + uu_x^2)\,dx\right). \tag{3.2.28}$$

Hence, through the recursive relationship,

---

[7]The term $r^{(n)}$ is a recursive formula omitted for brevity.

$$m_t^{k+1} = -\left(\partial_x - \partial_x^3\right)\frac{\delta H_k}{\delta m} = -\left(m\partial_x + \partial_x m\right)\frac{\delta H_{k-1}}{\delta m}, \ \ k \in \mathbb{N}, \tag{3.2.29}$$

we conclude there exists an infinite sequence of quantities,

$$..., H_{-2}, H_{-1}, H_0, H_1, H_2, ... \tag{3.2.30}$$

which are exactly the conservation laws for the Camassa–Holm equation. It should be kept in mind that both equations are bi-Hamiltonian, the above demonstrations being for the Degasperis–Procesi and Camassa–Holm cases respectively, and that most of the conservation laws are nonlocal in nature making them ineffective in a standard energy estimates approach, thus demonstrating the similarities of the two equations. Now we will consider some of the differences between the Degasperis–Procesi and Camassa–Holm equations.

We first note the trivial differences between the two sets of conservation laws, e.g. the Camassa–Holm having densities $(u^2 + u_x^2), (u^3 + uu_x^2)$ and the Degaperis-Procesi equation having densities mv, u^3. However, we seek more subtle differences. The only difference that was pointed out by Liu et al. [26] stated that the conservation laws for the Degasperis–Procesi equation are much weaker than the Camassa–Holm conservation laws. In particular, we observe that the conservation law for the Degasperis–Procesi equation is $E_2(u) \sim \|u\|_{L^2}^2$ while the conservation law for the Camassa–Holm equation is $F_2(u) = \|u\|_{H^1}^2$, i.e. the two having particularly different norms and one being asymptotically equivalent while the the other possesses strict equality. In fact, by a Fourier transform, we have

$$E_2(u) = \int_{\mathbb{R}} mv \, dx = \int_{\mathbb{R}} \frac{1+\xi^2}{4+\xi^2}|\hat{u}(\xi)|^2 \sim \|\hat{u}\|_{L^2}^2 = \|u\|_{L^2}^2. \tag{3.2.31}$$

Thus, the issue of orbital stability of Degasperis–Procesi peaked soliton solutions is more subtle, the case of course being that we have less control in L^2 than H^1.

CHAPTER 4

LOCAL WELL POSEDNESS OF B FAMILY OF EQUATIONS

4.1 Preamble

We establish the local well-posedness for the peakon b-family of equations which includes both the Camassa-Holm equation and Degasperis-Procesi equation as special cases.

4.2 Introduction

Recently, Holm and Staley studied a one-dimensional version of active fluid transport that is described by the following family of 1+1 evolutionary equations

$$m_t + \underbrace{um_x}_{convection} + \underbrace{bu_x m}_{streching} = 0, \quad with \ u = g * m \tag{4.1.1}$$

where the fluid velocity $u(t,x)$ is defined on the real line vanishing at spatial infinity and $u = g * m$ denotes the convolution (or filtering)

$$u(x) = \int_{\mathbb{R}} g(x - y)m(y)dy,$$

which relates velocity u to momentum density m by integration against kernel $g(x)$ over the real line.

It was shown by Degasperis and Procesi[6] using the method of asymptotic integrability that Eq.(4.1.1) cannot be completely integrable unless $b = 2$ or $b = 3$, [5], [6].

If $b = 2$, Eq.(1.1) becomes the Camassa-Holm(CH) equation of the form

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad t > 0, \ x \in \mathbb{R}, \tag{4.1.2}$$

modeling the unidirectional propagation of shallow water waves over a flat bottom.

If $b = 3$, Eq.(1.1) becomes the Degasperis-Procesi(DP) equation of the form

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad t > 0, \ x \in \mathbb{R}, \tag{4.1.3}$$

The goal of the present chapter is to study global existence of solutions and blow-up phenomena for Eq.(4.1.1) to better understand the common properties of these two integrable equations the Camassa-Holm equation (4.1.2) and the Degasperis-Procesi equation (4.1.3).

We have found it is convenient to rewrite Eq.(4.1.1) as the following form

$$u_t - u_{txx} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \quad t > 0, \ x \in \mathbb{S}, \tag{4.1.4}$$

for a real parameter b, which includes both the Camassa-Holm equation (4.1.2) ($b = 2$) and the Degasperis-Procesi equation (4.1.3) ($b = 3$) as special cases. Since it arises from (4.1.1) when the peakon kernel $g(x) = \frac{1}{2}e^{-|x|}$ is chosen, we refer to (4.1.4) as the peakon b-family of equations.

Notation. As above and henceforth, we denote the norm of the Lebesgue space L^p by $\| \cdot \|_{L^p}$, $1 \leq p \leq \infty$ and the norm in the Sobolev space H^s, $s \in \mathbb{R}$ by $\| \cdot \|_s$. We denote by $*$ the spatial convolution on $\mathbb{R}$. We also use $(\cdot, \cdot)$ to represent the standard inner product in $L^2(\mathbb{R})$, and $(\cdot, \cdot)_s$, the standard inner product in $H^s(\mathbb{R})$.

4.3 Local well-posedness

In this section, we apply Kato's theory to establish local well-posedness for the Cauchy problem of (4.1.4). For convenience, we state here Kato's theorem in a form suitable for our purpose. Consider the abstract quasi-linear evolution equation

$$\begin{cases} \frac{dv}{dt} + A(v)v = f(v), & t \geq 0, \\ v(0) = v_0 \end{cases} \tag{4.2}$$

Let X and Y be Hilbert spaces such that Y is continuously and densely embedded in X, and let $Q : Y \to X$ be a topological isomorphism. Let $L(Y, X)$ denote the space of all bounded linear operators from Y to X. If $X = Y$, we denote this space by $L(X)$. The linear operator A belongs to $G(X, 1, \beta)$ where β is a real number, that is, $-A$ generates a C_0-semigroup such that $\|e^{-sA}\|_{L(X)} \le e^{\beta s}$. We make the following assumptions, where μ_1, μ_2, μ_3, and μ_4 are constants depending only on $\max\{\|y\|_Y, \|z\|_Y\}$.

(i) $A(y) \in L(Y, X)$ for $y \in X$ with

$$\| (A(y) - A(z)) \, w\|_X \le \mu_1 \|y - z\|_X \|w\|_Y, \quad y, z, w \in Y$$

and $A(y) \in G(X, 1, \beta)$ (i.e., $A(y)$ is quasi-m-accretive), uniformly on bounded sets in Y.

(ii) $QA(y)Q^{-1} = A(y) + B(y)$, where $B(y) \in L(X)$ is bounded, uniformly on bounded sets in Y. Moreover,

$$\| (B(y) - B(z)) \, w\|_X \le \mu_2 \|y - z\|_Y \|w\|_X, \quad y, z \in Y, w \in X.$$

(iii) $f : Y \to Y$ extends to a map from X into X, is bounded on bounded sets in Y , and satisfies

$$\|f(y) - f(z)\|_Y \le \mu_3 \|y - z\|_Y, \quad y, z \in Y,$$

$$\|f(y) - f(z)\|_X \le \mu_4 \|y - z\|_X, \quad y, z \in Y.$$

Lemma 4.3.1. *(Kato,[22]) Assume that (i), (ii), and (iii) hold. Given $v_0 \in Y$, there is a maximal $T > 0$ depending only on $\|v_0\|_Y$, and a unique solution v to (4.2) such that $v = v(\cdot, v_0) \in C([0, T), Y) \cap C^1([0, T), X)$. Moreover, the map $v_0 \mapsto v(\cdot, v_0)$ is a continuous map from Y to $C([0, T), Y) \cap C^1([0, T), X)$.*

We now provide a framework in which we shall reformulate the problem (4.1.2). We begin by fixing some notations. All spaces of functions are assumed to be over

$\mathbb{S}, S = XnR$. If A is an unbounded operator, we denote by $D(A)$ the domain of A. $[A, B]$ denotes the commutator of two linear operators A and B.

With $m = u - u_{xx}$, we consider the Cauchy problem

$$\begin{cases} m_t + um_x + bu_x m = 0, & t > 0, \ x \in \mathbb{S}, \\ m(0, x) = u_0(x) - u_{0,xx}(x), & x \in \mathbb{S}. \end{cases} \tag{4.3}$$

Note that if $g(x) := \frac{1}{2}e^{-|x|}$, $x \in \mathbb{S}$, then $(1 - \partial_x^2)^{-1}f = g * f$ for all $f \in L^2(\mathbb{S})$ and $g * m = u$, where $*$ denotes convolution. Using this identity, we can rewrite (4.3) as a quasi-linear evolution equation of hyperbolic type:

$$\begin{cases} u_t + uu_x + \partial_x g * (\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2) = 0, & t > 0, \ x \in \mathbb{S}, \\ u(0, x) = u_0(x), & x \in \mathbb{S}, \end{cases} \tag{4.4}$$

Definition 4.3.1. If $u \in C([0, T), H^s(\mathbb{R})) \cap C^1([0, T), H^{s-1}(\mathbb{R}))$ with $s > \frac{3}{2}$ satisfies (4.4), then u is called a strong solution to (4.4). If u is a strong solution on $[0, T)$ for every $T > 0$, then it is called global strong solution to (4.4).

The local well-posedness of the Cauchy problem of (4.4) with initial data $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$ can be obtained by applying Kato's theorem[22]. More precisely, we have the following well-posedness result.

Theorem 4.3.2. *For any constant b, given $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, there exist a maximal $T = T(u_0) > 0$ and a unique strong solution u to (4.3.3), such that*

$$u = u(\cdot, u_0) \in C([0, T), H^s(\mathbb{R})) \cap C^1([0, T), H^{s-1}(\mathbb{R})).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $u_0 \mapsto u(\cdot, u_0) : H^s(\mathbb{S}) \to C([0, T), H^s(\mathbb{S})) \cap C^1([0, T), H^{s-1}(\mathbb{R}))$ is continuous.

To prove this theorem, we apply Lemma 4.3.1 with $A(u) = u\partial_x$, $f(u) = -\partial_x g * (\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2)$, $Y = H^s$, $X = H^{s-1}$, $\Lambda = (1 - \partial_x^2)^{\frac{1}{2}}$, and $Q = \Lambda^1$. Obviously, Q is an

isomorphism of H^s onto H^{s-1}. Thus, in order to derive Theorem 4.3.3 from Lemma 4.3.1, we only need to verify that $A(u)$ and $f(u)$ satisfy the conditions (i)-(iii). We break the argument into several lemmas.

Lemma 4.3.3. (***Yin***) *The operator* $A(u) = u\partial_x$, *with* $u \in H^s, s > \frac{3}{2}$, *belongs to* $G(H^{s-1}, 1, \beta)$.

Lemma 4.3.4. *Let the operator* $A(u) = u\partial_x$ *with* $u \in H^s, s > \frac{3}{2}$. *Then* $A(u) \in L(H^s, H^{s-1})$ *for* $u \in H^s$. *Moveover,*

$$\| (A(u) - A(z)) w\|_{s-1} \le \mu_1 \|u - z\|_{s-1} \|w\|_s, \quad u, z, w \in H^s. \tag{4.5}$$

Proof. Let $u, z, w \in H^s, s > \frac{3}{2}$. Note that H^{s-1} is a Banach algebra. Then we have

$$\| (A(u) - A(z)) w\|_{s-1} \le c\|u - z\|_{s-1} \|\partial_x w\|_{s-1} \le \mu_1 \|u - z\|_{s-1} \|w\|_s.$$

Taking $z = 0$ in the above inequality, we obtain $A(u) \in L(H^s, H^{s-1})$. This completes the proof of Lemma 4.3.4. □

Lemma 4.3.5. (***Yin***) $B(u) = [\Lambda^1, u\partial_x]\Lambda^{-1} \in L(H^{s-1})$ *for* $u \in H^s$, $s > 3/2$. *Moveover,*

$$\| (B(u) - B(z)) w\|_{s-1} \le \mu_2 \|u - z\|_s \|w\|_{s-1}, \quad u, z \in H^s, w \in H^{s-1}.$$

Lemma 4.3.6. *Let* $f(u) = -\partial_x g * (\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2)$. *Then,* f *is bounded on bounded set in* H^s *and satisfies*

$$\begin{aligned}
&(a) \ \|f(y) - f(z)\|_s \le \mu_3 \|y - z\|_s, \quad y, z \in H^s, \\
&(b) \ \|f(y) - f(z)\|_{s-1} \le \mu_4 \|y - z\|_{s-1}, \quad y, z \in H^s.
\end{aligned} \tag{4.6}$$

Proof. Let $y, z \in H^s$, $s > \frac{3}{2}$. Since H^{s-1} is a Banach algebra, it follows that

$$\|f(y) - f(z)\|_s$$

$$= \left\| -\partial_x (1 - \partial_x^2)^{-1} \left(\frac{b}{2}(y^2 - z^2) + \frac{3-b}{2}(y_x^2 - z_x^2) \right) \right\|_s$$

$$\leq \frac{|b|}{2} \|(y-z)(y+z)\|_{s-1} + \frac{|3-b|}{2} \|(y_x - z_x)(y_x + z_x)\|_{s-1}$$

$$\leq \frac{|b|}{2} \left(\|y-z\|_{s-1} \|y+z\|_{s-1} \right) + \frac{|3-b|}{2} \left(\|y_x - z_x\|_{s-1} \|y_x + z_x\|_{s-1} \right)$$

$$\leq c\|y-z\|_s \left(\|y\|_s + \|z\|_s \right)$$

This proves (a). Taking $z = 0$ in the above inequality, we obtain that f is bounded on any bounded set in H^s.

On the other hand, let $y, z \in H^s$, $s > \frac{3}{2}$. Then we have

$$\|f(y) - f(z)\|_{s-1}$$

$$= \left\| -\partial_x (1 - \partial_x^2)^{-1} \left(\frac{b}{2}(y^2 - z^2) + \frac{3-b}{2}(y_x^2 - z_x^2) \right) \right\|_{s-1}$$

$$\leq \frac{|b|}{2} \|(y-z)(y+z)\|_{s-1} + \frac{|3-b|}{2} \|(y_x - z_x)(y_x + z_x)\|_{s-2}$$

$$\leq \frac{|b|}{2} \left(\|y-z\|_{s-1} \|y+z\|_{s-1} \right) + \frac{|3-b|}{2} \left(\|y_x - z_x\|_{s-2} \|y_x + z_x\|_{L^\infty} \right)$$

$$\leq \frac{|b|}{2} \left(\|y-z\|_{s-1} \|y+z\|_{s-1} \right) + \frac{|3-b|}{2} \left(\|y_x - z_x\|_{s-2} \|y_x + z_x\|_{s-1} \right)$$

$$\leq c\|y-z\|_{s-1} \left(\|y\|_s + \|z\|_s \right).$$

where use has been made of the Sobolev imbedding theorem $H^{s-1} \hookrightarrow L^\infty$. This completes the proof of Lemma.

$\square$

Proof of Theorem 4.3.2. Theorem 4.3.2 follows from Lemma 4.1.1 and Lemmas 4.3.2-4.3.7.

$\square$

Theorem 4.3.7. *The maximal T in Theorem 4.3.2 may be chosen independent of s in the following sense. If $u = u(\cdot, u_0) \in C\left([0, T), H^s\right) \cap C^1\left([0, T), H^{s-1}\right)$ to (2.3) and*

$u_0 \in H^{s'}$ for some $s' \neq s$, $s' > \frac{3}{2}$, then $u \in C\left([0,T), H^{s'}\right) \cap C^1\left([0,T), H^{s'-1}\right)$ and with the same T. In particular, if $u_0 \in H^\infty = \bigcap_{s \geq 0} H^s$, then $u \in C\left([0,T), H^\infty\right)$.

To prove this theorem, we need the following lemma.

Lemma 4.3.8. (Kato) *Let* s, t *be real number such that* $-s < t \leq s$. *Then* $L^p_s \bigcap L^\infty$ *is an algebra. Moreover,*

$$\|fg\|_t \leq c\|f\|_s |g|_t, \quad if \quad s > \frac{1}{2}$$

$$\|fg\|_{s+t-\frac{1}{2}} \leq c\|f\|_s |g|_t, \quad if \quad s < \frac{1}{2}$$

where c *is a positive constant depending on* s *and* t.

Lemma 4.3.9. (Kato) *Let* $f \in H^r, r > \frac{3}{2}$ *and let* M_f *be the multiplication operator by* f. *Then* $\Lambda^{-\tilde{s}}[\Lambda^{\tilde{s}+\tilde{t}+1}, M_f]\Lambda^{-\tilde{t}} \in L\left(L^2(\mathbb{R}^2)\right)$, *if* $|\tilde{s}|, |\tilde{t}| \leq r - 1$. *Moreover,*

$$\|\Lambda^{-\tilde{s}}[\Lambda^{\tilde{s}+\tilde{t}+1}, M_f]\Lambda^{-\tilde{t}}\|_{L(L^2)} \leq c\|\partial f\|_{r-1}.$$

Proof of Theorem . It suffices to consider the case $s' > s$, since the case $s' < s$ is obvious from the uniqueness of solutions which is guaranteed by Theorem 4.3.2. Also we can suppose that $s < s' \leq s + 1$. Since if $s' > s + 1$, we obtain the result by iterated application of the below argument.

If we apply operator Λ^2 to (4.4), we obtain the following evolution equation for $m(t) = \Lambda^2 u(t) = u - u_{xx}$:

$$
\begin{cases}
\frac{d}{dt}m(t) + A(t)m + B(t)m = 0 \\
m(0) = \Lambda^2 u(0)
\end{cases}
\tag{4.7}
$$

where $A(t)m = \partial_x(um)$, $B(t)m = (b-1)u_x m$ and $u \in C\left([0,T), H^s\right)$ is viewed as a known function. Note also that $m \in C\left([0,T), H^{s-2}\right)$ and $m(0) = \Lambda^2 u(0) \in H^{s'-2}$. Our objective is to prove that $m \in C\left([0,T), H^{s'-2}\right)$ which will imply $u \in C\left([0,T), H^{s'}\right)$, because $(1 - \partial_x^2)$ is an isomorphism from $H^{s'}$ to $H^{s'-2}$. This will complete the proof of the Theorem .

Note that $u \in C\left([0, T), H^s\right), u_x \in H^{s-1}$, and that H^{s-1} is a Banach algebra. Then we obtain $B(t) \in L(H^{s-1})$.

To accomplish this, following the argument in Lemmas 3.1-3.3 in [?]) we first need to prove that the family $A(t)$ has a unique evolution operator $\{U(t, \tau)\}$ associated with the spaces $X = H^h$ and $Y = H^k$, where $-s \leq h \leq s-2, 1-s \leq k \leq s-1$, and $k \geq h+1$. Therefore, according to the proof of Lemma 3.1 in [22], we need to verify the following three conditions.

(i) $A(t) \in G(H^h, 1, \beta)$.

(ii) $\Lambda^h \partial_x [\Lambda^{k-h}, u] \Lambda^{-k}$ is uniformly bounded on L^2.

(iii) $A(t) \in L(H^k, H^h)$ is strongly continuous in t.

Let us begin verifying (i). Due to H^h being a Hilbert space, $A(t) \in G(H^h, 1, \beta)$ that is, we will show the following conditions([22])

(a) $(A(t)w, w)_h \geq -\beta \|w\|_h^2$,

(b) $-A(t)$ is the infinitesimal generator of a C_0-semigroup on H^h, for some (or all)$\lambda > \beta$.

To prove (a), note that

$$\left(\Lambda^h \partial_x(u(t)w), \Lambda^h w\right)_0 \geq -\beta \|w\|_h^2. \tag{4.8}$$

We begin estimating the term on the left-hand side of this inequality, which can be written of the following way

$$-\left(\Lambda^h(uw), \partial_x \Lambda^h w\right)_0, \tag{4.9}$$

but we have

$$\Lambda^h(uw) = \Lambda^h\left(u\Lambda^{-h}(\Lambda^h w)\right) = \Lambda^h\left(\Lambda^{-h}(u\Lambda^h w) - [\Lambda^{-h}, u]\Lambda^h w\right)$$
$$= u\Lambda^h w - \Lambda^h[\Lambda^{-h}, u]\Lambda^h w.$$

(4.10)

Using the identity in (4.9), we obtain

$$-\left(\Lambda^h(uw), \partial_x\Lambda^h w\right)_0$$
$$= \left(\Lambda^h[\Lambda^{-h}, u]\Lambda^h w, \partial_x\Lambda^h w\right)_0 - \left(u\Lambda^h w, \partial_x\Lambda^h w\right)_0$$
$$= \left(\Lambda^{h+1}[\Lambda^{-h}, u]\Lambda^h w, \partial_x\Lambda^{h-1} w\right)_0 - \left(u\Lambda^h w, \partial_x\Lambda^h w\right)_0.$$

(4.11)

The second term on the right-hand side of (4.11) can be easily estimated by applying the Cauchy-Schwartz inequality and integration by parts. For the other we apply the Cauchy-Schwartz inequality and Lemma 4.3.9 (with $r = s > \frac{3}{2}, \tilde{s} = -h - 1, \tilde{t} = 0$). Thus (4.8) is proved.

Next we verify (b). Let $S = \Lambda^{s-1-h}$. Note that S is an isomorphism of H^{s-1} onto H^h and that H^{s-1} is continuously and densely embedded in H^h as $-s \leq h \leq s - 2$. Define

$$A_1(t) := SA(t)S^{-1} = \Lambda^{s-1-h}A(t)\Lambda^{1+h-s},$$

$$B_1(t) := A_1(t) - A(t) = [S, A(t)]S^{-1}.$$

Let $w \in H^h$ and $u \in H^s, s > \frac{3}{2}$. Then it follows from Lemma 4.3.9 with $\tilde{s} = -(h+1), \tilde{t} = s - 1$ that

$$\|B_1(t)w\|_h = \left\|\Lambda^h\partial_x[\Lambda^{s-1-h}, u]\Lambda^{1+h-s}w\right\|_0$$
$$\leq \left\|\Lambda^h\partial_x[\Lambda^{s-1-h}, u]\Lambda^{1-s}\right\|_{L(L^2)}\|\Lambda^h w\|_0$$
$$\leq \|u\|_s\|w\|_h.$$

Therefore, we have $B_1(t) \in L\left(H^h\right)$. Since $A(t)m = \partial_x(um) = u\partial_x m + \partial_x um$, and $\partial_x u \in L\left(H^{s-1}\right)$, by applying Lemma 4.3.5 and a perturbation theorem for semigroups, we see that H^{s-1} is $A(t)-$admissible. Further, applying Lemma 5.3 in [28] with

$Y = H^{s-1}, X = H^h$ and $S = \Lambda^{s-1-h}$, we obtain that $-A_1(t)$ is the infinitesimal generator of a C_0−semigroup on H^h. Since $A_1(t) = A(t) + B_1(t)$ and $B_1(t) \in L\left(H^h\right)$, by a perturbation theorem for semigroups it follows that $-A(t)$ is the infinitesimal generator of a C_0−semigroup on H^h. This proves (b).

Next, we show (ii). But this is again a consequence from Lemma 4.3.9, since H^{k-h} is the isomorphism of Y onto X and we have the following estimate

$$\left\|\Lambda^h \partial_x [\Lambda^{k-h}, u]\Lambda^{-k}w\right\|_0 \leq c\|u\|_s\|w\|_0.$$

Finally, we verify (iii). In fact, consider the continuity of u and the following inequality

$$
\begin{aligned}
\|\left(A(t+\Delta t) - A(t)\right)w\|_h &= \|\partial_x\left((u(t+\Delta t) - u(t))w\right)\|_h \\
&\leq \|\left(u(t+\Delta t) - u(t)\right)w\|_{h+1} \\
&\leq \|u(t+\Delta t) - u(t)\|_{s-1}\|w\|_{h+1} \\
&\leq \|u(t+\Delta t) - u(t)\|_s\|w\|_k.
\end{aligned}
\tag{4.12}
$$

In view of the second inequality in Lemma 4.3.8, it is easy to see (iii) holds. Thus, the above three conditions imply the existence and uniqueness of evolution operator $U(t,\tau)$ for the family $A(t)$. In particular $U(t,\tau)$ maps H^r into itself for $-s \leq r \leq s-1$.

Choose $Y = H^{s-2}, X = H^{s-3}$ and note that $m \in C\left([0,T), H^{s-1}\right) \cap C^1\left([0,T), H^{s-2}\right)$. By the properties of evolution operator $U(t,\tau)$, we deduce that

$$\frac{d}{d\tau}\left(U(t,\tau)m(\tau)\right) = -U(t,\tau)B(\tau)m(\tau) \tag{4.13}$$

An integration in $\tau \in [0, t]$ yields

$$m(t) = U(t,0)m(0) - \int_0^t U(t,\tau)B(\tau)m(\tau)d\tau. \tag{4.14}$$

If $s < s' \leq s + 1$, then we have that $B(t) = \partial_x u \in L(H^{s'-2})$ is strongly continuous in $[0, t)$, and that $H^{s-1} H^{s'-2} \subset H^{s'-2}$ by $s - 1 > \frac{1}{2}$ (this is a consequence of Lemma 2.9). Due to $-s < s - 2 < s' - 2 \leq s - 1$, the family $\{U(t, \tau)\}$ is strongly continuous on $H^{s'-2}$ to itself. Observe that $m(0) \in H^{s'-2}$, we have only see (2.13) as an integral equation of Volterra type, which can be solved for m by successive approximation. This completes the proof of Theorem 4.3.7.

$\square$

CHAPTER 5

WAVE BREAKING AND BLOW UP

Wave breaking is one of the most intriguing long-standing problems of water wave theory [7]. For models describing water waves we say that wave breaking holds if the wave profile remains bounded, but its slope becomes unbounded in finite time [7]. Breaking waves are commonly observed in the ocean and important for a variety of reasons, but surprisingly little is known about them. Indeed, breaking waves place large hydrodynamic loads on man-made structure, transfer horizontal momentum to surface currents, provide a source of turbulent energy to mix the upper layers of the ocean, move sediment in shallow water, and enhance the air-sea exchange of gases and particulate matter . To further understand why waves break and what happens during and after breaking themselves, we must first investigate the dynamics of wave breaking. Research work on breaking waves can be divided into three categories: those concerning waves (1) before, (2) during, and (3) after breaking. Although we are now understanding much about the processes leading up to breaking, there are still some aspects of these questions unanswered, in particular, question (3), what happens after breaking of those waves. In this review we shall concentrate on some of the latest results for the DP equation in the first two categories.

The KdV equation is well-known a model for water-motion on shallow water with a flat bottom and admits interaction for its solitary waves. It, however, does not describe breaking of wave as physical water waves do (the KdV equation is globally well-posed for

initial data in L^2 [9]). On the other hand, wave-breaking phenomena have been observed for certain solutions to the Whitham equation [7],

$$u_t + uu_x + \int_{\mathbb{R}} K(x - \xi)u_x(t, \xi)d\xi = 0$$

with the singular kernel

$$K(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{\tanh \xi}{\xi}\right)^{1/2} e^{i\xi x}d\xi,$$

(see [3] for a rigorous proof). However, the numerical calculations carried out for the Whitham equation do not support any strong claim that soliton interaction can be expected [5]. As mentioned by Whitham , it is intriguing to know which mathematical models for shallow water waves exhibit both phenomena of soliton interaction and wave breaking. It is found that both of the CH equation and DP equation could be first such equations and have the potential to become the new master equations for shallow water wave theory [8], modeling the soliton interaction of peaked traveling waves, wave breaking, admitting as solutions permanent waves, and being integrable Hammiltonian systems. For the CH equation, a procedure to understand the continuation of solutions past wave breaking has been recently presented by Bressan and Constantin .

As far as we know, the case of the Camassa-Holm equation (first derived by Fokas and Fuchssteiner [8] using the method of recursion operators as an abstract bi-Hamiltonian equation) is well understood by now and the citations therein, while the Degasperis-Procesi equation case is the subject of this article. The main mathematical questions concerning with the DP equation are the well-posedness of the initial-value problem, wave-breaking phenomena, existence of global weak solutions, and stability of peakons and their role in the dynamics.

Since its discovery, there has been considerable interest in the Deasperis-Procesi equation, [6] and the citations therein. For example, Lundmark and Szmigielski [15]

presented an inverse scattering approach for computing n-peakon solutions to Eq.(1.5). Holm and Staley studied stability of solitons and peakons numerically to those equations. More recently, Liu and Yin [26] proved that the first blow-up for D-P Eq. must occur as wave breaking and shock waves possibly appear afterwards. It is shown in [26] that the lifespan of solutions of the DP equation is not affected by the smoothness and size of initial profiles, but affected by the shape of initial profiles. This can be viewed as a significant difference between the DP equation (or the CH equation) and the KdV equation.

5.1 Blow-up

By using the local well-posedness result and energy estimates, the following precise blow-up scenario of strong solutions to (4.4) can be obtained.

Theorem 5.1.1. *Assume $b \geq 1$ and $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$. Then blow up of the strong solution $u = u(\cdot, u_0)$ in finite time $T < +\infty$ occurs if and only if*

$$\liminf_{t \uparrow T} \{\inf_{x \in \mathbb{S}} [u_x(t, x)]\} = -\infty.$$

Proof. Applying a simple density argument, we only need to show that the above theorem with some $s \geq 3$. Here we assume $s = 3$ to prove the above theorem. Multiplying (4.1.1) with m and integrating on $\mathbb{S}$ with respect to x, we obtain

$$\frac{d}{dt} \int_{\mathbb{S}} m^2 dx = -(2b - 1) \int_{\mathbb{S}} m^2 u_x dx \qquad (5.1)$$

On the other hand, differentiating (4.1.1) with respect to x and multiplying with m_x, integrating on $\mathbb{R}$ with respect to x, and integrating by parts yield

$$\frac{d}{dt} \int_{\mathbb{S}} m_x^2 dx = -(2b + 1) \int_{\mathbb{S}} m_x^2 u_x dx + b \int_{\mathbb{S}} m^2 u_x dx. \qquad (5.2)$$

It is thereby inferred from (5.1) and (5.2) that

$$\frac{d}{dt}\int_{\mathbb{S}}(m^2 + m_x^2)dx = -(2b+1)\int_{\mathbb{S}}m_x^2 u_x dx - (b-1)\int_{\mathbb{S}}m^2 u_x dx. \tag{5.3}$$

If u_x is bounded from below on $[0, T) \times \mathbb{R}$, i.e., there exists $M > 0$ such that

$$-u_x(t, x) \leq M \quad \text{on } [0, T) \times \mathbb{R},$$

then the relation (5.3) implies

$$\frac{d}{dt}\int_{\mathbb{S}}\left(m^2 + m_x^2\right)dx \leq (2b+1)M\int_{\mathbb{S}}\left(m^2 + m_x^2\right)dx.$$

And by means of Gronwall's inequality, we deduce that

$$\int_{\mathbb{S}}m^2 + m_x^2 dx \leq \left(\int_{\mathbb{S}}m_0^2 + m_{0x}^2 dx\right)e^{(2b+1)Mt}, \quad \forall t \in [0, T). \tag{5.4}$$

Noting that

$$\|u(t)\|_3 \leq \left(\int_{\mathbb{R}}m^2 + m_x^2 dx\right)^{1/2}$$

and in view of (3.4), it follows that if $\{u_x(t)\}$ is bounded from below on $[0, T)$, then the $H^3(\mathbb{R})$-norm of the solution to Eq.(2.3) is said not to have broken in finite time. This completes the proof of Theorem 5.1.1. $\qquad\qquad\square$

Let us now consider the following differential equation

$$\begin{cases} q_t = u(t, q), & t \in [0, T), \\ q(0, x) = x, & x \in \mathbb{R}. \end{cases} \tag{5.5}$$

Applying classical results in the theory of ordinary differential equations, we have the following properties of q which are crucial in the proof of global existence. The consideration of (5.5) is geometrically motivated for the Camassa-Holm equation [13]. Such a geometric interpretation is lacking for the peakon b-family of equations with $b \neq 2$, but nevertheless some important invariance properties can be deduced (see below).

Lemma 5.1.2. *Let $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, and let $T > 0$ be the maximal existence time of the corresponding strong solution u to the b-family Eq. Then the Eq.(5.5) has a unique solution $q \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$ such that the map $q(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$q_x(t,x) = \exp\left(\int_0^t u_x(s, q(s,x))ds\right) > 0, \quad \forall (t,x) \in [0,T) \times \mathbb{R}. \tag{5.6}$$

Furthermore, setting $m = u - u_{xx}$, we have

$$m(t, q(t,x))q_x^b(t,x) = m_0(x), \quad \forall (t,x) \in [0,T) \times \mathbb{R}. \tag{5.7}$$

Proof. Since $u \in C^1([0,T), H^{s-1}(\mathbb{R}))$ and $H^s(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$, we see that both functions $u(t,x)$ and $u_x(t,x)$ are bounded, Lipschitz in the space variable x, and of class C^1 in time. Therefore, for fixed $x \in \mathbb{R}$, Eq.(3.5) is an ordinary differential equation. Then well-known classical results in the theory of ordinary differential equation yield that Eq.(3.5) has a unique solution $q(t,x) \in C^1([0,T) \times \mathbb{R}; \mathbb{R})$.

Differentiation of Eq.(5.5) with respect to x yields

$$\begin{cases} \frac{d}{dt}q_x = u_x(t,q)q_x, & t \in [0,T), \\ q_x(0,x) = 1, x \in \mathbb{R}. \end{cases} \tag{5.8}$$

The solution to Eq.(3.8) is given by

$$q_x(t,x) = \exp\left(\int_0^t u_x(s, q(s,x))ds\right), \quad (t,x) \in [0,T) \times \mathbb{R}. \tag{5.9}$$

For every $T' < T$, it follows from the Sobolev imbedding theorem that

$$\sup_{(s,x)\in[0,T')\times\mathbb{R}} |u_x(s,x)| < \infty.$$

We infer from (5.9) that there exists a constant $K > 0$ such that $q_x(t, x) \geq e^{-Kt}$, $(t, x) \in [0, T) \times \mathbb{R}$, which implies that the map $q(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with

$$q_x(t, x) = \exp\left(\int_0^t u_x(s, q(s, x))ds\right) > 0, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

On the other hand, combining (5.8) with (4.1.1), we have

$$\frac{d}{dt}\left(m(t, q(t, x))q_x^b(t, x)\right) = (m_t + m_x q_x) q_x^b(t, x) + bm q_x^{b-1} q_{xt}$$
$$= q_x^b(m_t + m_x u + bu_x m) = 0. \tag{3.10}$$

So,

$$m(t, q(t, x))q_x^b(t, x) = m_0(x), \quad \forall (t, x) \in [0, T) \times \mathbb{R}. \tag{3.11}$$

This completes the proof of Lemma 5.2. $\qquad\square$

Remark. Lemma 5.2 shows that, if $m_0 = u_0 - u_{0xx}$ does not change sign, then $m(t)$ ($\forall\, t$) will not change sign, as long as $m(t)$ exists.

It is observed that if $u(t, x)$ is a solution to the periodic b-family eq. with $u(0, x) = u_0(x)$, then $-u(t, -x)$ is also a solution to the same with initial datum $-u_0(-x)$. Hence, due to the uniqueness of the solutions, the solution is odd as long as the initial datum $u_0(x)$ is odd. So the first blow-up main theorem is concerned with this type of initial data.

Theorem 5.1.3. *Let $1 < b \leq 3$ and $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$ be odd and nonzero. If $u_{0x} \leq 0$, then the corresponding solution $u(t)$ with initial value $u(0) = u_0$ blows up in finite time.*

Proof. Again, applying a simple density argument, we only need to show that the above theorem with some $s \geq 3$. Since u_0 is odd, then $u(t, x)$ is odd and $u(t, 0) = u_{xx}(t, 0) = 0$. Taking derivatives with respect to x on both sides of the b-family eq., we obtain

$$u_{xt} = -u_x^2 - uu_{xx} - \partial_x^2 g * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right)$$
$$= \frac{b}{2}u^2 - \frac{b-1}{2}u_x^2 - uu_{xx} - g * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right) \tag{5.12}$$

Taking values of (5.12) at $x = 0$ and letting $h(t) = u_x(t, 0)$, it is deduced that

$$\frac{dh}{dt} = -\frac{b-1}{2}h^2 - g * (\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2).$$

If $h(0) = u_{0x} < 0$, then

$$\frac{dh}{dt} \leq -\frac{b-1}{2}h^2.$$

So,

$$h(t) \leq \frac{1}{\frac{b-1}{2}t + \frac{1}{h(0)}},$$

which tends to $-\infty$ as t goes to $-\frac{1}{h(0)}$.

If $h(0) = u_{0x} = 0$, by the continuity of the ordinary differential equation and the uniqueness, we have

$$\frac{dh}{dt} \leq -g * (\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2),$$

and consequently $h(t) < 0$, for all $t > 0$. So we can choose some time $t_0 > 0$, such that

$$\frac{dh}{dt} \leq -\frac{b-1}{2}h^2$$

for $t > t_0$, and $h(t_0) < 0$. Then we get the finite time blow-up result by the previous discussion. $\square$

Remark. It was shown by McKean that the solutions of the CH equation breaks down if and only if some portion of the positive part of $y = u - \partial_x^2 u$ initially lies to the left of some portion of its negative part. The problem whether or not the DP equation has these wave breaking phenomena still remains open. Because of the structural difference between these two equations, it is difficult to use the machinery of McKean in study of the associated spectral problem with the corresponding eigenvalues. The issue of if and how these particular initial data generate a global solution or blow-up in finite time is more subtle.

In contrast with the conditions of the blow-up solution of the DP equation defined on the line $\mathbb{R}$, one can see that the criteria of blow-up for periodic solutions of the DP equation are quite different. Let us consider periodic solutions, i.e, $u : \mathbb{S} \times [0, T) \to \mathbb{R}$ where $\mathbb{S}$ is the unit circle and $T > 0$ is the maximal existence time of the solution. The interest in periodic solutions is motivated by the observation that the majority of the waves propagating on a channel are approximately periodic.

Define $G(x)$ by $G(x) = \frac{\cosh(x-[x]-\frac{1}{2})}{2\sinh(\frac{1}{2})}$, where $[x]$ stands for the integer part of $x \in \mathbb{R}$, then $(1 - \partial_x^2)^{-1} f = G * f$ for all $f \in L^2(\mathbb{S})$. Using this identity, we can rewrite b-family eq. as a quasi-linear evolution equation of hyperbolic type, namely,

$$\begin{cases} u_t + uu_x + \partial_x G * (\frac{3}{2}u^2 + \frac{3-b}{2}u_x^2) = 0, & t > 0, \ x \in \mathbb{S}, \\ u(0, x) = u_0(x), & x \in \mathbb{S}, \\ u(t, x) = u(t, x+1), & t \geq 0, \ x \in \mathbb{S}, \end{cases} \qquad (5.1.1)$$

Theorem 5.1.4. *Let $\frac{5}{3} < b \leq 3$ and $\int_{\mathbb{S}} u_x^3(0) \, dx < 0$. Assume that $u_0 \in H^s(\mathbb{S}), s > \frac{3}{2}$, $u_0 \not\equiv 0$, and the corresponding solution $u(t)$ of (5.1.1) has a zero for any time $t \geq 0$. Then, the solution $u(t)$ of(5.1.1) blows up in finite time.*

Proof. Proof of blow-up solution for the periodic case is quite different from that of the line case. By assumption, for each $t \in [0, T)$ there is a $\xi_t \in [0, 1]$ such that $u(t, \xi_t) = 0$. Then for $\forall x \in \mathbb{S}$ we have

$$u^2(t, x) = \left(\int_{\xi_t}^x u_x \, dx \right)^2 \leq (x - \xi_t) \int_{\xi_t}^x u_x^2 \, dx, \quad x \in \left[\xi_t, \xi_t + \frac{1}{2} \right]. \qquad (5.1.2)$$

Hence the above relation and an integration by parts yield

$$\int_{\xi_t}^{\xi_t+\frac{1}{2}} u^2 u_x^2 \, dx \leq \int_{\xi_t}^{\xi_t+\frac{1}{2}} (x - \xi_t) u_x^2 \left(\int_{\xi_t}^x u_x^2 \right) dx \leq \frac{1}{4} \left(\int_{\xi_t}^{\xi_t+\frac{1}{2}} u_x^2 \, dx \right)^2.$$

Combining this estimate with a similar estimate on $[\xi_t + \frac{1}{2}, \xi_t + 1]$, we obtain

$$\int_{\mathbb{S}} u^2 u_x^2 \, dx \leq \frac{1}{4} \left(\int_{\mathbb{S}} u_x^2 \, dx \right)^2. \tag{5.1.3}$$

We also have

$$\sup_{x \in \mathbb{S}} u^2(t, x) \leq \frac{1}{2} \int_{\mathbb{S}} u_x^2 \, dx. \tag{5.1.4}$$

Let us assume that the solution $u(t, x)$ exists globally in time. Note that $G(x) \geq \frac{1}{2 \sinh(\frac{1}{2})}$

for all $x \in \mathbb{S}$. Then

$$\frac{d}{dt} \int_{\mathbb{S}} u_x^3 \, dx = 3 \int_{\mathbb{S}} u_x^2 \left(-\frac{b-1}{2} u_x^2 - u u_{xx} + \frac{b}{2} u^2 - G * \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 \right) \right) dx$$

$$= -3\frac{b-1}{2} \int_{\mathbb{S}} u_x^4 \, dx - 3 \int_{\mathbb{S}} u_x^2 u u_{xx} \, dx + \frac{3b}{2} \int_{\mathbb{S}} u_x^2 u^2 \, dx - 3 \int_{\mathbb{S}} u_x^2 G * \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 \right) dx$$

$$= -3\frac{b-1}{2} \int_{\mathbb{S}} u_x^4 \, dx + \int_{\mathbb{S}} u_x^4 \, dx + \frac{3b}{8} \left(\int_{\mathbb{S}} u_x^2 \, dx \right)^2 - 3 \int_{\mathbb{S}} u_x^2 G * \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 \right) dx$$

$$\tag{5.1.5}$$

Since $G \geq \frac{1}{2sh(\frac{1}{2})} > 0$, and $0 \leq b \leq 3$, we have

$$- \int_{\mathbb{S}} u_x^2 G * \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 \right) dx$$

$$\leq -\frac{3}{2sh(\frac{1}{2})} \int_{\mathbb{S}} u_x^2 \, dx \int_{\mathbb{S}} \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 \right) dx \tag{5.1.6}$$

$$\leq -\frac{3}{4} \frac{3-b}{sh(\frac{1}{2})} \int_{\mathbb{S}} u_x^2 \, dx \int_{\mathbb{S}} u_x^2 \, dx$$

Combining (5.1.5) and (5.1.6) we obtain

$$\frac{d}{dt} \int_{\mathbb{S}} u_x^3 \, dx \leq -\frac{3b-5}{2} \int_{\mathbb{S}} u_x^4 \, dx + \left(\frac{3b}{8} - \frac{3}{4} \frac{3-b}{sh(\frac{1}{2})} \right) \left(\int_{\mathbb{S}} u_x^2 \, dx \right)^2 \tag{5.1.7}$$

Case(i): $\frac{5}{3} < b \leq \frac{6}{2+sh(\frac{1}{2})}$

Since

$$\begin{cases} b \leq \frac{6}{2+sh(\frac{1}{2})} \\[2mm] 3bsh(\frac{1}{2}) \leq 18 - 6b \\[2mm] \frac{3b}{2} \leq \frac{3(3-b)}{sh(\frac{1}{2})} \\[2mm] \frac{3b}{2} \leq \frac{3(3-b)}{sh(\frac{1}{2})} \\[2mm] \frac{3b}{8} - \frac{3}{4}\frac{3-b}{sh(\frac{1}{2})} \leq 0 \end{cases}$$

And this implies parenthesied term in the second expression is less than zero and therefore the term could be dropped altogether and we would have

$$\frac{d}{dt}\int_{\mathbb{S}} u_x^3 \, dx \leq -\frac{3b-5}{2}\int_{\mathbb{S}} u_x^4 \, dx \tag{5.1.8}$$

If we define $V(t) := \int_{\mathbb{S}} u_x^3(t,x)\,dx$ for all $t \geq 0$, then

$$V(t) \leq V(0), \quad t \geq 0.$$

Since $V(0) < 0$, the above inequality implies that $V(t) < 0$ for all $t \geq 0$. It is then inferred that

$$\frac{d}{dt}V(t) \leq -\frac{3b-5}{2}\left(V(t)\right)^{\frac{4}{3}}, \quad t > 0.$$

Thus we have

$$\left(\frac{(3b-5)t}{6} + \frac{1}{(V(0))^{\frac{1}{3}}}\right)^3 \leq \frac{1}{V(t)} < 0, \quad t \geq 0.$$

Since $V(0) < 0$, the above inequality will lead to a contradiction as $t \geq 0$ is big enough, which implies $T < \infty$.

Case(ii): $\frac{6}{2+sh\left(\frac{1}{2}\right)} \leq b < 3$

Since

$$
\begin{cases}
b \geq \frac{6}{2+sh\left(\frac{1}{2}\right)} \\[2mm]
3bsh\left(\frac{1}{2}\right) \geq 18 - 6b \\[2mm]
\frac{3b}{2} \geq \frac{3(3-b)}{sh\left(\frac{1}{2}\right)} \\[2mm]
\frac{3b}{2} \geq \frac{3(3-b)}{sh\left(\frac{1}{2}\right)} \\[2mm]
\frac{3b}{8} - \frac{3}{4}\frac{3-b}{sh\left(\frac{1}{2}\right)} \geq 0
\end{cases}
$$

Therefore this enables us to apply the Cauchy-Schwartz inequality and we consequently obtain

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{S}} u_x^3\, dx &\leq -\frac{3b-5}{2}\int_{\mathbb{S}} u_x^4\, dx + \left(\frac{3b}{8} - \frac{3}{4}\frac{3-b}{sh\left(\frac{1}{2}\right)}\right)\left(\int_{\mathbb{S}} u_x^4\, dx\right) \\[2mm]
&= -\left[\frac{3b-5}{2} + \frac{3b}{8} - \frac{3(3-b)}{4sh\left(\frac{1}{2}\right)}\right]\int_{\mathbb{S}} u_x^4\, dx \\[2mm]
&= \left[\frac{-12b+20+3b}{8} - \frac{3(3-b)}{4sh\left(\frac{1}{2}\right)}\right]\int_{\mathbb{S}} u_x^4\, dx \\[2mm]
&= \left[\frac{-9b+20}{8} - \frac{3(3-b)}{4sh\left(\frac{1}{2}\right)}\right]\int_{\mathbb{S}} u_x^4\, dx \\[2mm]
&= -\left[\frac{9b-20}{8} + \frac{3(3-b)}{4sh\left(\frac{1}{2}\right)}\right]\int_{\mathbb{S}} u_x^4\, dx
\end{aligned}
\tag{5.1.9}
$$

We claim that $\frac{9b-20}{8} + \frac{3(3-b)}{4sh\left(\frac{1}{2}\right)} > 0$ and hence we obtain another bound for b which is $b < \frac{18-20sh\left(\frac{1}{2}\right)}{6-9sh\left(\frac{1}{2}\right)}$

But $3 < \frac{18-20sh\left(\frac{1}{2}\right)}{6-9sh\left(\frac{1}{2}\right)}$ and $b \leq 3$. This implies that $\frac{9b-20}{8} + \frac{3(3-b)}{4sh\left(\frac{1}{2}\right)} > 0$. Hence an application of Hölder's inequality on (5.1.9) yields

$$
\frac{d}{dt}\int_{\mathbb{S}} u_x^3\, dx \leq -K\left(\int_{\mathbb{S}} u_x^3\, dx\right)^{\frac{4}{3}}, \quad t \geq 0.
\tag{5.1.10}
$$

where $K = \frac{9b-20}{8} + \frac{3(3-b)}{4sh\left(\frac{1}{2}\right)} > 0$.

If we define $V(t) := \int_{\mathbb{S}} u_x^3(t,x)\, dx$ for all $t \geq 0$, then

$$V(t) \leq V(0), \quad t \geq 0.$$

Since $V(0) \leq 0$, the above inequality implies that $V(t) < 0$ for all $t \geq 0$. It is then inferred that

$$\frac{d}{dt}V(t) \leq -\mathbb{K}\,(V(t))^{\frac{4}{3}}, \quad t > 0.$$

Thus we have

$$\left(\frac{Kt}{3} + \frac{1}{(V(0))^{\frac{1}{3}}}\right)^3 \leq \frac{1}{V(t)} < 0, \quad t \geq 0.$$

Since $V(0) < 0$, the above inequality will lead to a contradiction as $t \geq t_0$ is big enough, which implies $T < \infty$. $\qquad\square$

As immediate consequences of Theorem 5.1.4, we have

Corollary 5.1.5. *If $u_0 \in H^3(\mathbb{S})$, $u_0 \not\equiv 0$ and $\int_{\mathbb{S}} u_0\, dx = 0$ or $\int_{\mathbb{S}} y_0\, dx = 0$, then the corresponding solution u to (5.1.1) blows up in finite time.*

Proof. Note

$$\int_{\mathbb{S}} u(t,x)\, dx = \int_{\mathbb{S}} y(t,x)\, dx = \int_{\mathbb{S}} y_0(x)\, dx = \int_{\mathbb{S}} u_0(x)\, dx = 0.$$

The above relation shows that $u(t,x)$ has a zero for all $t \in \mathbb{S}$. It follows from Theorem 5.1.4 that the solution u to (5.1.1.) blows up in finite time. $\qquad\square$

CHAPTER 6

GLOBAL EXISTENCE

6.1 Preamble

We plan to show that there exists global strong solutions to periodic b-family equations provided initial data u_0 satisfies sign conditions.Throughout this chapter $*$ stands for convolution and let $\mathbb{S} := \mathbb{R}\backslash\mathbb{Z}$ be the circle of unit length. Let us consider periodic solutions of (3.1), i.e, $u : \mathbb{S} \times [0,T) \to \mathbb{R}$ where $\mathbb{S}$ is the unit circle and $T > 0$ is the maximal existence time of the solution. The interest in periodic solutions is motivated by the observation that the majority of the waves propagating on a channel are approximately periodic.

Define $G(x)$ by $G(x) = \frac{\cosh(x-[x]-\frac{1}{2})}{2\sinh(\frac{1}{2})}$, where $[x]$ stands for the integer part of $x \in \mathbb{R}$, then $(1 - \partial_x^2)^{-1}f = G * f$ for all $f \in L^2(\mathbb{S})$.

Lemma 6.1.1. *If $u_0 \in H^r(\mathbb{S})$, $r \geq \frac{3}{2}$, then as long as the solution $u(t,x)$ is given by local well posedness theorem exists, we have*

$$\int_{\mathbb{S}} u(t,x) \, dx = \int_{\mathbb{S}} u_0 \, dx = \int_{\mathbb{S}} y(t,x) \, dx = \int_{\mathbb{S}} y_0 \, dx$$

Proof. u, G are both peridic in spatial variable,hence rewriting the original b-family equation

$$\begin{cases} u_t + uu_x = -\partial_x G * (\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2), & t > 0, \ x \in \mathbb{S}, \\ u(0,x) = u_0(x), & x \in \mathbb{S}, \\ u(t,x) = u(t,x+1), & t \geq 0, \ x \in \mathbb{S}, \end{cases}$$

and integrating by parts respectively we obtain

$$\frac{d}{dt}\int_{\mathbb{S}} u \, dx = -\int_{\mathbb{S}} uu_x \, dx - \int_{\mathbb{S}} -\partial_x G * (\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2) \, dx = 0$$

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$$\int_{\mathbb{S}} y\,(t,x)\ dx = \int_{\mathbb{S}} u\,dx - \int_{\mathbb{S}} u_{xx}\,dx = \int_{\mathbb{S}} u\,dx$$

Hence the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 6.1.2. *If $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, is such that $(u_0 - u_{0,xx})$ is a nonnegative or nonpositive function then the above b-family equation has a global strong solution*

$$u = u\,(\cdot, u_0) \in \mathbb{C}([0, \infty); H^r(\mathbb{S})) \cap \mathbb{C}^1([0, \infty); H^{r-1}(\mathbb{S})).$$

Moreover,$I(u) = \int_{\mathbb{S}} u\,dx$ is a conservation law and if $y(t,\cdot) := u(t,\cdot) - u_{xx}(t,\cdot)$ then $\forall t \in \Re_+$ we have

(i) $y(t,\cdot) \in L^1(\mathbb{S})$ and $y(t,\cdot) \geq 0$ a.e.,$u(t,\cdot) \geq 0$ and $|u_x(t,\cdot)| \leq u(t,\cdot)$ on $\mathbb{S}$,

(ii) $\|y_0\|_{L^1(\mathbb{S})} = \|y_t\|_{L^1(\mathbb{S})} = \|u(t,\cdot)\|_{L^1(\mathbb{S})}$ and $\|u_x(t,\cdot)\|_{L^\infty(\mathbb{S})} \leq \|u_0\|_{L^1(\mathbb{S})}$,

(iii) $\|u(t,\cdot)\|^2_{H^1(\mathbb{S})} \leq \|u_0\|^2_{H^1(\mathbb{S})} + t\,\|u_0\|^3_{L^1(\mathbb{S})}$.

Proof. By earlier discussion from chapter 4 we know

$u \in C([0, T), H^s(\mathbb{S})) \cap C^1([0, T), H^{r-1}(\mathbb{S}))$.

By Lemma 6.1.1,we know that $I(u)$ is a conservation law. In view of the same lemma,we have that $y(t,\cdot) \in L^1(\mathbb{S})$ and is a.e. nonnegative for every fixed $t \geq 0$,. Note that $u = G * y$. Using the positivity of G, it is infered that $u(t,\cdot) \geq 0$ forall $t \geq 0$,

$$u(t,x) = \frac{e^{x-\frac{1}{2}}}{4sinh(\frac{1}{2})} \int_0^x e^{-\xi} y d\xi + \frac{e^{-x+\frac{1}{2}}}{4sinh(\frac{1}{2})} \int_0^x e^{\xi} y d\xi$$
$$+ \frac{e^{x+\frac{1}{2}}}{4sinh(\frac{1}{2})} \int_0^x e^{-\xi} y d\xi + \frac{e^{-x-\frac{1}{2}}}{4sinh(\frac{1}{2})} \int_0^x e^{\xi} y d\xi$$

and

$$u_x(t,x) = \frac{e^{x-\frac{1}{2}}}{4sinh(\frac{1}{2})} \int_0^x e^{-\xi} y d\xi - \frac{e^{-x+\frac{1}{2}}}{4sinh(\frac{1}{2})} \int_0^x e^{\xi} y d\xi$$
$$+ \frac{e^{x+\frac{1}{2}}}{4sinh(\frac{1}{2})} \int_0^x e^{-\xi} y d\xi - \frac{e^{-x-\frac{1}{2}}}{4sinh(\frac{1}{2})} \int_0^x e^{\xi} y d\xi$$

From the above two equations it is easy to deduce that $|u_x(t, \cdot)| \leq u(t, \cdot)$ on $\mathbb{S}$, for all $t \geq 0$. This proves (i).

Given $t \in [0, T)$, due to the periodicity of $u(t, x)$ in the $x-$ variable, there exists $a_t \in (0, 1)$ such that $u_x(t, a_t) = 0$ since u_x is continuous. In view of $y \geq 0$, for every $x \in [a_t, a_{t+1}]$ we obtain

$$u_x(t, x) - \int_{a_t}^x u \, dx = -\int_{a_t}^x (u - u_{xx}) \, dx = -\int_{a_t}^x y \, dx \leq 0$$

By $u \geq 0$ and the above inequality, we get that

$$u_x(t, x) \leq \int_{a_t}^x u \, dx \leq \int_{a_t}^{a_{t+1}} u \, dx = \int_0^1 u_0 \, dx = \|u_0\|_{L^1(\mathbb{S})}$$

Next we claim that If $u_0 \in H^r(\mathbb{S})$, $r > \frac{3}{2}$, is such that $(u_0 - u_{0,xx})$ is a nonnegative/nonpositive function in $L^1(\mathbb{S})$ and doesn't change sign then $u_x(t, x) \geq -\|u_0\|_{L^1(\mathbb{S})}$.

Proof: **Case(i):** $y_0 = (u_0 - \partial_x^2 u_0) \geq 0$.

(a) Let $u_0 \in H^r(\mathbb{S}), r \geq 3$ and $T > 0$ be the maximal existence time of the solution u with initial data u_0. If $y = u - u_{xx}$ then by earlier result proved in chapter 5 ensures that if $y \geq 0$ initially, then this property will persist on $[0, T)$.

Given $t \in [0, T)$, due to the periodicity of $u(t, x)$ in the $x-$ variable, there exists $a_t \in (0, 1)$ such that $u_x(t, a_t) = 0$. Therefore applying lemma 6.1.1, for every $x \in [a_t, a_t+1]$ we obtain

$$-u_x(t, x) + \int_{a_t}^x u \, dx \leq \int_{a_t}^x (u - u_{xx}) dx = \int_{a_t}^x y \, dx \leq \int_{a_t}^{a_t+1} y \, dx = \int_0^1 y_0 \, dx$$

Note that $u = G * y$ and $y \geq 0$ on $[0, T)$, so that we can also infer $u \geq 0$ on $[0, T)$. With $K = \int_0^1 y_0 \, dx$ from above we conclude $u_x(t, x) \geq -K, (t, x) \in [0, T) \times \mathbb{S}$.

(b) If $u_0 \in H^r(\mathbb{S}), \frac{3}{2} < r < 3$, then we will use a simple density argument to prove the theorem. If $u_0^n = e^{\frac{\partial_x^2}{n}} u_0$, then $u_0^n \in H^3$ and $u_0^n - \partial_x^2 u_0^n \geq 0$. In view of the above arguments, we get

$$u_x^n(t, x) \geq -\int_0^1 u_0^n dx \geq -\int_0^1 u_0 dx \quad \forall (t, x) \in [0, T) \times \mathbb{S}.$$

By the previously mentioned theorem, we have for every $T' < T$,

$\|u_x^n - u_x\|_{L^\infty} \leq \sup_{[0,T']} \|u^n - u\|_r \to 0$ as $n \to \infty$. It follows then

$$u_x(t, x) \geq -\int_0^1 u_0 dx \quad = -\|u_0\|_{L^1(\mathbb{S})} \forall (t, x) \in [0, T) \times \mathbb{S}.$$

In view of the blow-up result 5.1.1 we conclude $T = \infty$.

Case(ii): $(u_0 - \partial_x^2 u_0) \leq 0$.

(a) Let $u_0 \in H^r(\mathbb{S}), r \geq 3$ and $T > 0$ be the maximal existence time of the solution u with initial data u_0. If $y = u - u_{xx}$ then by earlier result proved in chapter 5 ensures that if $y \leq 0$ initially,then this property will persist on $[0, T)$.

Using identical notations as above

$$-u_x(t, x) + \int_{a_t}^x u dx = \int_{a_t}^x (u - u_{xx}) dx$$
$$= \int_{a_t}^x y dx \leq 0$$

Note that $u = G * y$ and $y \leq 0$ on $[0, T)$, so that we can also infer $u \leq 0$ on $[0, T)$. With $K = \int_0^1 y_0 dx$ using identical arguments from case(i) we conclude $u_x(t, x) \geq K, (t, x) \in [0, T) \times \mathbb{S}$.

(b) If $u_0 \in H^r(\mathbb{S}), \frac{3}{2} < r < 3$, then we will use a simple density argument to prove the theorem. If $u_0^n = e^{\frac{\partial_x^2}{n}} u_0$, then $u_0^n \in H^3$ and $u_0^n - \partial_x^2 u_0^n \leq 0$. In view of the above arguments, we get

$$u_x^n(t, x) \geq \int_0^1 u_0^n dx \geq \int_0^1 u_0 dx \quad \forall (t, x) \in [0, T) \times \mathbb{S}.$$

By the theorem we just proved, we have for every $T' < T$,

$\|u_x^n - u_x\|_{L^\infty} \leq \sup_{[0,T']} \|u^n - u\|_r \to 0$ as $n \to \infty$. It follows then

$$u_x(t,x) \geq \int_0^1 u_0 dx = \|u_0\|_{L^1(\mathbb{S})} \forall (t,x) \in [0,T) \times \mathbb{S}.$$

$\square$

In view of the blow-up result 5.1.1 we conclude $T = \infty$. Henceforth, combining Case(i) and (ii) we obtain $u_x(t,\cdot) \geq -\|u_0\|_{L^1(\mathbb{S})}$.

This proves (ii) in view of the conservation law $I(u)$ and the existence of a global strong solution.

Multiplying the periodic b-family equation by u and integrating by parts, we get in view of the periodicity of u and (ii)

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}}\left(u^2 + u_x^2\right)dx = -(b+1)\int_{\mathbb{S}}\left(u^2 u_x\right)dx + b\int_{\mathbb{S}}(uu_x u_{xx})dx - 2\int_{\mathbb{S}}(uu_x u_{xx})dx$$

$$= (b-2)\int_{\mathbb{S}}(uu_x u_{xx})dx$$

$$= \frac{b-2}{2}\int_{\mathbb{S}}u(u_x^2)_x dx$$

$$= -\frac{b-2}{2}\int_{\mathbb{S}}u_x^3 dx \leq \frac{|b-2|}{2}\|u_0\|_{L^1(\mathbb{S})}^3$$

From the above inequality, it follows that $\|u(t,\cdot)\|_{H^1(\mathbb{S})}^2 \leq \|u_0\|_{H^1(\mathbb{S})}^2 + t\|u_0\|_{L^1(\mathbb{S})}^3$. Hence (iii) is proved and the unique strong solution is defined globally in time.

CHAPTER 7

SNAPSHOT OF SCHOLASTIC CONTRIBUTION

7.1 Hydrodynamical relevance of b family of shallow water wave equations

Consider the following partial differential equations.

(1)
$$m_t + c_0 m_x + \gamma u_{xxx} + u m_x + b m u_x = 0,$$

with $m = u - \alpha^2 u_{xx}$. Eq. (1) can be derived from nonlinear shallow water equations. $u(x,t)$ is the fluid velocity, $m = u - \alpha^2 u_{xx}$ is the momentum density, α^2 and $\frac{\gamma}{c_0}$ are squares of length scales, and c_0 is related to the critical shallow water wave speed. Suppose that the water flow is incompressible, irrotational and inviscid. Then the water wave equations for one-dimensional surfaces read, in nondimensionalized form

$$
\begin{cases}
\mu \partial_x^2 \psi + \partial_z \Psi^2 = 0, & in\ \Omega_t, \\[2mm]
\partial_z \Psi = 0, & at\ z = -1, \\[2mm]
\partial_t \xi - \frac{1}{\mu}(-\mu \partial_x \xi \partial_x \Phi + \partial_z \Psi) = 0, & at\ z = \epsilon\xi, \\[2mm]
\partial_t \Psi + \frac{\epsilon}{2}(\partial_x \Psi)^2 + \frac{\epsilon}{2\mu}(\partial_z \Psi)^2 = 0, & at\ z = \epsilon\xi,
\end{cases}
$$

$x \to \epsilon\xi(t,x):$ the free surface, $\Omega_t = \{(x,z);\ -1 < z < \epsilon\xi(t,x)\}:$ the fluid domain, $\Psi(t,\cdot):$ the velocity potential associated to the flow, and

$$\epsilon = \frac{a}{h}, \quad \mu = \frac{h^2}{\lambda^2},$$

where h is the mean depth, a is the typical amplitude, and λ is the typical wavelength of the waves.

Define the vertically averaged horizontal component of the velocity by

$$u(t,x) = \frac{1}{1+\epsilon\xi}\int_{-1}^{\epsilon\xi} \partial_x \Psi(t,x,z)dz.$$

under the shallow-water scaling: $\mu \ll 1$, $\epsilon = O(1)$: The Green-Naghdi equations read

$$\begin{cases} \xi_t + ((1+\epsilon\xi)u)_x = 0 \\ u_t + \xi_x + \epsilon u u_x = \frac{\mu}{3}\frac{1}{1+\epsilon\xi}\left((1+\epsilon\xi)^3(u_{xt} + \epsilon u u_{xx} - \epsilon u_x^2)\right)_x, \end{cases}$$

where $O(\mu^2)$ terms have been neglected. In the long-wave regime

$$\mu \ll 1, \qquad \epsilon = O(\mu),$$

the right-going wave should satisfy the KdV equation

$$u_t + u_x + \epsilon\frac{3}{2}u u_x + \mu\frac{1}{6}u_{xxx} = 0$$

with $\xi = u + O(\epsilon, \mu)$. the BBM equations (the regularized long-wave equations), which provide an approximation of the exact water wave equations of the same accuracy as the KdV equation are as follows:

$$u_t + u_x + \frac{3}{2}\epsilon u u_x + \mu(\alpha u_{xxx} + \beta u_{xxt}) = 0, \quad with\ \alpha - \beta = \frac{1}{6}.$$

Consider the so-called Camassa-Holm scaling,

$$\mu \ll 1, \quad \epsilon = O(\sqrt{\mu}).$$

With this scaling, one still has $\epsilon \ll 1$. The dimensionless parameter is, however, larger here than in the long wave scaling, and the nonlinear effects are therefore stronger and

it is possible that a stronger nonlinearity could allow the appearance of breaking waves.

Define the horizontal velocity u^θ ($\theta \in [0,1]$) at the level line θ of the fluid domain:

$$v \equiv u^\theta(x) = \partial_x \Psi \big|_{z=(1+\epsilon\xi)\theta-1}.$$

Let $p \in \mathbb{R}$ and $\lambda = \frac{1}{2}(\theta^2 - \frac{1}{3})$, with $\theta \in [0,1]$. Assume

$$\alpha = p + \lambda, \ \beta = p - \frac{1}{6} + \lambda, \ \gamma = -\frac{2}{3}p - \frac{1}{6} - \frac{3}{2}\lambda, \ \delta = -\frac{9}{2}p - \frac{23}{24} - \frac{3}{2}\lambda.$$

Under the Camassa-Holm scaling, one should have the following class of equations for $v \equiv u^\theta$ ($\theta \in [0,1]$),

$$(\star) \qquad v_t + v_x + \frac{3}{2}\epsilon v v_x + \mu(\alpha v_{xxx} + \beta v_{xxt}) = \epsilon\mu(\gamma v v_{xxx} + \delta v_x v_{xx}),$$

where $O(\epsilon^4, \mu^2)$ terms have been discarded. The averaged horizontal velocity u and the free surface ξ satisfy

$$u = u^\theta + \mu\lambda u^\theta_{xx} + 2\mu\epsilon\lambda u^\theta u^\theta_{xx},$$

$$\xi = u + \frac{\epsilon}{4}u^2 + \mu\frac{1}{6}u_{xt} - \epsilon\mu\left(\frac{1}{6}uu_{xx} + \frac{5}{48}u_x^2\right).$$

By rescaling, shifting the dependent variable, and applying a Galilean transformation, then the Camassa-Holm equation

$$U_t + \kappa U_x + 3UU_x - U_{txx} = 2U_x U_{xx} + UU_{xxx}$$

can be obtained from $(\star)$ if the following conditions hold

$$\beta < 0, \ \alpha \neq \beta, \ \beta = -2\gamma, \ \delta = 2\gamma,$$

where

$$p = -\frac{1}{3}, \qquad \theta^2 = \frac{1}{2}.$$

The solution u^θ of $(\star)$ is transformed to the solution U of the CH equation by

$$U(t,x) = \frac{1}{a}u^\theta\left(\frac{x}{b} + \frac{\nu}{c}t, \frac{t}{c}\right),$$

with $a = \frac{2}{\epsilon\kappa}(1-\nu)$, $b^2 = -\frac{1}{\beta}\mu$, $\nu = \frac{\alpha}{\beta}$, and $c = \frac{b}{\kappa}(1-\nu)$.

The Degasperis-Procesi equation

$$U_t + \kappa U_x + 4UU_x - U_{txx} = 3U_x U_{xx} + UU_{xxx}$$

can also be derived if the following conditions hold

$$\beta < 0, \quad \alpha \neq \beta, \quad \beta = -\frac{8}{3}\gamma, \quad \delta = 3\gamma,$$

where

$$p = -\frac{77}{216}, \qquad \theta^2 = \frac{23}{36}.$$

The solution u^θ of $(\star)$ is transformed to the solution U of the DP equation by

$$U(t,x) = \frac{1}{a}u^\theta\left(\frac{x}{b} + \frac{\nu}{c}t, \frac{t}{c}\right),$$

with $a = \frac{8}{3\epsilon\kappa}(1-\nu)$, $b^2 = -\frac{1}{\beta}\mu$, $\nu = \frac{\alpha}{\beta}$, and $c = \frac{b}{\kappa}(1-\nu)$.

Generalized b family Model from Water Waves: My first shot at Fame! The correct generalization of BBM equations under the appropriate scaling $v \equiv u^\theta$ ($\theta \in [0,1]$),

$(\star)$ $\qquad v_t + v_x + \frac{3}{2}\epsilon v v_x + \mu(\alpha v_{xxx} + \beta v_{xxt}) = \epsilon\mu(\gamma v v_{xxx} + \delta v_x v_{xx}),$

whereas the class of generalized b family equations

$$U_t + \kappa U_x + (b+1)UU_x - U_{txx} = bU_x U_{xx} + UU_{xxx}$$

could be obtained by rescaling, shifting the dependent variable, and applying a Galilean transformation,

$$U(t,x) = \frac{1}{a}u^\theta\left(\frac{x}{b_1} + \frac{\nu}{c}t, \frac{t}{c}\right),$$

with $a = \frac{2}{3\epsilon\kappa}(1-\nu)(b+1)$, $b_1^2 = -\frac{1}{\beta}\mu$, $\nu = \frac{\alpha}{\beta}$, and $c = \frac{b}{\kappa}(1-\nu)$.

Generalized b family Model from Water Waves: CH and DP as special cases

if the following conditions hold

$$\beta < 0, \quad \alpha \neq \beta, \quad \beta = -\frac{2}{3}\gamma(b+1), \quad \delta = b\gamma,$$

$b = 2 \Rightarrow \delta = 2\gamma \Rightarrow$ **CH**;

$\mathrm{b} = 3 \Rightarrow \delta = 3\gamma \Rightarrow$ **DP**

7.2 Local Existence results for b family

The peakon b family can be rewritten in the following form

$$u_t - u_{txx} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \quad t > 0, \ x \in \mathbb{S},$$

Then in order to apply Kato's semi-group theory to establish local well-posedness

Consider the abstract quasi-linear evolution equation

$$\begin{cases} \frac{dv}{dt} + A(v)v = f(v), & t \geq 0, \\ v(0) = v_0 \end{cases}$$

Setting up the framework: Kato's theorem (i) X,Y Hilbert spaces such that Y is continuously and densely embedded in X,.

(ii) $Q : Y \to X$ be a topological isomorphism. The linear operator $A \in$G$(X, 1, \beta)$,i.e $-A$ generates a C_0-semigroup such that $\|e^{-sA}\|_{L(X)} \leq e^{\beta s}$.

(iii) μ_1, μ_2, μ_3, and μ_4 are constants depending only on $\max\{\|y\|_Y, \|z\|_Y\}$.

Assumptions (i) $A(y) \in L(Y, X)$ for $y \in X$ with

$$\| (A(y) - A(z)) w\|_X \leq \mu_1 \|y - z\|_X \|w\|_Y, \quad y, z, w \in Y$$

and $A(y) \in G(X, 1, \beta)$, uniformly on bounded sets in Y.

(ii) $QA(y)Q^{-1} = A(y) + B(y)$, where $B(y) \in L(X)$ is bounded, uniformly on bounded sets in Y. Moreover,

$$\| \left(B(y) - B(z)\right) w\|_X \leq \mu_2 \|y - z\|_Y \|w\|_X, \quad y, z \in Y, w \in X.$$

(iii) $f : Y \to Y$ extends to a map from X into X, is bounded on bounded sets in Y , and satisfies

$$\|f(y) - f(z)\|_Y \leq \mu_3 \|y - z\|_Y, \quad y, z \in Y,$$

$$\|f(y) - f(z)\|_X \leq \mu_4 \|y - z\|_X, \quad y, z \in Y.$$

My contribution: Now *the* theorem

Lemma 7.2.1. *(Kato) Assume that (i), (ii), and (iii) hold. Given $v_0 \in Y$, there is a maximal $T > 0$ depending only on $\|v_0\|_Y$, and a unique solution v to the quasi linear evolution such that $v = v(\cdot, v_0) \in C([0, T), Y) \cap C^1 ([0, T), X)$. Moreover, the map $v_0 \mapsto v(\cdot, v_0)$ is a continuous map from Y to $C([0, T), Y) \cap C^1 ([0, T), X)$.*

reformulate the problem

(i) All spaces of functions are assumed to be over $\mathbb{S}, S = R \setminus Z$.

(ii) If A is an unbounded operator, $D(A)$ denotes the domain of A.

(iii) $[A, B]$ denotes the commutator of two linear operators A and B.

My contribution: Cauchy problem and Kato's theorem With $m = u - u_{xx}$, we reconsider the Cauchy problem

$$\begin{cases} m_t + um_x + bu_x m = 0, & t > 0, x \in \mathbb{S}, \\ m(0, x) = u_0(x) - u_{0,xx}(x), & x \in \mathbb{S}. \end{cases}$$

Note: if $g(x) := \frac{1}{2}e^{-|x|}$, $x \in \mathbb{S}$, then $(1 - \partial_x^2)^{-1}f = g * f$ for all $f \in L^2(\mathbb{S})$ and $g * m = u$,

. Using this identity, we can rewrite the Cauchy problem as a quasi-linear evolution equation of hyperbolic type:

My contribution: The quasi-linear evolution equation: Applying Kato framework

$$\begin{cases} u_t + uu_x + \partial_x g * (\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2) = 0, & t > 0, \ x \in \mathbb{S}, \\ u(0, x) = u_0(x), & x \in \mathbb{S}, \end{cases}$$

where $A(u) = u\partial_x$, $f(u) = -\partial_x g * (\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2)$, $Y = H^s$, $X = H^{s-1}$, $\Lambda = (1 - \partial_x^2)^{\frac{1}{2}}$, and $Q = \Lambda^1$.

My contribution: Consequence of Kato's theorem

Definition 7.2.1. If $u \in C([0, T), H^s(\mathbb{S})) \cap C^1([0, T), H^{s-1}(\mathbb{S}))$ with $s > \frac{3}{2}$ satisfies the evolution equation, then u is called a strong solution to the Cauchy problem.

If u is a strong solution on $[0, T)$ for every $T > 0$, then it is called global strong solution to the Cauchy problem. The local well-posedness of the Cauchy problem with initial data $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$ can be obtained by applying Kato's theorem. More precisely, we have the *well-posedness* result.

My contribution: First theorem on Local Wellposedness:

Theorem 7.2.2. *For any constant b, given $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, there exist a maximal $T = T(u_0) > 0$ and a unique strong solution u to the Cauchy problem, such that*

$$u = u(\cdot, u_0) \in C([0, T), H^s(\mathbb{S})) \cap C^1([0, T), H^{s-1}(\mathbb{S})).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $u_0 \mapsto u(\cdot, u_0) : H^s(\mathbb{S}) \to C([0, T), H^s(\mathbb{S})) \cap C^1([0, T), H^{s-1}(\mathbb{R}))$ is continuous.

Sketch of proof: apply Kato's theorem with $A(u) = u\partial_x$, $f(u) = -\partial_x g * (\frac{b}{2}u^2 +$

$\frac{3-b}{2}u_x^2)$, $Y = H^s$, $X = H^{s-1}$, $\Lambda = (1-\partial_x^2)^{\frac{1}{2}}$, and $Q = \Lambda^1$. Obviously, Q is an isomorphism

of H^s onto H^{s-1}. Only need to verify that $A(u)$ and $f(u)$ satisfy the conditions (i)-(iii).

My contribution: Several lemmas along the way

Lemma 7.2.3. *Let the operator* $A(u) = u\partial_x$ *with* $u \in H^s, s > \frac{3}{2}$. *Then* $A(u) \in$

$L(H^s, H^{s-1})$ *for* $u \in H^s$. *Moveover,*

$$\| (A(u) - A(z)) w\|_{s-1} \le \mu_1 \|u - z\|_{s-1} \|w\|_s, \quad u, z, w \in H^s.$$

Lemma 7.2.4. *Let* $f(u) = -\partial_x g * (\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2)$. *Then,* f *is bounded on bounded set in*

H^s *and satisfies*

$$(a) \; \|f(y) - f(z)\|_s \le \mu_3 \|y - z\|_s, \quad y, z \in H^s,$$

$$(b) \; \|f(y) - f(z)\|_{s-1} \le \mu_4 \|y - z\|_{s-1}, \quad y, z \in H^s.$$

My contribution: The point! (i)Lemmas lead to the proof of the theorem!

(ii) Local wellposedness results lead to....

(iii) Blow up solutions and.......

(iv) Global existence results ....

7.3 Global Existence results for b family

Lemma 7.3.1. *If* $u_0 \in H^r(\mathbb{S})$, $r \ge \frac{3}{2}$, *then as long as the solution* $u(t,x)$ *is given by*

local well posedness theorem exists, we have

$$\int_{\mathbb{S}} u(t,x) \; dx = \int_{\mathbb{S}} u_0 \; dx = \int_{\mathbb{S}} y(t,x) \; dx = \int_{\mathbb{S}} y_0 \; dx$$

Sketch of proof: rewrite the original b-family equation

$$\begin{cases} u_t + uu_x = -\partial_x G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right), & t > 0, \ x \in \mathbb{S}, \\[2mm] u(0,x) = u_0(x), & x \in \mathbb{S}, \\[2mm] u(t,x) = u(t,x+1), & t \geq 0, \ x \in \mathbb{S}, \end{cases}$$

and use integration by parts.

Theorem on global strong solutions

Theorem 7.3.2. *If $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, is such that $(u_0 - u_{0,xx})$ is a nonnegative or nonpositive function then the above b-family equation has a global strong solution*

$$u = u(\cdot, u_0) \in \mathbb{C}([0,\infty); H^r(\mathbb{S})) \cap \mathbb{C}^1([0,\infty); H^{r-1}(\mathbb{S})).$$

Moreover,$I(u) = \int_{\mathbb{S}} u\, dx$ is a conservation law and if $y(t,\cdot) := u(t,\cdot) - u_{xx}(t,\cdot)$ then $\forall t \in \Re_+$ we have

(i) $y(t,\cdot) \in L^1(\mathbb{S})$ and $y(t,\cdot) \geq 0$ a.e.,$u(t,\cdot) \geq 0$ and $|u_x(t,\cdot)| \leq u(t,\cdot)$ on $\mathbb{S}$,

(ii) $\|y_0\|_{L^1(\mathbb{S})} = \|y_t\|_{L^1(\mathbb{S})} = \|u(t,\cdot)\|_{L^1(\mathbb{S})}$ and $\|u_x(t,\cdot)\|_{L^\infty(\mathbb{S})} \leq \|u_0\|_{L^1(\mathbb{S})}$,

(iii) $\|u(t,\cdot)\|^2_{H^1(\mathbb{S})} \leq \|u_0\|^2_{H^1(\mathbb{S})} + t\,\|u_0\|^3_{L^1(\mathbb{S})}$.

Blow up Theorems: Tools used to obtain strong blow up conditions:

(i) Local wellposedness results

(ii)Energy estimates

First Blow up Theorem

Theorem 7.3.3. *Assume $b \geq 1$ and $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$. Then blow up of the strong solution $u = u(\cdot, u_0)$ in finite time $T < +\infty$ occurs if and only if*

$$\liminf_{t\uparrow T}\{\inf_{x\in\mathbb{S}}[u_x(t,x)]\} = -\infty.$$

Classical ODE theory Consider the following differential equation

$$\begin{cases} q_t = u(t,q), & t \in [0,T), \\ q(0,x) = x, & x \in \mathbb{S}. \end{cases}$$

Lemma 7.3.4. *Let $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, and let $T > 0$ be the maximal existence time of the corresponding strong solution u to the b-family Eq. Then the preceeding equation has a unique solution $q \in C^1([0,T) \times \mathbb{S}, \mathbb{S})$ such that the map $q(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{S}$ with*

$$q_x(t,x) = \exp\left(\int_0^t u_x(s, q(s,x))ds\right) > 0, \;\; \forall(t,x) \in [0,T) \times \mathbb{S}.$$

Furthermore, setting $m = u - u_{xx}$, we have

$$m(t, q(t,x))q_x^b(t,x) = m_0(x), \quad \forall(t,x) \in [0,T) \times \mathbb{S}.$$

Remark to the preceeding lemma *Remark I.* Lemma shows that, if $m_0 = u_0 - u_{0xx}$ does not change sign, then $m(t)$ ($\forall$ t) will not change sign, as long as $m(t)$ exists. *Remark II.* It is observed that if $u(t,x)$ is a solution to the periodic b-family eq. with $u(0,x) = u_0(x)$, then $-u(t,-x)$ is also a solution to the same with initial datum $-u_0(-x)$. Hence, due to the uniqueness of the solutions, the solution is odd as long as the initial datum $u_0(x)$ is odd. So the first blow-up main theorem is concerned with this type of initial data.

A significant theorem

Theorem 7.3.5. *Let $1 < b \leq 3$ and $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$ be odd and nonzero. If $u_{0x} \leq 0$, then the corresponding solution $u(t)$ with initial value $u(0) = u_0$ blows up in finite time.*

The unifying theorem

Theorem 7.3.6. *Let $\frac{5}{3} < b \leq 3$ and $\int_{\mathbb{S}} u_x^3(0)\,dx < 0$. Assume that $u_0 \in H^s(\mathbb{S}), s > \frac{3}{2}$, $u_0 \not\equiv 0$, and the corresponding solution $u(t)$ of the b family has a zero for any time $t \geq 0$. Then, the solution $u(t)$ of the b family equation blows up in finite time.*

An immediate corollary

Corollary 7.3.7. *If $u_0 \in H^3(\mathbb{S})$, $u_0 \not\equiv 0$ and $\int_{\mathbb{S}} u_0\,dx = 0$ or $\int_{\mathbb{S}} y_0\,dx = 0$, then the corresponding solution u to the b family equation blows up in finite time.*

Remark:

This shows that $u(t,x)$ has a zero for all $t \in \mathbb{S}$. It follows from the preceeding theorem that the solution u to the b family equation blows up in finite time.

Comments

A number of significant results(*Local well posedness,global existence and blow up*) regarding b family is proved which doesn't take into account only the specific values of $b = 2$ or $b = 3$. The expectation is if we are able to find another integrable PDE belonging to the b family the qualtitative prpperties are already studied and researchers have the motivation at least that would drive them to find that equation.

Some Proposals for further Research   (i) Numerical Simulation of Shockpeakons. Does there exist a numerical simulation scheme that models the exact solution to the peakon periodic b family and their subsequent decay into toshockpeakon?

(ii) Orbital Stability of Solutions to b family equations. Are the solutions stable under small intial perturbations?

(iii) Weak solutions to the b family.

(iv) The Shape of Initial Data. What about $n \geq 2$ zeros for m_0? Is there a relation between the number of zeros and the shape of the solutions?

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BIOGRAPHICAL STATEMENT

Snehanshu was born on October 29, 1973, in Calcutta, India. He grew up mostly in Calcutta but his professional life was scattered across the country. After earning his degrees in Mathematics and Computer Science he worked briefly in two Indian corporate houses. Soon after,he got an offer from Clemson University,South Carolina to pursue Masters in Mathematical Sciences. Graduating in Fall 2003 Snehanshu spent one semester in Mississippi Sate University and joined the the doctoral program in the Department of Mathematics,University of Texas @ Arlington in Spring 2005. Apart from Mathematical research Snehanshu takes a keen interest in Physics and space sciences. He is a very passionate teacher and encourages young people to study the sciences. Snehanshu would like to join a corporate sector actively engaged in mathematics and implementation of complex mathematical models aimed toward solving real time problems. Snehanshu's research intersets are, but not limited to, uniqueness and existence of solutions to differential equations and dynamical systems.