COOPERATIVE CONTROL OF MULTI-AGENT SYSTEMS

STABILITY, OPTIMALITY AND ROBUSTNESS

by

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Presented to the Faculty of the Graduate School of

The University of Texas at Arlington in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT ARLINGTON

August 2013

ACKNOWLEDGEMENTS

I would like to thank everyone who, in one way or another, provided me with invaluable support, guidance, counsel, comfort and love, during these past years at the University of Texas at Arlington. The time spent here, working towards this degree, has in many ways been a life fulfilling and shaping experience. At the beginning I could hardly foresee the scope of changes this experience was to bring. I can only hope the changes are for the better. First of all, I express my everlasting gratitude to my parents, Karmen and Tomica, who have continuously encouraged me to follow the path I have chosen, and supported the choices I made along the way. Without your selfless, loving dedication and support it would had been a much more difficult and lonelier path to traverse. You have been there, besides me, every step of the way. I am grateful to all my teachers. In particular to dr. Frank L. Lewis, who thought me how to maintain research rigor and address problems in a most direct and efficient way, and to dr. Barbara A. Shipman, who thought me the value of precise, austere, logic and the great importance of a definition. I shall strive to keep your lessons and advice alive through the work that follows. I wish to thank Kasia Sturlan, for making me see in a way different than before, for making me realize so much of that, which, otherwise, would have remained ignored, mistaken for irrelevant, dormant, and silenced. Thank you for compelling me to be. Your gift is cherished, more than I can express, and much more than I could ever hope to reciprocate. Without all of you, I would not be in a position to consider myself so fortunate. Words do not suffice to express my full gratitude. Also, this work is supported by NSF grant ECCS-1128050, AFOSR EOARD Grant #13-3055, China NNSF grant 61120106011, and China Education Ministry Project 111 (No.B08015).

July 03, 2013

ABSTRACT

COOPERATIVE CONTROL OF MULTI-AGENT SYSTEMS; STABILITY, OPTIMALITY AND ROBUSTNESS

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The University of Texas at Arlington, 2013

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In this work design methods are given for distributed synchronization control of multi-agent systems on directed communication graphs. Conditions are derived based on the relation of the graph eigenvalues to a region in a complex plane that depends on the single-agent system and the solution of the local Riccati equation. The *synchronizing region* concept is used. Cooperative observer design, guaranteeing convergence of the local estimates to their true values, is also proposed. The notion of *convergence region* for distributed observers is introduced. A duality principle is shown to hold for distributed observers and controllers on balanced graph topologies. Application of cooperative observers is made to the distributed synchronization problem. Three dynamic regulator architectures are proposed for cooperative synchronization.

In the second part this work brings together stability and optimality theory to design distributed cooperative control protocols, which guarantee consensus and are globally optimal with respect to a structured performance criterion. Here an *inverse optimality* approach is used together with *partial stability* to consider cooperative consensus and synchronization algorithms.

A new class of digraphs is defined admitting a distributed solution to the global optimal control problem.

The third part of this work investigates cooperative control performance under disturbances, and distributed static output-feedback control. Control design for the state consensus in presence of disturbances is investigated. Derived results are also applicable to multi-agent systems with heterogeneous agents. If, on the other hand, one constrains the control to be of the static output-feedback form, one needs to redefine the synchronizing region as the output-feedback synchronizing region.

Contributions to Discrete-time Multi-agent Consensus Problem

The main contribution to the discrete-time multi-agent consensus problem is the proposed design method based on local Riccati feedback gains, guaranteeing cooperative stability and convergence to consensus.

Contributions to Globally Optimal Distributed Control Problem

The globally optimal distributed synchronization control protocols are investigated. The main contribution is in merging the notions of inverse optimality and partial stability to guarantee robust stabilization to the noncompact consensus manifold. Furthermore, second contribution is the introduction of the class of digraphs that gives a distributed solution to a structured global optimal control problem.

Contributions to Cooperative Robustness of Multi-agent Systems

The robustness properties of asymptotic and exponential stability are applied in the context of cooperative stability for consensus. The results are based on Lyapunov functions for noncompact manifolds, and the pertinent stability and robustness properties are further elaborated. Distributed and local observers are utilized for disturbance compensation.

Contributions to Distributed Output-feedback for State Synchronization

An application of the cooperative stability analysis, via synchronizing region, to the distributed output-feedback is presented. It is shown that the guaranteed synchronizing region for output-feedback can be both bounded and unbounded.

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CHAPTER 1

INTRODUCTION

This chapter presents a brief introduction to multi-agent systems considered in this work and the control problems as they appear throughout this dissertation. Concepts of partial stability applied to synchronization are introduced as well as inverse optimality of cooperative control protocols. The issue of the robustness of cooperative control systems and the effect that the disturbances exert on a multi-agent system are introduced. Possibly utilization of static output-feedback is also addressed. Topics and contributions of the following chapters are briefly summarized here, providing a layout of the body of work expounded in this dissertation.

1.1 Multi-Agent Systems and Synchronization

The main topic of this dissertation is cooperative control of multi-agent systems for synchronization. In a multi-agent system each *agent* is a subsystem, granted with some form of autonomy that justifies the decomposition of a total system into its agent subsystems. It is assumed that agents are dynamical systems with inputs whose states reside in a state space $x_i \in X_i$, and their dynamics is described by differential or difference equations. The state space of the total system is naturally taken as the direct product of state spaces of individual agents, $X_{tot} = X_1 \times X_2 \times \cdots \times X_N$, with the total state being $x = (x_1, x_2, \cdots, x_N)$. In case of all agents having the same state space, X, the total space is $X_{tot} = X^N$.

State consensus is a dynamic property of a multi-agent system, with states as elements of *de facto* the same state space, $x_i \in X$, where $\lim_{t\to\infty} d(x_i, x_j) = 0$, $\forall (i, j)$, where $d : X \times X \to \mathbb{R}$ is the distance function on the state space. A subset $S \subset X_{tot}$ characterized by $x_i = x_j$, $\forall (i, j)$, is an embedding of X into X_{tot}, and we call it the consensus subset, for if the total state of the system is in that subset then the multi-agent system is in the state of consensus. If all the state spaces are manifolds, so is the total space and the consensus subset becomes the consensus manifold. Via an embedding the consensus manifold has all the topological properties of an individual agent state space, in particular if X is noncompact so will be the consensus manifold as well.

In case of agents having different state spaces the state consensus is generally not defined, but one can define output consensus if all agent outputs are elements of the same output space, Y. Such problems do not appear when one considers collections of identical agents, where state and output spaces are the same for every agent, together with their dynamics.

1.2 Distributed Control

Achieving asymptotic stability of a consensus manifold is the control goal, and the control to achieve that aim is assumed *distributed*. If one were to consider centralized control then the multi-agent system control problem would reduce to controlling a single system, albeit a large one. However, the assumption, we choose to adopt here, is that each agent's input signal can depend only on states of some proper subset of the collection of all agents. This assumption is made due to inherently distributed nature of multi-agent systems, where one cannot expect the total state to be available for the purpose of controlling a single agent. Even if this were possible one would be faced with the difficulty of increasing complexity of a controller with the increasing number of agents. This would mean greater communication load and price of the controller.

If the controller of each agent is constrained to use only the states of those agents that are in some neighborhood of the respective agent, then one has a distributed control structure. This makes distributed control *eo ipso* a constrained control problem. Such controller does not suffer from increasing complexity when increasing the number of agents, since it depends only on the limited number of agents, those that are in some local neighborhood.

This work focuses primarily on identical agents, having linear time-invariant dynamics, and their consensus problem, though references are made to heterogeneous agents.

1.3 Discrete-time Cooperative Control Design for Synchronization

The last two decades have witnessed an increasing interest in multi-agent network cooperative systems, inspired by natural occurrence of flocking and formation forming. These systems are applied to formations of spacecrafts, unmanned aerial vehicles, mobile robots, distributed sensor networks etc. 1. Early work with networked cooperative systems in continuous and discrete time is presented in 2,3,4,5,6,7. These papers generally referred to consensus without a leader. By adding a leader that pins to a group of other agents one can obtain synchronization to a command trajectory using a virtual leader 5, also named pinning control 8,9. The graph properties complicate the design of synchronization controllers due to the interplay between the eigenvalues of the graph Laplacian matrix and the required stabilizing gains. Necessary and sufficient conditions for synchronization are given by the master stability function, and the related concept of the synchronizing region, in 9,10,11. For continuous-time systems synchronization was guaranteed 9,12,13 using optimal state feedback derived from the continuous time Riccati equation. It was shown that, using Riccati design for the feedback gain of each node guarantees an unbounded right-half plane region in the *s*-plane. For discrete-time systems such general results are still lacking, though 14 deals with single-input systems and undirected graph topology and 15 deals with multivariable systems on digraphs. These were originally inspired by the earlier work of 16,17, concerning optimal logarithmic quantizer density for stabilizing discrete time systems.

1.4 Distributed Observer Design in Discrete-time

Results from cooperative control design in discrete-time can be applied with needed modifications to the problem of distributed observation. Output measurements are assumed and cooperative observers are specially designed for the multi-agent systems. Potential applications are distributed observation, sensor fusion, dynamic output regulators for synchronization, *etc*. For the needs of consensus and synchronization control we employ the cooperative tracker, or pinning control 7,23,6. The key difference between systems in continuous-time, 11,9,12, and discrete-time, 25,26,28, is in the form of their stability region. More precisely, in continuous-time the stability region, as the open left-half *s*-plane, is unbounded by definition, so the synchronizing region can also be made unbounded. On the other hand, the discrete-time stability region, as the interior of the unit circle in the *z*-plane, is inherently bounded and, therefore, so are the synchronizing regions. This makes conditions for achieving discrete-time stability more strict than the continuous-time counterparts.

1.5 Optimal Cooperative Control

Optimal cooperative control was recently considered by many authors-33,34,36,37,38,39, to name just a few. Optimality of a control protocol gives rise to desirable characteristics such as gain and phase margins, that guarantee robustness in presence of some types of disturbances 40,41. The common difficulty, however, is that in the general case optimal control is not distributed 34,36. Solution of a global optimization problem generally requires centralized, *i.e.* global, information. In order to have local control that is optimal in some sense it is possible *e.g.* to consider each agent optimizing its own, local, performance index. This is done for receding horizon control in 33, implicitly in 13, and for distributed games on graphs in 35, where the notion of optimality is Nash equilibrium. In 37 the LQR problem is phrased as a maximization problem of LMI's under the constraint of the communication graph topology. This is a constrained optimization taking into account the local character of interactions among agents. It is also possible to use a local observer to obtain the global information needed for the solution of the global optimal problem, as is done in 34. In the case of agents with identical linear time-invariant dynamics, 38 presents a suboptimal design that is distributed on the graph topology.

Optimal control for multi-agent systems is complicated by the fact that the graph topology interplays with system dynamics. The problems caused by the communication topology in the design of global optimal controllers with distributed information can be approached using the notion of inverse optimality, 41. There, one chooses an optimality criterion related to the communication graph topology to obtain distributed optimal control, as done for the single-integrator cooperative regulator in 36. This connection between the graph topology and the structure of the performance criterion can allow for the distributed optimal control. In the case that the agent integrator dynamics contains topological information, 39 shows that there is a performance criterion such that the original distributed control is optimal with respect to it.

1.6 Multi-agent Systems with Disturbances

When disturbances act on the multi-agent system the control law designed for the undisturbed system cannot generally guarantee that the control goal shall be attained. However, building on the classical results on the existence of Lyapunov functions for asymptotically stable systems, and their use in assessing the effect that disturbances exert on those systems, 46, it is possible to extend such reasoning to partially stable systems, 41, in particular those systems that reach consensus or synchronization. With Lyapunov functions for asymptotic stability one is able to ascertain the effect of disturbances on the multi-agent system, and to derive conditions on those disturbances that allow for asymptotic stability or uniform ultimate boundedness along the target set. Cooperative asymptotic stability is in that sense robust to this specific class of disturbances. Furthermore, with the means to quantify the effect of disturbances one also gains the ability to compensate it by an appropriate control law.

Robustness of the distributed synchronization control for the nominal, *i.e.* undisturbed, system guarantees cooperative asymptotic stability of consensus, or cooperative ultimate uniform boundedness with respect to consensus, in presence of certain disturbances, 46. This inherent property can be exploited in special cases of heterogeneous and nonlinear agents. Nevertheless, in case of general disturbances one needs to compensate for their effects in order to retain the qualitative behavior of the nominal system. For that purpose disturbance estimates are used. In Chapter 6, disturbances are assumed to act on both the leader and the following agents. Therefore both the leader's and the agents' disturbances need to be estimated and compensated. Local and distributed estimation schemes are employed for that purpose.

1.7 Output-feedback-Stability and Optimality

In realistic applications one generally cannot use the full state of each agent for feedback control purposes. This difficulty is usually overcome by using some observer to estimate the full state from the system's inputs and outputs. This way one obtains a dynamic output controller. However this controller has its dynamics and is more complicated then the static state controller. Bearing this in mind, it behooves one to investigate conditions under which the static output control suffices. Static output control combines the simplicity of static full state controllers with the availability of the system outputs. No additional state observation is needed. It was shown, in 60, that passive systems are able to output synchronize using static output feedback under very mild assumptions. However, here, one is primarily interested in the state synchronization, and therefore one can apply the concept of the synchronizing region to static output feedback. This line of thought reduces the multi-agent cooperative stability of consensus using output distributed feedback to robust output stabilization for a single agent. Furthermore, under certain conditions, one can parameterize the cooperatively stabilizing output distributed controllers by a quadratic optimality criterion, as was done for the static state distributed control in Chapter 4. Chapter 7 brings results along those lines in a more general setting of two-player zero-sum games. It is shown that, under additional conditions, the proposed distributed output-feedback is a solution of the specially structured two-player zero-sum game.

1.8 Outline of the Dissertation

The dissertation is organized as follows. Chapter 2 introduces the discrete-time multi-agent systems and synchronization problem. A short survey of graph theoretic results that are used throughout the dissertation is given in this chapter. Discrete-time synchronizing region is introduced and discrete-time Lyapunov method gives a part of the synchronizing region.

Discrete-time Riccati equation (DARE) is used with its guaranteed complex gain margin region to yield a sufficient condition on graph eigenvalues for synchronization. The work of Keyou You and Lihua Xie uses a modified Riccati inequality, *i.e.* H_∞-type Riccati inequality, while the author uses H_2-type Riccati equation. Both methods are presented for the sake of completeness.

Chapter 3 deals with distributed observation and output-feedback cooperative control. Contrary to the case presented in Chapter 2,26 and 28, perfect information on the state of the neighbouring systems is not presumed. Output measurements are assumed and cooperative observers are specially designed for the multi-agent systems. Potential applications are distributed observation, sensor fusion, dynamic output regulators for synchronization, *etc*. Conditions for cooperative observer convergence and for synchronization of the multi-agent system are shown to be related by a duality concept for distributed systems on directed graphs. Sufficient conditions are derived that guarantee observer convergence as well as synchronization. This derivation is facilitated by the concept of convergence region for a distributed observer, which is analogous, and in a sense dual, to the synchronization region defined for a distributed synchronization controller. Furthermore, the proposed observer and controller feedback designs have a robustness property like the one originally presented in Chapter 2, 28, for controller design.

Chapter 4 introduces globally optimal distributed synchronization protocols. In this chapter are considered fixed topology directed graphs and linear time-invariant agent dynamics. First, theorems are provided for partial stability and inverse optimality of a form useful for applications to cooperative control, where the synchronization manifold may be noncompact. In our first contribution, using these results, we solve the globally optimal cooperative regulator and cooperative tracker problems for both single-integrator agent dynamics and also agents with identical linear time-invariant dynamics. It is found that globally optimal linear quadratic regulator (LQR) performance cannot be achieved using distributed linear control protocols on arbitrary digraphs. A necessary and sufficient condition on the graph topology is given for the existence of distributed linear protocols that solve a global optimal LQR control problem. In our second contribution, we define a new class of digraphs, namely, those whose Laplacian matrix is simple, that is, has a diagonal Jordan form. On these graphs, and only on these graphs, does the globally optimal LQR problem have a distributed linear protocol solution. If this condition is satisfied, then distributed linear protocols exist that solve the global optimal LQR problem only if the performance indices are of a certain form that captures the topology of the graph. That is, the achievable optimal performance depends on the graph topology.

Chapter 5, extends results of Chapter 4 to slightly more general but still quadratic performance indices. The constraint on graph topology of Chapter 4 can be relaxed, and optimal cooperative controllers developed for arbitrary digraphs, containing a spanning tree, by allowing state-control cross-weighting terms in the performance criterion. Then, requiring that the performance criterion be positive (semi) definite leads to conditions on the matrix P in $V(x) = x^T P x$, or equivalently on the control Lyapunov function, which should be satisfied for the existence of distributed globally optimal controller. This condition is milder than the conditions in Chapter 4, where the performance index is taken without the state-control cross-weighting term.

Chapter 6 discusses the effects of disturbances on the multi-agent systems. Those effects are quantified by a Lyapunov function for stability as presented in Chapter 4. Properties of relevant types of stability are further elaborated for the needs of this chapter. In particular, the class of coordinate transformations preserving the stability properties is investigated and characterized. It is shown that systems that are asymptotically or exponentially stable with respect to some manifold share a robustness property to a certain class of disturbances. This class is characterized by growth bounds with respect to the target manifold. Since most of the disturbances are no expected to pertain to this specific class one resorts to disturbance estimation and compensation. Leader's and agents' disturbances are estimated and compensated using local and distributed observers. A number of special applications were considered; the case of there being an input driving the leader, which needs to be distributively observed by all the following agents, the case of second-order double-integrator systems with disturbances acting on the leader and the agents, and the case of heterogeneous agents.

Chapter 7 brings an extension of the cooperative stability analysis via synchronizing region, used originally for full-state feedback, to the case of output-feedback. It examines static distributed output-feedback control for state synchronization of identical linear time-invariant agents. Static output-feedback, guaranteeing the control goal of multi-agent system synchronization, is easy to implement, without any need for state observation. Cooperative stability of state synchronization is addressed using results derived from the single agent robust stability properties of the local output-feedback gain. It is shown that the guaranteed synchronizing region for output-feedback can be both bounded and unbound. The chosen distributed output-feedback control is also a solution of the specially structured two-player zero-sum game problem, under appropriate additional stipulations. Conditions for that are more conservative than those for simple cooperative synchronization. In a special case this game theoretic framework reduces to the optimal output-feedback control. Conditions are found, under which the distributed output-feedback control is optimal with respect to a quadratic performance criterion. These imply state synchronization.

Chapter 8 outlines future work directions. Current results are briefly mentioned. Application of the design methods developed in this dissertation to multi-agent systems with fixed control signal time-delay is partially developed. Identical agents are described by linear delay-differential equations. This brings one even closer to reality, since distributed communication causes delays in control signals.

Also, a related field of output synchronization is addressed. Output synchronization is practically important in that one usually requires synchronization of only some states, or just outputs of the system, while other states need only remain bounded.

CHAPTER 2

DISCRETE TIME RICCATI DESIGN OF LOCAL FEEDBACK GAINS FOR SYNCHRONIZATION

2.1 Introduction

The last two decades have witnessed an increasing interest in multi-agent network cooperative systems, inspired by natural occurrence of flocking and formation forming. These systems are applied to formations of spacecrafts, unmanned aerial vehicles, mobile robots, distributed sensor networks etc., 1. Early work with networked cooperative systems in continuous and discrete time is presented in 2,3,4,5,6,7. These papers generally referred to consensus without a leader. By adding a leader that pins to a group of other agents one can obtain synchronization to a command trajectory using a virtual leader 5, also named pinning control 8,9. Necessary and sufficient conditions for synchronization are given by the master stability function, and the related concept of the synchronizing region, in 9,10,11. For continuous-time systems synchronization was guaranteed 9,12,13 using optimal state feedback derived from the continuous time Riccati equation. It was shown that, using Riccati design for the feedback gain of each node guarantees an unbounded right-half plane region in the *s*-plane. For discrete-time systems such general results are still lacking, though 14 deals with single-input systems and undirected graph topology and 15 deals with multivariable systems on digraphs. These were originally inspired by the earlier work of 16,17, concerning optimal logarithmic quantizer density for stabilizing discrete time systems.

In this chapter we are concerned with synchronization for agents described by linear time-invariant discrete-time dynamics. The interaction graph is directed and assumed to contain a directed spanning tree. For the needs of consensus and synchronization to a leader or control node we employ pinning control 5,8. The concept of synchronizing region 9,10,11 is instrumental in analyzing the synchronization properties of cooperative control systems. The synchronizing region is the region in the complex plane within which the graph Laplacian matrix eigenvalues must reside to guarantee synchronization. The crucial difference between systems in continuous time and discrete time is the form of the stability region. For continuous-time systems the stability region is the left half *s*-plane, which is unbounded by definition, and a feedback matrix can be chosen, 9,13 such that the synchronizing region for a matrix pencil is also unbounded. On the other hand the discrete-time stability region is the interior of the unit circle in the *z*-plane, which is inherently bounded. Therefore, the synchronizing regions are bounded as well. This accounts for stricter synchronizability conditions in discrete-time, such as those presented in 14,15.

In the seminal paper, 18, is given an algorithm based on H_∞-type Riccati equation for synchronization control of linear discrete-time systems that have no poles outside the unit circle. The case of consensus without a leader is considered.

This chapter extends results in 15 to provide conditions for achieving synchronization of identical discrete-time state space agents on a directed communication graph structure. It extends results in 18 to the case of unstable agent dynamics. This work considers synchronization to a leader dynamics. The concept of discrete time synchronizing region in the *z*-plane is used.

The graph properties complicate the design of synchronization controllers due to the interplay between the eigenvalues of the graph Laplacian matrix and the required stabilizing gains. Two approaches to testing for synchronizability are given which decouple the graph properties from the feedback design details. Both give methods for selecting the feedback gain matrix to yield synchronization. The first result, based on an H_∞-type Riccati inequality, gives a milder condition for synchronization in terms of a circle whose radius is generally difficult to compute. The second result is in terms of a circle whose radius is easily computed from an H_2-type Riccati equation solution, but gives a stricter condition. Both are shown to yield known results in the case of single-input systems on undirected graphs. Based on the given designs, results are given on convergence and robustness of the design. An example illustrates the usefulness and effectiveness of the proposed design.

2.2 Graph Properties and Notation

Consider a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ with a nonempty finite set of N vertices $\mathcal{V}=\{v_1,\cdots,v_N\}$ and a set of edges or arcs $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$. It is assumed that the graph is simple, *i.e.* there are no repeated edges or self-loops $(v_i,v_i)\notin\mathcal{E},\forall i$. General directed graphs (*digraphs*) are considered, and it is taken that information propagates through the graph along directed arcs. Denote the connectivity matrix as $E=[e_{ij}]$ with $e_{ij}>0$ if $(v_j,v_i)\in\mathcal{E}$ and $e_{ij}=0$ otherwise. Note that diagonal elements satisfy $e_{ii}=0$. The set of neighbors of node v_i is denoted as $\mathcal{N}_i=\{v_j:(v_j,v_i)\in\mathcal{E}\}$, *i.e.* the set of nodes with arcs coming into v_i. Define the in-degree matrix as the diagonal matrix $D=diag\,(d_1\ldots d_N)$ with $d_i=\sum_j e_{ij}$ the (weighted) in-degree of node i (*i.e.* the *i-th* row sum of E). Define the graph Laplacian matrix as $L=D-E$, which has all row sums equal to zero.

A *path* from node v_{i_1} to node v_{i_k} is a sequence of edges $(v_{i_1},v_{i_2}),(v_{i_2},v_{i_3}),\ldots(v_{i_{k-1}},v_{i_k})$, with $(v_{i_{j-1}},v_{i_j})\in\mathcal{E}$ or $(v_{i_j},v_{i_{j-1}})\in\mathcal{E}$ for $j=\{2,\ldots,k\}$. A *directed path* is a sequence of edges $(v_{i_1},v_{i_2}),(v_{i_2},v_{i_3}),\ldots,(v_{i_{k-1}},v_{i_k})$, with $(v_{i_{j-1}},v_{i_j})\in\mathcal{E}$ for $j=\{2,\ldots,k\}$. The graph is said to be *connected* if every two vertices can be joined by a path. A graph is said to be *strongly connected* if every two vertices can be joined by a directed path. The graph is said to contain a (directed) *spanning tree* if there exists a vertex, v_0, such that every other vertex in $\mathcal{V}$ can be connected to v_0 by a (directed) path starting from v_0. Such a special vertex, v_0, is then called a root node.

The Laplacian matrix L has a simple zero eigenvalue if and only if the undirected graph is connected. For directed graphs, the existence of a directed spanning tree is necessary and sufficient for L to have a simple zero eigenvalue.

A *bidirectional* graph is a graph satisfying $e_{ij} > 0 \Leftrightarrow e_{ji} > 0$. A *detailed balanced graph* is a graph satisfying $\lambda_i e_{ij} = \lambda_j e_{ji}$ for some positive constants $\lambda_1 \dots \lambda_N$. By summing over the index *i* it is seen that then $[\lambda_1 \quad \dots \quad \lambda_N]$ is a left zero eigenvector of L.

The Laplacian matrix of a detailed balanced graph satisfies $L = \Lambda P$, for some diagonal matrix $\Lambda > 0$ and Laplacian matrix $P = P^T$ of some undirected graph. That is, the Laplacian is symmetrizable. In fact $\Lambda = diag(1/\lambda_i)$. The concept of a detailed balanced graph is related to the reversibility of an associated Markov process.

Given the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the reversed graph $\mathcal{G}'$ is a graph having the same set of vertices $\mathcal{V}' = \mathcal{V}$ and the set of edges with the property $E' = \left[e'_{ij}\right] = E^T = \left[e_{ji}\right]$. That is, the edges of $\mathcal{G}$ are reversed in $\mathcal{G}'$.

We denote the real numbers by $\mathbb{R}$, the positive real numbers by $\mathbb{R}^+$, and the complex numbers by $\mathbb{C}$.

$C(O,r)$ denotes an open circle in the complex plane with its center at $O \in \mathbb{C}$ and radius r. The corresponding closed circle is denoted as $\bar{C}(O,r)$.

For any matrix A, $\sigma_{\min}(A), \sigma_{\max}(A)$, are the minimal and the maximal singular values of A respectively. For any square matrix A, the spectral radius $\rho(A) = \max\left|eig(A)\right|$ is the maximal magnitude of the eigenvalues of A. For a positive semidefinite matrix A,

$\sigma_{>0\min}(A)$ denotes the minimal nonzero singular value. Note that for symmetric L, $\sigma_{>0\min}(L)=\lambda_2(L)$, the graph Fiedler eigenvalue.

2.3 State Feedback for Synchronization of Multi-Agent Systems

Given a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, endow each of its N nodes with a state vector $x_i \in \mathbb{R}^n$ and a control input, $u_i \in \mathbb{R}^m$, and consider at each node the discrete-time dynamics

$$x_i(k+1) = Ax_i(k) + Bu_i(k). \tag{1}$$

Assume that (A,B) is stabilizable and B has full column rank m. Consider also a leader node, *i.e.* control node or command generator

$$x_0(k+1) = Ax_0(k), \tag{2}$$

with $x_0 \in \mathbb{R}^n$. For example, if $n=2$ and A has imaginary poles, then the leader's trajectory is a sinusoid.

The *cooperative tracker* or *synchronization problem* is to select the control signals u_i, using the relative state of node i to its neighbors, such that all nodes synchronize to the state of the control node, that is, $\lim_{k\to\infty}\|x_i(k)-x_0(k)\|=0, \forall i$. These requirements should be fulfilled for all initial conditions, $x_i(0)$. If the trajectory $x_0(k)$ approaches a fixed point, this is usually called the consensus problem.

To achieve synchronization, define the local neighborhood tracking errors

$$\varepsilon_i = \sum_{j\in N_i} e_{ij}(x_j - x_i) + g_i(x_0 - x_i), \tag{3}$$

where the pinning gain, $g_i \geq 0$, is nonzero if node v_i can sense the state of the control node. The intent here is that only a small percentage of nodes have $g_i > 0$, yet all nodes should synchronize to the trajectory of the control node using local neighbor control protocols, 8. It is assumed that at least one pinning gain is nonzero. Note that the local neighborhood tracking error represents the information available to agent *i* for control purposes.

Choose the input of agent *i* as the weighted local control protocol

$$u_i = c\left(1+d_i+g_i\right)^{-1} K\varepsilon_i \,, \tag{4}$$

where $c \in \mathbb{R}^+$ is a coupling gain to be detailed later. Then, the closed-loop dynamics of the individual agents are given by

$$x_i(k+1) = Ax_i(k) + c(1+d_i+g_i)^{-1} BK\varepsilon_i(k). \tag{5}$$

Defining the global tracking error and state vector $\varepsilon = \begin{bmatrix} \varepsilon_1^T & \dots & \varepsilon_N^T \end{bmatrix}^T \in \mathbb{R}^{nN}$, $x = \begin{bmatrix} x_1^T & \dots & x_N^T \end{bmatrix}^T \in \mathbb{R}^{nN}$, one may write

$$\varepsilon(k) = -(L+G)\otimes I_n x(k) + (L+G)\otimes I_n \overline{x}_0(k)\,, \tag{6}$$

where $G = diag\left(g_1,\dots,g_N\right)$ is the diagonal matrix of pinning gains and $\overline{x}_0(k) = \underline{1} \otimes x_0(k)$ with $\underline{1} \in \mathbb{R}^N$ the vector of 1's. The global dynamics of the N -agent system is given by

$$x(k+1) = \left[I_N \otimes A - c\left(I+D+G\right)^{-1}\left(L+G\right)\otimes BK\right]x(k) + c\left(I+D+G\right)^{-1}\left(L+G\right)\otimes BK\overline{x}_0(k). \tag{7}$$

Define the *global disagreement error*, $\delta(k) = x(k) - \overline{x}_0(k)$, 3. Then one has the global error dynamics

$$\delta(k+1) = A_c\delta(k)\,, \tag{8}$$

where the closed-loop system matrix is

$$A_c = \left[I_N \otimes A - c\left(I+D+G\right)^{-1}\left(L+G\right)\otimes BK\right]. \tag{9}$$

We shall refer to the matrix

$$\Gamma = (I+D+G)^{-1}(L+G) \tag{10}$$

as the (weighted) *graph matrix* and to its eigenvalues, Λ_k, $k=1...N$, as the *graph matrix eigenvalues*. Assume the graph contains a directed spanning tree and has at least one nonzero pinning gain, g_i, connecting into the root node *i.* The graph matrix Γ is nonsingular since $L+G$ is nonsingular, if at least one nonzero pinning gain, g_i, connects into the root node, 19.

The following result shows the importance of using weighting by $(I+D+G)^{-1}$.

Lemma 2.1. Given the control protocol (4), the eigenvalues of Γ satisfy $\Lambda_k \subseteq \bar{C}(1,1)$, $k=1...N$, for any graph.

Proof: This follows directly from the Geršgorin circle criterion applied to $\Gamma=(I+D+G)^{-1}(L+G)$, which has Geršgorin circles $\bar{C}\left(\frac{d_i+g_i}{1+d_i+g_i}, \frac{d_i}{1+d_i+g_i}\right)$. These are all contained in $\bar{C}(1,1)$. ■

Lemma 2.1 reveals that weighting restricts the possible positions of the graph eigenvalues to the bounded known region $\bar{C}(1,1)$. The non-weighted protocol $u_i = cK\varepsilon_i$ yields graph eigenvalues in the region $\bigcup_i \bar{C}(d_i+g_i, d_i)$, which is larger than $\bar{C}(1,1)$. Examples 2.1 and 2.2 show situations where synchronization using control law of the form $u_i = cK\varepsilon_i$ cannot be guaranteed, whereas using weighted protocol (4) achieves synchronization.

The next result is similar to results in 6, 9, 14.

Lemma 2.2. The multi-agent systems (5) synchronize if and only if $\rho(A - c\Lambda_k BK) < 1$ for all eigenvalues Λ_k, $k=1...N$, of graph matrix (10).

Proof: Let J be a Jordan form of Γ. Then there exists a nonsingular matrix $R \in \mathbb{R}^{N\times N}$, such that $R\Gamma R^{-1} = J$. By (9), it is obtained that

$$(R\otimes I_n)A_c(R^{-1}\otimes I_n)=I_N\otimes A-cJ\otimes BK$$
$$=\begin{bmatrix} A-c\Lambda_1 BK & \times & \times \\ 0 & \ddots & \times \\ 0 & 0 & A-c\Lambda_N BK \end{bmatrix}, \tag{11}$$

where '×' denotes possibly nonzero elements. By (11), one has $\rho(A_c)<1$ if and only if $\rho(A-c\Lambda_k BK)<1$ for all Λ_k. From (8), the multi-agent systems (5) synchronize if and only if $\rho(A_c)<1$. Thus, the multi-agent systems (5) synchronize if and only if $\rho(A-c\Lambda_k BK)<1$ for all Λ_k. ■

For synchronization, one requires asymptotic stability of the error dynamics (8). It is assumed that (A,B) is stabilizable. If the matrix A is unstable or marginally stable, then Lemma 2.2 requires that $\Lambda_k \neq 0$, $k=1...N$, which is guaranteed if the interaction graph contains a spanning tree with at least one nonzero pinning gain into the root node.

Definition 2.1: Let Γ be a graph matrix, and let Λ_k, $k=1...N$, be the eigenvalues of Γ. A *covering circle*, $C(c_0,r_0)$, of the eigenvalues of Γ is an open circle centered at $c_0\in\mathbb{R}$ containing all the eigenvalues, Λ_k, $k=1,...,N$.

Definition 2.2: For a matrix pencil, $A-sBK$ with $s\in\mathbb{C}$, the *synchronizing region* of the matrix pencil is a subset $S_c\subseteq\mathbb{C}$ such that $S_c=\{s\in\mathbb{C}\,|\,\rho(A-sBK)<1\}$.

Given the choice of K, the synchronizing region S_c of the matrix pencil $A-sBK$ is determined. The synchronizing region was discussed in 9, 10, 11, 13.

Definition 2.3: The *complex gain margin region*, $U_c\subseteq\mathbb{C}$, for some stabilizing feedback matrix K, given a system (A,B), is a connected region containing 1, such that $\rho(A-sBK)<1, \forall s\in U_c$.

Definitions 2.2 and 2.3 express the connection between the synchronizing region S_c of a matrix pencil $A - sBK$ and the complex gain margin region U_c, given that the choice of K stabilizes the system (A, B), *i.e.* $\rho(A - BK) < 1$. The complex gain margin region, defined as a connected region, is contained in the synchronizing region, which may be generally disconnected. In 18, an example of a disconnected synchronizing region was given.

Lemma 2.3. Matrices $A - c\Lambda_k BK$ are stable for all eigenvalues Λ_k if and only if they satisfy $c\Lambda_k \in S, \forall k \in \{1, \ldots, N\}$.

Proof: This follows from Definition 2.2. ■

Based on these constructions, the main results of this chapter may now be stated. We develop two approaches that provide sufficient conditions for synchronization and give formal design methods for the synchronizing control (4), one based on an H_∞-type Riccati inequality and one based on an H_2-type Riccati equality.

2.4 Design Based on H_∞-type Riccati Inequality

In what follows, the design of K in (4) will be studied based on a modified H_∞-type Riccati inequality. Without loss of generality, we further assume that $A = \mathrm{diag}(A_s, A_u)$, where $A_s \in \mathbb{R}^{n_s \times n_s}$ is stable and all the eigenvalues of $A_u \in \mathbb{R}^{n_u \times n_u}$ lie on or outside the unit circle. Accordingly, write $B^T = [B_s^T \;\; B_u^T]$.

Let us first analyse the structure of the system. Since (A, B) is stabilizable, the unstable part (A_u, B_u) is controllable. From the multi-input reachable canonical form, 43, there exists a non-singular real matrix V such that $\tilde{A} = V^{-1} A_u V$ and $\tilde{B} = V^{-1} B_u$ take the form

$$\tilde{A} = \begin{bmatrix} A_1 & \times & \dots & \times \\ 0 & A_2 & \dots & \times \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_m \end{bmatrix}, \tilde{B} = \begin{bmatrix} b_1 & \times & \dots & \times \\ 0 & b_2 & \dots & \times \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_m \end{bmatrix}, \tag{12}$$

where '×' denotes possibly nonzero elements and (A_j, b_j) is controllable. Note that elements of matrices in (12) need not be scalars, but rather blocks in general. Using the above form, a result on the solution to a modified Riccati inequality can be developed.

Lemma 2.4. Assume that $M_1 \in \mathbb{R}^{n_1 \times n_1}$ and $M_2 \in \mathbb{R}^{n_2 \times n_2}$ are two symmetric positive definite matrices. Given any $M_{12} \in \mathbb{R}^{n_1 \times n_2}$, let

$$\beta > \frac{\sigma_{\max}(M_{12}^T M_1^{-1} M_{12})}{\sigma_{\min}(M_2)}. \tag{13}$$

Then $M \triangleq \begin{bmatrix} M_1 & M_{12} \\ M_{12}^T & \beta M_2 \end{bmatrix}$ is a positive definite matrix.

Proof: According to the Schur complement condition, a symmetric block matrix $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$ is positive definite if both M_{11} and $M_{22} - M_{12}^T M_{11}^{-1} M_{12}$ are positive definite. Now, it follows from (13) that

$$\beta M_2 - M_{12}^T M_1^{-1} M_{12} \geq \beta \sigma_{\min}(M_2) I - \sigma_{\max}(M_{12}^T M_1^{-1} M_{12}) I > 0.$$

Together with $M_1 > 0$, one has from Schur complement that $M > 0$. ■

Lemma 2.5. Consider a single input state-space system with A being unstable and (A, B) stabilizable. Then, a necessary and sufficient condition for the existence of a positive definite matrix $P > 0$ solving

$$P > A^T PA - (1-\delta^2) A^T PB (B^T PB)^{-1} B^T PA \tag{14}$$

is

$$\left|\delta\right| < 1 / \prod_i |\lambda_i^u(A)|.$$

Proof: (Necessity) In conformity with the partition $A = \operatorname{diag}(A_s, A_u)$, partition P as $P = \begin{bmatrix} P_s & P_{12} \\ P_{12}^T & P_u \end{bmatrix}$. Pre- and post-multiplying (14) by

$$\begin{bmatrix} I & -P_{12}P_u^{-1} \\ 0 & I \end{bmatrix} \text{ and } \begin{bmatrix} I & 0 \\ -P_u^{-1}P_{12}^T & I \end{bmatrix}, \text{ yields}$$

$$P_u > A_u^T P_u A_u - (1-\delta^2) A_u^T P_u B_u' (B_u'^T P_u B_u')^{-1} B_u'^T P_u A_u,$$

where $B_u' = B_u + P_u^{-1} P_{12}^T B_s$. By continuity, there exists a sufficiently small $\varepsilon > 0$ such that

$$\begin{aligned} P_u &> A_u^T P_u A_u - (1-\delta^2) A_u^T P_u B_u' (B_u'^T P_u B_u' + \varepsilon)^{-1} B_u'^T P_u A_u \\ &= A_u^T (P_u^{-1} + B_u' B_u'^T / \varepsilon)^{-1} A_u + \delta^2 A_u^T P_u B_u' (B_u'^T P_u B_u' + \varepsilon)^{-1} B_u'^T P_u A_u \\ &> 0. \end{aligned}$$

The equality in the second line follows by applying the Kailath variant. Since both sides of the inequality are positive definite, taking the determinant of both sides gives

$$\begin{aligned} \det(P_u) &> \det(A_u)^2 \det(P_u) \det(I - (1-\delta^2) P_u B_u' (B_u'^T P_u B_u' + \varepsilon)^{-1} B_u'^T) \\ &= \left[1 - (1-\delta^2) \frac{B_u'^T P_u B'}{B_u'^T P_u B_u' + \varepsilon} \right] \det(A_u)^2 \det(P_u) \\ &> \delta^2 \det(A_u)^2 \det(P_u), \end{aligned}$$

where the equality holds due to the identity $\det(I - MN) = \det(I - NM)$ for compatible matrices M, N. The fact that $\delta^2 < 1$ is used in the last inequality. Note that $\det(P_u) > 0$ implies $\left|\delta\right| < \det(A_u)^{-1} = 1 / \prod_i |\lambda_i^u(A)|$. Sufficiency follows from Lemma 5.4 of 21. ■

For multi-input systems, one has the following result. Results of Lemma 2.4 and Lemma 2.5 are used in the proof.

Lemma 2.6. Given $\delta \in (0,1]$, consider the modified H_∞-type Riccati inequality

$$P > A^T PA - (1-\delta^2) A^T PB(B^T PB)^{-1} B^T PA. \tag{15}$$

Assume that A is unstable and (A, B) is stabilizable. Then there exists a critical value $\delta_c \in (0,1)$ such that if $\delta < \delta_c$, then there exists a positive definite solution P to (15). Moreover, δ_c is bounded below by

$$\delta_c \geq 1 / \prod_i |\lambda_i^u(A_{m^*})| \triangleq \delta_{c'}, \tag{16}$$

where A_j is given in (12), $\lambda_i^u(A_j)$ are the unstable eigenvalues of A_j, and the index m^* is defined by $m^* = \arg\max_{j \in \{1,\ldots,m\}} \left(\prod_i |\lambda_i^u(A_j)| \right)$. Furthermore, if A is stable, one has that $\delta_c = 1$.

Proof: Since (A, B) is stabilizable, there exists a $P > 0$ solving the inequality

$$P > A^T PA - A^T PB(B^T PB)^{-1} B^T PA.$$

Then, a sufficiently small $\delta > 0$ exists such that $P > A^T PA - (1-\delta^2) A^T PB(B^T PB)^{-1} B^T PA.$ In addition, for any positive $\zeta \leq \delta$, one has that

$$P > A^T PA - (1-\zeta^2) A^T PB(B^T PB)^{-1} B^T PA.$$

Hence, the existence of a positive δ_c, as required in the lemma, is guaranteed. In fact, it can be obtained as

$$\delta_c = \sup_{\delta > 0} \left\{ \delta \mid \exists P > 0 \text{ s.t. } P > A^T PA - (1-\delta^2) A^T PB(B^T PB)^{-1} B^T PA \right\} \tag{17}$$

Let m be the number of inputs.

a) If $m = 1$, which corresponds to B being a vector, it follows from Lemma 2.5 that $\delta_c = 1 / \prod_i |\lambda_i^u(A)| = \delta_{c'}$. Therefore, (16) holds.

For the multi-input case, the result for single-input system, *i.e.* Lemma 2.5, is used in steps together with (12) and Lemma 2.4 to construct the following result.

b) Assume that $m > 1$. Case a) implies that given any positive $\delta < \delta_{c'}$, there exists a $P_j > 0$ such that

$$
\begin{aligned}
&P_j - A_j^T P_j A_j + (1-\delta^2) A_j^T P_j b_j (b_j^T P_j b_j)^{-1} b_j^T P_j A_j \\
&= P_j - \delta^2 A_j^T P_j A_j - (1-\delta^2)(A_j - b_j K_j)^T P_j (A_j - b_j K_j) > 0,
\end{aligned}
\tag{18}
$$

with the control gain $K_j = (b_j^T P_j b_j)^{-1} b_j^T P_j A_j$. Note that for any $\beta > 0,\ \tilde{P}_j = \beta P_j > 0$ also solves the modified Riccati inequality (18). Now, let $\tilde{P}_1 = \mathrm{diag}(P_1, \beta P_2)$ and $\tilde{K}_1 = \mathrm{diag}(K_1, K_2)$. It follows that

$$
\begin{aligned}
&\tilde{P}_1 - \delta^2 \tilde{A}_1^T \tilde{P}_1 \tilde{A}_1 - (1-\delta^2)(\tilde{A}_1 - \tilde{B}_1 \tilde{K}_1)^T \tilde{P}_1 (\tilde{A}_1 - \tilde{B}_1 \tilde{K}_1) \\
&\triangleq \begin{bmatrix} \tilde{M}_1 & \tilde{M}_{12} \\ \tilde{M}_{12}^T & \beta \tilde{M}_2 \end{bmatrix} \triangleq \tilde{M},
\end{aligned}
$$

where $\tilde{M}_j > 0$ and $\tilde{M}_{12}$ are independent of β and

$$
\tilde{A}_1 = \begin{bmatrix} A_1 & \times \\ 0 & A_2 \end{bmatrix}, \tilde{B}_1 = \begin{bmatrix} b_1 & \times \\ 0 & b_2 \end{bmatrix}.
$$

By Lemma 2.4, there exists a sufficiently large β such that $\tilde{M} > 0$. Continuing in the same fashion, one can find $\tilde{P}$ and $\tilde{K}$ such that

$$
\tilde{P} - \delta^2 \tilde{A}^T \tilde{P} \tilde{A} - (1-\delta^2)(\tilde{A} - \tilde{B}\tilde{K})^T \tilde{P} (\tilde{A} - \tilde{B}\tilde{K}) > 0. \tag{19}
$$

Let $P_u = V^{-T} \tilde{P} V^{-1}$ and $K_u = \tilde{K} V^{-1}$. It follows from (19) that

$$
P_u - \delta^2 A_u^T P_u A_u - (1-\delta^2)(A_u - B_u K_u)^T P_u (A_u - B_u K_u) > 0.
$$

Since A_s is stable, there exists a positive definite matrix P_s such that $P_s - A_s^T P_s A_s > 0$. Denoting $K = \mathrm{diag}(0, K_u)$ and $P = \mathrm{diag}(P_s, \beta_1 P_u)$, one can similarly establish that there exists a positive β_1 such that

$$
P - \delta^2 A^T P A - (1-\delta^2)(A - BK)^T P (A - BK) > 0. \tag{20}
$$

Note that for any $K \in \mathbb{R}^{m\times n}$, one can verify that

$$(A-BK)^T P(A-BK) \geq A^T PA - A^T PB(B^T PB)^{-1} B^T PA.$$

Whence, together with (20), it follows that

$$P > A^T PA - (1-\delta^2) A^T PB(B^T PB)^{-1} B^T PA.$$

Thus, given any $\delta < \delta_{c'}$, there exists a $P > 0$ which solves the modified Riccati inequality (15). This implies that $\delta_c \geq \delta_{c'}$. If A is stable, it follows that $\delta_c = 1$ since for any positive $\delta < \delta_c$, there always exists a positive definite solution to (15). ■

In general, δ_c can be found by solving an LMI, see Proposition 3.1 of 22.

The term appearing in Lemma 2.6 involving a product of unstable eigenvalues, $\prod_i \left|\lambda_i^u(A)\right|$, deserves further attention. It is related to the intrinsic entropy rate of a system, $\sum_i \log_2 \left|\lambda_i^u(A)\right|$, describing the minimum data-rate in networked control system that enables stabilization of an unstable system, 20. The product of unstable eigenvalue magnitudes of matrix A itself is the Mahler measure of the respective characteristic polynomial of A.

It is well recognized that Lemma 2.6 is of importance in the stability analysis of Kalman filtering with intermittent observations, 21, and the quadratic stabilization of an uncertain linear system (cf. Theorem 2.1 of 17). In fact, it is also useful for the design of a control gain in (4) to solve the synchronization problem, which is delivered as follows.

Theorem 2.1. *H_∞-type Riccati Inequality Design for Synchronization.* Given systems (1) with protocol (4), assume that the interaction graph contains a spanning tree with at least one pinning gain nonzero that connects into a root node. Then $\Lambda_i > 0, \forall i$. If there exists $\omega \in \mathbb{R}$ such that

$$\delta(\omega) \triangleq \max_{j\in\{1,\ldots,N\}} |1-\omega\Lambda_j| < \delta_c, \tag{21}$$

where δ_c is obtained by (17), then there exists $P>0$ solving (15) with $\delta=\delta(\omega)$. Moreover, the control gain

$$K=(B^T PB)^{-1}B^T PA\,,$$

and coupling gain, $c=\omega$, guarantee synchronization.

Proof: By Lemma 2.6, there exists $P>0$ solving (15). Thus, only the second part needs to be shown. Denote $\delta_j=1-\omega\Lambda_j$, it is found that

$$\begin{aligned} &P-(A-\omega\Lambda_j BK)^* P(A-\omega\Lambda_j BK) \\ &= P-A^T PA+(1-|\delta_j|^2)A^T PB(B^T PB)^{-1}B^T PA \\ &\geq P-A^T PA+(1-\delta^2)A^T PB(B^T PB)^{-1}B^T PA>0, \end{aligned}$$

where * denotes complex conjugate transpose. Thus, it follows that $\rho(A-\omega\Lambda_j BK)<1$. The rest of the result follows from Lemma 2.2. ■

Remark 2.1. For the single input case, *i.e.*, $\operatorname{rank}(B)=1$, the sufficient condition given in Theorem 2.1 is also necessary. This follows directly from Theorems 3.1 and 3.2 of 14.

In 18, an equation similar to (15) is used for synchronization design in the case where there is no leader, and no poles of the agent dynamics are outside the unit circle.

The conditions of Lemma 2.2 are awkward in that the graph properties, as reflected by the eigenvalues Λ_k, and the stability of the local node systems, as reflected in $(A-BK)$, are coupled. Theorem 2.1 is important because it shows that feedback design based on the modified Riccati inequality allows a separation of the feedback design problem from the properties of the graph, as long as it contains a spanning tree with a nonzero pinning gain into the root node. Specifically, it reveals that if Riccati-based design is used for the feedback K at each node, then synchronization is guaranteed for a class of communication graphs satisfying condition (21). This condition is appealing because it allows for a disentanglement of the properties of the individual agents' feedback gains, as reflected by δ_c, and the graph topology, described by Λ_k, relating these through an inequality.

2.5 Design Based on H_2-type Riccati Equation

An H_2-type Riccati design method for finding gain K in (4) that guarantees synchronization is now presented.

Theorem 2.2. H_2*-type Riccati Design for Synchronization.* Assume that the interaction graph contains a spanning tree with at least one pinning gain nonzero that connects into the root node. Then $\Lambda_i > 0, \forall i$. Let $P > 0$ be a solution of the discrete-time Riccati-like equation

$$A^T PA - P + Q - A^T PB(B^T PB)^{-1} B^T PA = 0 \tag{22}$$

for some prescribed $Q = Q^T > 0$. Define

$$r := \left[\sigma_{\max}(Q^{-1/2} A^T PB(B^T PB)^{-1} B^T PAQ^{-1/2})\right]^{-1/2} . \tag{23}$$

Then the protocol (4) guarantees synchronization of multi-agent systems (5) for some *K* if there exists a covering circle $C(c_0, r_0)$ of the graph matrix eigenvalues Λ_k, $k = 1 \ldots N$ such that

$$\frac{r_0}{c_0} < r . \tag{24}$$

Moreover, if condition (24) is satisfied then the choice of feedback matrix,

$$K = \left(B^T PB\right)^{-1} B^T PA, \tag{25}$$

and coupling gain,

$$c = \frac{1}{c_0}, \tag{26}$$

guarantee synchronization. ■

Theorem 2.2 is motivated by geometrical considerations as shown in Figure 2.1. Condition (24) means that the covering circle $C(c_0, r_0)$ is homothetic to a circle concentric with and contained in the interior of $C(1,r)$. It will be shown in Lemma 2.7 that the synchronizing region of the H_2-type Riccati feedback gain selected as in Theorem 2.2 contains $C(1,r)$. Therefore, synchronization is guaranteed by any value of c radially projecting $C(c_0, r_0)$ to the interior of $C(1,r)$. One such value is given by (26). Homothety, as understood in this work, refers to a radial projection with respect to the origin, *i.e.* zero in complex plane.

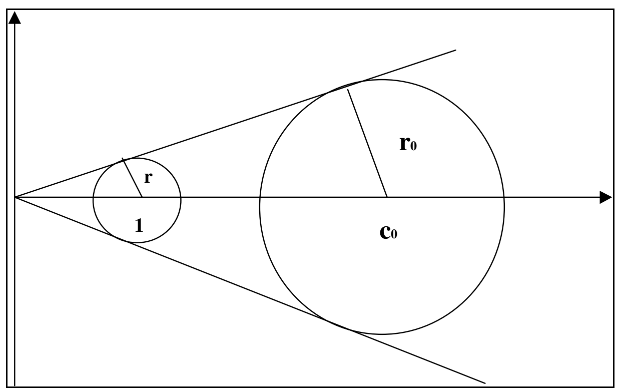


Figure 2.1 Motivation for proof of Theorem 2.2. Riccati design circle $C(1,r)$ and covering circle $C(c_0, r_0)$ of graph matrix eigenvalues.

The following technical lemmas are needed for the proof of Theorem 2.2.

Lemma 2.7. The synchronizing region for the choice of K given in Theorem 2.2 contains the open circle $C(1,r)$.

Proof: By choosing the state feedback matrix as (25), with $P > 0$ a solution of equation (22) one has

$$A^T PA - P + Q - K^T \left(B^T PB\right) K = 0. \tag{27}$$

From this equation one obtains by completing the square

$$A^T PA - P + Q - K^T B^T PBK = (A - BK)^T P(A - BK) - P + Q = 0. \tag{28}$$

Therefore, given the quadratic Lyapunov function $V(y) = y^T Py$, $y \in \mathbb{R}^n$, the choice of feedback gain stabilizes the system (A, B). At least a part of the synchronizing region S can be found as

$$(A - sBK)^* P(A - sBK) = (A - \mathrm{Re}\, sBK)^T P(A - \mathrm{Re}\, sBK) + \mathrm{Im}^2 s(BK)^T PBK < P, \tag{29}$$

where * denotes complex conjugate transpose (Hermitian adjoint), $s \in \mathbb{C}$. Therefore from equations (27) and (29) one has

$$\begin{aligned}
&(A - sBK)^* P(A - sBK) - P \\
&= A^T PA - P - 2\,\mathrm{Re}\, sK^T (B^T PB)K + \mathrm{Re}^2 s(BK)^T PBK + \mathrm{Im}^2 s(BK)^T PBK \\
&= A^T PA - P - (1 - |s-1|^2)(BK)^T PBK \\
&= -Q + K^T (B^T PB)K - (1 - |s-1|^2)(BK)^T PBK \\
&= -Q + |s-1|^2 K^T B^T PBK
\end{aligned} \tag{30}$$

This is the condition of stability if

$$Q - |s-1|^2 K^T B^T PBK > 0, \tag{31}$$

which gives a simple bounded complex region, more precisely a part of the complex gain margin region U,

$$Q > |s-1|^2 K^T B^T PBK \Rightarrow 1 > |s-1|^2 \sigma_{\max}\left(Q^{-1/2}(BK)^T P(BK)Q^{-1/2}\right). \tag{32}$$

This is an open circle $C(1, r)$ specified by

$$|s-1|^2 < \frac{1}{\sigma_{\max}\left(Q^{-1/2} K^T B^T PBKQ^{-1/2}\right)}. \tag{33}$$

Furthermore expressing K as the H_2-type Riccati equation state feedback completes the proof. ■

In 18, it is shown that in case of all the agent dynamics being marginally stable, that is, no unstable poles and only nonrepeated poles on the unit circle, the synchronizing region (in the sense of this chapter) is $C(1,1)$.

Lemma 2.8. Given the circle $C(1,r)$ in the complex plane, which is contained in the synchronizing region S for the H_2-type Riccati choice of gain (25), the system is guaranteed to synchronize for some value of $c>0$ if the graph matrix eigenvalues Λ_k, $k=1...N$, are located in such a way that

$$\left|c\Lambda_k - 1\right| < r, \forall k , \tag{34}$$

for that particular c in (26).

Proof: It follows from Lemmas 2.2, 2.3, and 2.7. ■

Based on these constructions the proof of Theorem 2.2 can now be given.

Proof of Theorem 2.2.

Given $C(1,r)$ contained in the synchronizing region S of matrix pencil $A-sBK$, and the properties of dilation (homothety), and assuming there exists a directed spanning tree in the graph with a nonzero pinning gain into the root node, it follows that synchronization is guaranteed if all eigenvalues Λ_k are contained in a circle $C(c_0,r_0)$ similar with respect to homothety to a circle concentric with and contained within $C(1,r)$.

The center of the covering circle c_0 can be taken on the real axis due to symmetry and the radius equals $r_0 = \max_k \left|\Lambda_k - c_0\right|$. Taking these as given, one should have

$$\frac{r_0}{c_0} < \frac{r}{1} . \tag{35}$$

If this equation is satisfied then choosing $c = 1/c_0$ maps with homothety the covering circle of all eigenvalues $C(c_0, r_0)$ into a circle $C(1, r_0/c_0)$ concentric with and, for $r > r_0/c_0$, contained in the interior of the circle $C(1, r)$. ■

If there exists a solution $P > 0$ to Riccati equation (22), B must have full column rank. Assuming B has full column rank, there exists a positive definite solution P to (22) only if (A, B) is stabilizable.

2.6 Relation Between the Two Design Methods

In this section the relation between Theorem 2.1 and Theorem 2.2 is drawn. By comparing (34) to (21) it is seen that a similar role is played by δ_c in Theorem 2.1 and *r* in Theorem 2.2. The latter can be explicitly computed using (23), but it depends on the selected Q. On the other hand, δ_c is found by numerical LMI techniques. See Lemma 2.6. Further, in Theorem 2.1, $\omega \in \mathbb{R}$ plays a role analogous to $1/c_0$ in Theorem 2.2. Theorem 2.1 relies on analysis based on the circle $C(1, \delta_c)$, whereas Theorem 2.2 uses $C(1, r)$. Both circles are contained in the synchronizing region of the respective designed feedbacks.

More importantly, Theorem 2.2 gives synchronization conditions in terms of the radius *r* easily computed as (23) in terms of an H_2-type Riccati equation solution. Computing the radius δ_c used in Theorem 2.1 must generally be done via an LMI. However, the condition in Theorem 2.1 is milder than that in Theorem 2.2. That is, $C(1, r)$ is generally contained in $C(1, \delta_c)$, so that systems that fail to meet the condition of Theorem 2.2 may yet be found to be synchronizable when tested according to the condition in Theorem 2.1. This result is summarized in the following theorem.

Theorem 2.3. Define *r* by (23) and δ_c as in (17). Then $r \leq \delta_c$.

Proof: Assume there exists a positive definite solution to (15) for some δ, denoted as P_δ, then it follows that

$$A^T P_\delta A - P_\delta - A^T P_\delta B(B^T P_\delta B)^{-1} B^T P_\delta A < -\delta^2 A^T P_\delta B(B^T P_\delta B)^{-1} B^T P_\delta A$$

This means, $\exists Q_\delta ; Q_\delta > \delta^2 A^T P_\delta B(B^T P_\delta B)^{-1} B^T P_\delta A \geq 0$ such that

$$A^T P_\delta A - P_\delta - A^T P_\delta B(B^T P_\delta B)^{-1} B^T P_\delta A = -Q_\delta .$$

Therefore, there exists a positive definite solution of the Riccati equation (22) for such Q_δ. According to the existence of positive definite solution to (15) for $0 < \delta < \delta_c$ there also exists the solution of (22) with the corresponding Q_δ. Since the guaranteed synchronization region for each P_δ is $|s-1| < \delta$ choosing $Q_\delta ; \delta < \delta_c$, although guaranteeing the existence of positive definite P_δ, thus providing sufficient condition for synchronization, does not necessarily give the largest guaranteed circular synchronization region, *i.e.* generally $|s-1| < r \leq \delta_c$. ■

Note that following the design proposed by Theorem 2.2 the matrix Q is given, and it explicitly determines the size of the guaranteed synchronizing region (23) if there exists a positive definite solution of (22) for that given choice of Q. Finding general $Q > 0$ that maximizes r would involve an LMI just as finding the value of δ_c does.

2.7 Robustness of the H_2 Riccati Design

The following lemma is motivated by the conjecture that if the conditions of Theorem 2.2 hold there is in fact an open interval of admissible values of the coupling constant c. This has the interpretation of robustness for the H_2-type Riccati design in sense that synchronization is still guaranteed under small perturbations of c from value given by (26).

Lemma 2.9. Taking $\alpha_{\max} := \arccos\sqrt{1-r^2} \geq 0$ and denoting the angle of an eigenvalue of the graph matrix Γ by $\phi_k := \arg\Lambda_k$, a necessary condition for the existence of at least one admissible value of c is given by

$$\frac{\operatorname{Re}\Lambda_k}{|\Lambda_k|} > \sqrt{1-r^2}, \quad \forall k, \tag{36}$$

which is equivalent to

$$|\phi_k| < \alpha_{\max}, \quad \forall k. \tag{37}$$

Furthermore, the interval of admissible values of c, if non-empty, is an open interval given as a solution to a set of inequalities

$$\frac{\cos\phi_k - \sqrt{\cos^2\phi_k - \cos^2\alpha_{\max}}}{|\Lambda_k|} < c < \frac{\cos\phi_k + \sqrt{\cos^2\phi_k - \cos^2\alpha_{\max}}}{|\Lambda_k|}, \forall k. \tag{38}$$

Finally if (24) holds then the solution of (38) is non-empty and contains the value (26).

Proof: If solution c exists, the equations

$$|c\Lambda_k - 1| < r \Rightarrow c^2|\Lambda_k|^2 - 2c\operatorname{Re}\Lambda_k + 1 - r^2 < 0 \tag{39}$$

are satisfied simultaneously for every k for at least one value of c. Therefore $\forall k$ (39) has an interval of real solutions for c. From that, and the discriminant of (39), relation (36) follows.

One therefore has $\cos\phi_k = \frac{\operatorname{Re}\Lambda_k}{|\Lambda_k|} > \sqrt{1-r^2} = \cos\alpha_{\max}$ meaning $|\phi_k| < \alpha_{\max}$. Bearing that in mind, expression (36) can further be equivalently expressed as

$$c^2|\Lambda_k|^2 - 2c|\Lambda_k|\cos\phi_k + \cos^2\alpha_{\max} < 0,$$

Solving this equation for $c|\Lambda_k|$ yields N intervals in (38). The intersection of these N open intervals is either an open interval or an empty set. Now given that $\Lambda_k \in C(c_0, r_0)$ one finds that

$$\frac{\cos\phi_k - \sqrt{\cos^2\phi_k - \cos^2\alpha_{\max}}}{|\Lambda_k|} < \frac{\cos\phi_k - \sqrt{\cos^2\phi_k - \cos^2\alpha_{\max}}}{|\Lambda_k|_{\min}}$$
$$\frac{\cos\phi_k + \sqrt{\cos^2\phi_k - \cos^2\alpha_{\max}}}{|\Lambda_k|_{\max}} < \frac{\cos\phi_k + \sqrt{\cos^2\phi_k - \cos^2\alpha_{\max}}}{|\Lambda_k|} \tag{40}$$

where $|\Lambda_k|_{\min}, |\Lambda_k|_{\max}$ are extremal values for fixed ϕ_k, as determined by $C(c_0, r_0)$. Namely, for $|\Lambda_k|_{\min}, |\Lambda_k|_{\max}$ one has

$$|\Lambda_k - c_0| = r_0 \Rightarrow \frac{|\Lambda_k|^2}{c_0^2} - 2\frac{|\Lambda_k|}{c_0}\cos\phi_k + 1 - \frac{r_0^2}{c_0^2} = 0$$

which gives $|\Lambda_k|_{\max,\min} = c_0\left[\cos\phi_k \pm \sqrt{\cos^2\phi_k - \left(1 - \frac{r_0^2}{c_0^2}\right)}\right]$. If (24) is satisfied then, $\cos^2\alpha_{\max} < 1 - \frac{r_0^2}{c_0^2}$, used in (40) together with expression for $|\Lambda_k|_{\min}, |\Lambda_k|_{\max}$ implies the non-emptyness of the interval solution of (38), and guarantees that $c = 1/c_0$ is a member of that interval. Furthermore, if the assumptions of Theorem 2.2 are satisfied with equality, then the lower and higher limit of every subinterval (40) in fact become equal to $1/c_0$. ■

Condition (37) means that the graph matrix eigenvalues Λ_k must be inside the cone in complex plane shown in Figure 2.1. It is evident from the geometry of the problem that eigenvalues Λ_k located outside the cone determined by (36), (37) cannot be made to fit into the region $C(1, r)$ by scaling with real values of c. It should be noted that condition (24) is stronger than condition (37) and condition (38) has also been derived in 14.

2.8 Application to Real Graph Matrix Eigenvalues and Single-input Systems

In this section condition (24) of Theorem 2.2 is studied in several special simplified cases and it is shown how this condition relates to known results. The special case of real eigenvalues of Γ and single input systems at each node allows one to obtain necessary and sufficient condition for synchronization.

Corollary 2.1. Let graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ have a spanning tree with pinning into the root node, and all eigenvalues of Γ be positive, that is, $\Lambda_k > 0$ for all k. Define $0<\Lambda_{\min} \leq \cdots \leq \Lambda_k \leq \cdots \leq \Lambda_{\max}$. A covering circle for Λ_k that also minimizes r_0 / c_0, is $C(c_0, r_0)$ with

$$\frac{r_0}{c_0} = \frac{\Lambda_{\max} - \Lambda_{\min}}{\Lambda_{\max} + \Lambda_{\min}}. \tag{41}$$

Then, condition (24) in Theorem 2.2 becomes

$$\frac{\Lambda_{\max} - \Lambda_{\min}}{\Lambda_{\max} + \Lambda_{\min}} < r. \tag{42}$$

Moreover, given that this condition is satisfied, the coupling gain choice (26) reduces to

$$c = \frac{2}{\Lambda_{\min} + \Lambda_{\max}}. \tag{43}$$

Proof: For graphs having a spanning tree with pinning into the root node, and all eigenvalues of Γ real, one has $0 < \Lambda_{\min}$. Note that in that case $\varphi_k = 0 \quad \forall k$ and $\cos^2 \alpha_{\max} = 1 - r^2$ so that necessary condition (37) is satisfied. Furthermore inequalities (38) become

$$\frac{1-r}{|\Lambda_k|} < c < \frac{1+r}{|\Lambda_k|}.$$

From this, a necessary and sufficient condition for the existence of a nonempty intersection of these N intervals is

$$\frac{1-r}{\Lambda_{\min}} < \frac{1+r}{\Lambda_{\max}},$$

where the extremal values $\Lambda_{\min} = |\Lambda_k|_{\min}, \Lambda_{\max} = |\Lambda_k|_{\max}$ $\Lambda_{\min,\max} = |\Lambda_k|_{\min,\max}$ are the same for every k and are determined as in the proof of Lemma 2.9.

The sought interval is given as $c \in \left\langle \frac{1-r}{\Lambda_{min}}, \frac{1+r}{\Lambda_{max}} \right\rangle$. This interval implies condition (42). Therefore, the sufficient condition (24) is equivalent to (42). Examining the positions of eigenvalues on the real axis it is found that for the minimal covering circle $C(r_0, c_0)$ one has

$$c_0 = \frac{\Lambda_{min} + \Lambda_{max}}{2},$$

$$r_0 = \Lambda_{max} - c_0 = c_0 - \Lambda_{min} = \frac{\Lambda_{max} - \Lambda_{min}}{2}.$$

So

$$\frac{\Lambda_{max} - \Lambda_{min}}{\Lambda_{max} + \Lambda_{min}} = \frac{r_0}{c_0} < r.$$

Note that in the case of real eigenvalues Λ_k their minimal covering circle has also the minimal ratio r_0/c_0 of all covering circles $C(c_0, r_0)$. This shows that (42) expresses the sufficient condition (24) specialized to Γ having all real eigenvalues. Theorem 2.2 then also gives the choice of c (43). ■

This special case concerning real eigenvalues of the graph matrix Γ bears strong resemblance to the case presented and studied in 14, concerning undirected graphs. For, note that if one uses the non-weighted protocol, $u_i = cK\varepsilon_i$, instead of (4), then, the eigenvalues Λ_k used in the analysis of this chapter are those of $(L+G)$, not Γ in (10). However, the condition that the eigenvalues of $(L+G)$ be real is related to the graph being undirected. It is noted that, even if all eigenvalues of $(L+G)$ are real, the eigenvalues of Γ may be complex, and vice versa. The importance of weighting is discussed in Lemma 2.1.

The radius r in (23) for Riccati-based design is important since it is instrumental in determining sufficient condition (24) in Theorem 2.2 as well as in Lemma 2.8. In the case of single-input systems the expression (23) simplifies.

Remark 2.2. If the node dynamics (1) are single-input, define *r* by (23) with the choice of $Q = P^* - A^T P^* A + \frac{A^T P^* B B^T P^* A}{B^T P^* B}$, where $P^* > 0$ is a positive definite solution of the Riccati equation

$$P^* - A^T P^* A + \frac{A^T P^* B B^T P^* A}{B^T P^* B + 1} = 0. \tag{44}$$

Then

$$r = \frac{1}{\prod_u \left|\lambda^u(A)\right|}. \tag{45}$$

where $\lambda^u(A)$ are the unstable eigenvalues of the system matrix A indexed by u.

This follows from Theorem 2.2 of 16. Namely putting Q as defined into (22) yields the solution $P = P^*$. In case of single-input systems $B \in \mathbb{R}^n$ and u is a scalar, so the result in 16 applies. Note that if the solution to (44) is only positive semi-definite, a regularizing term must be added to (44) as is done in 16.

Note that P^* defined as a solution to Riccati equation (44) becomes the solution of (22) for the choice of Q given in Remark 2.2. This means, in the context of this chapter, that H_2-type Riccati feedback gain for that specific choice of Q, applied to single-input systems, maximizes the value of r considered here to be the radius of $C(1, r)$ in the complex plane, rather than just the real interval 16. The following remark combines the above results.

Remark 2.3. If systems (1) are single-input and the Γ matrix of the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ has all eigenvalues real, selecting Q as in Remark 2.2, gives the condition (24) in the form

$$\prod_u \left|\lambda^u(A)\right| < \frac{\Lambda_{\max} + \Lambda_{\min}}{\Lambda_{\max} - \Lambda_{\min}}. \tag{46}$$

Moreover this condition is necessary and sufficient for synchronization for any choice of the feedback matrix K if all the eigenvalues of A lie on or outside the unit circle. Sufficiency follows by Corollary 2.1 and Remark 2.2, assuming the conditions of Theorem 2.2 hold. Necessity follows from 14.

The next result shows another case where a simplified form of *r* in Theorem 2.2 can be given. It follows directly from simplification of (23).

Corollary 2.2. Let matrix B be invertible and $Q = I$, then one has

$$r = \frac{1}{\sigma_{\max}\left(A^T P A\right)^{1/2}}. \tag{47}$$

2.9 Numerical Example

This example shows the importance of weighting by $u_i = c(1+d_i+g_i)^{-1} K\varepsilon_i$ in protocol (4), and features unstable individual agent dynamics. The graph is directed and connected. The Laplacian L and the pinning gains matrix G are given as

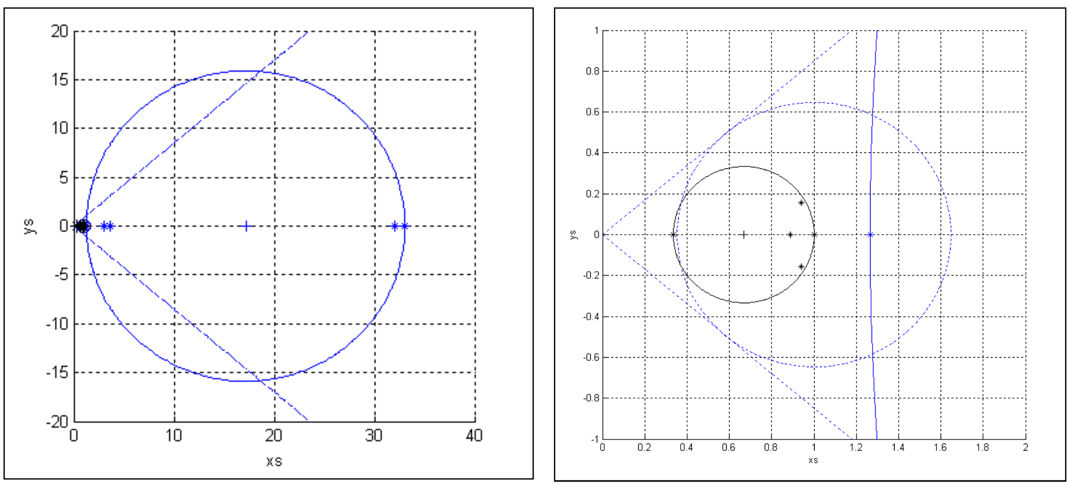
$$L = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}, \quad G = \mathrm{diag}(30,0,0,0,30).$$

The individual agent dynamics are given by (1), with (A, B) in controllable canonical form,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.2 & 0.2 & 1.1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvalues of A are $eig(A) = 1.1190,\ 0.4134,\ -0.4324$, therefore the uncontrolled systems are unstable.

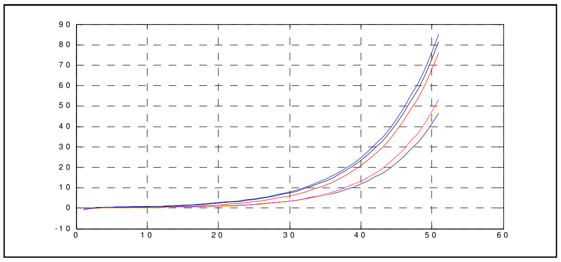
Figure 2.2 shows the non-weighted graph eigenvalues, that is the eigenvalues of $(L+G)$, and the weighted counterparts, that is the eigenvalues of $\Gamma=(I+D+G)^{-1}(L+G)$, with the synchronizing region $C(1,r)$ also displayed. Clearly the non-weighted graph eigenvalues do not satisfy the sufficient condition, (24) while the weighted ones do. Therefore the protocol $u_i = cK\varepsilon_i$ cannot be guaranteed to yield synchronization, while the weighted protocol (4) does. It is interesting that the non-weighted graph eigenvalues are real, while the weighted ones may be complex.



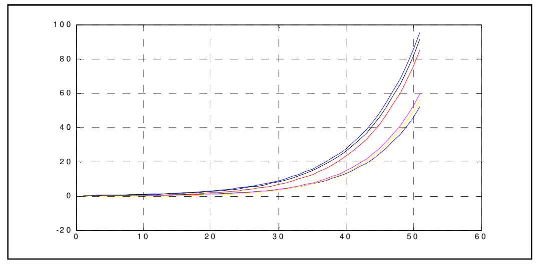
(a) (b)

Figure 2.2 (a) Non-weighted graph eigenvalues and their covering circle. (b) Weighted graph matrix eigenvalues, their covering circle, and the synchronization region (dashed circle).

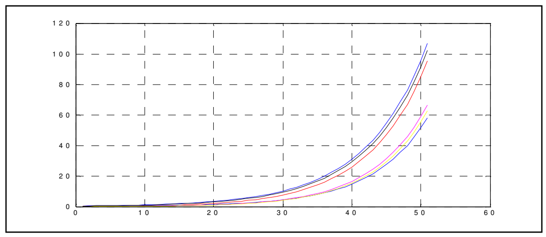
The gain matrix K for the protocols (4) was designed using Theorem 2.2. Relevant values for the design are $r = 0.5083,\ r_0 = 0.3333,\ c_0 = 0.6684$, $r_0/c_0 = 0.4987 < r$. We selected $Q = 0.20I_3$. Figure 2.3 shows the dynamics of a multi-agent system in the case that the coupling gain *c* is improperly chosen. Synchronization is not achieved.



(a)



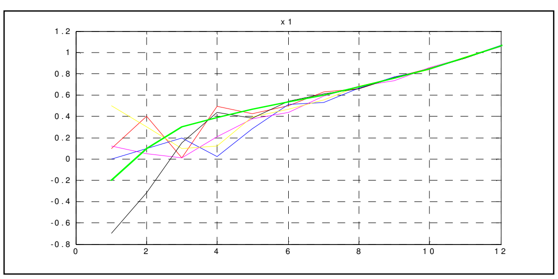
(b)



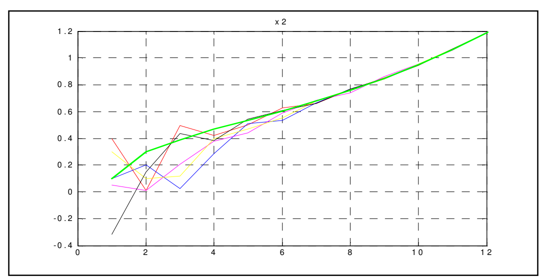
(c)

Figure 2.3 State trajectories of 5 agents; (a) First, (b) Second and (c) Third state component, for an improper choice of c .

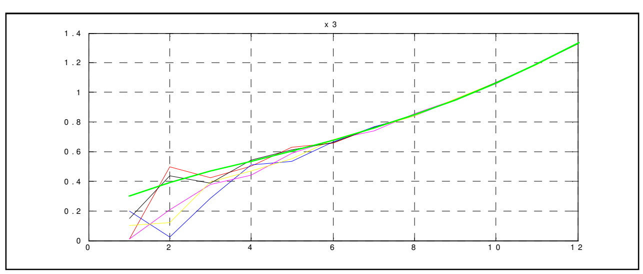
Simulated trajectories with the coupling gain chosen by (25), (26) are depicted in Figure 2.4. All nodes synchronize to the control node trajectory.



(a)



(b)



(c)

Figure 2.4 State space trajectories of 5 agents; (a) First, (b) Second and (c) Third state component, for proper value of c .

2.10 Conclusion

This chapter provides conditions for achieving synchronization of identical discrete time state space agents on a communication graph structure. The concept of discrete synchronizing region in the *z*-plane is used. Two conditions for synchronization are given which decouple the graph properties from the feedback design details. The first result, based on an H_∞-type Riccati inequality, gives a milder condition for synchronization in terms of a radius that is generally difficult to compute. The second result is in terms of a radius easily computed from an H_2-type Riccati equation solution, but may give too conservative a condition. Both are shown to yield known results in the single-input case. Example is given to illustrate the proposed feedback design and simulations justify the used approach.

CHAPTER 3

DISTRIBUTED OBSERVATION AND DYNAMIC OUTPUT COOPERATIVE FEEDBACK

3.1 Introduction

The last two decades have witnessed an increasing interest in multi-agent network cooperative systems, inspired by natural occurrences of flocking and formation forming in nature. In the technical world these systems are applied in formations of spacecrafts, unmanned aerial vehicles, mobile robots, distributed sensor networks etc. 9. Early work with networked cooperative systems in continuous and discrete time is presented in 1,3,4,7. These papers referred to consensus without a leader, *i.e.* the cooperative regulator problem, where the final consensus value is determined solely by the initial conditions. Necessary and sufficient conditions for the distributed systems to synchronize are given in 5,6,10. On the other hand by adding a leader that pins to a group of agents one can have synchronization to a command trajectory through pinning control, 7,23, for all initial conditions. We term this the cooperative tracker problem. An elegant approach for investigating sufficient conditions for a system to synchronize is provided by the concept of synchronization region, 6,8,11. This concept allows for the control design problem to be decoupled from the graph topology, yielding simplified formulations of sufficient conditions for synchronization to the leader trajectory.

For continuous-time systems it is shown in 9,12, that, by using state feedback derived from the Riccati equation, synchronization can be achieved for a broad class of communication graphs. Synchronization in continuous time using dynamic compensators or output-feedback is considered in 11,12,24,24. For discrete-time cooperative systems such general results are still lacking, although there are references to special cases of single-input systems and undirected graph topology, 25, and its extension to directed graphs, 26. These results were originally inspired by the earlier work of 15,16, concerning optimal logarithmic quantizer density for stabilizing discrete-time systems.

In this chapter we are concerned with systems of individual agents described by identical linear time-invariant discrete-time dynamics. The graph is assumed directed, of fixed topology, and containing a directed spanning tree. Contrary to the case presented in 26 and 28, perfect information on the state of the neighbouring systems is not presumed. Output measurements are assumed and cooperative observers are specially designed for the multi-agent systems. Potential applications are distributed observation, sensor fusion, dynamic output regulators for synchronization, *etc*. For the needs of consensus and synchronization control we employ the cooperative tracker, or pinning control, 7,23,6. The key difference between systems in continuous-time, 11,9,12 and discrete-time, 25,26,28 is in the form of their stability region. More precisely, in continuous-time the stability region, as the open left-half *s*-plane, is unbounded by definition, so the synchronizing region can also be made unbounded. On the other hand, the discrete-time stability region, as the interior of the unit circle in the *z*-plane, is inherently bounded and, therefore, so are the synchronizing regions. This makes conditions for achieving discrete-time stability more strict than the continuous-time counterparts.

Conditions for cooperative observer convergence and for synchronization of the multi-agent system are shown to be related by a duality concept for distributed systems on directed graphs. It is also shown that cooperative control design and cooperative observer design can both be approached by decoupling the graph structure from the design procedure by using Riccati-based design. Sufficient conditions are derived that guarantee observer convergence as well as synchronization. This derivation is facilitated by the concept of convergence region for a distributed observer, which is analogous, and in a sense dual, to the synchronization region defined for a distributed synchronization controller. Furthermore, the proposed observer and controller feedback designs have a robustness property like the one originally presented in 28 for controller design.

This chapter is organized as follows. Section 3.2 to 3.4. detail a local neighborhood observer structure. Theorem 3.1 gives a Riccati-based design method that decouples the design of the observer gains from the graph topology. The theorem gives sufficient conditions for the convergence of the proposed distributed observer design. Section 3.5 demonstrates that the distributed observer and controller schemas are dual to one another by using the concept of the reverse graph. Section 3.6 presents three observer/controller regulator architectures, together with proofs guaranteeing that the agents synchronize and the observer errors converge to zero. This demonstrates a separation principle for design of dynamic output feedback regulators on graph topologies. Numerical examples are given in Section 3.7. and conclusions are presented in Section 3.8.

3.2 Cooperative Observer Design

In this section we define cooperative observers on the interaction graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ and provide a design method for the observer gains that guarantees convergence of the observer errors to zero if certain conditions hold.

The agents are coupled on a graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ and have dynamics given as the discrete-time systems

$$\begin{aligned} x_i(k+1) &= Ax_i(k)+Bu_i(k) \\ y_i(k) &= Cx_i(k) \end{aligned} \tag{48}$$

A control or leader node 0 has the command generator dynamics

$$\begin{aligned} x_0(k+1) &= Ax_0(k), \\ y_0(k) &= Cx_0(k). \end{aligned} \tag{49}$$

The state, input, and output vectors are $x_i, x_0 \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i, y_0 \in \mathbb{R}^p$. Note that matrix *A* may not be stable. This autonomous command generator describes many reference trajectories of interest including periodic trajectories, ramp-type trajectories, *etc.*

3.3 Observer Dynamics

It is desired to design cooperative observers that estimate the states $x_i(k)$ given measurements of the outputs $y_j(k)$. Given an estimated output, $\hat{y}_i = C\hat{x}_i$, at node *i*, define the *local output error*

$$\tilde{y}_i = y_i - \hat{y}_i \,.$$

To construct an observer that takes into account information from the neighbors of node *i*, define the *local neighborhood output disagreement*

$$\varepsilon_i^o = \sum_j e_{ij}\left(\tilde{y}_j - \tilde{y}_i\right) + g_i\left(\tilde{y}_0 - \tilde{y}_i\right) \tag{50}$$

where the observer pinning gains are, $g_i \geq 0$, with $g_i > 0$ only for a small percentage of nodes that have direct measurements of the output error $\tilde{y}_0$ of the control node. With the global state vector defined as $x = \begin{bmatrix} x_1^T & \dots & x_N^T \end{bmatrix}^T \in \mathbb{R}^{nN}$, global output $y = \begin{bmatrix} y_1^T & \dots & y_N^T \end{bmatrix}^T \in \mathbb{R}^{pN}$, the global output disagreement error $\varepsilon^o(k) \in \mathbb{R}^{pN}$ is

$$\varepsilon^o(k) = -(L+G) \otimes I_m \tilde{y}(k) + (L+G) \otimes I_m \underline{1}\, \tilde{y}_0(k) \,. \tag{51}$$

Throughout this chapter it is assumed that $x_0 = \hat{x}_0 \Rightarrow \tilde{y}_0 \equiv 0$, *i.e.* the control node state is accurately known. Then, define the *distributed observer* dynamics

$$\hat{x}_i(k+1) = A\hat{x}_i(k) + Bu_i(k) - c_1\left(1 + d_i + g_i\right)^{-1} F\varepsilon_i^o(k) \,. \tag{52}$$

where F is an observer gain matrix and c_1 a coupling gain. This algorithm has local neighborhood output disagreement (50) weighted by $(1 + d_i + g_i)^{-1}$. The importance of using this weighting is shown in Example 3.1. The global distributed observer dynamics then has the form

$$\hat{x}(k+1) = I_N \otimes A\hat{x}(k) + I_N \otimes Bu(k) + c_1\left(I + D + G\right)^{-1}\left(L + G\right) \otimes F\tilde{y}(k) \,,$$

$$\hat{x}(k+1)=\left[I_N\otimes A-c_1\left(I+D+G\right)^{-1}\left(L+G\right)\otimes FC\right]\hat{x}(k)+I_N\otimes Bu(k)+c_1\left(I+D+G\right)^{-1}\left(L+G\right)\otimes Fy(k),$$

where $G=diag\left(g_1,\ldots,g_N\right)$ is the diagonal matrix of pinning gains.

Defining the *i*-th *state observer error*

$$\eta_i(k)=x_i(k)-\hat{x}_i(k),$$

the global observer error $\eta=\left[\eta_1^T \quad \ldots \quad \eta_N^T\right]^T\in\mathbb{R}^{nN}$ has the dynamics

$$\eta\left(k+1\right)=\left[I_N\otimes A-c_1\left(I+D+G\right)^{-1}\left(L+G\right)\otimes FC\right]\eta\left(k\right). \tag{53}$$

Denote the global observer system matrix as

$$A_o=I_N\otimes A-c_1(I+D+G)^{-1}(L+G)\otimes FC. \tag{54}$$

Definition 3.1: The matrix appearing in (53) and (54) is referred to as the *graph matrix*

$$\Gamma=(I+D+G)^{-1}(L+G). \tag{55}$$

The eigenvalues Λ_k, $k=1...N$, of Γ are instrumental in studying the stability of systems on the graph $\mathcal{G}$.

Lemma 3.1. If there exists a spanning tree in the communication graph $\mathcal{G}$ with pinning into a root node, then $L+G$, and hence Γ, is nonsingular, and both have all their eigenvalues in the open right half plane.

Proof: 11, Lemma 5. ■

3.4 Design of Observer Gains

It is now desired to design the observer gains *F* to guarantee stability of the observer errors and hence convergence of the estimates $\hat{x}_i(k)$ to the true states $x_i(k)$. Unfortunately, the stability of the observer error dynamics depends on the graph topology through the graph matrix eigenvalues, $\Lambda_k, k=1...N$, as shown by the following result.

Lemma 3.2. The global observer error dynamics (53) are asymptotically stable if and only if $\rho(A - c_1 \Lambda_k FC) < 1$ for all eigenvalues Λ_k, $k = 1...N$ of the graph eigenvalue matrix $\Gamma = (I + D + G)^{-1}(L + G)$.

Proof: Perform on (53) a state-space transformation $z^o = (T \otimes I_n)\eta$, where T is a matrix satisfying $T^{-1}\Gamma T = \Lambda$ with Λ being a block triangular matrix. The transformed global observer error system is

$$z^o(k+1) = [I_N \otimes A - c_1 \Lambda \otimes FC] z^o(k) \tag{56}$$

Therefore, (53) is stable if and only if $I_N \otimes A - c_1 \Lambda \otimes FC$ is stable. This is equivalent to matrices $A - c_1 \Lambda_k FC$ being stable for every k since Λ_k are the diagonal elements of matrix Λ. ■

The necessary condition for the stability of (53) is detectability of (A, C). Stability of $A - c_1 \Lambda_k FC$ implies detectability of (A, C).

Lemma 3.2 is dual to a result in 5. Also, compare to 11. This result is not convenient to use for the design of the observer gains *F* because it intermingles the effects of the observer dynamics and the graph matrix eigenvalues. It is desired to provide here an observer gain design method that is independent of the graph eigenvalues Λ_k. The following notions facilitate the development of such a design technique.

Definition 3.2: A *covering circle* $\bar{C}(c_0, r_0)$ of the graph matrix eigenvalues Λ_k, $k = 1...N$, is a closed circle in the complex plane centered at $c_0 \in \mathbb{R}$ and containing all graph matrix eigenvalues, Λ_k, $k = 1...N$, some possibly on the boundary.

The notion of synchronization region, 23, 6, 11, 12, has proven important for the design of state feedback gains for continuous-time systems that guarantee stability on arbitrary graph topologies. The synchronization region was defined for discrete-time systems in 28. The next definition provides a dual concept for observer design.

Definition 3.3: For a matrix pencil $A - sFC$, $s \in \mathbb{C}$, the *convergence region* is a subset, $S_o \subseteq \mathbb{C}$, such that $S_o = \left\{ s \in \mathbb{C} \middle| \rho\left(A - sFC\right) < 1 \right\}$.

The convergence region might not be connected, but it is intimately related to the complex gain margin region defined next.

Definition 3.4: The *complex gain margin* $U_o \subseteq \mathbb{C}$ for some stabilizing observer gain F, given a system (A, C) is a simply connected region in $\mathbb{C}$, $U_o = \left\{ s \in \mathbb{C} \middle| \rho\left(A - sFC\right) < 1 \right\}$ containing $s = 1$.

Note that for a stabilizing observer gain F one has a relation $U_o \subseteq S_o$.

Lemma 3.3. Matrices $A - c_1 \Lambda_k FC$ are stable for all eigenvalues Λ_k if and only if $c_1 \Lambda_k \in S_o; \forall k$.

Proof: follows from Definition 3.3 and 3.4. ■

The main result of this section is a design technique for the distributed observer gain F that does not depend on the graph eigenvalues and guarantees observer stability under a certain condition. It is presented in the following theorem.

Theorem 3.1. *Riccati Design of Distributed Observer Gains.* Given systems (48) assume the interaction graph contains a spanning tree with at least one pinning gain nonzero that connects into the root node. Let $P > 0$ be a solution of the discrete-time observer Riccati equation,

$$APA^T - P + Q - APC^T \left(CPC^T\right)^{-1} CPA^T = 0, \tag{57}$$

where $Q = Q^T > 0$. Choose the observer gain matrix as

$$F = APC^T \left(CPC^T\right)^{-1}. \tag{58}$$

Define

$$r_{obs} := \left[\sigma_{\max}\left(Q^{-1/2}APC^T\left(CPC^T\right)^{-1}CPA^TQ^{-1/2}\right)\right]^{-1/2}. \tag{59}$$

Then the observer error dynamics (53) are stable if there exists a covering circle $\bar{C}\left(c_0, r_0\right)$ of the graph matrix eigenvalues Λ_k, $k = 1...N$ such that

$$\frac{r_0}{c_0} < r_{obs}. \tag{60}$$

If (60) is satisfied then taking the coupling gain

$$c_1 = \frac{1}{c_0} \tag{61}$$

makes the observer error dynamics (53) stable. ■

The next technical lemma is needed in the proof of Theorem 3.1.

Lemma 3.4. The convergence region, S_o, for the observer gain F given in Theorem 3.1., contains the circle $C\left(1, r_{obs}\right)$.

Proof: The observer Riccati equation (57) can be written as

$$APA^T - P + Q - F\left(CPC^T\right)^{-1}F^T = 0. \tag{62}$$

Select any symmetric matrix $P > 0$. Then

$$\begin{aligned}
&(A^T - sC^TF^T)^* P(A^T - sC^TF^T) - P \\
&= APA^T - P - s^*FCPA^T - sAPC^TF^T + s^*sFCPC^TF^T \\
&= APA^T - P - 2\operatorname{Re} s APC^T(CPC^T)^{-1}CPA^T + |s|^2 APC^T(CPC^T)^{-1}CPA^T \\
&= APA^T - P - (2\operatorname{Re} s - |s|^2)FCPC^TF^T \\
&= APA^T - P - (1 - |s-1|^2)FCPC^TF^T
\end{aligned} \tag{63}$$

A sufficient condition for the stability of $(A - sFC)$ is that this be less than zero. Inserting (62) this is equivalent to

$$-Q + |s-1|^2 FCPC^TF^T < 0. \tag{64}$$

This equation furnishes the bound

$$|s-1|^2 < \frac{1}{\sigma_{\max}\left(Q^{-1/2}F(CPC^T)F^TQ^{-1/2}\right)}. \tag{65}$$

Furthermore, expressing F as (58) one obtains

$$|s-1|^2 < \frac{1}{\sigma_{\max}\left(Q^{-1/2}APC^T\left(CPC^T\right)^{-1}CPA^TQ^{-1/2}\right)} = r_{obs}^2. \tag{66}$$

This is an open circle $C(1, r_{obs})$, and for $s \in \mathbb{C}$ in that circle $\rho(A^T - sC^TF^T) < 1$, but this also means that $\rho(A - sFC) < 1$ since transposition does not change the eigenvalues of a matrix. ■

The guaranteed circle of convergence $C(1, r_{obs})$, being simply connected in the complex plane and containing $s = 1$ is contained within the observer gain margin region U_o.

Proof of Theorem 3.1:

Given $C(1, r_{obs})$ contained in the convergence region S_o of matrix pencil $A - sFC$, and the properties of dilation (homothety), assuming there exists a directed spanning tree in the graph with a nonzero pinning gain into the root node, it is clear that synchronization is guaranteed if all eigenvalues Λ_k are contained in a circle $C(c_0, r_0)$ similar with respect to homothety to a circle concentric with and contained within $C(1, r_{obs})$. Homothety, as understood in this chapter, refers to a radial projection with respect to the origin, *i.e.* zero in complex plane. That is, it must be possible to bring $C(c_0, r_0)$ into $C(1, r_{obs})$ by scaling the radii of all graph eigenvalues by the same constant multiplier.

The center of the covering circle c_0 can be taken on the real axis due to symmetry and the radius equals $r_0 = \max_k |\Lambda_k - c_0|$. Taking these as given, it is straightforward that one should have

$$\frac{r_0}{c_0} < \frac{r_{obs}}{1}.$$

If this equation is satisfied then choosing $c = 1/c_0$ maps with homothety the covering circle of all eigenvalues $C(c_0, r_0)$ into a circle $C(1, r_0/c_0)$ concentric with and, for $r > r_0/c_0$, contained in the interior of the circle $C(1, r_{obs})$. ■

Note that in (57) there is a condition $P > 0$, and one must have C of full row rank. A necessary condition for the existence of a solution $P > 0$ to (57) is detectability of (A, C).

For the sake of computational simplicity, the covering circle $C(1, r_{obs})$ is used to prove sufficiency. If, however, this covering circle is found not to satisfy (60), there could be other covering circles satisfying this condition, *e.g.* those having smaller r_0/c_0 ratio.

The following lemma highlights the robustness property of the proposed distributed observer design. It formalizes the fact that under the sufficient condition of Theorem 3.1 observer convergence is robust to small changes of the coupling gain value.

Lemma 3.5. If the sufficient condition (60) of Theorem 3.1 is satisfied then there exists an open interval of values for the observer coupling gain c_1 guaranteeing convergence, obtained as an intersection of a finite family of open intervals, as in Chapter 2, 28. The value given by (61) is contained in all the intervals, whence it is contained in their intersection.

Proof: The same as for the analogous result of Chapter 2. ■

3.5 Duality Between Cooperative Observer and Controller Design

In this section a duality property is given for cooperative observers and cooperative controllers on communication graphs, where $c_1 = c = c_2$. It proves to be convenient to consider non-weighted observer and controller protocols given by

$$x_i(k+1) = Ax_i(k) + c_2 BK\left[\sum_{j\in N_i} e_{ij}(x_j - x_i) + g_i(x_0 - x_i)\right](k), \tag{67}$$

$$\hat{x}_i(k+1) = A\hat{x}_i(k) + Bu_i(k) - c_1 F\left[\sum_j e_{ij}(\tilde{y}_j - \tilde{y}_i) + g_i(\tilde{y}_0 - \tilde{y}_i)\right](k). \quad (68)$$

The next result details the duality property for the distributed controllers (67) and observers (68). For this result, it is required that the interaction graph of the observer be the same as the interaction graph for the control design, as assumed throughout this chapter, but in addition in needs to be balanced.

Theorem 3.2. Consider a networked system of N identical linear agents on a balanced communication graph $\mathcal{G}$ with dynamics (A, B, C) given by (48). Suppose the feedback gain K in protocol (67) stabilizes the global tracking error dynamics (8) having closed-loop system matrix

$$A_c = I_N \otimes A - c_2(L+G) \otimes BK. \quad (69)$$

Then, assuming $c_1 = c_2 = c$, the observer gain $F = K^T$ in protocol (68) stabilizes the global observer error dynamics (53) having closed loop observer matrix

$$A_o = I_N \otimes A^T - c_1(L+G)^T \otimes FB^T, \quad (70)$$

for a networked dual system of N identical linear agents (A^T, C^T, B^T) on the reverse communication graph $\mathcal{G}'$.

Proof: Similar to the proof given in 13 for continuous-time systems. ■

Though weighting is not used in this section the importance of weighting is detailed in Example 3.1, subsection 3.7.1.

3.6 Three Regulator Configurations of Observer and Controller

In this section are presented three different methods of connecting observers and controllers at node i into a cooperative dynamic output feedback regulator for cooperative tracking. The observer graph and controller graph need not be the same. In the case that the observer and controller graph are not the same the sufficient conditions of Theorem 2.1 of

Chapter 2 and Theorem 3.1 of Chapter 3 apply separately to the respective graphs. The following results are derived under a reasonable simplifying assumption of having those two graphs equal, but apply similarly to the more general case. The development of this section is a discrete-time version of the continuous-time results of 13. The cooperative regulators should have an observer at each node and a control law based on the observed outputs, and the separation principle should hold. This is the case for the following three system architectures.

3.6.1 Distributed Observers and Controllers

In this dynamic regulator architecture, the controller uses a distributed feedback law

$$u_i = c_2 \left(1 + d_i + g_i\right)^{-1} K\hat{\varepsilon}_i \, , \tag{71}$$

where

$$\hat{\varepsilon}_i = \sum_j e_{ij}\left(\hat{x}_j - \hat{x}_i\right) + g_i\left(\hat{x}_0 - \hat{x}_i\right), \tag{72}$$

is the estimated local neighborhood tracking error. The observer is also distributed and of the form (52). Thus, both the controller and the observer depend on the neighbor nodes. The global state and estimate dynamics are given as

$$x(k+1) = \left(I_N \otimes A\right)x(k) - c_2\left(I + D + G\right)^{-1}\left(L + G\right) \otimes BK\left(\hat{x} - \overline{x}_0\right)(k) \tag{73}$$

$$\begin{aligned} \hat{x}(k+1) = &\left[I_N \otimes A - c_1\left(I + D + G\right)^{-1}\left(L + G\right) \otimes FC\right]\hat{x}(k) + c_1\left(I + D + G\right)^{-1}\left(L + G\right) \otimes Fy(k) \\ &- c_2\left(I + D + G\right)^{-1}\left(L + G\right) \otimes BK\left(\hat{x} - \overline{x}_0\right)(k) \end{aligned} \tag{74}$$

Note that the assumption, $\hat{x}_0 = x_0$, was used in (73). The global state error then follows from

$$x(k+1) = \left[I_N \otimes A - c_2\left(I + D + G\right)^{-1}\left(L + G\right) \otimes BK\right]x(k) + c_2\left(I + D + G\right)^{-1}\left(L + G\right) \otimes BK\left(\eta + \overline{x}_0\right)(k) \, ,$$

which further yields the global error dynamics

$$\begin{aligned} \delta(k+1) &= (I_N \otimes A - c_2\left(I + D + G\right)^{-1}\left(L + G\right) \otimes BK)\delta(k) + c_2\left(I + D + G\right)^{-1}\left(L + G\right) \otimes BK\eta(k) \\ &= A_c\delta(k) + B_c\eta(k), \end{aligned} \tag{75}$$

and

$$\begin{aligned}\eta(k+1) &= (I_N \otimes A - c_1(I+D+G)^{-1}(L+G)\otimes FC)\eta(k) \\ &= A_o\eta(k)\end{aligned} . \tag{76}$$

The entire error system is then

$$\begin{bmatrix}\delta \\ \eta\end{bmatrix}(k+1) = \begin{bmatrix}A_c & B_c \\ 0 & A_o\end{bmatrix}\begin{bmatrix}\delta \\ \eta\end{bmatrix}(k) . \tag{77}$$

Under the conditions detailed in Theorem 2.1 of Chapter 2 and Theorem 3.1 of Chapter 3, the error system, (77), is asymptotically stable, implying $\delta, \eta \to 0$ asymptotically. This guarantees synchronization of all the agents' states to the control node state.

3.6.2 Local Observers and Distributed Controllers

In this architecture, the controller is the distributed form (71), (72). On the other hand, the observers are now local

$$\hat{x}_i(k+1) = A\hat{x}_i(k) + Bu_i(k) + F\tilde{y}_i(k) . \tag{78}$$

and do not use information from the neighbors. Then, the state observation error $\eta_i = x_i - \hat{x}_i$ dynamics is

$$\eta_i(k+1) = (A - FC)\eta_i(k),$$

or globally

$$\eta(k+1) = I_n \otimes (A - FC)\eta(k). \tag{79}$$

Then, the overall error dynamics are

$$\begin{bmatrix}\delta \\ \eta\end{bmatrix}(k+1) = \begin{bmatrix}A_c & B_c \\ 0 & I_N \otimes (A-FC)\end{bmatrix}\begin{bmatrix}\delta \\ \eta\end{bmatrix}(k). \tag{80}$$

Now, the observer gain *F* is easily selected using, for instance, Riccati design so that $(A - FC)$ is asymptotically stable. The feedback gain is selected by Theorem 3.2. Then, under the hypotheses of Theorem 3.2, synchronization is guaranteed.

3.6.3 Distributed Observers and Local Controllers

In this architecture the controller is local of the form

$$u_i = -K\hat{x}_i \,, \tag{81}$$

which does not depend on the neighbors. On the other hand, the observer is distributed and given by protocol (52), with

$$\varepsilon_i^o = \sum_j e_{ij}\left(\tilde{y}_j - \tilde{y}_i\right) + g_i\left(y_0 - \tilde{y}_i\right).$$

Note that instead of $\tilde{y}_0$ in (50), which is assumed identically equal to zero, here one uses the control node output y_0. In global form this yields

$$\begin{aligned} x(k+1) &= I_N \otimes Ax(k) - I_N \otimes BK\hat{x}(k) \\ \hat{x}(k+1) &= I_N \otimes (A-BK)\hat{x}(k) + c_1\Gamma \otimes FC\eta(k) - c_1\Gamma \otimes FC\overline{x}_0(k) \end{aligned}. \tag{82}$$

Expressing $x, \hat{x}$ through $x, \eta = x - \hat{x}$ gives

$$\begin{aligned} & x(k+1) = I_N \otimes (A-BK)x(k) + I_N \otimes BK\eta(k), \\ & \eta(k+1) = I_N \otimes (A-BK)x(k) + I_N \otimes BK\eta(k) - I_N \otimes (A-BK)\hat{x}(k) - c_1\Gamma \otimes FC\eta(k) + c_1\Gamma \otimes FC\overline{x}_0(k) \\ & = I_N \otimes (A-BK)\eta(k) + I_N \otimes BK\eta(k) - c_1\Gamma \otimes FC(\eta - \overline{x}_0)(k) \\ & = (I_N \otimes A - c_1\Gamma \otimes FC)\eta(k) + c_1\Gamma \otimes FC\overline{x}_0(k) \\ & = A_o\eta(k) + c_1\Gamma \otimes FC\overline{x}_0(k), \end{aligned} \tag{83}$$

Global tracking error dynamics now follows as

$$\delta(k+1) = (I_N \otimes A)\delta(k) - (I_N \otimes BK)\hat{x}(k). \tag{84}$$

Using that $\hat{x} = x - \eta = x - \overline{x}_0 - \eta + \overline{x}_0 = \delta - (\eta - \overline{x}_0)$ one finds

$$\delta(k+1) = I_N \otimes (A-BK)\delta(k) + (I_N \otimes BK)(\eta(k) - \overline{x}_0(k)). \tag{85}$$

Neither (83) nor (85) are autonomous systems since an exogenous input is present in form of control node state $\overline{x}_0(k)$. However, if one looks at the dynamics of $\vartheta(k) = \eta(k) - \overline{x}_0(k)$ it follows that $\vartheta(k+1) = A_o\vartheta(k)$, and $\delta(k+1) = I_N \otimes (A-BK)\delta(k) + (I_N \otimes BK)\vartheta(k)$. Or, more clearly in matrix form

$$\begin{bmatrix} \delta \\ \vartheta \end{bmatrix}(k+1) = \begin{bmatrix} I_N \otimes (A-BK) & I_N \otimes BK \\ 0 & A_o \end{bmatrix} \begin{bmatrix} \delta \\ \vartheta \end{bmatrix}(k). \tag{86}$$

The control gain K is easily selected using, *e.g.* Riccati design, so that $(A-BK)$ is asymptotically stable. The observer gain is selected by Theorem 3.1. Then, under the hypotheses of Theorem 3.1, synchronization, $\delta \to 0$, is guaranteed.

Note that the observer in (46) is biased since $\vartheta \to 0$ implies $\eta \to \overline{x}_0$. Thus, the observers effectively estimate the tracking errors, converging to $\hat{x}_i(k) = x_i(k) - x_0(k) = \delta_i(k)$.

Remark 3.1: The three proposed observer/controller architectures differ in the amount of information that must be exchanged between neighbors for observation and control. When both the observer and controller are distributed, each agent requires a significant amount of computation and communication to produce the estimate of its own state, since neighbor outputs need to be measured and state, *i.e.* output, estimates need to be communicated between neighbors. Also the control input requires all the estimated neighbor states. Local observers and distributed controller architecture require less computations and communication, since the state of each agent is estimated using only its own inputs and outputs, and only the state estimates need to be communicated between neighbors. The architecture using distributed estimation and local controller is interesting because for control purposes agents do not need to communicate with their neighborhoods, though communication is needed for distributed estimation. The specific eigenvalues of $A-BK$, $A-FC$, A_o, A_c that appear in the augmented state equations (77), (80), (86) determine the time constants in each of the three architectures.

<u>3.7 Numerical Examples</u>

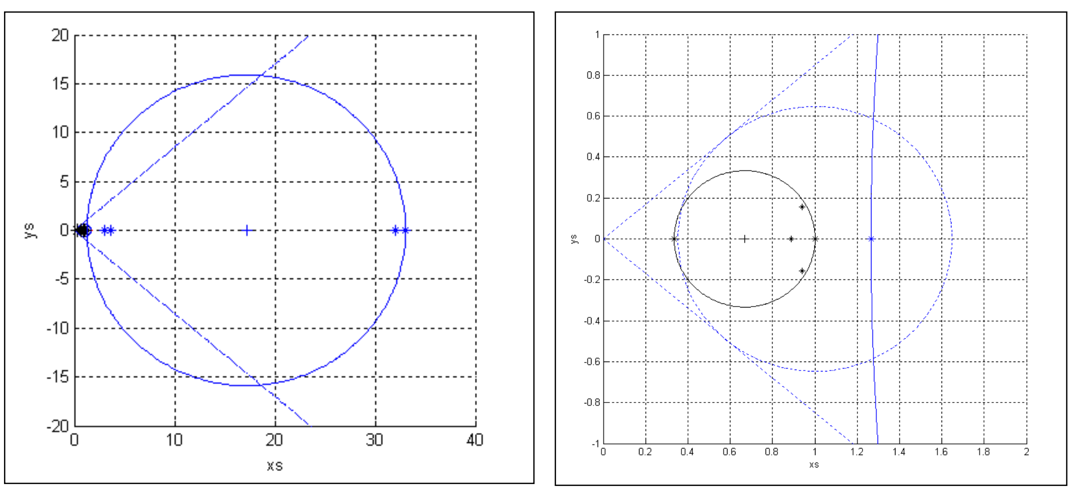
3.7.1 Example 1

This example shows the importance of using the weighting in observer algorithm (4) and the control law (21). Consider a set of 5 agents with dynamics given by (48) with (A, B, C) in controllable canonical form

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.2 & 0.2 & 1.1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

The control node has the dynamics (49). The control node is pinned into two of the nodes, and the graph structure is given by

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix},\ G = \operatorname{diag}(30, 0, 0, 0, 30).$$



(a) (b)

Figure 3.1 (a) Non-weighted graph eigenvalues and their covering circle. (b) Weighted graph matrix eigenvalues, their covering circle, and convergence region of observer (dashed circle).

Figure 3.1a shows the non-weighted graph eigenvalues, that is, the eigenvalues of $(L+G)$, along with their covering circle. Figure 3.1b shows the magnified part of Figure 3.1a containing the weighted counterparts, that is the eigenvalues of $\Gamma=(I+D+G)^{-1}(L+G)$, their covering circle and the observer convergence region $C(1,r_{obs})$ displayed in dashes. Dashed lines marking a sector in complex plane depict the region wherein the eigenvalue covering circles should lie so that they be scalable to the observer convergence region. It can be verified that the non-weighted eigenvalues in Figure 3.1a do not satisfy the sufficient condition (60) given in Theorem 3.1, while the weighted eigenvalues in Figure 3.1b do.

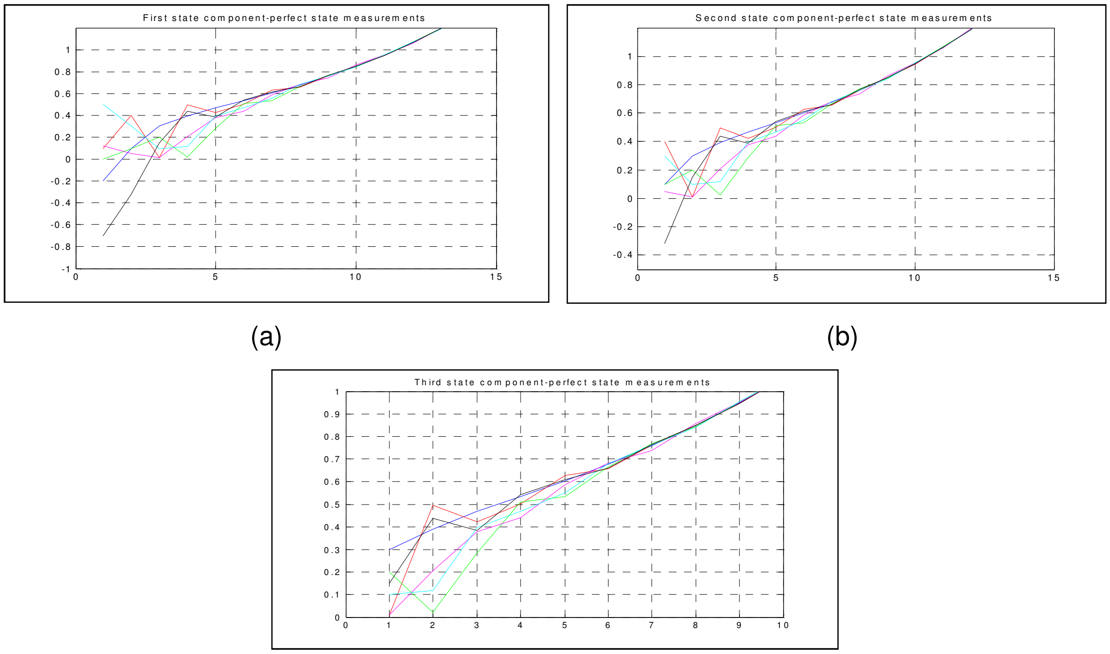
It is interesting that the non-weighted eigenvalues are real, while the weighted eigenvalues here are complex.

3.7.2 Example 2

This example gives a set of numerical simulations for the three different observer-controller architectures in Section 3.6. The dynamics and the communication graph used are the same as in Example 1.

Example 2a. Perfect state measurements

In this simulation it is assume that perfect state measurements are available. That is, the cooperative control law is used assuming full state feedback, as in Chapter 2. This provides a baseline for comparison of the performance of the three dynamic regulator designs given in Section 3.6. Figure 3.2 shows the state components of the 5 nodes and the leader control node. Figure 3.3 shows the first components of the state tracking errors. All nodes synchronize to the state of the leader.



(a) (b)

(c)

Figure 3.2 States of the nodes and the control node with perfect state measurements (a) First state components, (b) Second state components, (c) Third state components.

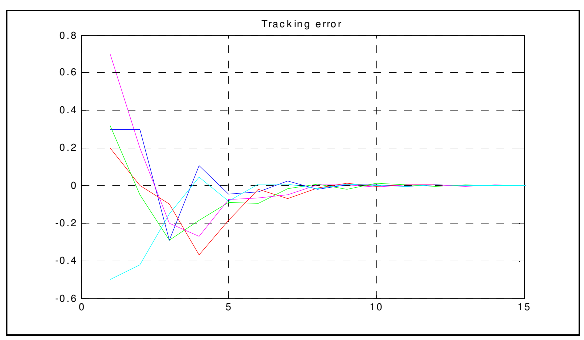
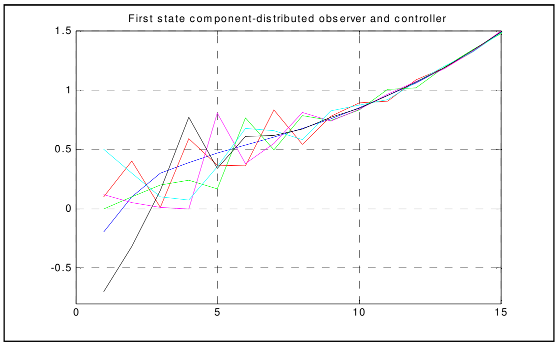


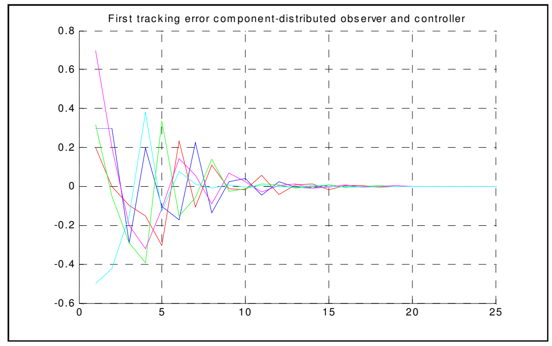
Figure 3.3 First state tracking error component with perfect state measurements.

Example 2b. Distributed observers and controllers

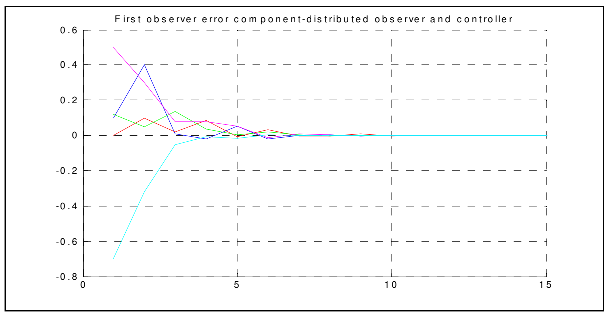
This simulation is for the case of distributed observers and controllers given as design 3.6.1 in Section 3.6. Figure 3.4 depicts the first state components of all 5 agents and the control node, as well as the first components of the tracking errors δ and the first components of the observer errors η. Synchronization is achieved.



(a)



(b)

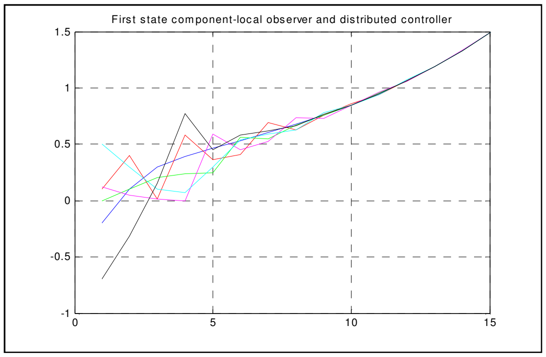


(c)

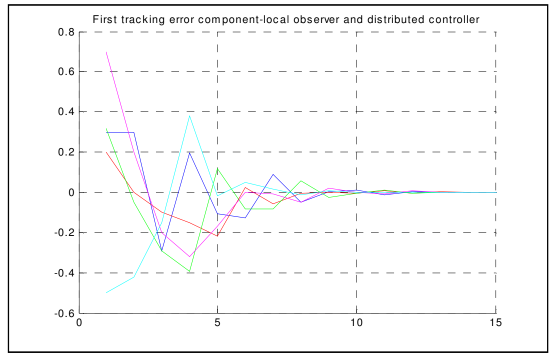
Figure 3.4 Distributed observers and controllers. (a) First state components, showing synchronization, (b) First tracking error components, (c) First observer error components.

Example 2c. Local observers and distributed controllers

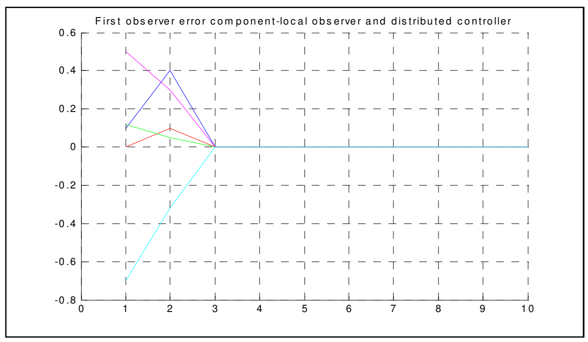
This simulation is for the case of local observers and distributed controllers given as design 3.6.2 in Section 3.6. Figure 2.5 depicts the first state components of all 5 agents and the control node, as well as the first components of the tracking errors δ and the first components of the observer errors η. Synchronization is achieved.



(a)



(b)



(c)

Figure 3.5 Local observers and distributed controllers. (a) First state components, showing synchronization, (b) First tracking error components, (c) First observer error components.

Example 2d. Distributed observers and local controllers

This simulation is for the case of distributed observers and local controllers given as design 3.6.3 in Section 3.6. Figure 3.6 depicts the first state components of all 5 agents and the control node, as well as the first components of the tracking errors δ and the first components of the observer errors η. Synchronization is achieved.

Note that the observer errors in Figure 3.6c do not converge to zero since the observers estimate the tracking errors, converging to $\hat{x}_i(k) = x_i(k) - x_0(k) = \delta_i(k)$.

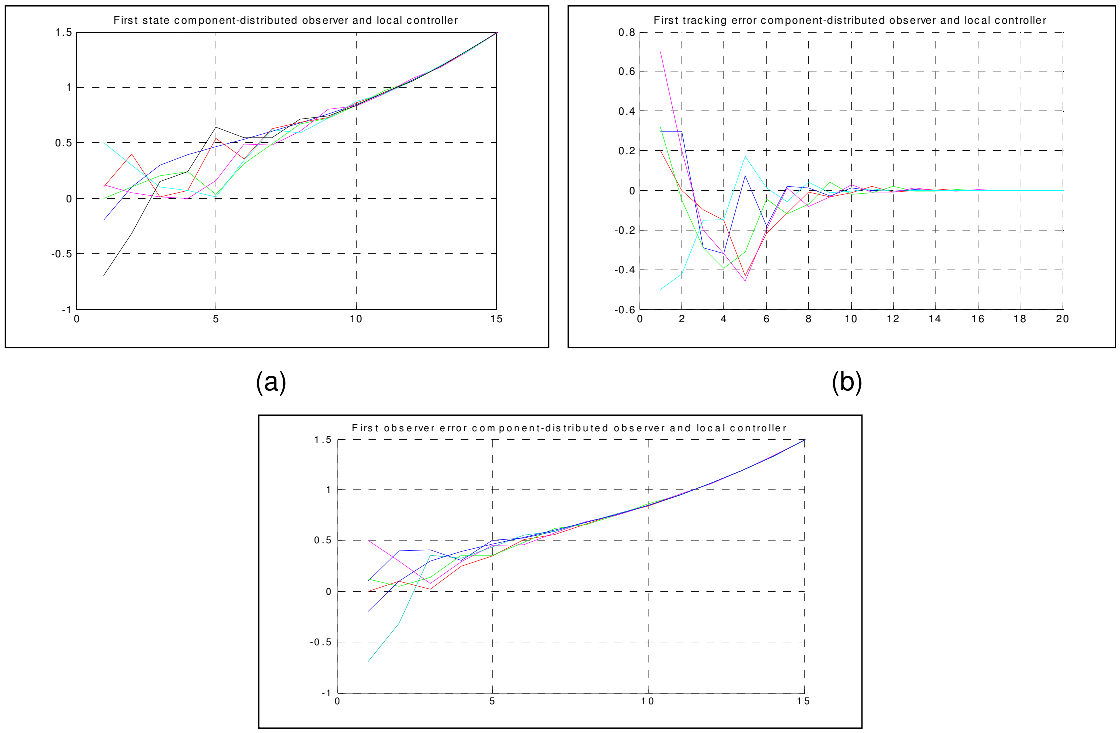


Figure 3.6 Distributed observers and local controllers. (a) First state components, showing synchronization, (b) First tracking error components, (c) First observer error components.

Remark 3.2: It is interesting to compare the time constants of state synchronization and estimate convergence for the three dynamic regulator architectures. The observations are in accordance with the remark made after Section 3.6. Compared to perfect state measurement, distributed observers and controllers took almost twice as long for state synchronization, and almost the same for estimate convergence alone. The case of distributed controller and local observer needed the shortest time for state synchronization, shorter even than in case of local controller and distributed observer. For the system considered, it appears to be more advantageous to make observation faster by using a local observer, then to simplify the control law. Depending on the specific eigenvalues, perhaps for some other system it would pay more to use a local controller with distributed observer.

3.8 Conclusion

Algorithms that do not depend on the topological properties of the graph were given to design observer gains and feedback control gains for cooperative systems on directed communication graphs. Riccati design for the observer and controller gains guarantees stability if a certain condition in terms of the graph eigenvalues holds. A duality principle for distributed systems on balanced graphs was given. Three cooperative regulator designs were presented, each using cooperative observers and feedback controllers, which guarantee synchronization of multi-agent systems to the state of a control node using only output feedback. Furthermore, the separation principle is shown to be valid in all three cases for multi-agent systems on graphs. These results are an extension of results for continuous-time systems in 11, 13, and, by duality, of the results obtained for distributed feedback control systems in 28. Presented results could be extended to time-varying graphs in future work.

CHAPTER 4

OPTIMAL DISTRIBUTED CONTROL

4.1 Introduction

The last two decades have witnessed an increasing interest in multi-agent network cooperative systems, inspired by natural occurrences of flocking and formation forming. Early work with networked cooperative systems in continuous and discrete time is presented in 1,5,3,4,7,6. Early work generally referred to consensus without a leader, where the final consensus value is determined solely by the initial conditions. We term this the *cooperative regulator problem*. Necessary and sufficient conditions for the distributed systems to synchronize were given. On the other hand, by adding a leader that pins to a group of other agents one can have synchronization to a command trajectory through pinning control, 5, 23, 8 for all initial conditions. We term this the *cooperative tracker problem*. It is shown in 12, 13 that, by using state feedback derived from the local algebraic Riccati equation, synchronization can be achieved for a broad class of communication graphs. Synchronization using dynamic compensators or output feedback is considered in 9,13,25.

Cooperative optimal control was recently considered by many authors-33,34,36,37,38,39, to name just a few. Optimality of a control protocol gives rise to desirable characteristics such as gain and phase margins, that guarantee robustness in presence of some types of disturbances, 40,41. The common difficulty, however, is that in the general case optimal control is not distributed, 34,36. Solution of a global optimization problem generally requires centralized, *i.e.* global, information. In order to have local distributed control that is optimal in some sense it is possible *e.g.* to consider each agent optimizing its own, local, performance index. This is done for receding horizon control in 33, implicitly in 13, and for distributed games on graphs in 35, where the notion of optimality is Nash equilibrium. In 37 the LQR problem is phrased as a maximization problem of LMI's under the constraint of the communication graph topology. This is a constrained optimization taking into account the local character of interactions among agents. It is also possible to use a local observer to obtain the global information needed for the solution of the global optimal problem, as is done in 34. In the case of agents with identical linear time-invariant dynamics, 38 presents a suboptimal design that is distributed on the graph topology.

Optimal control for multi-agent systems is complicated by the fact that the graph topology interplays with system dynamics. The problems caused by the communication topology in the design of global optimal controllers with distributed information can be approached using the notion of inverse optimality, 41. There, one chooses an optimality criterion related to the communication graph topology to obtain distributed optimal control, as done for the single-integrator cooperative regulator in 36. This connection between the graph topology and the structure of the performance criterion allows for the distributed optimal control. In the case that the agent integrator dynamics contains topological information, 39 shows that there is a performance criterion such that the original distributed control is optimal with respect to it.

In this chapter are considered fixed topology directed graphs and linear time-invariant agent dynamics. First, theorems are provided for partial stability and inverse optimality of a form useful for applications to cooperative control, where the synchronization manifold may be noncompact. In our first contribution, using these results, we solve the globally optimal cooperative regulator and cooperative tracker problems for both single-integrator agent dynamics and also agents with identical linear time-invariant dynamics. It is found that globally optimal linear quadratic regulator (LQR) performance cannot be achieved using distributed linear control protocols on arbitrary digraphs. A necessary and sufficient condition on the graph topology is given for the existence of distributed linear protocols that solve a global optimal LQR control problem. In our second contribution, we define a new class of digraphs, namely, those whose Laplacian matrix is simple, *i.e.* has a diagonal Jordan form. On these graphs, and only on these graphs, does the globally optimal LQR problem have a distributed linear protocol solution. If this condition is satisfied, then distributed linear protocols exist that solve the global optimal LQR problem only if the performance indices are of a certain form that captures the topology of the graph. That is, the achievable optimal performance depends on the graph topology.

The structure of this chapter is the following; Section 4.2 deals with stability of possibly noncompact invariant manifolds, motivated by partial stability. Section 4.3 introduces conditions of optimality and inverse optimality and the connection with stability. These results are applied in Sections 4.4 and 4.5. Section 4.4 discuses optimal leaderless consensus and pinning control for single-integrator agent dynamics, and Section 4.5 gives the optimality results for general linear time-invariant agent dynamics.

In Sections 4.4 and 4.5 it is found that optimality for a standard LQR performance index is only possible for graphs whose Laplacian matrix satisfies a certain condition. In Section 4.6 this condition is further discussed and shown to be satisfied by certain classes of digraphs, specifically for those graphs whose Laplacian matrix has a diagonal Jordan form. This condition holds for undirected graphs. Conclusions are given in Section 4.7.

4.2 Asymptotic Stability of the Consensus Manifold

This section presents concepts of partial stability applied to noncompact manifolds. The definitions are based on neighbourhoods of a manifold and avoid the need for global coordinates, in contrast to usual formulation of partial stability, *e.g.* 41. The fact that a manifold to which convergence is to be guaranteed is noncompact means that usual approaches for compact manifolds based on proper Lyapunov functions are not applicable as such. Furthermore when dealing with noncompact manifolds one makes the distinction between non-uniform and uniform stability.

In the cooperative control consensus problem, where there are N agents with states $x_i \in \mathbb{R}$, $i = 1 \ldots N$, it is desired for all agents to achieve the same state, that is $\left\| x_i(t) - x_j(t) \right\| \to 0$

as $t \to \infty$, $\forall i, j$. Then the consensus manifold, $S := span(\underline{1})$, where $\underline{1} = [1 \;\; 1 \;\; \cdots \;\; 1]^T \in \mathbb{R}^N$, is noncompact.

Let the system dynamics be given as

$$\dot{x} = f(x,u) = f(x) + g(x)u \,,$$

where $f(x)$, $g(x)$, and $u(t)$ are assumed to be such that the existence of a unique solution to the initial value problem is guaranteed.

Definition 4.1: Let S be a manifold embedded in a topological space X. A neighborhood $\mathcal{D}(S)$ of S is an open set in X containing the manifold S in its interior.

Definition 4.2: Let S be a manifold embedded in a metric space (X, d), and let $\mathcal{D}(S)$ be a neighborhood of S. An ε*-neighborhood* of $S \subset \mathcal{D}(S)$ is defined as $\mathcal{U}_\varepsilon(S) = \{ x \in X \,|\, d(x,S) < \varepsilon \}$, where $d(x,S) := \inf_{y \in S} d(x,y)$ is the distance from $x \in X$ to S as given by the metric d.

Note that in the case of compact manifolds any neighborhood $\mathcal{D}(S)$ contains some ε-neighborhood, but in the case of noncompact manifolds this need not be true. For the needs of defining stability of noncompact manifolds, one uses neighborhoods that contain some ε-neighborhood. We call such neighborhoods *regular*.

Definition 4.3: A manifold S is said to be (*Lyapunov*) *stable* if there exists a regular neighborhood $\mathcal{D}(S)$ of S, such that for every ε-neighborhood $\mathcal{U}_\varepsilon(S)$ contained in it, there exists a subneighborhood $\mathcal{V}(S)$ satisfying the property $x(0) \in \mathcal{V}(S) \Rightarrow x(t) \in \mathcal{U}_\varepsilon(S) \;\; \forall t \geq 0$. If $\mathcal{D}(S)$ can be taken as the entire space X, then the stability is *global*.

If S is Lyapunov stable and furthermore there exists a neighborhood $\mathcal{W}(S)$ of S satisfying the property $x(0) \in \mathcal{W}(S) \Rightarrow d(x(t),S) \to 0$ as $t \to \infty$, then S is *asymptotically stable*. If $d(x(t),S) \to 0$ we say that the *trajectory* $x(t)$ *converges to* S.

If S is Lyapunov stable and for every ε-neighborhood $\mathcal{U}_\varepsilon(S)$ of S, there exists a subneighborhood $\mathcal{V}(S) \subseteq \mathcal{U}_\varepsilon(S)$ containing a δ-neighborhood $\mathcal{V}_\delta(S)$ then S is *uniformly stable*. In this case the stability conclusion can be phrased as $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$d(x(0), S) < \delta \Rightarrow d(x(t), S) < \varepsilon \;\; \forall t \ge 0.$$

If a manifold is uniformly stable and asymptotically stable so that some neighborhood $\mathcal{W}(S)$ contains a δ-neighborhood $\mathcal{W}_\delta(S)$ satisfying the property $x(0) \in \mathcal{W}_\delta(S) \Rightarrow d(x(t), S) \to 0$ as $t \to \infty$, uniformly[1] on $\mathcal{W}_\delta(S)$, then S is *uniformly asymptotically stable*.

If a manifold is uniformly asymptotically stable and there exist constants $K, \sigma > 0$ such that $d(x(t), S) \le K d(x(0), S) e^{-\sigma t}$, for all $x(0)$ in some δ-neighborhood $\mathcal{W}_\delta(S)$ then the partial stability is *exponential*. ■

A sufficient condition for the various types of stability of a manifold, possibly noncompact, is given as the following theorem, motivated by 41, Chapter 4.

Theorem 4.1. (*Partial Stability*). Given a manifold S, possibly noncompact, contained in a neighborhood $\mathcal{D} \supset S$, if there exists a C^1 function $V : \mathcal{D} \to \mathbb{R}$ and a class $\mathcal{K}$ function α such that

$$\begin{aligned} &V(x) = 0 \Leftrightarrow x \in S \\ &V(x) \ge \alpha(d(x, S)) \;\; \text{for } x \in \mathcal{D} \setminus S \\ &\dot{V}(x) \le 0 \;\; \text{for } x \in \mathcal{D} \end{aligned}$$

then S is Lyapunov stable. If furthermore there is a class $\mathcal{K}$ function β such that

$$\alpha(d(x, S)) \le V(x) \le \beta(d(x, S))$$

[1] In this context, uniform convergence means $\forall \varepsilon > 0, \exists T > 0 \; s.t. \; d(x(t), S) < \varepsilon \; \forall t \ge T$, where T depends on ε and $d(x_0, S)$, but not on x_0 itself.

then the stability is uniform. If in addition there is a class $\mathcal{K}$ function γ such that

$$\dot{V}(x) < -\gamma(d(x,S))$$

then the stability is uniformly asymptotic. If there exist $a, b, c, p > 0$ such that

$$ad(x,S)^p \leq V(x) \leq bd(x,S)^p$$

$$\dot{V}(x) < -cd(x,S)^p$$

then the stability is exponential.

Proof: Take any $\mathcal{U}_\varepsilon(S)$, then on the boundary $\partial\mathcal{U}_\varepsilon(S)$ one has $d(x,S) = \varepsilon$ yielding a bound $V(x) \geq \alpha(\varepsilon)$. Taking $\eta = \alpha(\varepsilon)$ and defining $\mathcal{D}_\eta = \{x \in \mathcal{D}(S) | V(x) < \eta\}$ one has for $x \in \mathcal{D}_\eta$ that $\alpha(d(x,S)) < \eta$ meaning that $d(x,S) < \varepsilon$ which implies that $\mathcal{D}_\eta \subseteq \mathcal{U}_\varepsilon(S)$. Taking $x(0) \in \mathcal{D}_\eta$, owing to the nonincreasing property of $V(x)$, one obtains

$$\alpha(d(s(t),S) \leq V(x(t)) \leq V(x(0)) < \alpha(\varepsilon), \ \forall t \geq 0,$$

guaranteeing $x(t) \in \mathcal{U}_\varepsilon(S) \ \ \forall t \geq 0$, proving the Lypapunov stability of S.

However there is no guarantee that, due to noncompactness, one does not have $\inf_{x \in \partial\mathcal{D}_\eta} d(x,S) = 0$ which means that the η level set of $V(x)$ is not bounded away from the manifold S. This is remedied by the uniformity requirement since then one can choose $\delta > 0$ such that $\beta(\delta) = \alpha(\varepsilon)$, and by the previous argument for $x(0) \in \mathcal{V}_\delta(S)$

$$\alpha(d(s(t),S) \leq V(x(t)) \leq V(x(0)) < \beta(\delta) = \alpha(\varepsilon)$$

Uniform asymptotic and exponential stability follow from the proof analogous to partial stability results in 41. ∎

The purpose of class $\mathcal{K}$ functions is to bind the level sets of the Lyapunov function away from the manifold and away from infinity, allowing the interpretation of a value of the Lyapunov function as a measure of the distance from the manifold. For global conclusions one would require $\mathcal{K}_\infty$, as an analog of properness, *i.e.* radial unboudedness.

Remark 4.1: Note that the uniformity property is always found in the case where the manifold S is compact. However, since we are dealing here with a noncompact manifolds, the difference needs to be emphasized. Theorem 4.1 will be used together with the inverse optimality approach of the next section, to guarantee asymptotic stabilization to a consensus manifold that is also optimal.

Remark 4.2: In case partial stability conditions are satisfied one does not necessarily have bounded trajectories. Trajectories are constrained to a noncompact set, and finite escape to infinity, while remaining in the noncompact neighborhood of S, is thereby not precluded. Therefore, in order to avoid such an occurrence one needs to independently impose that solutions can be extended for all time. Since linear systems are globally Lipschitz solutions cannot escape to infinity in finite time. However, in more general cases, with nonlinear dynamics, an additional, proper, Lyapunov function could be used to guarantee boundedness of trajectories to a compact set. Note that this does not contradict the stability of a noncompact set, since La Salle invariance principle can be used to guarantee convergence to (a subset of) such a set.

Note that the requirements on the Lyapunov function specialize in case of compact manifolds to familiar conditions, 41. In that case all bounded open neighborhoods are precompact. Specifically when one considers an equilibrium point the conditions specialize to

$$V(x) = 0 \Leftrightarrow x = 0, V(x) \geq \alpha(\|x\|) \text{ for } x \in \mathcal{D} \setminus \{0\}, \dot{V} \leq 0 \text{ for } x \in \mathcal{D}.$$

In case of linear systems and pertaining quadratic partial stability Lyapunov functions $V(x) = x^T P x \geq 0$ the target manifold S is the null space of the positive semidefinite matrix P.

Also the uniformity condition is automatically satisfied since $\sigma_{\min>0}(P)\|y\|^2 \le y^T P y \le \sigma_{\max}(P)\|y\|^2$, where $y \in \ker P^{\perp}$, and $\|y\|$ serves as the distance from the null space.

4.3 Inverse Optimal Control

This section presents inverse optimality notions for specific application to the consensus problem. Inverse optimality is a property of a stabilizing control $u=\phi(x)$, *e.g.* guaranteed by a control Lyapunov function $V(x)$, that is optimal with respect to some positive (semi) definite performance index. Let the system be given by the affine-in-control form

$$\dot{x} = f(x,u) = f(x) + g(x)u \,. \tag{90}$$

Certain results of optimal control can be explicitly derived for systems in this form, *e.g.* Hamilton-Jacobi-Bellman equation.

4.3.1 Optimality

The following lemma details the conditions for optimality of a stabilizing control under an infinite horizon criterion, 41.

Lemma 4.1. (*Optimality*) Consider the control affine system (90). Let S be a target manifold, possibly noncompact. Given the infinite horizon optimality criterion

$$J(x_0,u) = \int_0^\infty \mathcal{L}(x,u)\,dt \,, \tag{91}$$

if there exist functions $V(x)$, $\phi(x)$, and class $\mathcal{K}$ functions α,γ satisfying the following conditions

$$\begin{aligned} &V(x)=0 \Leftrightarrow x\in S \\ &V(x)\ge \alpha(d(x,S)) \\ &\phi(x)=0 \Leftarrow x\in S \\ &\nabla V(x) f(x,\phi(x)) \le -\gamma(d(x,S)) \end{aligned}$$

$$\begin{aligned} &H(x,\phi(x))=0 \\ &H(x,u)\ge 0 \end{aligned}$$

with $$H(x,u) = \mathcal{L}(x,u) + \nabla V(x)^T f(x,u)$$

then the feedback control $u = \phi(x)$ is optimal with respect to the performance index (91) and asymptotically stabilizing with respect to the target set S. Furthermore the optimal value of the performance index is

$$J(x_0, \phi(x)) = V(x(0)). \qquad \blacksquare$$

Conditions 1, 2 and 4 show that $V(x)$ is a control Lyapunov function for the closed loop system guaranteeing asymptotic stability of the set S, by Theorem 4.1. If S is taken as a point, *i.e.* a compact manifold of dimension zero, then those conditions become the familiar conditions on the Lyapunov function stabilizing an equilibrium point

$$\begin{aligned} &V(0) = 0 \\ &V(x) \geq \alpha(\|x\|) \quad x \neq 0 \\ &\phi(0) = 0 \end{aligned}$$

$$\nabla V(x)^T f(x, \phi(x)) < 0.$$

4.3.2 Inverse Optimality

If the feedback control $u = \phi(x)$ is asymptotically stabilizing, then there exists a Lyapunov function $V(x)$ satisfying the conditions of Lemma 4.1 by the inverse Lyapunov theorems 41,42. In inverse optimality settings an asymptotically stabilizing control law $u = \phi(x)$ is given. Then the performance integrand $\mathcal{L}(x,u)$ and Lyapunov function $V(x)$ are to be determined. In this work we are concerned with nonnegative performance index integrands, so this is to be contrasted with the more general inverse optimality where the performance integrand need not be nonnegative, as long as it guarantees a stabilizing control. That the performance integrand must be positive (semi-) definite imposes constraints on $V(x)$.

Lemma 4.2a. (*Inverse optimality*) Consider the control affine system (1). Let $u = \phi(x)$ be a stabilizing control, with respect to a manifold S. If there exist scalar functions $V(x)$ and $L_1(x)$ satisfying the following conditions

$$V(x) = 0 \Leftrightarrow x \in S$$
$$V(x) \geq \alpha(d(x,S))$$
$$L_1(x) \geq \gamma(d(x,S))$$

$$L_1(x) + \nabla V(x)^T f(x) - \frac{1}{4}\nabla V(x)^T g(x) R^{-1} g(x)^T \nabla V(x) = 0 \qquad (92)$$

$$\varphi(x) = -\frac{1}{2} R^{-1} g(x)^T \nabla V(x)$$

then $u = \phi(x)$ is optimal with respect to the performance index with the integrand $\mathcal{L}(x,u) = L_1(x) + u^T R u$. Moreover the optimal value of the performance criterion equals $J(x_0, \phi(x)) = V(x_0)$.

Proof: Assume one has an optimal control problem (90), (91) with the optimality performance integrand in (91) $\mathcal{L}(x,u) = L_1(x) + u^T R u$. Then Lemma 4.1 states the solution of optimal problem can be obtained by forming the Hamiltonian

$$H(x,u) = L_1(x) + u^T R u + \nabla V(x)^T (f(x) + g(x)u).$$

This gives the optimal control in form of

$$\frac{\partial}{\partial u} H(x,u) = 2u^T R + \nabla V(x)^T g(x) = 0,$$

$$u = \phi(x) = -\frac{1}{2} R^{-1} g(x)^T \nabla V(x)$$

This feedback control vanishes on the set S since the gradient $\nabla V(x)$ equals zero there. The Hamiltonian evaluated at this optimal control equals zero since

$$H(x,\phi(x)) = L_1(x) + \nabla V(x)^T f(x) - \frac{1}{4}\nabla V(x)^T g(x) R^{-1} g(x)^T \nabla V(x) = 0.$$

This Hamiltonian can then be concisely written in quadratic form

$$
\begin{aligned}
H(x,u) &= L_1 + u^T R u + \underbrace{\nabla V^T f}_{=-L_1+\frac{1}{4}\nabla V^T g R^{-1} g^T \nabla V} + \nabla V^T g u \\
&= \underbrace{u^T R u + \frac{1}{4}\nabla V^T g R^{-1} g^T \nabla V}_{=\phi^T R \phi} + \underbrace{\nabla V^T g u}_{=-2\phi^T R u} \\
&= (u-\phi)^T R (u-\phi) \geq 0.
\end{aligned}
$$

The value of the performance criterion then follows as

$$
\begin{aligned}
J &= \int_0^\infty \mathcal{L}(x,u)dt = -\int_0^\infty \nabla V(x)^T f(x,u)dt + \int_0^\infty H(x,u)dt \\
&= -\int_0^\infty \frac{dV}{dt}dt + \underbrace{\int_0^\infty H(x,u)dt}_{\geq 0} = V(x_0) - \underbrace{V(x(t\to\infty))}_{\to 0} + \underbrace{\int_0^\infty H(x,u)dt}_{\geq 0}.
\end{aligned}
$$

The optimum is reached precisely at $u=\phi(x)$, for which the integral on the right side vanishes, and by asymptotic stability assumption $V(x(t)) \to 0$, thus the optimal value of the performance criterion equals $J^*(x_0) = J(x_0, \phi(x)) = V(x_0)$. ■

That this optimal feedback control $u=\phi(x)$ is stabilizing follows also from the Lyapunov equation

$$
\begin{aligned}
\dot{V}(x) &= \nabla V^T f + \nabla V^T g u = \nabla V^T f - \frac{1}{2}\nabla V^T g R^{-1} g^T \nabla V \\
&= -L_1 + \frac{1}{4}\nabla V^T g R^{-1} g^T \nabla V - \frac{1}{2}\nabla V^T g R^{-1} g^T \nabla V \\
&= -L_1(x) - \frac{1}{4}\nabla V^T g R^{-1} g^T \nabla V \leq -L_1(x) \leq -\gamma(d(x,S))
\end{aligned}
$$

By the assumptions of Lemma 4.2a the Lyapunov function $V(x)$ satisfies all the conditions of the Theorem 4.1 for partial stability.

In the linear quadratic case, *i.e.* $\dot{x} = Ax + Bu$, $L_1 = x^T Q x$, $V(x) = x^T P x$, the Hamilton Jacobi Bellman equation (HJB) (3) becomes the Algebraic Riccati equation (ARE)

$$
Q + A^T P + PA - PBR^{-1}B^T P = 0. \quad (93)
$$

The next more general result allows state-control cross-weighting terms in the performance integrand $\mathcal{L}$. This result is used in the following chapter.

Lemma 4.2b (*Inverse optimality with cross weighting terms)* Consider the control affine system (1). Let $u=\phi(x)$ be a stabilizing control, with respect to a manifold S. If there exist scalar functions $V(x)$ and $L_1(x)$, $L_2(x)$ satisfying the following conditions

$$
\begin{aligned}
&V(x)=0 \Leftrightarrow x\in S\\
&V(x)\geq \alpha(d(x,S))\\
&L_2(x)=0 \Leftarrow x\in S
\end{aligned}
$$

$$L_1(x)+\nabla V(x)^T f(x)-\frac{1}{4}\left[L_2(x)+\nabla V(x)^T g(x)\right]R^{-1}\left[L_2^T(x)+g(x)^T\nabla V(x)\right]=0$$

$$\phi(x)=-\frac{1}{2}R^{-1}(L_2^T(x)+g(x)^T\nabla V(x))$$

$$L_1(x)+L_2(x)\phi+\phi^T R\phi\geq\gamma(d(x,S))$$

then $u=\phi(x)$ is optimal with respect to the performance index with the integrand $\mathcal{L}(x,u)=L_1(x)+L_2(x)u+u^T Ru$. Moreover, the optimal value of the performance criterion equals $J(x_0,\phi(x))=V(x_0)$.

Proof: Assume one has an optimal control problem (90), (91) with the optimality performance integrand in (91) $\mathcal{L}(x,u)=L_1(x)+L_2(x)u+u^T Ru$. Then Lemma 4.1 states the solution of optimal problem can be obtained by forming the Hamiltonian

$$H(x,u)=L_1(x)+L_2(x)u+u^T Ru+\nabla V(x)^T(f(x)+g(x)u).$$

This gives the optimal control in form of

$$\frac{\partial}{\partial u}H(x,u)=2u^T R+L_2(x)+\nabla V(x)^T g(x)=0$$

$$u=\phi(x)=-\frac{1}{2}R^{-1}(L_2^T(x)+g(x)^T\nabla V(x))$$

This feedback control vanishes on the set S since the gradient $\nabla V(x)$ and $L_2(x)$ equal zero there. The Hamiltonian evaluated at this optimal control equals zero since

$$
\begin{aligned}
H(x,\phi(x)) &= L_1(x) + L_2(x)\left[-\frac{1}{2}R^{-1}(L_2^T(x) + g(x)^T\nabla V(x))\right] \\
&+\left[-\frac{1}{2}R^{-1}(L_2^T(x) + g(x)^T\nabla V(x))\right]^T R\left[-\frac{1}{2}R^{-1}(L_2^T(x) + g(x)^T\nabla V(x))\right] \\
&+\nabla V(x)^T f(x) + \nabla V(x)^T g(x)\left[-\frac{1}{2}R^{-1}(L_2^T(x) + g(x)^T\nabla V(x))\right] \\
&= L_1 + \nabla V^T f - \frac{1}{2}L_2R^{-1}L_2^T - \frac{1}{2}L_2R^{-1}g^T\nabla V + \frac{1}{4}L_2R^{-1}L_2^T \\
&+\frac{1}{4}\nabla V^T gR^{-1}g^T\nabla V + \frac{1}{2}L_2R^{-1}g^T\nabla V - \frac{1}{2}\nabla V^T gR^{-1}L_2^T - \frac{1}{2}\nabla V^T gR^{-1}g^T\nabla V \\
&= L_1 + \nabla V^T f - \frac{1}{4}L_2R^{-1}L_2^T - \frac{1}{2}L_2R^{-1}g^T\nabla V - \frac{1}{4}\nabla V^T gR^{-1}g^T\nabla V \\
&= L_1 + \nabla V^T f - \frac{1}{4}\left[L_2 + \nabla V^T g\right]R^{-1}\left[L_2^T + g^T\nabla V\right] = 0
\end{aligned}
$$

This Hamiltonian can then be concisely written in quadratic form

$$
\begin{aligned}
H(x,u) &= L_1 + L_2u + u^TRu + \\
&\underbrace{\nabla V^T f}_{=-L_1+\frac{1}{4}\nabla V^T gR^{-1}g^T\nabla V+\frac{1}{4}L_2R^{-1}L_2^T+\frac{1}{2}L_2R^{-1}g^T\nabla V} \qquad + \nabla V^T gu \\
&= u^TRu + \underbrace{\frac{1}{4}\nabla V^T gR^{-1}g^T\nabla V + \frac{1}{4}L_2R^{-1}L_2^T + \frac{1}{2}L_2R^{-1}g^T\nabla V}_{=\phi^T R\phi} \\
&+ \underbrace{L_2u + \nabla V^T gu}_{=-2\phi^T Ru} = (u-\phi)^T R(u-\phi) \geq 0.
\end{aligned}
$$

The optimal value of the performance criterion then follow as in Lemma 4.2a. ■

That this optimal feedback control $u = \phi(x)$ is stabilizing follows also from the Lyapunov equation

$$\dot{V}(x) = \nabla V^T f + \nabla V^T gu = \nabla V^T f - \frac{1}{2}\nabla V^T gR^{-1}(L_2^T + g^T \nabla V)$$
$$= -L_1 + \frac{1}{4}L_2R^{-1}L_2^T + \frac{1}{2}L_2R^{-1}g^T\nabla V + \frac{1}{4}\nabla V^T gR^{-1}g^T\nabla V$$
$$-\frac{1}{2}\nabla V^T gR^{-1}g^T\nabla V - \frac{1}{2}\nabla V^T gR^{-1}L_2^T$$

$$= -L_1(x) - \frac{1}{4}\nabla V^T gR^{-1}g^T\nabla V + \frac{1}{4}L_2R^{-1}L_2^T$$
$$= -L_1(x) - L_2(x)\phi - \phi^T R\phi \le -\gamma(d(x,S))$$

By the assumptions of Lemma 4.2b the Lyapunov function $V(x)$ satisfies all the conditions of the Theorem 4.1 for partial stability.

4.4 Optimal Cooperative Control for Quadratic Performance Index and Single-integrator Agent Dynamics

This section considers inverse optimality of consensus protocols for the leaderless and pinning control cases, 3,4,8. We call these respectively the cooperative regulator and cooperative tracker problem. This section considers only single integrator dynamics

$$\dot{x}_i = u_i \in \mathbb{R}, \tag{94}$$

or in global form

$$\dot{x} = u, \tag{95}$$

where $x = [x_1 \ \dots \ x_N]^T$ and $u = [u_1 \ \dots \ u_N]^T$. More general dynamics are considered in the next section.

4.4.1 Optimal Cooperative Regulator

In the leaderless consensus or cooperative regulator problem, 32, where there are N agents with states $x_i \in \mathbb{R}$, $i = 1..N$, it is desired for all agents to achieve the same state, that is $\|x_i(t) - x_j(t)\| \to 0$ as $t \to \infty$, $\forall(i,j)$. Then the consensus manifold, $S := span(\underline{1})$, where

$\underline{1} = [1 \;\; 1 \;\; \cdots \;\; 1]^T \in \mathbb{R}^N$, is noncompact. Therefore to apply inverse optimality to the cooperative regulator problem, 32, one needs the partial stability results of Theorem 4.1.

Define the local neighborhood error as

$$\eta_i = \sum_j e_{ij}(x_i - x_j)\,. \tag{96}$$

The overall local neighborhood error vector $e = [\eta_1 \;\; \ldots \;\; \eta_N]^T$ is equal to $e = Lx$. Then a local voting protocol that guarantees consensus is given as

$$u_i = -\eta_i\,,$$

1, or in global form

$$u = -Lx\,, \tag{97}$$

which gives the global closed loop system

$$\dot{x} = -Lx\,. \tag{98}$$

Lemma 4.3. If the graph has a spanning tree then 0 is a simple eigenvalue of the graph Laplacian matrix. Furthermore the control law $u = -Lx$ solves the cooperative regulator problem, 1, 5, for the system (95). ■

If the condition of Lemma 4.3 is satisfied then the measure of the distance from the consensus manifold is given by any norm of the local neighborhood disagreement vector $e = Lx$.

The following theorem states sufficient conditions for optimality of a distributed control law. This shows when the distributed control (97) is globally optimal on the graph.

Theorem 4.2. Let the system be given as (95), then for some $R = R^T > 0$ the control $u = -Lx$ is optimal with respect to the performance index

$$J(x_0, u) = \int_0^\infty (x^T L^T R L x + u^T R u)\,dt = \int_0^\infty (e^T R e + u^T R u)\,dt\,, \tag{99}$$

and is stabilizing to a manifold which is the null space of L if there exists a positive semidefinite matrix $P = P^T \geq 0$ satisfying

$$P = RL. \tag{100}$$

Proof: First note that since R is nonsingular the null space of P equals that of L. The Lyapunov function $V(x) = x^T P x$ equals zero on the null space of P and is greater than zero elsewhere since $P \geq 0$. If one introduces the orthogonal complement of the null space of P one has $V(x) = x^T P x \geq \sigma_{>0\min}(P)\|y\|^2$ where $x = x_0 + y$, $x_0 \in \ker P$, $y \in \ker P^{\perp}$. Since $\|y\|$ is a measure of the distance $d(x, \ker P)$ the Lyapunov function is found to satisfy the conditions of Lemma 4.2.

The part of the performance integrand $L_1(x) = x^T Q x = x^T L^T R L x \geq \sigma_{>0\min}(L^T RL)\|y\|^2$ satisfies the condition of Theorem 4.1. The Algebraic Riccati equation (93) given as

$$L^T RL - PR^{-1}P = 0$$

is satisfied by $P = RL = L^T R$, and $u = -Lx = -R^{-1}Px$. Therefore all the conditions of Lemma 4.2 are satisfied by this linear quadratic problem, which concludes the proof. ■

This result was first given in 36.

Note that the time derivative of Lyapunov function $V(x) = x^T P x$ equals

$$\dot{V}(x) = -x^T (PL + L^T P)x = -2x^T L^T RLx \leq -\sigma_{>0\min}(L^T RL)\|y\|^2,$$

guaranteeing asymptotic stability of the null space of L or equivalently that of P. Also, since $V(x) = x^T P x \leq \sigma_{>0\max}(P)\|y\|^2$, according to Theorem 4.1, the stability is uniform. If the graph has a spanning tree, by Lemma 4.3, zero is a simple eigenvalue of L and thus the null space of P is the consensus manifold $span(\underline{1})$. Then, consensus is reached.

Remark 4.3: The constraint $RL = P$ is a joint constraint on the graph topology and performance criterion. This can easily be met in the case of undirected graphs $L = L^T$ by selecting $R = I$, in which case $P = L$ and $Q = L^T L$. The value of the optimality criterion is then given as $J^*(x_0) = V(x_0) = x_0^T L x_0$ which is the graph Laplacian potential, 32. Moreover $\sigma_{>0\min}(L)$ in the proof equals $\lambda_2(L)$, the Fiedler eigenvalue of the graph. Also the constraint (100) is seen to be met by detail balanced graphs with $R = \Lambda^{-1}$ where $\Lambda^{-1} L = P$ is a symmetric Laplacian matrix.

For general digraphs, since P is symmetric, the constraint (100) implies the condition

$$RL = L^T R \geq 0 \,. \tag{101}$$

More discussion about condition (100) is given in Section 4.7.

4.4.2 Optimal Cooperative Tracker

In case of the optimal cooperative tracker for system (95) it is desired for all agents to synchronize to the state x_0 of a leader node. That is $x_i(t) \to x_0(t), \forall i$. It is assumed that the leader node has dynamics $\dot{x}_0 = 0$. A distributed control law that accomplishes this is given by

$$u_i = \sum_{j \in \mathcal{N}_i} e_{ij}(x_j - x_i) + g_i(x_0 - x_i) \tag{102}$$

with $g_i \geq 0$ the pinning gains which are nonzero only for a small number of agents *i*. This corresponds in global form to the distributed tracker control law

$$u = -(L+G)\delta, \tag{103}$$

where $\delta = x - \underline{1} x_0$ is the global tracking disagreement vector and $G = diag(g_1 \ldots g_N)$. This gives the global closed-loop dynamics

$$\dot{x} = -(L+G)\delta, \tag{104}$$

and the global closed-loop tracking disagreement dynamics system

$$\dot{\delta} = u = -(L+G)\delta. \tag{105}$$

This must be asymptotically stabilized to the origin. That means that partial stability notions need not be used, and the sought solution P of the Algebraic Riccati Equation (93) must be positive definite.

Lemma 4.4. If the graph has a spanning tree, given that there exists at least one non zero pinning gain connecting into a root node, then $L+G$ is nonsingular, and the control $u=-(L+G)\delta$ solves the cooperative tracking problem, 19, for the system (95).

If the conditions of Lemma 4.4 are satisfied, then the measure of distance from the target state $\underline{1}x_0$ is given by any norm of the local neighborhood error vector $e=(L+G)\delta$.

The next result shows when the distributed control (103) is globally optimal on the graph.

Theorem 4.3. Let the error dynamics be given as (105), and the conditions of Lemma 4.4 be satisfied. Then for some $R=R^T>0$ the control $u=-(L+G)\delta$ is optimal with respect to the performance index

$$\begin{aligned} J(\delta_0,u) &= \int_0^\infty (\delta^T (L+G)^T R(L+G)\delta + u^T Ru)dt \\ &= \int_0^\infty (e^T Re + u^T Ru)dt, \end{aligned} \tag{106}$$

and is stabilizing to the reference state $\underline{1}x_0$ if there exists a positive definite matrix $P=P^T>0$ satisfying

$$P=R(L+G). \tag{107}$$

Proof: The Lyapunov function $V(\delta)=\delta^T P\delta>0$, and $L_1(\delta)=\delta^T Q\delta=\delta^T (L+G)^T R(L+G)\delta>0$ satisfy the conditions of Lemma 4.2. The Algebraic Riccati equation

$$(L+G)^T R(L+G)-PR^{-1}P=0$$

is satisfied by P, and $u=-(L+G)\delta=-R^{-1}P\delta$ thus proving the theorem. ■

Similar remarks about the constraint between R and $L+G$ apply here as in Remark 4.3. Under conditions of Lemma 4.4 and (107) P is nonsingular since both R and $L+G$ are. For undirected graphs, one choice that satisfies (107) is $R=I$. Then $P=L+G$. A more general choice is any R that commutes with $L+G$. More discussion about condition (107) is relegated to Section 4.6.

Remark 4.4: Both in the optimal cooperative regulator and tracker performance criteria, the expressions Lx and $(L+G)\delta$, respectively, play a prominent role. In either case one can concisely write the performance index as

$$J(e_0,u)=\int_0^\infty (e^T Re+u^T Ru)dt .$$

<u>4.5 Optimal Cooperative Control for Quadratic Performance Index and LinearTime-invariant Agent Dynamics</u>

This section considers inverse optimality of consensus protocols for the leaderless and pinning control cases for agents having states $x_i \in \mathbb{R}^n$ with linear time-invariant dynamics

$$\dot{x}_i = Ax_i + Bu_i \in \mathbb{R}^n ,$$

or in global form

$$\dot{x}=(I_N \otimes A)x+(I_N \otimes B)u \tag{108}$$

4.5.1 Optimal Cooperative Regulator

In the cooperative regulator problem it is desired for all agents to achieve the same state, that is $\left\|x_i(t)-x_j(t)\right\| \to 0$ as $t \to \infty,\ \forall (i,j)$. Define the local neighborhood error as

$$\eta_i = \sum_j e_{ij}(x_i - x_j) . \tag{109}$$

where $\eta_i \in \mathbb{R}^n$. The overall local neighborhood error vector $e=\begin{bmatrix}\eta_1^T & \dots & \eta_N^T\end{bmatrix}^T$ is equal to

$$e=(L\otimes I_n)x .$$

Then, a distributed control that guarantees consensus is given as

$$u_i = -cK_2\eta_i ,$$

for an appropriate coupling gain $c > 0$ and local feedback matrix K_2, as shown in 13. In global form this is the local voting protocol

$$u = -c(L \otimes K_2)x , \tag{110}$$

which gives the global closed loop system

$$\dot{x} = (I_N \otimes A - cL \otimes BK_2)x \tag{111}$$

The next result shows when the distributed control (110) is globally optimal on the graph.

Theorem 4.4. Let the system be given as (108). Suppose there exist a positive semidefinite matrix $P_1 = P_1^T \geq 0$, and a positive definite matrix $P_2 = P_2^T > 0$ satisfying

$$P_1 = cR_1L , \tag{112}$$

$$A^T P_2 + P_2 A + Q_2 - P_2 B R_2^{-1} B^T P_2 = 0 , \tag{113}$$

for some $Q_2 = Q_2^T > 0, R_1 = R_1^T > 0, R_2 = R_2^T > 0$ and a coupling gain $c > 0$. Define the feedback gain matrix K_2 as

$$K_2 = R_2^{-1} B^T P_2 . \tag{114}$$

Then the control $u = -cL \otimes K_2 x$ is optimal with respect to the performance index

$$\begin{aligned} &J(x_0, u) = \\ &\int_0^\infty x^T \left[c^2 (L \otimes K_2)^T (R_1 \otimes R_2)(L \otimes K_2) - cR_1 L \otimes (A^T P_2 + P_2 A) \right] x + u^T (R_1 \otimes R_2) u dt, \end{aligned} \tag{115}$$

and is stabilizing to the null space of $L \otimes I_n$ for sufficiently high coupling gain c satisfying (117).

Proof: Form the matrix $P = P_1 \otimes P_2$. Since P_2 is nonsingular the null space of P equals the null space of $P_1 \otimes I_n$ which by nonsingularity of R_1 is the same as that of $L \otimes I_n$. Therefore by positive semidefiniteness the Lyapunov function $V(x)=x^T P x = x^T (P_1 \otimes P_2) x$ is zero on the null space of $L \otimes I_n$ and positive elsewhere. By the arguments similar to those presented in the proof of Theorem 4.2 this Lyapunov function satisfies the conditions of Theorem 4.1.

Also,

$$L_1(x) = x^T Q x = x^T (c^2 (L \otimes K_2)^T (R_1 \otimes R_2)(L \otimes K_2) - cR_1 L \otimes (A^T P_2 + P_2 A))x \qquad (116)$$

equals zero on the null space of $L \otimes I_n$, and satisfies the condition of Lemma 4.2 if the value of the coupling gain is taken as

$$c > \frac{\sigma_{\max}(R_1 L \otimes (Q_2 - K_2^T R_2 K_2))}{\sigma_{>0\min}(L^T R_1 L \otimes K_2^T R_2 K_2)} . \qquad (117)$$

Since

$$\begin{aligned} Q &= c^2 (L \otimes K_2)^T (R_1 \otimes R_2)(L \otimes K_2) - cR_1 L \otimes (A^T P_2 + P_2 A) \\ &= c^2 L^T R_1 L \otimes K_2^T R_2 K_2 + cR_1 L \otimes (Q_2 - K_2^T R_2 K_2), \end{aligned}$$

one has

$$c^2 \sigma_{>0\min}(L^T R_1 L \otimes K_2^T R_2 K_2) - c\sigma_{\max}(R_1 L \otimes (Q_2 - K_2^T R_2 K_2)) > 0 \Rightarrow Q \geq 0 ,$$

which gives the lower bound on the coupling gain c. The algebraic Riccati equation for system, (108),

$$(I_N \otimes A)^T P + P(I_N \otimes A) + Q - P(I_N \otimes B) R^{-1} (I_N \otimes B)^T P = 0 ,$$

written as,

$$\begin{aligned} & P_1 \otimes A^T P_2 + P_1 \otimes P_2 A + Q - (P_1 \otimes P_2 B) R_1^{-1} \otimes R_2^{-1} (P_1 \otimes B^T P_2) = 0 \\ & P_1 \otimes (A^T P_2 + P_2 A) + Q - P_1 R_1^{-1} P_1 \otimes (P_2 B R_2^{-1} B^T P_2) = 0 \end{aligned}$$

is satisfied by the choice of $P = P_1 \otimes P_2$ since

$$\begin{aligned} Q &= c^2 (L \otimes K_2)^T (R_1 \otimes R_2)(L \otimes K_2) - cR_1 L \otimes (A^T P_2 + P_2 A) \\ &= c^2 L^T R_1 L \otimes K_2^T R_2 K_2 + cR_1 L \otimes (Q_2 - P_2 B R_2^{-1} B^T P_2) \\ &= P_1 R_1^{-1} P_1 \otimes P_2 B R_2^{-1} B^T P_2 + P_1 \otimes (Q_2 - P_2 B R_2^{-1} B^T P_2) \\ &= Q_1 \otimes P_2 B R_2^{-1} B^T P_2 + P_1 \otimes (Q_2 - P_2 B R_2^{-1} B^T P_2), \end{aligned}$$

where $Q_1 = c^2 L^T R_1 L = P_1 R_1^{-1} P_1$ is introduced for notational simplicity. It follows by the conditions (112), (113) of the Theorem that

$$P_1 \otimes (A^T P_2 + P_2 A^T + Q_2 - P_2 B R_2^{-1} B^T P_2) + (Q_1 - P_1 R_1^{-1} P_1) \otimes (P_2 B R_2^{-1} B^T P_2) = 0.$$

By conditions (112), (113) and (114) the control satisfies

$$\begin{aligned} &u = -cL \otimes K_2 x = -R^{-1}(I_N \otimes B)^T P x = \\ &-(R_1^{-1} \otimes R_2^{-1})(I_N \otimes B^T)(P_1 \otimes P_2)x = -R_1^{-1} P_1 \otimes R_2^{-1} B^T P_2 x \end{aligned}.$$

Therefore, all the conditions of the Theorem 4.1 are satisfied, completing the proof. ■

Note that the time derivative of the Lyapunov function $V(x) = x^T P x$ equals

$$\begin{aligned} &\dot{V}(x) = 2x^T P \dot{x} = 2x^T P(I_N \otimes Ax - cL \otimes BK_2 x) \\ &= x^T (P_1 \otimes (P_2 A + A^T P_2) - 2P(I_N \otimes B)(R_1^{-1} \otimes R_2^{-1})(I_N \otimes B^T)P)x \\ &= -x^T (Q + c^2 (L \otimes K_2)^T (R_1 \otimes R_2)(L \otimes K_2))x \le -x^T Q x \end{aligned}$$

implying asymptotic stability to the nullspace of Q, which equals the null space of $L \otimes I_n$ (*c.f.* (116)).

If the conditions of Lemma 4.3 hold then $\ker L = span(\underline{1})$, and under the conditions of Theorem 4.4 consensus is reached.

Remark 4.5: The graph topology constraint (112) is similar to the constraint (100), and comments similar to those stated in Remark 4.3 apply here as well. There exists a $P_2 = P_2^T > 0$ satisfying (113) if (A, B) is stabilizable and $(A, \sqrt{Q_2})$ is observable. The value of the coupling gain *c* must be sufficiently great to overpower the, generally, indefinite terms stemming from the drift dynamics as in condition (117). If the communication graph satisfies conditions of Lemma 4.3 the null space of $L \otimes I_n$ equals the synchronization manifold $S := span(\underline{1} \otimes \alpha)$, $\alpha \in \mathbb{R}^n$, therefore synchronization is asymptotically achieved in the optimal way.

4.5.2 Optimal Cooperative Tracker

In the pinning control or cooperative tracker problem, 8,9 it is desired for all the agents to reach the state x_0 of a leader or control node 0, that is $\|x_i(t)-x_0(t)\|\to 0$ as $t\to\infty,\ \forall i$. It is assumed that $\dot{x}_0 = Ax_0$.

Consider the optimal cooperative tracker problem for the system (108) and define the local neighborhood tracking error as

$$\eta_i = \sum_j e_{ij}(x_i - x_j) + g_i(x_i - x_0) \tag{118}$$

where $\eta_i \in \mathbb{R}^n$. The pinning gains $g_i \ge 0$ are nonzero only for a few nodes directly connected to a leader node. The overall local neighborhood tracking error is equal to $e=(L+G)\otimes I_n\delta$, where the global disagreement error is $\delta = x - \underline{1}\otimes x_0$. Then a distributed control that guarantees synchronization is given as $u_i = -cK_2\eta_i$, for an appropriate coupling gain $c>0$ and local feedback matrix K_2. In global form this is

$$u = -c(L+G)\otimes K_2\delta, \tag{119}$$

with $G = diag(g_1 ... g_N)$ the diagonal matrix of pinning gains. This gives the global dynamics

$$\dot{x} = I_N \otimes Ax - c(L+G)\otimes BK_2\delta. \tag{120}$$

To achieve synchronization the global disagreement system

$$\dot{\delta} = (I_N \otimes A)\delta + (I_N \otimes B)u, \tag{121}$$

or, with control $u = -c(L+G)\otimes K_2\delta$,

$$\dot{\delta} = (I_N \otimes A - c(L+G)\otimes BK_2)\delta, \tag{122}$$

must be asymptotically stabilized to the origin. This means that partial stability notions need not be used, and the sought solution P of the Algebraic Riccati Equation (93) must be positive definite. Stability of (122) to the origin is equivalent to asymptotic reference tracking and synchronization. This disagreement system can be stabilized using the distributed control law (119), 13.

The next result shows when the distributed control (119) is globally optimal on the graph. Conclusions of Lemma 4.2a allow for the following result stated as a theorem.

Theorem 4.5. Let the error dynamics be given as (122), and conditions of Lemma 4.4 be satisfied. Suppose there exist a positive definite matrix $P_1 = P_1^T > 0$, and a positive definite matrix $P_2 = P_2^T > 0$ satisfying

$$P_1 = cR_1(L+G), \tag{123}$$

$$A^T P_2 + P_2 A + Q_2 - P_2 B R_2^{-1} B^T P_2 = 0, \tag{124}$$

for some $Q_2 = Q_2^T > 0, R_1 = R_1^T > 0, R_2 = R_2^T > 0$ and a coupling gain $c > 0$. Define the feedback gain matrix K_2 as

$$K_2 = R_2^{-1} B^T P_2. \tag{125}$$

Then the control $u = -c(L+G)\otimes K_2\delta$ is optimal with respect to the performance index

$$J(\delta_0,u) = \int_0^\infty \delta^T \left[c^2((L+G)\otimes K_2)^T (R_1\otimes R_2)((L+G)\otimes K_2) - cR_1(L+G)\otimes(A^T P_2 + P_2 A) \right]\delta + u^T (R_1 \otimes R_2) u\,dt, \tag{126}$$

and is stabilizing to the origin of (121) for sufficiently high coupling gain c satisfying (127).

Proof: Form the matrix $P = P_1 \otimes P_2$. Since P_1 and P_2 are nonsingular so is P. The Lyapunov function $V(\delta) = \delta^T P \delta > 0$, by the arguments similar to those presented in the proof of Theorem 4.3. This Lyapunov function satisfies the conditions of Theorem 4.1, specialized to a single equilibrium point. Also,

$$\begin{aligned} L_1(\delta) &= \delta^T Q \delta \\ &= \delta^T [c^2((L+G)\otimes K_2)^T (R_1 \otimes R_2)((L+G)\otimes K_2) + cR_1(L+G)\otimes(A^T P_2 + P_2 A)]\delta \end{aligned}$$

is positive definite and satisfies the condition of Theorem 4.1 if the value of the coupling gain is taken as

$$c > \frac{\sigma_{\max}(R_1(L+G)\otimes(Q_2 - K_2^T R_2 K_2))}{\sigma_{\min}((L+G)^T R_1(L+G)\otimes K_2^T R_2 K_2)}. \tag{127}$$

Since

$$\begin{aligned}Q &= c^2((L+G)\otimes K_2)^T(R_1\otimes R_2)((L+G)\otimes K_2) - cR_1(L+G)\otimes(A^T P_2 + P_2 A)\\ &= c^2(L+G)^T R_1(L+G)\otimes K_2^T R_2 K_2 + cR_1(L+G)\otimes(Q_2 - K_2^T R_2 K_2)\end{aligned}$$

one has

$$c^2\sigma_{\min}((L+G)^T R_1(L+G)\otimes K_2^T R_2 K_2) - c\sigma_{\max}(R_1(L+G)\otimes(Q_2 - K_2^T R_2 K_2)) > 0 \Rightarrow Q > 0,$$

which gives the lower bound on the coupling gain c. The Algebraic Riccati equation (93) for system (121),

$$(I_N\otimes A)^T P + P(I_N\otimes A) + Q - P(I_N\otimes B)R^{-1}(I_N\otimes B)^T P = 0,$$

written as

$$\begin{aligned}&P_1\otimes A^T P_2 + P_1\otimes P_2 A + Q - (P_1\otimes P_2 B)R_1^{-1}\otimes R_2^{-1}(P_1\otimes B^T P_2) = 0\\ &P_1\otimes(A^T P_2 + P_2 A) + Q - P_1 R_1^{-1} P_1\otimes(P_2 B R_2^{-1} B^T P_2) = 0,\end{aligned}$$

is satisfied by the choice of $P = P_1\otimes P_2$ since

$$\begin{aligned}Q &= c^2((L+G)\otimes K_2)^T(R_1\otimes R_2)((L+G)\otimes K_2) - cR_1(L+G)\otimes(A^T P_2 + P_2 A)\\ &= c^2(L+G)^T R_1(L+G)\otimes K_2^T R_2 K_2 + cR_1(L+G)\otimes(Q_2 - P_2 B R_2^{-1} B^T P_2)\\ &= P_1 R_1^{-1} P_1\otimes P_2 B R_2^{-1} B^T P_2 + P_1\otimes(Q_2 - P_2 B R_2^{-1} B^T P_2)\\ &= Q_1\otimes P_2 B R_2^{-1} B^T P_2 + P_1\otimes(Q_2 - P_2 B R_2^{-1} B^T P_2)\end{aligned}$$

where $Q_1 = c^2(L+G)^T R_1(L+G) = P_1 R_1^{-1} P_1$ is introduced for notational simplicity. It follows by the conditions (123), (124) of the theorem that

$$P_1\otimes(A^T P_2 + P_2 A^T + Q_2 - P_2 B R_2^{-1} B^T P_2) + (Q_1 - P_1 R_1^{-1} P_1)\otimes(P_2 B R_2^{-1} B^T P_2) = 0$$

By conditions (123), (124) and (125), the control u satisfies

$$u = -c(L+G)\otimes K_2\delta = -R^{-1}(I_N \otimes B)^T P\delta$$
$$= -(R_1^{-1}\otimes R_2^{-1})(I_N \otimes B^T)(P_1 \otimes P_2)\delta = -R_1^{-1}P_1 \otimes R_2^{-1}B^T P_2\delta.$$

Therefore, all the conditions of the Theorem 4.1 are satisfied, completing the proof. ■

Note that the time derivative of the Lyapunov function $V(\delta) = \delta^T P\delta$ equals

$$\dot{V}(\delta) = 2\delta^T P\dot{\delta} = 2\delta^T P(I_N \otimes A - c(L+G)\otimes BK_2)\delta$$
$$= \delta^T(P_1 \otimes (P_2 A + A^T P_2) - 2P(I_N \otimes B)(R_1^{-1}\otimes R_2^{-1})(I_N \otimes B^T)P)\delta$$
$$= -\delta^T(Q + c^2((L+G)\otimes K_2)^T(R_1 \otimes R_2)((L+G)\otimes K_2))\delta \le -\delta^T Q\delta < 0,$$

implying asymptotic stability of (122) to the origin.

Remark 4.6: The graph topology constraint (123) is similar to the constraint (107), and comments similar to those stated there apply here as well. This constraint is further discussed in Section 4.6. There always exists a $P_2 = P_2^T > 0$ satisfying (124) if (A, B) is stabilizable and $(A, \sqrt{Q_2})$ observable. The value of the coupling gain *c* must be sufficiently great to overpower the, generally, indefinite terms stemming from the drift dynamics Ax_i, as in condition (127).

4.6 Constraints on Graph Topology

In this section we introduce a new class of digraphs which, to our knowledge, has not yet appeared in the cooperative control literature. This class of digraphs admits a distributed solution to an appropriately defined global optimal control problem.

The essential conditions for global optimality of the distributed control (97) or (103) for the cooperative regulator and the cooperative tracker, respectively, are respectively (100) and (107). Both of these have the same form involving either the Laplacian matrix, L, or the pinned graph Laplacian, $L+G$.

This section investigates classes of graphs that satisfy those conditions. Generally one can express these conditions as

$$RL = P, \tag{128}$$

where $R = R^T > 0$, L is a singular (*i.e.* the graph Laplacian) or nonsingular (*i.e.* the pinned graph Laplacian) *M*-matrix, and $P = P^T \geq 0$, a singular or nonsingular positive definite matrix. Equivalently

$$RL = L^T R \geq 0 \tag{129}$$

For the following classes of graph topologies one can satisfy this condition.

4.6.1 Undirected Graphs

Given that the graph is undirected, then L is symmetric, *i.e.* $L = L^T$ so the condition (128) becomes a commutativity requirement

$$RL = L^T R = LR. \tag{130}$$

See Remark 4.3, where it is shown that the choice $R = I$ always satisfies this condition for undirected graphs. Then $P = L$.

More generally, condition (130) is satisfied by symmetric matrices R and L if and only if R and L have all eigenvectors in common. Since L is symmetric it has a basis of orthogonal eigenvectors, and one can construct R satisfying (130) as follows. Let T be an orthogonal matrix whose columns are eigenvectors of L, then $L = T\Lambda T^T$ with Λ a diagonal matrix of real eigenvalues. Then for any positive definite diagonal matrix Θ one has that $R = T\Theta T^T > 0$ commutes with L and satisfies the commutativity requirement (130). Note that the R so constructed depends on all the eigenvectors of the Laplacian L in (128) (*i.e.* the graph Laplacian or the pinned Laplacian as appropriate).

4.6.2 Detailed Balanced Graphs

Given a detail balanced graph (*cf.* Section 2.2), its Laplacian matrix is symmetrizble, that is there exists a positive diagonal matrix $\Lambda > 0$ such that $L = \Lambda P$ where P a symmetric graph Laplacian matrix. Then, condition (128) holds with $R = \Lambda^{-1}$. Recall that for detailed balanced graphs, the diagonal elements of R in the leaderless case are then the elements of the left eigenvector of the Laplacian for the eigenvalue of zero.

4.6.3 Directed Graphs with Simple Laplacian

In this section we introduce a new class of digraphs which, to our knowledge, has not yet appeared in the cooperative control literature. This class of digraphs admits a distributed solution to an appropriately defined global optimal control problem.

Given a directed graph, let it be such that Laplacian matrix L (either the graph Laplacian or the pinned graph Laplacian) in (128) is simple, *i.e.* there exists a basis of eigenvectors so that its Jordan form is diagonal. Then the Laplacian matrix is diagonalizable, so there exists an invertible matrix T such that $TLT^{-1} = \Lambda$ is diagonal. Then one has that

$$TLT^{-1} = \Lambda = \Lambda^T = T^{-T} L^T T^T ,$$

whence it follows that

$$T^T TL = L^T T^T T .$$

Therefore, $R = T^T T = R^T > 0$ satisfies the condition (129), 45. Note that this R depends on all the eigenvectors of the Laplacian L in (128). This discussion motivates the following theorem.

Theorem 4.6. Let L be a Laplacian matrix (generally not symmetric). Then there exists a positive definite symmetric matrix $R = R^T > 0$ such that $RL = P$ is a symmetric positive semidefinite matrix if and only if L is simple, *i.e*. there exists a basis of eigenvectors of L.

Proof: (*i*) Let L be simple. Then it is diagonalizable, *i.e.* there exists a transformation matrix T such that $TLT^{-1} = \Lambda$, where Λ is a diagonal matrix of eigenvalues of L. Then

$$TLT^{-1} = \Lambda = \Lambda^T = T^{-T} L^T T^T ,$$

implying $T^T TLT^{-1} T^{-T} = L^T$, which implies that $(T^T T)L = L^T (T^T T)$. Let $R = T^T T$. Obviously, $R = R^T > 0$ and $P = RL = T^T TL = T^T \Lambda T \geq 0$ since $\Lambda \geq 0$ ($\forall x \;\; 0 \leq x^T P x = x^T T^T \Lambda T x = y^T \Lambda y \;\; \forall y$), 45.

(*ii*) Let L be a Laplacian matrix. Suppose there exists $R = R^T > 0$ satisfying the condition $RL = P$ is a symmetric positive semidefinite matrix. Then one needs to show that L is simple. To show this, we will prove the contrapositive by contradiction. So we suppose the negation of the contrapositive. That is, suppose L is not simple but that there exists $R = R^T > 0$ satisfying the condition $RL = P$ is a symmetric positive semidefinite matrix.

Since L is not simple, there exists a coordinate transformation bringing L to a Jordan canonical form

$$T^{-1}LT = J ,$$

with nonzero superdiagonal (otherwise L would be simple). Then one has $RL = RTJT^{-1} = P = P^T = T^{-T}J^T T^T R$. But then $(T^T RT)J = J^T (T^T RT)$. Therefore there exists $R_2 = T^T RT = R_2^T > 0$ such that $R_2 J = J^T R_2$. Without loss of generality let us assume that the first Jordan block is not simple, and with a slight abuse of notation $R_{2,11}$ will refer to the corresponding block in R_2. Then one has that $R_{2,11}(\lambda I + E) = (\lambda I + E^T)R_{2,11} \Rightarrow R_{2,11}E = E^T R_{2,11}$, where E is a nilpotent matrix having ones on the superdiagonal. This identity means that the first row and first column of $R_{2,11}$ are zero, except for the last entry. However then R_2 cannot be positive definite, since there are vanishing principal minors (Sylvester's test). ■

Since the simplicity of the Laplacian matrix is a necessary and sufficient condition for the topology constraint (128) to be satisfied, one can state the following theorem relating the graph Laplacian matrix (unpinned or pinned) to the optimality of decentralized cooperative control as given in previous sections of this chapter.

Theorem 4.7. The distributed control in cooperative regulator and tracker problems for single integrator agent dynamics (95) is optimal under optimality criterion

$$J(e_0,u)=\int_0^\infty (e^T Re+u^T Ru)dt\,,$$

for some $R>0$, with pertaining error functions, if and only if the Laplacian matrix in (128) is simple.

Proof: If the control law is optimal then the constraint (128) on P,R,L is satisfied, hence, by Theorem 4.6, L is simple. If, conversely L is simple, then there is a quadratic performance criterion with respect to which the cooperative distributed control given by L is inversely optimal. ■

Remark 4.7: These results place conditions on the graph topology that guarantee that the cooperative stabilizing distributed control is optimal with respect to a structured performance criterion, which in turn guarantees asymptotic consensus or synchronization. The following section will allow for a more general form of performance criterion which yields optimal control for general directed graphs.

4.7 Conclusion

This chapter presents inverse optimality of consensus and pinning control algorithms in cases of single-integrator and general linear time-invariant agent dynamics. A partial stability motivated result on stability of noncompact invariant manifolds is used in optimality and inverse optimality conditions to guarantee optimality of cooperative control achieving asymptotic consensus or synchronization under a positive semidefinite performance integrand. The optimality requirement imposes constraint between communication graph topology and the structure of performance integrand.

CHAPTER 5

OPTIMAL COOPERATIVE CONTROL FOR GENERAL DIGRAPHS: PERFORMANCE INDEX WITH CROSS-WEIGHTING TERMS

5.1 Introduction

When dealing with general directed graphs the constraint (100) on the graph Laplacian L, or (107) on the pinned graph Laplacian $L+G$, is too restrictive. These constraints only hold for a special class of digraphs whose Laplacian matrix (or pinned Laplacian $L+G$) has diagonal Jordan form. This constraint can be relaxed, and optimal cooperative controllers developed for arbitrary digraphs, by allowing state-control cross-weighting terms in the performance criterion, 44. Then, requiring that the performance criterion be positive (semi) definite leads to conditions on kernel matrix P in $V(x)=x^T Px$, or equivalently on the control Lyapunov function, which should be satisfied for the existence of distributed globally optimal control law. This condition is milder than the conditions (100), (107) where no cross-weighting term is allowed in the performance index.

In the following subsections we treat the optimal cooperative regulator and tracker for single-integrator dynamics, including a state-control cross weighting term in the performance index. In Section 5.4. we discuss the resulting constraint conditions on the graph topology that must be satisfied by the graph Laplacian for existence of an optimal controller of distributed form. It is shown that, unlike conditions (100) and (107), these new conditions can be satisfied for arbitrary directed graphs, containing a spanning tree, by proper selection of the performance index weighting matrices.

Sections 5.5 and 5.6 treat the optimal cooperative regulator and tracker for linear time-invariant agent dynamics. Again, it is shown that, if cross-weighting terms are allowed in the performance index, then the conditions required for existence of a distributed optimal controller can be satisfied on arbitrary digraphs. Conclusion is provided in section 5.7.

5.2 Optimal Cooperative Regulator- Single-integrator Agent Dynamics

This section considers the cooperative regulator problem for single-integrator agent dynamics $\dot{x}_i = u_i$ or in global form (95). Since the consensus manifold is noncompact, it is necessary to use the partial stability results in Theorem 4.1

Using the distributed control protocol $u = -Lx$ gives the closed-loop system

$$\dot{x} = -Lx .$$

The conclusions of Lemma 4.2b allow for the following result.

Theorem 5.1. Let the system be given as (95). Then for some $R = R^T > 0$ the control $u = \phi(x) = -Lx$ is optimal with respect to the performance index $J(x_0, u) = \int_0^\infty \mathcal{L}(x,u)dt$ with the performance integrand

$$\mathcal{L}(x,u) = x^T L^T RLx + u^T Ru + 2x^T (L^T R - P)u \geq 0 ,$$

and is stabilizing to a manifold which is the null space of of L if there exists a positive semidefinite matrix $P = P^T \geq 0$, having the same kernel as L, satisfying the inequality condition

$$L^T P + PL - PR^{-1}P \geq 0 . \tag{131}$$

Proof: Let $V(x) = x^T Px$ be a partial stability Lyapunov function. Then

$$\dot{V}(x) = -2x^T PLx = -x^T (PL + L^T P)x \leq -x^T PR^{-1}Px \leq \sigma_{>0\min}(PR^{-1}P)\|y\|^2$$

where $x = x_0 + y$, $y \in \ker P^{\perp}$. Thus the control is stabilizing to the null space of P. Also the performance integrand equals

$$\mathcal{L}(x,u) = \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} L^T RL & L^T R - P \\ RL - P & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} .$$

One has $\mathcal{L}(x,u) \ge 0$ if $R > 0$ and the Schur complement inequality (131) holds. The control is optimal since

$$
\begin{aligned}
u = \phi(x) &= -\frac{1}{2} R^{-1} (L_2^T + g^T \nabla V) \\
&= -\frac{1}{2} R^{-1} (2(RL - P)x + 2Px) = -\frac{1}{2} R^{-1} (2RL)x \\
&= -Lx
\end{aligned}
$$

The performance integrand evaluated at the optimal control satisfies

$$
\begin{aligned}
\mathcal{L}(x,\phi(x)) &= x^T L^T RLx + x^T LRLx - 2x^T (L^T R - P)Lx \\
&= 2x^T L^T RLx - 2x^T L^T RLx + 2x^T PLx \\
&= x^T (PL + L^T P)x \ge x^T PR^{-1} Px \\
&\ge \sigma_{\min} (PR^{-1}P) \left\| y \right\|^2
\end{aligned}
$$

Hence, all the conditions of Lemma 4.2b of Ch. 4 are satisfied, which concludes the proof. ■

If the graph has a spanning tree, then by Lemma 4.3, zero is a simple eigenvalue of L and thus the null space of L is the consensus manifold $span(\underline{1})$. Then, consensus is reached by all agents.

Note that if the constraint $P = RL$ in (100) is met as in previous sections, the cross-weighting term in $\mathcal{L}(x,u)$ in the proof vanishes, so that and the performance criterion is quadratic and therefore $\mathcal{L}(x,u) \ge 0$. That is, substituting $P = RL$ into (131) shows that (131) is satisfied.

Remark 5.1: The kernels of P and L in (131) need not be the same, but under condition (131) their relation is given by the following result.

Proposition 5.1. Let there exist a symmetric positive semidefinite P satisfying (131) for some L, and $R > 0$ then $\ker L \subseteq \ker P$.

The proof of this proposition uses a result on symmetric positive semidefinite matrices summarized in the following lemma, which is not difficult to prove.

Lemma 5.1. Given a positive semidefinite symmetric matrix $Q \geq 0$ the kernel of Q equals the set where the pertaining quadratic form satisfies $x^T Qx = 0$.

Note that this result, albeit straightforward, is not trivial since a nonsymmetric matrix Q generally does not share this property.

Now, to prove the proposition, notice that under (131) $L^T P + PL - PR^{-1}P$ is a symmetric positive semidefinite matrix. Consider the quadratic form

$$x^T (L^T P + PL - PR^{-1}P)x \geq 0.$$

Assume $x \in \ker L$ but $x \notin \ker P$, then for such an x

$$\underbrace{x^T L^T Px + x^T PLx}_{=0} - x^T PR^{-1}Px \geq 0 \Rightarrow -x^T PR^{-1}Px = -(Px)^T R^{-1}(Px) \geq 0.$$

However, since $R > 0$ this forces $Px = 0$, whence it follows $x \in \ker P$, which is a contradiction. Therefore, one has $x \in \ker L \Rightarrow x \in \ker P$ meaning $\ker L \subseteq \ker P$, which proves the proposition. ■

More discussion about the condition (131) is provided in Section 5.4.

5.3 Optimal Cooperative Tracker- Single-integrator Agent Dynamics

In case of the optimal cooperative tracker for the single-integrator agent dynamics system (95), one has the global disagreement error dynamics (105), which must be stabilized to the origin. This means that partial stability notions need not be used, and the sought matrix P must be positive definite. Therefore, for the optimal cooperative tracker problem the applied logic is the same, the only difference being that positive semidefiniteness is replaced by positive definiteness in the performance index and Lyapunov function.

Theorem 5.2. Let the error dynamics be given as (105), and the conditions of Lemma 4.4 be satisfied. Then for some $R=R^T>0$ the control $u=\phi(\delta)=-(L+G)\delta$ is optimal with respect to the performance index

$$J(\delta_0,u)=\int_0^\infty \mathcal{L}(\delta,u)dt$$

with the performance integrand

$$\mathcal{L}(\delta,u)=\delta^T(L+G)^T R(L+G)\delta+u^T Ru+2\delta^T((L+G)^T R-P)u>0,$$

and is stabilizing the origin for (105) if there exists a positive definite matrix $P=P^T\geq 0$, satisfying the following inequality

$$(L+G)^T P+P(L+G)-PR^{-1}P>0 \tag{132}$$

Proof: Let $V(\delta)=\delta^T P\delta$ be a Lyapunov function. Then

$$\dot{V}(\delta)=-2\delta^T P(L+G)\delta=-\delta^T(P(L+G)+(L+G)^T P)\delta<-\delta^T PR^{-1}P\delta<0.$$

Thus the control is stabilizing to the origin. Also the performance integrand equals

$$\mathcal{L}(\delta,u)=\begin{bmatrix}\delta^T & u^T\end{bmatrix}\begin{bmatrix}(L+G)^T R(L+G) & (L+G)^T R-P\\ R(L+G)-P & R\end{bmatrix}\begin{bmatrix}\delta\\ u\end{bmatrix}>0,$$

and $\mathcal{L}(x,u)>0$ if $R>0$ and inequality (132) holds (by Schur complement). The control is optimal since

$$\begin{aligned}u&=\phi(\delta)=-\frac{1}{2}R^{-1}(L_2^T+g^T\nabla V)\\&=-\frac{1}{2}R^{-1}(2(R(L+G)-P)\delta+2P\delta)=-\frac{1}{2}R^{-1}(2R(L+G))\delta\\&=-(L+G)\delta.\end{aligned}$$

The performance integrand evaluated at the optimal control satisfies

$$\mathcal{L}(\delta,\phi(\delta)) = \delta^T (L+G)^T R(L+G)\delta + \delta^T (L+G)^T R(L+G)\delta - 2\delta^T ((L+G)^T R - P)(L+G)\delta$$
$$= 2x\delta^T (L+G)^T R(L+G)\delta - 2\delta^T (L+G)^T R(L+G)\delta + 2\delta^T P(L+G)\delta$$
$$= \delta^T (P(L+G) + (L+G)^T P)\delta > \delta^T PR^{-1}P\delta > 0.$$

Hence all the conditions of Lemma 4.2b of Ch. 4 are satisfied, which concludes the proof. ■

Note that if the constraint $P = R(L+G)$ in (107) is met as in previous sections, the cross-weighting term in $\mathcal{L}(\delta,u)$ in the proof vanishes, so that the performance criterion is quadratic and therefore $\mathcal{L}(x,u) > 0$. That is, $P = R(L+G)$ guarantees that (132) is satisfied. More discussion about the condition (132) is provided in Section 5.4.

5.4 Condition for Existence of Distributed Optimal Control With Cross-Weighting Terms in the Performance Index

Conditions (131) and (132), which allow state-control cross-weighting terms in the performance index, express joint constraints on the graph matrices L, or $L+G$, and the choice of matrix R that are less strict than the conditions (100) and (107), respectively, that guarantee the existence of an optimal controller of distributed form when no cross-weighting term is allowed. In particular, note that if condition (100) or (107) hold then, respectively, (131) and (132) hold as well, and the cross-weighting terms equal zero. The additional freedom stemming from allowing the presence of cross-weighting terms in the performance integrand allows for more general graph topologies excluded under the Riccati conditions (100) or (107) to be inverse optimal.

Conditions (100) and (107) are only satisfied by digraphs that have a simple Jordan form for L or $L+G$, respectively. This includes undirected graphs and the detail balanced digraphs. It is now shown that conditions (131) and (132) can be satisfied for arbitrary digraphs.

For arbitrary digraphs, a matrix P having the same kernel as L and satisfying the optimal cooperative regulator condition (131) can be constructed as follows. To find

$P = P^T \geq 0$ such that $L^T P + PL - PR^{-1}P \geq 0$ and $\ker P = \ker L$ one can solve the equivalent inequality

$$-L^T P - PL \leq -PR^{-1}P ,$$

by solving an intermediate Lyapunov equation

$$-L^T P - PL = -Q \leq -PR^{-1}P ,$$

with $Q = Q^T \geq 0$, $\ker Q = \ker L$. The solution P to this Lyapunov equation exists since $-L$ is stabilizing to its nullspace, and equals

$$P = \int_0^\infty e^{-L^T \tau} Q e^{-L\tau} d\tau \geq 0 .$$

That $P = P^T$ follows from the construction, and if L has a simple zero eigenvalue, which is necessary for consensus, $\ker P = \ker Q = \ker L$. Since P is bounded for any fixed Q, and the solution P does not depend on R, choosing R big enough, in the sense of any matrix norm, will make $PR^{-1}P \leq Q$. And such P satisfies condition (131).

A similar method can be used to construct, for any digraph, the required matrix P for the optimal tracker condition (132).

5.5 General Linear Time-invariant Systems- Cooperative Regulator

This section deals with agents having states $x_i \in \mathbb{R}^n$ with drift dynamics $\dot{x}_i = Ax_i + Bu_i \in \mathbb{R}^n$, or in global form (108). Since the synchronization manifold, null space of $L \otimes I_n$, is noncompact, the partial stability Theorem 4.1 must be used.

Theorem 5.3. Let the system be given as (108). Define the local feedback matrix to be K_2 such that the cooperative feedback control $u = -cL \otimes K_2 x$, with a scalar coupling gain $c > 0$, makes (108) asymptotically converge to the consensus manifold, *i.e.* achieve consensus. Then there exists a positive semidefinite matrix $P = P^T \geq 0$, satisfying

$$c^2 (L \otimes K_2)^T R(L \otimes K_2) - P(I_N \otimes A) - (I_N \otimes A^T)P \geq 0 \quad (133)$$

$$P(cL \otimes BK_2 - (I_N \otimes A)) + (cL \otimes BK_2 - (I_N \otimes A))^T P - P(I_N \otimes B)R^{-1}(I_N \otimes B^T)P \geq 0 \quad (134)$$

for some $R = R^T > 0$. And the control $u = -cL \otimes K_2 x$ is optimal with respect to the performance index $J(x_0, u) = \int_0^\infty \mathcal{L}(x,u)dt$ with the performance integrand

$$\mathcal{L}(x,u) =$$
$$x^T \left[c^2 (L \otimes K_2)^T R(L \otimes K_2) - P(I_N \otimes A) - (I_N \otimes A^T)P \right] x + u^T R u + 2x^T \left[c(L \otimes K_2)^T R - P(I_N \otimes B) \right] u \geq 0,$$

and is stabilizing to the null space of P for sufficiently high coupling gain c.

Proof: Since the cooperative feedback is assumed to make (108) synchronize the closed loop system matrix $A_{cl} = I_N \otimes A - cL \otimes BK_2$ defines a partially stable system with respect to the consensus manifold. This manifold equals the kernel of $L \otimes I_n$ if L contains a spanning tree. Then for any positive semidefinite matrix $Q = Q^T \geq 0$, having the kernel equal to the consensus manifold, there exists a solution $P = P^T \geq 0$ to the Lyapunov equation

$$PA_{cl} + A_{cl}^T P = -Q$$

$$P = \int_0^\infty e^{A_{cl}^T \tau} Q e^{A_{cl} \tau} d\tau \geq 0$$

with kernel equal to the consensus manifold.

The inequality (133) is satisfied for sufficiently high values of the coupling gain $c > 0$, and inequality (134) is satisfied via Lyapunov equation by choosing $R = R^T > 0$ sufficiently big, in the sense of matrix norms, such that $0 \le P(I_N \otimes B)R^{-1}(I_N \otimes B^T)P \le Q$.

Let $V(x) = x^T Px$ be a partial stability Lyapunov function. Then

$$
\begin{aligned}
&\dot{V}(x) = 2x^T P\dot{x} = 2x^T P((I_N \otimes A)x - c(L \otimes BK_2)x) \\
&= x^T (P(I_N \otimes A) + (I_N \otimes A^T)P)x - cx^T P(L \otimes BK_2)x - cx^T (L \otimes BK_2)^T Px \\
&= x^T \left[P(I_N \otimes A - cL \otimes BK_2) + (I_N \otimes A - cL \otimes BK_2)^T P \right] x \\
&= x^T \left[PA_{cl} + A_{cl}^T P \right] x = -x^T Qx \\
&\le -x^T P(I_N \otimes B)R^{-1}(I_N \otimes B^T)Px \le 0
\end{aligned}
$$

The quadratic form $x^T Qx \ge 0$ serves as a measure of distance from the consensus manifold. Thus the control is stabilizing to the consensus manifold, as assumed. Also the performance integrand equals

$$
\mathcal{L}(x,u) =
$$
$$
\begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} c^2 (L \otimes K_2)^T R(L \otimes K_2) - P(I_N \otimes A) - (I_N \otimes A^T)P & c(L \otimes K_2)^T R - P(I_N \otimes B) \\ cR(L \otimes K_2) - (I_N \otimes B)P & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \ge 0
$$

and $\mathcal{L}(x,u) \ge 0$ if inequalities (133) and (134) hold (by Schur complement). The control is optimal since

$$
\begin{aligned}
&u = \phi(x) = -\frac{1}{2} R^{-1}(L_2^T + g^T \nabla V) \\
&= -\frac{1}{2} R^{-1}(2(cR(L \otimes K_2) - (I_N \otimes B^T)P)x + 2(I_N \otimes B^T)Px) \\
&= -\frac{1}{2} R^{-1}(2cR(L \otimes K_2))x = -c(L \otimes K_2)x
\end{aligned}
$$

The performance integrand evaluated at the optimal control satisfies

$$\mathcal{L}(x,\phi(x)) = x^T\left[c^2(L\otimes K_2)^T R(L\otimes K_2) - P(I_N\otimes A) - (I_N\otimes A^T)P\right]x$$
$$+x^T c^2(L\otimes K_2)^T R(L\otimes K_2)x - 2cx^T\left[c(L\otimes K_2)^T R - P(I_N\otimes B)\right](L\otimes K_2)x$$
$$= -x^T\left[P(I_N\otimes A) + (I_N\otimes A^T)P\right]x + 2cx^T P(I_N\otimes B)(L\otimes K_2)x$$
$$= -x^T\left[P(I_N\otimes A) + (I_N\otimes A^T)P\right]x + 2cx^T P(L\otimes BK_2)x$$
$$= x^T\left[(cL\otimes BK_2 - I_N\otimes A)^T P + P(cL\otimes BK_2 - I_N\otimes A)\right]x$$
$$= x^T\left[-A_{cl}^T P - PA_{cl}\right]x = x^T Qx \ge x^T P(I_N\otimes B)R^{-1}(I_N\otimes B^T)Px \ge 0$$

Hence all the conditions of Lemma 4.2b of Chapter 4 are satisfied, which concludes the proof. ■

Specifying the form of P and R as $P = P_1\otimes P_2$, $R = R_1\otimes R_2$ and assuming further that $K_2 = R_2^{-1}B^T P_2$, where P_2 is a solution of local Riccati equation $P_2A + A^T P_2 + Q_2 - P_2BR_2^{-1}B^T P_2 = 0$ as used in Theorems 4 and 5, 12,32 the inequality (134) becomes

$$cP_1L\otimes P_2BK_2 - P_1\otimes(P_2A + A^T P_2) + cL^T P_1\otimes K_2^T B^T P_2 - P_1R_1^{-1}P_1\otimes P_2BR_2^{-1}B^T P_2 \ge 0\,.$$

Equivalently, since

$$P_2BK_2 = K_2^T B^T P_2 = P_2BR_2^{-1}B^T P_2 = K_2^T R_2K_2 \ge 0\,,$$

one has

$$c(P_1L + L^T P_1 - P_1R_1^{-1}P_1)\otimes K^T R_2K - P_1\otimes(K^T R_2K - Q_2) \ge 0\,.$$

Note the similarity to condition (131) introduced in single integrator consensus problem Section 5.1. If condition (131) is satisfied then for sufficiently high value of the coupling gain *c* this constraint can be met.

Clearly if Riccati conditions (112), (113) are satisfied then the cross-weighting term vanishes

$$cR(L\otimes K_2) = (I_N\otimes B^T)P \Leftrightarrow c(L\otimes K_2) = R^{-1}(I_N\otimes B^T)P\,.$$

One should also note that choosing $K_2 = R_2^{-1}B^T P_2$ affords an infinite interval of positive values for coupling gain c that achieve synchronization, 32, which allows one to find a sufficiently high value of c to satisfy inequality (133) without worrying about loosing synchronization. The same applies whenever one needs to choose c to be, for any purpose, sufficiently big.

If the communication graph is connected, then by Lemma 4.3 the null space of $L \otimes I_n$ equals the synchronization manifold $S := span(\underline{1} \otimes \alpha)$, $\alpha \in \mathbb{R}^n$, therefore synchronization is asymptotically achieved in the optimal way.

5.6 General Linear Time-invariant Systems- Cooperative Tracker

If the goal is to synchronize dynamics (108) to the leader's trajectory $\dot{x}_0 = Ax_0$, *i.e.* solve the cooperative tracking problem, then one should use the global error system $\delta = x - \underline{1} \otimes x_0$, with the same global error dynamics

$$\dot{\delta} = (I_N \otimes A)\delta + (I_N \otimes B)u \ . \tag{135}$$

Theorem 5.4. Let the system be given as (135). Define the local feedback matrix to be K_2 such that the cooperative feedback control $u = -c(L+G) \otimes K_2\delta$, with a scalar coupling gain $c > 0$, makes (135) asymptotically converge to the origin. Then there exists a positive definite matrix $P = P^T > 0$, satisfying

$$c^2((L+G)\otimes K_2)^T R((L+G)\otimes K_2) - P(I_N \otimes A) - (I_N \otimes A^T)P > 0 \tag{136}$$

$$P(c(L+G)\otimes BK_2 - (I_N \otimes A)) + (c(L+G)\otimes BK_2 - (I_N \otimes A))^T P - P(I_N \otimes B)R^{-1}(I_N \otimes B^T)P > 0 \tag{137}$$

for some $R = R^T > 0$. And the control $u = -c(L+G) \otimes K_2\delta$ is optimal with respect to the performance index $J(\delta_0, u) = \int_0^\infty \mathcal{L}(\delta, u)dt$ with the performance integrand

$$\mathcal{L}(\delta,u)=\delta^T\left[c^2((L+G)\otimes K_2)^T R((L+G)\otimes K_2)-P(I_N\otimes A)-(I_N\otimes A^T)P\right]\delta+u^T Ru$$
$$+2\delta^T\left[c((L+G)\otimes K_2)^T R-P(I_N\otimes B)\right]u>0$$

Proof: Since the cooperative feedback is assumed to stabilize (135) to the origin the closed loop system matrix $A_{cl}=I_N\otimes A-c(L+G)\otimes BK_2$ defines a stable system. Then for any positive definite matrix $Q=Q^T>0$, there exists a solution $P=P^T>0$ to the Lyapunov equation

$$PA_{cl}+A_{cl}^T P=-Q$$

$$P=\int_0^\infty e^{A_{cl}^T\tau}Qe^{A_{cl}\tau}d\tau>0$$

The inequality (136) is satisfied for sufficiently high values of the coupling gain $c>0$, and inequality (137) is satisfied via Lyapunov equation by choosing $R=R^T>0$ sufficiently big, in the sense of matrix norms, such that $0\le P(I_N\otimes B)R^{-1}(I_N\otimes B^T)P<Q$.

Let $V(\delta)=\delta^T P\delta>0$ be a Lyapunov function. Then

$$\dot{V}(\delta)=2\delta^T P\dot{\delta}=2\delta^T P((I_N\otimes A)\delta-c((L+G)\otimes BK_2)\delta)$$
$$=\delta^T(P(I_N\otimes A)+(I_N\otimes A^T)P)\delta-c\delta^T P((L+G)\otimes BK_2)\delta-c\delta^T((L+G)\otimes BK_2)^T P\delta$$
$$=\delta^T(P(I_N\otimes A-c(L+G)\otimes BK_2)+(I_N\otimes A-c(L+G)\otimes BK_2)^T P)\delta$$
$$=\delta^T(PA_{cl}+A_{cl}^T P)\delta=-\delta^T Q\delta<-\delta^T P(I_N\otimes B)R^{-1}(I_N\otimes B^T)P\delta\le 0.$$

Thus the control is stabilizing to the origin, as assumed. Also the performance integrand equals

$$\mathcal{L}(\delta,u)=\begin{bmatrix}\delta & u\end{bmatrix}^T\begin{bmatrix}c^2((L+G)\otimes K_2)^T R((L+G)\otimes K_2)-P(I_N\otimes A)-(I_N\otimes A^T)P & c((L+G)\otimes K_2)^T R-P(I_N\otimes B)\\ cR((L+G)\otimes K_2)-(I_N\otimes B^T)P & R\end{bmatrix}\begin{bmatrix}\delta\\ u\end{bmatrix}$$
$$>0$$

and $\mathcal{L}(\delta,u)>0$ if inequalities (136), (137) hold (by Schur complement). The control is optimal since

$$
\begin{aligned}
u=\phi(\delta)&=-\frac{1}{2}R^{-1}(L_2^T+g^T\nabla V)\\
&=-\frac{1}{2}R^{-1}(2(cR((L+G)\otimes K_2)-(I_N\otimes B^T)P)\delta+2(I_N\otimes B^T)P\delta)\\
&=-\frac{1}{2}R^{-1}(2cR((L+G)\otimes K_2))\delta\\
&=-c(L+G)\otimes K_2\delta
\end{aligned}
$$

The performance integrand evaluated at the optimal control satisfies

$$
\begin{aligned}
\mathcal{L}(\delta,\phi(\delta))=\delta^T\left[c^2((L+G)\otimes K_2)^T R((L+G)\otimes K_2)-P(I_N\otimes A)-(I_N\otimes A^T)P\right]\delta+\delta^T c^2((L+G)\otimes K_2)^T R((L+G)\otimes K_2)\delta\\
-2c\delta^T\left[c((L+G)\otimes K_2)^T R-P(I_N\otimes B)\right]((L+G)\otimes K_2)\delta\\
=-\delta^T\left[P(I_N\otimes A)+(I_N\otimes A^T)P\right]\delta+2c\delta^T P(I_N\otimes B)((L+G)\otimes K_2)\delta\\
=-\delta^T\left[P(I_N\otimes A)+(I_N\otimes A^T)P\right]\delta+2c\delta^T P((L+G)\otimes BK_2)\delta\\
=\delta^T\left[(c(L+G)\otimes BK_2-I_N\otimes A)^T P+P(c(L+G)\otimes BK_2-I_N\otimes A)\right]\delta\\
=\delta^T\left[-A_d^T P-PA_d\right]\delta=\delta^T Q\delta>\delta^T P(I_N\otimes B)R^{-1}(I_N\otimes B^T)P\delta\geq 0
\end{aligned}
$$

Hence all the conditions of Lemma 4.2b of Chapter 4 are satisfied, which concludes the proof. ■

Specifying the form of P and R as $P=P_1\otimes P_2$, $R=R_1\otimes R_2$, and assuming further that $K_2=R_2^{-1}B^T P_2$, where P_2 is a solution of the local Riccati equation $P_2A+A^TP_2+Q_2-P_2BR_2^{-1}B^TP_2=0$, as used in Theorems 4.4 and 4-5 of Chapter 4, the inequality (137) becomes

$$
cP_1(L+G)\otimes P_2BK_2-P_1\otimes(P_2A+A^TP_2)+c(L+G)^TP_1\otimes K_2^TB^TP_2-P_1R_1^{-1}P_1\otimes P_2BR_2^{-1}B^TP_2>0\,.
$$

Equivalently, since

$$
P_2BK_2=K_2^TB^TP_2=P_2BR_2^{-1}B^TP_2=K_2^TR_2K_2\geq 0\,,
$$

one has

$$
c(P_1(L+G)+(L+G)^TP_1-P_1R_1^{-1}P_1)\otimes K_2^TR_2K_2-P_1\otimes(K_2^TR_2K_2-Q_2)>0\,.
$$

Note the similarity to condition (132) introduced in single-integrator consensus problem Section 5.2. If condition (132) is satisfied then for sufficiently high value of the coupling gain *c* this constraint can be met.

Remark 5.2: Due to the connection of this inequality and the one in the earlier section with inequalities for single-integrator systems (131) and (132) respectively, conclusions in Remark 5.1. on constructing the suitable P_1 apply here as well.

Clearly if Riccati conditions (123), (124) are satisfied then the cross-weighting term vanishes

$$cR((L+G)\otimes K_2) = (I_N \otimes B^T)P \Leftrightarrow c((L+G)\otimes K_2) = R^{-1}(I_N \otimes B^T)P \ .$$

One should also note that choosing $K_2 = R_2^{-1}B^T P_2$ affords an infinite interval of positive values for coupling gain c that achieve stabilization, 32, which allows one to find a sufficiently high value of c to satisfy inequality (136) without worrying about stability of the closed-loop system.

5.7 Conclusion

In order to lift the constraints on the graph topology detailed in Chapter 4 one can use a more general form of a performance index, such as one containing state-control cross-weighting terms. It is found that multi-agent systems with agents having considered dynamics on any directed graph, achieving consensus or synchronization, are inverse optimal with respect to this more general performance index. Inverse optimality gives favorable properties to a partially stabilizing control law such as robustness and guaranteed gain margins, 41.

CHAPTER 6

MULTI-AGENT SYSTEMS WITH DISTURBANCES

6.1 Introduction

The last two decades have witnessed an increasing interest in multi-agent network cooperative systems, inspired by natural occurrence of flocking and formation forming. These systems are applied to formations of spacecrafts, unmanned aerial vehicles, mobile robots, distributed sensor networks etc., 1. Early work with networked cooperative systems in continuous and discrete time is presented in 2,3,4,5,6,7. These papers generally refer to consensus without a leader. We call this the *cooperative regulator problem*. There the final state of consensus depends on initial conditions. By adding a leader that pins to a group of other agents one can obtain synchronization to a command trajectory using a virtual leader 5, also named pinning control 8, 9. We call this the *cooperative tracker problem*. In the cooperative tracker problem all the agents synchronize to the leader's reference trajectory. Necessary and sufficient conditions for synchronization are given by the master stability function, 10, and the related concept of the synchronizing region, in 9,10,11. For continuous-time systems, synchronization was guaranteed, 9,12,13 using local optimal state feedback derived from the continuous time algebraic Riccati equation. It was shown that using Riccati design of the feedback gain for each node guarantees an unbounded right-half plane synchronization region in the *s*-plane. This allows for synchronization under mild conditions on the directed communication topology.

Dual to the distributed synchronization control problem is the distributed estimation problem, 13,30. Distributed estimators are used when agents are only able to measure relative information in their local neighborhoods. Output measurements are assumed and cooperative disturbance observers can be designed for the multi-agent systems. Potential applications are distributed observation, distributed disturbance estimation, sensor fusion, and dynamic output regulators for synchronization. Conditions for cooperative observer convergence are shown in

13 to be related by a duality concept to distributed synchronization control conditions for systems on directed graphs.

This chapter is concerned with the effects of disturbances on multi-agent systems. Building on the classical results on the existence of Lyapunov functions for asymptotically stable systems and their use in assessing the effect the disturbances exert on those systems, 46, it is possible to extend such reasoning to partially stable systems, in particular those systems that reach consensus or synchronization. With Lyapunov functions for partial stability one is able to ascertain the effect of disturbances on the partial stability of systems, and to derive conditions on those disturbances that allow for asymptotic partial stability or uniform ultimate boundedness along the target set. Partial asymptotic stability is in that sense robust to this specific class of disturbances. Furthermore, with the means to quantify the effect of disturbances one also gains the ability to compensate for it by an appropriate control law. In order to construct such a compensating control disturbances need to be known. However, the fact that disturbances are usually inaccessible to direct measurement introduces the need for disturbance estimation.

The unified approach to disturbance estimation of the leader's and agents' disturbances in multi-agent systems is presented. Disturbance estimation and related compensation is used to guarantee asymptotic state synchronization of agents that are influenced by disturbances. The interaction graph is directed and assumed to contain a directed spanning tree. For the needs of consensus and synchronization to a leader or control node we employ pinning control. However, with disturbances present, the distributed feedback pinning control designed for disturbance free systems no longer guarantees synchronization. Robustness of the distributed synchronization control for the nominal system indeed guarantees asymptotic cooperative stability or cooperative ultimate uniform boundedness in presence of some disturbances, 46. This property can be exploited in special cases of heterogeneous and nonlinear agents. Nevertheless, in the general case of disturbances one needs to compensate for their effect in order to retain the qualitative behavior of the nominal system. For that purpose disturbance estimates are used. In this chapter disturbances are assumed to act on both the leader and the following agents. Therefore both the leader's and the agents' disturbances need to be estimated and compensated. Leader's disturbance estimate is obtained by all agents via a distributed estimator, while the local agents' disturbances can be estimated by local observers or, in a special instance, a distributed observer.

The structure of this chapter is the following; Section 6.2 presents stability definitions and properties in a topological way which is more general than usually presented in the literature, 41, under the name of partial stability. Converse Lyapunov result for partial stability is also given in this section for the sake of completeness since inverse Lyapunov functions are used in later sections. Coordinate transformations that preserve the introduced stability notions are also briefly addressed. Section 6.3 gives the means to quantitatively assess the effect disturbances have on multi-agent systems and also allows characterizing the classes of disturbances to which the multi-agent system under consideration is robust. Section 6.4 introduces the leader's and the agents' disturbance estimation schemes. The results of Section 6.4 are applied in Section 6.5 and Section 6.6. Section 6.5 is concerned with the case of a multi-agent system having a leader driven by an input. Section 6.6 applies the same results to the multi-agent system comprised of second-order double-integrator agents with agents' disturbances as well as the leader's input present. Section 6.7 and 6.8 discuss how the robustness property of the nominal system and disturbance compensation can be used to address cases of multi-agent systems with heterogeneous and nonlinear agents. The example of a Lienard system was investigated in Section 6.8 since those systems are known to have an attractive limit cycle, furnishing an example of a nontrivial invariant set for an isolated agent. Section 6.9 gives a numerical example based on the case presented in Section 6.6 that justifies the proposed observer schemes and control laws. Concluding remarks are given in Section 6.10.

6.2 Stability Definitions and Properties

This section introduces partial stability properties and basic results that are used later in the chapter. Definitions given here describe partial stability in a general coordinate free way, in terms of neighborhoods of a target set.

Definition 6.1: Let S be a manifold embedded in a Euclidean space X. A neighborhood $\mathcal{D}(S)$ of S is an open set in X containing the manifold S in its interior.

Definition 6.2: Let S be a manifold embedded in a metric space (X,d), and let $\mathcal{D}(S)$ be a neighborhood of S. An ε*-neighborhood* of $S\subset\mathcal{D}(S)$ is defined as $\mathcal{U}_\varepsilon(S)=\{x\in X \,|d(x,S)<\varepsilon\}$, where $d(x,S):=\inf_{y\in S} d(x,y)$ is the distance from $x\in X$ to S as given by the metric d.

Note that in the case of compact manifolds S, any neighborhood $\mathcal{D}(S)$ contains some ε-neighborhood of S, but in the case of noncompact manifolds this need not be true. For the needs of defining stability of noncompact manifolds S, one uses neighborhoods of S that contain some ε-neighborhood of S. We call such neighborhoods *regular*.

Definition 6.3: A manifold S is said to be (*Lyapunov*) *stable* if there exists a regular neighborhood $\mathcal{D}(S)$ of S, such that for every ε-neighborhood $\mathcal{U}_\varepsilon(S)$ contained in it, there exists a subneighborhood $\mathcal{V}(S)$ satisfying the property $x(0)\in\mathcal{V}(S)\Rightarrow x(t)\in\mathcal{U}_\varepsilon(S) \quad \forall t\geq 0$. If $\mathcal{D}(S)$ can be taken as the entire space X, then the stability is *global*.

If S is Lyapunov stable and furthermore there exists a neighborhood $\mathcal{W}(S)$ of S satisfying the property $x(0)\in\mathcal{W}(S)\Rightarrow d(x(t),S)\to 0$ as $t\to\infty$, then S is *asymptotically stable*. If $d(x(t),S)\to 0$ we say that the *trajectory* $x(t)$ *converges to* S.

If S is Lyapunov stable and for every ε-neighborhood $\mathcal{U}_\varepsilon(S)$ of S, there exists a subneighborhood $\mathcal{V}(S) \subseteq \mathcal{U}_\varepsilon(S)$ containing a δ-neighborhood $\mathcal{V}_\delta(S)$ then S is *uniformly stable*. In this case the stability conclusion can be phrased as $\forall\varepsilon > 0$, $\exists\delta > 0$ such that

$$d(x(0),S) < \delta \Rightarrow d(x(t),S) < \varepsilon \;\; \forall t \geq 0.$$

If a manifold is uniformly stable and asymptotically stable so that some neighborhood $\mathcal{W}(S)$ contains a δ-neighborhood $\mathcal{W}_\delta(S)$ satisfying the property $x(0) \in \mathcal{W}_\delta(S) \Rightarrow d(x(t),S) \to 0$ as $t \to \infty$, uniformly[2] on $\mathcal{W}_\delta(S)$, then S is *uniformly asymptotically stable*.

If a manifold is uniformly asymptotically stable and there exist constants $K, \sigma > 0$ such that $d(x(t),S) \leq Kd(x(0),S)e^{-\sigma t}$, for all $x(0)$ in some δ-neighborhood $\mathcal{W}_\delta(S)$ then the partial stability is *exponential*. ■

The following results address generally time-varying systems, where qualitative properties depend on initial time t_0; $x(t_0) = x_0$. The existence of a Lyapunov function in case of uniform-in-time[3] asymptotic stability of the origin is a familiar result guaranteed under mild assumptions on the closed loop dynamics

$$\dot{x} = f(t,x), \tag{138}$$

46, and for the case of uniform asymptotic stability of an invariant manifold S the existence follows from an analogous consideration, using the Massera's lemma (Lemma 6.2). To present the proof, one needs the following definitions and results. Results are presented in a slightly more general form than needed here; in particular, they are suited for time-varying systems.

---

[2] In this context, uniform convergence means $\forall\varepsilon > 0, \exists T > 0 \; s.t. \; d(x(t),S) < \varepsilon \; \forall t \geq T$, where T depends on ε and $d(x_0,S)$, but not on x_0 itself.

[3] Uniform-in-time means that stability and convergence do not depend on initial time t_0.

Proposition 6.1. Given a possibly noncompact invariant manifold S and the dynamical system (138), S is uniformly stable if and only if there exists a class $\mathcal{K}$ or $\mathcal{K}_\infty$ function α and $c>0$ such that

$$d(x(t),S) \le \alpha(d(x(0),S))$$

$\forall x(0)$ such that $d(x(0),S) < c$.

The manifold S is uniformly asymptotically stable if there exist a class $\mathcal{KL}$ function β and $c>0$ such that

$$d(x(t),S) \le \beta(d(x(t_0),S), t-t_0), \tag{139}$$

$\forall x(0)$ satisfying $d(x(0),S) < c$.

If the system is time-invariant one can simply write $d(x(t),S) \le \beta(d(x(0),S),t)$ picking initial time arbitrary.

Proof follows by analogy with results presented in 46, where the usual Euclidean norm, $\|\cdot\|$, is replaced with a more general distance, $d(\cdot,S)$. ■

The qualification of stability types in Proposition 6.1 refers to local stability. For global stability one needs $\mathcal{K}_\infty$ functions and the condition to hold $\forall c>0$.

Note that the use of class $\mathcal{KL}$ function guarantees uniformity of asymptotic stability in this case for different reason than presented in 46. There, uniformity of convergence with respect to the initial time was a consequence of the dependence of the $\mathcal{KL}$ class function on $t-t_0$, while here it is because of the dependence on $d(x,S)$ and $t-t_0$ both, guaranteeing uniformity of convergence to S with respect to initial time as well as uniformity of convergence along the manifold S, i.e. depending on $d(x_0,S)$, which is the one we are mainly interested here. The proof of the Proposition 6.1 follows by analogy to the reasoning presented in 46.

We are motivated to prove the following result concerning the existence of a Lyapunov function for uniformly asymptotically partially stable systems.

Theorem 6.1. (*Converse Lyapunov theorem*) Suppose the system $\dot{x} = f(t,x)$ has its solutions defined for all future times $t \geq t_0$, where t_0 is the initial time, and is uniformly asymptotically stable with respect to the manifold $S \subset \mathcal{U}(S)$. If the system satisfies the following technical condition

$$\left|\nabla_x d(x,S) f(t,x)\right| \leq L d(x,S)\,, \tag{140}$$

on the neighborhood $\mathcal{U}(S)$, then there exists a Lyapunov function defined there satisfying

$$\begin{aligned} &\alpha_1(d(x,S)) \leq V(t,x) \leq \alpha_2(d(x,S)),\\ &\dot{V}(t,x) \leq -\alpha_3(d(x,S)), \end{aligned}$$

$$\left\|\nabla V(t,x)\right\| \leq \alpha_4(d(x,S)),$$

for $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ class $\mathcal{K}$ functions. ■

The following lemma presents a useful technical result. It serves in comparison systems involving Lyapunov functions, as well as providing an elegant proof of sufficiency of the Lyapunov condition for partial stability types as described in Proposition 6.1.

Lemma 6.1. Given a dynamical system, $\dot{y} = -\alpha(y)$, with $y(t_0) = y_0$, where α is a locally Lipschitz class $\mathcal{K}$ function defined on $[0,a)$, there exists a class $\mathcal{KL}$ function σ defined on $[0,a)\times[0,\infty)$ such that $y(t) = \sigma(y_0, t-t_0)$.

Lemma 6.2. Massera's lemma, 46. Let $g:[0,\infty) \to (0,\infty)$, be a continuous strictly decreasing function with $g(t) \to 0$ as $t \to \infty$. Let $h:[0,\infty) \to (0,\infty)$, be a continuous nondecreasing function. Then there exists a function $G(t)$ such that

$G(t)$ and $G'(t)$ are class $\mathcal{K}$ functions defined on $[0,\infty)$.

For every continuous $u(t)$, such that $0 \le u(t) \le g(t)$, $\forall t \ge 0$, there exist positive constants, K_1, K_2, independent of u, such that $\int_0^\infty G(u(t))dt \le K_1$ and $\int_0^\infty G'(u(t))h(t)dt \le K_2$. ■

In the following lemma the implicit assumption is made that the manifold S is embedded in the Euclidean space, with its usual differential and vector space structure. The neighborhood $\mathcal{U}(S)$ under consideration is thus a subset of the Euclidean space inheriting all the differential structure.

Lemma 6.3. The gradient of the distance function $\nabla_x d(x,S)$ is uniformly bounded.

Proof: Assume one chooses two arbitrary points $x, y \in \mathcal{U}(S)$, and that the infimum distance from the manifold S is actually attained at points $x_0, y_0 \in S$ such that

$$d(x,S) := \inf_{z\in S} d(x,z) = d(x,x_0),$$
$$d(y,S) := \inf_{z\in S} d(y,z) = d(y,y_0).$$

Then $d(y,S) \le d(y,x_0) \le d(y,x) + d(x,x_0) = d(y,x) + d(x,S)$, wherefrom it follows that

$$d(y,S) - d(x,S) \le d(y,x).$$

Exchanging x and y one obtains $-d(x,y) \le d(x,S) - d(y,S) \le d(x,y)$, therefore

$$\left| d(x,S) - d(y,S) \right| \le d(x,y),$$

proving Lipschitz continuity of the function $d(\cdot,S)$. This also means that each directional derivative satisfies

$$\lim_{t\to 0} \left| \frac{d(x+tv,S) - d(x,S)}{\|tv\|} \right| \le \frac{d(x+tv,x)}{\|tv\|} = 1,$$

implying boundedness of the gradient, $\nabla_x d(x,S)$, on $\mathcal{U}(S)$. ■

Remark 6.1: Here the vector space structure of $\mathcal{U}(S)$ is implicitly used. Otherwise the term $x+tv$ makes no sense. Also, the assumption that the infimum distance is attained for some point on the manifold S can be made redundant if one invokes the property of infimum, that it can be approached arbitrarily closely. Then one can use an approximating point on the manifold S, introducing arbitrarily small deviations from the actual infimum. However, since the deviations are allowed to be arbitrarily small, the derived inequalities hold just as well.

Proof of Theorem 6.1.

Let us denote the solution to the dynamics $\dot{x} = f(t,x)$ with initial state x at time t_0 as $x(t) = \phi(t,t_0,x)$. Since $d(x(t),S) \leq \beta(d(x(t_0),S),t-t_0)$ one has $d(\phi(t,t_0,x),S) \leq \beta(d(x,S),t-t_0)$.

Let us make use of the following three technical conditions

The solution $x(t) = \phi(t,t_0,x)$ can be extended $\forall t \geq t_0$,

$\|\nabla_x d(x,S)\| < K$, $\forall x \in \mathcal{U}(S)$,

$|\nabla_x d(x,S) f(t,x)| \leq L d(x,S)$, $\forall x \in \mathcal{U}(S)$.

Since $\frac{d}{dt} d(x,S) = \nabla_x d(x,S) f(t,x)$ one has

$$\left| \frac{d}{dt} d(x,S) \right| = |\nabla_x d(x,S) f(t,x)| \leq L d(x,S),$$

$$\Rightarrow -L d(x,S) \leq \frac{d}{dt} d(x,S) \leq L d(x,S)$$

$$d(x(t_0),S) e^{-L(t-t_0)} \leq d(x,S) \leq d(x(t_0),S) e^{L(t-t_0)}$$

establishing a different bound on $d(x,S)$. Let the Lyapunov function be defined as

$$V(t,x) = \int_t^{t+T} G(d(\phi(\tau,t,x),S)) d\tau,$$

with function G whose existence is guaranteed by Massera's lemma. Then

$$\int_t^{t+T} G(d(x,S)e^{-L(\tau-t)})d\tau \le \int_t^{t+T} G(d(\phi(\tau,t,x),S))d\tau \le \int_t^{t+T} G(\beta(d(x,S),\tau-t)d\tau ,$$

$$\alpha_1(d(x,S)) := \int_t^{t+T} G(d(x,S)e^{-L(\tau-t)})d\tau = \int_0^T G(d(x,S)e^{-Ls})ds ,$$

$$\alpha_2(d(x,S)) := \int_t^{t+T} G(\beta(d(x,S),\tau-t))d\tau = \int_0^T G(\beta(d(x,S),s)ds .$$

Further, one makes use of the assumption that the orbit can be extended to all future times. For compact manifolds, local Lipschitz continuity on a compact neighborhood guarantees extendibility of solutions to all future times, but for neighborhoods of general noncompact manifolds this condition needs to be imposed since an escape to infinity in finite time along the noncompact manifold remains a possibility.

The time derivative of the Lyapunov function equals

$$\begin{aligned}&\frac{d}{dt}V(t,x) = \frac{\partial}{\partial t}V(t,x) + \nabla V(t,x) f(t,x)\\ &= G(d(\phi(t+T,t,x),S)) - G(d(x,S)) + \int_t^{t+T} G'(d(\phi(\tau,t,x),S)\nabla_\phi d(\phi,S)\left[\phi_t(\tau,t,x)+\phi_x(\tau,t,x)f(t,x)\right]d\tau.\end{aligned}$$

Owing to the identity $\phi_t(\tau,t,x)+\phi_x(\tau,t,x)f(t,x)\equiv 0,\ \forall\,\tau\ge t$, one has

$$\frac{d}{dt}V(t,x) = G(d(\phi(t+T,t,x),S) - G(d(x,S)) \le G(\beta(d(x,S),T)) - G(d(x,S)) .$$

Since, by assumption on the existence of solution for all future times, T can be chosen arbitrarily large, so that as $\beta(\cdot,T)\to 0$ when $T\to\infty$, one will have, by continuity, that $G\circ\beta(\cdot,T)\to 0$. Hence

$$\frac{d}{dt}V(t,x) \le -G(d(x,S)) := -\alpha_3(d(x,S)) .$$

One also has the bound on the gradient

$$\nabla_x V(t,x) = \int_t^{t+T} G'(d(\phi,S))\nabla_\phi d(\phi,S)\phi_x d\tau ,$$

which implies $\|\nabla_x V(t,x)\| \le \int_t^{t+T} G'(d(\phi,S))\|\nabla_\phi d(\phi,S)\|\|\phi_x\| d\tau$.

If $\|\nabla_x f(t,x)\| < M$, which is satisfied in particular by linear systems, then $\|\phi_x\|_2 < e^{M(\tau-t)}$, furnishing the bound

$$\begin{aligned}
&\|\nabla_x V(t,x)\| \le \int_t^{t+T} G'(d(\phi,S))\|\nabla_\phi d(\phi,S)\| e^{M(\tau-t)} d\tau \\
&\le \int_t^{t+T} G'(d(\phi,S))Ke^{M(t-\tau)} d\tau \\
&\le \int_t^{t+T} G'(\beta(d(x,S),\tau-t))Ke^{M(\tau-t)} d\tau \\
&\le \int_t^{t+T} G'(\beta(d(x,S),s))Ke^{Ms} ds := \alpha_4(d(x,S)).
\end{aligned}$$

The existence of a class $\mathcal{K}$ function α_4 is guaranteed by Massera's lemma (Lemma 6.2).

In case of time-invariant dynamics the solution satisfies $x(t) = \phi(t,t_0,x) = \tilde{\phi}(t-t_0,x)$, with no explicit dependence on the initial time t_0. This means that the expression for the Lyapunov function becomes

$$\begin{aligned}
V(t,x) &= \int_t^{t+T} G(d(\phi(\tau,t,x),S))d\tau = \int_t^{t+T} G(d(\tilde{\phi}(\tau-t,x),S))d\tau \\
V(x) &= \int_0^T G(d(\tilde{\phi}(s,x),S))ds,
\end{aligned}$$

with no explicit time dependence. Since solutions are assumed to exist for all future times, one can use the above expression with $T \to \infty$, yielding

$$V(x) = \int_0^\infty G(d(\tilde{\phi}(s,x),S))ds .$$

The above integral exists by Massera's lemma (Lemma 6.2). ■

The special case of uniform asymptotic stability, in which one has exponential stability, where the $\mathcal{KL}$ class function $\beta(d(x,S),t-t_0)$ is the exponential, $d(\phi(t,t_0,x),S) \le Kd(x,S)e^{-\sigma(t-t_0)}$, proves to be sufficiently general for our purposes. In that case, a Lyapunov function can be chosen as

$$V(t,x) = \int_t^{t+T} d^2(\phi(\tau,t,x))d\tau .$$

The class $\mathcal{K}$ functions $\alpha_1,\alpha_2,\alpha_3$ become quadratic functions of $d(x,S)$, and α_4 is a linear function of $d(x,S)$, 46.

$$\begin{aligned} &c_1 d^2(x,S) \le V(t,x) \le c_2 d^2(x,S), \\ &\dot{V}(t,x) \le -c_3 d^2(x,S), \end{aligned}$$

$$\left\|\nabla V(t,x)\right\| \le c_4 d(x,S) .$$

The following developments necessitate a precise definition of the neighborhoods under consideration.

Definition 6.4: A constrained regular neighborhood of S is a regular neighborhood of S contained in some ε-neighborhood of S.

Therefore, constrained regular neighborhoods of S are those that contain some ε-neighborhood and are contained in some ε-neighborhood of S. Note that ε-neighborhoods themselves are constrained regular neighborhoods by definition. These are precisely the neighborhoods relevant for our sense of stability.

One should bear in mind that the stability properties of Definition 6.3 are not invariant with respect to all homeomorphisms or even diffeomorphisms. One can have a homeomorphism mapping a constrained regular neighborhood onto one that is not constrained regular. An example is easily constructed; the diffeomorphism $(x,y) \mapsto (x',y') = (x, ye^x)$, preserves the *x*-axis and maps a constrained regular neighborhood of the *x*-axis to a neighborhood of the *x*'-axis that is not constrained regular. The dynamical system

$$\dot{x} = 2$$
$$\dot{y} = -y,$$

which is exponentially stable with respect to the *x*-axis, is transformed to

$$\dot{x}' = 2$$
$$\dot{y}' = y',$$

which is clearly not stable with respect to the *x*'-axis, although the *x*'-axis remains invariant. Linear coordinate transformations, being uniformly continuous, with uniformly continuous inverse, indeed preserve the stability notions of Definition 6.3. This is elaborated in the following lemma.

Lemma 6.4. Let a mapping $h : \mathcal{U}(S) \to X$, defined on a neighborhood, $\mathcal{U}(S) \subseteq X$, of S, be a homeomorpism onto its image, $\mathcal{V}(h(S)) = h(\mathcal{U}(S))$. The property of neighborhoods of S being constrained regular, and subsequently the sense of distance from S, are preserved under the mapping h if there exist class $\mathcal{K}$ functions α, β such that

$$\alpha(d(x,S)) \le d(h(x), h(S)) \le \beta(d(x,S)).$$

Such coordinate transformations also preserve the convergence of the trajectory to the manifold S. Likewise, any Lyapunov function, $V(x)$, for uniform stability with respect to the manifold S retains its uniformity properties in the transformed coordinates.

Proof: It suffices to show that such a mapping maps constrained regular neighborhoods of S onto constrained regular neighborhoods of the image of S. The same needs to hold for the inverse mapping.

If $\mathcal{U}(S)$ satisfies $d(x,S) \le c, \forall x \in \mathcal{U}(S)$, then $d(h(x), h(S)) \le \beta(c) \quad \forall x \in \mathcal{U}(S)$. Hence $\forall y \in h(\mathcal{U}(S))$, $d(y, h(S)) \le \beta(c) = c_2$. If a neighborhood $\mathcal{U}(S)$ of S is regular, *i.e.* it contains some ε-neighborhood $\mathcal{U}_\varepsilon(S)$, then the exterior of $\mathcal{U}_\varepsilon(S)$ in $\mathcal{U}(S)$, $\mathcal{U}(S)\setminus\mathcal{U}_\varepsilon(S)$, gets mapped to the exterior of $\mathcal{U}_{\alpha(\varepsilon)}(h(S))$ in $h(\mathcal{U}(S))$. So, the image $h(\mathcal{U}(S))$ contains $\mathcal{U}_{\alpha(\varepsilon)}(h(S))$; therefore it is a regular neighborhood. If the same property does not hold for the inverse mapping, one would have h violating one of its bounds imposed by functions α,β. Also, if $d(x(t),S)\to 0$ as $t\to\infty$, one has $d(h(x(t)),h(S))\le\beta(d(x(t),S)\to 0$ as $t\to\infty$, so that the property of the convergence of trajectories is preserved under the mapping h.

As for the Lyapunov function $V(x)$, let it satisfy the condition

$$\alpha_1(d(x,S))\le V(x)\le\alpha_2(d(x,S))$$

for some class $\mathcal{K}$ functions α_1,α_2. Let $x=h(y)$ be the mapping satisfying the condition of the Lemma,

$$\alpha(d(y,S))\le d(h(y),h(S))\le\beta(d(y,S)).$$

Then one finds that

$$\alpha_1(\alpha(d(y,h^{-1}(S)))\le\alpha_1(d(h(y),S))\le V(h(y))\le\alpha_2(d(h(y),S))\le\alpha_2(\beta(d(y,h^{-1}(S))).$$

Defining class $\mathcal{K}$ functions $\tilde{\alpha}_1=\alpha_1\circ\alpha$, $\tilde{\alpha}_2=\alpha_2\circ\beta$ and $\tilde{V}=V\circ h$ one has

$$\tilde{\alpha}_1(d(y,h^{-1}(S)))\le\tilde{V}(y)\le\tilde{\alpha}_2(d(y,h^{-1}(S))),$$

which is the uniformity condition in the transformed coordinates, y, owing to the fact that the composition of class $\mathcal{K}$ functions is again a class $\mathcal{K}$ function, 46. A similar argument applies to the condition on the derivative of the Lyapunov function, completing the proof. ■

Remark 6.2: Uniformly continuous homeomorphisms having uniformly continuous inverse satisfy the condition of Lemma 6.4. Uniform continuity of the mapping guarantees the upper bound, and uniform continuity of the inverse guarantees the lower bound. Linear transformations, in particular, belong to this class. This requirement should be contrasted with the simpler continuity condition that is a defining property of a homeomorphism. Homeomorphisms preserve open sets, while homeomorphisms satisfying the more stringent condition of Lemma 6.4 preserve constrained regular neighborhoods, describing the presented notions of stability. These are preserved for compact manifolds and their precompact neighborhoods, by virtue of compactness, under usual homeomorphisms, but for general noncompact manifolds the additional condition of Lemma 6.4 is needed.

Therefore, when using coordinate transformations on dynamical systems exhibiting stability with respect to a noncompact manifold S, one should exercise care since homeomorphisms, and even diffeomorphisms, do not preserve the stability properties unless they satisfy the more stringent condition of Lemma 6.4. This, in particular, applies to feedback linearization methods, when transformations of single-agent state-space comprise the transformation of the total state-space X, and the target manifold S embedded in it. More subtly, one can have topologically equivalent systems that do not exhibit the same stability properties if the target set is noncompact.

6.3 Assessing the Effects of Disturbances on Multi-agent Systems Using Lyapunov Functions

This section introduces the dynamics of the multi-agent system with an optional presence of the leader. The agents are assumed to be in control affine form, and of the same order. Here the Lyapunov functions are used to quantify the effect that disturbances have on the uniformly asymptotically stable closed-loop undisturbed system. Specified to linear time-invariant systems, bounds on disturbances are expressed in a more precise way, using quadratic Lyapunov functions.

6.3.1 Multi-agent System Dynamics with Disturbances

Consider a multi-agent system with an optional presence of a leader. Agents are described as nodes of a directed graph endowed with dynamics. Edges of the graph represent the communication structure, *i.e.* denote which agents' states are available for the purpose of the cooperative feedback to a given agent. It is assumed that the full state of neighboring agents is available for feedback purposes. Assume that the leader node dynamics is

$$\dot{x}_0 = f(x_0) + \xi_0 , \tag{141}$$

and the N identical agents have dynamics in the control affine form

$$\dot{x}_i = f(x_i) + g(x_i)u_i + \xi_i , \tag{142}$$

with $x_0, x_i \in \mathbb{R}^n \ \forall i$ and $u_0, u_i \in \mathbb{R}^p \ \forall i$. For the purposes of this chapter, functions f, g describe a *nominal* system. Signals ξ_i and ξ_0 are the agent's and leader's disturbances. The nature of the disturbances can be various. Disturbance terms can even come from unmodelled dynamics, allowing different drift dynamics of each agent, as long as the dynamical systems are of the same order. This furnishes an instance of a state dependent disturbance $\xi_i(x_i)$.

Definition 6.5: The distributed consensus or synchronization problem is to find distributed feedback controls $u_i(x)$ for agents that guarantee $\lim_{t\to\infty}\left\|x_i(t) - x_j(t)\right\| = 0 \ \forall i, j$ when there is no leader, and $u_i(\delta)$, where $\delta_i = x_i - x_0$, that guarantee $\lim_{t\to\infty}\left\|x_i(t) - x_0(t)\right\| = 0 \ \forall i$ when the leader x_0 is present. We call the former *cooperative regulator problem* and the latter *cooperative tracker problem*. The target sets are the consensus manifold $S : x_i = x_j, \forall (i, j)$ in the total state-space $X = \mathbb{R}^{Nn}$ and $S : \delta_i = 0, \forall i$, subset of the (δ, x_0) space $X = \mathbb{R}^{(N+1)n}$.

Assumption 6.1. One has found the cooperative feedback control law $u(x)$ that guarantees asymptotically stable consensus for the nominal system, *i.e.* solves the cooperative regulator problem. Alternatively, one has the cooperative feedback control law $u(\delta)$ that guarantees asymptotically stable synchronization of the nominal system, *i.e.* solves the cooperative tracker problem for the nominal system. Asymptotic stability is assumed uniform.

The closed loop nominal systems are given as

$$\dot{x} = F_{cl1}(x) := F(x) + G(x)u(x), \tag{143}$$

$$\begin{aligned} \dot{\delta} &= F_{cl2}(\delta, x_0) := F(x) - F(\overline{x}_0) + G(x)u(\delta), \\ \dot{x}_0 &= f(x_0), \end{aligned} \tag{144}$$

where $F(x) = \begin{bmatrix} f(x_1)^T & \cdots & f(x_N)^T \end{bmatrix}^T$, $G(x) = diag(g(x_1),..., g(x_N))$, and x in (144) is replaced by $\delta + \overline{x}_0$. Under Assumption 6.1, by the converse Lyapunov theorem (Theorem 6.1), there exist Lyapunov function for stability of the consensus manifold, $V(x)$, and a Lyapunov function $V(\delta, x_0)$ for the error signals $\delta_i = x_i - x_0$, for the closed loop nominal systems (143), (144), respectively. Actually, the latter case is a classic case of partial stability, 41, in which $\|\delta\|$ measures the distance from $\delta = 0$ subspace of the total (δ, x_0) space. Note that a special case of the Lyapunov function, $V(\delta, x_0)$, for partial stability, having the form $V(\delta)$, suffices in some instances. The effects of disturbances can be assessed using those Lyapunov functions. Just as in the classical case, 46, one has asymptotic stability with respect to the consensus manifold or uniformly ultimately bounded consensus[4], depending on the kind of bounds that the disturbances satisfy. To elaborate further on different types of disturbance bounds, let us start with the following result formulated as a theorem. The Lyapunov functions under consideration are allowed to depend explicitly on time, for the sake of generality and because this changes nothing conceptually, as long as they satisfy the same time independent uniformity bounds. This generalization allows one to treat time-varying agents as well.

Theorem 6.2. Let the closed loop nominal systems (143), (144), satisfy the Assumption 6.1. Let the disturbances satisfy the uniform growth bound

$$\|\xi(x)\| \le \alpha_5(d(x, S)), \tag{145}$$

for the cooperative regulator problem, and

[4] The uniformly ultimately bounded consensus means that $(\exists \varepsilon > 0)\ (\exists T > 0)\ s.t.\ d(x(t), S) < \varepsilon\ \forall t > T$.

$$\left\|\xi(x)-\overline{\xi}_0(x_0)\right\| \le \alpha_5(\|\delta\|)\,,\ \|\xi_0\| < M\ , \tag{146}$$

for the cooperative tracker problem. Then, for a proper choice of class $\mathcal{K}$ function α_5, one has asymptotic stability in presence of disturbances. If the disturbances and the Lyapunov functions satisfy the uniform bound for some M_1, M_2

$$\left|\nabla V(t,x)\xi\right| \le M_1\,, \tag{147}$$

for the cooperative regulator problem, and

$$\left|\nabla_\delta V(t,\delta,x_0)(\xi-\overline{\xi}_0)\right| \le M_1\,,\ \left|\nabla_{x_0} V(t,\delta,x_0)\xi_0\right| < M_2 \tag{148}$$

for the cooperative tracker problem, then the resulting convergence to consensus is uniformly ultimately bounded.

Proof: With the cooperative feedback control, $u(x)$, satisfying Assumption 6.1 for the nominal system (143), the closed loop nominal system

$$\dot{x} = F(x) + G(x)u(x) = F_{cl1}(x)\,,$$

is uniformly asymptotically stable with respect to the consensus manifold S. Therefore, by Theorem 6.1, there exists a Lyapunov function, $V(t,x)$, for uniform asymptotic stability with respect to the consensus manifold $S \subset \mathbb{R}^{Nn}$.

With the cooperative feedback control, $u(\delta)$, satisfying Assumption 6.1 for the nominal system (144), the closed-loop nominal system

$$\begin{aligned}\dot{\delta} &= F(x) - F(\overline{x}_0) + G(x)u(\delta) = F_{cl2}(\delta, x_0),\\ \dot{x}_0 &= f(x_0),\end{aligned}$$

is uniformly asymptotically stable to the $\delta = 0$ subspace. Therefore, by Theorem 6.1, there exists a Lyapunov function, $V(t,\delta,x_0)$, for uniform asymptotic stability with respect to $\delta = 0$ subspace. Those functions satisfy the following

$$V(t,x)\ge 0;\;\; V(t,x)=0 \Leftrightarrow x\in S$$
$$\alpha_1(d(x,S)) \le V(t,x) \le \alpha_2(d(x,S))$$
$$\frac{d}{dt}V(t,x)=\frac{\partial}{\partial t}V(t,x)+\nabla V(t,x)^T F_{cl1}(x) \le -\alpha_3(d(x,S))$$

for the cooperative regulator and

$$V(t,\delta,x_0)>0;\;\; V(t,\delta,x_0)=0 \Leftrightarrow \delta=0$$
$$\alpha_1(\|\delta\|) \le V(t,\delta,x_0) \le \alpha_2(\|\delta\|)$$
$$\frac{d}{dt}V(t,\delta,x_0)=\frac{\partial}{\partial t}V(t,\delta,x_0)+\nabla_\delta V(t,\delta,x_0)^T F_{cl2}(\delta,x_0)+\nabla_{x_0}V(t,\delta,x_0)f(x_0) \le -\alpha_3(\|\delta\|),$$

for the cooperative tracker. The $\alpha_1,\alpha_2,\alpha_3$ are some class $\mathcal{K}$ functions. In the presence of disturbances the global system (143) takes the form

$$\dot{x}=F_{cl1}(x)+\xi\,, \tag{149}$$

and the error system (144) equals

$$\begin{aligned}\dot{\delta}&=F_{cl2}(\delta,x_0)+\xi-\overline{\xi}_0,\\ \dot{x}_0&=f(x_0)+\xi_0.\end{aligned} \tag{150}$$

In that case Lyapunov functions for nominal cooperative regulator and tracker systems, respectively, satisfy

$$\frac{d}{dt}V(t,x) \le -\alpha_3(d(x,S))+\nabla V(t,x)\xi\,, \tag{151}$$

and

$$\frac{d}{dt}V(t,\delta,x_0) \le -\alpha_3(\|\delta\|)+\nabla_\delta V(t,\delta,x_0)(\xi-\overline{\xi}_0)+\nabla_{x_0}V(t,\delta,x_0)\xi_0\,. \tag{152}$$

If there exists a class $\mathcal{K}$, or globally $\mathcal{K}_\infty$, function $\tilde{\alpha}_3(d(x,S))$ such that

$$-\alpha_3(d(x,S))+\nabla V(x,t)\xi \le -\tilde{\alpha}_3(d(x,S))\,, \tag{153}$$

then (151) implies that the consensus is uniformly asymptotically achieved in spite of the disturbances. The same applies to the asymptotic stability for the cooperative tracker problem (152) if there exists $\tilde{\alpha}_3(\|\delta\|)$ such that

$$-\alpha_3(\|\delta\|) + \nabla_\delta V(t,\delta,x_0)(\xi - \overline{\xi}_0) + \nabla_{x_0} V(t,\delta,x_0)\xi_0 \leq -\tilde{\alpha}_3(\|\delta\|) . \tag{154}$$

Gradient of the Lyapunov function satisfies the growth bound $\|\nabla V(t,x)\| \leq \alpha_4(d(x,S))$, in the case of the cooperative regulator problem, or the bound $\|\nabla_\delta V(t,\delta,x_0)\| \leq \alpha_4(\|\delta\|)$, and $\|\nabla_{x_0} V(t,\delta,x_0)\| \leq \alpha_4(\|\delta\|)$, in the case of the cooperative tracker problem. If the disturbances satisfy bounds $\|\xi(x)\| \leq \alpha_5(d(x,S))$, or $\|\xi(x) - \overline{\xi}_0(x_0)\| \leq \alpha_5(\|\delta\|)$ and $\|\xi_0\| < M$, it is possible that for a proper choice of class $\mathcal{K}$ functions $\alpha_3, \alpha_4, \alpha_5$ and bound M one can guarantee uniform asymptotic stability in presence of disturbances, 46. In that case one has a sought $\tilde{\alpha}_3$, class $\mathcal{K}$ function, such that

$$\frac{d}{dt}V(x,t) \leq -\alpha_3(d(x,S)) + \alpha_4(d(x,S))\alpha_5(d(x,S)) := -\tilde{\alpha}_3(d(x,S)) ,$$

for the cooperative regulator and

$$\frac{d}{dt}V(\delta,t,x_0) \leq -\alpha_3(\|\delta\|) + \alpha_4(\|\delta\|)(\alpha_5(\|\delta\|) + M) := -\tilde{\alpha}_3(\|\delta\|) ,$$

for the cooperative tracker. In the case of the uniform bounds (147), (148), the resulting convergence to consensus is uniformly ultimately bounded, since for sufficiently large $d(x,S)$ or $\|\delta\|$ the derivative of a Lyapunov function is negative, 46. The numerical value of the ultimate bound depends on the choice of the Lyapunov function. This concludes the proof. ■

Disturbances coming from unmodelled dynamics are likely to satisfy the former case, and generally unknown disturbances that come from the environment are more likely to fit into the latter case. If the nominal cooperative tracker admits a Lyapunov function $V(t,\delta,x_0)$ having a special form, $V(t,\delta)$, the robustness conditions of Theorem 6.2 simplify. This shall be the case in the following subsection.

6.3.2 Specification to Linear Time-invariant Systems

Having agents and the leader in a form of linear time-invariant systems, one can derive more specific bounds on the disturbances and the region of attraction, using quadratic Lyapunov functions. The dynamics of the leader and N identical agents is given as

$$\dot{x}_0 = Ax_0 + \xi_0, \tag{155}$$

$$\dot{x}_i = Ax_i + Bu_i + \xi_i, \tag{156}$$

with $x_0, x_i \in \mathbb{R}^n; \forall i$, $u_0, u_i \in \mathbb{R}^p; \forall i$. For the purposes of this chapter matrices A, B describe the *nominal* system. Signals ξ_i and ξ_0 are the agent and leader disturbances. The global systems for cooperative regulator and tracker problems with disturbances are respectively

$$\dot{x} = (I \otimes A)x + (I \otimes B)u + \xi, \tag{157}$$

and defining the global disagreement vector $\delta = \begin{bmatrix} \delta_1^T & \cdots & \delta_N^T \end{bmatrix}^T$, where $\delta_i = x_i - x_0$ one has the system

$$\dot{\delta} = (I \otimes A)\delta + (I \otimes B)u + \xi - \overline{\xi}_0. \tag{158}$$

Here $\overline{\xi}_0 = \begin{bmatrix} \xi_0^T & \cdots & \xi_0^T \end{bmatrix}^T$. Define the *local neighborhood error* as

$$e_i = \sum_j e_{ij}(x_j - x_i) + g_i(x_0 - x_i). \tag{159}$$

The linear cooperative feedback control is chosen as

$$u_i = cKe_i. \tag{160}$$

In global form this is $u = -cL \otimes Kx$ or $u = -c(L+G) \otimes K\delta$, depending on whether there are nonzero pinning terms or not. The closed loop nominal systems are

$$\dot{x} = A_{cl1}x + \xi, \tag{161}$$

$$\dot{\delta} = A_{cl2}\delta + \xi - \overline{\xi}_0 . \tag{162}$$

The closed-loop system matrices $A_{cl1} = I_N \otimes A - cL \otimes BK$ and $A_{cl2} = I_N \otimes A - c(L+G) \otimes BK$ need to be asymptotically stable with respect to the consensus manifold, or with respect to $\delta = 0$ subspace, respectively. Note that, because of the linearity of the drift dynamics, x_0 does not explicitly appear in (162). This, in particular, motivates our choice of $u(\delta)$ for the cooperative tracker feedback control instead of a more general control law $u(\delta, x_0)$, as well as our choice of the Lyapunov function $V(\delta)$, instead of $V(\delta, x_0)$. These choices reduce the linear cooperative tracker problem to a classical problem of asymptotic stability of the origin. One possible choice of the local feedback gain K that stabilizes A_{cl1} and A_{cl2} is given by the following lemma.

Lemma 6.5. Assume that a graph has a spanning tree with at least one pinning gain nonzero, connecting into the root node when pinning is present. Let the pair (A, B) be stabilizable. Choose the local feedback gain K as

$$K = R^{-1}B^T P , \tag{163}$$

where P is the solution to the algebraic Riccati equation

$$A^T P + PA + Q - PBR^{-1}B^T P = 0 . \tag{164}$$

Then the closed loop system matrix A_{cl1} is asymptotically, exponentially, stable to the consensus manifold if the coupling gain is sufficiently great so that

$$c \geq \frac{1}{2\min_j \operatorname{Re} \lambda_{j>0}} , \tag{165}$$

where $\lambda_{j>0}$ are the positive eigenvalues of the L matrix, and the closed loop matrix A_{cl2} is asymptotically, exponentially, stable if the coupling gain is sufficiently great so that

$$c \geq \frac{1}{2\min_j \operatorname{Re}\lambda_j}, \tag{166}$$

where λ_j are the eigenvalues of the $L+G$ matrix.

Proof is given in 13. ■

With this result, one can guarantee that the linear multi-agent systems, (155), (156), satisfy Assumption 6.1. The following conclusions apply when disturbances are assumed to act on the system.

Corollary 6.1. If disturbances, ξ_i, satisfy the uniform growth bound, $d(\xi,S) < Cd(x,S)$ [5], given that a growth constant C is sufficiently small, one has uniform asymptotic stability of consensus. If disturbances satisfy the bound $d(\xi,S) < M$, one has uniformly ultimately bounded convergence to consensus. For the cooperative tracker, the disturbance bound $\|\xi - \bar{\xi}_0\| < C\|\delta\|$ implies asymptotic stability, given that a growth constant C is sufficiently small. On the other hand, the disturbance bound $\|\xi - \bar{\xi}_0\| < M$ implies uniform ultimate boundedness of the error dynamics.

Proof: The relevant Lyapunov functions for the linear systems (161), (162), satisfying Assumption 6.1, exist in quadratic form as $V_1(x) = x^T P_1 x \geq 0$ and $V_2(\delta) = \delta^T P_2 \delta > 0$ with matrices $P_1 \geq 0$, $P_2 > 0$, where $\ker P_1 = S$; the consensus manifold. Therefore one has a Lyapunov function for the uniform asymptotic stability satisfying $\dot{V}_1(x) = -x^T Q_1 x \leq -\sigma_{>0\min}(Q_1) d(x,S)^2$ and the Lyapunov function satisfying $\dot{V}_2(\delta) = -\delta^T Q_2 \delta < 0$.

In presence of disturbances one finds the following identities

[5] This is a slight abuse of notation. The disturbance ξ, as a part of the vector field defining dynamics of the system, is not an element of the state space X, but of the tangent space TX_x, and $d(\xi,S)$ is a distance from the consensus manifold in that tangent space. However for linear systems on Euclidean vector spaces, $\mathbb{R}^{Nn}$, this ought to cause no confusion.

$$\dot{V}_1(x) = -x^T Q_1 x + 2x^T P_1 \xi \tag{167}$$

and

$$\dot{V}_2(\delta) = -\delta^T Q_2 \delta + 2\delta^T P_2 (\xi - \underline{\xi}_0)\ , \tag{168}$$

where $Q_1 \geq 0$, with $\ker Q_1 = \ker P_1$, and $Q_2 > 0$.

Owing to the fact that both quadratic forms in (167), $x^T P_1 x$ and $x^T Q_1 x$, measure the distance from the consensus manifold, they are equivalent in the sense of distance norms, *i.e.* they can be bounded by above and below by a scaled distance from the consensus manifold $d(x,S)$. Hence, for V_1 one can state

$$\begin{aligned}
&\dot{V}_1(x) \leq -\sigma_{>0\min}(Q) d(x,S)^2 + 2(\sqrt{P_1}x)\cdot(\sqrt{P_1}\xi) \\
&\leq -\sigma_{>0\min}(Q) d(x,S)^2 + 2\left\|\sqrt{P_1}x\right\|\left\|\sqrt{P_1}\xi\right\| \\
&\leq -\sigma_{>0\min}(Q) d(x,S)^2 + K d(x,S) d(\xi,S) \\
&\leq -(\sigma_{>0\min}(Q) d(x,S) - K d(\xi,S)) d(x,S),
\end{aligned} \tag{169}$$

for some real constant $K > 0$. If there exists a disturbance bound $d(\xi,S) < C d(x,S)$ or $d(\xi,S) < M$ one will have asymptotic convergence to consensus manifold S in the former case if a growth constant C is sufficiently small, or uniform ultimate boundedness in the latter case, 46.

For cooperative tracker, positive definiteness of P_2 and Q_2 allow for an application of the classical results on disturbances, 46. According to those results one has, respectively, asymptotic stability and uniform ultimate boundedness with respect to the origin of the synchronization error system. ■

Note that a more general bound on $d(\xi,S)$, by a $\mathcal{K}$ class function of $d(x,S)$ that satisfies the linear growth bound locally, also suffices.

In many realistic cases of disturbances the robustness alone, stemming naturally from the asymptotic or the exponential cooperative stability of the disturbance free systems, does not suffice. In those instances it becomes advantageous to measure or estimate the disturbances acting on the leader and the following agents, so that those could be compensated for, even if only in part. Disturbance estimation schemes for the leader's and agents' disturbances are presented in the next section.

6.4 Disturbance Estimators

In this section we introduce the disturbance estimation of the leader's and the agents' disturbance. The important fact is that in cooperative tracker problem, in addition to its own disturbance signal ξ_i, each agent also needs to know the leader's disturbance ξ_0. In reality those need to be replaced with their estimates. This introduces the need for estimators of the leader's and the agents' disturbances.

In the following subsections, local and distributed observers are introduced for the purpose of estimating local disturbances. Local observers are a natural choice for estimating the agents' disturbances if measurements of those disturbances are available. In the special case of all agents having the same disturbance generator one can estimate the local disturbances by a distributed observer. In fact if only relative disturbance measurements are available to all agents, except to a few that know the absolute reference value, one needs to use the distributed observer. Leader's disturbance observer naturally has a distributed form.

6.4.1 Estimating Disturbances on the Agents

This and the following subsection are concerned with estimating the agents' disturbances. Apart from the leader's disturbance estimate $\hat{\xi}_0$ each agent needs to know only its own disturbance estimate $\hat{\xi}_i$, not at all its neighbor's disturbances. This is a consequence of the fact that disturbances acting on agents are assumed independent, so ξ_i acts solely on agent i.

Assume that the disturbance acting on agent i is modelled by the following linear disturbance signal generator

$$\begin{aligned} \dot{\xi}_i &= \Gamma_i \xi_i \\ \varsigma_i &= F_i \xi_i \end{aligned} \tag{170}$$

The disturbance acting on the agent ξ_i cannot be measured directly,but a related quantity ς_i, the output of the disturbance generator, is accessible to measurements. Thus one uses the pertaining disturbance generator to design a local estimator.

$$\begin{aligned} \dot{\hat{\xi}}_i &= \Gamma_i \hat{\xi}_i + L_i(\varsigma_i - \hat{\varsigma}_i) \\ \hat{\varsigma}_i &= F_i \hat{\xi}_i \end{aligned} \tag{171}$$

If the matrix $\Gamma_i - L_i F_i$ is Hurwitz the estimate error $\xi_i - \hat{\xi}_i$ will converge to zero. This is a classical local Luenberger estimator. If and only if the pair (Γ_i, F_i) is detectable one can design the observer gain L_i that guarantees stability of $\Gamma_i - L_i F_i$, *i.e.* convergence of estimate to the true value of the disturbance.

6.4.2 Case of Identical Disturbance Generators for all Agents

In the special case when the disturbance generators for all agents are the same, a distributed disturbance observer can make all disturbance estimates converge to the true values of the disturbance signals, as long as there is a pinned reference value of 0 to some agents. This amounts to some agents knowing the absolute zero reference, while all others rely on relative information only. Hence, agents estimate their own disturbances, but the estimation is performed in a distributed fashion. Therefore, the entire multi-agent system estimates the total disturbance vector ξ. This is made possible in such a form precisely because all the disturbance generators are assumed to be the same

$$\begin{aligned} \dot{\xi}_i &= \Gamma \xi_i \\ \varsigma_i &= F \xi_i \end{aligned} \quad i = 0, 1, ..., N. \tag{172}$$

The distributed disturbance estimator, 13, has the form

$$
\begin{aligned}
\dot{\hat{\xi}}_i &= \Gamma \hat{\xi}_i - c_2 L_1 \sum_j e_{ij} (\tilde{\varsigma}_j - \tilde{\varsigma}_i) - g_i \tilde{\varsigma}_i \\
\tilde{\varsigma}_i &= \varsigma_i - \hat{\varsigma}_i \\
\hat{\varsigma}_i &= F \hat{\xi}_i ,
\end{aligned}
\tag{173}
$$

and guarantees estimate convergence to the true disturbance value if the observer system matrix

$$
I_N \otimes \Gamma - c_2 (L+G) \otimes L_1 F \tag{174}
$$

is Hurwitz.

Lemma 6.6. Assume that the graph has a spanning tree with at least one pinning gain nonzero, connecting into the root node, and that the pair (Γ, F) is detectable. Choose the local observer gain L_1 as

$$
L_1 = P F^T R^{-1} , \tag{175}
$$

where P is the solution to the observer algebraic Riccati equation

$$
\Gamma P + P \Gamma^T + Q - P F^T R^{-1} F P = 0 . \tag{176}
$$

Then the observer system matrix (174) is asymptotically, exponentially, stable if the coupling gain is sufficiently great

$$
c_2 \geq \frac{1}{2 \min_j \operatorname{Re} \lambda_j} , \tag{177}
$$

where λ_j are the eigenvalues of the $L+G$ matrix.

Proof is given in 13. ■

Local estimation remains an option in this case as in 29, with the same choice of the observer gain (175),(176), guaranteeing convergence of local observer (171). It should be noted that an assumption on identical disturbance generators for distinct agents might be too naïve for most real cases.

6.4.3 Estimating Disturbance on the Leader with Known Leader's Disturbance Generator

In this subsection the distributed leader's disturbance observers are introduced. All the following agents need to have a leader's disturbance estimator $\hat{\xi}_{0i}$. In case all agents know the leader's disturbance generating system

$$\begin{aligned} \dot{\xi}_0 &= \Gamma_0 \xi_0 \\ \varsigma_0 &= F_0 \xi_0 \end{aligned}, \tag{178}$$

if one pins the measurement ς_0 into a selected few nodes then the following synchronization type distributed estimator architecture is available. The distributed leader's disturbance estimators take the classical cooperative tracker form with local neighborhood output disagreement feedback

$$\begin{aligned} \dot{\hat{\xi}}_{0i} &= \Gamma_0 \hat{\xi}_{0i} + c_3 L_0 \sum_j e_{ij} (\hat{\varsigma}_{0j} - \hat{\varsigma}_{0i}) + g_i (\varsigma_0 - \hat{\varsigma}_{0i}) \\ \dot{\hat{\xi}}_{0i} &= \Gamma_0 \hat{\xi}_{0i} + c_3 L_0 F_0 \sum_j e_{ij} (\hat{\xi}_{0j} - \hat{\xi}_{0i}) + g_i (\xi_0 - \hat{\xi}_{0i}), \end{aligned} \tag{179}$$

where

$$\hat{\varsigma}_{0i} = F_0 \hat{\xi}_{0i}. \tag{180}$$

If the observer system matrix $I \otimes \Gamma_0 - c_3 (L+G) \otimes L_0 F_0$ is Hurwitz then all local estimates $\hat{\xi}_{0i}$ converge to the true value of the leader's disturbance ξ_0 asymptotically. The design of c_3, L_0 guaranteeing stability of the observer system matrix is detailed in Lemma 6.6.

6.4.4 Estimating Disturbance on the Leader without the Leader's Disturbance Generator

This subsection introduces the distributed leader's disturbance observers without having the leader's disturbance generating system. If the leader's disturbance generator is not known but there exists a finite *k* such that $\xi_0^{(k)} = 0$ identically then a *k*-th order pinned consensus algorithm for the cooperative observer,

$$\hat{\xi}_{0i}^{(k)} = \sum_{j=0}^{k-1} c_j \frac{d^j}{dt^j}\left[\sum_l e_{il}(\hat{\xi}_{0l} - \hat{\xi}_{0i}) + g_i(\xi_0 - \hat{\xi}_{0i})\right], \tag{181}$$

can guarantee convergence of all leader's disturbance estimates to the true value. Note that only the pinned nodes can measure the leader's disturbance ξ_0 directly. Others rely on their local neighborhood estimates $\hat{\xi}_{0j}$, or more precisely on local neighborhood relative estimate disagreement. All disturbances and their estimates are assumed to be elements of $\mathbb{R}^p$, $p = n$ or m. The coefficients c_k are chosen to guarantee stability, *i.e.* so that $\delta_\xi = \hat{\xi}_0 - \overline{\xi}_0$ converges to zero asymptotically. Indeed, by canonically assigning state variables $y_1 = \delta_\xi, y_2 = \dot{\delta}_\xi, \cdots y_k = \delta_\xi^{(k-1)}$, $y = \begin{bmatrix} y_1^T & \cdots & y_k^T \end{bmatrix}^T$ one finds the system in block controllable canonical form

$$\frac{d}{dt}y = \begin{bmatrix} 0 & I_{Np} & 0 & \cdots & 0 \\ 0 & 0 & I_{Np} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & I_{Np} \\ -c_0(L+G)\otimes I_p & -c_1(L+G)\otimes I_p & \cdots & \cdots & -c_{k-1}(L+G)\otimes I_p \end{bmatrix} y \,. \tag{182}$$

Lemma 6.7. The characteristic polynomial of the block controller canonical form matrix (182) equals

$$P(s) = \det(sI_{kNp} - M_k) = \det(s^k I_{Np} + (c_{k-1}s^{k-1} + c_{k-2}s^{k-2} + \cdots + c_0)(L+G)\otimes I_p)\,, \tag{183}$$

Proof: Let the bases of the induction be $m = 2$

$$\begin{aligned} \det(sI_{2Np} - M_2) &= \det\begin{bmatrix} sI_{Np} & -I_{Np} \\ c_0(L+G)\otimes I_p & sI_{Np} + c_1(L+G)\otimes I_p \end{bmatrix} \\ &= \det(sI_{Np})\det(sI_{Np} + c_1(L+G)\otimes I_p + \frac{1}{s}c_0(L+G)\otimes I_p) \\ &= \det(s^2 I_{Np} + (c_1 s + c_0)(L+G)\otimes I_p) \end{aligned},$$

verifying the assertion for the basis. Let M_m be the block matrix of the form

$$M_m = \begin{bmatrix} 0 & I_{Np} & 0 & \cdots & 0 \\ 0 & 0 & I_{Np} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & I_{Np} \\ -c_0(L+G)\otimes I_p & -c_1(L+G)\otimes I_p & \cdots & \cdots & -c_{m-1}(L+G)\otimes I_p \end{bmatrix}$$

Assume for m that $\det(sI_{mNp} - M_m) = \det(s^m I_{Np} + (c_{m-1}s^{m-1} + \cdots + c_0)(L+G)\otimes I_p)$, then one has the following identity for $m+1$

$$M_{m+1} = \begin{bmatrix} 0 & I_{Np} & 0 & \cdots & 0 \\ 0 & 0 & I_{Np} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & I_{Np} \\ -c_0(L+G)\otimes I_p & -c_1(L+G)\otimes I_p & \cdots & -c_{m-1}(L+G)\otimes I_p & -c_m(L+G)\otimes I_p \end{bmatrix}$$

and

$$\det(sI_{(m+1)Np} - M_{m+1}) = \begin{bmatrix} sI_{Np} & -I_{Np}\ 0\cdots 0 \\ 0 & \\ \vdots & sI_{mNp} - M_m \\ 0 & \\ c_0(L+G)\otimes I_p & \end{bmatrix}$$

where the coefficients in the last block row of M_m go from $c_1, \cdots, c_m$.

$$\det(sI_{(m+1)Np} - M_{m+1}) = \det sI_{Np} \det((sI_{mNp} - M_m) - \frac{1}{s}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ c_0(L+G)\otimes I_p \end{bmatrix}\begin{bmatrix} -I_{Np} & 0 & \cdots 0 \end{bmatrix})$$

$$= s^{Np} \det \begin{bmatrix} sI_{Np} & -I_{Np} & & \\ & sI_{Np} & \ddots & \\ & & sI_{Np} & -I_{Np} \\ (c_1 + \frac{1}{s}c_0)(L+G)\otimes I_p & c_2(L+G)\otimes I_p & \cdots & sI_{Np} + c_m(L+G)\otimes I_p \end{bmatrix}$$

By induction assumption, this equals

$$= s^{Np}\det(s^m I_{Np} + (c_m s^{m-1} + \cdots + c_1 + \frac{1}{s}c_0)(L+G)\otimes I_p)$$
$$= \det(s^{m+1} I_{Np} + (c_m s^m + \cdots c_1 s + c_0)(L+G)\otimes I_p)$$

completing the induction proof. ■

Remark 6.3: It should be noted that for higher order *k* the proposed algorithm for corrections $\hat{\xi}_{0i}$ becomes increasingly complicated. Necessary order is determined by the constraint on the disturbance signal ξ_0. If ξ_0 is modeled by *e.g.* spline functions (of preferably lower order) then such an algorithm suffices. Also, if the higher order derivatives are not identically zero but can be considered negligible, or otherwise small in some sense, compared to a finite number of lower order ones, the application of such an observer is still justified owing to the fact that those small terms introduce a comparatively small perturbation of the asymptotically stable system, resulting in a small steady state error.

The choice of the coefficients c_j has to be made such that the asymptotic stability is guaranteed. This constitutes the *higher order observer design problem*. In case stability is guaranteed one has $\lim_{t\to\infty} y = 0$, meaning as $t \to \infty$, $\hat{\xi}_{0i}^{(m)} = \xi_0^{(m)}$ for all $m = 0,\ldots,k-1$. Most importantly the steady state error δ_ξ converges to zero, which means the convergence of the estimate to the true value of the disturbance. One way of choosing the coefficients c_j in relation to the graph matrix $L+G$ eigenvalues is detailed in the following theorem.

Theorem 6.3. Let A_C, B_C be the single input *k*-th order controller canonical form matrices with all characteristic polynomial coefficients equal to zero

$$A_C = \begin{bmatrix} 0 & 1 & 0\cdots & 0 \\ 0 & 0 & 1\cdots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix},\ B_C = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{184}$$

The optimal feedback gain

$$K = R^{-1} B_C^T P = \begin{bmatrix} c_0 & \cdots & c_{k-1} \end{bmatrix}, \tag{185}$$

where P is the solution of the algebraic Riccati equation

$$A_C^T P + P A_C + Q - P B_C R^{-1} B_C^T P = 0\,, \tag{186}$$

gives coefficients c_j that guarantee stability if all the eigenvalues λ_j of the matrix $L+G$ are in the complex gain margin region, 29, for the Riccati feedback (185).

Proof: In order to simplify the determinant of a matrix polynomial

$$P(s) = \det(sI_{kNp} - M) = \det(s^k I_{Np} + (c_{k-1}s^{k-1} + c_{k-2}s^{k-2} + \cdots + c_0)(L+G) \otimes I_p)\,,$$

one can apply the linear transformation T reducing $(L+G) \otimes I_p$ to the upper triangular form, *i.e.* if $T^{-1}(L+G)T = \Lambda$, with Λ upper triangular, then $T \otimes I_p$ is the desired transformation for $(L+G) \otimes I_p$. Applying such a transformation on the matrix polynomial

$$s^k I_{Np} + (c_{k-1}s^{k-1} + c_{k-2}s^{k-2} + \cdots + c_0)(L+G) \otimes I_p$$

does not change the determinant and yields the simplified form

$$\begin{aligned} &T^{-1}(s^k I_{Np} + (c_{k-1}s^{k-1} + c_{k-2}s^{k-2} + \cdots + c_0)(L+G) \otimes I_p)T = s^k I_{Np} + (c_{k-1}s^{k-1} + c_{k-2}s^{k-2} + \cdots + c_0)\Lambda \otimes I_p \\ &= (s^k I_N + (c_{k-1}s^{k-1} + c_{k-2}s^{k-2} + \cdots + c_0)\Lambda) \otimes I_p \end{aligned},$$

which is itself in an upper triangular form. The determinant is now simply the product of diagonal elements having the following form

$$\begin{aligned} P_j(s) &= s^k + (c_{k-1}s^{k-1} + c_{k-2}s^{k-2} + \cdots + c_0)\lambda_j \\ &= s^k + c_{k-1}\lambda_j s^{k-1} + c_{k-2}\lambda_j s^{k-2} + \cdots + c_0 \lambda_j \end{aligned}. \tag{187}$$

For this polynomial product to be stable all the factor polynomials $P_j(s)$ need to be stable, *i.e.* Hurwitz, for every λ_j, eigenvalue of $(L+G)$. Using the algebraic Riccati equation

$$A_C^T P + P A_C + Q - P B_C R^{-1} B_C^T P = 0\,,$$

for the *k-th* order system with matrices A_C and B_C

$$A_C = \begin{bmatrix} 0 & 1 & 0\cdots & 0 \\ 0 & 0 & 1\cdots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B_C = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

the optimal feedback (185) will stabilize the system (184). Next we use the complex gain margin region assessed using the Lyapunov equation, where P is the solution of the Riccati equation

$$\begin{aligned} &(A_C + zB_C K)^\dagger P + P(A_C + zB_C K) = A_C^T P + PA_C + z^* K^T B_C^T P + zPB_C K \\ &= A_C^T P + PA_C - 2\operatorname{Re} zPB_C R^{-1} B_C^T P \\ &= -Q + (1 - 2\operatorname{Re} z) PB_C R^{-1} B_C^T P < 0 \end{aligned},$$

to guarantee an unbounded gain margin region in complex plane, 13. Therefore, for $\operatorname{Re} z$ sufficiently large one has stability, meaning that the characteristic polynomial of $A_C - zB_C K$ is Hurwitz. But the characteristic polynomial of $A_C - zB_C K$ equals (187), with z instead of λ_j. This concludes the proof. ■

Remark 6.4: The observer coefficients $c_0 \cdots c_{k-1}$ can be read off from the optimal single input feedback. This allows for a fairly simple design procedure for the system (181). Also, this means that the complex gain margin region for the Riccati feedback is contained in a region of complex plane defined as

$$S_c = \left\{ z \in \mathbb{C} \middle| P_z(s) = s^n + c_{n-1} z s^{n-1} + \ldots + c_0 z \text{ stable} \right\}.$$

generalizing the concept of synchronizing region to systems described by polynomials.

6.5 Synchronization to the Leader Driven by an Unknown Input

This section gives a specific application of the results of previous sections on a multi-agent system with linear time-invariant agents having a leader driven by an input. Adding an input to the leader enlarges the set of possible tracking commands $x_0(t)$ one achieves synchronization to. Also, because of the way this input effects the system it satisfies the matching condition, 46, and can be ideally completely compensated using the appropriate control.

6.5.1 System Dynamics

Assume the leader node dynamics to be of the form

$$\dot{x}_0 = Ax_0 + Bu_0 , \tag{188}$$

and the set of N identical agents having dynamics

$$\dot{x}_i = Ax_i + Bu_i , \tag{189}$$

with $x_0, x_i \in \mathbb{R}^n; \forall i$, $u_0, u_i \in \mathbb{R}^p; \forall i$. The global synchronization error dynamics reads

$$\begin{aligned} \dot{\delta}_i &= Ax_i - Ax_0 + Bu_i - Bu_0 \\ &= A\delta_i + B(u_i - u_0) \end{aligned}, \tag{190}$$

which in global form equals

$$\dot{\delta} = (I_N \otimes A)\delta + I_N \otimes B(u - \overline{u}_0) . \tag{191}$$

Pick the distributed cooperative linear feedback control (160) $u_i = cKe_i$, (164). Such choice of distributed control guarantees synchronization without the leader's input, $u_0 = 0$, for $c > 0$ sufficiently large, as detailed in Lemma 6.5 and 13. In global form this leads to

$$u = -c(L+G) \otimes K(x - \overline{x}_0) = -c(L+G) \otimes K\delta , \tag{192}$$

and the global synchronization error dynamics equals

$$\begin{aligned} \dot{\delta} &= (I_N \otimes A)\delta - c(L+G) \otimes BK\delta - I_N \otimes B\overline{u}_0 \\ \dot{\delta} &= A_c\delta - I_N \otimes B\overline{u}_0 \end{aligned}. \tag{193}$$

Even though A_c can be assumed stable under conditions of Lemma 6.5, the system (193) is not autonomous so one cannot guarantee that $\lim_{t \to \infty} \delta = 0$, *i.e.* that the systems (189) reach synchronization asymptotically. In fact the second term in (193) can be interpreted as disturbance $\xi_0 = Bu_0$, arising solely from the leader's input. That disturbance acts only on the leader, but its effect is distributed to all the agents. If u_0 is uniformly bounded, asymptotic stability of A_c guarantees uniformly ultimately bounded convergence of the system (189), 46.

6.5.2 Estimating the Leader's Input

One solution to the leader's input problem is adding an additional compensating input, equal to u_0, to each agent, in order to compensate for the disturbance. This leads to the control law

$$u_i = cKe_i + u_0, \tag{194}$$

so that the i-th agent's dynamics equals

$$\dot{x}_i = Ax_i + Bu_0 + cBKe_i. \tag{195}$$

Such a choice makes the "drift" part of the agents dynamics $Ax_i + Bu_0$ equal to that of the leader node. This results in a δ-system

$$\begin{aligned} \dot{\delta} &= (I_N \otimes A)\delta - c(L+G) \otimes BK\delta + I_N \otimes B\bar{u}_0 - I_N \otimes B\bar{u}_0 \\ \dot{\delta} &= A_c\delta \end{aligned}, \tag{196}$$

which is an autonomous system that, under the assumption that A_c be asymptotically stable, leads to synchronization; $\lim_{t\to\infty} \delta = 0$. The problem with such a choice is that it requires u_0 to be known to all agents. That means, either pinning u_0 to each agent or off-line stored u_0 at each agent. Neither of these choices seems appealing.

However, a modified control law involving the estimator of the type introduced in Section 6.4, with a correction for every agent in the form

$$u_i = cKe_i + w_i \tag{197}$$

leads in global form to

$$\begin{aligned}\dot{\delta} &= (I_N \otimes A)\delta - c(L+G)\otimes BK\delta + I_N \otimes Bw - I_N \otimes B\bar{u}_0 \\ \dot{\delta} &= A_c\delta + I_N \otimes B(w-\bar{u}_0) \\ \dot{\delta} &= A_c\delta + I_N \otimes B\delta_w\end{aligned} \tag{198}$$

where $w = \begin{bmatrix} w_1 & \cdots & w_N \end{bmatrix}^T \in \mathbb{R}^{pN}$, and $\delta_w = w - \bar{u}_0$ is the estimation error. The augmented system can be written as

$$\frac{d}{dt}\begin{bmatrix} \delta \\ \delta_w \\ \vdots \\ \delta_w^{n-1} \end{bmatrix} = \begin{bmatrix} A_c & I_N \otimes B & 0 \cdots 0 \\ 0 & & \\ \vdots & A_w & \\ 0 & & \end{bmatrix}\begin{bmatrix} \delta \\ \delta_w \\ \vdots \\ \delta_w^{n-1} \end{bmatrix}, \tag{199}$$

where it is implicitly assumed that δ_w satisfies a, possibly high order, autonomous linear dynamics described by the system matrix A_w. Given the stability of A_c, one can guarantee $\lim_{t\to\infty}\delta = 0$ if also $\lim_{t\to\infty}\delta_w = 0$. But the latter is the consensus problem for the estimates $w_i \to u_0 \ \forall i$. Crucial problem is assuring the convergence of all observation w_i to the common value equal to u_0.

In order to solve this observation consensus problem one uses the control communication graph, or more generally some subgraph of it, still containing a directed spanning tree since this is necessary for convergence. Here we practically use the results of Section 4 to estimate the leader's disturbance signal.

Under the condition $u_0^{(k)} = 0$ identically the higher order observer (181)

$$w_i^{(k)} = \sum_{j=0}^{k-1} c_j \frac{d^j}{dt^j}\left[\sum_l e_{il}(w_l - w_i) + g_i(u_0 - w_i)\right], \tag{200}$$

or in global form

$$w^{(k)} = -\sum_{j=0}^{k-1} c_j (L+G)\otimes I_p \delta_w^{(j)}, \tag{201}$$

gives asymptotic estimate convergence for the proper choice of coefficients c_j *e.g.* as detailed in Theorem 6.3. One should note that the increasing order *k* requires observers of increasing complexity. The estimation error dynamics δ_w (201) satisfies the assumption on the linear form of the estimation error system (199).

If, however, one has the leader's input u_0 modeled as an output of an autonomous linear system, *i.e.* the command generator, given in state-space form as

$$\begin{aligned} \dot{v}_0 &= F v_0 \\ u_0 &= H v_0 \end{aligned},$$

with the system matrix F, and the output matrix H, of an appropriate order, one can use the observation-synchronization algorithm (179) in the following form

$$\dot{v}_i = F v_i + c_1 L_1 \left[\sum_j e_{ij} (w_j - w_i) + g_i (u_0 - w_i) \right].$$

$$w_i = H v_i$$

Synchronization of v_i, defined for every agent i, implies the output synchronization of w_i, meaning $\delta_w = w - \bar{u}_0 = I_N \otimes H(v - \bar{v}_0) \to 0$, which in turn guarantees the state synchronization; $\lim_{t\to\infty} \delta = 0$. One has the synchronization of estimates v_i if the matrix $I_N \otimes F - c_1(L+G) \otimes L_1 H$ is Hurwitz. In this case also the assumption on the linear form of the estimation error $\delta_v = v - \bar{v}_0$ dynamics (199) is satisfied.

Again it should be remarked that for more complicated form of the control input u_0, requiring increasing dimension of the state-space model for u_0, the correction algorithm becomes increasingly complicated.

6.6 Application to Second-order Double-integrator Systems With Disturbances

This section presents another application of disturbance estimation. Here we are concerned with the second-order double-integrator systems with disturbances. This model of primarily motion systems is often studied in the literature, 48,49,49. Let the multi-agent system consist of a leader

$$\begin{aligned}\dot{x}_0 &= v_0 \\ \dot{v}_0 &= u_0 + \xi_0\end{aligned}, \tag{202}$$

and N agents of the form

$$\begin{aligned}\dot{x}_i &= v_i \\ \dot{v}_i &= u_i + \xi_i\end{aligned}. \tag{203}$$

The disturbances in this case satisfy the matching condition.

Let the agents' disturbances ξ_i be modeled as (172), or more precisely as an output of (172) that is not directly measurable, *e.g.* as detailed in the numerical example, Section 6.9. Note that this is more general than the case presented in Subsections 6.4.1 and 6.4.2 where it is assumed that the entire disturbance state acts on the agent dynamics. Measurable output of the disturbance generator (172), ς_i, is used for disturbance system state estimation. Let the leader's disturbance be modeled as (178) or have the identically vanishing derivative of a finite order, $\xi_0^{(k)} = 0$.

Here one applies the results of Section 6.4, using the distributed observer (179) or (181) for the leader's disturbance and local observers (171) for agents' disturbances. The distributed observer (173) is applicable if some agents, root nodes in a spanning tree, know the fixed zero reference, 30. The agents that have the leader state pinned into them are considered here to know the fixed zero reference, 30. The estimates of ξ_i and ξ_0 are computed from (171) or (173) and (179) or (181) respectively. Those are then used as the compensating signal producing the total control signal

$$u_i = \sum_j e_{ij} \left\{ c_1 \left[(x_j - x_i) + g_i (x_0 - x_i) \right] + c_2 \left[(v_j - v_i) + g_i (v_0 - v_i) \right] \right\} - \hat{\xi}_i + \hat{\xi}_0 . \qquad (204)$$

The control gains c_1, c_2 are determined by the local Riccati design, as detailed in Lemma 6.5, $K = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$, and the observer gains are designed as detailed in Lemma 6.6, and Theorem 6.3. So by results of Lemma 6.5, 6.6, and Theorem 6.3, one has estimate convergence for the agents' and the leader's disturbances together with the asymptotic state synchronization under control (204).

6.7 Application of Disturbance Compensation to Dynamical Difference Compensation

This section investigates the effects of unmodelled dynamics and possibly heterogeneous agents' dynamics both described as state dependent disturbances acting on the nominal system. Given the agent's dynamics in control affine form

$$\dot{x}_i = f_i(x_i) + g_i(x_i) u_i ,$$

with all agents having the same order as dynamical systems, one can assume the existence of a nominal drift dynamics $f(x_i)$ having the same order as $f_i(x_i), \forall i$. Each agent can then be described as

$$\dot{x}_i = \underbrace{f(x_i) + g_i(x_i) u_i}_{nominal\ sistem} + \underbrace{\left[f_i(x_i) - f(x_i) \right]}_{\xi_i(x_i)} . \qquad (205)$$

In the case a leader is present, having dynamics given by

$$\dot{x}_0 = f_0(x_0) , \qquad (206)$$

the choice of the nominal dynamics is suggested by (206),

$$\dot{x}_i = \underbrace{f_0(x_i) + g_i(x_i) u_i}_{nominal\ sistem} + \underbrace{\left[f_i(x_i) - f_0(x_i) \right]}_{\xi_i(x_i)} . \qquad (207)$$

The error system in that case equals

$$\dot{\delta}_i = f_0(x_i) - f_0(x_0) + g_i(x_i) u + \xi_i(x_i) . \qquad (208)$$

Assuming Lipshitz continuity of f_0, or assuming differentiability and applying contraction analysis, allows one to assess the contribution of the nominal nonlinear drift part, $f_0(x_i)-f_0(x_0)$, in terms of δ_i. Note that the dynamical difference of the leader and an agent under consideration is decomposed into two parts

$$f_i(x_i)-f_0(x_0)=f_0(x_i)-f_0(x_0)+(f_i(x_i)-f_0(x_i)),$$

first part begin due to the state discrepancy from consensus, *i.e.* δ_i and the second one due to the inherent difference in dynamics.

If one can find distributed feedback control laws, $u_i(x)$, for all agents such that the nominal agents reach consensus asymptotically then the earlier conclusions can be applied to the disturbance term

$$\xi_i(x_i)=f_i(x_i)-f(x_i) \tag{209}$$

Furthermore, the compensating input can be used, since the nominal system is again affine in control $u_i=\tilde{u}_i+\omega_i$, such that the size of the effective disturbance $g_i(x_i)\omega_i+\xi_i$ is minimized. It is clear that if the dynamical difference $\xi_i(x_i)=f_i(x_i)-f(x_i)$ satisfies the matching condition, 46, then the compensating term ω_i can cancel it completely. In real situations dynamic difference observers should be used, 47.

An example of such a case is a set of feedback linearizable agents all having the same (full) relative order. As long as the same agent-state-space transformation, preserving partial stability notions as detailed in Lemma 6.4, brings all agents to the nonlinear controllable canonical form, the dynamical differences can be cancelled by using transformed control signals. Such an action preserves the consensus manifold S.

Yet another interesting example is furnished by the special case of agents having the following form

$$\dot{x}_i = Ax_i + Bu_i + \xi_i(x_i) .$$

Thus, all agents have identical linear dynamics (A, B), but the nonlinear unmodelled dynamics may be different. One can use linear distributed synchronization control (160) to guarantee the synchronization of the nominal system. If the disturbances are uniformly bounded $\xi_i(x_i) < M$, $\forall i$ the uniformly ultimately bounded consensus result readily follows. If, on the other hand, the unmodelled dynamics is the same, *i.e.* $\xi_i(x_i) = \xi_j(x_i)$, $\forall(i, j)$, then it satisfies

$$x \in S \Rightarrow \xi(x) \in TS_x ,$$

where S is the consensus affine manifold. Thus the consensus manifold S remains an invariant manifold for the multi-agent system. If the contribution of the unmodelled dynamics satisfes uniform bounds of Theorem 6.2 robustness of the linear part of the system guarantees synchronization for the nonlinear system. Such is the particular case of uniformly continuous functions, though this condition is not necessary. State dependent uniformly continuous disturbances $\xi(x_i)$, lumped together into a global vector, preserve the invariance of the consensus manifold S and satisfy the growth bound uniformly along the manifold. A specific example of this instance is the topic of the next section.

6.8 Example of State Dependent Disturbances-Agents' Dynamics Satisfying the Uniform Growth Bound

This section presents an example of a multi-agent system with identical agents having Lienard dynamics. Such systems are guaranteed to have limit cycles. The Lienard dynamical system describes an oscillator, having 2-dimensional state space $\mathbb{R}^2$, with an input. The corresponding differential equation has the form

$$\ddot{y} + \alpha\phi(y)\dot{y} + \psi(y) = u \tag{210}$$

In the state space $\mathbb{R}^2$ with state variables $x_1 = y, x_2 = \dot{y} + \alpha\Phi(y)$, where $\Phi(x_1) = \int_0^{x_1} \phi(\xi) d\xi$, the equation (210) can be written as a system,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 - \alpha\Phi(x_1) \\ -\psi(x_1) + u \end{bmatrix}. \tag{211}$$

The limit cycle exists if $\psi(y) > 0$ for $y > 0$, and $\lim_{y\to\infty} \Phi(y) = \infty$. If $\psi(y) = y$ one can separate the linear and nonlinear parts of the system as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \alpha\begin{bmatrix} \Phi(x_1) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \tag{212}$$

If the nonlinear function $\Phi(x_1)$ is uniformly continuous, which is accomplished for uniformly bounded $\phi(y)$, negative for small y, and positive for large y, conclusions of previous paragraphs become applicable. Taking for example

$$\phi(y) = \frac{y^2 - 1}{y^2 + 1}$$

one has $\Phi(x_1) = \int_0^{x_1} \frac{\xi^2 - 1}{\xi^2 + 1} d\xi$. A linear synchronization algorithm designed for linear part of the system, 13, guarantees exponential partial stability with respect to the consensus manifold, with the pertaining robustness property. As the nonlinear parts are the same for all agents their contribution to the partial stability Lyapunov function vanishes on the consensus manifold, and since the lumped vector of the nonlinear terms comprises a uniformly continuous function, all components being uniformly continuous, one can state that there exists a growth bound valid uniformly along the consensus manifold.

The single agent system can be concisely written as

$$\dot{x}_i = Ax_i + \alpha\xi(x_i) + Bu_i(x). \tag{213}$$

Multi-agent system has N agents of the form (213), with $u_i(x)$ the feedback cooperative control for agent i. One can choose the linear local neighborhood error cooperative control (160), with the feedback gain K designed using local Riccati design for the linear system (A,B). Since $\Phi(x_1)$ is uniformly continuous with a bounded derivative one has the following bound

$$\begin{aligned}&\sum_{i=1}^{N}\left\|\Phi(x_{1i})-\Phi(x_0)\right\|^2 \le \sum_{i=1}^{N} M^2\left\|x_{1i}-x_0\right\|^2\\&d^2(\overline{\Phi}(x),S) \le \sum_{i=1}^{N}\left\|\Phi(x_{1i})-\Phi(x_0)\right\|^2 \le \sum_{i=1}^{N} M^2\left\|x_{1i}-x_0\right\|^2 = M^2 d^2(x,\underline{x}_0) = M^2 d^2(x,S)\end{aligned}, \quad (214)$$

if x_0 is chosen so that it minimizes the distance of the state from the consensus manifold, *i.e.* $d(x,S)=d(x,x_0)$. M is a positive constant satisfying $M \le 1$. So the disturbance stemming from the nonlinear part of the dynamics satisfies the uniform growth bound (145).

If one had a leader of the form

$$\dot{x}_0 = Ax_0 + \alpha\xi(x_0) \quad (215)$$

The same conclusion applies, but now the system in question is the error system $\delta_i = x_i - x_0$

$$\dot{\delta}_i = A\delta_i + Bu(\delta_i) + \alpha(\xi(x_i) - \xi(x_0)) \quad (216)$$

Using again (160) with the local neighborhood error e_i modified appropriately for the pinning terms one obtains the closed loop system in global form

$$\dot{\delta} = A_{cl}\delta + \alpha(\Xi(x) - \Xi(\overline{x}_0)), \quad (217)$$

where $\Xi(x) = \begin{bmatrix}\xi^T(x_1) & \cdots & \xi^T(x_N)\end{bmatrix}^T$. Since one can assume the nominal system to be exponentially stable there is a classical quadratic Lyapunov function $V = \delta^T P \delta$ and with the growth bound $\left\|\Xi(x) - \Xi(\overline{x}_0)\right\| \le M\left\|\delta\right\|$ one has exponential stability given that the convergence rate of the nominal system is sufficiently large. The proof of the growth bound (146) in this case is identical to (214) with x_0 being the leader's state.

$$\sum_{i=1}^{N} \left\| \Phi(x_{1i}) - \Phi(x_0) \right\|^2 \le \sum_{i=1}^{N} M^2 \left\| x_{1i} - x_0 \right\|^2$$

$$\left\| \overline{\Phi}(x) - \overline{\Phi}(\overline{x}_0) \right\|^2 = \sum_{i=1}^{N} \left\| \Phi(x_{1i}) - \Phi(x_0) \right\|^2 \le \sum_{i=1}^{N} M^2 \left\| x_{1i} - x_0 \right\|^2 = M^2 \left\| \delta \right\|^2$$

In the case of cooperative regulator problem designing the linear distributed synchronization algorithm $u = -cL \otimes Kx$ for the system $\dot{x}_i = Ax_i + Bu_i$ and finding the quadratic partial stability Lyapunov function $V_1(x) = x^T P_1 x$ for it, with distance convergence sufficiently large to overpower the disturbance contribution (this begin measured by the matrix Q_1) (167), guarantees synchronization of the original nonlinear system globally.

Note that these results are different than conclusions based simply on the linearization, since the linearization guarantees distance convergence, *i.e.* synchronization, only locally, while the Lyapunov method gives an estimate of the attraction region to an appreciable distance from the consensus manifold S. For all initial states in this region of attraction, the trajectory converges to the consensus manifold, and in particular to the limit cycle there. For the feedback control signals $u(x)$ vanishing on the consensus manifold the dynamics on the consensus manifold equals the dynamics of a single free, *i.e.* uncontrolled, agent.

Since all the individual free agents' limit cycles, homeomorphic to S^1, comprise the *N*-torus $T^N = S^1 \times S^1 \times \cdots \times S^1$ in the product total space $\mathbb{R}^{2N}$ this is an invariant set for the non-interacting agents. The resulting limit set for interacting agents will be the intersection of the T^N with the consensus manifold S. This is a subset of the invariant set for non-interacting agents T^N. The fact that a dissipative distributed consensus algorithm, $u(x)$, changes the topological nature of the invariant sets for the closed loop system, as compared to the invariant sets for the aggregation of free (not controlled) systems, can be considered an onset of complex behaviour. This qualitatively different, complex, behaviour was absent when the agents were not interacting.

6.9 Numerical Example

This section gives numerical examples of the disturbance observation schemes and the control protocols introduced in this chapter, in particular Section 6.6. Consider the leader and 5 agents having the double integrator form (202), (203),

$$\frac{d}{dt}\begin{bmatrix} x_{0,i} \\ v_{0,i} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{0,i} \\ v_{0,i} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{0,i} + \begin{bmatrix} 0 \\ \xi_{0,i} \end{bmatrix}.$$

The leader's disturbance is modelled as $\xi_0(t) = 3\sin 0.0063t \sin 0.063t$, so $\xi_0^{(5)} \approx 10^{-9}$. The agents' disturbances ξ_i are modelled as an output of the disturbance generator that is not directly measurable. It is assumed that the output ς_i is directly measurable and it is used in the disturbance observer protocols.

$$\begin{aligned} \dot{y}_i &= 0.05\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} y_i \\ \xi_i &= \begin{bmatrix} 1 & 0 \end{bmatrix} y_i \\ \varsigma_i &= \begin{bmatrix} 0 & 1 \end{bmatrix} y_i \end{aligned}$$

All agents' disturbances are modelled by the same model, thought with different, random, initial conditions, so both local (171) and distributed (173) observer shall be applied. The interconnection graph and pinning gains are given by

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}, \quad G = \mathrm{diag}(1,0,0,0,0).$$

First we observe the behavior of the multi-agent system when disturbances are not compensated, that is under the distributed synchronization control for the nominal systems.

Then we compensate for the leader's disturbances assuming it alone acts on the system. Further, the agents' disturbances are allowed to act as well. They are estimated using both local and distributed observer, comparing the synchronization performance in both cases. For the design of the synchronization control law (204) we chose the local feedback gain using the Riccati design (163) $K = [0.2236 \quad 0.7051]$, with the coupling constant $c = 0.5$. This corresponds to the choice $Q = 0.05I_2, R = 1$ in (164). For the observers we made the following choices. The local observer gain is $L = [-0.9512 \quad 1.0465]^T$ which is obtained from the observer Riccati equation (175), (176). This corresponds to the choice $Q = I_2, R = 1$ in (176). The same observer gain was used for the distributed agents' disturbance observer (173) with the coupling constant $c_2 = 2$. The distributed leader input observer coefficients were chosen according to the Riccati design detailed in Theorem 6.3, with $Q = 0.01I_5, R = 1$. This gives the necessary coefficients as $10[0.1 \quad 0.5362 \quad 1.3876 \quad 2.1756 \quad 2.0884]$. Coefficients were deliberately chosen to give slow convergence so that the effects of leader's input estimation would be discernible.

Figure 6.1 depicts the generated agents' disturbance signals, while Figure 6.2 and 6.3 show the observer errors for local and distributed observers. The observers are seen to converge to the true values of agents' disturbances since their observer errors converge to zero. Small periodic deviations of observer errors from zero seen in Figure 6.2 and 6.3 are artifacts of computing errors.

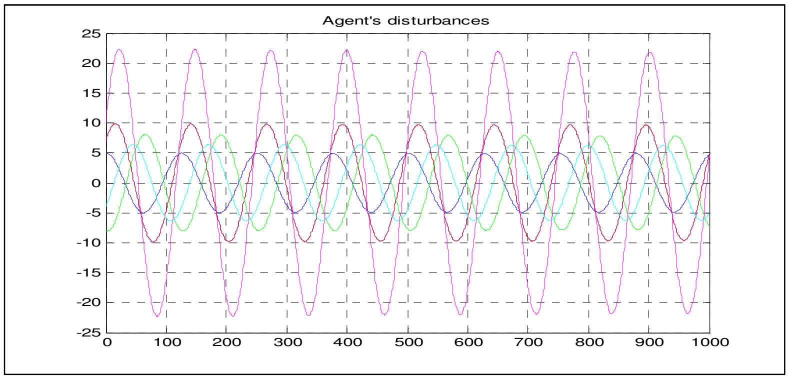


Figure 6.1 Disturbances acting on agents $\xi_1 \ldots \xi_5$

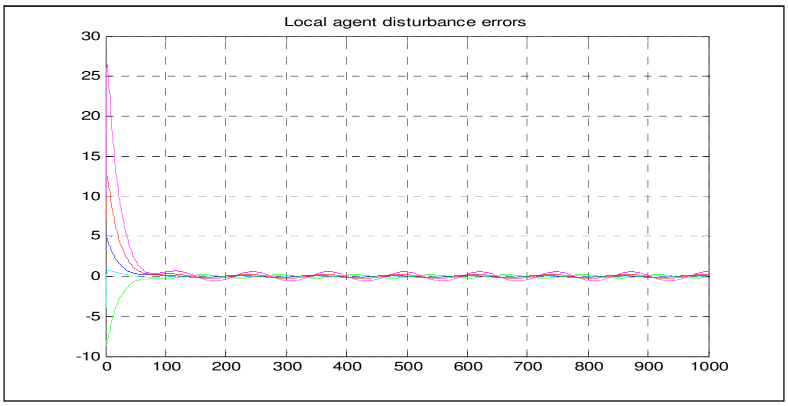


Figure 6.2 Agents' disturbance local estimate errors

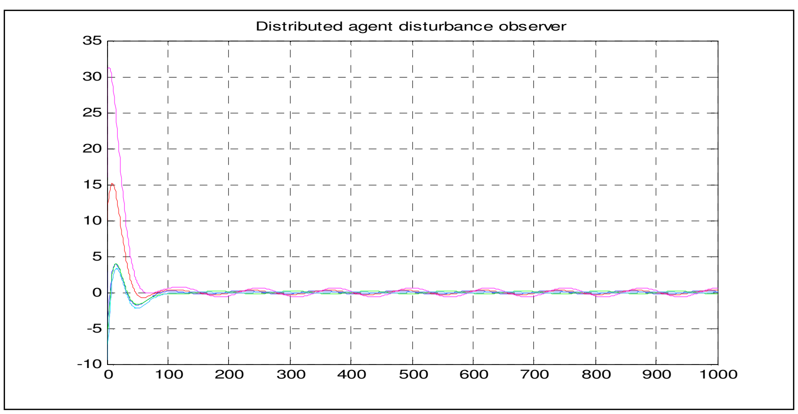


Figure 6.3 Agents' disturbance distributed estimate errors

Figure 6.4 gives a comparison of observer convergence for local and distributed observers of agents' disturbances. In this particular instance the local observers converge faster then the distributed observer. Though generally this need not be the case, and the use of a local versus a distributed observer is dictated by whether it is possible to measure absolute or only relative quantities.

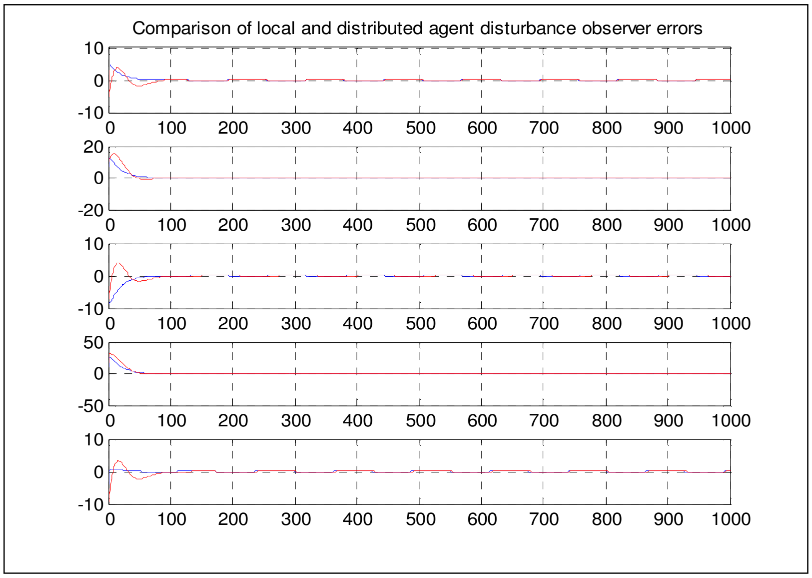


Figure 6.4 Comparison of local and distributed agents' disturbance estimate errors

The leader's disturbance signal, which is also interpreted as an input, is given together with the agents' estimates of the same in Figure 6.5. All agents' estimates are seen to converge to the leader's input signal.

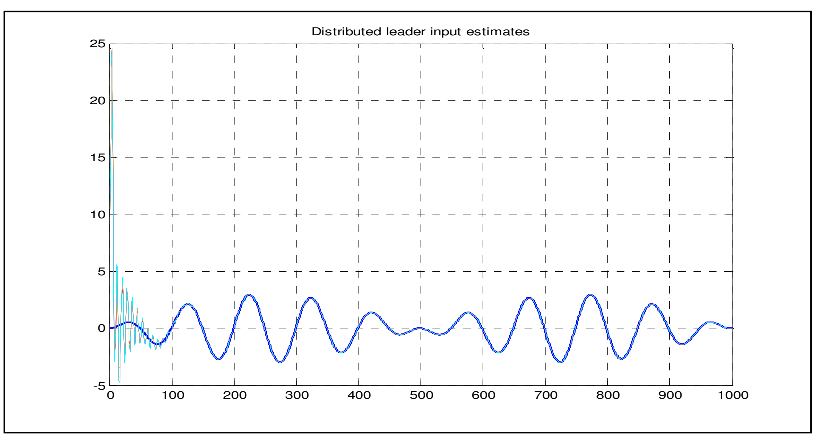


Figure 6.5 Leader's input distributed estimates

The remaining figures show the dynamics of the multi-agent system, *i.e.* the first states of all agents, in different circumstances. Figure 6.6 shows the case where the leader's input is not compensated, and the agents' disturbances do not act on the system. Synchronization is not achieved since the leader's disturbance prevents it. Notice that the effect of leader's disturbance gets distributed to all the following agents. Then, the leader's disturbance is estimated and compensated which results in synchronization to the leader's trajectory as shown in Figure 6.7.

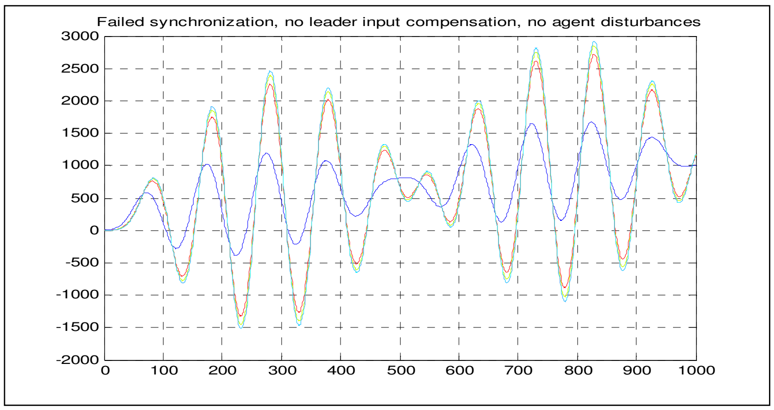


Figure 6.6 Agents' first states, agents disturbances do not act, there is no leader input compensation- synchronization to the leader is not achieved

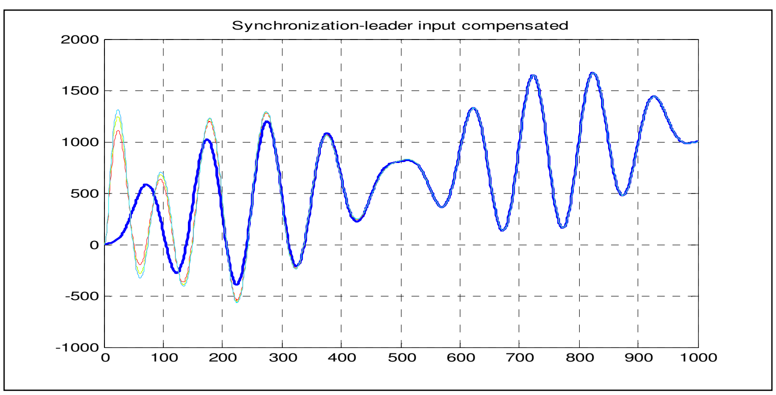


Figure 6.7 Agents' first states, leader's input is estimated and compensated, agent disturbances do not act, synchronization to the leader is achieved

However, when the disturbances act on the agents the synchronization is not achieved, as depicted in Figure 6.8. Figures 6.9 and 6.10 show cases of synchronization when the agents' disturbances are also estimated and compensated. First by the local observers, depicted in Figure 6.9, then by the distributed observers, depicted in Figure 6.10.

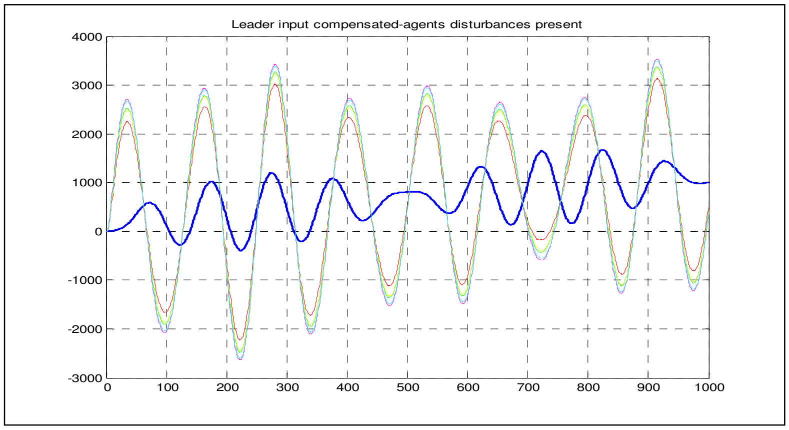


Figure 6.8 Agents' first states, leader's input is compensated, but disturbances acting on the agents are not compensated. The synchronization is not achieved

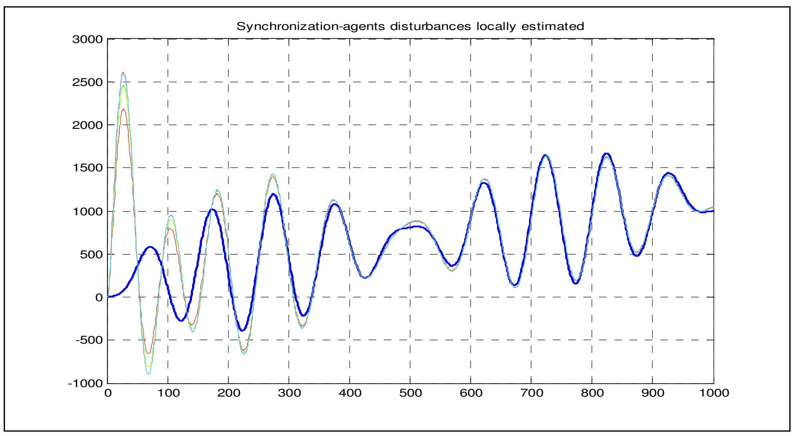


Figure 6.9 Agents' first states, leader's input is compensated, agents' disturbances are locally estimated and compensated. Synchronization is achieved.

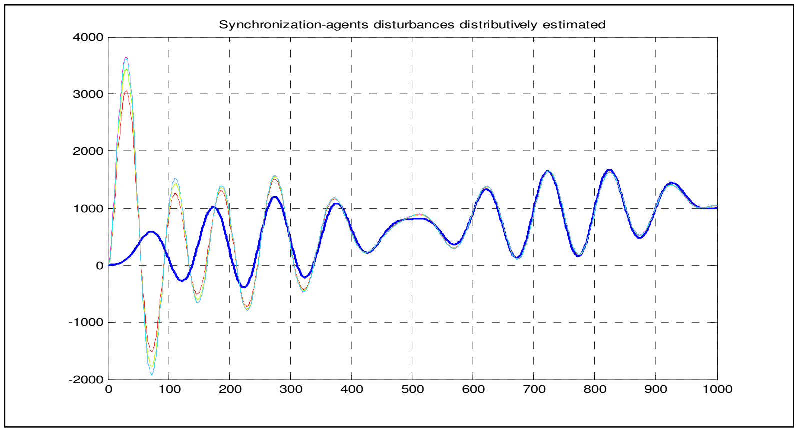


Figure 6.10 Agents' first states, leader's input is compensated, agents' disturbances are distributively estimated and compensated. Synchronization is achieved.

6.10 Conclusion

To conclude, this chapter presents various methods of disturbance estimation in multi-agent systems for the disturbances acting on the leader and on the following agents. If the nominal system reaches synchronization exponentially then it possesses a type of cooperative robustness which can be quantified by a Lyapunov function. However, in many cases such robustness alone does not guaranty that the control goal of synchronization is attained when disturbances are present. Disturbance estimators are used to guarantee the control goal, *i.e.* synchronization, even in presence of disturbances. Special applications were considered; the case of there being an input driving the leader, which needs to be distributively observed by all the following agents, the case of second-order double-integrator systems with disturbances acting on the leader and the agents, and the case of heterogeneous agents. Computer simulations show effectiveness of the proposed observation schemes and control design methods on the example of second-order double-integrator systems. This offers a comparison of multi-agent system performance with and without disturbance compensation. It can be seen from the presented example that good quality disturbance estimation and compensation is a necessary prerequisite for high performance multi-agent systems.

CHAPTER 7

COOPERATIVE OUTPUT FEEDBACK FOR STATE SYNCHRONIZATON

7.1 Introduction

The last two decades have witnessed an increasing interest in multi-agent network cooperative systems, inspired by natural occurrence of flocking and formation forming. These systems are applied to formations of spacecrafts, unmanned aerial vehicles, mobile robots, distributed sensor networks etc. 1,2,3,4,5,6,7,8,9. Early work with networked cooperative systems in continuous and discrete time is presented in 1,2,3,4,6,7. These papers generally referred to consensus without a leader. We call this the *cooperative regulator problem*. There, the final state of consensus depends on initial conditions. By adding a leader that pins to a group of other agents one can obtain synchronization to a command trajectory using a virtual leader, also named pinning control, 8,13,29. We call this the *cooperative tracker problem*. In the cooperative tracker problem all the agents synchronize to the leader's reference trajectory. Necessary and sufficient conditions for synchronization are given by the master stability function 10, and the related concept of the synchronizing region, 8,11,12. For continuous-time systems synchronization was guaranteed, 12,13, using local optimal state-feedback derived from the algebraic Riccati equation. It was shown that, using Riccati design of the local feedback gain for each node guarantees an unbounded right-half plane synchronization region in the *s*-plane. This allows for synchronization under mild conditions on the directed communication topology.

However, the entire state is not always available for the feedback control purposes. In such instances one possible solution is to use a dynamic output-feedback, as detailed in,13,28,30. There, the observer based dynamic cooperative regulator was used to guarantee state synchronization. This chapter investigates conditions under which state synchronization is asymptotically achieved by static output-feedback. Static output-feedback is simpler, and more easily implementable than dynamic output-feedback. No additional state observation is needed.

The structure of the chapter is as follows, Section 7.2 presents the multi-agent system dynamics and defines the control problem. Section 7.3 derives sufficient conditions on the distributed output control that guarantee the control goal. Following sections expand on the results of Section 7.3. Namely, Section 7.4 describes the distributed output-feedback in two-player zero-sum game context. Sufficient conditions are derived that allow for a solution of the global two-player zero-sum game by the distributed output-feedback. Section 7.5 specializes results of Section 7.4 to the case where the disturbance signal is absent, *i.e.* the cooperative globally optimal output-feedback control problem. Conditions are shown, under which the considered distributed output-feedback control is optimal with respect to a quadratic performance criterion, implying state synchronization. Section 7.5 can be seen as complementing the work presented in Chapter 4 that discusses global optimality of cooperative control laws for consensus and synchronization in the full-state feedback case. Section 7.6 examines the condition on the graph topology appearing in Sections 7.4 and 7.5, and identifies the key property of the graph matrix that satisfies that condition. Conclusions are presented in Section 7.7.

7.2 Synchronization With Distributed Output Feedback Control

The multi-agent system under consideration is comprised of N identical agents and a leader, having linear time-invariant dynamics. Let the leader system be given as

$$\begin{aligned} \dot{x}_0 &= Ax_0, \\ y_0 &= Cx_0, \end{aligned} \tag{218}$$

and the following agents as

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i, \\ y_i &= Cx_i. \end{aligned} \tag{219}$$

Definition 7.1: The *distributed synchronization control problem*, for a multi-agent system (219) with a leader (218), is to find distributed feedback controls, u_i, for agents, that guarantee $\lim_{t\to\infty}\|x_i(t)-x_0(t)\|=0, \forall i$. We call this the *cooperative tracker problem*.

Define the *local neighborhood error*

$$e_i = \sum_j e_{ij}(x_j - x_i) + g_i(x_0 - x_i)\,, \tag{220}$$

The feedback control signal for agent *i* is chosen as

$$u_i = cKCe_i\,. \tag{221}$$

The expression

$$e_{yi} = Ce_i = \sum_j e_{ij}(y_j - y_i) + g_i(y_0 - y_i)\,, \tag{222}$$

is the *local neighborhood output error*. Let $\delta_i = x_i - x_0$ be the synchronization error. In global form one has $\delta = \begin{bmatrix}\delta_1^T & \cdots & \delta_N^T\end{bmatrix}^T$, so

$$e = -(L+G)\otimes I_n\delta\,, \tag{223}$$

which appeared in 6,8. Given that the graph contains a spanning tree with at least one non zero pinning gain connecting into a root node, the matrix $L+G$ is nonsingular, 6, therefore $e=0 \Leftrightarrow \delta=0$. In global form, the local neighborhood output error, (222), equals

$$e_y = -(L+G)\otimes C\delta\,. \tag{224}$$

Expression (224) reflects the constraint on the information available for distributed feedback control. The communication topology, determined by the matrix $L+G$, and the structure of the output matrix C determine which states, and which parts thereof, can be used for distributed control purposes.

The choice of feedback (221) gives the closed loop agent system as

$$\dot{x}_i = Ax_i + cBKCe_i\,, \tag{225}$$

which yields the dynamics in global form

$$\dot{x} = (I_N \otimes A)x - c(L+G) \otimes BKC\delta . \tag{226}$$

The synchronization error global dynamics follows as

$$\dot{\delta} = (I_N \otimes A - c(L+G) \otimes BKC)\delta . \tag{227}$$

Lemma 7.1. The matrix $I_N \otimes A - c(L+G) \otimes BKC$ is Hurwitz if and only if all the matrices $A - c\lambda_j BKC$ are stable, where λ_j are the eigenvalues of the graph matrix $(L+G)$.

Proof: Upon applying the state transformation, $T \otimes I_n$, where $T^{-1}(L+G)T = \Lambda$ is a triangular matrix, to $I_N \otimes A - c(L+G) \otimes BKC$, one obtains the system matrix

$$I_N \otimes A - c\Lambda \otimes BKC . \tag{228}$$

The block-diagonal elements of (228), $A - c\lambda_j BKC$, determine the stability of the original matrix, $I_N \otimes A - c(L+G) \otimes BKC$. ■

The result of Lemma 7.1. allows for the interpretation of the stability properties of system (227) in the context of robust stabilization for a single agent system. Before proceeding further, a useful concept of a synchronizing region for output feedback is introduced. The notion of the synchronizing region for output feedback appeared in 28.

Definition 7.2: Given matrices (A, B, C), and the feedback gain K, the synchronizing region for the matrix pencil $A - \sigma BKC$ is a subset of the complex plane, $S_y = \{\sigma \in \mathbb{C} : A - \sigma BKC \text{ is stable}\}$. This we call the *synchronizing region for output feedback*.

Hence, the matrix (228) is stable if and only if all the scaled graph matrix eigenvalues, $c\lambda_j$, are in the synchronizing region for output feedback of the matrix pencil

$$A - \sigma BKC . \tag{229}$$

With the system (A,B,C) considered as given, the synchronizing region for output feedback, S_y, depends on the choice of the local output feedback gain K.

7.3 Local Output Feedback Design

This section investigates a specific choice of the local output feedback gain, K, and the pertaining synchronizing region for output feedback. The chosen local output feedback gain is determined by the solution of the output algebraic Riccati-type equation, 66, as detailed in the following subsection.

7.3.1 Output Feedback Gain

Let the local output feedback gain K satisfy the following relation for some matrix M,

$$KC = R^{-1}(B^T P + M), \tag{230}$$

where the matrix $P = P^T > 0$ solves the output algebraic Riccati-type equation

$$A^T P + PA + Q - PBR^{-1}B^T P + M^T R^{-1} M = 0. \tag{231}$$

First we show that the output-feedback (230) is stabilizing for the system (A,B). This result, presented here as a proposition, was mentioned in 66.

Proposition 7.1. Let the linear time-invariant system be given by matrices (A,B,C). Let there exists a positive definite solution, $P = P^T$, of the output algebraic Riccati-type equation

$$A^T P + PA + Q - PBR^{-1}B^T P + M^T R^{-1} M = 0. \tag{232}$$

Then, the output-feedback,

$$u = Ky = KCx, \tag{233}$$

with the output gain, K, satisfying

$$KC = R^{-1}(B^T P + M), \tag{234}$$

for some matrix M, is a stabilizing output feedback for the matrix $A - BKC$.

Proof: Take the quadratic Lyapunov function, $V(x) = x^T Px$, for the system determined by the matrix $A - BKC$. The matrix $P > 0$ is chosen as a solution of the output algebraic Riccati-type equation (232). The time derivative of this Lyapunov function is determined by the matrix

$$(A-BKC)^T P + P(A-BKC) = A^T P + PA - C^T K^T B^T P - PBKC$$
$$= A^T P + PA + (KC - R^{-1}B^T P)^T R^{-1}(KC - R^{-1}B^T P) - C^T K^T RKC - PBR^{-1}B^T P.$$

With the choice of the output feedback gain satisfying (234), this becomes

$$(A-BKC)^T P + P(A-BKC) = A^T P + PA + M^T R^{-1} M - C^T K^T RKC - PBR^{-1}B^T P$$
$$= -Q - C^T K^T RKC < 0,$$

hence the stability of the matrix $A - BKC$ is guaranteed. ■

7.3.2 The Guaranteed Synchronizing Region for Output Feedback

For the complex matrix pencil, $A - \sigma BKC$, one adopts a similar approach to find the guaranteed synchronizing region for output feedback. This motivates the following theorem.

Theorem 7.1. Let the multi-agent system be given by (218), (219). Let the graph have a spanning tree with at least one non-zero pinning gain connecting to a root note. Choose the distributed static output feedback control (221), where the output feedback gain, K, satisfies (234). Let the following abbreviations be introduced

$$a := \lambda_{\min}(Q^{-T/2} C^T K^T RKCQ^{-1/2}) > 0, \tag{235}$$

where $\lambda_{\min}$ denotes the smallest eigenvalue on the complement of the kernel of $KCQ^{-1/2}$,

$$2b := \left\| Q^{-T/2} C^T K^T MQ^{-1/2} \right\| + \left\| Q^{-T/2} M^T KCQ^{-1/2} \right\| \geq 0, \tag{236}$$

and

$$A_1 := 4(a^2 - b^2),\ A_2 := 4(a - a^2 + 2b^2),\ A_3 := 4b^2,\ A_4 := (1-a)^2 - (2b)^2. \tag{237}$$

The form of the guaranteed synchronizing region for output feedback depends on the matrix M, in that *i.* The guaranteed synchronizing region for output feedback is an interior of an elliptic region in $\mathbb{C}$ if $A_1 < 0$ and $A_4 - \frac{A_2^2}{4A_1} > 0$.

ii. The guaranteed synchronizing region for output feedback reduces to the empty set if $A_1 < 0$ and $A_4 - \frac{A_2^2}{4A_1} < 0$.

iii. The guaranteed synchronizing region for output feedback is a hyperbolic region in $\mathbb{C}$, having two connected components, if $A_1 > 0$ and $A_4 - \frac{A_2^2}{4A_1} < 0$.

iv. The guaranteed synchronizing region for output feedback is a hyperbolic region in $\mathbb{C}$, having a single connected component if $A_1 > 0$ and $A_4 - \frac{A_2^2}{4A_1} > 0$.

Proof: Take the quadratic Lyapunov function, $V(x) = x^\dagger Px$, for the complex system (229). The positive definite real matrix, $P = P^T$, is chosen as a solution of the output algebraic Riccati-type equation (231). The time derivative of this Lyapunov function is determined by the matrix

$$(A-\sigma BKC)^\dagger P + P(A-\sigma BKC) = A^T P + PA - \bar{\sigma} C^T K^T B^T P - \sigma PBKC,$$

which can be further written, by completing the squares, as

$$\begin{aligned}&A^T P + PA - \bar{\sigma} C^T K^T B^T P - \sigma PBKC \\ &= A^T P + PA + (\sigma KC - R^{-1}(B^T P + M))^\dagger R(\sigma KC - R^{-1}(B^T P + M)) \\ &-|\sigma|^2 C^T K^T RKC + \bar{\sigma} C^T K^T M + \sigma M^T KC - C^T K^T RKC.\end{aligned}$$

The choice of the output feedback satisfying (230) makes this expression equal to

$$A^T P + PA + |\sigma - 1|^2 C^T K^T RKC - |\sigma|^2 C^T K^T RKC + \bar{\sigma} C^T K^T M + \sigma M^T KC - C^T K^T RKC.$$

Using the output Riccati-type equation (231), this can be written as

$$
\begin{aligned}
&-Q + PBR^{-1}B^{T}P - M^{T}R^{-1}M - C^{T}K^{T}RKC \\
&+(1-2\operatorname{Re}\sigma)C^{T}K^{T}RKC + \bar{\sigma}C^{T}K^{T}M + \sigma M^{T}KC \\
&= -Q - (B^{T}P+M)^{T}R^{-1}(B^{T}P+M) - M^{T}R^{-1}M + PBR^{-1}B^{T}P \\
&+(1-2\operatorname{Re}\sigma)C^{T}K^{T}RKC + \bar{\sigma}C^{T}K^{T}M + \sigma M^{T}KC.
\end{aligned}
$$

The sufficient condition guaranteeing the synchronizing region for output-feedback then becomes

$$
\begin{aligned}
&-Q - (B^{T}P+M)^{T}R^{-1}(B^{T}P+M) - M^{T}R^{-1}M + PBR^{-1}B^{T}P \\
&+(1-2\operatorname{Re}\sigma)C^{T}K^{T}RKC + \bar{\sigma}C^{T}K^{T}M + \sigma M^{T}KC < 0,
\end{aligned}
$$

allowing the assessment of the output synchronizing region in $\mathbb{C}$. In particular, this expression is equivalent to

$$
-Q - |\sigma|^{2}C^{T}K^{T}RKC + |\sigma-1|^{2}C^{T}K^{T}RKC + (\bar{\sigma}-1)C^{T}K^{T}M + (\sigma-1)M^{T}KC < 0
$$

whence one finds the sufficient condition for the synchronizing region for output-feedback to be

$$
\begin{aligned}
&-Q - (2\operatorname{Re}\sigma - 1)C^{T}K^{T}RKC + (\bar{\sigma}-1)C^{T}K^{T}M + (\sigma-1)M^{T}KC < 0 \\
&\Leftrightarrow (I + 2(\operatorname{Re}\sigma - 1)Q^{-T/2}C^{T}K^{T}RKCQ^{-1/2}) > (\bar{\sigma}-1)Q^{-T/2}C^{T}K^{T}MQ^{-1/2} + (\sigma-1)Q^{-T/2}M^{T}KCQ^{-1/2}.
\end{aligned} \quad (238)
$$

This inequality is certainly satisfied if

$$
\begin{aligned}
&\lambda_{\min}(I + 2(\operatorname{Re}\sigma - 1)Q^{-T/2}C^{T}K^{T}RKCQ^{-1/2}) \\
&> \left\|(\sigma-1)^{*}Q^{-T/2}C^{T}K^{T}MQ^{-1/2} + (\sigma-1)Q^{-T/2}M^{T}KCQ^{-1/2}\right\|,
\end{aligned}
$$

where $\lambda_{\min}$ denotes the smallest eigenvalue on the subspace complement of the kernel of $KCQ^{-1/2}$, since on the $\ker(KCQ^{-1/2})$ the inequality (238) reduces to the tautology, $I > 0$. The properties of the matrix norm for Hermitan matrices give an inequality,

$$
\begin{aligned}
&\left\|(\sigma-1)^{*}Q^{-T/2}C^{T}K^{T}MQ^{-1/2} + (\sigma-1)Q^{-T/2}M^{T}KCQ^{-1/2}\right\| \\
&\le |\sigma-1|(\left\|Q^{-T/2}C^{T}K^{T}MQ^{-1/2}\right\| + \left\|Q^{-T/2}M^{T}KCQ^{-1/2}\right\|),
\end{aligned}
$$

which lends itself to give a conservative condition guaranteeing the synchronizing region for output-feedback,

$$\lambda_{\min}(I+2(\operatorname{Re}\sigma-1)Q^{-T/2}C^TK^TRKCQ^{-1/2})$$
$$=1+2(\operatorname{Re}\sigma-1)\lambda_{\min}(Q^{-T/2}C^TK^TRKCQ^{-1/2})\geq|\sigma-1|(\|Q^{-T/2}C^TK^TMQ^{-1/2}\|+\|Q^{-T/2}M^TKCQ^{-1/2}\|). \tag{239}$$

The conservative nature of the stability condition (239), in this case, is the price to pay for simplicity. Introducing the abbreviations (235),(236), renders the condition (239) for the synchronizing region for output-feedback to a considerably simpler form

$$1+(2\operatorname{Re}\sigma-1)a>2|\sigma-1|b. \tag{240}$$

The geometry in $\mathbb{C}$ determined by this expression can be elucidated by explicitly calculating the bound for $\sigma=x+jy\in\mathbb{C}$. Assuming $1+(2\operatorname{Re}\sigma-1)a>0$, which, given positiveness of *b*, is necessary for the inequality (240) to hold, one can square both sides to obtain an equivalent inequality

$$(1+(2\operatorname{Re}\sigma-1)a)^2>4|\sigma-1|^2b^2.$$

This equals

$$(1+(2x-1)a)^2>4((x-1)^2+y^2)b^2,$$
$$1+2a(2x-1)+(2x-1)^2a^2-4b^2(x-1)^2-4b^2y^2>0,$$

and after some straightforward algebraic manipulations one obtains an expression

$$4(a^2-b^2)x^2+4(a-a^2+2b^2)x-4b^2y^2+(1-a)^2-(2b)^2>0.$$

At this point, for the sake of simplicity, further abbreviations, (237), are introduced yielding the expression, $A_1x^2+A_2x-A_3y^2+A_4>0$. Finally, by completing the squares in x, one finds the geometrical meaning of the inequality (240), expounded as

$$A_1(x+\frac{A_2}{2A_1})^2-A_3y^2>-A_4+\frac{A_2^2}{4A_1}. \tag{241}$$

Depending on the signs of A_1, $A_4-\dfrac{A_2^2}{4A_1}$ this is either an unbounded hyperbolic region in the complex plane or the interior of an elliptic region. Since $A_3\geq0$, if $A_1=4(a^2-b^2)<0$ one has an interior of the elliptic region for $A_4 - \frac{A_2^2}{4A_1} > 0$, or an empty set for $A_4 - \frac{A_2^2}{4A_1} < 0$. If, alternatively, $A_1 = 4(a^2 - b^2) > 0$ one has a hyperbolic region. This region has two connected components if $A_4 - \frac{A_2^2}{4A_1} < 0$, or a single connected component if $A_4 - \frac{A_2^2}{4A_1} > 0$. ■

In case *ii*) of Theorem 7.1, the guaranteed synchronizing region for output-feedback reduces to an empty set. However, the stability condition (240) is conservative, hence this does not mean that the actual synchronizing region is empty. By the result of Proposition 7.1, given the output-feedback gain (234), the complex number $\sigma = 1$ is an element of a synchronizing region for output-feedback, therefore the latter cannot empty.

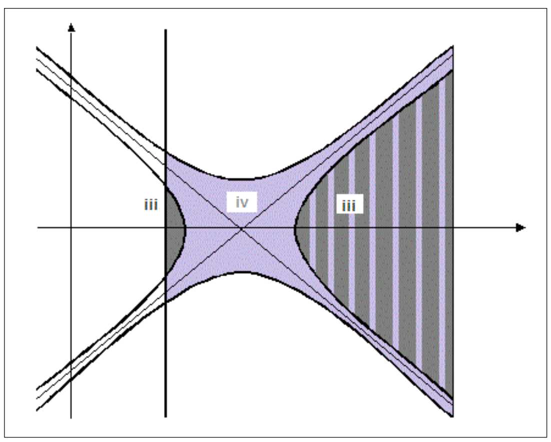


Figure 7.1 The connected and disconnected hyperbolic regions of Theorem 7.1.

Remark 7.1: Notice that in case $M = 0$ one recovers the unbounded synchronizing region characteristic of the full-state feedback, 12,13,

$$-Q + (1 - 2\operatorname{Re}\sigma)C^T K^T RKC < 0 .$$

A key observation that one should make is, $M=0 \Rightarrow b=0$, and by continuity, b grows with M. Therefore, for M small enough, in terms of the matrix norm, one has a hyperbolic unbounded synchronizing region for output-feedback. As M grows this hyperbola can turn into an ellipse, making the synchronizing region for output-feedback bounded. In the case of unbounded hyperbolic region the asymptotes of the hyperbolae bounding the region in $\mathbb{C}$ give an angle within which all the graph matrix, $L+G$, eigenvalues should reside in order that they can be scaled into the synchronizing region by radial projection; achieved by a coupling gain $c>0$. Of course, having an unbounded synchronizing region is preferable, but the existence of it is determined by the size of M. Depending on the system matrices, (A,B,C), the choice, $M=0$, generally cannot be made. The size of M in any particular case can be interpreted as a measure of how different the output-feedback is from the full-state feedback.

7.4 H_∞ Synchronization of Multi-agents Systems on Graphs

This section discusses the two-player zero-sum game framework for output-feedback and gives conditions under which the cooperative distributed static output-feedback is a solution of the global two-player zero-sum game. The multi-agent system has a leader

$$\begin{aligned} \dot{x}_0 &= Ax_0, \\ y_0 &= Cx_0, \end{aligned} \tag{242}$$

and the following agents,

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i + Dd_i, \\ y_i &= Cx_i, \end{aligned} \tag{243}$$

where d_i are the disturbance signals acting on the agents. One is interested in the dynamics of the synchronization error , $\delta_i = x_i - x_0$. In global form this dynamics reads

$$\begin{aligned} \dot{\delta} &= (I_N \otimes A)\delta + (I_N \otimes B)u + (I_N \otimes D)d, \\ y &= (I_N \otimes C)\delta. \end{aligned} \tag{244}$$

The output of (244) has been redefined bearing in mind its use in the control law (221). The system (244) has two inputs, and the theory of two-player zero-sum games, 44, can be applied.

Given a performance criterion, $J(x_0, u, d)$, and a dynamical system with two inputs, u, d, the two-player zero-sum game puts the control u and disturbance d at odds with one another. The objective of the control is to minimize the performance criterion J, while the objetive of the disturbance is to maximize it. Hence, anything one input gains in its objective is lost by the other. For games in general, the concept of Nash equilibrium is of central importance, 35,44.

Definition 7.3: Given a performance criterion, $J(x_0, u, d)$, the policies u^*, d^* are in the Nash equilibrium if

$$J(x(0), u^*, d) \leq J(x(0), u^*, d^*) \leq J(x(0), u, d^*).$$

According to the Nash condition, if both players are at equilibrium, then neither has any incentive to change the policy unilaterally, since unilateral changes make one's performance worse. The two player zero sum game has a unique solution if a game theoretic saddle point, (u^*, d^*), exists, that is if the value of the performance criterion at the Nash equilibrium, $V(x_0) = J(x_0, u^*, d^*)$, satisfies

$$V^*(x_0) = \min_u \max_d J(x(0), u, d) = \max_d \min_u J(x(0), u, d)$$

Having introduced this much of the dynamic game theory, 44, the main result of this section is presented as the following theorem.

Theorem 7.2. Let the synchronization error dynamics with disturbances acting upon the agents be given in global form as (244). Let the graph have a spanning tree with at least one non-zero pinning gain connecting to a root node. Suppose there exist matrices P_1, P_2, symmetric and positive definite, satisfying

$$P_1 = cR_1 (L+G) \tag{245}$$

$$A^T P_2 + P_2 A + Q_2 - P_2 B R_2^{-1} B^T P_2 + \gamma^{-2} P_2 D D^T P_2 + M_2^T R_2^{-1} M_2 = 0 \tag{246}$$

for some $Q_2 = Q_2^T > 0, R_1 = R_1^T > 0, R_2 = R_2^T > 0$, and the coupling gain, $c > 0$. Define the output-feedback gain matrix, K_2^*, as the one satisfying

$$K_2^* C = R_2^{-1}(B^T P_2 + M_2), \tag{247}$$

with matrix M_2 satisfying

$$C^T(K_2 - K_2^*)^T R_2 (K_2 - K_2^*)C + M_2^T (K_2 - K_2^*)C + C^T (K_2 - K_2^*)^T M_2 \geq 0. \tag{248}$$

for any $K_2 \neq K_2^*$. Then the control

$$u^* = -c(L+G) \otimes K_2^* C\delta, \tag{249}$$

and the worst case disturbance

$$d^* = \gamma^{-2} T^{-1} D^T P\delta, \tag{250}$$

are the solution of the two-player zero-sum game with respect to the performance index

$$J = \int_0^\infty (\delta^T Q\delta + u^T Ru - \gamma^2 d^T Td)dt, \tag{251}$$

where matrices Q, R, T are given by

$$R = R_1 \otimes R_2,$$

and either

$T = P_1 \otimes I = cR_1(L+G) \otimes I$, with

$$Q = c^2(L+G)^T R_1(L+G) \otimes (Q_2 + A^T P_2 + P_2 A + \gamma^2 P_2 DD^T P_2) - cR_1(L+G) \otimes (A^T P_2 + P_2 A + \gamma^2 P_2 DD^T P_2) > 0 \tag{252}$$

or

$T = R_1 \otimes I$, with

$$Q = c^2(L+G)^T R_1(L+G) \otimes (Q_2 + A^T P_2 + P_2 A) - cR_1(L+G) \otimes (A^T P_2 + P_2 A) > 0. \tag{253}$$

Proof: Let the assumed output-feedback control be

$$u = K(I_N \otimes C)\delta = K_1 \otimes K_2 C\delta.$$

The Hamiltonian of the problem, with the quadratic value function, $V(\delta)=\delta^T P\delta$, equals

$$H=\delta^T(P(A-BKC)+(A-BKC)^T P)\delta+\delta^T PDd+d^T D^T P\delta+\delta^T(Q+C^T K^T RKC)\delta-\gamma^2 d^T Td \quad (254)$$

The worst case disturbance, d^*, is obtained from the stationarity condition

$$\frac{\partial H}{\partial d}=0\Rightarrow d^*=\gamma^{-2}T^{-1}D^T P\delta\,, \quad (255)$$

and with such a disturbance the Hamiltonian becomes

$$\begin{aligned}H&=\delta^T(P(A-BKC)+(A-BKC)^T P)\delta+\gamma^{-2}\delta^T PDT^{-1}D^T P\delta\\&+\delta^T(Q+C^T K^T RKC)\delta\end{aligned}.$$

This quadratic form is determined by the matrix

$$\begin{aligned}&P(A-BKC)+(A-BKC)^T P+Q+C^T K^T RKC+\gamma^{-2}PDT^{-1}D^T P\\&=PA+A^T P+Q+\gamma^{-2}PDT^{-1}D^T P-PBKC-C^T K^T B^T P+C^T K^T RKC\\&=A^T P+PA+Q+\gamma^{-2}PDT^{-1}D^T P-PBR^{-1}B^T P+(KC-R^{-1}B^T P)^T R(KC-R^{-1}B^T P)\\&=A^T P+PA+Q+\gamma^{-2}PDT^{-1}D^T P-PBR^{-1}B^T P+M^T R^{-1}M.\end{aligned} \quad (256)$$

With system (244) at hand, the structured value function $P=P_1\otimes P_2$, and structured matrices R,T of the Theorem, one finds that the matrix (256) is equal to

$$\begin{aligned}&(I_N\otimes A^T)(P_1\otimes P_2)+(P_1\otimes P_2)(I_N\otimes A)+Q\\&+\gamma^{-2}(P_1\otimes P_2)(I_N\otimes D)(T_1^{-1}\otimes T_2^{-1})(I_N\otimes D^T)(P_1\otimes P_2)\\&-(P_1\otimes P_2)(I_N\otimes B)(R_1^{-1}\otimes R_2^{-1})(I_N\otimes B^T)(P_1\otimes P_2)+(M_1^T\otimes M_2^T)(R_1^{-1}\otimes R_2^{-1})(M_1\otimes M_2)\end{aligned}$$

Choosing $T_2=I$, of an appropriate dimension, one has $T=T_1\otimes I$ and

$$P_1\otimes(A^T P_2+P_2A)+Q+\gamma^{-2}P_1T_1^{-1}P_1\otimes P_2DD^T P_2-P_1R_1^{-1}P_1\otimes P_2BR_2^{-1}B^T P_2+M_1^T R_1^{-1}M_1\otimes M_2^T R_2^{-1}M_2\,.$$

For the output-feedback one has

$$\begin{aligned}KC&=(K_1\otimes K_2)(I_N\otimes C)=K_1\otimes K_2C\\&=(R_1^{-1}\otimes R_2^{-1})(P_1\otimes B^T P_2+M_1\otimes M_2)\\&=R_1^{-1}P_1\otimes R_2^{-1}(B^T P_2+M_2)\\&=c(L+G)\otimes K_2C\end{aligned},$$

which is satisfied, given the global topology conditions, (245), and the local output feedback condition, (247),

$$\begin{aligned} R_1^{-1}P_1 &= c(L+G) \\ R_2^{-1}(B^T P_2 + M_2) &= K_2 C \end{aligned}.$$

Nothing in the conditions for output feedback, (245), (247), specifies the matrix T. Choosing $T_1 = P_1 > 0$, and using $M_1 = P_1$ gives

$$\begin{aligned} &P_1 \otimes (A^T P_2 + P_2 A) + Q + \gamma^{-2} P_1 P_1^{-1} P_1 \otimes P_2 D D^T P_2 - P_1 R_1^{-1} P_1 \otimes P_2 B R_2^{-1} B^T P_2 + P_1 R_1^{-1} P_1 \otimes M_2^T R_2^{-1} M_2 \\ &= P_1 \otimes (A^T P_2 + P_2 A) + Q + \gamma^{-2} P_1 \otimes P_2 D D^T P_2 - P_1 R_1^{-1} P_1 \otimes (P_2 B R_2^{-1} B^T P_2 - M_2^T R_2^{-1} M_2) \\ &= P_1 \otimes (A^T P_2 + P_2 A + \gamma^{-2} P_2 D D^T P_2) + Q - P_1 R_1^{-1} P_1 \otimes (P_2 B R_2^{-1} B^T P_2 - M_2^T R_2^{-1} M_2) \end{aligned}.$$

If one takes the Q matrix to be

$$Q = P_1 \otimes (Q_2 - P_2 B R_2^{-1} B^T P_2 + M_2^T R_2^{-1} M_2) + Q_1 \otimes (P_2 B R_2^{-1} B^T P_2 - M_2^T R_2^{-1} M_2),$$

there follows that

$$\begin{aligned} &P_1 \otimes (A^T P_2 + P_2 A + \gamma^{-2} P_2 D D^T P_2 + Q_2 - P_2 B R_2^{-1} B^T P + M_2^T R_2^{-1} M_2) \\ &+ (Q_1 - P_1 R_1^{-1} P_1) \otimes (P_2 B R_2^{-1} B^T P_2 - M_2^T R_2^{-1} M_2) = 0. \end{aligned}$$

That is certainly satisfied if

$$Q_1 = P_1 R_1^{-1} P_1,$$

$$A^T P_2 + P_2 A + Q_2 - P_2 B R_2^{-1} B^T P_2 + \gamma^{-2} P_2 D D^T P_2 + M_2^T R_2^{-1} M_2 = 0.$$

Alternatively, the choice $T_1 = R_1$ gives

$$P_1 \otimes (A^T P_2 + P_2 A) + Q + P_1 R_1^{-1} P_1 \otimes (\gamma^{-2} P_2 D D^T P_2 - P_2 B R_2^{-1} B^T P_2 + M_2^T R_2^{-1} M_2) = 0,$$

suggesting the choice of

$$Q = P_1 \otimes (Q_2 - P_2 B R_2^{-1} B^T P_2 + M_2^T R_2^{-1} M_2 + \gamma^{-2} P_2 D D^T P_2) + Q_1 \otimes (P_2 B R_2^{-1} B^T P_2 - M_2^T R_2^{-1} M_2 - \gamma^{-2} P_2 D D^T P_2) \quad (257)$$

or equivalently

$$Q = -P_1 \otimes (A^T P_2 + P_2 A) + Q_1 \otimes (Q_2 + A^T P_2 + P_2 A), \quad (258)$$

where $Q_1 = P_1 R_1^{-1} P_1 = c^2 (L+G)^T R_1 (L+G)$. The global topology condition, (245), used in expressions (257), (258), gives matrices (252), (253).

Hence, with the choice of the form of matrices Q, R, P, T, detailed in the Theorem one has the global output algebraic Riccati-type equation decoupled into local and global parts, expressing the conditions on the graph topology and the local output feedback.

In order for the output feedback control law, (249), and the worst case disturbance, (255), to be in the Nash equilibrium for the two player zero sum game , the Hamiltonian, (254), needs to satisfy an additional requirement 66. With the quadratic value function, $V(\delta) = \delta^T P \delta > 0$, one has the Hamiltonian

$$\begin{aligned} H &= \delta^T (P(A-BKC) + (A-BKC)^T P)\delta + \delta^T PDd + d^T D^T P\delta \\ &+\delta^T (Q + C^T K^T RKC)\delta - \gamma^2 d^T T d \end{aligned},$$

where the worst case disturbance is given as $d^* = \gamma^{-2} T^{-1} D^T P \delta$. With this one finds that

$$\begin{aligned} H(\delta, \nabla V, K, d^*) &= \delta^T (P(A-BKC) + (A-BKC)^T P)\delta + 2\gamma^{-2} \delta^T PDT^{-1} D^T P\delta \\ &+\delta^T (Q + C^T K^T RKC)\delta - \gamma^{-2} \delta^T PDT^{-1} D^T P\delta \\ &= \delta^T (P(A-BKC) + (A-BKC)^T P + Q + C^T K^T RKC + \gamma^{-2} PDT^{-1} D^T P)\delta \end{aligned}.$$

The matrix determining this quadratic form can be further written as

$$\begin{aligned} & P(A-BKC) + (A-BKC)^T P + Q + C^T K^T RKC + \gamma^{-2} PDT^{-1} D^T P \\ &= A^T P + PA + Q + \underbrace{C^T K^T RKC - PBKC - C^T K^T B^T P}_{(KC - R^{-1} B^T P)^T R (KC - R^{-1} B^T P) - PBR^{-1} B^T P} + \gamma^{-2} PDT^{-1} D^T P \\ &= A^T P + PA + Q + (KC - R^{-1} B^T P)^T R (KC - R^{-1} B^T P) - PBR^{-1} B^T P + \gamma^{-2} PDT^{-1} D^T P \end{aligned}.$$

Therefore, one finds for K^*, satisfying the output feedback condition $K^* C = R^{-1}(B^T P + M)$, that

$$\begin{aligned} & H(\delta, \nabla V, K^*, d^*) = \\ &= \delta^T (PA + A^T P + Q - PBR^{-1} B^T P + \gamma^{-2} PDT^{-1} D^T P + M^T R^{-1} M)\delta \end{aligned}.$$

The original Hamiltonian is now expressed as

$$
\begin{aligned}
&H(\delta,\nabla V,K,d)=\delta^T(P(A-BKC)+(A-BKC)^T P)\delta+\delta^T PDd+d^T D^T P\delta+\delta^T(Q+C^T K^T RKC)\delta-\gamma^2 d^T Td \\
&\delta^T(PA+A^T P+Q+\gamma^{-1}PDT^{-1}D^T P-PBR^{-1}B^T P+M^T R^{-1}M)\delta \\
&-\delta^T(PBKC+C^T K^T B^T P-C^T K^T RKC+\gamma^{-1}PDT^{-1}D^T P-PBR^{-1}B^T P+M^T R^{-1}M)\delta \\
&+\delta^T PDd+d^T D^T P\delta-\gamma^2 d^T Td \\
&=H(\delta,\nabla V,K^*,d^*)+\delta^T(-PB(KC-R^{-1}B^T P)-C^T K^T(B^T P-RKC)-M^T R^{-1}M)\delta \\
&-\underbrace{\gamma^{-2}\delta^T PDT^{-1}D^T P\delta}_{\gamma^2 d^{*T}Td^*}+\underbrace{\delta^T PDd}_{\gamma^2 d^{*T}Td}+d^T D^T P\delta-\gamma^2 d^T Td \\
&=H(\delta,\nabla V,K^*,d^*)-\gamma^2(d-d^*)^T T(d-d^*)+\delta^T(-PB(KC-R^{-1}B^T P)-C^T K^T(B^T P-RKC)-M^T R^{-1}M)\delta
\end{aligned}
$$

Hence one has an optimal solution, or the game theoretic saddle point equilibrium if the matrix M is chosen such that

$$-PB(KC-R^{-1}B^T P)-C^T K^T(B^T P-RKC)-M^T R^{-1}M\geq 0\,.$$

Taking into account that $K^* C=R^{-1}B^T P+R^{-1}M$, the matrix condition in question can be written as

$$C^T(K-K^*)^T R(K-K^*)C+M^T(K-K^*)C+C^T(K-K^*)^T M\geq 0\,. \tag{259}$$

This is clearly satisfied in the special case of $M=0$.

Given that $K=K_1\otimes K_2, K^*=K_1\otimes K_2^*$ since K_1 is fixed by the graph structure, $L+G$, this general condition, for the purposes of the Theorem, reduces to

$$
\begin{aligned}
&(I_N\otimes C^T)K_1^T\otimes(K_2-K_2^*)^T(R_1\otimes R_2)K_1\otimes(K_2-K_2^*)(I_N\otimes C) \\
&+(M_1^T\otimes M_2^T)K_1\otimes(K_2-K_2^*)(I_N\otimes C)+(I_N\otimes C^T)K_1^T\otimes(K_2-K_2^*)^T(M_1\otimes M_2)>0\,.
\end{aligned}
$$

With $M_1=P_1$ one obtains

$$
\begin{aligned}
&K_1^T R_1 K_1\otimes C^T(K_2-K_2^*)^T R_2(K_2-K_2^*)C+P_1K_1\otimes M_2^T(K_2-K_2^*)C+K_1^T P_1\otimes C^T(K_2-K_2^*)^T M_2 \\
&=P_1R_1^{-1}P_1\otimes C^T(K_2-K_2^*)^T R_2(K_2-K_2^*)C+P_1R_1^{-1}P_1\otimes(M_2^T(K_2-K_2^*)C+C^T(K_2-K_2^*)^T M_2)\,. \\
&=P_1R_1^{-1}P_1\otimes(C^T(K_2-K_2^*)^T R_2(K_2-K_2^*)C+M_2^T(K_2-K_2^*)C+C^T(K_2-K_2^*)^T M_2)
\end{aligned}
$$

The first term being positive definite and equal to $P_1R_1^{-1}P_1=c^2(L+G)^T R_1(L+G)$ from (245), positive definiteness of the second term guarantees the positive definiteness of the entire expression. Therefore, if M_2 is chosen such that

$$C^T(K_2-K_2^*)^T R_2(K_2-K_2^*)C + M_2^T(K_2-K_2^*)C + C^T(K_2-K_2^*)^T M_2 \geq 0,$$

then the entire global expression (259) is positive semidefinite. This implies that if the local output feedback K_2^*C is the solution of the local optimal problem, or the game theoretic problem, then the same holds for the global problem. ■

Remark 7.2: Conditions in Theorem 7.2, that the Q matrix be positive definite, (252),(253), are not trivial conditions. For those to be satisfied, in case *i*) one needs to have

$$Q_2 + A^T P_2 + P_2 A + \gamma^2 P_2 DD^T P_2 \geq 0. \tag{260}$$

According to the output Riccati-type equation (260) is equivalent to

$$P_2 BR_2^{-1} B^T P_2 - M_2^T R_2^{-1} M_2 \geq 0, \tag{261}$$

which is a condition on the size of M_2. Also, since $R_2 > 0$, (261) implies that $\ker(B^T P_2) \subseteq \ker M_2$. Similarly in the case *ii*), one must have

$$Q_2 + A^T P_2 + P_2 A \geq 0, \tag{262}$$

which is equivalent to

$$P_2 BR_2^{-1} B^T P_2 - \gamma^{-2} P_2 DD^T P_2 - M_2^T R_2^{-1} M_2 \geq 0. \tag{263}$$

The expression (263) is a condition on the size of M_2, D. Both conditions (261), (263) are satisfied in the case $M_2 = 0, D = 0$. Given conditions (261), (263), choosing the constant c sufficiently great will make the matrix Q in (251), positive definite.

7.5 Case of Optimal Output Feedback

This section addresses the special case of the two-player zero-sum framework, when the noise is absent. Setting $D = 0$ in (244), one recovers the system in global form

$$\begin{aligned} \dot{x} &= (I_N \otimes A)x + (I_N \otimes B)u, \\ y &= (I_N \otimes C)x, \end{aligned} \tag{264}$$

or, equivalently, the synchronization error system with an output, $y_i = C(x_i - x_0)$,

$$\begin{aligned} \dot{\delta} &= (I_N \otimes A)x + (I_N \otimes B)u, \\ y &= (I_N \otimes C)\delta. \end{aligned} \tag{265}$$

The game theoretic problem reduces to the optimal control problem for output-feedback.

Theorem 7.3. Let the synchronization error dynamics be given in global form as (265). Let the graph have a spanning tree with at least one non-zero pinning gain connecting to a root node. Suppose there exist matrices P_1, P_2, symmetric and positive definite, satisfying

$$P_1 = cR_1\,(L+G) \tag{266}$$

$$A^T P_2 + P_2 A + Q_2 - P_2 B R_2^{-1} B^T P + M_2^T R_2^{-1} M_2 = 0 \tag{267}$$

for some $Q_2 = Q_2^T > 0,\ R_1 = R_1^T > 0,\ R_2 = R_2^T > 0$, and the coupling gain, $c > 0$. Define the output-feedback gain matrix, K_2^*, as the one satisfying

$$K_2^* C = R_2^{-1}(B^T P_2 + M_2)\,, \tag{268}$$

with matrix M_2 satisfying

$$C^T(K_2 - K_2^*)^T R_2 (K_2 - K_2^*)C + M_2^T (K_2 - K_2^*)C + C^T (K_2 - K_2^*)^T M_2 \geq 0\,. \tag{269}$$

for any $K_2 \neq K_2^*$. Then the control

$$u^* = -c(L+G) \otimes K_2^* C\delta\,, \tag{270}$$

is optimal with respect to the performance index

$$J = \int_0^\infty (\delta^T Q \delta + u^T R u)\,dt\,, \tag{271}$$

where the matrices Q, R are given by

$$R = R_1 \otimes R_2\,,$$

and

$$Q = c^2 (L+G)^T R_1 (L+G) \otimes (Q_2 + A^T P_2 + P_2 A) - cR_1 (L+G) \otimes (A^T P_2 + P_2 A) > 0 , \quad (272)$$

for the coupling gain $c > 0$ sufficiently great, if $Q_2 + A^T P_2 + P_2 A \geq 0$. ■

The proof is analogous to the proof of Theorem 7.2, noting that $D=0$ makes the choice of T immaterial, and both variants for Q, presented in Theorem 7.2, reduce to a single choice,(272).

Note that if the graph does not contain the spanning tree, with at least one non-zero pinning gain connecting to a root node, then, with all other conditions of the Theorem 7.3 satisfied, one has optimality, with positive semidefinite Q in (271). This implies convergence to the null space of Q. The existence of the spanning tree, *i.e.* the non-singularity of the graph matrix, $L+G$, is necessary for state synchronization.

Remark 7.3: For (271) to be a reasonable performance criterion, the matrix Q needs to be positive definite. This is indeed the case, with the coupling gain $c > 0$ sufficiently great, if the graph has a spanning tree with at least one non zero pinning gain connecting to the root node and $Q_2 + A^T P_2 + P_2 A \geq 0$. That this is positive semidefinite follows from the output Riccati-type equation if and only if $P_2 B R_2^{-1} B^T P_2 - M_2^T R_2^{-1} M_2 \geq 0$. This is a bound on the size of M_2 as well as the condition on its kernel, $\ker B^T P_2 \subseteq \ker M_2$. Note also that one needs to have an unbounded synchronizing region for output feedback, as detailed in Theorem 7.1, in order to be able to increase the coupling gain c without an *a priori* upper bound, which is convenient fo guarantee that $Q > 0$. Namely, optimality with positive definite Q in (271) implies synchronization, but synchronization need not be optimal. Hence, the conditions on c for synchronization, imposed by optimality requirements of Theorem 7.3, are conservative.

7.6 Conditions on Graph Topology

In this section we introduce a new class of digraphs which, to our knowledge, has not yet appeared in the cooperative control literature. This class of digraphs admits a distributed solution to an appropriately defined global optimal control problem. The essential conditions for global optimality of the distributed control (268),(270), is (266). Generally one can express that condition as

$$R_1(L+G)=P_1, \tag{273}$$

where $R_1 = R_1^T > 0$, $L+G$ is a nonsingular (the pinned graph Laplacian) *M*-matrix, and $P_1 = P_1^T > 0$. Equivalently,

$$R_1(L+G)=(L+G)^T R_1 > 0. \tag{274}$$

For the following classes of graph topologies one can satisfy this condition.

7.6.1 Undirected Graphs

Given that the graph is undirected, then L is symmetric, *i.e.* $L=L^T$, so the condition (128) becomes a commutativity requirement

$$R_1(L+G)=(L+G)^T R_1 = (L+G)R_1. \tag{275}$$

Then, $P_1 = L+G$. More generally, condition (130) is satisfied by symmetric matrices R_1 and $L+G$ if and only if R_1 and $L+G$ have all eigenvectors in common. Since $L+G$ is symmetric it has a basis of orthogonal eigenvectors, and one can construct R_1 satisfying (130) as follows. Let T be an orthogonal matrix whose columns are eigenvectors of $L+G$, then $L+G=T\Lambda T^T$ with Λ a diagonal matrix of real eigenvalues. Then for any positive definite diagonal matrix, Θ, one has that $R_1 = T\Theta T^T > 0$ commutes with $L+G$ and satisfies the commutativity requirement (130). Note that the R_1 matrix thus constructed depends on all the eigenvectors of the graph matrix $L+G$ in (128)

7.6.2 Directed Graphs with Simple Graph Matrix

Given a directed graph, let it be such that the graph matrix, $L+G$, in (128) is simple, then there exists a matrix $R_1 > 0$ depending on all the eigenvectors of the graph matrix $L+G$ that satisfies the condition (128). This result is given in the form of the following theorem.

Theorem 7.4. Let $L+G$ be a graph matrix (generally not symmetric). Then there exists a positive definite symmetric matrix $R_1 = R_1^T > 0$ such that $R_1(L+G) = P_1$ is a symmetric positive semidefinite matrix if and only if the matrix $L+G$ is simple, *i.e.* there exists a basis of eigenvectors of $L+G$.

Proof: (*i*) Let $L+G$ be simple. Then it is diagonalizable, *i.e.* there exists a transformation matrix T such that $T(L+G)T^{-1} = \Lambda$, where Λ is a diagonal matrix of eigenvalues of $L+G$. Then

$$T(L+G)T^{-1} = \Lambda = \Lambda^T = T^{-T}(L+G)^T T^T \ ,$$

implying $T^T T(L+G)T^{-1}T^{-T} = (L+G)^T$, which further implies that $(T^T T)(L+G) = (L+G)^T (T^T T)$. Let $R_1 = T^T T$. Obviously, $R_1 = R_1^T > 0$ and $P_1 = R_1(L+G) = T^T T(L+G) = T^T \Lambda T \geq 0$ since $\Lambda \geq 0$ ($\forall x$ $0 \leq x^T P_1 x = x^T T^T \Lambda T x = y^T \Lambda y \ \ \forall y$), 45.

(*ii*) Let $L+G$ be a graph matrix. Suppose there exists $R_1 = R_1^T > 0$ satisfying the condition that $R_1(L+G) = P_1$ be a symmetric positive semidefinite matrix. Then one needs to show that $L+G$ is simple. To do this, we will prove the contrapositive by contradiction. So we suppose the negation of the contrapositive. That is, suppose $L+G$ is not simple but that there exists $R_1 = R_1^T > 0$ satisfying the condition $R_1(L+G) = P_1$ is a symmetric positive semidefinite matrix.

Since $L+G$ is not simple, there exists a coordinate transformation bringing $L+G$ to a Jordan canonical form $T^{-1}(L+G)T = J$, with nonzero superdiagonal (otherwise $L+G$ would be simple). Then one has $R_1(L+G)=R_1TJT^{-1}=P_1=P_1^T=T^{-T}J^TT^TR_1$. But then $(T^TR_1T)J=J^T(T^TR_1T)$. Therefore there exists $R'_1=T^TR_1T=R'^T_1>0$ such that $R'_1J=J^TR'_1$. Without loss of generality let us assume that the first Jordan block is not simple, and with a slight abuse of notation $R'_{1,11}$ will refer to the corresponding block in R'_1. Then one has that $R'_{1,11}(\lambda I+E)=(\lambda I+E^T)R'_{1,11}\Rightarrow R'_{1,11}E=E^TR'_{1,11}$, where E is a nilpotent matrix having ones on the superdiagonal. This identity means that the first row and first column of $R'_{1,11}$ are zero, except for the last entry. However, then R'_1 cannot be positive definite, since there are vanishing principal minors (Sylvester's test). ■

7.7 Conclusion

This chapter examines distributed output-feedback control for state synchronization of identical linear time-invariant agents. Cooperative stability of state synchronization is addressed using results derived from the single-agent robust stability properties of the local output-feedback gain. This development is an extension of the synchronizing region methodology, well known in the full-state feedback case, to the case of output-feedback. It is shown that the guaranteed synchronizing region for output-feedback can be both bounded and unbound. In the second part of the chapter the proposed distributed output-feedback leads to a solution of a specially structured two-player zero-sum game problem. Conditions for that are more strict than those for simple cooperative synchronization. As a special case of the two-player zero-sum game problem, the optimal control problem is addressed. Conditions are found under which the proposed distributed output-feedback control is optimal with respect to a quadratic performance criterion. These imply state synchronization.

CHAPTER 8

FUTURE WORK

8.1 Introduction

This chapter gives a brief account of some future work directions and current results on basis of which a future development, in the opinion of the author, should be pursued. Prior chapters presented results on static and dynamic, state and output, distributed feedback control for the state consensus. The standing assumptions are that all agents be identical and that the states of neighboring agents be instantly available for the purposes of distributed feedback control. In realistic applications of multi-agent systems one can expect both assumptions to fail. Therefore, it becomes of interest to investigate multi-agent system with communication delays, and heterogeneous systems, where agents are not dynamically identical. Heterogeneous multi-agent systems are briefly mentioned in Chapter 6, in the context of the robustness of cooperative stability.

Two extensions of the hitherto introduced concepts are considered. The state consensus problem of identical agents that have a control signal delay is mentioned first. The second research direction deals with the output consensus problem for, generally, heterogeneous multi-agent systems.

Distributed communication always imparts certain communication delays, and since multi-agent systems are assumed to function via distributed communication and control one needs to take the communication delays into account. Section 8.1. extends the spirit of single agent analysis presented in Chapters 2 and 3, as well as in 12,13,28,29,30. To summarize, this approach uses single agent robustness to determine the synchronizing region. This then gives robustness with respect to the graph topology. It is shown here that the stability analysis based on Lyapunov-Razuminkhin functions, 51,52, allows one to extend the notion of conventional synchronizing region of a matrix pencil to a delay dependent synchronizing region for matrix pencils with delays. Such a result can be used to investigate the dependence of robustness to the graph topology in relation to control delays, and vice versa. One is looking at the robustness of a single agent system to communication graph topology as well as to a time-delay in controls.

Heterogeneous agent output synchronization is the second field of future work where results of Chapters 3 and 6 could be applied. An approach to output synchronization, present in the current literature 61,62,63,64, involves output tracking via a reference model, and synchronization of the reference generators. In practical terms this means one has a distributed dynamic regulator, and agent's regulators need to communicate their states in a distributed fashion. Separation principle is found in a sense that output tracking of local reference generators is independent from the synchronization of those generators, 61,63. Therefore, one solves the distributed synchronization problem for reference generators and the reference output tracking problem for each system separately. A simplification in the structure of such dynamic regulators is achieved if, instead of having regulators communicating their states, one can use the local output neighborhood errors as inputs to the regulators. It is plausible that with good output reference tracking, the difference between such a novel, simpler, algorithm and the more complicated ones already present, 61,62,63,64, would not change cooperative stability and convergence of the entire system.

8.2 Multi-agent System Synchronization With Control Delays

This section expounds the dynamics of multi-agent systems consisting of identical agents, where the agents' systems have a control signal delay. In that case, each agent is described by a single-delay retarded functional differential equation. The stability of retarded functional differential equations is a problem well known in the literature 51,52,53,54,55,56,57,58,59, motivated by the frequent use of such equations as models in process control, 51. Classical results for stability of linear retarded differential equations are given here together with the generalization to complex equations resembling the results in Chapter 2, and 12, 13, for matrix pencils.

8.2.1 Dynamics of the Multi-agent System with Constant, Uniform, Input Signal Delay

Let the single-agent system be given as a linear time-invariant system with control signal delay, which is constant in time and the same for all agents,

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t-\tau)\,. \tag{276}$$

Let there be a leader, whose system is given as,

$$\dot{x}_0(t) = Ax_0(t)\,. \tag{277}$$

The control is calculated as a linear feedback of the local neighborhood error

$$e_i = \sum_j e_{ij}(x_j - x_i) + g_i(x_0 - x_i)\,,$$

as

$$u_i(t) = cKe_i(t)\,,$$

where K is the local linear feedback gain matrix to be detailed later. This gives the closed loop single-agent system dynamics

$$\dot{x}_i(t) = Ax_i(t) + cBKe_i(t-\tau)\,. \tag{278}$$

Introducing the synchronization error, $\delta_i = x_i - x_0$, one has in global form,

$$\dot{x}(t) = (I_N \otimes A)x(t) - c(L+G) \otimes BK\delta(t-\tau)\,, \tag{279}$$

And the synchronization error system is given as,

$$\dot{\delta}(t) = (I_N \otimes A)\delta(t) - c(L+G) \otimes BK\delta(t-\tau)\,. \tag{280}$$

This is a linear retarded functional differential equation of the form,

$$\dot{y}(t) = Ay(t) + A_d y(t-\tau)\,. \tag{281}$$

Existence of solutions to the initial condition problem for this equation is given by the Schauder fixed point theorem and uniqueness follows from Lipschitz continuity of the functional, 51.

However, instead of analyzing the entire global system, (280), the analysis can be simplified as follows. In the Laplace domain the synchronization error system, (280), can be written as

$$s\delta(s)-\delta(0)=\left[(I_N\otimes A)-c(L+G)\otimes BKe^{-s\tau}\right]\delta(s)$$
$$\left[sI_{Nn}-(I_N\otimes A)+c(L+G)\otimes BKe^{-s\tau}\right]\delta(s)=\delta(0)$$

So, the crucial object for consideration is the transfer function from the initial condition,

$$\left[sI_{Nn}-(I_N\otimes A)+c(L+G)\otimes BKe^{-s\tau}\right]^{-1}, \tag{282}$$

that needs to be asymptotically stable. The characteristic polynomial of the transfer function (282) is given as

$$\Delta(s)=\det\left[sI_{Nn}-(I_N\otimes A)+c(L+G)\otimes BKe^{-s\tau}\right]. \tag{283}$$

Knowing that coordinate transformations do not change the determinant of a matrix, one can find a coordinate transformation, $T\otimes I_n$, where $T^{-1}(L+G)T=\Lambda$ is an upper triangular matrix, thereby simplifying the expression (283) considerably. Thus one obtains

$$\begin{aligned}&\det\left[sI_{Nn}-(I_N\otimes A)+c(L+G)\otimes BKe^{-s\tau}\right]\\&=\det\left[(T\otimes I_n)^{-1}(sI_{Nn}-(I_N\otimes A)+c(L+G)\otimes BKe^{-s\tau})(T\otimes I_n)\right]\\&=\det\left[sI_{Nn}-(I_N\otimes A)+c\Lambda\otimes BKe^{-s\tau}\right]\\&=\prod_i\det\left[sI_n-A+c\lambda_jBKe^{-s\tau}\right]\end{aligned},$$

where λ_i are eiganvalues of Λ. Hence, each factor in the product, $\det(sI_n-A+c\lambda_jBKe^{-s\tau})$, needs to be stable. This is equivalent to the stability of a set of linear retarded functional differential equations with a single delay

$$\dot{y}(t)=Ay(t)-c\lambda_jBKy(t-\tau)\,. \tag{284}$$

These are of the form, $\dot{y}(t)=Ay(t)+A_dy(t-\tau)$, 52,55, and are of the order of a single agent, (276). It is more advantageous, however, to separate the dependence from the, possibly complex, scaled eigenvalues, $c\lambda_j$, of the graph matrix $(L+G)$, and instead investigate the complex retarded functional differential equation

$$\dot{y}(t) = Ay(t) + \sigma A_d y(t-\tau), \tag{285}$$

$\sigma \in \mathbb{C}$, $A_d = -BK$. The aim is to guarantee a synchronizing region in the complex plane in dependence of the delay, τ, so that all scaled eigenvalues, $c\lambda_j$, are in that region, which in turn shall guarantee stability, that is synchronization, with an admissible delay. For that purpose, one first looks into the classical stability results for the retarded functional differential equation,

$$\dot{y}(t) = Ay(t) + A_d y(t-\tau), \tag{286}$$

and applies those results, *mutatis mutandum*, to the case of the complex retarded functional differential equation (285).

8.2.2 Stability Results for Linear Single-delay Retarded Functional Differential Equation

The following theorems will be given in the most general setting for retarded functional differential equations, 51,52. These are sufficient conditions for stability in presence of delays.

Definition 8.1: The *retarded functional differential equation* is the equation of the form

$$\dot{x}(t) = f(t, x_t), \tag{287}$$

where $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$, $t \geq 0$ is generally a time dependent functional from the space of continuous functions on a compact interval; $x_t \in \mathcal{C},\ x_t = x(t+\vartheta),\ \vartheta \in [-\tau, 0]$. The initial condition x_0 is also an element of that space.

Definition 8.2: The trivial solution of the initial value problem is assumed to be a fixed point; $f(t,0) = 0$, $\forall t$. Then the trivial solution of the retarded functional differential equation is s*table* (uniformly) if $\forall \varepsilon > 0 \ \ \exists \delta(\varepsilon) > 0$ such that

$$\sup \|x_0\| < \delta \Rightarrow \|x_t(t)\| < \varepsilon \ \ \forall t \geq 0$$

and *asymptotically stable* (uniformly) if it is stable and $\forall \eta > 0 \quad \exists T(\eta) > 0$ and $\exists \delta_0 > 0$ such that

$$\sup \|x_0\| < \delta_0 \Rightarrow \sup \|x_t\| < \eta \quad \forall t \geq T(\eta) . \qquad \blacksquare$$

Note that, since we are dealing here with time-invariant functional, all stability properties hold uniformly in time.

The next two theorems give sufficient conditions for (asymptotic) stability of the fixed point solution of retarded functional differential equations.

Theorem 8.1. (*Lyapunov-Krasovskii theorem*). Let $f(x_t)$ be a bounded functional, meaning it takes bounded (in sup norm) $x_t \in \mathcal{C}$ to bounded vectors in $\mathbb{R}^n$. Further let α, β, γ be class $\mathcal{K}$ functions; that is non-decreasing positive functions having value of 0 at 0. Let there exist a continuous functional $V(x_t)$ on $\mathcal{C}$ such that for some $c > 0$ for all functions $x_t \in \mathcal{C}$ satisfying $V(x_t) \leq c$ one has

$$\alpha(\|x_t(t)\|) \leq V(x_t) \leq \beta(\sup \|x_t\|) , \tag{288}$$

$$\frac{d}{dt} V(x_t) \leq -\gamma(\|x_t(t)\|) . \tag{289}$$

Then, the trivial solution, $x_t \equiv 0$, is uniformly asymptotically stable. Furthermore, the preimage in $\mathcal{C}$ of $V(x_t) \leq c$ is an invariant subset in the region of attraction in the space $\mathcal{C}$. $\blacksquare$

Theorem 8.2. (*Lyapunov-Razuminkhin theorem*). Let α, β, γ be class $\mathcal{K}$ functions, and let $p(s) > s \ \forall s > 0$ be a scalar continuous non-decreasing function. If there exists a continuous function $V(x) : \mathbb{R}^n \to \mathbb{R}$ such that for some $c > 0$, for all $x_t \in \mathcal{C}$ satisfying $V(x_t(\vartheta)) \leq c, \ \forall \vartheta \in [t - \tau, t]$, one has

$$\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|) , \tag{290}$$

$$\frac{d}{dt}V(x(t)) \le -\gamma(\|x(t)\|), \ \ if \ V(x(\vartheta)) < p(V(x(t)), \forall \vartheta \in [t-\tau, t]. \tag{291}$$

Then, the trivial solution, $x(t) \equiv 0$, is uniformly asymptotically stable. Furthermore, the set in $\mathcal{C}$ where $V(x_t(\vartheta)) \le c, \ \forall \vartheta \in [t-\tau, t]$ is an invariant subset in the region of attraction in the space $\mathcal{C}$. ■

Sufficient conditions for asymptotic stability are now derived for the special case of a linear single delay retarded functional differential equation as an application of general theorems of Krasovskii and Razuminkhin. Let the system under consideration be described by the linear single-delay retarded functional differential equation,

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) \, . \tag{292}$$

The following theorems give sufficient conditions for asymptotic stability, 52,54,55.

Theorem 8.3. (*Lyapunov-Krasovskii delay independent result*). Let there exist positive definite symmetric matrices $P > 0, Q > 0$ such that

$$\begin{bmatrix} A^T P + PA + Q & PA_d \\ A_d^T P & -Q \end{bmatrix} < 0 \, , \tag{293}$$

Or equivalently, by Schur complement,

$$A^T P + PA + Q + PA_d Q^{-1} A_d^T P < 0 \, . \tag{294}$$

Then the trivial solution of the system (292), $x(t) \equiv 0$, is uniformly asymptotically stable.

Proof: Choose the Lyapunov-Krasowskii functional as

$$V(x_t) = x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(\vartheta)Qx(\vartheta)d\vartheta \, .$$

Then one finds the bounds $\lambda_{\min}(P)\|x_t(t)\|^2 \le V(x_t) \le (\lambda_{\max}(P) + \tau\lambda_{\max}(Q)) \sup\|x_t\|^2$ and the derivative turns out to be

$$\frac{d}{dt}V(x_t) = x^T(t)(A^T P + PA)x(t) + x^T(t)PA_d x(t-\tau) + x^T(t-\tau)A_d^T Px(t)$$
$$+ x^T(t)Qx(t) - x^T(t-\tau)Qx(t-\tau) \qquad ,$$
$$= \begin{bmatrix} x^T(t) & x^T(t-\tau) \end{bmatrix} \begin{bmatrix} A^T P + PA + Q & PA_d \\ A_d^T P & -Q \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} < -\delta \|x_t(t)\|^2$$

where δ is the greatest eigenvalue of the negative definite block matrix defining the quadratic form. This completes the proof. ■

Theorem 8.4. (*Lyapunov-Razuminkhin delay independent result*). Let there exist positive definite symmetric matrices $P > 0, Q > 0$ such that

$$P \geq Q^{-1}, \tag{295}$$

$$A^T P + PA + PA_d Q A_d^T P + P < 0. \tag{296}$$

Then the trivial solution of the system (292), $x(t) \equiv 0$, is uniformly asymptotically stable.

Proof: Choose the Lyapunov-Razuminkhin function as $V(x) = x^T Px > 0$. The lower and upper bound are satisfied

$$\lambda_{\min}(P)\|x\|^2 \leq x^T Px \leq \lambda_{\max}(P)\|x\|^2,$$

and the derivative equals

$$\frac{d}{dt}V(x(t)) = x^T(t)(A^T P + PA)x(t) + x^T(t)PA_d x(t-\tau) + x^T(t-\tau)A_d^T Px(t).$$

Given any vectors x, y and a matrix $M = M^T > 0$, one has that

$$x^T y + y^T x \leq x^T Mx + y^T M^{-1} y,$$

(follows from $(x - M^{-1}y)^T M(x - M^{-1}y) \geq 0$). Therefore, one finds that

$$x^T(t)PA_d x(t-\tau) + x^T(t-\tau)A_d^T Px(t) \leq x^T(t)PA_d Q A_d^T Px(t) + x^T(t-\tau)Q^{-1}x(t-\tau)$$
$$\leq x^T(t)PA_d Q A_d^T Px(t) + x^T(t-\tau)Px(t-\tau) \qquad ,$$
$$= x^T(t)PA_d Q A_d^T Px(t) + V(x(t-\tau))$$

whence it follows,

$$\frac{d}{dt}V(x(t)) \le x^T(t)(A^T P + PA + PA_d Q A_d^T P)x(t) + V(x(t-\tau)).$$

According to the Lyapunov-Razuminkhin stability condition, if there exist $\varepsilon > 1, \delta > 0$ such that $\dot{V}(x(t)) \le -\delta V(x(t))$ whenever $V(x(\vartheta)) < \varepsilon V(x(t)), \forall \vartheta \in [t-\tau, t]$ one has asymptotic stability. By the condition of the Theorem, there exists a $\delta > 0$ such that

$$A^T P + PA + PA_d Q A_d^T P + (1+2\delta)P < 0.$$

Let $\varepsilon = 1+\delta$, then if $V(x(\vartheta)) < \varepsilon V(x(t)), \forall \vartheta \in [t-\tau, t]$ one has

$$\begin{aligned}\frac{d}{dt}V(x(t)) &\le x^T(t)(A^T P + PA + PA_d Q A_d^T P + \varepsilon P)x(t) \\ &= x^T(t)(A^T P + PA + PA_d Q A_d^T P + (1+\delta)P)x(t), \\ &< -\delta x^T(t)Px(t) = -\delta V(x(t))\end{aligned}$$

completing the proof. ■

Theorem 8.5. (*Lyapunov-Razuminkhin delay dependent result*). Let there exist positive definite symmetric matrices $P, P_1, P_2 > 0$, and $\tau_0 > 0$. Let the following conditions be satisfied

$$P_1^{-1} < P,\ P_2^{-1} < P, \tag{297}$$

$$(A + A_d)^T P + P(A + A_d) + \tau_0 P A_d A P_1 A^T A_d^T P + \tau_0 P A_d^2 P_2 (A_d^2)^T P + 2\tau_0 P < 0, \tag{298}$$

(equivalently $(A + A_d)^T P + P(A + A_d) + \tau_0 P A_d (A P_1 A^T + A_d P_2 A_d^T) A_d^T P + 2\tau_0 P < 0$).

Then the trivial solution of the system (292), $x(t) \equiv 0$, is uniformly asymptotically stable.

Proof: Observe that, since the solution $x(t)$ of (292) is continuously differentiable, one has

$$x(t-\tau) = x(t) - \int_{t-\tau}^{t} \dot{x}(s)ds = x(t) - \int_{t-\tau}^{t} Ax(s) + A_d x(s-\tau)ds$$

So the original equation can be written as

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) = Ax(t) + A_d x(t) - \int_{t-\tau}^{t} A_d A x(s) + A_d^2 x(s-\tau) ds$$
$$\dot{x}(t) = (A + A_d) x(t) - \int_{t-\tau}^{t} A_d A x(s) + A_d^2 x(s-\tau) ds \, .$$

Choosing the Lyapunov-Razuminkhin function as $V(x(t)) = x^T(t) P x(t) > 0$ one finds its derivative to equal

$$\begin{aligned}
&\frac{d}{dt} V(x(t)) = \dot{x}^T(t) P x(t) + x^T(t) P \dot{x}(t) \\
&= \left[(A + A_d) x(t) - \int_{t-\tau}^{t} A_d A x(s) + A_d^2 x(s-\tau) ds \right]^T P x(t) + x^T(t) P \left[(A + A_d) x(t) - \int_{t-\tau}^{t} A_d A x(s) + A_d^2 x(s-\tau) ds \right]. \\
&= x^T(t) \left[(A + A_d)^T P + P(A + A_d) \right] x(t) \\
&- \int_{t-\tau}^{t} x^T(s) A^T A_d^T P x(t) + x^T(t) P A_d A x(s) ds - \int_{t-\tau}^{t} x^T(s-\tau) (A_d^2)^T P x(t) + x^T(t) P A_d^2 x(s-\tau) ds
\end{aligned}$$

We denote the integral terms in the last line of the above expression as

$$\eta_1 = -\int_{t-\tau}^{t} x^T(s) A^T A_d^T P x(t) + x^T(t) P A_d A x(s) ds = -\int_{-\tau}^{0} x^T(t+\vartheta) A^T A_d^T P x(t) + x^T(t) P A_d A x(t+\vartheta) d\vartheta,$$

$$\eta_2 = -\int_{t-\tau}^{t} x^T(s-\tau) (A_d^2)^T P x(t) + x^T(t) P A_d^2 x(s-\tau) ds = -\int_{-\tau}^{0} x^T(t-\tau+\vartheta) (A_d^2)^T P x(t) + x^T(t) P A_d^2 x(t-\tau+\vartheta) d\vartheta.$$

One finds by completing the squares with $P_1 > 0, P_2 > 0$ that

$$\begin{aligned}
&(A^T A_d^T P x(t) + P_1^{-1} x(t+\vartheta))^T P_1 (A^T A_d^T P x(t) + P_1^{-1} x(t+\vartheta)) \geq 0 \\
&x^T(t) P A_d A P_1 A^T A_d^T P x(t) + x^T(t) P A_d A x(t+\vartheta) + x^T(t+\vartheta) A^T A_d^T P x(t) + x^T(t+\vartheta) P_1^{-1} x(t+\vartheta) \geq 0 \\
&-\left[x^T(t) P A_d A x(t+\vartheta) + x^T(t+\vartheta) A^T A_d^T P x(t) \right] \leq x^T(t) P A_d A P_1 A^T A_d^T P x(t) + x^T(t+\vartheta) P_1^{-1} x(t+\vartheta)
\end{aligned}$$

Hence,

$$\eta_1 = -\int_{-\tau}^{0} x^T(t+\vartheta) A^T A_d^T P x(t) + x^T(t) P A_d A x(t+\vartheta) d\vartheta \leq \int_{-\tau}^{0} x^T(t) P A_d A P_1 A^T A_d^T P x(t) + x^T(t+\vartheta) P_1^{-1} x(t+\vartheta) d\vartheta$$

$$\eta_1 \leq \tau x^T(t) P A_d A P_1 A^T A_d^T P x(t) + \int_{-\tau}^{0} x^T(t+\vartheta) P_1^{-1} x(t+\vartheta) d\vartheta$$

In a similar manner,

$$((A_d^2)^T Px(t)+P_2^{-1}x(t-\tau+\vartheta))^T P_2((A_d^2)^T Px(t)+P_2^{-1}x(t-\tau+\vartheta))\geq 0$$
$$x^T(t)PA_d^2P_2(A_d^2)^T Px(t)+x^T(t)PA_d^2x(t-\tau+\vartheta)+x^T(t-\tau+\vartheta)(A_d^2)^T Px(t)+x^T(t-\tau+\vartheta)P_2^{-1}x(t-\tau+\vartheta)\geq 0$$
$$-\left[x^T(t)PA_d^2x(t-\tau+\vartheta)+x^T(t-\tau+\vartheta)(A_d^2)^T Px(t)\right]\leq x^T(t)PA_d^2P_2(A_d^2)^T Px(t)+x^T(t-\tau+\vartheta)P_2^{-1}x(t-\tau+\vartheta)$$

Hence,

$$\eta_2=-\int_{-\tau}^{0}x^T(t-\tau+\vartheta)(A_d^2)^T Px(t)+x^T(t)PA_d^2x(t-\tau+\vartheta)d\vartheta\leq\int_{-\tau}^{0}x^T(t)PA_d^2P_2(A_d^2)^T Px(t)+x^T(t-\tau+\vartheta)P_2^{-1}x(t-\tau+\vartheta)d\vartheta$$
$$\eta_2\leq\tau x^T(t)PA_d^2P_2(A_d^2)^T Px(t)+\int_{-\tau}^{0}x^T(t-\tau+\vartheta)P_2^{-1}x(t-\tau+\vartheta)d\vartheta$$
.

Inserting those bounds in the expression for the time derivative of the Lyapunov-Razumnkhin function, one finds that

$$\frac{d}{dt}V(x(t))=x^T(t)\left[(A+A_d)^T P+P(A+A_d)\right]x(t)+\eta_1+\eta_2$$
$$\frac{d}{dt}V(x(t))\leq x^T(t)\left[(A+A_d)^T P+P(A+A_d)+\tau PA_dAP_1A^TA_d^TP+\tau PA_d^2P_2(A_d^2)^T P\right]x(t)\cdot$$
$$+\int_{-\tau}^{0}x^T(t+\vartheta)P_1^{-1}x(t+\vartheta)d\vartheta+\int_{-\tau}^{0}x^T(t-\tau+\vartheta)P_2^{-1}x(t-\tau+\vartheta)d\vartheta$$

Under the condition that $P_1^{-1}<P,\ P_2^{-1}<P$ one finds

$$\frac{d}{dt}V(x(t))\leq x^T(t)\left[(A+A_d)^T P+P(A+A_d)+\tau PA_dAP_1A^TA_d^TP+\tau PA_d^2P_2(A_d^2)^T P\right]x(t)$$
$$+\int_{-\tau}^{0}x^T(t+\vartheta)P_1^{-1}x(t+\vartheta)d\vartheta+\int_{-\tau}^{0}x^T(t-\tau+\vartheta)P_2^{-1}x(t-\tau+\vartheta)d\vartheta$$
$$\leq x^T(t)\left[(A+A_d)^T P+P(A+A_d)+\tau PA_dAP_1A^TA_d^TP+\tau PA_d^2P_2(A_d^2)^T P\right]x(t)$$
$$+\int_{-\tau}^{0}x^T(t+\vartheta)Px(t+\vartheta)d\vartheta+\int_{-\tau}^{0}x^T(t-\tau+\vartheta)Px(t-\tau+\vartheta)d\vartheta$$
$$=x^T(t)\left[(A+A_d)^T P+P(A+A_d)+\tau PA_dAP_1A^TA_d^TP+\tau PA_d^2P_2(A_d^2)^T P\right]x(t)$$
$$+\int_{-\tau}^{0}V(x(t+\vartheta))d\vartheta+\int_{-\tau}^{0}V(x(t-\tau+\vartheta))d\vartheta$$
.

Following Razuminkhin stability condition one has that if $V(x(\vartheta)) < \varepsilon V(x(t))$, $\forall \vartheta \in [t-2\tau, t]$, for some $\varepsilon > 1$, the time derivative of the Lyapunov-Razuminkin function satisfies

$$\frac{d}{dt}V(x(t)) \le x^T(t)\left[(A+A_d)^T P + P(A+A_d) + \tau P A_d A P_1 A^T A_d^T P + \tau P A_d^2 P_2 (A_d^2)^T P + 2\tau\varepsilon P\right]x(t).$$

To find an appropriate $\varepsilon > 1$ and $\delta > 0$, such that for $V(x(\vartheta)) < \varepsilon V(x(t)), \forall \vartheta \in [t-2\tau, t]$ one has $\dot{V}(x(t)) \le -\delta V(x(t))$, let us proceed similarly as in the proof of Theorem 8.4. If the condition (298) of the Theorem is satisfied then there exists $\delta_1 > 0$ such that

$$(A+A_d)^T P + P(A+A_d) + \tau_0 P A_d A P_1 A^T A_d^T P + \tau_0 P A_d^2 P_2 (A_d^2)^T P + 2\tau_0(1+2\delta_1)P < 0.$$

Take $\varepsilon = 1+\delta_1$, then

$$(A+A_d)^T P + P(A+A_d) + \tau_0 P A_d A P_1 A^T A_d^T P + \tau_0 P A_d^2 P_2 (A_d^2)^T P + 2\tau_0 \varepsilon P < -2\delta_1 \tau_0 P$$

so $\dot{V}(x(t)) \le -2\delta_1\tau_0 V(x(t))$, completing the proof. ■

Notice that if stability is guaranteed for some τ_0 then it is also guaranteed $\forall \tau; 0 \le \tau \le \tau_0$, since the delay, τ, multiplies only positive (semi)definite terms and the matrix inequality is indeed satisfied for all smaller values as well.

Corollary 8.1. The conditions of Theorem 8.5 can be replaced by

$$A^T P_1^{-1} A \le P,\ A_d^T P_2^{-1} A_d \le P, \tag{299}$$

$$(A+A_d)^T P + P(A+A_d) + \tau_0 P A_d (P_1+P_2) A_d^T P + 2\tau_0 P < 0. \tag{300}$$

Proof: Using a different square completion one obtains the bounds for the integral terms in the proof as,

$$(A_d^T P x(t) + P_1^{-1} A x(t+\vartheta))^T P_1 (A_d^T P x(t) + P_1^{-1} A x(t+\vartheta)) \ge 0$$
$$x^T(t) P A_d P_1 A_d^T P x(t) + x^T(t) P A_d A x(t+\vartheta) + x^T(t+\vartheta) A^T A_d^T P x(t) + x^T(t+\vartheta) A^T P_1^{-1} A x(t+\vartheta) \ge 0$$
$$-\left[x^T(t) P A_d A x(t+\vartheta) + x^T(t+\vartheta) A^T A_d^T P x(t)\right] \le x^T(t) P A_d P_1 A_d^T P x(t) + x^T(t+\vartheta) A^T P_1^{-1} A x(t+\vartheta)$$

Hence,

$$\eta_1 = -\int_{-\tau}^{0} x^T(t+\vartheta)A^T A_d^T Px(t) + x^T(t)PA_d Ax(t+\vartheta)d\vartheta \leq \int_{-\tau}^{0} x^T(t)PA_d P_1 A_d^T Px(t) + x^T(t+\vartheta)A^T P_1^{-1} Ax(t+\vartheta)d\vartheta$$

$$\eta_1 \leq \tau x^T(t)PA_d P_1 A_d^T Px(t) + \int_{-\tau}^{0} x^T(t+\vartheta)A^T P_1^{-1} Ax(t+\vartheta)d\vartheta$$

In a similar manner

$$(A_d^T Px(t) + P_2^{-1}A_d x(t-\tau+\vartheta))^T P_2(A_d^T Px(t) + P_2^{-1}A_d x(t-\tau+\vartheta)) \geq 0$$

$$x^T(t)PA_d P_2 A_d^T Px(t) + x^T(t)PA_d^2 x(t-\tau+\vartheta) + x^T(t-\tau+\vartheta)(A_d^2)^T Px(t) + x^T(t-\tau+\vartheta)A_d^T P_2^{-1} A_d x(t-\tau+\vartheta) \geq 0$$

$$-\left[x^T(t)PA_d^2 x(t-\tau+\vartheta) + x^T(t-\tau+\vartheta)(A_d^2)^T Px(t)\right] \leq x^T(t)PA_d P_2 A_d^T Px(t) + x^T(t-\tau+\vartheta)A_d^T P_2^{-1} A_d x(t-\tau+\vartheta)$$

Hence,

$$\eta_2 = -\int_{-\tau}^{0} x^T(t-\tau+\vartheta)(A_d^2)^T Px(t) + x^T(t)PA_d^2 x(t-\tau+\vartheta)d\vartheta \leq \int_{-\tau}^{0} x^T(t)PA_d P_2 A_d^T Px(t) + x^T(t-\tau+\vartheta)A_d^T P_2^{-1} A_d x(t-\tau+\vartheta)d\vartheta$$

$$\eta_2 \leq \tau x^T(t)PA_d P_2 A_d^T Px(t) + \int_{-\tau}^{0} x^T(t-\tau+\vartheta)A_d^T P_2^{-1} A_d x(t-\tau+\vartheta)d\vartheta$$

Inserting those bounds in the expression for the derivative of the Lyapunov-Razuminkhin function and applying the conditions of corollary in the same manner as in the proof of Theorem 8.5 one finds guaranteed asymptotic stability of the trivial solution. ■

Corollary 8.2. 55. Assume that one has the asymptotic stability of the delay-free system characterized by the Lyapunov inequality

$$(A + A_d)^T P + P(A + A_d) \leq -Q, \tag{301}$$

Choose $\beta_1^{-1} \leq 1, \beta_2^{-1} \leq 1$. Then one has asymptotic stability for delays less than

$$\tau_0 = \left\| Q^{-1/2}\left[\beta_1 PA_d AP^{-1}A^T A_d^T P + \beta_2 PA_d^2 P^{-1}(A_d^2)^T P + 2P\right]Q^{-1/2}\right\|^{-1}. \tag{302}$$

Proof: In conditions of Theorem 8.5, choose $P_1 = \beta_1 P^{-1}, P_2 = \beta_2 P^{-1}$. With $\beta_1^{-1} \leq 1, \beta_2^{-1} \leq 1$ one finds the bounds, $P_1^{-1} < P, P_2^{-1} < P$, are satisfied. Then it follows that the condition of the Theorem 8.5,

$$(A+A_d)^T P + P(A+A_d) + \tau_0 PA_d AP_1 A^T A_d^T P + \tau_0 PA_d^2 P_2 (A_d^2)^T P + 2\tau_0 P < 0,$$

is satisfied if

$$-Q + \tau_0 \beta_1 PA_d AP^{-1} A^T A_d^T P + \tau_0 \beta_2 PA_d^2 P^{-1} (A_d^2)^T P + 2\tau_0 P < 0.$$

This is surely so if

$$\begin{aligned} -I &< -\tau_0 Q^{-1/2}\left[\beta_1 PA_d AP^{-1} A^T A_d^T P + \beta_2 PA_d^2 P^{-1}(A_d^2)^T P + 2P\right]Q^{-1/2} \\ \tau_0 Q^{-1/2}&\left[\beta_1 PA_d AP^{-1} A^T A_d^T P + \beta_2 PA_d^2 P^{-1}(A_d^2)^T P + 2P\right]Q^{-1/2} < I \end{aligned},$$

meaning $\tau_0 \left\| Q^{-1/2}\left[\beta_1 PA_d AP^{-1} A^T A_d^T P + \beta_2 PA_d^2 P^{-1}(A_d^2)^T P + 2P\right]Q^{-1/2} \right\| < 1$, whence the condition (302) on the delay follows, completing the proof. ■

It should be remarked that delay independent results, both Krasovskii and Razuminkhin, tend to give conservative stability conditions. These do not account for the actual value of the delay τ and it is difficult to compare the effects of the delay on the known behavior of the delay free system. Delay dependent result, on the other hand, offers such a comparison, and allows for less strict stability conditions.

8.2.3 Application to the Complex Retarded Linear Single-delay Functional Differential Equation

Applying the results of the last section to a complex system

$$\dot{x}(t) = Ax(t) + \sigma A_d x(t-\tau) \tag{303}$$

one obtains estimates of the synchronizing region, that is the region of $\sigma \in \mathbb{C}$ for which the complex system, (303), is asymptotically stable. The functions and functionals appearing in stability conditions are modified to accommodate for the fact that the system under consideration is now complex. State vectors x are allowed to be complex and the dagger, $\dagger$, denotes the hermitian adjoint, that is transposition and complex conjugation. This gives real valued Lyapunov-Razuminkhin functions and Lyapunov-Krasovskii functionals. Therefore one has

$$V(x) = x^{\dagger} P x > 0,$$

and

$$V(x_t) = x^{\dagger}(t)Px(t) + \int_{t-\tau}^{t} x^{\dagger}(\vartheta)Qx(\vartheta)d\vartheta .$$

The change, $A_d \mapsto \sigma A_d$, $A_d^T \mapsto (\sigma A_d)^{\dagger} = \bar{\sigma} A_d^T$, in the stability conditions is justified by proving the results for the complex system (303), analogous to Theorem 8.3, 8.4, 8.5,. This shall be exemplified by presenting the proof of Theorem 8.5'. The sufficient conditions for the asymptotic stability of the complex system, (303), are given as the following theorems.

Theorem 8.3'. (*Lyapunov-Krasovskii delay independent result for the complex system*). Let there exist real positive definite symmetric matrices $P>0, Q>0$ such that

$$\begin{bmatrix} A^T P + PA + Q & \sigma PA_d \\ \bar{\sigma} A_d^T P & -Q \end{bmatrix} < 0, \tag{304}$$

or equivalently, by Schur complement,

$$A^T P + PA + Q + |\sigma|^2 PA_d Q^{-1} A_d^T P < 0 . \tag{305}$$

Then, the trivial solution of (303), $x(t) \equiv 0$, is uniformly asymptotically stable. ■

Theorem 8.4'. (*Lyapunov-Razuminkhin delay independent result for the complex equation*). Let there exist real positive definite symmetric matrices $P>0, Q>0$ such that

$$P \ge Q^{-1}, \tag{306}$$

$$A^T P + PA + |\sigma|^2 PA_d Q A_d^T P + P < 0 . \tag{307}$$

Then, the trivial solution of (303), $x(t) \equiv 0$, is uniformly asymptotically stable. ■

Observing that in both the delay independent cases $|\sigma|^2$ multiplies a positive (semi) definite term, the guaranteed synchronization region shall surely be of the form $|\sigma| < r$ for some positive bound $r > 0$. Even in classical cases the delay independent stability conditions are conservative, and especially this bounded synchronizing region is found unsatisfactory if one compares it with the unbounded synchronizing region for the continuous time cooperative systems without delays, 13. Furthermore, it is not easy to draw comparison with the delay free case (case of $\tau \to 0$) from these conditions since they do not include any information on the delay. These reasons compel one to adapt the delay dependent stability condition of Theorem 8.5 and its corollaries to the case of complex linear single delay retarded functions differential equation. This result provides a clear connection with the delay free system. For the sake of clarity the modified theorem is presented with the proof.

Theorem 8.5'. (*Lyapunov-Razuminkhin delay dependent result for the complex equation*). Let there exist real positive definite symmetric matrices $P, P_1, P_2 > 0$, and $\tau_0 > 0$. Let the following conditions be satisfied

$$P_1^{-1} < P,\; P_2^{-1} < P, \tag{308}$$

$$(A+\sigma A_d)^\dagger P + P(A+\sigma A_d) + \tau_0 |\sigma|^2 PA_d AP_1 A^T A_d^T P + \tau_0 |\sigma|^4 PA_d^2 P_2 (A_d^2)^T P + 2\tau_0 P < 0 . \tag{309}$$

(equivalently: $(A+\sigma A_d)^\dagger P + P(A+\sigma A_d) + \tau_0 |\sigma|^2 PA_d (AP_1 A^T + |\sigma|^2 A_d P_2 A_d^T) A_d^T P + 2\tau_0 P < 0$)

Then the trivial solution of (303), $x(t) \equiv 0$, is uniformly asymptotically stable for any delay $\tau \le \tau_0$.

Proof: Observe that, since the solution, $x(t)$, of (303) is continuously differentiable, one has

$$x(t-\tau) = x(t) - \int_{t-\tau}^{t} \dot{x}(s)ds = x(t) - \int_{t-\tau}^{t} Ax(s) + \sigma A_d x(s-\tau)ds .$$

So, the original equation can be written as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sigma A_d x(t-\tau) = Ax(t) + \sigma A_d x(t) - \int_{t-\tau}^{t} \sigma A_d Ax(s) + \sigma^2 A_d^2 x(s-\tau)ds \\ \dot{x}(t) &= (A+\sigma A_d)x(t) - \int_{t-\tau}^{t} \sigma A_d Ax(s) + \sigma^2 A_d^2 x(s-\tau)ds \end{aligned} .$$

Choosing the Lyapunov-Razuminkhin function as $V(x(t)) = x^{\dagger}(t)Px(t) > 0$ one finds its derivative to equal

$$\frac{d}{dt}V(x(t)) = \dot{x}^{\dagger}(t)Px(t) + x^{\dagger}(t)P\dot{x}(t)$$
$$= \left[(A+\sigma A_d)x(t) - \int_{t-\tau}^{t} \sigma A_d Ax(s) + \sigma^2 A_d^2 x(s-\tau)ds\right]^{\dagger} Px(t) + x^{\dagger}(t)P\left[(A+\sigma A_d)x(t) - \int_{t-\tau}^{t} \sigma A_d Ax(s) + \sigma^2 A_d^2 x(s-\tau)ds\right].$$
$$= x^{\dagger}(t)\left[(A+\sigma A_d)^{\dagger}P + P(A+\sigma A_d)\right]x(t)$$
$$-\int_{t-\tau}^{t} \bar{\sigma}x^{\dagger}(s)A^T A_d^T Px(t) + \sigma x^{\dagger}(t)PA_d Ax(s)ds - \int_{t-\tau}^{t} \bar{\sigma}^2 x^{\dagger}(s-\tau)(A_d^2)^T Px(t) + \sigma^2 x^{\dagger}(t)PA_d^2 x(s-\tau)ds$$

Looking at the integral terms in the last line of the above expression, we denote them

$$\eta_1 = -\int_{t-\tau}^{t} \bar{\sigma}x^{\dagger}(s)A^T A_d^T Px(t) + \sigma x^{\dagger}(t)PA_d Ax(s)ds = -\int_{-\tau}^{0} \bar{\sigma}x^{\dagger}(t+\vartheta)A^T A_d^T Px(t) + \sigma x^{\dagger}(t)PA_d Ax(t+\vartheta)d\vartheta,$$

$$\eta_2 = -\int_{t-\tau}^{t} \bar{\sigma}^2 x^{\dagger}(s-\tau)(A_d^2)^T Px(t) + \sigma^2 x^{\dagger}(t)PA_d^2 x(s-\tau)ds = -\int_{-\tau}^{0} \bar{\sigma}^2 x^{\dagger}(t-\tau+\vartheta)(A_d^2)^T Px(t) + \sigma^2 x^{\dagger}(t)PA_d^2 x(t-\tau+\vartheta)d\vartheta,$$

One finds by completing the squares with $P_1 > 0, P_2 > 0$ that

$$(\bar{\sigma}A^T A_d^T Px(t) + P_1^{-1}x(t+\vartheta))^{\dagger} P_1(\bar{\sigma}A^T A_d^T Px(t) + P_1^{-1}x(t+\vartheta)) \geq 0$$
$$|\sigma|^2 x^{\dagger}(t)PA_d AP_1 A^T A_d^T Px(t) + \sigma x^{\dagger}(t)PA_d Ax(t+\vartheta) + \bar{\sigma}x^{\dagger}(t+\vartheta)A^T A_d^T Px(t) + x^{\dagger}(t+\vartheta)P_1^{-1}x(t+\vartheta) \geq 0$$
$$-\left[\sigma x^{\dagger}(t)PA_d Ax(t+\vartheta) + \bar{\sigma}x^{\dagger}(t+\vartheta)A^T A_d^T Px(t)\right] \leq x^{\dagger}(t)|\sigma|^2 PA_d AP_1 A^T A_d^T Px(t) + x^{\dagger}(t+\vartheta)P_1^{-1}x(t+\vartheta)$$

Hence,

$$\eta_1 = -\int_{-\tau}^{0} \bar{\sigma}x^{\dagger}(t+\vartheta)A^T A_d^T Px(t) + \sigma x^{\dagger}(t)PA_d Ax(t+\vartheta)d\vartheta$$
$$\leq \int_{-\tau}^{0} |\sigma|^2 x^{\dagger}(t)PA_d AP_1 A^T A_d^T Px(t) + x^{\dagger}(t+\vartheta)P_1^{-1}x(t+\vartheta)d\vartheta$$
$$\eta_1 \leq \tau|\sigma|^2 x^{\dagger}(t)PA_d AP_1 A^T A_d^T Px(t) + \int_{-\tau}^{0} x^{\dagger}(t+\vartheta)P_1^{-1}x(t+\vartheta)d\vartheta.$$

In a similar manner,

$$(\bar{\sigma}^2(A_d^2)^T Px(t)+P_2^{-1}x(t-\tau+\vartheta))^\dagger P_2(\bar{\sigma}^2(A_d^2)^T Px(t)+P_2^{-1}x(t-\tau+\vartheta))\geq 0$$
$$|\sigma|^4 x^\dagger(t)PA_d^2P_2(A_d^2)^T Px(t)+\sigma^2 x^\dagger(t)PA_d^2x(t-\tau+\vartheta)+\bar{\sigma}^2x^\dagger(t-\tau+\vartheta)(A_d^2)^T Px(t)+x^\dagger(t-\tau+\vartheta)P_2^{-1}x(t-\tau+\vartheta)\geq 0$$
$$-\left[\sigma^2 x^\dagger(t)PA_d^2x(t-\tau+\vartheta)+\bar{\sigma}^2x^\dagger(t-\tau+\vartheta)(A_d^2)^T Px(t)\right]\leq x^\dagger(t)|\sigma|^4 PA_d^2P_2(A_d^2)^T Px(t)+x^\dagger(t-\tau+\vartheta)P_2^{-1}x(t-\tau+\vartheta)$$

Hence,

$$\eta_2=-\int_{-\tau}^{0}\bar{\sigma}^2x^\dagger(t-\tau+\vartheta)(A_d^2)^T Px(t)+\sigma^2x^\dagger(t)PA_d^2x(t-\tau+\vartheta)d\vartheta$$
$$\leq\int_{-\tau}^{0}x^\dagger(t)|\sigma|^4 PA_d^2P_2(A_d^2)^T Px(t)+x^\dagger(t-\tau+\vartheta)P_2^{-1}x(t-\tau+\vartheta)d\vartheta$$
$$\eta_2\leq\tau|\sigma|^4 x^\dagger(t)PA_d^2P_2(A_d^2)^T Px(t)+\int_{-\tau}^{0}x^\dagger(t-\tau+\vartheta)P_2^{-1}x(t-\tau+\vartheta)d\vartheta$$

Inserting these inequalities in the expression for the time derivative of the Lyapunov-Razuminkhin function one finds that

$$\frac{d}{dt}V(x(t))=x^\dagger(t)\left[(A+\sigma A_d)^\dagger P+P(A+\sigma A_d)\right]x(t)+\eta_1+\eta_2$$
$$\frac{d}{dt}V(x(t))\leq x^\dagger(t)\left[(A+\sigma A_d)^\dagger P+P(A+\sigma A_d)+\tau|\sigma|^2 PA_dAP_1A^TA_d^TP+\tau|\sigma|^4 PA_d^2P_2(A_d^2)^TP\right]x(t)\cdot$$
$$+\int_{-\tau}^{0}x^\dagger(t+\vartheta)P_1^{-1}x(t+\vartheta)d\vartheta+\int_{-\tau}^{0}x^\dagger(t-\tau+\vartheta)P_2^{-1}x(t-\tau+\vartheta)d\vartheta$$

Under the condition that $P_1^{-1}<P,\ P_2^{-1}<P$, one has

$$\frac{d}{dt}V(x(t))\leq x^\dagger(t)\left[(A+\sigma A_d)^\dagger P+P(A+\sigma A_d)+\tau|\sigma|^2 PA_dAP_1A^TA_d^TP+\tau|\sigma|^4 PA_d^2P_2(A_d^2)^TP\right]x(t)$$
$$+\int_{-\tau}^{0}x^\dagger(t+\vartheta)P_1^{-1}x(t+\vartheta)d\vartheta+\int_{-\tau}^{0}x^\dagger(t-\tau+\vartheta)P_2^{-1}x(t-\tau+\vartheta)d\vartheta$$
$$\leq x^\dagger(t)\left[(A+\sigma A_d)^\dagger P+P(A+\sigma A_d)+\tau|\sigma|^2 PA_dAP_1A^TA_d^TP+\tau|\sigma|^4 PA_d^2P_2(A_d^2)^TP\right]x(t)$$
$$+\int_{-\tau}^{0}x^\dagger(t+\vartheta)Px(t+\vartheta)d\vartheta+\int_{-\tau}^{0}x^\dagger(t-\tau+\vartheta)Px(t-\tau+\vartheta)d\vartheta$$
$$=x^\dagger(t)\left[(A+\sigma A_d)^\dagger P+P(A+\sigma A_d)+\tau|\sigma|^2 PA_dAP_1A^TA_d^TP+\tau|\sigma|^4 PA_d^2P_2(A_d^2)^TP\right]x(t)$$
$$+\int_{-\tau}^{0}V(x(t+\vartheta))d\vartheta+\int_{-\tau}^{0}V(x(t-\tau+\vartheta))d\vartheta$$

Following Razuminkhin stability condition, (*c.f.* Theorem 8.2) one has that if $V(x(\vartheta)) < \varepsilon V(x(t))$, $\forall \vartheta \in [t-2\tau, t]$, for some $\varepsilon > 1$, the time derivative of the Lyapunov-Razuminkin function satisfies

$$\frac{d}{dt}V(x(t)) \le x^\dagger(t)\Big[(A+\sigma A_d)^\dagger P + P(A+\sigma A_d) + \tau|\sigma|^2 PA_d AP_1 A^T A_d^T P + \tau|\sigma|^4 PA_d^2 P_2 (A_d^2)^T P + 2\tau\varepsilon P\Big]x(t).$$

To find an appropriate $\varepsilon > 1$ and $\delta > 0$, such that for $V(x(\vartheta)) < \varepsilon V(x(t)), \forall \vartheta \in [t-2\tau, t]$ one has $\dot{V}(x(t)) \le -\delta V(x(t))$, let us proceed similarly as in the proof of Theorem 8.4 and 8.5. If the condition (309) of the Theorem is satisfied then there exists $\delta_1 > 0$ such that

$$(A+\sigma A_d)^\dagger P + P(A+\sigma A_d) + \tau_0|\sigma|^2 PA_d AP_1 A^T A_d^T P + \tau_0|\sigma|^4 PA_d^2 P_2 (A_d^2)^T P + 2\tau_0(1+2\delta_1)P < 0.$$

Take $\varepsilon = 1 + \delta_1$, then

$$(A+\sigma A_d)^\dagger P + P(A+\sigma A_d) + \tau_0|\sigma|^2 PA_d AP_1 A^T A_d^T P + \tau_0|\sigma|^4 PA_d^2 P_2 (A_d^2)^T P + 2\tau_0\varepsilon P < -2\delta_1\tau_0 P$$

so $\dot{V}(x(t)) \le -2\delta_1\tau_0 V(x(t))$, completing the proof. ■

The corollaries of Theorem 8.5 can also be similarly adapted to serve the present purpose.

Corollary 8.1'. The conditions of Theorem 8.5' can be replaced by

$$A^T P_1^{-1} A \le P, \ |\sigma|^2 A_d^T P_2^{-1} A_d \le P, \tag{310}$$

$$(A+\sigma A_d)^\dagger P + P(A+\sigma A_d) + \tau_0|\sigma|^2 PA_d(P_1+P_2)A_d^T P + 2\tau_0 P < 0. \tag{311}$$

Proof: Using a different square completion one obtains the bounds for the integral terms in the proof as

$$(\bar{\sigma}A_d^T Px(t) + P_1^{-1}Ax(t+\vartheta))^\dagger P_1(\bar{\sigma}A_d^T Px(t) + P_1^{-1}Ax(t+\vartheta)) \ge 0$$
$$|\sigma|^2 x^\dagger(t)PA_d P_1 A_d^T Px(t) + \sigma x^\dagger(t)PA_d Ax(t+\vartheta) + \bar{\sigma}x^\dagger(t+\vartheta)A^T A_d^T Px(t) + x^\dagger(t+\vartheta)A^T P_1^{-1}Ax(t+\vartheta) \ge 0$$
$$-\Big[\sigma x^\dagger(t)PA_d Ax(t+\vartheta) + \bar{\sigma}x^\dagger(t+\vartheta)A^T A_d^T Px(t)\Big] \le x^\dagger(t)|\sigma|^2 PA_d P_1 A_d^T Px(t) + x^\dagger(t+\vartheta)A^T P_1^{-1}Ax(t+\vartheta)$$

Hence,

$$\eta_1 = -\int_{-\tau}^{0} \bar{\sigma} x^{\dagger}(t+\vartheta) A^T A_d^T P x(t) + \sigma x^{\dagger}(t) P A_d A x(t+\vartheta) d\vartheta$$

$$\leq \int_{-\tau}^{0} x^{\dagger}(t) \left|\sigma\right|^2 P A_d P_1 A_d^T P x(t) + x^{\dagger}(t+\vartheta) A^T P_1^{-1} A x(t+\vartheta) d\vartheta \quad .$$

$$\eta_1 \leq \tau \left|\sigma\right|^2 x^{\dagger}(t) P A_d P_1 A_d^T P x(t) + \int_{-\tau}^{0} x^{\dagger}(t+\vartheta) A^T P_1^{-1} A x(t+\vartheta) d\vartheta$$

In a similar manner,

$$(\bar{\sigma} A_d^T P x(t) + \sigma P_2^{-1} A_d x(t-\tau+\vartheta))^{\dagger} P_2 (\bar{\sigma} A_d^T P x(t) + \sigma P_2^{-1} A_d x(t-\tau+\vartheta)) \geq 0$$

$$\left|\sigma\right|^2 x^{\dagger}(t) P A_d P_2 A_d^T P x(t) + \sigma^2 x^{\dagger}(t) P A_d^2 x(t-\tau+\vartheta) + \bar{\sigma}^2 x^{\dagger}(t-\tau+\vartheta)(A_d^2)^T P x(t) + \left|\sigma\right|^2 x^{\dagger}(t-\tau+\vartheta) A_d^T P_2^{-1} A_d x(t-\tau+\vartheta) \geq 0 \ .$$

$$-\left[\sigma^2 x^{\dagger}(t) P A_d^2 x(t-\tau+\vartheta) + \bar{\sigma}^2 x^{\dagger}(t-\tau+\vartheta)(A_d^2)^T P x(t)\right] \leq x^{\dagger}(t)\left|\sigma\right|^2 P A_d P_2 A_d^T P x(t) + x^{\dagger}(t-\tau+\vartheta)\left|\sigma\right|^2 A_d^T P_2^{-1} A_d x(t-\tau+\vartheta)$$

Hence

$$\eta_2 = -\int_{-\tau}^{0} \bar{\sigma}^2 x^{\dagger}(t-\tau+\vartheta)(A_d^2)^T P x(t) + \sigma^2 x^{\dagger}(t) P A_d^2 x(t-\tau+\vartheta) d\vartheta$$

$$\leq \int_{-\tau}^{0} x^{\dagger}(t)\left|\sigma\right|^2 P A_d P_2 A_d^T P x(t) + x^{\dagger}(t-\tau+\vartheta)\left|\sigma\right|^2 A_d^T P_2^{-1} A_d x(t-\tau+\vartheta) d\vartheta$$

$$\eta_2 \leq \tau\left|\sigma\right|^2 x^{\dagger}(t) P A_d P_2 A_d^T P x(t) + \int_{-\tau}^{0} x^{\dagger}(t-\tau+\vartheta)\left|\sigma\right|^2 A_d^T P_2^{-1} A_d x(t-\tau+\vartheta) d\vartheta$$

Inserting those bounds in the expression for the derivative of the Lyapunov-Razuminkhin function and applying the conditions of the Corollary in the same manner as in the proof of Theorem 8.5' one finds guaranteed asymptotic stability of the trivial solution. ■

This corollary seemingly introduces the dependence of the condition only on $\left|\sigma\right|^2$, not on $\left|\sigma\right|^4$, but the down side is that now the dependence on $\left|\sigma\right|^2$ also appears in the condition for the matrix P_2 and this complicates any effort to utilize the results of this corollary.

Corollary 8.2'. Assume that one has the asymptotic stability of the delay free system characterized by the Lyapunov inequality

$$(A+\sigma A_d)^{\dagger} P + P(A+\sigma A_d) \leq -Q \ , \tag{312}$$

Choose $\beta_1^{-1} \le 1, \beta_2^{-1} \le 1$. Then one has asymptotic stability for delays less than

$$\tau_0 = \left\| Q^{-1/2} \left[\beta_1 |\sigma|^2 PA_d AP^{-1} A^T A_d^T P + \beta_2 |\sigma|^4 PA_d^2 P^{-1} (A_d^2)^T P + 2P \right] Q^{-1/2} \right\|^{-1}. \qquad (313)$$

Proof: In conditions of Theorem 8.5 choose $P_1 = \beta_1 P^{-1}, P_2 = \beta_2 P^{-1}$. The choice $\beta_1^{-1} \le 1, \beta_2^{-1} \le 1$ satisfies the condition $P_1^{-1} < P, P_2^{-1} < P$. Then it follows that the condition of the Theorem 8.5,

$$(A + \sigma A_d)^\dagger P + P(A + \sigma A_d) + \tau_0 |\sigma|^2 PA_d AP_1 A^T A_d^T P + \tau_0 |\sigma|^4 PA_d^2 P_2 (A_d^2)^T P + 2\tau_0 P < 0$$

is satisfied if

$$-Q + \tau_0 \beta_1 |\sigma|^2 PA_d AP^{-1} A^T A_d^T P + \tau_0 \beta_2 |\sigma|^4 PA_d^2 P^{-1} (A_d^2)^T P + 2\tau_0 P < 0.$$

This is surely so if

$$\begin{gathered} -I < -\tau_0 Q^{-1/2} \left[\beta_1 |\sigma|^2 PA_d AP^{-1} A^T A_d^T P + \beta_2 |\sigma|^4 PA_d^2 P^{-1} (A_d^2)^T P + 2P \right] Q^{-1/2} \\ \tau_0 Q^{-1/2} \left[\beta_1 |\sigma|^2 PA_d AP^{-1} A^T A_d^T P + \beta_2 |\sigma|^4 PA_d^2 P^{-1} (A_d^2)^T P + 2P \right] Q^{-1/2} < I \end{gathered},$$

Implying that $\tau_0 \left\| Q^{-1/2} \left[\beta_1 |\sigma|^2 PA_d AP^{-1} A^T A_d^T P + \beta_2 |\sigma|^4 PA_d^2 P^{-1} (A_d^2)^T P + 2P \right] Q^{-1/2} \right\| < 1$, whence the condition on the delay, (313), follows, completing the proof. ■

With the proofs of these analogous results one has a rigorous justification of using a simple change, $A_d \to \sigma A_d$, $A_d^T \to (\sigma A_d)^\dagger = \bar{\sigma} A_d^T$, in the classical stability conditions, detailed in Theorem 8.3, 8.4, 8.5, and their corollaries, to address the stability of complex delay differential equation, (303). Making this change in delay dependent stability conditions leads to a dependence of stability of (303) on both σ and τ. This can be used to investigate the dependence of the synchronizing region for (303) on the delay τ. Such a delay dependent synchronizing region limits the maximal delay in dependence on the graph topology, or limits the graph topology in dependence of the maximal delay.

8.3 Output Consensus of Heterogeneous Agents

This section discusses output consensus and output synchronization problems for multi-agent systems. The state consensus problem requires all the agents to have states in the same state space. Otherwise the equality of states of all agents is not defined. Thus, even if one considers heterogeneous agents they still need to be of the same dimension, *c.f.* Chapter 6, Section 6.7. Output consensus problem requires all the system outputs to be of the same dimension. This allows fairly general heterogeneous agents. In fact, 32, 60, deal with passive affine in control agents, and show that static local neighborhood output error control guarantees output synchronization. Assumption on passivity allows for such a great degree of generality on the structure of agents' systems. Here we do not consider passive systems, but general linear time-invariant systems, as presented in 61,62,63,64.

8.3.1 Multi-agent System Dynamics and the Control Goal

Let the multi-agent system consist of N agents of the form

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i, \\ y_i &= C_i x_i. \end{aligned} \tag{314}$$

Let there exist a leader, having the dynamics

$$\begin{aligned} \dot{x}_0 &= A x_0, \\ y_0 &= C x_0. \end{aligned} \tag{315}$$

Definition 8.3: The *distributed output consensus problem* for the multi-agent system (314) is to find distributed controls, u_i, such that as $t \to \infty$ $\|y_i - y_j\| \to 0$, $\forall (i, j)$. The *distributed output synchronization problem* for the multi-agent system (314), (315), is to find distributed controls u_i such that as $t \to \infty$ $\|y_i - y_0\| \to 0$, $\forall i$.

8.3.2 Distributed Dynamic Control for Output Synchronization

Define the *local neighborhood output error* as

$$e_{yi} = \sum_j e_{ij}(y_j - y_i) + g_i(y_0 - y_i). \tag{316}$$

This signal can be used for distributed control purposes. Such a distributed control is then called distributed output feedback control. One can have static distributed output control, as was used in Chapter 7 for state consensus, or dynamic distributed output control, which seems more appropriate for the output consensus and synchronization problems of heterogenous multi-agent systems, (314),(315).

Proposition 8.1. Let matrices A, B, C, A_0, C_0, where (A, B) is stabilizable, define the linear time-invariant system,

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{317}$$

and the linear time invariant reference generator,

$$\begin{aligned} \dot{x}_0 &= A_0 x_0, \\ y_0 &= C_0 x_0. \end{aligned} \tag{318}$$

Let there exist matrices Π, Γ solving the equations

$$\begin{aligned} \Pi A_0 &= A\Pi + B\Gamma \\ C\Pi &= C_0 \end{aligned}. \tag{319}$$

Let K be a feedback gain such that the matrix $A - BK$ is Hurwitz. Then the control

$$u = -K(x - \Pi x_0) + \Gamma x_0 \tag{320}$$

makes the output y of (317) asymptotically track the reference output y_0 of (318).

Proof: Under the control (320) the subspace of the total state space $\mathbb{R}^{(N+1)n}$, having states (x, x_0), defined by $x = \Pi x_0$ is invariant and stable, with respect to the dynamics (317), (318). On that subspace one finds that $y = Cx = C\Pi x_0 = C_0 x_0 = y_0$, hence $x \to \Pi x_0 \Rightarrow y \to y_0$. That this subspace is invariant and stable follows from the dynamics

$$\frac{d}{dt}(x-\Pi x_0) = \dot{x} - \Pi\dot{x}_0 = Ax + Bu - \Pi A_0 x_0$$
$$= Ax - BK(x-\Pi x_0) + B\Gamma x_0 - \Pi A_0 x_0 \tag{321}$$
$$= (A-BK)(x-\Pi x_0) + (A\Pi - \Pi A_0 + B\Gamma)x_0 .$$

Therefore, by (319), this gives

$$\frac{d}{dt}(x-\Pi x_0) = (A-BK)(x-\Pi x_0) \tag{322}$$

Showing both invariance and stability of the subspace $x-\Pi x_0 = 0$. That a feedback K, stabilizing $A-BK$, exists follows from stabilizability of the pair (A,B). ■

The above result on output reference tracking is called the internal model principle, 61, and it is used for synchronization of multi-agent systems, (314), (318), as follows, 61,62,63. Each of the agents is endowed with a leader's state observer.

$$\dot{\hat{x}}_{0i} = A_0\hat{x}_{0i} + cL_2\sum_j e_{ij}(\hat{y}_{0j} - \hat{y}_{0i}) + g_i(y_0 - \hat{y}_{0i}) , \tag{323}$$

where $\hat{y}_{0i} = C_0\hat{x}_{0i}$.

This is a distributed observation problem (*c.f.* Chapter 3, 6), 13,30. One can guarantee, under appropriate stipulations, that all the local estimates, $\hat{x}_{0i}$, converge to the true value of the leader's state, x_0, (*c.f.* Chapter 6). Those estimates are used in the control law instead of the leader's state, x_0. Hence, the distributed control takes the form

$$u_i = -K_i(x_i - \Pi_i\hat{x}_{0i}) + \Gamma_i\hat{x}_{0i} . \tag{324}$$

with Π_i, Γ_i, satisfying the equations

$$\begin{aligned} \Pi_i A_0 &= A_i\Pi_i + B_i\Gamma_i \\ C_i\Pi_i &= C_0 \end{aligned} , \tag{325}$$

and the feedback gain K_i, stabilizing the matrix $A_i - B_iK_i$.

Define the *output synchronization error*

$$\delta_i = x_i - \Pi_i x_0 , \tag{326}$$

the *local reference tracking error*,

$$\hat{\delta}_i = x_i - \Pi_i \hat{x}_{0i} , \tag{327}$$

and the *observer error*,

$$\eta_i = x_0 - \hat{x}_{0i} . \tag{328}$$

One has the relation, $\delta_i = \hat{\delta}_i - \Pi_i \eta_i$, connecting those different error signals, $\delta_i, \hat{\delta}_i, \eta_i$. Global dynamics of $\delta = \begin{bmatrix} \delta_1^T & \cdots & \delta_N^T \end{bmatrix}^T$, $\hat{\delta} = \begin{bmatrix} \hat{\delta}_1^T & \cdots & \hat{\delta}_N^T \end{bmatrix}^T$ and $\eta = \begin{bmatrix} \eta_1^T & \cdots & \eta_N^T \end{bmatrix}^T$ under the control protocol (323), (324), equals

$$\frac{d}{dt}\begin{bmatrix} \delta \\ \eta \end{bmatrix} = \begin{bmatrix} diag(A_i - B_i K_i) & -diag(B_i(K_i \Pi_i + \Gamma_i)) \\ 0 & I_N \otimes A - c(L+G) \otimes L_2 C_0 \end{bmatrix} \begin{bmatrix} \delta \\ \eta \end{bmatrix}. \tag{329}$$

It is evident from (329) that $\eta = 0$ subspace of the (δ, η) space is invariant, and that on this subspace the δ dynamics is asymptotically stable. In fact, because of the upper triangular structure of the system matrix in (329), asymptotic stability of $A_i - B_i K_i$, $\forall i$ and $I_N \otimes A - c(L+G) \otimes L_2 C_0$ is necessary and sufficient for the asymptotic stability of (329). This then solves the problem of heterogeneous agents output synchronization. All agents' outputs, y_i, asymptotically track the local output reference signals, $\hat{y}_{0i}$, which, due to synchronization of the local leader's state estimates, asymptotically synchronize. Hence, $y_i \to \hat{y}_{0i} \to y_0$.

The control (324) depends on the state on a single node and on the state of the local observer. The local observer dynamics, (323), depends on the local estimator output disagreement, requiering the local regulators to communicate their states, *i.e.* leader's state estimates, in a distributed fashion.

Bearing in mind the cascade structure of the proposed controller one might try to use a modified control law,

$$\dot{\hat{x}}_{0i} = A_0 \hat{x}_{0i} + cL_2 \sum_j e_{ij}(y_j - y_i) + g_i(y_0 - y_i)\,, \tag{330}$$

where the local neighborhood output error, (316), appears directly instead of (323). In that case the observer error dynamics equals

$$\dot{\eta}_i = A_0 \eta_i - cL_2 \sum_j e_{ij}(y_j - y_i) + g_i(y_0 - y_i)\,, \tag{331}$$

The local neighborhood output error couples the observer dynamics to the agents' dynamics. This term can be written as

$$\begin{aligned} &\sum_j e_{ij}(y_j - y_i) + g_i(y_0 - y_i) = \sum_j e_{ij}(C_j x_j - C_i x_i) + g_i(C_0 x_0 - C_i y_i) \\ &= \sum_j e_{ij}(C_j x_j - C_0 x_0 - (C_i x_i - C_0 x_0) + g_i(C_0 x_0 - C_i y_i) \end{aligned} \tag{332}$$

Since, by equations (325) $C_j x_j - C_0 x_0 = C_j(x_j - \Pi_j x_0) = C_j \delta_j$ one has that

$$e_y = -(L+G) \otimes I_p diag(C_i)\delta\,. \tag{333}$$

This then gives the global dynamics in form of

$$\frac{d}{dt}\begin{bmatrix}\delta \\ \eta\end{bmatrix} = \begin{bmatrix} diag(A_i - B_i K_i) & -diag(B_i(K_i \Pi_i + \Gamma_i)) \\ c(L+G)\otimes L_2 diag(C_i) & I_N \otimes A \end{bmatrix}\begin{bmatrix}\delta \\ \eta\end{bmatrix}. \tag{334}$$

It is more advantageous for our purposes to describe the dynamics of the system (334) in the transformed coordinates

$$\begin{bmatrix}\delta \\ \eta\end{bmatrix} = \begin{bmatrix} I & -diag(\Pi_i) \\ 0 & I \end{bmatrix}\begin{bmatrix}\hat{\delta} \\ \eta\end{bmatrix}. \tag{335}$$

Applying the transformation (335) to (334) one obtains

$$\frac{d}{dt}\begin{bmatrix}\hat{\delta} \\ \eta\end{bmatrix} = \begin{bmatrix} diag(A_i - B_i K_i) + c\,diag(\Pi_i)(L+G)\otimes L_2 diag(C_i) & -c\,diag(\Pi_i)(L+G)\otimes L_2 C_0 \\ c(L+G)\otimes L_2 diag(C_i) & I_N \otimes A - c(L+G)\otimes L_2 C_0 \end{bmatrix}\begin{bmatrix}\hat{\delta} \\ \eta\end{bmatrix}. \tag{336}$$

The system (336) can be interpreted as

$$\frac{d}{dt}\begin{bmatrix}\hat{\delta}\\ \eta\end{bmatrix}=\begin{bmatrix}diag(A_i-B_iK_i) & -c\,diag(\Pi_i)(L+G)\otimes L_2C_0\\ 0 & I_N\otimes A-c(L+G)\otimes L_2C_0\end{bmatrix}\begin{bmatrix}\hat{\delta}\\ \eta\end{bmatrix}$$
$$+c\begin{bmatrix}diag(\Pi_i)(L+G)\otimes L_2 diag(C_i)\\ (L+G)\otimes L_2 diag(C_i)\end{bmatrix}\hat{\delta} \qquad . \tag{337}$$

The system (337) can be concisely written as

$$\frac{d}{dt}\begin{bmatrix}\hat{\delta}\\ \eta\end{bmatrix}=\bar{A}\begin{bmatrix}\hat{\delta}\\ \eta\end{bmatrix}+cM\hat{\delta}. \tag{338}$$

The dynamics (338) can be interpreted as a linear system, determined by the system matrix $\bar{A}$, stabilized with the proper choice of K_i, L_2, c, together with a state dependent disturbance term, $cM\hat{\delta}$. Hence, the local output reference tracking error, $\hat{\delta}$, acts as a disturbance. This error is precisely the difference between the observer protocols (323) and (330). Depending on the growth bound of the disturbance term, limited by $\sigma_{\max}(M)$, one can have asymptotic stability of (337), or uniform ultimate boundedness. Asymptotic stability implies $(\hat{\delta},\eta)\to 0$, guaranteeing $y_i \to \hat{y}_{0i} \to y_0$, and, therefore, solves the output synchronization problem, while uniform ultimate boundedness ensures that the deviations from the output synchronization and leader's state perfect estimation remain bounded.

Clearly the proposed simplified control law, (330), requires the disturbance term to satisfy a certain growth bound, or equivalently, that the exponential convergence of the nominal system, $\bar{A}$, be sufficiently fast, (*c.f.* Chapter 6), 46. Essentially, one tries to make the local output reference tracking sufficiently fast, in order to guarantee the asymptotic stability of the system (337). Additional investigation is required to determine the conditions under which this is achieved. These more stringent conditions are the price to pay for simplifying the control law.

A concluding remark on dynamic feedback control is appropriate here. In order to solve the output synchronization problem one needs to stabilize the total multi-agent system on the output consensus manifold. The output consensus manifold, S_1, is a subspace of the state space $\mathbb{R}^{Nn}$, of the multi-agent system, determined by $y_i(x_i) = y_j(x_j)$, $\forall(i, j)$; in particular, for linear systems, (314), $C_i x_i = C_j x_j$. Augmenting the state space with the leader's state, x_0, one obtains the total state (x, x_0) in the space $\mathbb{R}^{Nn} \times \mathbb{R}^n$. In this augmented space one defines the output synchronization manifold as a submanifold, S_2, of $\mathbb{R}^{Nn} \times \mathbb{R}^n$, where $y_i = y_0$. However, we are interested in an even smaller manifold contained in S_2. We would like to consider the intersection of the submanifolds of $\mathbb{R}^{Nn} \times \mathbb{R}^n$, determined by $x_i - \Pi_i x_0 = 0$, for all *i*. We will denote this intersection by S_3. But, on S_3 one has the additional property that bounded x_0 implies bounded x_i. This property of state boundedness is not necessarily shared with the larger space obtained as the intersection of the submanifolds determined by $x_i - \Pi_i x_0 \in \ker C_i$. Hence, in the spirit of output regulation, as detailed in Proposition 8.1., one constrains the target set but obtains the property of state boundedness.

By adding dynamic regulators one further augments the state space, $\mathbb{R}^{Nn} \times \mathbb{R}^n$, to include the states of the regulators, $\hat{x}_0$. Thus, one ends up with states $(x, \hat{x}_0, x_0)$ in the augmented space $\mathbb{R}^{Nn} \times \mathbb{R}^{Nn} \times \mathbb{R}^n$. Within this augmented space, the control goal is to stabilize the invariant manifold determined by $\hat{\delta} = 0, \eta = 0$, (327),(328). This manifold is contained in the output synchronization manifold of $\mathbb{R}^{Nn} \times \mathbb{R}^{Nn} \times \mathbb{R}^n$. Note that the entire output synchronization manifold need not be stable nor invariant, but stability and invariance must hold for the target set, $\hat{\delta} = 0, \eta = 0$. Stability properties detailed in Chapter 4 and 6 are applicable to the problem of stabilizing such target sets. Together with the internal model principle, 61, the results of Chapter 6 can be used to guarantee this control goal.

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BIOGRAPHICAL INFORMATION

Kristian Hengster-Movric was born in Zagreb, Croatia, in 1986. Elementary and High School education was received in Zagreb. In 2004 Kristian graduated from XV. Gimnazija, mathematics and science oriented high school, in Zagreb, as one of the best students in his generation. He subsequently enrolled in the College of Electrical Engineering and Computing at The University of Zagreb in 2004, and received the M.S. degree in the field of Automatics in 2009. He received the state scholarship for being in the top 10% of his generation for the entire duration of the program. During the time in college he also worked on different projects that resulted in various publications, and served as a teaching assistant in Mathematics and Signals and Systems during his senior years. Shortly before graduation he was awarded the 2009. Rector's Prize for excellence in research work on multi-agent potential field control.

In 2009 Kristian was accepted to the University of Texas at Arlington, Electrical Engineering department for a PhD. Since the Fall semester that year he has been working towards the PhD at the UTARI institute, under the supervision of dr. Frank L. Lewis. In addition he has served in the capacity of a graduate teaching assistant for the courses in Control Systems, Circuits, Electronics, Linear Systems Optimal Control and Distributed Decision and Control. In 2010 Kristian was inducted into The Golden Key Honor Society, for academic achievement. In 2013 he was awarded second place prize, N.M. Stelmakh Outstanding Student Research Award, for the excellence of the research work conducted while in the PhD program, and the Dean's Fellowship for summer 2013.