ON A CLASS OF STRONGLY NONLINEAR DIRICHLET BOUNDARY-VALUE PROBLEMS: BEYOND POHOZAEV'S RESULTS*†

by

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1. INTRODUCTION.

Let $\Omega \subset \mathbb{R}^n \ni (x_1;\ldots;x_n) \equiv x$ with $n \geq 2$ be an open bounded domain with closure $\bar{\Omega}$ and smooth boundary $\partial\Omega$. Consider the class of nonlinear boundary-value problems defined by

$$\begin{aligned} \Delta z + \lambda\Phi(z) &= 0 \quad \text{in} \quad \Omega \\ z &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{I.1}$$

with Φ regular enough, $\Phi(0) = 0$ and where Δ denotes Laplacian. If $n = 2$, it is known that the boundary-value problem (I.1) possesses nontrivial, classical eigensolutions (with appropriate eigenvalues λ) even if Φ grows exponentially fast; if $n \geq 3$, it is also known that problem (I.1) has, with Ω starshaped, no nontrivial, classical solutions as soon as Φ grows as fast as $|t|^{n+2/n-2}$, thus in particular if Φ grows exponentially fast (loss of compactness in Sobolev's embedding Theorem, see [1] and [2]). On the basis of simple energy considerations, it is however natural to expect that, if Δz in (I.1) is replaced by some sufficiently strong nonlinear term in the first-order partial derivatives z_{x_i}, say $\sum_{i=1}^{n} \{\phi_i(z_{x_i})\}_{x_i}$ for some suitably chosen ϕ_i's, one can restore the existence of nontrivial eigenfunctions in the boundary-value problem (I.1) for any dimension $n \geq 3$, even with an exponential growth in Φ (for a certain appropriate class of Φ's, see below). In this paper, we announce new results which precisely go in that direction. The proofs are ommitted and we refer the reader to [3] and [4] for complete details.

II. COUNTABLY MANY SOLUTIONS FOR A CLASS OF SMOOTH ISOPERIMETRIC VARIATIONAL PROBLEMS.

Let Y be a $C^{(2)}$-Young function, namely a twice continuously differentiable, even convex $\mathbb{R}^+$-valued function on $\mathbb{R}$ such that $\lim_{t\to\pm\infty} t^{-1}Y(t) = \pm\infty$ and

$\lim_{t\to 0} t^{-1}Y(t) = 0$. The following four definitions will be repeatedly used in the sequel.

<u>Definition II.1.</u> The $C^{(2)}$-Young function Y is said to satisfy <u>property P</u> if it is convex in t^2 in the sense of [5]: there exists a $C^{(2)}$-function H on $\mathbb{R}^+$ such that $Y(t) = H(y)$ with $y = t^2$ and H convex in y.

<u>Definition II.2.</u> The $C^{(2)}$-Young function Y is said to satisfy <u>property (Q)</u> it satisfies the following Young condition: there exist $\nu > 2$ and $t_0 \geq 0$ su that

$$Y(2t) \leq \nu Y(t)$$ (

for each $t \in \mathbb{R}$ such that $|t| \geq t_0$.

<u>Definition II.3.</u> Let $2 \leq n \in \mathbb{N}^+$; the $C^{(2)}$-function Y is said to satisfy <u>property (R_n)</u> if there exist $\kappa(n) \equiv \kappa > 0$ and $t_0(n) \equiv t_0 \geq 0$ such that

$$|t|^n \leq Y(\kappa t)$$ (

for every $t \in \mathbb{R}$ with $|t| \geq t_0$.

Finally, for $2 \leq n \in \mathbb{N}^+$, consider the Young function

$$Y_n(t) = e^{|t|^{n/n-1}} - 1$$ (

<u>Definition II.4.</u> The $C^{(2)}$-Young function Y is said to satisfy <u>property (S_n)</u> if it grows essentially more slowly than Y_n, namely if

$$\lim_{t\to\pm\infty} \frac{Y_n(\kappa t)}{Y(t)} = +\infty$$

for each $\kappa \in \mathbb{R}^+$.

For instance, it is easily seen that the two Young functions

$$Y(t) = \frac{|t|^q}{q} \qquad 2 \le n \le q < +\infty$$
$$Y(t) = e^{|t|} - |t| - 1 \tag{II.4}$$

both satisfy properties (P), (R_n) and (S_n) while only the first one satisfies property (Q).

We shall denote by $\tilde{Y}$ the Legendre transform of Y, namely

$$\tilde{Y}(s) = \max_{t \ge 0} \{|s|t - Y(t)\} \tag{II.5}$$

for each $s \in \mathbb{R}$ and by $\tilde{Y}^{-1}$ its monotone inverse on $\mathbb{R}^+$. For $\rho \in C(\bar{\Omega};\mathbb{R}^+)$, let $K_{Y,\rho}(\Omega) \equiv K_{Y,\rho}$ be the Orlicz class on Ω associated with Y and ρ; let $E_{Y,\rho}$ and $L_{Y,\rho}$ be the corresponding Orlicz spaces, equipped with Luxemburg's norm

$$\|u\|_{Y,\rho} = \inf \{k > 0: \int_\Omega dx\rho(x)Y(\frac{u(x)}{k}) \le 1\} \tag{II.6}$$

(see for instance [6] or [7] for the relevant background information about these spaces). It is known that the inclusions $E_{Y,\rho} \subseteq K_{Y,\rho} \subseteq L_{Y,\rho}$ are in general proper and that neither $E_{Y,\rho}$ nor $L_{Y,\rho}$ is reflexive. It is also known that $E_{Y,\rho}$ is separable whereas $L_{Y,\rho}$ is, in general, not (these statements hold, for instance, if Y is the second example of (II.4)). The spaces $E_{\tilde{Y},\rho}$, $K_{\tilde{Y},\rho}$ and $L_{\tilde{Y},\rho}$ are defined similarly. Let F be a real-valued function on $\Omega \times \mathbb{R}$ which satisfies the following <u>property (T)</u>:

(T) $F:(x,\tau) \longrightarrow F(x,\tau)$ is a Carathéodory function of (x,τ) and is odd in τ; moreover, there exist $f \in E_{\tilde{Y},\rho}$ and a constant $\eta > 0$ such that

$$|F(x;\tau)| \le f(x) + \eta\tilde{Y}^{-1}(Y(\tau)) \tag{II.7}$$

for each τ and almost each x (with respect to the measure ρdx).

For $i \in \{1,2,\ldots,n\}$, we next pick $\rho_i \in C(\bar{\Omega};\mathbb{R}^+)$ and n Young functions Y_i of class $C^{(2)}$; with $\hat{\rho} \equiv (\rho_i)_{1\le i\le n}$ and $\hat{Y} \equiv (Y_i)_{1\le i\le n}$, define the Banach space

$B_{Y,\hat{Y},\hat{p},\rho}$ of all real-valued functions $z \in E_{Y,\rho}$ with distributional derivative $z_{x_i} \in E_{Y_i,p_i}$, with respect to the norm

$$\|z\|^2_{Y,\hat{Y},\hat{p},\rho} = \|z\|^2_{Y,\rho} + \sum_{i=1}^{n} \|z_{x_i}\|^2_{Y_i,p_i} \quad (I$$

Write $\overset{\circ}{B}_{Y,\hat{Y},\hat{p},\rho}$ for the closure of $C^\infty_\circ(\Omega)$ with respect to the norm (II.8). Wh $\overset{\circ}{B}_{Y,\hat{Y},\hat{p},\rho}$ is not reflexive in general, it is separable since it can be isometri embedded into $\bigoplus_{i=1}^{n} E_{Y_i,p_i} \oplus E_{Y,\rho}$. We define on $\overset{\circ}{B}_{Y,\hat{Y},\hat{p},\rho}$ the two functionals

$$V(z) = \sum_{i=1}^{n} \int_\Omega dx p_i(x) Y_i(z_{x_i}(x)) + \int_\Omega dx G(x;z(x)) \quad ($$

and

$$C_\mu(z) = \int_\Omega dx \rho(x) Y(z(x)) - \mu \quad ($$

where $\mu \in (1;+\infty)$ and

$$G(x;\tau) = \int_0^\tau d\xi F(x;\xi) \quad ($$

We also define the (in general unbounded) set of contraints

$$K_\mu = \{z \in \overset{\circ}{B}_{Y,\hat{Y},\hat{p},\rho} : C_\mu(z) = 0\} \quad ($$

While both V and C_μ are real-valued, $C^{(1)}$-Fréchet differentiable functiona on $\overset{\circ}{B}_{Y,\hat{Y},\hat{p},\rho}$, our main result is the following (see [3], [4], and also [8]-[12 for complementary approaches to related problems).

Theorem II.1. Assume that properties (P), (Q) and (R_n) hold for the $C^{(2)}$-Youn functions Y_i and that (P), (R_n) and (S_n) hold for the $C^{(2)}$-Young function Y, $n \geq 2$. Assume moreover that F satisfies property (T). Then the functional sesses at least a countable infinity of distinct one-parameter families of pai $\{\pm z^{(\alpha)}_\mu\}_{\alpha \in \mathbb{N}^+}$ of critical points on K_μ, with critical values of the form

$$d_m = \inf_{A \in \Gamma_m} \sup_{z \in A} V(z) \quad ($$

In (II.13), the sets $(\Gamma_m)_{m \in \mathbb{N}^+}$ denote the sets of all closed, balanced and bounded subsets of K_μ with genus no less than m. (We have written m in (II.13) instead of α because of possible degeneracy.)

The essential ideas of the proof. The main strategy is to set up a suitable bypass around the lack of reflexivity of $\overset{\circ}{B}_{Y,\hat{Y},\hat{p},\rho}$ and around the difficulty of having to deal with an unbounded set of contraints. This is done using, among other things, new inequalities valid for all $C^{(2)}$-Young functions satisfying property ($\mathcal{P}$) and an embedding result by Trudinger [2], which both allow one to prove compactness for the critical levels of V on K_μ. This eventually leads to a new deformation theorem.

Remarks. (1). It is easily seen that the critical points $\{\pm z_\mu^{(\alpha)}\}_{\alpha \in \mathbb{N}^+}$ are distributional solutions to the real-valued elliptic boundary-value problem

$$\begin{aligned} \sum_{i=1}^{n} \{p_i(x) Y_i'(z_{x_i}(x)\}_{x_i} + \lambda\rho(x) Y'(z(x)) &= F(x;z(x)) \quad \text{in} \quad \Omega \\ z &= 0 \quad\quad \text{on} \quad \partial\Omega \end{aligned} \tag{II.14}$$

In fact, with reinforced regularity assumptions on the coefficients p_i, ρ and F and replacing $(\mathcal{R}_n)$ by $(\mathcal{R}_{n+\varepsilon})$ for some $\varepsilon > 0$, we can prove that the $\{\pm z_\mu^{(\alpha)}\}_{\alpha\in\mathbb{N}^+}$ are $C^{(2)}(\Omega) \cap C(\bar{\Omega})$-eigensolutions of (II.14) with appropriate eigenvalues λ_μ^α (see [3] and [5]).

(2). With $p_i \equiv \rho \equiv 1$ and $F = 0$, (II.14) is reduced to

$$\begin{aligned} \sum_{i=1}^{n} \{Y_i'(z_{x_i}(x))\}_{x_i} + \lambda Y'(z(x)) &= 0 \quad \text{in} \quad \Omega \\ z &= 0 \quad\quad \text{on} \quad \partial\Omega \end{aligned} \tag{II.15}$$

which is the modifed version of (I.1) discussed in the introduction if we choose $\phi_i = Y_i'$ and $\phi = Y'$. Thus the significance of Theorem (II.1) is that for any dimension $n \geq 3$, one can restore the existence of at least an infinite number of eigensolutions to the boundary-value problem (II.15), even with an exponentia growth in Y (controlled by (II.3) and (II.4)), provided that ϕ_i grows at least as fast as $|t|^n$ but no faster than polynomially (because of property ($\mathcal{Q}$) satisfied by the Y_i's).

(3). Whether Theorem (II.1) remains valid without property ($\mathcal{Q}$) (that is, with possible exponential growth in the principal part of the equation) is an open question at this time. A similar remark holds if one replaces (R_n) by (R_q) with $2 < q < n$ when $n \geq 3$.

(4). For other extensions or perturbations around Pohožaev's results, see also Professor Brézis' lectures in these proceedings.

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