

**On the Image of the Totalling Functor**

by

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Dedicated to each person in my life who has inspired me to never sit still.

*Live as if your were to die tomorrow.*

*Learn as if you were to live forever.*

—Mahatma Gandhi

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## ABSTRACT

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Let  $A$  denote a DG algebra and  $k$  a field. The totalling functor  $\text{Tot} : \mathbf{ChG}(A) \rightarrow \mathbf{DG}(A)$  can be extended to a functor between the derived categories  $\mathbf{DG}(A)$  and  $\mathbf{DDG}(A)$ . If  $\text{Tot} : \mathbf{DG}(A) \rightarrow \mathbf{DDG}(A)$  were onto, then  $\mathbf{DDG}(A)$  would be superfluous. This paper investigates the image of  $\text{Tot}$  on its extension to the derived categories in the fundamental case when  $A = k[x_1, \dots, x_d]$ . It will be shown that when  $d \geq 2$ , there are semifree DG modules of rank  $n \geq 4$  that are not obtained from the totalling of any complex in  $\mathbf{DG}(A)$ . However when  $A = k[x]$ , we will find that every rank  $n$  semifree DG module over  $A$  is, in fact, in the image of  $\text{Tot}$ . Moreover, for a polynomial ring  $A = k[x_1, \dots, x_d]$  of arbitrary size, a special class of rank  $n$  semifree DG modules over  $A$  which are always equal to the totalling of some complex of graded  $A^{\natural}$ -modules will be defined.

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## CHAPTER 1

### Introduction

Category theory was first introduced in 1945 by Eilenberg and Mac Lane [3] as a means of comparing certain algebraic structures of topological spaces in algebraic topology. For about fifteen years, category theory remained merely a convenient language for algebraists and geometers. Books that utilized its methods helped new generations of mathematicians learn homological algebra and algebraic topology directly through categorical language. But, beginning in 1957, the use of category theory began to spread to other areas of mathematics. Grothendieck introduced certain abstract types of categories (ie. additive and abelian) and, by performing various constructions on them, was able to prove results for them [4]. This gave mathematicians the ability to see how the methods of category theory were applicable in other areas. For example, recognizing the category of sheaves over some topological space  $X$  as an abelian category, one can immediately incorporate the concepts of homological algebra into algebraic geometry. With these developments, category theory became not only a pervasive part of, but ultimately a universal framework for, mathematics.

The concept of the derived category of an abelian category did not come to light until the early 1960's, when Grothendieck was trying to formulate and prove the extension of Serre's duality theorem [5]. The many details of the construction of the derived category were worked out in the 1963 dissertation [6] of Jean-Louis Verdier, a student of Grothendieck. Since then, the methods developed by Grothendieck and Verdier starting spreading, and are today used throughout many disciplines of mathematics.

The motivation behind the necessity for the derived category lies in the fact that the constructions of homological algebra yield complexes with indeterminacy—namely, homology. For this purpose, Grothendieck defined the concept of *quasiisomorphism*. In particular, a quasiisomorphism between two complexes over some abelian category is defined to be a morphism between the complexes which induces an isomorphism in homology. Thus, homological constructions can be realized as complexes which are unique up to quasiisomorphism. As it turns out, objects of the derived category are themselves complexes over an abelian category, and the concept of isomorphism at the derived category level is actually the same as quasiisomorphism.

Once relevant structures are defined, a mathematician immediately desires to study the relationships between them. The same holds true for the derived categories. When we want to know about the relationship between certain categories, it is often useful to consider the functors between them. Later, we will see that if a functor  $F$  between two abelian categories  $\mathcal{C}$  and  $\mathcal{D}$  preserves quasiisomorphisms, then the functor can be extended to one of the derived categories  $\mathbf{D}(\mathcal{C})$  and  $\mathbf{D}(\mathcal{D})$ . Thus, to compare two derived categories, it is useful to look for quasiisomorphism-preserving functors between their underlying abelian categories.

This paper investigates the relationship between two particular derived categories: (i) that of  $\mathbf{G}(A)$ , the category of graded  $A$ -modules, and (ii) that of  $\mathbf{DG}(A)$ , the category of DG modules over  $A$ , where the algebra  $A$  is the polynomial ring  $A = k[x_1, \dots, x_d]$  over a field  $k$ . The functor  $\text{Tot}$ , called the *totalling functor*, is a relation between the abelian categories  $\mathbf{ChG}(A)$ , the category of chain complexes over  $\mathbf{G}(A)$ , and  $\mathbf{DG}(A)$ . We can extend  $\text{Tot}$  to the derived categories because it preserves quasiisomorphic complexes. Therefore, to understand the relationship between  $\mathbf{DG}(A)$  and  $\mathbf{DDG}(A)$ , we can study the functor  $\text{Tot} : \mathbf{DG}(A) \rightarrow \mathbf{DDG}(A)$ . In particular, it follows that if  $\text{Tot}$  is onto, then



the derived category  $\mathbf{DDG}(A)$  can be obtained from the derived category  $\mathbf{DG}(A)$ , and is therefore redundant.

It is this question—of the image of  $\text{Tot}$ —that the results of this paper are concerned with. The initial motivation for studying this topic was derived from the question posed by Avramov and Jorgensen in [2].

A note to the reader: This paper will assume a moderate knowledge of category theory. Since the notation used in category theory is not often standard, it will be useful to know that the categorical notation used in this paper will be consistent with that of Weibel's *An Introduction to Homological Algebra*. A thorough summary of the concepts of the category theory utilized throughout this paper is found in Appendix A of [7]. The paper will also assume a knowledge of basic differential graded algebra. An excellent reference for this topic is Avramov, Foxby, and Halperin's *Differential Graded Homological Algebra* [1].

## CHAPTER 2

### Background

Before looking at results, we first need to consider the some essential background material encompassing the nature of the problem. The ultimate goal of this chapter will be to construct the derived categories of interest and define the action of the totalling functor on them. However, there are a few basic concepts from homological algebra that need to be recalled first.

Throughout this chapter, unless otherwise stated, let  $R$  denote a commutative ring with identity, and  $A$  denote a DG algebra.

#### 2.1 Homology

Let  $X$  be a complex of  $R$ -modules. The sets

$$Z_d(X) = \{x \mid x \in \ker \partial_d^X\}$$

$$B_d(X) = \{x \mid x \in \operatorname{im} \partial_{d+1}^X\}$$

are called the  $d$ -cycles and  $d$ -boundaries of  $X$ , respectively. Clearly,  $Z(X) = (Z_d(X))_{d \in \mathbb{Z}}$  and  $B(X) = (B_d(X))_{d \in \mathbb{Z}}$  are each graded  $R$ -modules, and  $B_d(X) \subseteq Z_d(X)$  for all  $d \in \mathbb{Z}$  since  $\partial^X \circ \partial^X = 0$ . Thus we can consider the graded  $R$ -module

$$H(X) = (H_d(X))_{d \in \mathbb{Z}} = (Z_d(X)/B_d(X))_{d \in \mathbb{Z}} = Z(X)/B(X)$$

called the *homology module* of  $X$ . Clearly, if  $X$  is exact at  $X_i$  for some  $i \in \mathbb{Z}$ , then it follows that  $H_i(X) = 0$ . Thus,  $H(X)$  in effect measures the exactness of the complex  $X$ . A complex  $X$  is said to be *homologically bounded* if  $H_i(X) = 0$  for all  $|i| \gg 0$ .

**Fact 2.1.1.** If  $\mu : X \rightarrow Y$  is a morphism of complexes of  $R$ -modules, then  $\mu$  induces a well-defined degree 0 homomorphism of graded  $R$ -modules

$$H(\mu) : H(X) \rightarrow H(Y)$$

A chain map  $\mu : X \rightarrow Y$  is called a *quasiisomorphism* if  $H(\mu) : H(X) \rightarrow H(Y)$  is an isomorphism of  $R$ -modules.

**Fact 2.1.2.** Quasiisomorphism is an equivalence relation.

Thus  $X$  and  $Y$  are said to be *quasiisomorphic*, denoted  $X \simeq Y$ , if there exists a sequence of chain maps  $\mu^i$  linking  $X$  and  $Y$ , each of which is a quasiisomorphism. To illustrate this definition, consider the following diagram of chain maps between complexes:

$$V \xleftarrow{\mu_1} W \xrightarrow{\mu_2} X \xrightarrow{\mu_3} Y \xleftarrow{\mu_4} Z$$

If  $\mu_i$  is a quasiisomorphism for  $1 \leq i \leq 4$ , then  $V \simeq Z$ .

## 2.2 Homotopy

Let  $\mu, \lambda : X \rightarrow Y$  be morphisms of complexes. A *homotopy* between  $\mu$  and  $\lambda$  is a degree +1 map  $\sigma : X \rightarrow Y$  of complexes such that

$$\mu - \lambda = \partial^Y \circ \sigma + \sigma \circ \partial^X$$

Thus, for all  $n \in \mathbb{Z}$  we have

$$\mu_n - \lambda_n = \partial_{n+1}^Y \circ \sigma_n + \sigma_{n-1} \circ \partial_n^X$$

which, for clarity, is usually accompanied by the following diagram:

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & X_{n+1} & \xrightarrow{\partial_{n+1}^X} & X_n & \xrightarrow{\partial_n^X} & X_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow \mu_{n+1} & \swarrow \sigma_n & \downarrow \mu_n & \swarrow \sigma_{n-1} & \downarrow \mu_{n-1} & & \\
 \cdots & \longrightarrow & Y_{n+1} & \xrightarrow{\partial_{n+1}^Y} & Y_n & \xrightarrow{\partial_n^Y} & Y_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow \lambda_{n+1} & \swarrow \sigma_n & \downarrow \lambda_n & \swarrow \sigma_{n-1} & \downarrow \lambda_{n-1} & & 
 \end{array}$$

In the case that such a map  $\sigma$  exists, we say that  $\mu$  and  $\lambda$  are *homotopic*. A chain map  $\mu : X \rightarrow Y$  is said to be *null homotopic* if it is homotopic to 0, and the associated homotopy is called a *chain contraction* of  $\mu$ .

**Fact 2.2.1.** If two morphisms  $\mu, \lambda : X \rightarrow Y$  are homotopic, then  $H(\mu) = H(\lambda) : H(X) \rightarrow H(Y)$ .

*Proof.* Suppose that  $\mu, \lambda : X \rightarrow Y$  are homotopic chain maps, and consider the map  $H(\mu - \lambda) : H(X) \rightarrow H(Y)$ . By the definition of homotopy, we know that

$$H(\mu - \lambda) = H(\partial^Y \circ \sigma + \sigma \circ \partial^X)$$

for some degree +1 map  $\sigma : X \rightarrow Y$ . Linearity yields:

$$H(\mu) - H(\lambda) = H(\partial^Y \circ \sigma) + H(\sigma \circ \partial^X)$$

Since the domain of this function is a quotient of  $Z(X)$ , it is clear that  $H(\sigma \circ \partial^X)$  must be the zero map. Furthermore, since the image of  $H(\partial^Y \circ \sigma)$  is clearly contained in  $B(Y)$ , it follows that  $H(\partial^Y \circ \sigma)$  is also the zero map. Thus,  $H(\mu) = H(\lambda)$ .  $\square$

**Fact 2.2.2.** Homotopy is an equivalence relation.

*Remark.* In the case when we restrict the ring  $R$  to be a graded algebra (resp. DG algebra), we will find that all of the definitions and results of sections 2.1 and 2.2 will follow in a similar way for complexes of graded  $R$ -modules (resp. DG modules over  $R$ ).

### 2.3 Semifreeness

Let  $X = (X^\natural, \partial)$  be a DG module over  $A$ . A subset  $E$  of  $X$  is called a *semibasis* if it is a basis of  $X^\natural$  over  $A^\natural$ , and has a decomposition given by  $E = \bigsqcup_{d \geq 0} E^d$  as a union of disjoint graded sets  $E^d$  such that

$$\partial(E^d) \subseteq A \left( \bigsqcup_{i < d} E^i \right)$$

for all  $d \in \mathbb{Z}$ . A DG module that possesses such a semibasis is said to be *semifree*.

A *semifree filtration* of a DG module  $X$  is a sequence of DG submodules

$$\mathcal{X} = \{ \dots \subseteq X^{d-1} \subseteq X^d \subseteq \dots \}$$

with  $X = \bigcup_{d \in \mathbb{Z}} X^d$ ,  $X^{-1} = 0$ , and  $X^d/X^{d-1}$  free on a basis of cycles for every  $d \in \mathbb{Z}$ .

**Fact 2.3.1.** For a DG module  $X$  the following are equivalent:

- (i)  $X$  is semifree.
- (ii)  $X$  has a semifree filtration.
- (iii)  $X$  has a well-ordered basis  $E$  such that for each  $e \in E$

$$\partial(e) \in A(\{e' \in E \mid e' < e\})$$

*Proof.* (i)  $\Rightarrow$  (ii). Let  $E = \bigsqcup_{d \geq 0} E^d$  be a semibasis for  $X$  over  $A$ . Then for each  $d \in \mathbb{Z}$ ,  $X^d = A(\bigsqcup_{i \leq d} E^i)$  is a DG submodule of  $X$ . The inclusions  $X^{d-1} \subseteq X^d$  for all  $d \in \mathbb{Z}$  define a semifree filtration of  $X$ .

(ii)  $\Rightarrow$  (iii). Let  $\mathcal{X}$  be a semifree filtration of  $X$ . For each  $d \geq 0$  choose a basis of cycles for  $X^d/X^{d-1}$  over  $A$ , and lift this basis to a set  $E^d \subseteq X^d$ . Now clearly  $E = \bigsqcup_{d \geq 0} E^d$  is a basis for  $X^\natural$  over  $A^\natural$ . Now, if we impose a well-ordering on  $E^d$ , and suppose that each element of  $E^{d'}$  is smaller than any element of  $E^d$  if  $d' < d$ , then  $E$  has an ordering with the desired property.

(iii)  $\Rightarrow$  (i). Suppose that  $E$  is a well-ordered basis for  $X$  over  $A$  such that  $\partial(e) \in A(\{e' \in E \mid e' < e\})$  for every  $e \in E$ . Set  $E^{-1} = \emptyset$  and  $X^{-1} = 0$ . Now we will recursively define  $E^d$  and  $X^d$  by

$$E^d = \{e \in E \mid \partial(e) \in X^{d-1}\}$$

$$X^d = AE^d$$

Clearly,  $X^d$  is a DG submodule of  $X$  and  $\{e + X^{d-1} \mid e \in E^d\}$  is a basis of cycles for  $(X^d/X^{d-1})^\natural$  over  $A^\natural$ . This implies that  $E' = \bigcup_{d \geq 0} E^d$  generates a semifree submodule of  $X$ . If  $E' \neq E$ , then let  $e$  be the initial element in  $E \setminus E'$ . By the hypothesis on  $E$ , it follows that  $\partial(e) \in A(\{e' \in E \mid e' < e\})$ . However, by the way that  $e$  was chosen, we have that  $\{e' \in E \mid e' < e\} \subseteq E'$ , so  $\partial(e) \in E^d$  for some  $d \geq 0$ . But this implies that  $e \in E^{d+1}$ , which is a contradiction. Hence,  $E' = E$ , and the result follows.  $\square$

A *semifree resolution* of a DG module  $X$  is a quasiisomorphism  $\pi : F \rightarrow X$  of DG modules over  $A$ , where  $F$  is semifree.

**Fact 2.3.2.** Every DG module  $X$  has a semifree resolution.

*Proof.* See 8.3.3 and 8.3.4 of [1] for a construction and proof.  $\square$

Now that the required background has been established, it is time to start defining the categories that we will be working in.

## 2.4 The Categories

The DG modules over  $A$ , along with their degree 0 chain maps, are the objects and morphisms, respectively, of the category  $\text{DG}(A)$ . Letting  $A^\natural$  denote the underlying graded algebra of  $A$ , we can consider, for each DG module  $X$  over  $A$ , its associated graded  $A^\natural$ -module  $X^\natural$ . Such modules are the objects of the category  $\text{G}(A)$ , whose morphisms

are the degree 0  $A^{\natural}$ -linear maps. If we consider the fact that the trivial map defines a differential, it becomes obvious that for each object  $X$  in  $\mathrm{DG}(A)$  of the form  $X = (X^{\natural}, 0)$ ,  $X$  is also an object of  $\mathrm{G}(A)$ . Furthermore, for two such objects  $X$  and  $Y$ , it follows that  $\mathrm{Mor}_{\mathrm{G}(A)}(X, Y) = \mathrm{Mor}_{\mathrm{DG}(A)}(X, Y)$ . Thus,  $\mathrm{G}(A)$  is a full subcategory of  $\mathrm{DG}(A)$ , both of which are abelian categories.

**Fact 2.4.1.** As a consequence of Fact 2.1.1,  $\mathrm{H}(-)$  is a functor from  $\mathrm{DG}(A)$  to  $\mathrm{G}(A)$ , called the *homology functor*.

For any abelian category  $\mathcal{A}$ , we consider the category  $\mathbf{Ch}(\mathcal{A})$  of chain complexes over  $\mathcal{A}$ . The components of a complex in  $\mathbf{Ch}(\mathcal{A})$  are themselves objects in  $\mathcal{A}$ . Given two complexes  $C_{\bullet}, D_{\bullet}$  in  $\mathbf{Ch}(\mathcal{A})$ , the set of morphisms between them is given by the following:

$$\mathrm{Mor}_{\mathbf{Ch}(\mathcal{A})}(C_{\bullet}, D_{\bullet}) = \{(\mu_i)_{i \in \mathbb{Z}} \mid \mu_i \in \mathrm{Mor}_{\mathcal{A}}(C_i, D_i), \mu_i \circ \partial_{i+1}^{C_{\bullet}} = \partial_{i+1}^{D_{\bullet}} \circ \mu_{i+1}\}$$

Furthermore, given the category  $\mathbf{Ch}(\mathcal{A})$  of chain complexes over  $\mathcal{A}$ , we can form its quotient category  $\mathbf{K}(\mathcal{A})$  by equating homotopy equivalent maps of complexes in  $\mathbf{Ch}(\mathcal{A})$ . Thus,  $\mathrm{Obj}_{\mathbf{K}(\mathcal{A})} = \mathrm{Obj}_{\mathbf{Ch}(\mathcal{A})}$  and  $\mathrm{Mor}_{\mathbf{K}(\mathcal{A})} \subseteq \mathrm{Mor}_{\mathbf{Ch}(\mathcal{A})}$ .

The homologically bounded complexes in  $\mathbf{ChG}(A)$ , along with their morphisms, form a category, which we denote by  $\mathbf{G}_{\bullet}(A)$ . Clearly,  $\mathbf{G}_{\bullet}(A)$  is a full subcategory of  $\mathbf{ChG}(A)$ .

**Fact 2.4.2.** Let  $M_{\bullet}$  be a complex in  $\mathbf{ChG}(A)$ . There exists a complex  $F_{\bullet}$  of graded free  $A^{\natural}$ -modules such that  $M_{\bullet} \simeq F_{\bullet}$  in  $\mathbf{ChG}(A)$ .

**Fact 2.4.3.** Let  $M_{\bullet}$  be a complex in  $\mathbf{G}_{\bullet}(A)$ . There exists a complex  $F_{\bullet}$  of graded free  $A^{\natural}$ -modules with  $F_i = 0$  for all  $i \ll 0$  such that  $M_{\bullet} \simeq F_{\bullet}$  in  $\mathbf{G}_{\bullet}(A)$ .

We now have the proper machinery to give a basic construction of the particular derived categories that this paper will be concerned with.

## 2.5 The Derived Categories

The derived category  $\mathbf{D}(\mathcal{A})$ , often abbreviated to  $\mathbf{DA}$  for simplicity in notation, of an abelian category  $\mathcal{A}$  is constructed in three stages. First, we consider the category  $\mathbf{Ch}(\mathcal{A})$  of chain complexes in  $\mathcal{A}$ . We then construct its quotient category  $\mathbf{K}(\mathcal{A})$ . Finally, we localize  $\mathbf{K}(\mathcal{A})$  by inverting quasiisomorphisms through a calculus of fractions.

The results of this paper will, in particular, require familiarity with two derived categories:  $\mathbf{DG}(A)$  and  $\mathbf{DDG}(A)$ .

In lieu of Fact 2.4.2, the derived category  $\mathbf{DG}(A)$  can be constructed as the category whose objects are complexes of graded free  $A^{\natural}$ -modules, and whose morphisms are the homotopy classes of morphisms of these complexes.

Furthermore, as a result of Fact 2.3.2, the derived category  $\mathbf{DDG}(A)$  can be viewed as the category whose objects are semifree DG modules over  $A$ , and whose morphisms are the homotopy classes of morphisms of these DG modules.

## 2.6 Totalling

Let  $M_{\bullet} \in \mathbf{ChG}(A)$ . We define the *totalling* of  $M_{\bullet}$  to be the DG module  $((\mathrm{Tot} M_{\bullet})^{\natural}, \partial^{\mathrm{Tot} M_{\bullet}})$  given by

$$(\mathrm{Tot} M_{\bullet})^{\natural} = \bigoplus_{i \in \mathbb{Z}} \Sigma^i M_i \quad \text{and} \quad \partial^{\mathrm{Tot} M_{\bullet}}((\Sigma^i m_i)_{i \in \mathbb{Z}}) = (\Sigma^{i-1} \partial_i^{M_{\bullet}}(m_i))_{i \in \mathbb{Z}}$$

Likewise, if  $\mu : M_{\bullet} \rightarrow N_{\bullet}$  is a morphism of complexes of graded  $A$ -modules, we define  $\mathrm{Tot} \mu : \mathrm{Tot} M_{\bullet} \rightarrow \mathrm{Tot} N_{\bullet}$  by

$$\mathrm{Tot} \mu((\Sigma^i m_i)_{i \in \mathbb{Z}}) = (\Sigma^i \mu_i(m_i))_{i \in \mathbb{Z}}$$

Thus, totalling defines a functor  $\mathrm{Tot} : \mathbf{ChG}(A) \rightarrow \mathbf{DG}(A)$ .



**Fact 2.6.1.** The functor  $\text{Tot}$  preserves quasiisomorphism.

*Proof.* Suppose that  $M_\bullet$  and  $N_\bullet$  are quasiisomorphic complexes in  $\mathbf{ChG}(A)$ . Let  $\mu$  be a chain map from  $M_\bullet$  to  $N_\bullet$  which induces an isomorphism in homology. Define a chain map  $\lambda : \text{Tot } M_\bullet \rightarrow \text{Tot } N_\bullet$  by

$$\begin{aligned} \lambda : \text{Tot } M_\bullet &\rightarrow \text{Tot } N_\bullet \\ \Sigma^i m_i &\mapsto \Sigma^i (\mu_i(m_i)) \end{aligned}$$

Clearly  $\lambda$  induces an isomorphism in homology. The result follows.  $\square$

The previous fact implies that the functor  $\text{Tot}$  extends to a functor of derived categories:

$$\text{Tot} : \mathbf{DG}(A) \rightarrow \mathbf{DDG}(A)$$

**Fact 2.6.2.** For complexes  $L_\bullet, M_\bullet \in \mathbf{ChG}(A)$ , there exists a natural isomorphism

$$\text{Tot } L_\bullet \otimes_A \text{Tot } M_\bullet \cong \text{Tot} (L_\bullet \otimes_{\mathbf{G}(A)} M_\bullet)$$

**Fact 2.6.3.** If  $M_\bullet \in \mathbf{G}(A)$  is such that each  $M_i$  is free as a graded  $A$ -module, and  $M_i = 0$  for all  $i \ll 0$ , then  $\text{Tot } M_\bullet$  is semifree in  $\mathbf{DG}(A)$ .

*Proof.* Let  $\ell \in \mathbb{Z}$  be such that  $M_i = 0$  for all  $i < \ell$ . Now, for every  $j \geq \ell$ , let  $\mathbf{B}_j$  be a well-ordered basis for  $M_j$  over  $A$ , and define

$$\mathbf{B} = \bigsqcup_{j=0}^{\infty} \Sigma^j(\mathbf{B}_j)$$

For  $m_j \in M_j$  and  $m_{j'} \in M_{j'}$ , let  $\Sigma^j(m_j) < \Sigma^{j'}(m_{j'})$  if either  $j < j'$  or  $j = j'$  and  $m_j < m_{j'}$  in  $M_j = M_{j'}$ . Then, clearly,  $\mathbf{B}$  defines a well-ordered basis for  $\text{Tot } M_\bullet$ .  $\square$

**Fact 2.6.4.** For each  $L_\bullet \in \mathbf{ChG}(A^0)$  and  $M_\bullet \in \mathbf{G}_\bullet(A)$ , and every  $m \in \mathbb{Z}$ , there exists an isomorphism

$$\mathrm{Tor}_m^A(\mathrm{Tot} L_\bullet, \mathrm{Tot} M_\bullet) \cong \bigoplus_{i \in \mathbb{Z}} \mathrm{Tor}_i^{\mathbf{G}(A)}(L_\bullet, M_\bullet)_{m-i}$$

*Proof.* By Fact 2.4.3 we may assume that each  $M_i$  is free, and  $M_i = 0$  for  $i \ll 0$ . Then by Fact 3.2.2 we know that  $\mathrm{Tot} M_\bullet$  is semifree, and thus projective. With this assumption, along with the help of Fact 2.6.2 and the definition of totalling, we have the following isomorphisms:

$$\begin{aligned} \mathrm{Tor}_m^A(\mathrm{Tot} L_\bullet, \mathrm{Tot} M_\bullet) &\cong \mathrm{H}_m(\mathrm{Tot} L_\bullet \otimes_A \mathrm{Tot} M_\bullet) \\ &\cong \mathrm{H}_m(\mathrm{Tot}(L_\bullet \otimes_{\mathbf{G}(A)} M_\bullet)) \\ &\cong \bigoplus_{i \in \mathbb{Z}} \mathrm{H}_i(L_\bullet \otimes_{\mathbf{G}(A)} M_\bullet)_{m-i} \\ &\cong \bigoplus_{i \in \mathbb{Z}} \mathrm{Tor}_i^{\mathbf{G}(A)}(L_\bullet, M_\bullet)_{m-i} \end{aligned}$$

□

## CHAPTER 3

### Results

Now with a thorough understanding of the action of the totalling functor on the derived category  $\mathbf{DG}(A)$ , we would like to be able to describe its image. Of great importance is the fact that  $\text{Tot} : \mathbf{DG}(A) \rightarrow \mathbf{DDG}(A)$  is *not* a surjection of derived categories, as is illustrated in the following section.

As before, in this chapter we assume that  $A$  is an arbitrary DG algebra, unless otherwise stated.

#### 3.1 DG Modules not in the Image of Tot

**Fact 3.1.1.** If  $M_\bullet$  is a complex of graded  $A^{\mathbb{N}}$ -modules such that  $H(M_\bullet) = H_r(M_\bullet)$  for some  $r \in \mathbb{Z}$ , then  $M_\bullet \simeq H(M_\bullet)$  in  $\mathbf{DG}(A)$ .

*Proof.* Consider the following commutative diagram of complexes and chain maps:

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & M_{r+1} & \xrightarrow{\partial_{i+1}} & M_r & \xrightarrow{\partial_i} & M_{r-1} & \longrightarrow & \cdots \\
 & & \parallel & & \uparrow \iota & & \uparrow & & \\
 \cdots & \longrightarrow & M_{r+1} & \xrightarrow{\partial_{i+1}} & Z_r(M_\bullet) & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow \pi & & \downarrow & & \\
 \cdots & \longrightarrow & 0 & \longrightarrow & H_r(M_\bullet) & \longrightarrow & 0 & \longrightarrow & \cdots
 \end{array}$$

where  $\iota$  and  $\pi$  are the obvious inclusion and projection maps, respectively, and the unlabeled arrows represent zero maps. Clearly, these chain maps induce an isomorphism in homology. Thus,  $M_\bullet \simeq H(M_\bullet)$ . □

**Lemma 3.1.2.** *Let  $X$  be a semifree DG module over  $A$  such that  $X \simeq \text{Tot } M_\bullet$  for some complex  $M_\bullet$  of graded  $A$ -modules. If  $H(X)$  is indecomposable over  $A^\natural$ , then*

$$\sum_{i \in \mathbb{Z}} b_i = \text{rank}_A(X)$$

where  $b_i$  is the  $i^{\text{th}}$  Betti number of  $H(X)$ .

*Proof.* The fact that  $X \simeq \text{Tot } M_\bullet$  produces the following isomorphisms of graded  $A^\natural$ -modules:

$$H(X) \cong H(\text{Tot } M_\bullet) \cong \bigoplus_{i \in \mathbb{Z}} \Sigma^i H_i(M_\bullet) \quad (3.1)$$

Now  $H_i(M_\bullet)$  is an  $A^\natural$ -module for each  $i \in \mathbb{Z}$ . Since we assumed that  $H(X)$  does not decompose over  $A^\natural$ , it follows that there exists some  $r \in \mathbb{Z}$  with  $H_r(M_\bullet) \cong H(X)$  and  $H_i(M_\bullet) \cong 0$  for all  $i \neq r$ . This implies that  $H(M_\bullet) \cong H_r(M_\bullet)$ . By Fact 3.1.1, we have that  $M_\bullet \simeq H(M_\bullet)$ . Recalling our initial hypothesis on  $X$ , as well as Fact 2.6.1, we have the following quasiisomorphisms:

$$X \simeq \text{Tot } M_\bullet \simeq \text{Tot } (H(M_\bullet)) \quad (3.2)$$

Furthermore, the isomorphisms given by

$$\text{Tot } (H(M_\bullet)) \cong \text{Tot } (H_r(M_\bullet)) \cong \Sigma^r H_r(M_\bullet) \cong \Sigma^r H(X) \quad (3.3)$$

are due, respectively, to the fact that the homology of  $M_\bullet$  is concentrated in degree  $r$ , the definition of totalling, and the fact that  $H(X) \cong H_r(M_\bullet)$ . But notice that since the terms in (3.3) are merely graded  $A^\natural$ -modules, each of the isomorphisms is actually a quasiisomorphism, yielding  $\text{Tot } (H(M_\bullet)) \simeq \Sigma^r H(X)$ . Combining this result with (3.2)

gives us that  $X \simeq \Sigma^r H(X)$ . This quasiisomorphism and Fact 2.6.4 account for the isomorphisms below:

$$\begin{aligned}
X \otimes_A k &= H(X \otimes_A k) \\
&= \bigoplus_{m \in \mathbb{Z}} \mathrm{Tor}_m^A(X, k) \\
&\cong \bigoplus_{m \in \mathbb{Z}} \mathrm{Tor}_m^A(\Sigma^r H(X), k) \\
&\cong \bigoplus_{m, i \in \mathbb{Z}} \mathrm{Tor}_i^{\mathbf{G}(A)}(\Sigma^r H(X), k)_{m-i}
\end{aligned}$$

The equalities follow from the semifreeness of  $X$ . Hence the following is implied:

$$\begin{aligned}
\mathrm{rank}_A(X) &= \mathrm{rank}_k(X \otimes_A k) \\
&= \sum_{m, i \in \mathbb{Z}} \mathrm{rank}_k \left( \mathrm{Tor}_m^{\mathbf{G}(A)}(\Sigma^r H(X), k)_{m-i} \right) \\
&= \sum_{i \in \mathbb{Z}} b_i
\end{aligned}$$

□

**Theorem 3.1.3.** *Suppose that  $A = k[x_1, \dots, x_d]$  under the standard grading. Then the functor*

$$\mathrm{Tot} : \mathbf{DG}(A) \rightarrow \mathbf{DDG}(A)$$

*is not onto in the case when  $d \geq 2$ .*

*Proof.* Let  $k$  be a field, and consider the rank 4 semifree DG module  $X$  over  $A = k[x_1, \dots, x_d]$ , where  $d \geq 2$ , given by  $X^\natural = Ae_1 \oplus Ae_2 \oplus Ae_3 \oplus Ae_4$  and

$$\partial(e_1) = 0$$

$$\partial(e_2) = x_1 x_2 e_1$$

$$\partial(e_3) = x_2^3 e_1$$

$$\partial(e_4) = x_1^7 e_1 - x_2^4 e_2 + x_1 x_2^2 e_3$$

We will show that there does not exist a complex  $M_\bullet$  of graded  $A$ -modules such that  $\text{Tot } M_\bullet \simeq X$  in  $\mathbf{DDG}(A)$ .

Note that minimal generating sets of  $Z(X)$  and  $B(X)$  over  $A$  are  $\{e_1, x_2^2 e_2 - x_1 e_3\}$  and  $\{x_1 x_2 e_1, x_2^3 e_1, x_1^7 e_1 - x_2^4 e_2 + x_1 x_2^2 e_3\}$ , respectively. Thus the presentation matrix of  $H(X)$  is given by

$$\begin{pmatrix} x_1 x_2 & x_2^3 & x_1^7 \\ 0 & 0 & -x_2^2 \end{pmatrix}$$

From this, we see that  $H(X)$  is indecomposable as an  $A$ -module. Writing down its minimal graded free resolution yields

$$0 \rightarrow \Sigma^4 A \xrightarrow{\begin{pmatrix} x_2^2 \\ -x_1 \\ 0 \end{pmatrix}} \Sigma^2 A \oplus \Sigma^3 A \oplus \Sigma^7 A \xrightarrow{\begin{pmatrix} x_1 x_2 & x_2^3 & x_1^7 \\ 0 & 0 & -x_2^2 \end{pmatrix}} A \oplus \Sigma^5 A \rightarrow H(X) \rightarrow 0$$

Thus, we have that:

$$\sum_{i \in \mathbb{Z}} b_i = 2 + 3 + 1 = 6 \neq 4 = \text{rank}(X)$$

Since this is a contradiction to the previous lemma, we can conclude that  $X$  is not quasiisomorphic to  $\text{Tot } M_\bullet$  for any complex  $M_\bullet$  of graded  $A$ -modules.  $\square$

Note that the previous theorem does not imply that for  $d \geq 2$  there do not exist semifree DG modules over  $A = k[x_1, \dots, x_d]$  which lie in the image of  $\text{Tot}$ . The next section begins by investigating a special class of semifree DG modules of arbitrary rank over  $A = k[x_1, \dots, x_d]$  which are always obtained from the totalling of some complex in  $\mathbf{DG}(A)$ .

## 3.2 DG Modules in the Image of Tot

### 3.2.1 Crossing

**Definition.** Let  $A$  be a DG algebra, and consider an arbitrary rank  $n$  semifree DG module  $X$ , with ordered basis  $\{e_1, \dots, e_n\}$  over  $A$ . Now define a family of disjoint sets by:

$$S_0 = \{i \in \mathbb{Z}^+ \mid \partial(e_i) = 0\}$$

$$S_\ell = \left\{ i \in \mathbb{Z}^+ \mid 0 \neq \partial(e_i) \in \bigoplus_{j \in S_{\ell-1}} Ae_j \right\}$$

for all  $\ell \in \mathbb{Z}^+$ . If

$$\bigcup_{\ell \in \mathbb{N}} S_\ell = \{1, 2, \dots, n\}$$

then we say that the differential of  $X$  *has no crossing*. However, if

$$\bigcup_{\ell \in \mathbb{N}} S_\ell \subset \{1, 2, \dots, n\}$$

is a proper inclusion, we say that the differential of  $X$  *has crossing*.

Now, to see the application of this definition, we consider two examples.

**Example 3.2.1.** Let  $A = k[x, y, z]$ . Consider the rank 4 semifree DG module  $X$  over  $A$  given by  $X^\natural = Ae_1 \oplus Ae_2 \oplus Ae_3 \oplus Ae_4$  and

$$\partial(e_1) = 0$$

$$\partial(e_2) = xe_1$$

$$\partial(e_3) = yze_1$$

$$\partial(e_4) = xz^3e_1 + yze_2 - xe_3$$

Then  $S_0 = \{1\}$ ,  $S_1 = \{2, 3\}$ , and  $S_\ell = \emptyset$  for  $\ell \geq 2$ . Since

$$\bigcup_{\ell \in \mathbb{N}} S_\ell = S_0 \cup S_1 = \{1, 2, 3\} \subset \{1, 2, 3, 4\}$$

is a proper inclusion, the differential of  $X$  has crossing.

**Example 3.2.2.** Let  $A$  and  $X^{\natural}$  be as in the previous example, but consider the differential of  $X$  given by:

$$\partial(e_1) = 0$$

$$\partial(e_2) = xe_1$$

$$\partial(e_3) = yze_1$$

$$\partial(e_4) = yze_2 - xe_3$$

Now we have that  $S_0 = \{1\}$ ,  $S_1 = \{2, 3\}$ ,  $S_2 = \{4\}$ , and  $S_\ell = \emptyset$  for  $\ell \geq 3$ . Since this differential yields

$$\bigcup_{\ell \in \mathbb{N}} S_\ell = S_0 \cup S_1 \cup S_2 = \{1, 2, 3, 4\}$$

it does not have crossing.

**Theorem 3.2.3.** *Let  $A$  be a DG algebra. If  $X$  is a rank  $n$  semifree DG module over  $A$  such that its differential has no crossing, then there exists a complex  $M_\bullet$  of graded  $A^{\natural}$ -modules such that  $X = \text{Tot } M_\bullet$ .*

*Proof.* Let  $A$  be a given DG algebra, and let  $X$  be a rank  $n$  semifree DG module over  $A$  with ordered basis  $\{e_1, \dots, e_n\}$ . Note that there exists  $0 \leq N \leq n$  such that  $S_m = \emptyset$  for all  $m > N$ , while  $S_0, S_1, \dots, S_N \neq \emptyset$ . Now define a sequence  $M_\bullet$  of homomorphisms of graded  $A^{\natural}$ -modules by

$$M_\bullet : \quad 0 \rightarrow \bigoplus_{j \in S_N} (\Sigma^{-N} Ae_j) \xrightarrow{\partial_N^M} \dots \xrightarrow{\partial_2^M} \bigoplus_{j \in S_1} (\Sigma^{-1} Ae_j) \xrightarrow{\partial_1^M} \bigoplus_{j \in S_0} (Ae_j) \rightarrow 0$$

where, for each  $i_1 \leq m \leq i_\ell$  and each  $0 \leq \ell \leq N$ , we have  $\partial_\ell^{M_\bullet}((0, \dots, 0, e_m, 0, \dots, 0)) = \partial^X(e_m)$ .



Clearly,  $\partial_{i+1}^{M_\bullet} \circ \partial_i^{M_\bullet} = 0$  for all  $N \leq i \leq 0$ . To see that  $\text{Tot } M_\bullet = X$ , notice that

$$\begin{aligned}
(\text{Tot } M_\bullet)^\natural &= \bigoplus_{0 \leq i \leq N} (\Sigma^i M_i) \\
&= \bigoplus_{0 \leq i \leq N} \left( \Sigma^i \left( \bigoplus_{j \in S_i} (\Sigma^{-i} A e_j) \right) \right) \\
&\cong \bigoplus_{0 \leq i \leq N} \left( \bigoplus_{j \in S_i} A e_j \right) \\
&= A e_1 \oplus \cdots \oplus A e_n = X^\natural
\end{aligned}$$

and that  $\partial^{\text{Tot } M_\bullet}(e_i) = \partial^M(e_i) = \partial^X(e_i)$  for all  $1 \leq i \leq n$ . Therefore,

$$\text{Tot } M_\bullet = ((\text{Tot } M_\bullet)^\natural, \partial^{\text{Tot } M_\bullet}) = (X^\natural, \partial^X) = X$$

and the result follows.  $\square$

Notice the weight of the statement of the previous theorem. It not only states that any rank  $n$  semifree DG module over  $A$  without crossing in its differential is in the image of  $\text{Tot}$ , but it asserts that the DG module is *equal* to the totalling of some complex of graded  $A^\natural$ -modules.

Now we will illustrate the practical use of the proof of the theorem with an example.

**Example 3.2.4.** Let  $X$  be the rank 4 semifree DG module over  $A = k[x, y, z]$  given in Example 3.2.2. Since the differential of  $X$  has no crossing, it is equal to the totalling of some complex of graded  $A$ -modules. By the construction in the proof of the previous theorem, we see that

$$M_\bullet : \quad 0 \rightarrow \Sigma^{-2} A e_4 \xrightarrow{\begin{pmatrix} yz \\ -x \end{pmatrix}} \Sigma^{-1}(A e_2 \oplus A e_3) \xrightarrow{\begin{pmatrix} x & yz \end{pmatrix}} A e_1 \rightarrow 0$$

is a complex of graded  $A$ -modules such that  $\text{Tot } M_\bullet = X$ .

**Corollary 3.2.5.** *Let  $A$  be a DG algebra, and  $X$  be a rank  $n$  semifree DG module over  $A$ . If  $n \leq 3$  then there exists a complex  $M_\bullet$  of graded  $A$ -modules such that  $X = \text{Tot } M_\bullet$ .*

*Proof.* Let  $\{e_1, \dots, e_n\}$  be an ordered basis for  $X$  over  $A$ . Recalling that  $X$  is semifree, we obtain the following about its differential:

$$n = 1 \quad \Rightarrow \quad \partial(e_1) = 0$$

$$n = 2 \quad \Rightarrow \quad \partial(e_1) = 0$$

$$\partial(e_2) = f_{12}e_1$$

$$n = 3 \quad \Rightarrow \quad \partial(e_1) = 0 \qquad \partial(e_1) = 0$$

$$\partial(e_2) = f_{12}e_1 \quad \text{or} \quad \partial(e_2) = 0$$

$$\partial(e_3) = f_{13}e_1 \qquad \partial(e_3) = f_{13}e_1 + f_{23}e_2$$

where  $f_{ij} \in A$  for each  $i, j$ . Note that in each case, the differential of  $X$  has no crossing. The result follows by the previous theorem.  $\square$

### 3.2.2 When $A = k[x]$

**Lemma 3.2.6.** *Let  $X$  be a rank  $n$  semifree DG module over  $A = k[x]$ . There exist nonnegative integers  $m, s$  with  $1 \leq m \leq n$  and  $0 \leq s \leq m$ , along with integers  $r_i, c_j$  for  $1 \leq i \leq m$  and  $1 \leq j \leq s$  such that  $H(X)$  has graded minimal free resolution over  $A$  given by:*

$$0 \rightarrow \bigoplus_{j=1}^s \Sigma^{c_j} A \xrightarrow{\begin{pmatrix} h_1 & & & 0 \\ & \ddots & & \\ & & h_s & \\ \hline & & & 0 \end{pmatrix}} \bigoplus_{i=1}^m \Sigma^{r_i} A \rightarrow H(X) \rightarrow 0$$

where  $h_i = x^{c_i - r_i}$  and  $c_i - r_i \geq 1$  for  $1 \leq i \leq s$ .

*Proof.*  $B(X) \subseteq Z(X)$  are submodules of  $X$ , an  $n$ -generated module over  $A = k[x]$ . Since  $A$  is a principal ideal domain, it follows that  $Z(X)$  and  $B(X)$  are finitely generated over  $A$ . Thus  $H(X)$  is a finitely generated  $A$ -module. By Hilbert's syzygy theorem, any minimal free resolution of  $H(X)$  can be, at most, length 1. Therefore, we can write down a minimal free resolution of  $H(X)$  as:

$$0 \rightarrow \bigoplus_{j=1}^s \Sigma^{\tilde{c}_j} A \xrightarrow{\phi} \bigoplus_{i=1}^m \Sigma^{\tilde{r}_i} A \rightarrow H(X) \rightarrow 0$$

where  $\tilde{r}_i$  is the degree of the  $i^{\text{th}}$  minimal generator of  $H(X)$ . The exactness of the resolution tells us that  $s \leq m$ .

Notice that the homogeneity of  $\phi$ , along with the minimality of the resolution of  $H(X)$ , yields that each nonzero entry  $f_{i,j}$  in  $\phi$  is such that

$$\tilde{r}_i - \tilde{c}_j = |f_{i,j}| \geq 1 \tag{3.4}$$

Simple arithmetic involving (3.4) reveals the following relations between nonzero entries of  $\phi$ :

$$\begin{aligned} |f_{i+1,j}| - |f_{i,j}| &= |f_{i+1,j+1}| - |f_{i,j+1}| \\ |f_{i,j+1}| - |f_{i,j}| &= |f_{i+1,j+1}| - |f_{i+1,j}| \end{aligned}$$

for all  $1 \leq i \leq m-1$ ,  $1 \leq j \leq s-1$ . Therefore, we can rearrange the rows and columns of  $\phi$  so that its nonzero entries are such that

$$|f_{i,j}| \leq |f_{i',j}| \quad \text{and} \quad |f_{i,j}| \leq |f_{i,j'}| \tag{3.5}$$

for all  $1 \leq i \leq i' \leq m$  and  $1 \leq j \leq j' \leq s$ . In other words, any nonzero entry of  $\phi$  has only zeros and entries of higher degree in each spot below and to the right of it.

The proof will continue recursively for  $\ell = 1, 2, \dots, s$ . Notice that by the minimality of the resolution of  $H(X)$ , there must be a nonzero entry in column  $\ell$ . Let  $\ell \leq i' \leq m$

be the smallest integer such that  $f_{i',\ell} \neq 0$ . Shift the rows of  $\phi$  so that  $f_{i',\ell} \mapsto f_{\ell,\ell}$  and  $f_{i,\ell} \mapsto f_{i+1,\ell}$  for  $\ell \leq i \leq i' - 1$ . Now we have that  $f_{\ell,\ell} \neq 0$ , and each nonzero entry below and to the right of  $f_{\ell,\ell}$  still has degree at least  $|f_{\ell,\ell}|$ . Thus, using  $f_{\ell,\ell}$  as a pivot entry, we can perform elementary row and column operations on  $\phi$  to eliminate all entries below and to the right of  $f_{\ell,\ell}$ , so that  $f_{i,\ell} = f_{\ell,j} = 0$  for all  $\ell \leq i \leq m$ ,  $\ell \leq j \leq s$ . Notice that  $\phi$  now has the form:

$$\phi = \begin{pmatrix} f_{1,1} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & & & \vdots \\ \vdots & & f_{\ell,\ell} & 0 & \cdots & 0 \\ \vdots & & 0 & f_{\ell+1,\ell+1} & \cdots & f_{\ell+1,s} \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & f_{m,\ell+1} & \cdots & f_{m,s} \end{pmatrix}$$

Moreover, notice that the  $(m - \ell) \times (s - \ell)$  submatrix of  $\phi$  given by  $(f_{i,j})$  for  $\ell + 1 \leq i \leq m$ ,  $\ell + 1 \leq j \leq s$  still possesses the property in (3.5) for all of its nonzero entries.

Now once  $\phi$  has been reduced to an  $m \times s$  matrix of the form

$$\phi = \begin{pmatrix} f_{1,1} & & 0 \\ & \ddots & \\ 0 & & f_{s,s} \\ \hline & & 0 \end{pmatrix}$$

where, for each  $1 \leq i \leq s$ ,  $f_{i,i} = \alpha_i x^{c_i - r_i}$  for some  $\alpha_i \in k$ , it is easy to see that  $h_i = (\alpha_i)^{-1} f_{i,i}$  for  $1 \leq i \leq s$ . The result is immediate.  $\square$

**Theorem 3.2.7.** *Every rank  $n$  semifree DG module over  $A = k[x]$  is in the image of  $\text{Tot} : \mathbf{DG}(A) \rightarrow \mathbf{DDG}(A)$ .*

*Proof.* Let  $X$  be a rank  $n$  semifree DG module over  $A$ . By the previous lemma, we can assume the homology of  $X$  to have the form

$$H(X) \cong \bigoplus_{i=1}^s \frac{\Sigma^{r_i} A}{h_i \Sigma^{c_i} A} \oplus \bigoplus_{i=s+1}^m \Sigma^{r_i} A \quad (3.6)$$

where  $0 < m \leq n$  and  $0 \leq s \leq m$ , and  $r_i, c_j \in \mathbb{Z}$  for  $1 \leq i \leq m$  and  $1 \leq j \leq s$ . Now consider its deleted minimal graded free resolution  $M_\bullet$  given by

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Sigma^{c_1} A & \xrightarrow{h_1} & \Sigma^{r_1} A & \longrightarrow & 0 \\ & & \oplus & & \oplus & & \\ & & \vdots & & \vdots & & \\ & & \oplus & & \oplus & & \\ 0 & \longrightarrow & \Sigma^{c_s} A & \xrightarrow{h_s} & \Sigma^{r_s} A & \longrightarrow & 0 \\ & & \oplus & & \oplus & & \\ & & 0 & \longrightarrow & \Sigma^{r_{s+1}} A & \longrightarrow & 0 \\ & & \oplus & & \oplus & & \\ & & \vdots & & \vdots & & \\ & & \oplus & & \oplus & & \\ & & 0 & \longrightarrow & \Sigma^{r_m} A & \longrightarrow & 0 \end{array}$$

where  $M_0 = \Sigma^{r_1} A \oplus \cdots \oplus \Sigma^{r_m} A$  and  $M_1 = \Sigma^{c_1} A \oplus \cdots \oplus \Sigma^{c_s} A$ . To complete the proof, we will show that  $\text{Tot } M_\bullet \simeq X$  in  $\mathbf{DDG}(A)$ .

Fix  $1 \leq i \leq m$ . Let  $L_i$  be the complex which is given by the  $i^{\text{th}}$  summand of  $M_\bullet$ . Namely,

$$L_i : \begin{cases} 0 \rightarrow \Sigma^{c_i} A \xrightarrow{h_i} \Sigma^{r_i} A \rightarrow 0 & \text{if } 1 \leq i \leq s \\ 0 \rightarrow \Sigma^{r_i} A \rightarrow 0 & \text{if } s+1 \leq i \leq m \end{cases}$$

We will define a family  $\mu^i$  for  $1 \leq i \leq m$  of chain maps  $\mu^i : X \rightarrow \text{Tot } L_i$  such that the chain map  $\mu : X \rightarrow \text{Tot } M_\bullet$  given by

$$\mu := (\mu^i) : X \rightarrow \bigoplus_{i=1}^m \text{Tot } L_i = \text{Tot } M_\bullet \quad (3.7)$$

induces an isomorphism in homology.

By (3.6), there exists an isomorphism of graded  $A$ -modules

$$\varphi : H(X) \rightarrow \bigoplus_{i=1}^s \frac{\Sigma^{r_i} A}{h_i \Sigma^{c_i} A} \oplus \bigoplus_{i=s+1}^m \Sigma^{r_i} A$$

Let  $z_i$  be the cycle in  $X_{r_i}$  such that  $\text{cls}(z_i) = \varphi^{-1}(\Sigma^{r_i} 1)$ . Since  $\varphi$  is  $A$ -linear,  $\text{cls}(x^\ell z_i) = \varphi^{-1}(x^\ell \Sigma^{r_i} 1)$  for all nonnegative integers  $\ell$ . Define:

$$N_i = \begin{cases} |h_i| & \text{if } 1 \leq i \leq s \\ \infty & \text{if } s+1 \leq i \leq m \end{cases}$$

Note that  $\text{cls}(x^\ell z_i) \neq \text{cls}(0)$  in  $H(X)$  and  $\text{cls}(x^\ell \Sigma^{r_i} 1) \neq \text{cls}(0)$  in  $H(\text{Tot } L_i)$  for each  $0 \leq \ell < N_i$ .

Now for each  $j \in \mathbb{Z}$  we see that  $X_j$  is a finite-dimensional vector space over  $k$ . Let  $Y_j^i$  be the vector subspace of  $X_j$  given by

$$Y_j^i := \{y \in X_j \mid \partial_j^X(y) \in kx^{j-r_i-1}z_i\}$$

Clearly  $Y_j^i = \emptyset$  for all  $j \leq r_i$ . However, notice that for  $j > r_i$ ,  $Y_j^i \supseteq Z_j(X)$ . In fact, since  $x^\ell z_i$  is not homologous to 0 for  $1 \leq \ell < N_i$ , we have that  $Y_j^i = Z_j(X)$  whenever  $r_i + 1 \leq j < r_i + N_i + 1$ . Let  $j > r_i + N_i$ , and suppose that  $Y_j^i$  is a  $t$ -dimensional vector space over  $k$ . Since  $\dim_k(kx^{j-r_i-1}z_i) = 1$  for  $j \geq r_i + 1$ , it follows that the restriction of  $\partial_j^X$  to  $Y_j^i$  can be viewed as a  $1 \times t$  matrix over  $k$ . This, along with the fact that  $Z_j(X) \subseteq Y_j^i$ , clearly implies that  $\dim_k(Z_j(X)) = t - 1$ . Thus if  $j > r_i + N_i$ , we can choose a basis for  $Y_j^i$  over  $k$  which has a unique element which is mapped by  $\partial_j^X$  to  $x^{j-r_i-1}z_i$ . Now consider  $j = r_i + N_i + 1$ . Let  $y_i \in Y_{r_i+N_i+1}^i$  be such that  $\partial_{r_i+N_i+1}^X(y_i) = x^{N_i}z_i$ . Note that for each  $\ell \geq 0$ , the elements  $x^{N_i+\ell+1}z_i$  and  $(-1)^\ell x^\ell y_i$  are linearly independent over  $k$  in  $X_{r_i+N_i+\ell+1}$ . Thus, for each  $j \in \mathbb{Z}$  we will define a basis  $\mathbf{B}_j^i$  for  $X_j$  over  $k$  such that

$$\mathbf{B}_j^i \supseteq \{x^{j-r_i}z_i, (-1)^{j-c_i-1}x^{j-c_i-1}y_i\}$$

Now we will define  $\mu_j^i : X_j \rightarrow (\text{Tot } L_i)_{j-r_i}$  on the basis  $\mathbf{B}_j^i$  of  $X_j$  for each  $j \in \mathbb{Z}$ :

$$\mu_j^i(v) = \begin{cases} x^{j-r_i} \Sigma^{r_i} 1 & \text{if } v = x^{j-r_i} z_i \\ x^{j-c_i-1} \Sigma^{c_i+1} 1 & \text{if } v = (-1)^{j-c_i-1} x^{j-c_i-1} y_i \\ 0 & \text{otherwise} \end{cases}$$

Since  $\mu_j^i$  is defined from a basis of  $X_j$  to a basis of  $(\text{Tot } L_i)_{j-r_i}$ , it is clearly well-defined. Moreover,  $A$ -linearity is immediate from the definition of the bases. The fact that  $\mu^i$  commutes with the differentials of  $X$  and  $\text{Tot } L_i$  is a direct consequence of the Leibniz rule, along with the fact that  $A = k[x]$  is a graded algebra (with trivial differential).

Therefore,  $\mu^i$  is indeed a chain map from  $X$  to  $\text{Tot } L_i$ . This implies that the map  $\mu : X \rightarrow \text{Tot } M_\bullet$  given in (3.7) is also a chain map of DG modules. To see that  $\mu$  induces an isomorphism in homology, note that  $\mu^i$  establishes a one-to-one correspondence between generators of homology of  $X$  in degree  $j$  for  $r_i \leq j < r_i + N_i$ , and generators of homology in  $\text{Tot } L_i$ . Thus,  $\mu = (\mu^i)$  establishes a one-to-one correspondence between generators of homology in  $X$  and generators of homology in  $\text{Tot } M_\bullet = \bigoplus_{i=1}^m \text{Tot } L_i$ . The result is immediate.  $\square$

Now we will illustrate the practical use of the theorem with an example.

**Example 3.2.8.** Let  $X$  be a rank 5 semifree DG module with well-ordered basis  $\{e_1, e_2, e_3, e_4, e_5\}$  over  $A = k[x]$ , where  $|e_1| = 0$ ,  $|e_2| = 2$ ,  $|e_3| = 4$ ,  $|e_4| = 8$ , and  $|e_5| = 9$ . Suppose that the differential of  $X$  is given by

$$\partial(e_1) = 0$$

$$\partial(e_2) = 0$$

$$\partial(e_3) = 0$$

$$\partial(e_4) = x^7 e_1 + x^5 e_2$$

$$\partial(e_5) = x^4 e_3$$

Then we obtain the following decomposition in homology:

$$\begin{aligned} \mathbb{H}(X) &= \frac{Ae_1 \oplus Ae_2 \oplus Ae_3}{A(x^7e_1 + x^5e_2) \oplus Ax^4e_3} \\ &\cong \frac{A(x^2e_1 + e_2)}{A(x^7e_1 + x^5e_2)} \oplus \frac{Ae_3}{Ax^4e_3} \oplus Ae_1 \end{aligned}$$

So the deleted minimal free resolution  $M_\bullet$  of  $\mathbb{H}(X)$  is given by

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Sigma^7 A & \xrightarrow{x^5} & \Sigma^2 A & \longrightarrow & 0 \\ & & \oplus & & \oplus & & \\ 0 & \longrightarrow & \Sigma^8 A & \xrightarrow{x^4} & \Sigma^4 A & \longrightarrow & 0 \\ & & \oplus & & \oplus & & \\ & & 0 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

We will utilize the proof of the previous theorem to show that  $\text{Tot } M_\bullet \simeq X$  in  $\mathbf{DDG}(A)$ .

From  $M_\bullet$  we obtain:

$$\begin{aligned} L_1 : & \quad 0 \rightarrow \Sigma^7 A \xrightarrow{x^5} \Sigma^2 A \rightarrow 0 \\ L_2 : & \quad 0 \rightarrow \Sigma^8 A \xrightarrow{x^4} \Sigma^4 A \rightarrow 0 \\ L_3 : & \quad 0 \rightarrow A \rightarrow 0 \end{aligned}$$

Now, referring to the decomposition of homology, we find that the cycles generating  $\mathbb{H}(X)$  over  $A$  are  $z_1 = x^2e_1 + e_2$ ,  $z_2 = e_3$ , and  $z_3 = e_1$ . Furthermore, we obtain the following sets:

$$\begin{aligned} Y_j^1 &= \{y \in X_j \mid \partial(y) \in k(x^{j-1}e_1 + x^{j-3}e_2)\} \\ &= Z_j(X) \oplus kx^{j-8}e_4 \\ Y_j^2 &= \{y \in X_j \mid \partial(y) \in k(x^{j-5}e_3)\} \\ &= Z_j(X) \oplus kx^{j-9}e_5 \\ Y_j^3 &= \{y \in X_j \mid \partial(y) \in k(x^{j-1}e_1)\} \\ &= Z_j(X) \end{aligned}$$



Let  $\mathbf{B}_j^i$  be a basis for  $X_j$  over  $A$  such that

$$\mathbf{B}_j^1 \supseteq \{x^j e_1 + x^{j-2} e_2, (-1)^{j-8} x^{j-8} e_4\}$$

$$\mathbf{B}_j^2 \supseteq \{x^{j-4} e_3, (-1)^{j-9} x^{j-9} e_5\}$$

$$\mathbf{B}_j^3 \supseteq \{x^j e_1\}$$

Now, we define the chain map  $\mu_j^i$  on the above basis  $\mathbf{B}_j^i$  of  $X_j$  by:

$$\mu_j^1(v) = \begin{cases} x^{j-2} \Sigma^2 1 & \text{if } v = x^j e_1 + x^{j-2} e_2 \\ x^{j-8} \Sigma^8 1 & \text{if } v = (-1)^{j-8} x^{j-8} e_4 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_j^2(v) = \begin{cases} x^{j-2} \Sigma^2 1 & \text{if } v = x^{j-4} e_3 \\ x^{j-8} \Sigma^8 1 & \text{if } v = (-1)^{j-9} x^{j-9} e_5 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_j^3(v) = \begin{cases} x^{j-2} \Sigma^2 1 & \text{if } v = x^j e_1 \\ 0 & \text{otherwise} \end{cases}$$

Then  $\mu : X \rightarrow \text{Tot } M_\bullet$  is given by:

$$\mu(y) = (\mu^1(y), \mu^2(y), \mu^3(y))$$

for every  $y \in X$ .

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## BIOGRAPHICAL STATEMENT

Kristen A. Beck was born in Lakeland, Florida on May 28, 1978 to Richard and Karen Beck. The eldest of four children, she was raised and received her primary education throughout many parts of the U.S., but spent the majority of her childhood in the Dallas/Fort Worth metroplex. Kristen received her high school diploma from Covenant Christian Academy in May 1996, and enrolled at The University of Texas at Arlington in September of the same year. After spending time dabbling in a few disciplines, Kristen admitted her love for mathematics. She was awarded a B.S. in Mathematics, with a minor in Physics, in May 2002. Kristen decided to remain at U.T.A. for graduate studies, and under the supervision of David Jorgensen, received an M.S. in Mathematics in August 2005.