SCALAR EQUILIBRIA FOR n-PERSON GAMES

by

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ABSTRACT

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In this dissertation we develop a scalarization approach for one-shot, n-person games by defining the notion of Scalar Equilibria. We first show that existing solution concepts can be represented as Scalar Equlibria. For example, Regret, Disappointment, and Joint Equilibrium can be determined by defining Regret, Disappointment, and Joint Scalar Equilibria. These scalar equilibria are useful for finding pure strategies when pure Regret, Disappointment, and Joint Equilibria do not exist. Next, we present the Maximin Scalarization Equilibrium to yield maximin solution concept.

In addition, we propose other Scalar Equilibria for various notions of rationality. The Aspiration Scalar Equilibrium is developed for an aspiration criterion when players have specified payoff aspiration levels. Then Risk, Greedy and Cooperative Scalar Equilibrium are developed for risk, greed, and cooperative criteria, respectively.

Moreover, Sequential, Simultaneous, and Priority Scalar Equilibria are developed as well as Coalition Scalar Equilibria. In a Sequential Scalar Equilibrium we sequentially, in some chosen order, apply other scalarizations to Scalar Equilibrium of the game until we find a unique one if possible. In a Simultaneous Scalar Equilibrium we combine the criteria for various scalarizations into one. Effectively the multiple criteria are applied simultaneously. In a Priority Scalar Equilibrium players are prioritized as their ability to get their highest payoff. A Coalition Scalar Equilibrium consider fixed teams of players seek team payoffs that are then divided among the players. Finally, we presented examples to illustrate the usage and theoretical aspects of these equilibria.

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CHAPTER 1

INTRODUCTION

Decisions are made constantly by individuals, groups, companies, and societies. In particular, game theory is the study of strategic economic and social interactions among the agents, also known as players. In the existing decision-making models of game theory, the players are usually assumed to be rational in sense that they consistently pursue their own welfare and goals as they define them. However, in many of these cases the agents do not realize that they are playing a game. They have some inaccurate idea of their opponents, their environment, their goals, the actions available to them, and the payoffs associated with any combination of actions for all players. Therefore, the players often use heuristics that are generated from experience and that may not yield a perfect outcome for them. The proposed research will provide heuristics in the form of scalarizations to obtain pure strategy equilibria for one-shot games in normal form where a game is depicted by a matrix. In other words, the players will make collective decision by maximizing some scalar function.

In these games the players will be assumed to pursue their individual goals. That pursuit may involve pre-game negotiation. In one-shot games, each player makes a single decision resulting in a single outcome, as opposed to repeated games where the players can learn each other's tendencies. The proposed research focuses on new scalarizations that are applied to one-shot, n-person games. In this chapter we present relevant literatures, give basic concepts in game theory, and outline our contribution.

1.1 Literature review

A systematic study of game theory effectively began with von Neuman and Morgenstern [1], who studied zero-sum noncooperative games, as well as cooperative games where coalitions can be formed. Nash [2] developed their results to the n-person, non-zero-sum case for noncooperative games by notion of the Nash Equilibrium. In his solution concept, rational players are assumed to be selfish and act in their individual self-interest in the sense that each player considers his best responses to the possible joint actions of the other players. The result is that no player can improve his expected payoff in the Nash equilibrium by unilaterally changing his pure or mixed strategy. If he did so, the amount of payoff he would lose is called regret, which is an enforcement mechanism their essentially eliminates the distinction behavior cooperative and noncooperative games. Therefore, modern game theory [3-7] usually assumes that any joint rational action by the players must be a Regret Equilibrium (RE), as a Nash Equilibrium is called here, to be sustainable.

However, the RE has weaknesses as illustrated in social dilemmas, such well-known games as the Prisoner's Dilemma and Chicken games [8-10]. For example, the paradox of Prisoner's Dilemma is that the unique RE is strictly dominated by another outcome and thus may not present the actual cooperative behavior of players in such a situation. Moreover, multiple pure or mixed REs often exist for a game. Thus, two rational players could choose from different REs to yield an outcome that is not an RE.

To improve these weaknesses, Schelleing's focal point effect [11], Harsanyi's Bayesian equilibrium point [12], Harsanyi's purification of mixed equilibria [13], Selton's perfect equilibrium [14], Myerson's proper equilibrium [15], and van Damme's quasi-perfect equilibrium [16] have been proposed for refinement of REs.

Furthermore, Harsanyi and Selton [17] also used notions of payoff dominance and risk dominance to yield a unique RE. Aumann [18] considered possible epistemic ways what players'reason about their opponents to obtain REs, and also defined the concept of correlated equilibria [19]. Kahneman and Tversky [20] developed Prospect Theory as an alternative to expected utility theory. Brams [21] eliminated mixed strategies and defined a non-myopic pure equilibrium that always exists. Shalev [22] developed a non-Nash equilibrium emphasizing loss aversion with respect to reference points for the players. For only two-person games, Rabin [23] defined a fairness equilibrium that is not always an RE. Stirling [24] outlined an approach for satisficing solutions with respect to aspiration levels for the players.

Insuwan [25] and Corley [26] presented an alternative, the Disappointment Equilibrium (DE), for n-person nonzero-sum games in normal form for possible use when an RE is not satisfactory. It selects a player's best strategy based on the disappointment that the responses of his opponents would cause him for each of his strategies. The stability enforcement mechanism for a DE is that the strategies of every n-1 players maximize the expected payoff for the remaining players' DE strategies. For any player to change strategies would negate the situation. Finally, Charoensri [27][28] studied a new optimization criterion called compromise criterion. Then Corley [29] applied the compromise criteria to n-person noncooperative games and developed Compromise Equilibrium (CE), which is the prototype of the scalarizations here.

In most one-shot, n-person games players seek a decision that is a pure strategy of a game, not a mixed strategy. One reason is that pure DEs and REs do not always exist. Another reason is that even the concept of mixed strategies has been challenged as problematic [30]. Rubenstein [31] gave two different interpretation of mixed strategies. The first is that the mixed strategies interpretation lacks the knowledge of the players' information, so the random choices are made by less than rigorous unspecified factors; The second interpretation is that the game players stand for a large population of agents. The mixed strategy represents the distribution of pure strategies chosen by each population. However, this does not provide any justification for the case when players are individual agents. Game theorists' attitude about mixed strategies are thus now ambivalent.

The advantage of the Scalar Equilibrium (SE) approach is that each player can choose a pure strategy within the context of the players' individual decision criteria, which are here assumed identical for all players. These decision criteria, amount to notions of rationality, and a collective rationality may be either prescribed or may be due to common beliefs. This situation might occur in arbitration, or in online competition at a website according to specified rules, or in situations where a preliminary agreement is made by the players. The various decision criteria in this research consists of greed, cooperation, no risk, high risk, no aspiration, and aspiration. We therefore present here some new scalarizations for various notions of rationality.

1.2 Definitions and Notation

We follow the standard notation of [5]. Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be an n-person, one-shot games in normal form, where $N = \{1,...,n\}$ is the set of players, S_i is the finite set of pure strategies for player i, and $u_i(s_1,...,s_n)$ is the von Neumann - Morgenstern utility of player i for a pure strategy profile $(s_1,...,s_n) \in \mathsf{X}_{j\in N} S_j$. Write $\mathsf{X}_{j\in N} S_j = S$ and $\mathsf{X}_{j\in N-\{i\}} S_j = S_{-i}$. Player i's set of mixed strategies is denoted by ΔS_i. A mixed-strategy profile $\sigma = (\sigma_1,...,\sigma_n) \in \mathsf{X}_{j\in N} \Delta S_j = \Delta S$ is an n-tuple of individual mixed strategies, where $\sigma_i(s_i)$ is the probability that player i chooses pure strategy $s_i \in S_i$. Write $\mathsf{X}_{j\in N} \Delta S_j = \Delta S$ and $\mathsf{X}_{j\in N-\{i\}} \Delta S_j = \Delta S_{-i}$. When clear from context, s_i will also represent the unique $\sigma_i \in \Delta S_i$ for which $\sigma_i(s_i) = 1$. For any $\tau_i \in \Delta S_i$ the strategy profile $(\sigma_1,...,\sigma_{i-1}, \tau_i, \sigma_{i+1},...,\sigma_n)$ is abbreviated as (σ_{-i}, τ_i). For any $\sigma \in \Delta S$, we can thus write $\sigma = (\sigma_{-i}, \sigma_i), \forall i \in 1,...,n.$ If τ_i is identified with the pure strategy t_i, then (σ_{-i}, t_i) will be used. Similarly we may write (t_{-i}, σ_i). The utility function $u_i : S \to R$ is extended to expected utility over ΔS by writing $u_i(\sigma) = \sum_{(s_1,...,s_n)\in S} \prod_{j\in N} \sigma_j(s_j)\, u_i(s_1,...,s_n).$ In addition, vectors are represented by boldface lowercase Roman letter such as x and y, while x_i denotes the component i^{th} of vector x.

We next define the RE and DE for later reference here.

Definition 1.2.1 [4] For the game Γ, a strategy $\sigma^* \in \Delta S$ is an RE if and only if the payoff $u_i(\sigma^*)$ for player $i = 1,\dots,n$ satisfies

$$u_i(\sigma^*) = \max_{\sigma_i} u_i(\sigma_{-i}, \sigma_i^*).$$

Definition 1.2.2 [25] For the game Γ, a strategy $\sigma^* \in \Delta S$ is a DE if and only if the payoff $u_i(\sigma^*)$, $i = 1,\dots,n$ satisfies

$$u_i(\sigma^*) = \max_{\sigma_1,\dots,\sigma_{i-1},\sigma_{i+1},\dots,\sigma_n} u_i(\sigma_{-i}, \sigma_i^*) \text{ for } \sigma_j \in \Delta(S_j),\ j = 1,\dots n,\ j \neq i, i = 1,\dots,n.$$

Definition 1.2.3 [25] For the game Γ, a strategy $\sigma^* \in \Delta S$ is a Joint Equilibrium (JE) if and only if the strategy σ^* is both an RE and DE.

1.3 Dissertation Contributions

The proposed dissertation will make the following contributions:

1. Give theoretical results for the scalar approach to general n-person games;
2. Develop new scalarizations, based on various notions of rationality, for finding pure strategy solutions;
3. Present computational examples to explain (1) and (2).

In Chapter 2 of this proposed we explain the general approach of scalarizations and the prototype CE. Then in Chapter 3 we present scalarizations for existing solution concept. The Regret, Disappointment, Joint, and Maximin Scalar Equilibria are developed. We proposed a scalarization for aspiration levels of the players in Chapter 4.

In Chapter 5 the Risk Scalar Equilibrium, Greedy Scalar Equilibrium, and Cooperative Scalar Equilibrium is presented. We develop Risk, Greedy, and Cooperative Scalar Equilibrium in Chapter 6. In Chapter 7, we represent Coalition Scalar Equilibrium. In Chapter 8, we give an application of SEs. Finally, in Chapter 9, we discuss future research.

CHAPTER 2

SCALARIZATIONS

In this chapter the scalarization of n-person games is developed. Both a general approach and a prototype scalarization are presented.

2.1 General approach to scalarization

In this research we present the general scalar approach for one-shot n-person games. Let $s = (s_1,...,s_n) \in S$ and $u(s) = (u_1(s),...,u_n(s)),$ where $u_i(s)$ is the von Neumann - Morgenstern (VNM) utility for player $i, i = 1,...,n.$ We transform each vector $u(s), \forall s \in S,$ into a scalar value between 0 and 1 when a prescribed notion of rationality is either agreed upon by the players or enforced by external arbiters. Rationality means here acting consistently to achieve one's specified goal. The decision criteria are tantamount to notions of rationality. A list is shown in Table 2.1.

In Table 2.1, we indicate the spectrum of various criteria categories considered here. First, we have the scale from greed to cooperation. Greed occurs when a player wants a maximum payoff, and we develop a new scalarization that captures the essence of an RE. Similarly, one based on cooperation emulated DE. In the risk spectrum, we develop a scalarization that minimizes risk. The aspiration criterion ranges from a player having no target value to a definite one. For priority criteria, players are given different priority for games. Finally, the sequential criteria are used when the multiple solutions exist and we apply secondary, tertiary criteria, etc,. Simultaneous are compromise between several ones. Scalarizations are developed for range of criteria categories.

Table 2.1 Range of Criteria

Criteria		
Greed	⟷	Cooperation
No risk	⟷	Risk
No aspiration	⟷	Aspiration
No player priority	⟷	Player priority
Simultaneous	⟷	Sequential
Individual	⟷	Coalition

A scalarization considers payoff combinations of all possible strategies, and the result is termed as a Scalar Equilibrium (SE). We next state our scalar procedure.

2.1.1. Scalar Approach

a) For $i = 1,\ldots,n$ create a numerator which is $((u_i(s)) -$ (*a number based on all possible payoffs for the players and related to the notion of rationality*))

b) For $i = 1,\ldots,n$ create a denominator which is ((the maximum value of utility function $(u(s))$) $-$ (*a number based on all possible payoffs for the players and related to the notion of rationality*))

c) Form each ratio, $ratio_i = \dfrac{\text{numerator}_\text{i} \text{ from } 2.1.1(\text{a})}{\text{denominator}_\text{i} \text{ from } 2.1.1(\text{b})}$, for $i = 1,...,n,$ then make a product of all $ratio_i$.

Define a scalar transformation formula ($T(u(s)) = \prod_{i=1}^{n} ratio_i$) consistent with the notion of rationality being considered.

We next formally define Scalar Equilibrium (SE) for the game Γ.

Definition 2.1.1 Let $\Gamma = (N,(S_i)_{i \in N},(u_i)_{i \in N})$ be an one-shot n-person game. Let $s^* = (s_1^*,..., s_n^*)$ be a pure strategy profile of Γ maximizing the scalar transformation T specifying some notion of rationality. Then s^* is an SE of Γ if and only if for all $s \in S, T(u_1(s),..., u_n(s)) \le T(u_1(s^*),..., u_n(s^*))$.

An SE $s^* = (s_1^*,..., s_n^*)$ is termed an equilibrium for the following reason. No player would change his strategy since the scalar measure of rationality cannot be improved or because an arbitrator based his decision on the scalarization value. The prototype SE from Charoensri [27] is next presented.

2.2 Prototype SE: Compromise Equilibria

An optimization criterion for selecting compromise or fair solutions to a mathematical decision problem is shown here to one-shot n-person game theory. Charoensri [26][27] studied the compromise criterion to give reasonable and computationally tractable solutions for multidimensional objective function.

Corley [28] applied the compromise criterion to n-person games. The CE, a global scalarization, is an SE defined as follows. Let $u_i(s)$ be the associated von Neumann - Morgenstern (VNM) utility for player $i, i = 1,...,n$; and let $u(s) = (u_1(s),...,u_n(s))$. $T(u(s))$ assigns a single real number in (0,1] for each payoff in the utility matrix of n-person games. A compromise would be a vector s^* of pure strategies that gives each player a highest scalar value with the following property. All players achieve a payoff with similar ratios between their lowest and highest possible payoffs in the payoff matrix.

In particular, denote $M_i = \max_{s \in S} u_i(s)$ and $m_i = \min_{s \in S} u_i(s)$. Now define $T : u(S) \to R$ by

$$T(u(s)) = \left[(\frac{u_1(s) - m_1 + 1}{M_1 - m_1 + 1}) \times ... \times (\frac{u_n(s) - m_n + 1}{M_n - m_n + 1}) \right], \text{ for all } s \in S.$$

The CE s^* is the solution to the scalar optimization problem $\max_{s \in S} T(u(s))$. Moreover, it is shown in [26] that the set of CEs, defined as Compromise $u(S)$, satisfies Compromise $u(S) \subset$ Vmax $u(S)$, the set of Pareto maxima of $u(S)$.

We now determine the CE for an example game and compare the results to the game's REs and DEs in the following game.

Example 2.2.1 [27] Two-person game with 3 x 3 payoff matrix.

Player *II*

Player *I*	β_1	β_2	β_3
α_1	(3,4)	(2,2)	(2,1)
α_2	(2,3)	(7,1)	(7,4)
α_3	(2,1)	(5,6)	(6,5)

Figure 2.1 Payoff matrix of Example 2.2.1.

Calculate the $M_i = \max_{\substack{\alpha \in \{\alpha_1,\alpha_2,\alpha_3\} \\ \beta \in \{\beta_1,\beta_2,\beta_3\}}} u_i(\alpha,\beta)$ and $m_i = \min_{\substack{\alpha \in \{\alpha_1,\alpha_2,\alpha_3\} \\ \beta \in \{\beta_1,\beta_2,\beta_3\}}} u_i(\alpha,\beta)$ where $u_i(\alpha,\beta)$ is the payoff value for player i^{th}, i =1, 2. So, we obtain $M_1 = 7$, $m_1 = 2$. $M_2 = 6$, $m_2 = 1$.

We calculate the Compromise scalar value using the transformation,

$$T(\alpha_i,\beta_j) = \left[\frac{u_1(\alpha_i,\beta_j) - m_1 + 1}{M_1 - m_1 + 1}\right] \times \left[\frac{u_2(\alpha_i,\beta_j) - m_2 + 1}{M_2 - m_2 + 1}\right] \text{for all } i, j = 1,2,3.$$

Player *II*

Player *I*	β_1	β_2	β_3
α_1	0.2222	0.0555	0.0277
α_2	0.0833	0.1666	0.6666
α_3	0.0277	0.6666	0.6944

Figure 2.2 Compromise matrix of Example 2.2.1.

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	(0,0)	(5,2)	(5,3)
	α_2	(1,1)	(0,3)	(0,0)
	α_3	(1,5)	(2,0)	(1,1)

Figure 2.3 Regret matrix of Example 2.2.1

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	(0,0)	(1,4)	(1,4)
	α_2	(5,1)	(0,5)	(0,1)
	α_3	(4,3)	(1,0)	(0,0)

Figure 2.4 Disappointment matrix of Example 2.2.1

From Figure 2.2-2.4, the results are includes as follows. REs are at $(\alpha_1, \beta_1) = (3,4)$ and $(\alpha_2, \beta_3) = (7,4)$. DEs are at $(\alpha_1, \beta_1) = (3,4)$ and $(\alpha_3, \beta_3) = (6,5)$. The JE is at $(\alpha_1, \beta_1) = (3,4)$. CE is at $(\alpha_3, \beta_3) = (6,5)$. No RE is a CE. The CE is a DE, but one DE is not a CE. The reason that some DEs and REs are best solution is that DEs, like REs, are only local maxima in Definition 1.2.1 and 1.2.2. A JE in Definition 1.2.3 is also local but a smaller subset of ΔS.

CHAPTER 3

SCALAR EQUILIBRIA FOR EXISTING SOLUTION CONCEPTS

In this chapter we present new scalar equilibria based on various notions of rationality for finding pure strategy solutions.

3.1 Regret, Disappointment, and Joint Scalar Equilibria

In this section, it is shown that REs, DEs, and JEs can be determined by the scalarization. These scalarizations are useful for finding pure strategies approximating an RE, DE, or JE when such pure equilibria do not exist.

3.1.1 The concept of Regret, Disappointment, and Joint Matrices

We first define the concepts of Regret, Disappointment, and Joint Matrices [25][26] abbreviated respectively as RMs, DMs, and JMs.

3.1.1.1 The Regret Matrix

Let $R_i(s) = \max_{s_i \in S_i} u_i(s_{-i}, s_i) - u_i(s)$ be the regret function of any payoff for player $i, i = 1,\ldots,n,$ and $R(s) = (R_1(s),\ldots,R_n(s))$. Let $\max_{s_i \in S_i} u_i(s_{-i}, s_i)$ be a maximum value of expected payoff when all player strategies fixed except player $i, i = 1,...,n.$ $R_i(s)$ is a transformation of a player's payoff function for pure strategies to a loss function. In particular, a player's regret function gives the amount he would lose by not choosing his best response to fixed pure strategies of his opponent. For any normal form game, the regret function is completely described by a Regret Matrix (RM) obtained from the payoff matrix for the players.

Result 3.1.1.1.1 [25] A strategy s^* for Γ is a pure RE if and only if this strategy s^* yields $R_i(s^*) = 0$ for every player $i, i = 1,...,n,$ in the RM.

3.1.1.2 The Disappointment Matrix

Let $D_i(s) = \max_{s_{-i} \in S_{-i}} u_i(s_{-i}, s_i) - u_i(s)$ be the disappointment function of any payoff for player $i, i = 1,...,n,$ and $D(s) = (D_1(s),...,D_n(s)).$ Let $\max_{s_{-i} \in S_{-i}} u_i(s_{-i}, s_i)$ be a maximum value of expected payoff when we fixed only the strategies of player $i, i = 1,...,n.$ $D_i(s)$ gives the amount he would lose for a fixed pure strategy of the player if his opponents did not choose the pure strategies yielding his maximum payoff function. For any normal form games the disappointment function is completely described by a Disappointment Matrix (DM) obtained from the payoff matrix for the players.

Result 3.1.1.2.1[25] A strategy s^* for Γ is a pure DE if and only if this strategy s^* yields $D_i(s^*) = 0$ for every player $i, i = 1,...,n,$ in the DM.

3.1.1.3 The Joint Matrix

Let $J_i(s) = R_i(s) + D_i(s)$ be the joint function of any payoff for player $i, i = 1,...,n,$ and $J(s) = (J_1(s),...,J_n(s)).$ As before, the joint function is completely described by a Joint Matrix (JM) obtained from the RM and DM for the players. The next result immediately follows.

Result 3.1.1.3.1 A strategy s^* for Γ is a pure JE if and only if this strategy s^* yields $J_i(s^*) = 0$ for every player $i, i = 1,...,n,$ in the JM.

Proof.

Since a strategy s^* is a pure JE, we obtain that $R_i(s^*) = 0$ and $D_i(s^*) = 0$ for each player $i, i = 1,...,n.$ It follows that $J_i(s^*) = R_i(s^*) + D_i(s^*)$ =0 for every player $i, i = 1,...,n,$ in the JM. ■

3.1.2 Regret, Disappointment, and Joint Scalar Equilibria

The Regret, Disappointment, and Joint Scalar Equilibria are next presented.

3.1.2.1 Regret Scalar Equilibria

The Regret Scalar Equilibrium (RSE) is defined for Γ as follows. Let $R_i(s)$ be the regret function of any payoff for player $i, i = 1,...,n,$ and $R(s) = (R_1(s),...,R_n(s)).$ $T(R(s))$ assigns a single real number in [0,1] for each regret value in the RM of n-person games, where

$$T(R(s)) = \left[(\frac{1}{R_1(s)+1}) \times (\frac{1}{R_2(s)+1}) \times ... \times (\frac{1}{R_n(s)+1}) \right] \text{ for all } s \in S.$$

We seek s^* that solves the scalar optimization problem $\max_{s \in S} T(R(s))$.

The RSE is a pure strategy approximation to an RE, where no player can unilaterally change strategies and improve his payoff. It represent a greedy criteria. \

Theorem 3.1.2.1.1 The game Γ has an RE s^* if and only if $T(R(s^*)) = 1$.

Proof.

It is first shown that if s^* is RE, then $T(R(s^*)) = 1$. Since s^* is RE, there exists a strategy that gives $R_i(s^*) = \max_{s_i \in S_i} u_i(s_{-i}, s_i) - u_i(s^*) = 0$ for every player $i, i = 1,...,n.$ It follows that $T(R(s^*)) = \left[(\frac{1}{R_1(s^*)+1}) \times (\frac{1}{R_2(s^*)+1}) \times ... \times (\frac{1}{R_n(s^*)+1}) \right] = 1.$

It is next shown that if $T(R(s^*)) = 1,$ then an n-person noncooperative game Γ has an RE s^*. Since $T(R(s^*)) = 1,$ there exist $R_i(s^*) = \max_{s_i \in S_i} u_i(s_{-i}, s_i) - u_i(s^*) = 0$ for every player $i, i = 1,...,n.$ By the definition 3.1.1.1, the n-person noncooperative game Γ has a pure RE. It follows that an RSE is a pure RE. ■

It should be noted that a RSE for a game with no pure RE is not necessarily a good solution since the regret value for player i is local regret only, i.e., regret with respect to a fixed strategy s_{-i} for the remaining $n-1$ players.

3.1.2.2 Disappointment Scalar Equilibria

The Disappointment Scalar Equilibrium (DSE) is developed for Γ as follows. Let $D_i(s)$ be the disappointment function of any payoff for player $i, i = 1,...,n,$ and $D(s) = (D_1(s),...,D_n(s)).$ $T(D(s))$ assigns a single real number in [0,1] for each regret value in the DM of n-person games, where

$$T(D(s)) = \left[(\frac{1}{D_1(s)+1}) \times (\frac{1}{D_2(s)+1}) \times ... \times (\frac{1}{D_n(s)+1}) \right], \text{ for all } s \in S.$$

We seek s^* that solves the scalar optimization problem $\max_{s \in S} T(D(s)).$

A DSE is a pure strategy approximation of a DE and represent a cooperative criteria.

Theorem 3.1.2.2.1 The game Γ has an DE s^* if and only if $T(D(s^*)) = 1$.

Proof.

It is first shown that if s^* is DE, then $T(D(s^*)) = 1.$ Since s^* is DE, there exists a strategy that gives $D_i(s^*) = \max_{s_{-i} \in S_{-i}} u_i(s_{-i}, s_i) - u_i(s^*) = 0$ for all player $i, i = 1,...,n.$ It follows that $T(D(s^*)) = \left[(\frac{1}{D_1(s^*)+1}) \times (\frac{1}{D_2(s^*)+1}) \times ... \times (\frac{1}{D_n(s^*)+1}) \right] = 1.$

It is next shown that if $T(D(s^*)) = 1.$ then an n-person noncooperative game Γ has a DE s^*. Since $T(D(s^*)) = 1,$ there exist $D_i(s^*) = \max_{s_{-i} \in S_{-i}} u_i(s_{-i}, s_i) - u_i(s^*) = 0$ for all player $i, i = 1,...,n.$ By the definition 3.1.2.1, the n-person noncooperative game Γ has a pure DE. It follows that a DSE is a pure DE. ■

It should be noted that a DSE for a game with no pure DE is not necessarily a good solution since the disappointment value is local disappointment only, i.e., disappointment for player i with respect to a single fixed strategy s_i.

3.1.2.3 Joint Scalar Equilibria

The Joint Scalar Equilibrium (JSE) is developed for Γ as follows.

Let $J_i(s) = R_i(s) + D_i(s)$ be the joint function of any payoff for player $i, i = 1,\ldots,n,$ and $J(s) = (J_1(s),\ldots,J_n(s)).$ $T(J(s))$ assigns a single real number in [0,1] for each joint value in the Joint Matrix of n-person games, where

$$T(J(s)) = \left[(\frac{1}{J_1(s)+1}) \times (\frac{1}{J_2(s)+1}) \times \ldots \times (\frac{1}{J_n(s)+1}) \right], \text{ for all } s \in S.$$

We seek s^* as the JS that solves the scalar optimization problem $\max_{s \in S} T(J(s)).$

Theorem 3.1.2.3.1 The game Γ has an JE s^* if and only if $T(J(s^*)) = 1$.

Proof. It is first shown that if s^* is JE, then $T(J(s^*)) = 1.$ Since s^* is JE, there exists a strategy that gives $J_i(s^*) = R_i(s^*) + D_i(s^*) = 0.$ for all player $i, i = 1,\ldots,n.$ It follows that $T(J(s^*)) = \left[(\frac{1}{J_1(s^*)+1}) \times (\frac{1}{J_2(s^*)+1}) \times \ldots \times (\frac{1}{J_n(s^*)+1}) \right] = 1.$

It is next shown that if $T(J(s^*)) = 1,$ then a game Γ has an JE s^*. Since $T(J(s^*)) = 1,$ there exist $J_i(s^*) = R_i(s^*) + D_i(s^*) = 0$ for all player $i, i = 1,\ldots,n.$ By the definition 3.1.3.2, the n-person noncooperative game Γ has a pure JE. It follows that a JSE is a pure JE. ■

3.1.3 Examples of Regret, Disappointment, and Joint Scalar Equilibria

We now determine the RSE, DSE, and JSE for the following example games.

Example 3.1.3.1 Prisoner's Dilemma

Player *II*

Player *I*		$\beta_1(Defect)$	$\beta_2(Cooperate)$
	$\alpha_1(Defect)$	(1,1)	(5,0)
	$\alpha_2(Cooperate)$	(0,5)	(3,3)

Figure 3.1 Payoff matrix of Example 3.1.3.1.

Player *II*

Player *I*		$\beta_1(Defect)$	$\beta_2(Cooperate)$
	$\alpha_1(Defect)$	(0,0)	(0,1)
	$\alpha_2(Cooperate)$	(1,0)	(2,2)

Figure 3.2 Regret matrix of Example 3.1.3.1.

Player *II*

Player *I*		$\beta_1(Defect)$	$\beta_2(Cooperate)$
	$\alpha_1(Defect)$	(4,4)	(0,3)
	$\alpha_2(Cooperate)$	(3,0)	(0,0)

Figure 3.3 Disappointment matrix of Example 3.1.3.1.

		Player *II*	
		$\beta_1(Defect)$	$\beta_2(Cooperate)$
Player *I*	$\alpha_1(Defect)$	(4,4)	(0,4)
	$\alpha_2(Cooperate)$	(4,0)	(2,2)

Figure 3.4 Joint matrix of Example 3.1.3.1.

We calculate Regret Scalar values using the transformation

$$T(\alpha_i,\beta_j)=\left[(\frac{1}{R_1(\alpha_i,\beta_j)+1})\times(\frac{1}{R_2(\alpha_i,\beta_j)+1})\right],\text{ for all } i,j=1,2.$$

		Player *II*	
		$\beta_1(Defect)$	$\beta_2(Cooperate)$
Player *I*	$\alpha_1(Defect)$	1.0000	0.5000
	$\alpha_2(Cooperate)$	0.5000	0.1111

Figure 3.5 Regret scalar matrix of Example 3.1.3.1.

We calculate Disappointment Scalar values using the transformation

$$T(\alpha_i,\beta_j)=\left[(\frac{1}{D_1(\alpha_i,\beta_j)+1})\times(\frac{1}{D_2(\alpha_i,\beta_j)+1})\right],\text{ for all } i,j=1,2.$$

		Player *II*	
		$\beta_1(Defect)$	$\beta_2(Cooperate)$
Player *I*	$\alpha_1(Defect)$	0.0400	0.2500
	$\alpha_2(Cooperate)$	0.2500	1.0000

Figure 3.6 Disappointment scalar matrix of Example 3.1.3.1.

We calculate Joint Scalar values using the transformation

$$T(\alpha_i,\beta_j)=\left[(\frac{1}{J_1(\alpha_i,\beta_j)+1})\times(\frac{1}{J_2(\alpha_i,\beta_j)+1})\right],\text{ for all } i,j=1,2.$$

Player *II*

	$\beta_1(Defect)$	$\beta_2(Cooperate)$
Player *I* $\alpha_1(Defect)$	0.0400	0.2000
$\alpha_2(Cooperate)$	0.2000	0.1111

Figure 3.7 Joint scalar matrix of Example 3.1.3.1.

From Figures 3.2 – 3.7, we have the following results. The RSE is (*Defect*, *Defect*) with payoff (1,1) that is the same result as RE. The DSE is (*Cooperate*, *Cooperate*) with payoff (3,3) that is the same result as DE. There are no JE. The JSE are (*Cooperate*, *Defect*) and (*Defect*, *Cooperate*) with payoff (0,5) and (5,0), respectively. Note that the JSE is not a JE, however.

Example 3.1.3.2 Recall Example 2.2.1.

Player *II*

	β_1	β_2	β_3
α_1	(0,0)	(6,8)	(6,7)
Player *I* α_2	(6,2)	(0,8)	(0,1)
α_3	(5,8)	(3,0)	(1,1)

Figure 3.8 Joint matrix of Example 3.1.3.2.

We calculate Regret Scalar values using the transformation

$$T(\alpha_i,\beta_j)=\left[(\frac{1}{R_1(\alpha_i,\beta_j)+1})\times(\frac{1}{R_2(\alpha_i,\beta_j)+1})\right], \text{ for all } i,j=1,2,3.$$

	Player *II*		
	β_1	β_2	β_3
Player *I* α_1	1.0000	0.0555	0.0417
α_2	0.2500	0.2500	1.0000
α_3	0.0833	0.6667	0.2500

Figure 3.9 Regret scalar matrix of Example 3.1.3.2.

We calculate Disappointment Scalar values using the transformation

$$T(\alpha_i,\beta_j)=\left[\left(\frac{1}{D_1(\alpha_i,\beta_j)+1}\right)\times\left(\frac{1}{D_2(\alpha_i,\beta_j)+1}\right)\right], \text{ for all } i,j=1,2,3.$$

	Player *II*		
	β_1	β_2	β_3
Player *I* α_1	1.0000	0.1000	0.1000
α_2	0.0833	0.1667	0.5000
α_3	0.0500	0.5000	1.0000

Fig 3.10 Disappointment scalar matrix of Example 3.1.3.2.

We calculate Joint Scalar values using the transformation

$$T(\alpha_i,\beta_j)=\left[\left(\frac{1}{J_1(\alpha_i,\beta_j)+1}\right)\times\left(\frac{1}{J_2(\alpha_i,\beta_j)+1}\right)\right], \text{ for all } i,j=1,2,3.$$

From Figures 3.9 - 3.11, the results are includes as follows. RSEs are at $(\alpha_1,\beta_1)=(3,4)$ and $(\alpha_2,\beta_3)=(7,4)$ that is the same results as REs. DSEs are at

$(\alpha_1, \beta_1) = (3,4)$ and $(\alpha_3, \beta_3) = (6,5)$ that are the same results as DEs. JSE is at $(\alpha_1, \beta_1) = (3,4)$ that is the same results as JE.

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	1.0000	0.0204	0.0179
	α_2	0.0476	0.1111	0.5000
	α_3	0.0185	0.2500	0.2500

Fig 3.11 Joint scalar matrix of example 3.1.3.2.

Example 3.1.3.3 Two-person Payoff matrix with 3 x 3 with no pure RE and DE

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	(9,2)	(3,6)	(3,5)
	α_2	(1,5)	(5,4)	(4,6)
	α_3	(3,7)	(4,5)	(6,5)

Figure 3.12 Payoff matrix of Example 3.1.3.3.

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	(0,4)	(2,0)	(3,1)
	α_2	(8,1)	(0,2)	(2,0)
	α_3	(6,0)	(1,2)	(0,2)

Figure 3.13 Regret matrix of Example 3.1.3.3.

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	(0,5)	(6,0)	(6,1)
	α_2	(3,2)	(0,2)	(1,0)
	α_3	(3,0)	(2,1)	(0,1)

Figure 3.14 Disappointment matrix of Example 3.1.3.3.

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	(0,9)	(8,0)	(9,2)
	α_2	(11,3)	(0,4)	(3,0)
	α_3	(9,0)	(3,3)	(0,3)

Figure 3.15 Joint matrix of Example 3.1.3.3.

We calculate Regret Scalar values using the transformation

$$T(\alpha_i,\beta_j)=\left[(\frac{1}{R_1(\alpha_i,\beta_j)+1})\times(\frac{1}{R_2(\alpha_i,\beta_j)+1})\right], \text{ for all } i, j=1,2,3.$$

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	0.2000	0.3333	0.1250
	α_2	0.0556	0.3333	0.3333
	α_3	0.1429	0.1667	0.3333

Figure 3.16 Regret scalar matrix of Example 3.1.3.3.

We calculate Disappointment Scalar values using the transformation

$$T(\alpha_i,\beta_j)=\left[(\frac{1}{D_1(\alpha_i,\beta_j)+1})\times(\frac{1}{D_2(\alpha_i,\beta_j)+1})\right], \text{ for all } i,j=1,2,3.$$

Player II

Player I		β_1	β_2	β_3
	α_1	0.1667	0.1428	0.0714
	α_2	0.0833	0.3333	0.5000
	α_3	0.2500	0.1667	0.5000

Fig 3.17 Disappointment scalar matrix of Example 3.1.3.3.

We calculate Joint Scalar values using the transformation

$$T(\alpha_i,\beta_j)=\left[(\frac{1}{J_1(\alpha_i,\beta_j)+1})\times(\frac{1}{J_2(\alpha_i,\beta_j)+1})\right], \text{ for all } i,j=1,2,3.$$

Player II

Player I		β_1	β_2	β_3
	α_1	0.1000	0.1111	0.0333
	α_2	0.0208	0.2000	0.2500
	α_3	0.1000	0.0625	0.2500

Fig 3.18 Joint scalar matrix of example 3.1.3.3.

From Figures 3.13 - 3.18, the results are includes as follows. This payoff matrix has no RE, DE, and JE. However, the RSE, DSE, and JSE still exist. RSEs are at

$(\alpha_1, \beta_2) = (3,6)$, $(\alpha_2, \beta_2) = (5,4)$, $(\alpha_2, \beta_3) = (4,6)$, and $(\alpha_3, \beta_3) = (6,5)$. DSEs are at $(\alpha_2, \beta_3) = (4,6)$ and $(\alpha_3, \beta_3) = (6,5)$. JSEs are at $(\alpha_2, \beta_3) = (4,6)$ and $(\alpha_3, \beta_3) = (6,5)$.

Example 3.1.3.4 Three-person game payoff matrix

	γ_1			γ_2	
	β_1	β_2		β_1	β_2
α_1	(50, 260, 170)	(100, 200, 170)		(50, 260, 170)	(100, 200, 170)
α_2	(50, 160, 260)	(50, 170, 260)		(130, 160, 170)	(130,150,170)

Figure 3.19 Payoff matrix of Example 3.1.3.4.

	γ_1			γ_2	
	β_1	β_2		β_1	β_2
α_1	(0, 0, 0)	(0, 60, 0)		(80, 0, 0)	(30, 60, 0)
α_2	(0, 10, 0)	(50, 0, 0)		(0, 0, 90)	(0, 10, 90)

Figure 3.20 Regret matrix of Example 3.1.3.4.

	γ_1			γ_2	
	β_1	β_2		β_1	β_2
α_1	(50, 0, 90)	(0, 0, 90)		(50, 0, 0)	(0, 0, 0)
α_2	(80, 100, 0)	(80, 30, 0)		(0, 100, 0)	(0, 50, 0)

Figure 3.21 Disappointment matrix of Example 3.1.3.4.

	γ_1			γ_2	
	β_1	β_2		β_1	β_2
α_1	(50, 0, 90)	(0, 60, 90)		(130, 0, 0)	(30, 60, 0)
α_2	(80, 110, 0)	(130, 30, 0)		(0, 100, 90)	(0, 60, 90)

Figure 3.22 Joint matrix of Example 3.1.3.4.

We calculate Regret scalar values using the transformation

$$T(\alpha_i, \beta_j, \gamma_k) = \left[\left(\frac{1}{R_1(\alpha_i, \beta_j, \gamma_k) + 1}\right) \times \left(\frac{1}{R_2(\alpha_i, \beta_j, \gamma_k) + 1}\right) \times \left(\frac{1}{R_3(\alpha_i, \beta_j, \gamma_k) + 1}\right) \right],$$

for all $i, j, k = 1,2.$

	γ_1		γ_2	
	β_1	β_2	β_1	β_2
α_1	1.0000	0.0164	0.0123	0.0005
α_1	0.0909	0.0196	0.0110	0.0010

Figure 3.23 Regret scalar matrix of Example 3.1.3.4.

We calculate Disappointment scalar values using the transformation

$$T(\alpha_i, \beta_j, \gamma_k) = \left[\left(\frac{1}{D_1(\alpha_i, \beta_j, \gamma_k) + 1}\right) \times \left(\frac{1}{D_2(\alpha_i, \beta_j, \gamma_k) + 1}\right) \times \left(\frac{1}{D_3(\alpha_i, \beta_j, \gamma_k) + 1}\right) \right],$$

for all $i, j, k = 1,2.$

	γ_1		γ_2	
	β_1	β_2	β_1	β_2
α_1	0.0002	0.0110	0.0196	1.0000
α_2	0.0001	0.0004	0.0099	0.0196

Figure 3.24 Disappointment scalar matrix of Example 3.1.3.4.

We calculate Joint scalar values using the transformation

$$T(\alpha_i, \beta_j, \gamma_k) = \left[\left(\frac{1}{J_1(\alpha_i, \beta_j, \gamma_k) + 1}\right) \times \left(\frac{1}{J_2(\alpha_i, \beta_j, \gamma_k) + 1}\right) \times \left(\frac{1}{J_3(\alpha_i, \beta_j, \gamma_k) + 1}\right) \right],$$

for all $i, j, k = 1,2.$

	γ_1		γ_2	
	β_1	β_2	β_1	β_2
α_1	0.00022	0.00018	0.00760	0.00053
α_2	0.00010	0.00025	0.00011	0.00018

Figure 3.25 Joint scalar matrix of Example 3.1.3.4.

From Figures 3.20 – 3.25, the results are includes as follows. RSE is at $(\alpha_1, \beta_1, \gamma_1)$ with payoff $(50, 260, 170)$ that is the same result as RE. DSE is at $(\alpha_1, \beta_2, \gamma_1)$ with payoff $(100, 200, 170)$ that is the same result as DE. JSE is at $(\alpha_1, \beta_1, \gamma_2)$ with payoff $(50, 260, 170)$.

3.2 Maximin Scalar Equilibria

A scalarization for selecting maximin solutions to a mathematical decision problem is applied here to one-shot, n-person game theory. The maximin model maximizes the minimum gain of a player regardless of what the other player does. To select a pure strategy, each player chooses an action by determining the worst possible payoff of any of his actions for the various possible actions of his opponent, then selects an action yielding the best of these worst payoffs. If any player does not select his maximin strategy, his payoff could be worse. The Maximin Scalar Equilibrium is next developed.

3.2.1 Maximin Scalar Equilibria for n-person games

The Maximin Scalar Equilibrium (MSE) is developed for Γ as follows. Let $u_i(s)$ be the associated von Neumann - Morgenstern (VNM) utility for player i; and let $u(s) = (u_1(s),...,u_n(s))$. $T(u(s))$ assigns a single real number in [0,1] for each payoff in the utility matrix of n-person games.

Now define $T : f(S) \to R$ by

$$T(u(s)) = \left[\frac{\Delta_1}{M_1 - \hat{m}_1 + 1} \times \frac{\Delta_2}{M_2 - \hat{m}_2 + 1} \times \ldots \times \frac{\Delta_n}{M_n - \hat{m}_n + 1}\right]$$

where $\Delta_i = \begin{cases} u_i(s) - \hat{m}_i + 1, u_i(s) - \hat{m}_i \geq 0 \\ 0 \qquad\qquad , u_i(s) - \hat{m}_i < 0 \end{cases}$ for all $s \in S$.

Denote $M_i = \max_{s \in S} u_i(s)$, $\hat{m}_1 = \max_{s_1} \min_{\substack{s_2 \in S_2 \\ \ldots \\ s_n \in S_n}} f_1(s_{-1}, s_1)$, and $\hat{m}_2 = \max_{s_2} \min_{\substack{s_1 \in S_1 \\ \ldots \\ s_n \in S_n}} f_2(s_{-2}, s_2)$,..., and

$\hat{m}_n = \max_{s_n} \min_{\substack{s_1 \in S_1 \\ s_2 \in S_2 \\ \ldots \\ s_{n-1} \in S_{n-1}}} u_n(s_n, s_{-n})$.

Definition 3.2.1.1 The s^* is an Maximin Scalar Equilibrium (MSE) if and only if the s^* is the solution to the scalar optimization problem $\max_{s \in S} T(u(s))$.

A maximin solution would be a vector s^* of pure strategies that gives the best of the worst possible expected payoff for each player with the following property. All players achieve a payoff with similar ratios between their maximin and highest possible payoffs in the payoff matrix. A maximin solution and MSE are risk averse criteria.

3.2.2 Maximin Scalar Equilibria Examples

We now determine MSEs for some example games

Example 3.2.2.1

Calculate the $M_i = \max_{\substack{\alpha \in \{\alpha_1, \alpha_2, \alpha_3\} \\ \beta \in \{\beta_1, \beta_2, \beta_3\}}} u_i(\alpha, \beta)$, $\hat{m}_1 = \max_{\alpha} \min_{\beta \in \{\beta_1, \beta_2, \beta_3\}} u_1(\alpha, \beta)$, and $\hat{m}_2 = \max_{\beta} \min_{\alpha \in \{\alpha_1, \alpha_2, \alpha_3\}} u_2(\alpha, \beta)$ where $u_i(\alpha, \beta)$ is the payoff value for player *i*, *i* =1, 2.

Hence, we obtain $M_1 = 10,\ M_2 = 8,\ \hat{m}_1 = 4,\ \hat{m}_2 = 6.$

Player II

Player I

	β_1	β_2	β_3
α_1	(10,3)	(4,7)	(3,6)
α_2	(2,6)	(9,5)	(5,6)
α_3	(4,8)	(5,4)	(7,6)

Figure 3.26 Payoff matrix of Example 3.2.2.1.

We calculate Maximin scalar values using the transformation

$$T(\alpha_i, \beta_j) = \left[\frac{(\Delta_1)(\Delta_2)}{(M_1 - \hat{m}_1 + 1)(M_2 - \hat{m}_2 + 1)} \right]$$

where $\Delta_i = \begin{cases} u_i(\alpha_i, \beta_j) - \hat{m}_i + 1, u_i(\alpha_i, \beta_j) - \hat{m}_i \geq 0 \\ 0 \qquad\qquad , u_i(\alpha_i, \beta_j) - \hat{m}_i < 0 \end{cases}$ for all $i, j = 1,2,3.$

Player II

Player I

	β_1	β_2	β_3
α_1	0	0.0952	0
α_2	0	0	0.0952
α_3	0.1429	0	0.1905

Figure 3.27 Maximin scalar matrix of Example 3.2.2.1.

From Figure 3.26, we obtain that MSE is at (α_3, β_3) with payoff $(7,6)$.

Example 3.2.2.2

		Player II		
		β_1	β_2	β_3
Player I	α_1	(3,4)	(2,2)	(2,1)
	α_2	(2,3)	(7,1)	(7,4)
	α_3	(2,1)	(5,6)	(6,5)

Figure 3.28 Payoff matrix of Example 3.2.2.2.

Calculate the $M_i = \max_{\substack{\alpha\in\{\alpha_1,\alpha_2,\alpha_3\} \\ \beta\in\{\beta_1,\beta_2,\beta_3\}}} u_i(\alpha,\beta)$, $\hat{m}_1 = \max_{\alpha} \min_{\beta\in\{\beta_1,\beta_2,\beta_3\}} u_1(\alpha,\beta)$, and $\hat{m}_2 = \max_{\beta} \min_{\alpha\in\{\alpha_1,\alpha_2,\alpha_3\}} u_2(\alpha,\beta)$ where $u_i(\alpha,\beta)$ is the payoff value for player i, i =1, 2.

Thus, we obtain $M_1 = 7$, $M_2 = 6$, $\hat{m}_1 = 2$, $\hat{m}_2 = 1$.

		Player II		
		t_1	t_2	t_3
Player I	α_1	0.2222	0.0556	0.0278
	α_2	0.0833	0.1667	0.6667
	α_3	0.0278	0.6667	0.6944

Figure 3.29 Maximin scalar matrix of example 3.2.2.2.

We calculate Maximin scalar values using the transformation

$$T(\alpha_i,\beta_j) = \left[\frac{(\Delta_1)(\Delta_2)}{(M_1-\hat{m}_1+1)(M_2-\hat{m}_2+1)}\right]$$

where $\Delta_i = \begin{cases} u_i(\alpha_i,\beta_j)-\hat{m}_i+1, & u_i(\alpha_i,\beta_j)-\hat{m}_i \ge 0 \\ 0 & ,u_i(\alpha_i,\beta_j)-\hat{m}_i < 0 \end{cases}$ for all i, j = 1,2,3.

From Figure 3.29, we obtain that MSE is at (α_3, β_3) with payoff $(6,5)$. Note that $(6,5), (5,6)$, and $(7,4)$, are intuitively the best outcomes, with $(6,5)$ giving the best scalar value.

CHAPTER 4

ASPIRATION SCALAR EQUILIBRIA

A new scalar equilibrium for selecting fixed goal solutions to a mathematical decision problem is applied here to one-shot, n-person game theory. The satisficing approach of Stirling in [24], [32], [33], [34] and [35] uses expected epistemic utilities. Since Stirling's method requires some assumptions of the players' behavior for decision problem by using some probability parameters to decide the player preference. On the other hand, our method called Aspiration Scalar Equilibrium here uses only VNM utilities. Therefore our method does not need Stirling's questionable assumptions regarding players' behavior. In addition, our method is simpler. It is obviously a criterion target-value end of the aspiration spectrum.

4.1 The Scalar Aspiration Criterion

The scalar aspiration criterion is developed here as follows. Let $\mathbf{x}$ denote a Euclidean space, R^n. Let $f_i : R^m \to R^1$, $i = 1,...,n$ and consider the objective function $f(\mathbf{x}) = (f_1(\mathbf{x}), ..., f_n(\mathbf{x}))$ over the feasible region $A \subset R^m$. Let $p_i; i = 1, ..., n$ be the real-valued aspiration level for $f_i(\mathbf{x})$ on *A*. Let $p = (p_1, ..., p_n)$ and assume

$A_p = \{\mathbf{x} \in A \mid f_i(\mathbf{x}) \geq p_i, i = 1, ..., n\} \neq \phi.$ In other words, it is desired that $f_i(\mathbf{x}) \geq p_i, i = 1, ..., n$ for $\mathbf{x} \in A.$ Now for all $\mathbf{x} \in A$ define $T : f(A) \to R$ by

$$T(f(\mathbf{x})) = \prod_{i=1}^{n} \Delta_i(\mathbf{x}; p_i)$$

$$\text{where} \quad \Delta_i(\mathbf{x}; p_i) = \begin{cases} 1 & , f_i(\mathbf{x}) - p_i \geq 0 \\ \dfrac{1}{1 + p_i - f_i(\mathbf{x})}, & f_i(\mathbf{x}) - p_i < 0. \end{cases}$$

Consider the following the aspiration order $\leq_{Asp}$ on $f(A)$ for fixed $p_i, i = 1, ..., n.$ For any $f(\mathbf{x}_1), f(\mathbf{x}_2) \in f(A),$ write $f(\mathbf{x}_1) <_{Asp} f(\mathbf{x}_2)$ if and only if $T(f(\mathbf{x}_1)) < T(f(\mathbf{x}_2)).$ Write $f(\mathbf{x}_1) \leq_{Asp} f(\mathbf{x}_2)$ if and only if $f(\mathbf{x}_1) <_{Asp} f(\mathbf{x}_2)$ or $T(f(\mathbf{x}_1)) = T(f(\mathbf{x}_2)).$ The associated aspiration decision problem is denoted by $\mathrm{D}(p_1, ..., p_n) = \underset{\mathbf{x} \in A_p}{\mathrm{Asp}}[f(\mathbf{x})].$ This problem involves finding a vector $\mathbf{x}^* \in A \subset X$ for which there is no vector $\mathbf{x} \in A$ such that $f(\mathbf{x}^*) <_{Asp} f(\mathbf{x}).$ The set of such $\mathbf{x}^*$ is written as $\mathrm{Asp}\, f(A).$

Lemma 4.1.1 For any $f(\mathbf{x}), f(\mathbf{y}) \in f(A), p_i \in R,$ if $f(\mathbf{x}) <_{Pareto} f(\mathbf{y}),$ then $T_{Asp}(f(\mathbf{x})) < T_{Asp}(f(\mathbf{y})).$

Proof. Let $f(\mathbf{x}), f(\mathbf{y}) \in f(A),$ such that $f(\mathbf{x}) <_{Pareto} f(\mathbf{y}).$ We have the following cases.

<u>Case 1</u> $f_i(\mathbf{x}) \leq f_i(\mathbf{y}) < p_i$ for all $i = 1, ..., n$ and $f_j(\mathbf{x}) < f_j(\mathbf{y})$ for some index $j.$ We have $\dfrac{1}{1 + p_i - f_i(\mathbf{x})} \leq \dfrac{1}{1 + p_i - f_i(\mathbf{y})},$ for all $i = 1, ..., n$, and $\dfrac{1}{1 + p_j - f_j(\mathbf{x})} < \dfrac{1}{1 + p_j - b_j}$ for some index j. It follows that

$$T_{Asp}(f(\mathbf{x})) = \prod_{i=1}^{n} \frac{1}{1 + p_i - f_i(\mathbf{x})} < \prod_{i=1}^{n} \frac{1}{1 + p_i - f_i(\mathbf{y})} = T_{Asp}(f(\mathbf{y})).$$

<u>Case 2</u> $f_i(\mathbf{x}) \le p_i \le f_i(\mathbf{y})$ for all $i = 1, \ldots, n$..

<u>Case 2.1</u> $f_j(\mathbf{x}) \le p_j < f_j(\mathbf{y})$ for some index *j*. We have

$\Delta_i(\mathbf{x}; p_i) = 1 \le \Delta_i(\mathbf{y}; p_i) = 1$, for all $i = 1, ..., n$, and $\Delta_j(\mathbf{x}; p_j) = 1 \le \Delta_i(b_j; p_j) = 1$ for some index *j*.

<u>Case 2.2</u> $f_j(\mathbf{x}) < p_j \le f_j(\mathbf{y})$ for some index *j*. We have

$\Delta_i(\mathbf{x}; p_i) = 1 \le \Delta_i(\mathbf{y}; p_i) = 1$, for all $i = 1, ..., n$, and

$\Delta_j(\mathbf{x}; p_j) = \dfrac{1}{1 + p_j - f_j(\mathbf{x})} < \Delta_i(b_j; p_j) = 1$ for some index *j*.

From case 2.1 and 2.2, it follows that

$$T_{Asp}(f(\mathbf{x})) = \prod_{i=1}^{n} \Delta_i(\mathbf{x}; p_i) < \prod_{i=1}^{n} \Delta_i(\mathbf{y}; p_i) = T_{Asp}(f(\mathbf{y})).$$

<u>Case 3</u> $p_i \le f_i(\mathbf{x}) \le f_i(\mathbf{y})$ for all $i = 1, ..., n$.

<u>Case 3.1</u> $p_j \le f_j(\mathbf{x}) < f_j(\mathbf{y})$ for some index *j*.

We have $\Delta_i(\mathbf{x}; p_i) \le \Delta_i(\mathbf{y}; p_i) = 1$, for all $i = 1, ..., n$, and $\Delta_j(\mathbf{x}; p_j) = \Delta_i(b_j; p_j) = 1$ for some index *j*.

<u>Case 3.2</u> $p_j < f_j(\mathbf{x}) \le f_j(\mathbf{y})$ for some index *j*.

We have $\Delta_i(\mathbf{x}; p_i) \le \Delta_i(\mathbf{y}; p_i) = 1$, for all $i = 1, ..., n$, and $\Delta_j(\mathbf{x}; p_j) = \Delta_i(b_j; p_j) = 1$ for some index *j*.

From case 3.1 and 3.2, it follows that

$$T_{Asp}(f(\mathbf{x})) = \prod_{i=1}^{n} \Delta_i(\mathbf{x}; p_i) \le \prod_{i=1}^{n} \Delta_i(\mathbf{y}; p_i) = T_{Asp}(f(\mathbf{y})).$$

From Cases 1, 2, and 3, $T_{Asp}(f(\mathbf{x})) < T_{Asp}(f(\mathbf{y})).$ ■

The set of Pareto maxima of a set $f(A)$, also called the efficient frontier, and is written Par $f(A)$. Two results of the Aspiration Scalarization are now stated.

Theorem 4.1.2 For any $p_i \in A, i = 1, ..., n,$ Asp $f(A) \subset$ Par $f(A)$.

Proof. To obtain a contradiction, suppose that $f(\mathbf{x}) \notin$ Par $f(A)$. Then there exist $f(\mathbf{y}) \in f(A)$ such that $f(\mathbf{x}) <_{Pareto} f(\mathbf{y})$. By Lemma 4.1.1 it follows that $f(\mathbf{x}) <_{Asp} f(\mathbf{y})$ in contradiction to the optimality of $f(\mathbf{x})$. We conclude that $f(\mathbf{x}) \in$ Par $f(A)$ to give the result. ■

Theorem 4.1.3 The preference order $\le_{Asp}$ is a total order on $f(A)$.

Proof. We show that $\le_{Asp}$ is reflexive, transitive, antisymmetric, and comparable.

a. (Reflexive). Since $f(\mathbf{x}) = f(\mathbf{x})$, we have $f(\mathbf{x}) \le_{Asp} f(\mathbf{x})$ for any $f(\mathbf{x}) \in f(A)$.

b. (Transitive). Let $f(\mathbf{x}) \le_{Asp} f(\mathbf{y})$ and $f(\mathbf{y}) \le_{Asp} f(\mathbf{z})$ for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in A$.

<u>Case b-1:</u> $f(\mathbf{x}) \le_{Pareto} f(\mathbf{y})$ and $f(\mathbf{y}) \le_{Pareto} f(\mathbf{z})$.

Since Pareto order is transitive, we have that $f(\mathbf{x})$ comparable to $f(\mathbf{z})$ and in particular $f(\mathbf{x}) \le_{Pareto} f(\mathbf{z})$. Therefore, $f(\mathbf{x}) \le_{Asp} f(\mathbf{z})$.

<u>Case b-2</u>: $f(\mathbf{x}) \le_{Pareto} f(\mathbf{y})$ and $f(\mathbf{y})$ are not Pareto comparable with $f(\mathbf{z})$ with $T_{Asp}(f(\mathbf{y})) < T_{Asp}(f(\mathbf{z}))$.

Case b-2.1: $f(\mathbf{x})$ is Pareto comparable with $f(\mathbf{z})$.

We claim that $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{z})$. Suppose that $f(\mathbf{z}) <_{Pareto} f(\mathbf{x})$. By Lemma 4.1.1, we have $T_{Asp}(f(\mathbf{z})) < T_{Asp}(f(\mathbf{x}))$. Since $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{y})$ and by Lemma 4.1.1, we have $T_{Asp}(f(\mathbf{x})) < T_{Asp}(f(\mathbf{y}))$. Therefore we obtain $T_{Asp}(f(\mathbf{z})) < T_{Asp}(f(\mathbf{y}))$ in contradiction to the assumption that $T_{Asp}(f(\mathbf{y})) < T_{Asp}(f(\mathbf{z}))$. We conclude that $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{z})$. Thus $f(\mathbf{x}) \leq_{Asp} f(\mathbf{z})$.

Case b-2.2: $f(\mathbf{x})$ is not Pareto comparable with $f(\mathbf{z})$.

Since $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{y})$ by Lemma 1, we have $T_{Asp}(f(\mathbf{x})) \leq T_{Asp}(f(\mathbf{y}))$. Combining with $T_{Asp}(f(\mathbf{y})) < T_{Asp}(f(\mathbf{z}))$, we obtain $T_{Asp}(f(\mathbf{x})) \leq T_{Asp}(f(\mathbf{z}))$, i.e., $f(\mathbf{x}) \leq_{Asp} f(\mathbf{z})$.

Case b-2.3: $f(\mathbf{y}) \leq_{Pareto} f(\mathbf{z})$ and $f(\mathbf{x})$ are not comparable with $f(\mathbf{y})$ with $T_{Asp}(f(\mathbf{x})) < T_{Asp}(f(\mathbf{y}))$. The proof is similar to Case b-2.1.

From Cases b-1 and b-2, we obtain $f(\mathbf{x}) \leq_{Asp} f(\mathbf{z})$.

c. (Anti-Symmetric). Let $f(\mathbf{x}) \leq_{Asp} f(\mathbf{y})$ and $f(\mathbf{y}) \leq_{Asp} f(\mathbf{x})$. We must have $f(\mathbf{x}) = f(\mathbf{y})$. To obtain a contradiction, suppose that $f(\mathbf{x}) \neq f(\mathbf{y})$. Immediately we have $f(\mathbf{x}) <_{Asp} f(\mathbf{y})$ and $f(\mathbf{y}) <_{Asp} f(\mathbf{x})$.

Case c-1: $f(\mathbf{x})$ is Pareto comparable to $f(\mathbf{y})$.

Since $f(\mathbf{x}) <_{Asp} f(\mathbf{y})$, we obtain $f(\mathbf{x}) <_{Pareto} f(\mathbf{y})$. Since $f(\mathbf{y}) <_{Asp} f(\mathbf{x})$, we obtain $f(\mathbf{y}) <_{Pareto} f(\mathbf{x})$, which contradicts the previous conclusion.

Case c-2: $f(\mathbf{x})$ is not comparable to $f(\mathbf{y})$.

Since $f(\mathbf{x}) <_{Asp} f(\mathbf{y})$, we have $T_{Asp}(f(\mathbf{x})) < T_{Asp}(f(\mathbf{y}))$. Also, since $f(\mathbf{y}) <_{Asp} f(\mathbf{x})$, we have $T_{Asp}(f(\mathbf{y})) < T_{Asp}(f(\mathbf{x}))$, contradicting the fact that $T_{Asp}(f(\mathbf{x})) < T_{Asp}(f(\mathbf{y}))$. From Cases c-1 and c-2, we conclude that $f(\mathbf{x}) = f(\mathbf{y})$.

d.(Comparable). Let $f(\mathbf{x}), f(\mathbf{y}) \in f(A)$ either $f(\mathbf{x}) \leq_{Asp} f(\mathbf{y})$ or $f(\mathbf{y}) \leq_{Asp} f(\mathbf{x})$.

Case d-1: $f(\mathbf{x})$ is Pareto comparable to $f(\mathbf{y})$.

Give $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{y})$. Suppose that $f(\mathbf{y}) <_{pareto} f(\mathbf{x})$. By Lemma 4.1.1, we have $T_{Asp}(f(\mathbf{y})) < T_{Asp}(f(\mathbf{x}))$. Since $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{y})$ and by Lemma 4.1.1, we have $T_{Asp}(f(\mathbf{x})) \leq T_{Asp}(f(\mathbf{y}))$. Therefore we obtain $T_{Asp}(f(\mathbf{y})) < T_{Asp}(f(\mathbf{x}))$. in contradiction to the assumption that $T_{Asp}(f(\mathbf{x})) \leq T_{Asp}(f(\mathbf{y}))$. We conclude that $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{y})$. Thus $f(\mathbf{x}) \leq_{Asp} f(\mathbf{y})$.

Case d-2: $f(\mathbf{x})$ is not comparable to $f(\mathbf{y})$ with $T_{Asp}(f(\mathbf{x})) \leq T_{Asp}(f(\mathbf{y}))$.

Since $T_{Asp}(f(\mathbf{x})) \leq T_{Asp}(f(\mathbf{y}))$, we obtain $f(\mathbf{x}) \leq_{Asp} f(\mathbf{y})$. From Cases d-1 and d-2, we conclude that any $f(\mathbf{x}), f(\mathbf{y}) \in f(A)$ either $f(\mathbf{x}) \leq_{Asp} f(\mathbf{y})$ or $f(\mathbf{y}) \leq_{Asp} f(\mathbf{x})$.

It follows that $\leq_{Asp}$ is a total order on $f(A)$. ■

An obvious scalar equivalence of the aspiration optimization problem is

$$\begin{array}{ll} \max\limits_{x \in A_p} & T_{Asp}(f(\mathbf{x})) \\ \text{s.t.} & \mathbf{x} \in A. \end{array}$$

The next theorem shows that the payoff matrix determines the set of $(p_1,...,p_n)$ for which there is an $\mathbf{x}^* \in A$ yielding each component's aspiration level. The set of Pareto maxima of a set A, also called the efficient frontier, and is written Par A.

Theorem 4.1.4 Let P = Par$\{(p_1,...,p_n)$: There is an aspiration solution for D$(p_1,...,p_n)\}$ and let Q = Par$\{(f_1(\mathbf{x}),..., f_n(\mathbf{x})): f_i(\mathbf{x}) \in A, i = 1,...,n\}$. Then P = Q.

Proof. It is first shown that P$\subset$Q. Let $(p_1,...,p_n) \subset$ P. By definition there exists an aspiration solution to D$(p_1, ..., p_n)$. If $(p_1, ..., p_n) \notin$ Q, there exists $(f_1(\mathbf{x}^*), ..., f_n(\mathbf{x}^*))$ that dominates $(p_1, ..., p_2)$. But for $p_i^* = f_i(\mathbf{x}^*), i = 1,...,n$, thus $(p_1^*, ..., p_n^*)$ dominates $(p_1, ..., p_n)$ in contradiction to $(p_1, ..., p_n) \notin$ Q. Thus, P$\subset$Q.

It is next shown that Q$\subset$P. Let $(f_1(\hat{\mathbf{x}}), ..., f_2(\hat{\mathbf{x}})) \in$ Q in which case $(\hat{p}_1, ..., \hat{p}_n)$ for $\hat{p}_i = f_i(\hat{\mathbf{x}})$ that is in P. Because these aspiration levels can be achieved, Q$\subset$P. Hence we conclude that P = Q. ■

4.2 Aspiration Scalar Equilibria

We apply the aspiration criterion to n-person games when each player has a target value to be achieved. Let $s \in S$ be a vector of pure strategies for players $1, ..., n$; let $u_i(s)$ be the associated von Neumann - Morgenstern (VNM) utility for player i; and let $u(s) = (u_1(s),...,u_n(s))$. $T(u(s))$ assigns a single real number in (0,1] for each payoff in the utility matrix of n-person games. For all $s \in S$ define $T: u(S) \to R$ by

$$T(u(s)) = \prod_{i=1}^{n} \Delta_i(s; p_i)$$

where $\Delta_i(s; p_i) = \begin{cases} 1 & , u_i(s) - p_i \geq 0 \\ \dfrac{1}{1 + p_i - u_i(s)} & , u_i(s) - p_i < 0. \end{cases}$

For the aspiration transformation, when achieved payoff $u_i(s)$ is equal or greater than aspiration level p_i, of player i. Thus, we assigned 1 value to $ratio_i$ for player i. Otherwise, we assign the term $\dfrac{1}{1 + p_i - u_i(s)}$ to $ratio_i$ for player i. The smaller $u_i(s)$ is less than p_i, the smaller the $ratio_i$ is.

Definition 4.2.1 The s^* is an Aspiration Scalar Equilibrium (ASE) if and only if the s^* is the solution to the scalar optimization problem $\max_{s \in S} T(u(s))$.

We now determine the ASE for some example games in the following games.

Example 4.2.2

	(a) Player 2				(b) Player 2		
	β_1	β_2	β_3		β_1	β_2	β_3
Player 1 α_1	(3,4)	(2,2)	(2,1)	α_1	0.2222	0.0555	0.0277
Player 1 α_2	(2,3)	(7,1)	(7,4)	α_2	0.0833	0.1666	0.6666
Player 1 α_3	(2,1)	(5,6)	(6,5)	α_3	0.0277	0.6666	0.6944

Figure 4.1 (a) Payoff, (b) compromise matrices for the Example 4.2.2.

Note: (8,8) and (7,5) is not possible for both players. No value of 1 is attained for any (α, β). From Figures 4.1 – 4.2, the results are as follows. CE at $(\alpha_3, \beta_3) = (6,5)$. The ASE for $p_1 = 8, p_2 = 8$ is at $(\alpha_2, \beta_3) = (7,4)$. The ASEs for $p_1 = 7, p_2 = 5$ are at

$(\alpha_2, \beta_3) = (7,4)$ and $(\alpha_3, \beta_3) = (6,5)$. The ASE for $p_1 = 6, p_2 = 5$ are at $(\alpha_3, \beta_3) = (6,5)$.

The ASEs for $p_1 = 7, p_2 = 5$ are at $(\alpha_2, \beta_3) = (7,4)$ and $(\alpha_3, \beta_3) = (6,5)$.

Player 1 (rows) / Player 2 (columns)

(a)	β_1	β_2	β_3
α_1	0.0333	0.0204	0.0179
α_2	0.0238	0.0625	0.1000
α_3	0.0179	0.0833	0.0833

(b)	β_1	β_2	β_3
α_1	0.1000	0.0417	0.0333
α_2	0.0556	0.2000	0.5
α_3	0.0333	0.333	0.5

(c)	β_1	β_2	β_3
α_1	0.1250	0.0500	0.4000
α_2	0.0667	0.2000	0.5000
α_3	0.0400	0.5000	1.0000

(d)	β_1	β_2	β_3
α_1	0.1667	0.0625	0.0500
α_2	0.0833	0.200	0.500
α_3	0.0500	1.000	1.000

Figure 4.2 Aspiration matrices of the Example 4.2.2 (a) for $p_1 = 8, p_2 = 8$ (b) $p_1 = 7, p_2 = 5$ (c) $p_1 = 6, p_2 = 5$ (d) $p_1 = 5, p_2 = 5$.

Player I (rows) / Player II (columns)

(a)	β_1	β_2	β_3
α_1	0.1111	0.0500	0.0417
α_2	0.0625	0.1667	0.3333
α_3	0.0417	1.0000	0.5000

(b)	β_1	β_2	β_3
α_1	0.1250	0.0500	0.4000
α_2	0.0667	0.2000	0.5000
α_3	0.0400	0.5000	1.0000

(c)	β_1	β_2	β_3
α_1	0.2000	0.0556	0.0417
α_2	0.0833	0.2500	1.0000
α_3	0.0417	0.3333	0.5000

Figure 4.3 Aspiration matrices of Example 3.3.2.2 (a) for $p_1 = 5, p_2 = 6$ (b) $p_1 = 6, p_2 = 5$ (c) $p_1 = 7, p_2 = 4$.

From Figure 4.1(a) and Theorem 4.1.4, we can get the Pareto maximum aspiration levels {(5,6), (6,5), (7,4)} for which there exist a solution to $D(p_1, ..., p_n)$. Then, we find Aspiration matrices for each aspiration levels. From Figure 4.3, we can get an aspiration solution $(\max T_{Asp}(u(s)) = 1)$ for the Pareto set of payoffs.

Example 4.2.3

	(a)	β_1	β_2	β_3	(b)	β_1	β_2	β_3
		Player 2				Player 2		
Player 1	α_1	(10,3)	(4,7)	(4,6)	α_1	0.1666	0.2777	0.2222
	α_2	(2,6)	(9,5)	(5,7)	α_2	0.0741	0.4444	0.3703
	α_3	(4,8)	(5,6)	(7,5)	α_3	0.3333	0.2963	0.3333

Figure 4.4 (a) Payoff, (b) compromise matrices for Example 4.2.3.

From Figures 4.4 – 4.5 the results show that CE is (α_2, β_2) with payoff of (9,5). The ASE for aspiration level $p_1 = 11, p_2 = 11$ and $p_1 = 10, p_2 = 10$ is at (α_1, β_1) $= (10,3)$, while the ASE for aspiration level $p_1 = 8, p_2 = 8$ is at $(\alpha_1, \beta_1) = (10,3)$. The ASE when $p_1 = 8, p_2 = 5$ is at $(\alpha_2, \beta_2) = (7,4)$.

From Figure 4.4 (a), we can get the Pareto maximum aspiration levels {(4,8), (5,7), (9,5), (10,3)} for which there exist a solution to $D(p_1, ..., p_n)$. Then, we find Aspiration matrices for each aspiration levels. From Figure 3.34, we can get an aspiration solution $(\max T_{Asp}(u(s)) = 1)$ for the Pareto set of payoff.

			Player II				Player II		
	(a)	β_1	β_2	β_3	(b)	β_1	β_2	β_3
	α_1	0.05556	0.0250	0.0208	α_1	0.1250	0.0357	0.0285
Player I	α_2	0.01667	0.0476	0.0333	α_2	0.0222	0.0833	0.0417
	α_3	0.03125	0.0238	0.0333	α_3	0.0476	0.0333	0.0417
	(c)	β_1	β_2	β_3	(d)	β_1	β_2	β_3
	α_1	0.1667	0.1000	0.0667	α_1	0.3333	0.2000	0.2000
Player I	α_2	0.0476	0.2500	0.1250	α_2	0.1429	1.0000	0.2500
	α_3	0.2000	0.0833	0.1250	α_3	0.2000	0.2500	0.5000

Figure 4.5 Aspiration matrices of Example 4.2.3 (a) for $p_1 = 11, p_2 = 11$ (b) $p_1 = 10, p_2 = 10$ (c) $p_1 = 8, p_2 = 8$ (d) $p_1 = 8, p_2 = 5$.

			Player II				Player II		
	(a)	β_1	β_2	β_3	(b)	β_1	β_2	β_3
	α_1	0.1667	0.5000	0.3333	α_1	0.2000	0.5000	0.2500
Player I	α_2	0.1111	0.2500	0.5000	α_2	0.1250	0.3333	1.000
	α_3	1.0000	0.3333	0.2500	α_3	0.5000	0.5000	0.333
	(c)	β_1	β_2	β_3	(d)	β_1	β_2	β_3
	α_1	0.3333	0.1667	0.1667	α_1	1.0000	0.1429	0.1429
Player I	α_2	0.1250	1.0000	0.2000	α_2	0.1111	0.5000	0.1667
	α_3	0.1667	0.2000	0.2222	α_3	0.1429	0.1667	0.2500

Figure 4.6 Aspiration matrices of Example 4.2.3 (a) for $p_1 = 4, p_2 = 8$ (b) $p_1 = 5, p_2 = 7$ (c) $p_1 = 9, p_2 = 5$ (d) $p_1 = 10, p_2 = 3$.

We next show the comparison our method with Stirling et al.'s method [31].

4.3 Comparison of ASE with Stirling's method

Example 4.3.1 Prisoner's Dilemma Payoff Matrix [34]

Player *II*

Player *I*		*Defect (D)*	*Cooperate (C)*
	Defect (D)	(2,2)	(4,1)
	Cooperate (C)	(1,4)	(3,3)

Figure 4.7 Payoff matrix of Prisoner's Dilemma.

We now demonstrate the complexity of Stirling's approach [24] for comparison. In this method, we need to create the interdependence mass function from selectability and rejectability mass function for each joint option. In addition, we have to know the characteristics of game for creating assumption to define joint rejectability mass function. For Prisoner's dilemma example, two attitudes in players' minds may affect their decision. First, dissociation is that the agents go their separate ways without regarding cooperation. Second, vulnerability is that the agents expose themselves to individual risk in the hope of improving the joint outcome. Let $\alpha \in [0,\ 1]$ be a dissociation index and a measure of the joint value the players place on rejecting the joint option (C, C). Let $\beta \in [0,\ 1]$ be a vulnerability index and a measure of the joint value the players place on rejecting the joint option (D, D). With these assumptions and constraints on α and β, we define the joint rejectability mass function: $p_{R_1R_2}(C,C) = \alpha$, $p_{R_1R_2}(C,D) = \dfrac{1-\alpha-\beta}{2}$, $p_{R_1R_2}(C,D) = \dfrac{1-\alpha-\beta}{2}$, $p_{R_1R_2}(D,D) = \beta$. Then, we find the conditional selectability on joint rejectability, $p_{S_1S_2|R_1R_2}(v_1, v_2 \mid w_1 w_2)$ for all (v_1, v_2) and (w_1, w_2) in action space $\{(C,C),(C,D), \quad (D,C),(D,D)\}$, from which the interdependence mass function may be obtained by the product rule: $p_{S_1S_2R_1R_2}(v_1,v_2,w_1,w_2) = p_{S_1S_2|R_1R_2}(v_1,v_2 \mid w_1,w_2) \cdot p_{R_1R_2}(w_1,w_2)$. Note that no aspiration levels are used in Stirling's method. From [24][34], we get conditional selectability function in Table 4.1.

Table 4.1 Conditional selectability for satisficing Prisoner's Dilemma

$p_{S_1S_2\|R_1R_2}(v_1,v_2 \mid w_1,w_2)$				
	(w_1,w_2)			
(v_1,v_2)	(C,C)	(C,D)	(D,C)	(D,D)
(C,C)	0	0	0	1
(C,D)	0	0	0	0
(D,C)	0	0	0	0
(D,D)	1	1	1	0

We obtain the joint selectability by substituting the conditional selectability interdependence function given by Table 4.1 and joint rejectability mass function into $p_{S_1S_2R_1R_2}(v_1,v_2,w_1,w_2) = p_{S_1S_2|R_1R_2}(v_1,v_2 \mid w_1,w_2) \cdot p_{R_1R_2}(w_1,w_2)$. The joint selectability function is following: $p_{S_1S_2}(C,C) = \beta$, $p_{S_1S_2}(C,D) = 0$, $p_{S_1S_2}(C,D) = 0$, $p_{S_1S_2}(D,D) = 1-\beta$. After finding joint rejectability mass function, selectability mass function and conditional credibility for the satisficing Prisoner's Dilemma, we compare the selectability function with rejectability following $p_{S_1S_2}(w_1,w_2) \geq b \cdot p_{R_1R_2}(w_1,w_2)$ of all decision pairs, w_1, w_2 in action space. We can get the satisficing solution shown as satisficing equilibrium set parameterized by α and β, for the special case b=1, is

$$\xi_b = \begin{cases} \{(C,C)\} & \text{for } \beta \geq \frac{1}{2} \\ \{(D,D)\} & \text{for } \beta \leq \alpha \\ \{(C,C),(D,D)\} & \text{for } \alpha < \beta < \frac{1}{2}. \end{cases}$$

In comparison, an ASE method yields {(C,C)}, {(D,C)}, and {(C,D)}for the aspiration levels $p_1 = 3, p_2 = 3,$ $p_1 = 4, p_2 = 1$ and $p_1 = 1, p_2 = 4$, respectively.

	Player II			Player II	
(a)	*D*	*C*	(b)	*D*	*C*
Player *I* *D*	0.2500	0.3333	*D*	0.3333	1.0000
C	0.3333	1.0000	*C*	0.2500	0.5000
(c)	*D*	*C*			
D	0.3333	0.2500			
C	1.0000	0.5000			

Figure 4.8 Aspiration matrices of Example 5.1.2.1 (a) for $p_1 = 3, p_2 = 3$ (b) $p_1 = 4, p_2 = 1$ (c) $p_1 = 1, p_2 = 4.$

From Figure 4.8, we obtain an aspiration solution $(\max T_{Asp}(u(s)) = 1)$ for each aspiration level. Although (D, C) and (C, D) strategies for $p_1 = 4, p_2 = 1$ and $p_1 = 1, p_2 = 4$, respectively, can make the aspiration solution, the expected payoffs for the strategies are undesired and unfair for players which one of them get the worst payoff. In addition, (C, C) for $p_1 = 3, p_2 = 3$ can also create aspiration solution which is the fairest strategy for the game. Moreover, if we reduced the aspiration levels from $p_1 = 3, p_2 = 3$ to $p_1 = 2, p_2 = 2$, we will also get strategy (D, D) that get desired level.

Example 4.3.2

	Player *II*		
	t_1	t_2	t_3
Player *I* s_1	(3,4)	(2,2)	(2,1)
s_2	(2,3)	(7,1)	(7,4)
s_3	(2,1)	(5,6)	(6,5)

Figure 4.9 Payoff matrix of Example 4.3.2.

From Stirling's method [31], we need to create the interdependence mass function from selectability and rejectability mass function for each joint option. In addition, we have to know the characteristics of game for creating assumption to define joint rejectability mass function. For this example, let $\alpha \in [0, 1]$ be an index measure of the joint value the players place on rejecting the joint option (s_1, t_1). Let $\beta \in [0, 1]$ be an index measure of the joint value the players place on rejecting the joint option (s_2, t_2). Let $\gamma \in [0, 1]$ be an index measure of the joint value the players place on rejecting the joint option (s_3, t_3). With these assumption, we define the joint rejectability mass function: $p_{R_1R_2}(s_1,t_1) = \alpha$, $p_{R_1R_2}(s_1,t_2) = \frac{1-\alpha-\beta}{2}$, $p_{R_1R_2}(s_1,t_3) = \frac{1-\alpha-\gamma}{2}$,

$p_{R_1R_2}(s_1,t_2) = \frac{1-\alpha-\beta}{2}$, $p_{R_1R_2}(s_1,t_2) = \beta$, $p_{R_1R_2}(s_2,t_3) = \frac{1-\alpha-\gamma}{2}$,

$p_{R_1R_2}(s_3,t_1) = \frac{1-\alpha-\gamma}{2}$, $p_{R_1R_2}(s_3,t_2) = \frac{1-\gamma-\beta}{2}$, $p_{R_1R_2}(s_3,t_3) = \gamma$.

Then we find the conditional selectability on joint rejectability, $p_{S_1S_2|R_1R_2}(v_1,v_2 \mid w_1w_2)$ for all (v_1,v_2) and (w_1,w_2) in action space $\{(s_1,t_1),(s_1,t_2),(s_2,t_1),(s_2,t_2),(s_2,t_3),$

$(s_3,t_1),(s_3,t_2),(s_3,t_3)\}$ from which the interdependence mass function may be obtained by the product rule: $p_{S_1S_2R_1R_2}(v_1,v_2,w_1,w_2) = p_{S_1S_2|R_1R_2}(v_1,v_2 \mid w_1,w_2)\cdot p_{R_1R_2}(w_1,w_2)$.

We obtain the joint selectability by substituting the conditional selectability interdependence function given by Table 4.2 and joint rejectability mass function into $p_{S_1S_2R_1R_2}(v_1,v_2,w_1,w_2) = p_{S_1S_2|R_1R_2}(v_1,v_2 \mid w_1,w_2)\cdot p_{R_1R_2}(w_1,w_2)$. The joint selectability function is following: $p_{S_1S_2}(s_1,t_1)=\beta+\gamma$, $p_{S_1S_2}(s_1,t_2)=0$, $p_{S_1S_2}(s_1,t_3)=0$, $p_{S_1S_2}(s_1,t_2)=0$, $p_{S_1S_2}(s_1,t_2)=1-\beta+\gamma$, $p_{S_1S_2}(s_2,t_3)=0$, $p_{S_1S_2}(s_3,t_1)=0$, $p_{S_1S_2}(s_3,t_2)=0$, $p_{S_1S_2}(s_3,t_3)=2-2\gamma$.

Table 4.2 Conditional selectability for Example 4.3.2

$p_{S_1S_2|R_1R_2}(v_1,v_2 \mid w_1,w_2)$									
	(w_1,w_2)								
(v_1,v_2)	(s_1,t_1)	(s_1,t_2)	(s_1,t_3)	(s_2,t_1)	(s_2,t_2)	(s_2,t_3)	(s_3,t_1)	(s_3,t_2)	(s_3,t_3)
(s_1,t_1)	0	0	0	0	1	0	0	0	1
(s_1,t_2)	0	0	0	0	0	0	0	0	0
(s_1,t_3)	0	0	0	0	0	0	0	0	0
(s_2,t_1)	1	1	1	0	0	0	0	0	0
(s_2,t_2)	0	0	0	0	0	0	0	0	0
(s_2,t_3)	1	1	0	1	0	0	0	0	1
(s_3,t_1)	0	0	0	0	0	0	0	0	0
(s_3,t_2)	0	0	0	0	0	0	0	0	0
(s_3,t_3)	1	0	1	0	1	1	1	1	0

After finding joint rejectability mass function, selectability mass function and conditional credibility for Example 4.3.2, we compare the selectability function with rejectability following $p_{S_1S_2}(w_1, w_2) \geq b \cdot p_{R_1R_2}(w_1, w_2)$ of all decision pairs, w_1, w_2 in action space. We can get the satisficing solution shown as satisficing equilibrium set parameterized by α, β and γ, for the special case b=1, is

$$\xi_{\mathrm{b}} = \begin{cases} \{(s_1,t_1)\} & \text{for } \alpha \leq \frac{1}{2} \\ \{(s_2,t_2)\} & \text{for } \alpha \geq \beta \\ \{(s_3,t_3)\} & \text{for } \beta + \alpha \geq \frac{1}{3} \\ \{(s_1,t_1),(s_2,t_2)\} & \text{for } \alpha > \frac{1}{2}, \alpha < \beta \\ \{(s_1,t_1),(s_3,t_3)\} & \text{for } \alpha > \frac{1}{2}, \beta + \alpha \geq \frac{1}{3} \\ \{(s_2,t_2),(s_3,t_3)\} & \text{for } \alpha < \beta, \beta + \alpha < \frac{1}{3}. \end{cases}$$

Player II (columns); Player I (rows)

(a)	t_1	t_2	t_3
s_1	0.1111	0.0500	0.0417
s_2	0.0625	0.1667	0.3333
s_3	0.0417	1.0000	0.5000

(b)	t_1	t_2	t_3
s_1	0.1250	0.0500	0.4000
s_2	0.0667	0.2000	0.5000
s_3	0.0400	0.5000	1.0000

(c)	t_1	t_2	t_3
s_1	0.2000	0.0556	0.0417
s_2	0.0833	0.2500	1.0000
s_3	0.0417	0.3333	0.5000

Figure 4.10 Aspiration matrices of Example 4.3.2 (a) for $p_1 = 5, p_2 = 6$ (b) $p_1 = 6, p_2 = 5$ (c) $p_1 = 7, p_2 = 4$.

In comparison, an ASE method yields $\{(s_3, t_2)\}$, $\{(s_3, t_3)\}$, and $\{(s_2, t_3)\}$ for the aspiration levels $p_1 = 5, p_2 = 6,$ $p_1 = 6, p_2 = 5,$ and $p_1 = 7, p_2 = 4,$ respectively. From Figure 4.10 above, we can get $\max T_{Asp}(u(s)) = 1$ for these values.

From these examples above, we can see that the Stirling's method required numerous steps to get the solutions and is difficult to interpret. On the other hand, an ASE is easily computable and interpreted.

CHAPTER 5

RISK, GREEDY, AND COOPERATIVE SCALAR EQUILIBRIA

5.1 Risk Scalar Equilibria

A new optimization criterion for selecting risk avoiding solutions to a mathematical decision problem is applied here to n-person game theory. We formulate a new scalarization using risk dominance [17], [37]. It seeks a least risky solution in the sense of minimizing each player's lost utility when the other players change strategies. The basic idea is that no player wants to decrease his utility much if other player changes strategies. This scalarization can be used as a second criterion for choosing a pure strategy when a game has more than one solution.

5.1.1 Risk Scalar Equilibria for n-person games

The Risk Scalar Equilibrium (RISE) is developed for Γ as follows. Let $u_i(s)$ be the associated von Neumann - Morgenstern (VNM) utility for player i, and let $u(s) = (u_1(s),...,u_n(s))$. $T(u(s))$ assigns a single real number in (0,1] for each payoff in the utility matrix of n-person games. Let

$$T(u(s)) = \left[\frac{1}{u_1(s) - v_1 + 1} \times \frac{1}{u_2(s) - v_2 + 1} \times ... \times \frac{1}{u_n(s) - v_n + 1} \right] \text{ for all } s \in S,$$

where $v_1 = v_1(s_1) = \min_{s'_{-1} \in S_{-1}} u_1(s'_{-1}, s_1),\ v_2 = v_2(s_2) = \min_{s'_{-2} \in S_{-2}} u_2(s'_{-2}, s_2), ..., v_n = v_n(s_n) =$

$\min_{s'_{-n} \in S_{-n}} u_1(s'_{-n}, s_n)$. In other words $v_i(s_i)$ is a mimimum value of expected payoff when the strategies of player $i, i = 1,...,n,$ are fixed.

Definition 5.1.1.1 The s^* is an Risk Scalar Equilibrium (RISE) if and only if the s^* is the solution to scalar optimization problem $\max_{s \in S} T(u(s))$.

5.1.2 Risk Scalar Equilibria Examples

We now determine RISE in the following games with RISE being the sole criteria to illustrate its purpose.

Example 5.1.2.1 Stag Hunt game

Player II

Player I		$\beta_1(Hunt)$	$\beta_2(Gather)$
	$\alpha_1(Hunt)$	(5,5)	(0,4)
	$\alpha_2(Gather)$	(4,0)	(2,2)

Figure 5.1 Payoff matrix of Example 5.1.2.1.

Here $v_1 = v_1(s_1) = v_1(\alpha) = \min_{\beta \in \{\beta_1, \beta_2\}} u_1(\alpha_i, \beta), i = 1,2.$ $v_2 = v_2(s_2) = v_2(\beta) =$

$\min_{\beta \in \{\beta_1, \beta_2\}} u_2(\alpha, \beta_j), j = 1,2,$ where $u_1(\alpha, \beta)$ and $u_2(\alpha, \beta)$ is the payoff value for player 1 and player 2. Then, we obtain $v_1 = 0; i = 1,\ v_1 = 2; i = 2,\ v_2 = 0; j = 1,\ v_2 = 2; j = 2.$

Risk scalar values are calculated using the transformation

$$T(\alpha_i, \beta_j) = \left[\frac{1}{u_1(\alpha_i, \beta_j) - v_1 + 1} \times \frac{1}{u_2(\alpha_i, \beta_j) - v_2 + 1} \right], \text{ for all } i, j = 1,2.$$

	Player *II*	
	$\beta_1(Hunt)$	$\beta_2(Gather)$
Player *I* $\alpha_1(Hunt)$	0.0277	0.3333
$\alpha_2(Gather)$	0.3333	1.0000

Figure 5.2 Risk scalar matrix of Example 5.1.2.1.

From Figures 5.1 and 5.2, we obtain that RISE are at (α_2, β_2) with payoff $(2,2)$. The payoff at strategies (α_1, β_1) is better that payoff at strategies (α_2, β_2). However, RISE is at (α_2, β_2) instead of (α_1, β_1) because the risk of (α_2, β_2) is less than the risk of (α_1, β_1). It should be noted that RISE obtains the risk dominance solution proposed in [17].

Example 5.1.2.2

Player *I* \ Player *II*	β_1	β_2	β_3
α_1	(3,4)	(2,2)	(2,1)
α_2	(2,3)	(7,1)	(7,4)
α_3	(2,1)	(5,6)	(6,5)

Figure 5.3 Payoff matrix of Example 5.1.2.2.

Calculate $v_1 = v_1(s_1) = v_1(\alpha) = \min_{\beta \in \{\beta_1, \beta_2, \beta_3\}} u_1(\alpha_i, \beta), i = 1,2,3,$ $v_2 = v_2(s_2) = v_2(\beta)$ $= \min_{\alpha \in \{\alpha_1, \alpha_2, \alpha_3\}} u_2(\alpha, \beta_j), j = 1,2,3,$ where $u_1(\alpha, \beta)$ and $u_2(\alpha, \beta)$ is the payoff value for player 1 and player 2. Then, we obtain $v_1 = 2; i = 1,$ $v_1 = 2; i = 2,$ $v_1 = 2; i = 3,$ $v_2 = 1; j = 1,$ $v_2 = 1; j = 2,$ $v_2 = 1; j = 3.$

		Player *II*	
	β_1	β_2	β_3
Player *I* α_1	0.1250	0.5000	1.0000
α_2	0.3333	0.1667	0.0417
α_3	1.0000	0.0333	0.0400

Figure 5.4 Risk scalar matrix of Example 5.1.2.2.

Risk scalar values are calculated using the transformation

$$T(\alpha_i, \beta_j) = \frac{1}{u_1(\alpha_i, \beta_j) - v_1 + 1} \times \frac{1}{u_2(\alpha_i, \beta_j) - v_2 + 1}, \text{ for all } i, j = 1,2,3.$$

From Figure 5.4, we obtain that RISEs are at (α_3, β_1) and (α_1, β_3) with the same payoff (2,1).

5.2 Greedy Scalar Equilibria

A scalarization for selecting greedy solutions to a decision problem is applied here to one-shot n-person games. The new scalar equilibrium finds a pure strategy when each player desires the maximum payoff.

5.2.1 Greedy Scalar Equilibria for n-person games

The Greedy Scalar Equilibrium (GSE) is developed for Γ as follows. Let $u_i(s)$ be the associated von Neumann - Morgenstern (VNM) utility for player $i, i = 1,...,n$; and let $u(s) = (u_1(s),...,u_n(s))$. $T(u(s))$ assigns a single real number in (0,1] for each payoff in the utility matrix of *n*-person games.

In particular denote $M_i = \max_{s \in S} u_i(s)$. Now define $T : u(S) \to R$ by

$$T(u(s)) = \left[(\frac{1}{M_1 - u_1(s) + 1}) \times (\frac{1}{M_2 - u_2(s) + 1}) \times ... \times (\frac{1}{M_n - u_n(s) + 1}) \right], \text{ for all } s \in S.$$

In the greedy transformation, each player i seeks his maximum payoff, so $ratio_i$ is the term $\frac{1}{M_i - u_i(s) + 1}$ to make it close to value 1 when payoff $u_i(s)$ is close to M_i. We next establish that a GSE is a Pareto solution.

Lemma 5.2.1.1 For any $u(s'), u(s'') \in R$, if $u(s') <_{Pareto} u(s'')$ then $T\ (u(s')) < T\ (u(s'')))$.

Proof. Let $u(s'), u(s'') \in R$, such that $u(s') <_{Pareto} u(s'')$. Then, $u_i(s') \le u_i(s'')$ for all $i = 1,...,n$ and $u_j(s') < u_j(s'')$ for some index j. We have

$$0 \le \frac{1}{M_i - u_i(s') + 1} \le \frac{1}{M_i - u_i(s'') + 1}, \text{ for all } i = 1,...,n, \text{ and}$$

$$0 \le \frac{1}{M_j - u_j(s') + 1} < \frac{1}{M_j - b_j + 1} \text{ for some index } j. \text{ It follows that}$$

$$T(u(s')) = \prod_{i=1}^{n} \frac{1}{M_i - u_i(s') + 1} < \prod_{i=1}^{n} \frac{1}{M_i - u_i(s'') + 1} = T(u(s'')). \blacksquare$$

Now let Gr $u(s)$ be the set of GSE's for Γ. The set of Pareto maxima of a set $u(S)$, also called the efficient frontier, and is written Par $u(S)$.

Theorem 5.2.1.2 For any $u(s) \in R$, $\text{Gr}\, u(S) \subset \text{Par}\, u(S)$.

Proof. To obtain a contradiction, suppose that $u(s') \notin \text{Par}\, u(S)$. Then there exist $u(s'') \in R$ such that $u(s') <_{Pareto} u(s'')$. By Lemma 5.2.1.1 it follows that $T(u(s')) < T(u(s''))$ in contradiction to the optimality of $u(s')$. We conclude that $u(s') \in \text{Par}\, u(S)$ to give the result. ■

Definition 5.2.1.3 The s^* is a Greedy Scalar Equilibrium (GSE) if and only if the s^* is the solution to scalar optimization problem $\max_{s \in S} T(u(s))$.

5.2.2 Greedy Scalar Equilibria Examples

We now determine GSEs for some example games.

Example 5.2.2.1 Two-person Payoff matrix with 3 x 3

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	(3,4)	(2,2)	(2,1)
	α_2	(2,3)	(7,1)	(7,4)
	α_3	(2,1)	(5,6)	(6,5)

Figure 5.5 Payoff matrix of Example 5.2.2.1.

Calculate the $M_i = \max_{\substack{\alpha \in \{\alpha_1, \alpha_2, \alpha_3\} \\ \beta \in \{\beta_1, \beta_2, \beta_3\}}} u_i(\alpha, \beta)$ where $u_i(\alpha, \beta)$ is the payoff value for player i^{th}, i =1, 2. So, we obtain $M_1 = 7$, $M_2 = 6$. We calculate greedy scalar values using the transformation

$$T(\alpha_i, \beta_j) = \left[\frac{1}{M_1 - u_1(\alpha_i, \beta_j) + 1}\right] \times \left[\frac{1}{M_2 - u_2(\alpha_i, \beta_j) + 1}\right] \text{ for all } i, j = 1,2,3.$$

Player II

Player I		β_1	β_2	β_3
	α_1	0.0667	0.0333	0.0278
	α_2	0.0417	0.1667	0.3333
	α_3	0.0278	0.3333	0.2500

Figure 5.6 Greedy scalar matrix of Example 5.2.2.1.

From Figures 5.5-5.6, the results are includes as follows. GSEs are at $(\alpha_2, \beta_3) = (7, 4)$ and $(\alpha_3, \beta_2) = (5, 6)$. One GSE is an RE, but the other GSE is not. Thus some REs are not greedy. In particular, the GSE $(\alpha_2, \beta_3) = (7, 4)$ dominates payoff $(3, 4)$, the other pure RE.

Example 5.2.2.2 Prisoner's Dilemma

Player II

Player I		$\beta_1(Defect)$	$\beta_2(Cooperate)$
	$\alpha_1(Defect)$	(1,1)	(5,0)
	$\alpha_2(Cooperate)$	(0,5)	(4,4)

Figure 5.7 Payoff matrix of Example 5.2.2.2.

Calculate the $M_i = \max_{\substack{\alpha \in \{\alpha_1, \alpha_2\} \\ \beta \in \{\beta_1, \beta_2\}}} u_i(\alpha, \beta)$ where $u_i(\alpha, \beta)$ is the payoff value for player i, i =1, 2. So, we obtain $M_1 = 5$, $M_2 = 5$. We calculate greedy scalar values using the transformation

$$T(\alpha_i, \beta_j) = \left[\frac{1}{M_1 - u_1(\alpha_i, \beta_j) + 1}\right] \times \left[\frac{1}{M_2 - u_2(\alpha_i, \beta_j) + 1}\right] \text{ for all } i, j = 1,2.$$

Player *II*

Player *I*		$\beta_1(Defect)$	$\beta_2(Cooperate)$
	$\alpha_1(Defect)$	0.0400	0.1667
	$\alpha_2(Cooperate)$	0.1667	0.2500

Figure 5.8 Greedy scalar matrix of Example 5.2.2.2.

Player *II*

Player *I*		$\beta_1(Defect)$	$\beta_2(Cooperate)$
	$\alpha_1(Defect)$	1.0000	0.2000
	$\alpha_2(Cooperate)$	0.2000	0.0400

Figure 5.9 Risk scalar matrix of Example 5.2.2.2.

From Figure 5.8, the results are includes as follows. GSEs are at $(\alpha_2, \beta_2) = (4, 4)$. Note that the GSE is not an RE even though it is based on greed. The pure RE is (1,1). In other words, greed can have more than one interpretation. On the other hand, the RISE value is 0.400 for (4,4) that is less than the value 1.0000 for (1,1) in Figure 5.9. Thus, in the GSE the lower risk of (1,1) is the determining factor.

5.3 Cooperative Scalar Equilibria

A Cooperative Scalar Equilibrium related to the GSE is now developed by using scalarization concepts.

5.3.1 Cooperative Scalar Equilibria for n-person game

The Cooperative Scalar Equilibrium (CSE) is developed for Γ as follows. Let $u_i(s)$ be the associated von Neumann - Morgenstern (VNM) utility for player i; and let $u(s)=(u_1(s),...,u_n(s))$. Let $\min_{s_{-i}\in S_{-i}} u_i(s_{-i},s_i)$ be a minimum value of expected payoff when we fixed only the strategies of player $i, i=1,...,n.$ Denote $M_i=\max_{s\in S} u_i(s)$ and $m_i=\min_{s\in S} u_i(s).$ Then we define the transformation

$$T(u(s))=\left[\frac{u_1(s)-v_1+1}{M_1-m_1+1}\times\frac{u_2(s)-v_2+1}{M_2-m_2+1}\times...\times\frac{u_n(s)-v_n+1}{M_n-m_n+1}\right] \text{for all } s\in S,$$

where $v_1=v_1(s_1)=\min_{s_{-1}\in S_{-1}} u_1(s_{-1},s_1),\ v_2=v_2(s_2)=\min_{s_{-2}\in S_{-2}} u_2(s_{-2},s_2),...,\ v_n=v_n(s_n)=$

$\min_{s_{-n}\in S_{-n}} u_1(s_{-n},s_n).$ $T(u(s))$ assigns a single real number in (0,1] for each payoff in the utility matrix of *n*-person games.

From cooperative transformation, every n-1 players chooses strategies to maximize the expected payoff for the remaining players' strategy, so we make the interval between $u_i(s)$ and v_i as the numerator term $u_i(s)-v_i+1.$ We make the interval between M_i and m_i as the denominator term M_i-m_i+1 for $ratio_i$ to make it close to value 1 when payoff $u_i(s)$ is close to M_i and far from the possible minimum value v_i when other players change their strategies.

Definition 5.3.1.1 The s^* is a Cooperative Scalar Equilibrium (CSE) if and only if the s^* is the solution to the scalar optimization problem $\max_{s\in S} T(u(s)).$

5.3.2 Cooperative Scalar Equilibria Examples

We now determine the CSE in the following example games.

Example 5.3.2.1 Recall Example 5.2.2.2 (Prisoner's dilemma)

Calculate $v_1 = v_1(s_1) = v_1(\alpha) = \min_{\beta \in \{\beta_1, \beta_2, \beta_3\}} u_1(\alpha_i, \beta), i = 1,2,$ $v_2 = v_2(s_2) = v_2(\beta) = \min_{\alpha \in \{\alpha_1, \alpha_2, \alpha_3\}} u_2(\alpha, \beta_j), j = 1,2,$ where $u_1(\alpha, \beta)$ and $u_2(\alpha, \beta)$ is the payoff value for player 1 and player 2, respectively. Cooperative Scalar values are calculated using the transformation

$$T(u(s)) = \left[\frac{u_1(\alpha_i, \beta_j) - v_1 + 1}{M_1 - m_1 + 1} \times \frac{u_2(\alpha_i, \beta_j) - v_2 + 1}{M_2 - m_2 + 1} \right], \text{ for all } i, j = 1,2.$$

Player *II*

Player *I*		$\beta_1(Defect)$	$\beta_2(Cooperate)$
	$\alpha_1(Defect)$	0.0278	0.1389
	$\alpha_2(Cooperate)$	0.1389	0.6944

Figure 5.10 Cooperative scalar matrix of Example 5.3.2.1.

Player *II*

Player *I*		$\beta_1(Defect)$	$\beta_2(Cooperate)$
	$\alpha_1(Defect)$	(4,4)	(0,4)
	$\alpha_2(Cooperate)$	(4,0)	(0,0)

Figure 5.11 Disappointment matrix of Example 5.3.2.1.

From Figures 5.10-5.11, the results are includes as follows. CSE is at $(\alpha_1, \beta_1) = (4,4)$ that is the same as DE. It should be noted that GSE and CSE are the same for Prisoner's dilemma. Both greed and cooperation yield the best payoff for such player. However, (4,4) is riskier as shown in the previous example.

Example 5.3.2.2 Stag Hunt game

Player *II*

Player *I*	$\beta_1(Hunt)$	$\beta_2(Gather)$
$\alpha_1(Hunt)$	(5,5)	(0,4)
$\alpha_2(Gather)$	(4,0)	(2,2)

Figure 5.12 Payoff matrix of Example 5.3.2.2.

Calculate $v_1 = v_1(s_1) = v_1(\alpha) = \min_{\beta\in\{\beta_1,\beta_2,\beta_3\}} u_1(\alpha_i,\beta), i=1,2,\ v_2 = v_2(s_2) = v_2(\beta) =$ $\min_{\alpha\in\{\alpha_1,\alpha_2,\alpha_3\}} u_2(\alpha,\beta_j), j=1,2,$ where $u_1(\alpha,\beta)$ and $u_2(\alpha,\beta)$ is the payoff value for player 1 and player 2, respectively. So, we obtain $v_1 = 0; i=1,\ v_1 = 2; i=2,\ v_2 = 0; j=1,$ $v_2 = 2; j=2.$

Cooperative Scalar values are calculated using the transformation

$$T(u(s)) = \left[\frac{u_1(\alpha_i,\beta_j) - v_1 + 1}{M_1 - m_1 + 1} \times \frac{u_2(\alpha_i,\beta_j) - v_2 + 1}{M_2 - m_2 + 1}\right], \text{ for all } i, j = 1,2.$$

Player *II*

Player *I*	$\beta_1(Hunt)$	$\beta_2(Gather)$
$\alpha_1(Hunt)$	1.0000	0.0833
$\alpha_2(Gather)$	0.0833	0.0277

Figure 5.13 Cooperative scalar matrix of Example 5.3.2.2.

	Player II	
	$\beta_1(Hunt)$	$\beta_2(Gather)$
Player I $\alpha_1(Hunt)$	(0,0)	(0,4)
$\alpha_2(Gather)$	(4,0)	(2,2)

Figure 5.14 Disappointment matrix of Example 5.3.2.2.

From Figures 5.13-5.14, the results are includes as follows. The CSE is the same as DE. CSE is at $(\alpha_1, \beta_1) = (5, 5)$.

Example 5.3.2.3 Recall Example 5.1.2.2

Calculate $v_1 = v_1(s_1) = v_1(\alpha) = \min_{\beta \in \{\beta_1, \beta_2, \beta_3\}} u_1(\alpha_i, \beta), i = 1,2,3,\ v_2 = v_2(s_2) = v_2(\beta)$ $= \min_{\alpha \in \{\alpha_1, \alpha_2, \alpha_3\}} u_2(\alpha, \beta_j),\ j = 1,2,3,$ where $u_1(\alpha, \beta)$ and $u_2(\alpha, \beta)$ is the payoff value for player 1 and player 2, respectively. Then, we obtain $v_1 = 2; i = 1,\ v_1 = 2; i = 2,$ $v_1 = 2; i = 3,\ v_2 = 1; j = 1,\ v_2 = 1; j = 2,\ v_2 = 1; j = 3.$

Cooperative Scalar values are calculated using the transformation

$$T(u(s)) = \left[\frac{u_1(\alpha_i, \beta_j) - v_1 + 1}{M_1 - m_1 + 1} \times \frac{u_2(\alpha_i, \beta_j) - v_2 + 1}{M_2 - m_2 + 1} \right], \text{ for all } i, j = 1,2.$$

		Player II		
		β_1	β_2	β_3
Player I	α_1	0.2222	0.0556	0.0277
	α_2	0.0833	0.1667	0.6667
	α_3	0.0277	0.6667	0.6944

Figure 5.15 Cooperative scalar matrix of Example 5.3.2.3.

Player II

	β_1	β_2	β_3
Player I α_1	(0,0)	(1,4)	(1,4)
α_2	(5,1)	(0,5)	(0,1)
α_3	(4,3)	(1,0)	(0,0)

Figure 5.16 Disappointment matrix of Example 5.3.2.3.

From Figures 5.15-5.16, we obtain that CSE is at (α_3, β_3) with payoff $(6,5)$. The CSE is a DE, but the other DE is not a CSE and has a poor metric value. Thus some DEs are not necessarily very cooperative. In particular, the GSE $(\alpha_2, \beta_3) = (6, 5)$ dominates payoff $(3, 4)$, the other pure DE. The significant point here is that the GSE and CSE matrices provide an approach to selecting among REs and DEs. The SE theory attempts to do so via the notion of refinements based on various approaches. Scalar matrices represent another approach to refining RE's and DE's.

CHAPTER 6

SEQUENTIAL, SIMULTANEOUS, AND PRIORITY SCALAR EQUILIBRIA

In this chapter, Sequential, Simultaneous, and Priority Scalar Equilibria are developed. In a Sequential Scalar Equilibrium we sequentially, in some chosen order, apply other scalarizations to SEs of the game until we find a unique one if possible. In a Simultaneous Scalar Equilibrium we combine the criteria for carious scalarizations into one. Effectively the multiple criteria are applied simultaneously. In a Priority Scalar Equilibrium players are prioritized as their ability to get their highest payoff.

6.1 Sequential Scalar Equilibria

When the SEs of Chapter 3, 4, and 5 give more than one pure strategies solution, we may use sequential criteria to find a pure strategy that satisfies secondary criteria. The procedure for determining a Sequential Scalar Equilibrium (SSE) is now presented.

6.1.1 Procedure of the Sequential Scalar Equlibrium

a) Some criteria is applied, and multiple pure strategy solutions exist.

b) Another criterion is then applied to the SE's of (a) that is agreed upon by all players to find the best solution among the solution.

c) If multiple solutions still exist, repeated step b) until a unique solution is obtained if possible.

6.1.2 Examples of Sequential Scalar Equilibria

We now determine SSE in the following games.

Example 6.1.2.1 Consider two person game with 3x3 payoff matrix that CE gives multiple solutions

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	(3,4)	(2,2)	(5,6)
	α_2	(2,3)	(4,7)	(7,4)
	α_3	(6,5)	(4,5)	(2,7)

Figure 6.1 Payoff matrix of Example 6.1.2.1.

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	0.1667	0.0277	0.5556
	α_2	0.0556	0.1667	0.5000
	α_3	0.5556	0.3333	0.1667

Figure 6.2 Compromise matrix of Example 6.1.2.1.

First we use find the best pure CE's. From Figure 6.2, there are multiple CE's: (α_3, β_1) and (α_1, β_3) with payoff (6,5) and (5,6), respectively. Next we apply the RISE as a secondary criterion to find the best pure strategies among them. In Figure 6.3, after we apply RISE the numerical terms and find the unique pure strategy (α_1, β_3).

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	xxxx	xxxx	0.0833
	α_2	xxxx	xxxx	xxxx
	α_3	0.0667	xxxx	xxxx

Figure 6.3 Risk scalar matrix of Example 6.1.2.1.

Example 6.1.2.2 Consider two person game with 3x3 payoff matrix that GSE gives multiple solutions

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	(3,5)	(2,1)	(6,7)
	α_2	(5,2)	(8,5)	(4,2)
	α_3	(4,5)	(7,5)	(2,6)

Figure 6.4 Payoff matrix of Example 6.1.2.2.

First we use the GSE to find the best pure strategies. From Figure 6.5, GSE gives the pure strategies (α_2, β_2) and (α_1, β_3) with payoff (8,5) and (6,7), respectively. Next we apply the CSE as a secondary criterion to find the best pure strategies among them. In Figure 6.6, after we use CSE to find the best solution among them, we get the unique pure strategy that is (α_1, β_3).

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	0.0556	0.0204	0.3333
	α_2	0.0417	0.3333	0.0333
	α_3	0.0667	0.1667	0.0714

Figure 6.5 Greedy scalar matrix of Example 6.1.2.2.

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	xxxx	xxxx	0.6122
	α_2	xxxx	0.5100	xxxx
	α_3	xxxx	xxxx	xxxx

Figure 6.6 Cooperative scalar matrix of Example 6.1.2.2.

Example 6.1.2.3 Consider two-person game with 3x3 payoff matrix

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	(3,5)	(2,1)	(6,7)
	α_2	(5,2)	(8,5)	(7,7)
	α_3	(8,6)	(4,5)	(2,1)

Figure 6.7 Payoff matrix of Example 6.1.2.3.

	Player II		
	β_1	β_2	β_3
Player I α_1	0.1634	0.0204	0.7143
α_2	0.0204	0.4082	0.4286
α_3	0.7143	0.3061	0.0204

Figure 6.8 Cooperative scalar matrix of Example 6.1.2.3.

First we use the CE to find the best pure strategies. From Figure 6.8, the CE gives the pure strategies (α_3, β_1) and (α_1, β_3) with payoff (8,6) and (6,7), respectively. Next we apply the GSE as a secondary criterion to find the best pure strategies among them. In Figure 6.9, after we use GSE to find the best solution among them, we get the unique pure strategy that is (α_3, β_1).

	Player II		
	β_1	β_2	β_3
Player I α_1	xxxx	xxxx	0.3333
α_2	xxxx	xxxx	xxxx
α_3	0.5000	xxxx	xxxx

Figure 6.9 Greedy scalar matrix of Example 6.1.2.3.

Second, we use the GSE to find the best pure strategies. From Figure 6.10, the GSE gives the pure strategies (α_3, β_1) and (α_2, β_3) with payoff (8,6) and (7,7), respectively. Next we apply the CSE as a secondary criterion to find the best pure strategies among them. In Figure 6.10, after we use the CSE to find the best solution among them, we get the unique pure strategy that is (α_3, β_1).

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	0.0556	0.0204	0.3333
	α_2	0.0417	0.3333	0.5000
	α_3	0.5000	0.0667	0.0204

Figure 6.10 Greedy scalar matrix of Example 6.1.2.3.

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	xxxx	xxxx	xxxx
	α_2	xxxx	xxxx	0.4286
	α_3	0.7143	xxxx	xxxx

Figure 6.11 Cooperative scalar matrix of Example 6.1.2.3.

6.2 Simultaneous Scalar Equilibria

The Simultaneous Scalar Equilibrium (SISE) may be considered as compromise between criteria. We next present two different methods to create the SISE.

6.2.1 SISE by multiplying between two SEs together

We create the first method of SISE by multiplying between greedy and risk criteria, cooperative and risk criteria, as well as greedy and cooperative criteria.

6.2.1.1 Greedy and Risk criteria

The SISE combining the greedy and risk criteria is next developed for Γ. Let $u_i(s)$ be the associated von Neumann - Morgenstern (VNM) utility for player $i, i=1,...,n$; and let $u(s)=(u_1(s),...,u_n(s))$. Let $v_i(s_i)$ be a mimimum value of expected payoff when we fixed only the strategies of player $i, i=1,...,n.$

In particular denote $M_i = \max_{s\in S} u_i(s)$. Now define $T: u(S)\to R$ by

$$T(u(s)) = \left[(\frac{1}{M_1 - u_1(s)+1})\times ...\times (\frac{1}{M_n - u_n(s)+1}) \right] \times \left[\frac{1}{u_1(s)-v_1+1}\times ...\times \frac{1}{u_n(s)-v_n+1} \right],$$

for all $s\in S$, where $v_1 = v_1(s_1) = \min_{s'_{-1}\in S_{-1}} u_1(s'_{-1}, s_1)$, $v_2 = v_2(s_2) = \min_{s'_{-2}\in S_{-2}} u_2(s'_{-2}, s_2),...,$ $v_n = v_n(s_n) = \min_{s'_{-n}\in S_{-n}} u_1(s'_{-n}, s_n)$. $T(u(s))$ assigns a single real number in (0,1] for each payoff in the utility matrix of n-person games.

Definition 6.2.1.1.1 The s^* is an SISE combining greedy and risk criteria if and only if the s^* is the solution to the scalar optimization problem $\max_{s\in S} T(u(s))$.

Example 6.2.1.1.2 Recall 5.2.2.2

		Player II	
		$\beta_1(Defect)$	$\beta_2(Cooperate)$
Player I	$\alpha_1(Defect)$	0.0400	0.0333
	$\alpha_2(Cooperate)$	0.3333	0.0100

Figure 6.12 Greedy and Risk scalar matrix of example 6.2.1.1.2.

From Figure 6.12, the SISE combining greedy and risk is at $(\alpha_1, \beta_1) = (1,1)$.

6.2.1.2 Cooperative and Risk criteria

The SISE combining the cooperative and risk criteria is developed for Γ. Let $u_i(s)$ be the associated von Neumann - Morgenstern (VNM) utility for player $i, i = 1,\ldots,n$; and let $u(s) = (u_1(s),\ldots,u_n(s))$. Let $\min_{s_{-i}\in S_{-i}} u_i(s_{-i}, s_i)$ be a minimum value of expected payoff when we fixed only the strategies of player $i, i = 1,\ldots,n.$ Denote $M_i = \max_{s\in S} u_i(s)$ and $m_i = \min_{s\in S} u_i(s)$. Let define

$$T(u(s)) = \left[\frac{u_1(s) - v_1 + 1}{M_1 - m_1 + 1} \times \ldots \times \frac{u_n(s) - v_n + 1}{M_n - m_n + 1}\right] \times \left[\frac{1}{u_1(s) - v_1 + 1} \times \ldots \times \frac{1}{u_n(s) - v_n + 1}\right] \text{ for}$$

all $s \in S$, where $v_1 = v_1(s_1) = \min_{s_{-1}\in S_{-1}} u_1(s_{-1}, s_1)$, $v_2 = v_2(s_2) = \min_{s_{-2}\in S_{-2}} u_2(s_{-2}, s_2),\ldots,$ $v_n = v_n(s_n) = \min_{s_{-n}\in S_{-n}} u_1(s_{-n}, s_n)$. $T(u(s))$ assigns a single real number in (0,1] for each payoff in the utility matrix of n-person games.

Definition 6.2.1.2.1 The s^* is an SISE combining cooperative and risk criteria if and only if the s^* is the solution to the scalar optimization problem $\max_{s\in S} T(u(s))$.

Example 6.2.1.2.2 Recall Example 5.1.2.1

From Figure 6.13, SISE's combining greedy and risk are at $(\alpha_3, \beta_1) = (2,1)$, $(\alpha_2, \beta_2) = (7,1)$, and $(\alpha_1, \beta_3) = (2,1)$.

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	0.0083	0.0167	0.0278
	α_2	0.0139	0.0278	0.0139
	α_3	0.0278	0.0111	0.0100

Figure 6.13 Cooperative and Risk Scalar matrix of Example 6.2.1.2.2.

6.2.1.3 Greedy and Cooperative criteria

The SISE of greedy and cooperative criteria is developed for Γ. Let $u_i(s)$ be the associated von Neumann - Morgenstern (VNM) utility for player $i, i = 1,...,n$; and let $u(s) = (u_1(s),...,u_n(s))$. Let $T(u(s))$ be the transformation

$$T(u(s)) = \left[\left(\frac{1}{M_1 - u_1(s) + 1}\right) \times ... \times \left(\frac{1}{M_n - u_n(s) + 1}\right) \right] \times \left[\frac{u_1(s) - v_1 + 1}{M_1 - m_1 + 1} \times ... \times \frac{u_n(s) - v_n + 1}{M_n - m_n + 1} \right],$$

for all $s \in S$, where $v_1 = v_1(s_1) = \min_{s_{-1} \in S_{-1}} u_1(s_{-1}, s_1)$, $v_2 = v_2(s_2) = \min_{s_{-2} \in S_{-2}} u_2(s_{-2}, s_2),...,$ $v_n = v_n(s_n) = \min_{s_{-n} \in S_{-n}} u_1(s_{-n}, s_n)$. Denote $M_i = \max_{s \in S} u_i(s)$ and $m_i = \min_{s \in S} u_i(s)$. Let $\min_{s_{-i} \in S_{-i}} u_i(s_{-i}, s_i)$ be a minimum value of expected payoff when we fixed only the strategies of player $i, i=1,...,n.$ $T(u(s))$ assigns a single real number in (0,1] for each payoff in the utility matrix of *n*-person games.

Definition 6.2.1.3.1 The s^* is an SISE combining greedy and cooperation criteria if and only if the s^* is the solution to the scalar optimization problem $\max_{s \in S} T(u(s))$.

Example 6.2.1.3.2 Recall 5.1.2.2

Player *II*

		β_1	β_2	β_3
Player *I*	α_1	0.0148	0.0019	0.0008
	α_2	0.0035	0.0278	0.2222
	α_3	0.0008	0.2222	0.1736

Figure 6.14 Greedy and Cooperative Scalar matrix of Example 6.2.1.3.2.

From Figure 6.14, the SISE's combining greedy and cooperation are $(\alpha_2, \beta_3) = (7, 4)$ and $(\alpha_3, \beta_2) = (5, 6)$.

6.2.2 SISE with weighted criteria

We create the second method of SISE by weighted the criteria. The SISE is developed for Γ as follows. Let $u_i(s)$ be the associated von Neumann - Morgenstern (VNM) utility for player $i, i=1,...,n$; and let $u(s) = (u_1(s),...,u_n(s))$. Denote $0 < \lambda < 1$. Let $T(u(s))$ be the transformation

$$T(u(s)) = \lambda(\text{a SE value of } u(s)) + (1-\lambda)(\text{another SE value of } u(s)),$$

for all $s \in S$. $T(u(s))$ assigns a single real number in (0,1] for each payoff in the utility matrix of *n*-person games. It should be noted that the well-known Hurwigz criterion [38] is in effect an SISE.

Definition 6.2.2.1 The s^* is an SISE if and only if the s^* is the solution to the scalar optimization problem $\max_{s \in S} T(u(s))$.

Example 6.2.2.2 Recall Example 5.1.2.2 and consider GSE and RISE at the same time.

Given $\lambda = 0.5$ calculate $T(u(s)) = 0.5(\text{GSE transformation value of } u(s)) + (1-0.5)(\text{RISE transformation value of } u(s))$,

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	0.0959	0.2667	0.5139
	α_2	0.1875	0.1667	0.1875
	α_3	0.5139	0.1833	0.1450

Figure 6.15 Simultaneous scalar matrix of Example 6.2.2.2 for $\lambda = 0.5$

From Figure 6.15, SISE are at $(\alpha_3, \beta_1) = (2,1)$ and $(\alpha_1, \beta_3) = (2,1)$.

Given $\lambda = 0.7$ calculate $T(u(s)) = 0.7(\text{GSE transformation value of } u(s)) + (1-0.7)(\text{RISE transformation value of } u(s))$. From Figure 6.16, SISE are at $(\alpha_3, \beta_1) = (2,1)$ and $(\alpha_1, \beta_3) = (2,1)$.

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	0.0842	0.1733	0.3195
	α_2	0.1292	0.1667	0.2458
	α_3	0.3195	0.2433	0.1870

Figure 6.16 Simultaneous scalar matrix of Example 6.2.2.2 for $\lambda = 0.7$.

Example 6.2.2.3 Recall Example 5.1.2.2 and consider GSE and CSE at the same time.

Given $\lambda = 0.5$ calculate $T(u(s)) = 0.5(\text{GSE transformation value of } u(s)) + (1 - 0.5)(\text{CSE transformation value of } u(s))$,

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	0.1445	0.0444	0.0278
	α_2	0.0625	0.1667	0.5000
	α_3	0.0278	0.5000	0.4722

Figure 6.17 Simultaneous scalar matrix of Example 6.2.2.3 for $\lambda = 0.5$.

From Figure 6.17, SISE are at $(\alpha_3, \beta_2) = (5, 6)$ and $(\alpha_2, \beta_3) = (7, 4)$.

Given $\lambda = 0.7$ calculate $T(u(s)) = 0.7(\text{GSE transformation value of } u(s)) + (1 - 0.7)(\text{CSE transformation value of } u(s))$. From Figure 6.18, SISE are at $(\alpha_3, \beta_2) = (5, 6)$ and $(\alpha_2, \beta_3) = (7, 4)$.

	Player II		
	β_1	β_2	β_3
Player I α_1	0.1134	0.0399	0.0277
α_2	0.0542	0.1667	0.4333
α_3	0.0277	0.4333	0.3833

Figure 6.18 Simultaneous scalar matrix of Example 6.2.2.3 for $\lambda = 0.7$.

We summarize the SSE and SISE solution in the following table.

Table 6.1 The SSE and SISE solution with different criteria

Example	Criteria	Solution
6.1.2.1	SSE with Compromise and Risk	(α_1, β_3) with payoff (5,6)
6.1.2.2	SSE with Greedy and Cooperation	(α_1, β_3) with payoff (6,7)
6.1.2.3	SSE with Cooperation and Greedy	(α_3, β_1) with payoff (8,6)
6.1.2.3	SSE with Greedy and Cooperation	(α_3, β_1) with payoff (8,6)
6.2.1.2.2	SISE's combining greedy and risk	(α_3, β_1), (α_2, β_2), and (α_1, β_3) with payoff (2,1), (7,1), (2,1)
6.2.1.3.2	SISE's combining greedy and cooperation	(α_2, β_3) and (α_3, β_2) with payoff (7,4) and (5,6),
6.2.2.2	SISE with $\lambda = 0.5$ of GSE and RISE	(α_3, β_1) and (α_1, β_3) with payoff (2,1), (2,1)
6.2.2.2	SISE with $\lambda = 0.7$ of GSE and RISE	(α_3, β_1) and (α_1, β_3) with payoff (2,1) and (2,1)
6.2.2.3	SISE with $\lambda = 0.5$ of GSE and CSE	(α_3, β_2) and (α_2, β_3) with payoff (5,6) and (7,8)
6.2.2.3	SISE with $\lambda = 0.7$ of GSE and CSE	(α_3, β_2) and (α_2, β_3) with payoff (5,6) and (7,8)

6.3 Priority Scalar Equilibria

Players are next prioritized as their ability to get their highest payoff. In other words, players are given preference much as in sense practical situations. The priority is not preemptive. It only influences the scalarization. We focus only on greediness for this criterion.

6.3.1 Priority Scalar Equilibria for n-person games

The Priority Scalar Equilibrium (PSE) is developed for Γ as follows. Let $u_i(s)$ be the associated von Neumann - Morgenstern (VNM) utility for player $i, i = 1,...,n$; and let $u(s) = (u_1(s),...,u_n(s))$. Let ρ_i be the integer-valued priority rank for each player $i, i = 1,...,n,$ where 1 denotes the highest priority. In particular denote $M_i = \max_{s \in S} u_i(s)$. Now define $T : u(S) \to R$ by

$$T(u(s)) = \left[\left(\frac{1}{M_1 - \frac{u_1(s)}{\rho_1} + 1}\right) \times \left(\frac{1}{M_2 - \frac{u_2(s)}{\rho_2} + 1}\right) \times ... \times \left(\frac{1}{M_n - \frac{u_n(s)}{\rho_n} + 1}\right) \right], \text{ for all}$$

$s \in S$. $T(u(s))$ assigns a single real number in (0,1] for each payoff in the utility matrix of *n*-person games.

Definition 6.3.1.1 The s^* is a Priority Scalar Equilibrium (PSE) if and only if the s^* is the solution to the scalar optimization problem $\max_{s \in S} T(u(s))$.

Note that we have used integer priorities above. In fact, we could have any positive numbers as priorities. A higher priority is assigned by a lower positive number. Therefore, some players could be given more favorable treatment for some reason by an arbiter.

6.3.2 Priority Scalar Equilibria Examples

We now determine PSEs in following example games.

Example 6.3.2.1 Two-person 3 x 3 payoff matrix where Player 2 has greater priority than Player 1.

Let $\rho_2 = 1, \rho_1 = 2.$ Calculate the $M_i = \max\limits_{\substack{\alpha \in \{\alpha_1, \alpha_2, \alpha_3\} \\ \beta \in \{\beta_1, \beta_2, \beta_3\}}} u_i(\alpha, \beta)$ where $u_i(\alpha, \beta)$ is the payoff value for player *i*, *i* =1, 2. So, we obtain $M_1 = 7,\ M_2 = 6.$

Player *II*

Player *I*

	β_1	β_2	β_3
α_1	(3,4)	(2,2)	(2,1)
α_2	(2,3)	(7,1)	(7,4)
α_3	(2,1)	(5,6)	(6,5)

Figure 6.19 Payoff matrix of Example 6.3.2.1.

We calculate priority scalar values using the transformation

$$T(u(s)) = \left[\left(\frac{1}{M_1 - \frac{u_1(s)}{\rho_1} + 1}\right) \times \left(\frac{1}{M_2 - \frac{u_2(s)}{\rho_2} + 1}\right) \right] \text{ for all } i, j = 1,2,3.$$

From Figure 6.20, we obtain that PSE is at (α_3, β_2) with payoff $(5,6)$. This solution is reasonable because Player 2 can get the highest payoff.

Player *II*

Player *I*		β_1	β_2	β_3
	α_1	0.0513	0.0286	0.0238
	α_2	0.0357	0.0370	0.0741
	α_3	0.0238	0.1818	0.1000

Figure 6.20 Priority scalar matrix of Example 6.3.2.1.

Example 6.3.2.2 Recall Example 6.3.2.1. Assume now that Player 1 has greater priority than Player 2.

Let $\rho_1 = 1, \rho_2 = 2.$ Calculate the $M_i = \max\limits_{\substack{\alpha \in \{\alpha_1, \alpha_2, \alpha_3\} \\ \beta \in \{\beta_1, \beta_2, \beta_3\}}} u_i(\alpha, \beta)$ where $u_i(\alpha, \beta)$ is the payoff value for player *i*, *i* =1, 2. So, we obtain $M_1 = 7,\ M_2 = 6.$ We calculate priority scalar values using the transformation

$$T(u(s)) = \left[\left(\frac{1}{M_1 - \frac{u_1(s)}{\rho_1} + 1} \right) \times \left(\frac{1}{M_2 - \frac{u_2(s)}{\rho_2} + 1} \right) \right] \text{for all } i, j = 1,2,3.$$

In Figure 6.21, we obtain that PSE is at (α_2, β_3) with payoff $(7,4)$. The result is different from Example 6.3.2.1 because the priority rank has changed.

Player *II*

		β_1	β_2	β_3
	α_1	0.0400	0.0278	0.0256
Player *I*	α_2	0.0303	0.1538	0.2000
	α_3	0.0256	0.0833	0.1111

Figure 6.21 Priority scalar matrix of Example 6.3.2.2.

CHAPTER 7

COALITION SCALAR EQUILIBRIA

Previous chapters have treated players individually. Here we consider given coalitions of players. In general, coalitions can be formed for any $n \geq 2.$ But effectively in the $n = 2$ case, we merely have cooperative behavior as in the DE. We limit coalitions to the greedy criterion. Effectively each coalition wishes to maximize its total payoff. After the game is played, each coalition divides its total payoff in any way it chooses, perhaps using some other game-theoretic approach. In fact, there is a topic in game theory known as fair division [39].

7.1 Coalition Scalar Equilibria for n-person games

The Coalition Scalar Equilibrium (COSE) is developed for Γ as follows. Let $u_i(s)$ be the associated von Neumann - Morgenstern (VNM) utility for player $i, i = 1,...,n$; and let $u(s) = (u_1(s),...,u_n(s)).$ Let k be the number of coalitions. Let n_j be the number of players in coalition $j, j = 1,2,...,k; \sum_{j=1}^{k} n_j = n.$ Let J_j be the set of players in coalitions $j, j = 1,2,...,k..$ Let $u^j(s)$ be the average value of players' payoff for each coalition $j, j=1,2,\ldots,k,$ $u^j(s)=\frac{\sum_{i\in J_j} u_i(s)}{n_j}$. Let $u^{M_j}(s)$ be the average maximum value of players' payoff for each coalition $j, j=1,2,\ldots,k,$ $u^{M_j}(s)=\frac{\sum_{i\in J_j} M_i}{n_j}$.

In particular denote $M_i=\max_{s\in S} u_i(s)$. Now define $T: u(S)\to R$ by

$$T(u(s))=\left[\left(\frac{1}{u^{M_1}(s)-u^1(s)+1}\right)\times\left(\frac{1}{u^{M_2}(s)-u^2(s)+1}\right)\times\ldots\times\left(\frac{1}{u^{M_k}(s)-u^k(s)+1}\right)\right],$$ for

all $s\in S$.

$T(u(s))$ assigns a single real number in (0,1] for each payoff in the utility matrix of *n*-person games.

Definition 7.1.1 The s^* is an Coalition Scalar Equilibrium (COSE) if and only if the s^* is the solution to the scalar optimization problem $\max_{s\in S} T(u(s))$.

7.2 Coalition Scalar Equilibria Examples

Example 7.2.1 Consider the four-person game with coalitions. Player 1 and 2 form the first coalition, and Player 2 and 3 form the second coalition.

Let each player have two strategies. In particular let Player 1 have strategies α_1,α_2, Player 2 have strategies β_1,β_2, Player 3 have strategies γ_1,γ_2, Player 4 have strategies φ_1,φ_2. Suppose $J_1=\{1,2\}$ and $J_2=\{3,4\}$. From Figure 7.1 we first obtain

$M_1 = 8,\ M_2 = 6,\ M_3 = 7,\ M_4 = 9,$ and then $u^{M_1}(s) = 7$ and $u^{M_2}(s) = 8$ from the formula $u^{M_j}(s) = \frac{\sum_{i \in J_j} M_i}{n_j},\ j = 1,2.$ We finally calculate coalition scalar values using the transformation $T(u(s)) = \left[(\frac{1}{u^{M_1}(s) - u^1(s) + 1}) \times (\frac{1}{u^{M_2}(s) - u^2(s) + 1}) \right]$, for all $s \in S$.

Set of strategies	Payoff		Set of strategies	Payoff
$(\alpha_1, \beta_1, \gamma_1, \varphi_1)$	(3,5,6,4)		$(\alpha_2, \beta_1, \gamma_1, \varphi_1)$	(8,4,7,3)
$(\alpha_1, \beta_1, \gamma_1, \varphi_2)$	(8,3,5,7)		$(\alpha_2, \beta_1, \gamma_1, \varphi_2)$	(3,2,4,4)
$(\alpha_1, \beta_1, \gamma_2, \varphi_1)$	(3,5,7,9)		$(\alpha_2, \beta_1, \gamma_2, \varphi_1)$	(8,6,6,5)
$(\alpha_1, \beta_2, \gamma_1, \varphi_1)$	(4,2,4,8)		$(\alpha_2, \beta_2, \gamma_1, \varphi_1)$	(6,6,7,5)
$(\alpha_1, \beta_1, \gamma_2, \varphi_2)$	(7,1,5,7)		$(\alpha_2, \beta_2, \gamma_1, \varphi_2)$	(4,3,4,3)
$(\alpha_1, \beta_2, \gamma_1, \varphi_2)$	(3,5,6,9)		$(\alpha_2, \beta_2, \gamma_2, \varphi_1)$	(2,5,2,9)
$(\alpha_1, \beta_2, \gamma_2, \varphi_1)$	(6,2,4,7)		$(\alpha_2, \beta_1, \gamma_2, \varphi_2)$	(4,2,5,6)
$(\alpha_1, \beta_2, \gamma_2, \varphi_2)$	(3,4,2,2)		$(\alpha_2, \beta_2, \gamma_2, \varphi_2)$	(8,4,6,7)

Figure 7.1 Payoff matrix of Example 7.2.1.

Set of strategies	Transformation value		Set of strategies	Transformation value
$(\alpha_1, \beta_1, \gamma_1, \varphi_1)$	0.0625		$(\alpha_2, \beta_1, \gamma_1, \varphi_1)$	0.1250
$(\alpha_1, \beta_1, \gamma_1, \varphi_2)$	0.1333		$(\alpha_2, \beta_1, \gamma_1, \varphi_2)$	0.0364
$(\alpha_1, \beta_1, \gamma_2, \varphi_1)$	0.2500		$(\alpha_2, \beta_1, \gamma_2, \varphi_1)$	0.2857
$(\alpha_1, \beta_2, \gamma_1, \varphi_1)$	0.0667		$(\alpha_2, \beta_2, \gamma_1, \varphi_1)$	0.1667
$(\alpha_1, \beta_1, \gamma_2, \varphi_2)$	0.0833		$(\alpha_2, \beta_2, \gamma_1, \varphi_2)$	0.0404
$(\alpha_1, \beta_2, \gamma_1, \varphi_2)$	0.1667		$(\alpha_2, \beta_2, \gamma_2, \varphi_1)$	0.0634
$(\alpha_1, \beta_2, \gamma_2, \varphi_1)$	0.0714		$(\alpha_2, \beta_1, \gamma_2, \varphi_2)$	0.0571
$(\alpha_1, \beta_2, \gamma_2, \varphi_2)$	0.0317		$(\alpha_2, \beta_2, \gamma_2, \varphi_2)$	0.2000

Figure 7.2 Coalition scalar matrix of Example 7.2.1.

These transformation values are shown in Figure 7.2, where we obtain that COSE is at $(\alpha_2, \beta_1, \gamma_2, \varphi_1)$ with payoff $(8,6,6,5)$ when Player 1 and 2 form the first coalition and Player 2 and 3 form the second coalition.

Example 7.2.2 Recall Example 7.2.1 and Figure 7.1. Now let Player 1 and 4 form the first coalition and Player 2 and 3 form the second coalition. Hence $J_1 = \{1,4\}$ and $J_2 = \{2,3\}$. From Figure 7.1 we first obtain $M_1 = 8, M_2 = 6, M_3 = 7, M_4 = 9,$ and then $u^{M_1}(s) = 8.5$ and $u^{M_2}(s) = 6.5$ from the formula $u^{M_j}(s) = \frac{\sum_{i \in J_j} M_i}{n_j}, j = 1,2.$ We finally calculate coalition scalar values using the transformation

$$T(u(s)) = \left[\left(\frac{1}{u^{M_1}(s) - u^1(s) + 1}\right) \times \left(\frac{1}{u^{M_2}(s) - u^2(s) + 1}\right) \right], \text{ for all } s \in S.$$

Set of strategies	Transformation value		Set of strategies	Transformation value
$(\alpha_1, \beta_1, \gamma_1, \varphi_1)$	0.0833		$(\alpha_2, \beta_1, \gamma_1, \varphi_1)$	0.1250
$(\alpha_1, \beta_1, \gamma_1, \varphi_2)$	0.1429		$(\alpha_2, \beta_1, \gamma_1, \varphi_2)$	0.0370
$(\alpha_1, \beta_1, \gamma_2, \varphi_1)$	0.1905		$(\alpha_2, \beta_1, \gamma_2, \varphi_1)$	0.2222
$(\alpha_1, \beta_2, \gamma_1, \varphi_1)$	0.0635		$(\alpha_2, \beta_2, \gamma_1, \varphi_1)$	0.2500
$(\alpha_1, \beta_1, \gamma_2, \varphi_2)$	0.0889		$(\alpha_2, \beta_2, \gamma_1, \varphi_2)$	0.0417
$(\alpha_1, \beta_2, \gamma_1, \varphi_2)$	0.1429		$(\alpha_2, \beta_2, \gamma_2, \varphi_1)$	0.0625
$(\alpha_1, \beta_2, \gamma_2, \varphi_1)$	0.0741		$(\alpha_2, \beta_1, \gamma_2, \varphi_2)$	0.0556
$(\alpha_1, \beta_2, \gamma_2, \varphi_2)$	0.0317		$(\alpha_2, \beta_2, \gamma_2, \varphi_2)$	0.2000

Figure 7.3 Coalition scalar matrix of Example 7.2.2.

From Figure 7.3, we obtain that COSE is now $(\alpha_2, \beta_2, \gamma_1, \varphi_1)$ with payoff $(6,6,7,5)$ when Player 1 and 4 form the first coalition, and Player 2 and 3 form the second coalition.

Example 7.2.3 Recall Example 7.2.1 and Figure 7.1. Now let Player 1 form the first coalition and Player 2, 3, and 4 form the second coalition. Hence $J_1 = \{1\}$ and $J_2 = \{2,3.4\}$. From Figure 7.1 we first obtain $M_1 = 8, M_2 = 6, M_3 = 7, M_4 = 9,$ and then $u^{M_1}(s) = 8$ and $u^{M_2}(s) = 7.33$ from the formula $u^{M_j}(s) = \frac{\sum_{i \in J_j} M_i}{n_j}, j = 1,2.$ We finally calculate coalition scalar values using the transformation

$$T(u(s)) = \left[\left(\frac{1}{u^{M_1}(s) - u^1(s) + 1}\right) \times \left(\frac{1}{u^{M_2}(s) - u^2(s) + 1}\right) \right], \text{ for all } s \in S.$$

Set of strategies	Transformation value		Set of strategies	Transformation value
$(\alpha_1, \beta_1, \gamma_1, \varphi_1)$	0.0501		$(\alpha_2, \beta_1, \gamma_1, \varphi_1)$	0.2730
$(\alpha_1, \beta_1, \gamma_1, \varphi_2)$	0.3003		$(\alpha_2, \beta_1, \gamma_1, \varphi_2)$	0.0334
$(\alpha_1, \beta_1, \gamma_2, \varphi_1)$	0.1253		$(\alpha_2, \beta_1, \gamma_2, \varphi_1)$	0.3755
$(\alpha_1, \beta_2, \gamma_1, \varphi_1)$	0.0546		$(\alpha_2, \beta_2, \gamma_1, \varphi_1)$	0.1431
$(\alpha_1, \beta_1, \gamma_2, \varphi_2)$	0.1251		$(\alpha_2, \beta_2, \gamma_1, \varphi_2)$	0.0400
$(\alpha_1, \beta_2, \gamma_1, \varphi_2)$	0.1002		$(\alpha_2, \beta_2, \gamma_2, \varphi_1)$	0.0477
$(\alpha_1, \beta_2, \gamma_2, \varphi_1)$	0.0834		$(\alpha_2, \beta_1, \gamma_2, \varphi_2)$	0.0500
$(\alpha_1, \beta_2, \gamma_2, \varphi_2)$	0.0295		$(\alpha_2, \beta_2, \gamma_2, \varphi_2)$	0.3755

Figure 7.4 Coalition scalar matrix of Example 7.2.3.

From Figure 7.4, we obtain the COSE are $(\alpha_2,\beta_1,\gamma_2,\varphi_1)$ with payoff $(8,6,6,5)$ and another COSE $(\alpha_2,\beta_2,\gamma_2,\varphi_2)$ with payoff $(8,4,6,7)$ when Player 1 form the first coalition and Player 2,3, and 4 form the second coalition.

Example 7.2.4 Recall Example 7.2.1 and Figure 7.1. Now let Player 1 form the first coalition, Player 2 form the second coalition, and Player 3 and 4 form the third coalition. Hence $J_1=\{1\}$, $J_2=\{2\}$, and $J_3=\{3,4\}$. From Figure 7.1 we first obtain $M_1=8, M_2=6, M_3=7, M_4=9,$ and then $u^{M_1}(s)=8, u^{M_2}(s)=6,$ and $u^{M_3}(s)=8$ from the formula $u^{M_j}(s)=\frac{\sum_{i\in J_j} M_i}{n_j}, j=1,2,3.$ We finally calculate coalition scalar values using the transformation

$$T(u(s))=\left[(\frac{1}{u^{M_1}(s)-u^1(s)+1})\times(\frac{1}{u^{M_2}(s)-u^2(s)+1})\times(\frac{1}{u^{M_3}(s)-u^3(s)+1})\right], \text{ for all}$$

$s\in S.$

Set of strategies	Transformation value		Set of strategies	Transformation value
$(\alpha_1,\beta_1,\gamma_1,\varphi_1)$	0.02083		$(\alpha_2,\beta_1,\gamma_1,\varphi_1)$	0.08333
$(\alpha_1,\beta_1,\gamma_1,\varphi_2)$	0.08333		$(\alpha_2,\beta_1,\gamma_1,\varphi_2)$	0.00667
$(\alpha_1,\beta_1,\gamma_2,\varphi_1)$	0.08333		$(\alpha_2,\beta_1,\gamma_2,\varphi_1)$	0.28571
$(\alpha_1,\beta_2,\gamma_1,\varphi_1)$	0.01333		$(\alpha_2,\beta_2,\gamma_1,\varphi_1)$	0.11111
$(\alpha_1,\beta_1,\gamma_2,\varphi_2)$	0.02778		$(\alpha_2,\beta_2,\gamma_1,\varphi_2)$	0.00909
$(\alpha_1,\beta_2,\gamma_1,\varphi_2)$	0.05556		$(\alpha_2,\beta_2,\gamma_2,\varphi_1)$	0.02041
$(\alpha_1,\beta_2,\gamma_2,\varphi_1)$	0.01905		$(\alpha_2,\beta_1,\gamma_2,\varphi_2)$	0.01143
$(\alpha_1,\beta_2,\gamma_2,\varphi_2)$	0.00794		$(\alpha_2,\beta_2,\gamma_2,\varphi_2)$	0.13333

Figure 7.5 Coalition scalar matrix of Example 7.2.4.

From Figure 7.5, we obtain the COSE $(\alpha_2, \beta_1, \gamma_2, \varphi_1)$ with payoff $(8,6,6,5)$ when Player 1 forms the first coalition, Player 2 form the second coalition, and Player 3 and 4 form the third coalition.

Example 7.2.5 Recall Example 7.2.1 and Figure 7.1. Now let Player 1 form the first coalition, Player 2 and Player 3 form the second coalition, and Player 4 form the third coalition. Hence $J_1 = \{1\}$, $J_2 = \{2,3\}$, and $J_3 = \{4\}$. From Figure 7.1 we first obtain $M_1 = 8,\ M_2 = 6,\ M_3 = 7,\ M_4 = 9,$ and then $u^{M_1}(s) = 8,\ u^{M_2}(s) = 6.5,$ and $u^{M_3}(s) = 9$ from the formula $u^{M_j}(s) = \dfrac{\sum_{i \in J_j} M_i}{n_j},\ j = 1,2,3.$ We finally calculate coalition scalar values using the transformation

$$T(u(s)) = \left[\left(\frac{1}{u^{M_1}(s) - u^1(s) + 1}\right) \times \left(\frac{1}{u^{M_2}(s) - u^2(s) + 1}\right) \times \left(\frac{1}{u^{M_3}(s) - u^3(s) + 1}\right) \right],$$ for all $s \in S$.

These transformation values are shown in Figure 7.6, we obtain the COSE $(\alpha_2, \beta_1, \gamma_2, \varphi_1)$ with payoff $(8,6,6,5)$ and $(\alpha_2, \beta_2, \gamma_2, \varphi_2)$ with payoff $(8,4,6,7)$ when Player 1 forms the first coalition, Player 2 and Player 3 form the second coalition, and Player 4 form the third coalition.

Set of strategies	Transformation value		Set of strategies	Transformation value
$(\alpha_1,\beta_1,\gamma_1,\varphi_1)$	0.0139		$(\alpha_2,\beta_1,\gamma_1,\varphi_1)$	0.0714
$(\alpha_1,\beta_1,\gamma_1,\varphi_2)$	0.0952		$(\alpha_2,\beta_1,\gamma_1,\varphi_2)$	0.0062
$(\alpha_1,\beta_1,\gamma_2,\varphi_1)$	0.1111		$(\alpha_2,\beta_1,\gamma_2,\varphi_1)$	0.1333
$(\alpha_1,\beta_2,\gamma_1,\varphi_1)$	0.0222		$(\alpha_2,\beta_2,\gamma_1,\varphi_1)$	0.0667
$(\alpha_1,\beta_1,\gamma_2,\varphi_2)$	0.0370		$(\alpha_2,\beta_2,\gamma_1,\varphi_2)$	0.0071
$(\alpha_1,\beta_2,\gamma_1,\varphi_2)$	0.0833		$(\alpha_2,\beta_2,\gamma_2,\varphi_1)$	0.0357
$(\alpha_1,\beta_2,\gamma_2,\varphi_1)$	0.0247		$(\alpha_2,\beta_1,\gamma_2,\varphi_2)$	0.0125
$(\alpha_1,\beta_2,\gamma_2,\varphi_2)$	0.0046		$(\alpha_2,\beta_2,\gamma_2,\varphi_2)$	0.1333

Figure 7.6 Coalition Scalar matrix of Example 7.2.5.

Example 7.2.6 Recall Example 7.2.1 and Figure 7.1. Now let Player 1 form the first coalition, Player 2 form the second coalition, and Player 3 form the third coalition, and Player 4 form the fourth coalition. Hence $J_1 = \{1\}$, $J_2 = \{2,\}$, $J_3 = \{3\}$, and $J_4 = \{4\}$. From Figure 7.1 we first obtain $M_1 = 8$, $M_2 = 6$, $M_3 = 7$, $M_4 = 9$, and then $u^{M_1}(s) = 8$, $u^{M_2}(s) = 6$, $u^{M_2}(s) = 7$, and $u^{M_3}(s) = 9$ from the formula $u^{M_j}(s) = \frac{\sum_{i \in J_j} M_i}{n_j}$, $j = 1,2,3,4.$

We finally calculate coalition scalar values using the transformation

$$T(u(s)) = \left[\left(\frac{1}{u^{M_1}(s) - u^1(s) + 1}\right) \times \left(\frac{1}{u^{M_2}(s) - u^2(s) + 1}\right) \times \left(\frac{1}{u^{M_3}(s) - u^3(s) + 1}\right) \right], \text{ for all}$$

$s \in S$.

These transformation values are shown in Figure 7.7, we obtain the COSE $(\alpha_2,\beta_1,\gamma_2,\varphi_1)$ with payoff $(8,6,6,5)$ when Player 1 forms the first coalition, Player 2 forms the second coalition, Player 3 forms the third coalition, and Player 4 form the fourth coalition.

Set of strategies	Transformation value		Set of strategies	Transformation value
$(\alpha_1,\beta_1,\gamma_1,\varphi_1)$	0.0069		$(\alpha_2,\beta_1,\gamma_1,\varphi_1)$	0.0476
$(\alpha_1,\beta_1,\gamma_1,\varphi_2)$	0.0278		$(\alpha_2,\beta_1,\gamma_1,\varphi_2)$	0.0014
$(\alpha_1,\beta_1,\gamma_2,\varphi_1)$	0.0833		$(\alpha_2,\beta_1,\gamma_2,\varphi_1)$	0.1000
$(\alpha_1,\beta_2,\gamma_1,\varphi_1)$	0.0050		$(\alpha_2,\beta_2,\gamma_1,\varphi_1)$	0.0667
$(\alpha_1,\beta_1,\gamma_2,\varphi_2)$	0.0093		$(\alpha_2,\beta_2,\gamma_1,\varphi_2)$	0.0018
$(\alpha_1,\beta_2,\gamma_1,\varphi_2)$	0.0417		$(\alpha_2,\beta_2,\gamma_2,\varphi_1)$	0.0120
$(\alpha_1,\beta_2,\gamma_2,\varphi_1)$	0.0056		$(\alpha_2,\beta_1,\gamma_2,\varphi_2)$	0.0033
$(\alpha_1,\beta_2,\gamma_2,\varphi_2)$	0.0012		$(\alpha_2,\beta_2,\gamma_2,\varphi_2)$	0.0556

Figure 7.7 Coalition Scalar matrix of Example 7.2.6.

Note that the total sum of payoffs of COSE for $J_1=\{1,2\}$, $J_2=\{3,4\}$ is the same as for $J_1=\{1\}$, $J_2=\{2,3,4\}$, for $J_1=\{1\}$, $J_2=\{2\}$, $J_3=\{3,4\}$, for $J_1=\{1\}$, $J_2=\{2,3\}$, $J_3=\{4\}$, and for $J_1=\{1\}$, $J_2=\{2\}$, $J_3=\{3\}$, $J_4=\{4\}$. Moreover, the total sum of payoffs of COSE for $J_1=\{1,2\}$, $J_2=\{3,4\}$, for $J_1=\{1\}$, $J_2=\{2,3,4\}$, for $J_1=\{1\}$, $J_2=\{2\}$, $J_3=\{3,4\}$, for $J_1=\{1\}$, $J_2=\{2,3\}$, $J_3=\{4\}$, and for $J_1=\{1\}$, $J_2=\{2\}$, $J_3=\{3\}$, $J_4=\{4\}$ is greater than for $J_1=\{1,4\}$, $J_2=\{2,3\}$. In addition, the transformation value of COSE for $J_1=\{1\}$, $J_2=\{2,3,4\}$, has the highest value.

CHAPTER 8

APPLICATION

In this chapter we present a real life situation where SEs provide solutions for determining market strategies in the car industry.

8.1 Scenario

In this application, we consider three competitors Mercedes, BMW, and Audi, each of which is trying to obtain the business from a finite number of customers. Each competitor has a fixed advertising percentage of its total budget (40%, 80%, 100%) that must be allocated among the potential customers. We assume here that a car company will get more market share that a competitor when the company allocates a larger advertising budget than the competitors, and the market share will be equal when all companies allocate the same budget of advertising. In addition, the utility function is market share values to Mercedes, BMW, and Audi that are shown in the Figure 8.1. We use Scalar Equilibria to provide pure strategies solutions for determining the advertising strategies for Mercedes, BMW, and Audi.

8.2 Calculation of SE's

Each company have three strategies. In particular let Mercedes (Player 1) have strategies $\alpha_1, \alpha_2, \alpha_3$, BMW (Player 2) have strategies $\beta_1, \beta_2, \beta_3$, and Audi (Player 3) have strategies $\gamma_1, \gamma_2, \gamma_3$. We calculate the RM and DM that are shown in Figures 8.2 - 8.3 for comparison with SEs.

	40%			80%		
	40%	80%	100%	40%	80%	100%
40%	(13,13,13)	(10,20,10)	(4,40,4)	(10,10,20)	(10,20,20)	(4,35,15)
80%	(20,10,10)	(20,20,10)	(15,35,4)	(20,10,20)	(14,14,14)	(10,25,10)
100%	(40,4,4)	(35,15,4)	(30,30,8)	(35,4,15)	(25,10,10)	(25,25,12)

		100%	
	40%	80%	100%
40%	(4,4,40)	(4,15,35)	(8,30,30)
80%	(15,4,35)	(10,10,25)	(12,25,25)
100%	(30,8,30)	(25,12,25)	(20,20,20)

Figure 8.1 Payoff matrix of three car brands in the car industry.

	γ_1			γ_2		
	β_1	β_2	β_3	β_1	β_2	β_3
α_1	(27,27,27)	(25,20,25)	(26,0,26)	(25,25,20)	(15,15,15)	(21,0,15)
α_2	(20,25,25)	(15,15,15)	(15,0,21)	(15,15,15)	(11,11,11)	(15,0,15)
α_3	(0,26,26)	(0,15,21)	(0,0,12)	(0,21,15)	(0,15,15)	(0,10,8)

		γ_3	
	β_1	β_2	β_3
α_1	(26,26,0)	(27,15,0)	(27,0,0)
α_2	(15,21,0)	(20,15,0)	(20,0,0)
α_3	(0,12,0)	(0,8,0)	(0,0,0)

Figure 8.2 Regret matrix of three car brands in the car industry.

	γ_1			γ_2		
	β_1	β_2	β_3	β_1	β_2	β_3
α_1	(0,0,0)	(3,0,3)	(9,0,9)	(3,3,0)	(3,0,0)	(9,5,5)
α_2	(5,3,3)	(5,0,3)	(10,5,9)	(5,3,0)	(9,6,6)	(0,15,10)
α_3	(0,9,9)	(5,5,9)	(10,10,5)	(5,9,5)	(15,10,10)	(15,15,8)

		γ_3	
	β_1	β_2	β_3
α_1	(26,9,0)	(27,5,5)	(27,10,10)
α_2	(15,9,5)	(20,10,15)	(20,15,15)
α_3	(10,5,10)	(15,8,15)	(20,20,10)

Figure 8.3 Disappointment matrix of three car brands in the car industry.

Note that the RE of Figure 8.2 dominates the DE of Figure 8.3. We next calculate other SEs that are shown in following figures.

	γ_1			γ_2		
	β_1	β_2	β_3	β_1	β_2	β_3
α_1	0.0000456	0.0000496	0.0007305	0.0000496	0.0000731	0.0001733
α_2	0.0000496	0.0000731	0.0001733	0.0000731	0.0000508	0.0000650
α_3	0.0007305	0.0001733	0.0002504	0.0001733	0.0000650	0.0001347

		γ_3	
	β_1	β_2	β_3
α_1	0.0007305	0.0001733	0.0002504
α_2	0.0001733	0.0000650	0.0001347
α_3	0.0002504	0.0001347	0.0001080

Figure 8.4 Greedy scalar matrix of three car brands in the car industry.

Note that the GSEs of Figure 8.4 are only good for one car company.

	γ_1			γ_2		
	β_1	β_2	β_3	β_1	β_2	β_3
α_1	0.0197422	0.0106410	0.0004146	0.0106410	0.0167216	0.0006120
α_2	0.0106410	0.0167216	0.0018952	0.0167216	0.0024678	0.0001185
α_3	0.0004146	0.0018952	0.0119440	0.0018952	0.0001185	0.0078179

		γ_3	
	β_1	β_2	β_3
α_1	0.0030798	0.0018952	0.0119440
α_2	0.0018952	0.0001185	0.0021322
α_3	0.0119440	0.0021322	0.0000197

Figure 8.5 Cooperative scalar matrix of three car brands in the car industry.

In Figure 8.5 the CSE yields cooperate DE as expected.

	γ_1			γ_2		
	β_1	β_2	β_3	β_1	β_2	β_3
α_1	0.0010000	0.0018553	0.0476190	0.0018553	0.0011806	0.0322581
α_2	0.0018553	0.0011806	0.0104167	0.0011806	0.0080000	0.1666667
α_3	0.0476190	0.0104167	0.0016529	0.0104167	0.1666667	0.0025253

		γ_3	
	β_1	β_2	β_3
α_1	0.0064103	0.0104167	0.0016529
α_2	0.0104167	0.1666667	0.0092593
α_3	0.0016529	0.0092593	1.0000000

Figure 8.6 Risk scalar matrix of three car brands in the car industry.

In Figure 8.6 the RSE is actually the RE with its stability enforcement.

In summary, from Figures 8.2-8.5, RE is (100%,100%,100%) with payoff (20,20,20). DE is (40%,40%,40%) with payoff (13,13,13). GSEs are (100%,40%,40%), with a payoff of (40,4,4), (4,4,40), and (4,40,4), respectively. The CSE is (40%,40%,40%) with a payoff of (13,13,13). The RE is not a GSE. Thus, the RE is not greedy in this case. Moreover, the DE is a CSE. From Figure 8.6, RSE is (100%,100%,100%) with a payoff of (20,20,20). The RSE is same as RE.

	γ_1			γ_2		
	β_1	β_2	β_3	β_1	β_2	β_3
α_1	0.12500	0.04000	0.00826	0.04000	0.20000	0.00826
α_2	0.04000	0.20000	0.09091	0.20000	1.00000	0.04000
α_3	0.00826	0.09091	0.14286	0.09091	0.04000	0.33333

		γ_3	
	β_1	β_2	β_3
α_1	0.09091	0.09091	0.14286
α_2	0.09091	0.04000	0.33333
α_3	0.14286	0.33333	1.00000

Figure 8.7 Aspiration scalar matrix of three car brands in car industry for $p_1 = 14, p_2 = 14, p_3 = 14$.

Now assume that all companies have an aspiration level of market share payoff, suppose the target payoff of Mercedes is 14, the target payoff of BMW is 14, and the target payoff of Audi is 14. Hence, $p_1 = 14, p_2 = 14, p_3 = 14$. In this case, we find the ASE for $p_1 = 14, p_2 = 14, p_3 = 14$. From Figure 8.7 the ASEs for $p_1 = 14, p_2 = 14, p_3 = 14$ are (80%,80%,80%) and (100%,100%,100%) with payoff (14,14,14) and (20,20,20), respectively. Thus the three companies should select

(80%,80%,80%) and (100%,100%,100%) strategies to achieved their market share target payoff.

Next suppose the target payoff of Mercedes is 24, the target payoff of BMW is 10, and the target payoff of Audi is 10. Hence, Then, we calculate the ASE for $p_1 = 24, p_2 = 10, p_3 = 10$. From Figure 8.8, the ASEs for $p_1 = 24, p_2 = 10, p_3 = 10$ are (100%,80%,80%), (100%,100%,80%) and (100%,80%,100%) with payoff (25,12,12) (25,12,25), and (25,12,25), respectively. Thus, the three companies should select these strategies to achieved their market share target.

	γ_1			γ_2		
	β_1	β_2	β_3	β_1	β_2	β_3
α_1	0.08333	0.01333	0.00433	0.06667	0.06667	0.04762
α_2	0.20000	0.20000	0.00909	0.20000	0.09091	0.08333
α_3	0.00826	0.09091	0.14286	0.09091	1.00000	1.00000

		γ_3	
	β_1	β_2	β_3
α_1	0.04762	0.04762	0.05882
α_2	0.00909	0.06667	0.07692
α_3	0.14286	1.00000	0.20000

Figure 8.8 Aspiration scalar matrix of three car brands in the car industry for $p_1 = 24, p_2 = 10, p_3 = 10$.

Now assume that Mercedes has greater priority than BMW, and BMW has greater priority than Audi. Then, we calculate the priority scalar value matrix. From Figure 8.9, we obtain that PSE is at are (100%,40%,40%) with payoff (40,4,4).

	γ_1			γ_2		
	β_1	β_2	β_3	β_1	β_2	β_3
α_1	0.0000282	0.0000276	0.0000324	0.0000261	0.0000303	0.0000250
α_2	0.0000351	0.0000408	0.0000413	0.0000385	0.0000300	0.0000461
α_3	0.0006464	0.0001254	0.0000912	0.0001187	0.0000300	0.0000593

		γ_3	
	β_1	β_2	β_3
α_1	0.0000319	0.0000275	0.0000376
α_2	0.0000336	0.0000274	0.0000370
α_3	0.0000793	0.0000547	0.0000447

Figure 8.9 Priority scalar matrix of three car brands in the car industry when Mercedes has greater priority than BMW and BMW has greater priority than Audi.

We next assume that Mercedes forms one coalition into itself, while BMW and Audi form another coalition. We then calculate coalition scalar values in Figure 8.10. In Figure 8.10, we obtain the COSE are (100%,40%,40%) with payoff (40,4,4).

In summary, our method may applied to the real world situation to find the solution. We can use our SEs approach for finding solutions in pure strategies that depend on the criteria considered.

	γ_1			γ_2		
	β_1	β_2	β_3	β_1	β_2	β_3
α_1	0.00128	0.00124	0.00142	0.00124	0.00154	0.00169
α_2	0.00154	0.00183	0.00179	0.00183	0.00137	0.00137
α_3	0.02703	0.00529	0.00413	0.00529	0.00202	0.00278

		γ_3	
	β_1	β_2	β_3
α_1	0.00142	0.00169	0.00275
α_2	0.00179	0.00137	0.00216
α_3	0.00413	0.00278	0.00227

Figure 8.10 Coalition scalar matrix of three car brands in the car industry when Mercedes form the first coalition, and BMW and Audi form the second coalition.

CHAPTER 9

CONCLUSION

9.1 Summary

In the preceding chapters, a general scalarization approach for one-shot, n-person games has been presented by defining the notion of a Scalar Equilibrium. New scalar equilibria for existing solution concepts based on various notions of rationality have been presented for finding pure strategy solutions. We showed that REs, DEs, and JEs can be determined by defining Regret, Disappointment, and Joint Scalar Equilibria. These scalar equilibria are useful for finding pure strategies when pure REs, DEs, and JEs do not exist. Next, we presented the Maximin Scalarization Equilibria for the standard maximin solution concept. Computational examples were presented for these cases.

In addition, we proposed new Scalar Equilibria with various notions of rationality. Aspiration Scalar Equilibria are developed for an aspiration criterion when a player has a target payoff goal. Risk, Greedy and Cooperative Scalar Equilibrium were developed for risk, greed, and cooperative criteria, respectively. On the other hand, Sequential and Simultaneous Scalar Equilibria combine decision criteria. In a Sequential Scalar Equilibrium we sequentially, in some chosen order, apply other scalarizations to SEs of the game until we find a unique one if possible. In a Simultaneous Scalar Equilibrium we combine the criteria for various scalarizations into one. Effectively the multiple criteria are applied simultaneously.

Two further Scalar Equilibria were then presented. In a Priority Scalar Equilibrium, players are prioritized so that higher priority players get better payoffs than lower priority players. A Coalition Scalar Equilibrium next considers fixed teams of players seek team payoffs that are then divided among the players. Examples illustrated the usage and theoretical aspects of all Scalar Equilibria defined here.

9.2 Future work

Future work will develop further theory for the SEs of this dissertation. Further SEs will be developed for other notions of rationality, i.e., decision criteria. Scalarizations should also be developed so that each player may pursue an individual criterion possibly different from the other players' criteria. Empirical experiments should be presented to see how people accept the decision determined by Scalar Equilibria. Finally a computer program should be developed for the Scalar Equilibria so that SEs can be computed for games with large number of players.

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BIOGRAPHICAL INFORMATION

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