A GENERALIZED APPROACH TO DARBOUX TRANSFORMATIONS FOR DIFFERENTIAL EQUATIONS

by

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Presented to the Faculty of the Graduate School of
The University of Texas at Arlington in Partial Fulfillment
of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT ARLINGTON

May 2014

Acknowledgements

I would like to thank my supervising professor Dr. Tuncay Aktosun for his guidance and encouragement throughout my research. I deeply appreciate his expertise and professionalism. He has inspired me all the time toward my degree. I would also like to thank Dr. Cornelis van der Mee of University of Cagliari for his help.

I would like to thank everyone in the Mathematics Department at the University of Texas at Arlington along with my committee members . I would also like to thank the University for the support through assistantship and Dean's Dissertation Fellowship granted by the Office of Graduate Studies.

I want to thank my wife, Songul Unlu, for her love and support. Finally, I would like to thank my mother and father for their encouragement, patience, and love. Only they know the challenges of this effort and always encourage me to complete this degree.

April 9, 2014

Abstract

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The University of Texas at Arlington, 2014

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A Darboux transformation is a mathematical procedure to produce a solution to a differential equation when the solution to a related differential equation is known. The basic idea behind a Darboux transformation is to change the discrete spectrum of a linear differential operator in a controlled way without changing its continuous spectrum. For example, by using a Darboux transformation one can describe the change in a quantum mechanical system when some of its quantum levels are removed or some extra quantum levels are added. Darboux transformation formulas for various differential equations have been developed, but such formulas seem to be specific to those particular equations without much connection among them. In our method, we develop a generalized and unified approach for Darboux transformations that is applicable to a large class of differential equations. This approach uses the solution to a linear integral equation where the kernel and nonhomogeneous terms coincide. We apply our unified approach to some specific differential equations such as the Schrödinger equation and the Zakharov-Shabat system, and we relate our results to the existing results in the literature. We also apply our results to obtain exact

solutions to various integrable nonlinear partial differential equations such as the Korteweg-de Vries equation and the nonlinear Schrödinger equation.

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Chapter 1

Introduction

The idea behind the Darboux transformation was originated in the work [17] by Théodore Florentin Moutard, a French mathematician, in 1875. Later in 1882, Jean Gaston Darboux presented the transformation in his paper [8] as a proposition to the second order differential equation today known as the Schrödinger equation. In 1889, Darboux published a book [9] and the transformation was shown in that book as well. In 1955, M. M. Crum [7] came up with a similar transformation for differential equations on a finite interval. In 1957, M. G. Krein [12] extended the ideas and he came up with a similar transformation for the Schrödinger equation on the half line. In 1979, V. B. Matveev [16] applied such transformations to integrable nonlinear partial differential equations.

The basic idea behind a Darboux transformation is finding a solution to a differential equation if a solution to a related differential equation is known, finding a solution to an integrable evolution equation if a solution to a related integrable evolution equation is known, or finding another solution to an integrable evolution equation if a solution to the same equation is known.

In this thesis we consider the spectral problem $\mathcal{L}\,\Psi = \lambda\,\Psi$, where $\mathcal{L}$ is a linear ordinary differential operator acting on some function space and λ is the spectral parameter. The spectrum of $\mathcal{L}$ consists of all λ-values for which there exists a nonzero solution Ψ, called a *wave function*. The operator $\mathcal{L}$ also usually contains a function $u(x)$ as a coefficient, called the *potential*.

The spectrum of $\mathcal{L}$ usually consists of two parts: the discrete spectrum and the continuous spectrum. We perturb the operator $\mathcal{L}$ into $\tilde{\mathcal{L}}$ in a such a way that the continuous spectra of $\mathcal{L}$ and $\tilde{\mathcal{L}}$ coincide while their discrete spectra differ by a finite number of eigenvalues. Under the transformation $\mathcal{L} \mapsto \tilde{\mathcal{L}}$, the potential is transformed as $u(x) \mapsto \tilde{u}(x)$ and the wave function changes as $\Psi(\lambda, x) \mapsto \tilde{\Psi}(\lambda, x)$. Note that we use a tilde to denote the corresponding perturbed quantity.

After the perturbation, the unperturbed spectral problem $\mathcal{L}\,\Psi = \lambda\,\Psi$ is changed to the perturbed problem $\tilde{\mathcal{L}}\,\tilde{\Psi} = \lambda\,\tilde{\Psi}$. One-to-one correspondences for $\mathcal{L}\,\Psi = \lambda\,\Psi$ and $\tilde{\mathcal{L}}\,\tilde{\Psi} = \lambda\,\tilde{\Psi}$ are outlined in the following diagram by using two-sided arrows:

linear differential operator	$\mathcal{L}$	$\tilde{\mathcal{L}}$
$\updownarrow$	$\updownarrow$	$\updownarrow$
potential	$u(x)$	$\tilde{u}(x)$
$\updownarrow$	$\updownarrow$	$\updownarrow$
scattering data	$S(\lambda)$	$\tilde{S}(\lambda)$
$\updownarrow$	$\updownarrow$	$\updownarrow$
fundamental integral operator	$\Omega(x)$	$\tilde{\Omega}(x)$
$\updownarrow$	$\updownarrow$	$\updownarrow$
solution to the fundamental integral equation	$\alpha(x, y)$	$\tilde{\alpha}(x, y)$
$\updownarrow$	$\updownarrow$	$\updownarrow$
wave function	$\Psi(\lambda, x)$	$\tilde{\Psi}(\lambda, x)$

Diagram 1: Equivalent quantities for the unperturbed and perturbed problems

The Darboux transformation has two parts: at the potential level and at the wave function level. At the potential level, the Darboux transformation consists of determining $\tilde{u}(x)$ in terms of $u(x)$ and of the quantities evaluated at the discrete λ-eigenvalues appearing in the perturbation. At the wave function level, the Darboux transformation consists of determining $\tilde{\Psi}(\lambda, x)$ in terms of $\Psi(\lambda, x)$ and of the quantities evaluated at the discrete λ-eigenvalues appearing in the perturbation.

Our main goal in this thesis is to develop a unified and fundamental approach to Darboux transformations for a wide class of linear differential operators. This is done with help of a fundamental integral equation, related to the so-called Marchenko integral equations or the Gel'fand-Levitan integral equation.

This thesis is organized as follows. In Chapter 2 we obtain the Darboux transformation with the help of the (left) Marchenko integral equation [4, 6, 13, 14]. In Chapter 3 we obtain the Darboux transformation with the help of the (right) Marchenko integral equation [4, 6, 13, 14]. In Chapter 4 we obtain the Darboux transformation with the help of the Gel'fand-Levitan equation [4, 6, 13, 14].

The (left) Marchenko integral equation is used when the wave function $\Psi(\lambda, x)$ is specified through some asymptotic conditions at $x = +\infty$. Hence, the procedure given in Chapter 2 is suitable to obtain the Darboux transformation at the wave function level for wave functions defined via asymptotic conditions at $x = +\infty$.

Similarly, the (right) Marchenko integral equation is used when the wave function $\Psi(\lambda, x)$ is specified through some asymptotic conditions at $x = -\infty$. Hence, the procedure given in Chapter 3 is suitable to obtain the Darboux transformation at the wave function level for wave functions defined via asymptotic conditions at $x = -\infty$.

As for remaining wave functions, such wave functions are usually specified by using some initial values at a specific point on the x-axis. Without loss of any generality, that point can be chosen as $x = 0$. In Chapter 4 we obtain the Darboux transformation with the help of the Gel'fand-Levitan equation [4, 6, 13, 14]. The Gel'fand-Levitan equation is used when when the wave function $\Psi(\lambda, x)$ is specified via some initial conditions at $x = 0$. Hence, the procedure given in Chapter 4 is suitable to obtain the Darboux transformation at the wave function level for wave functions defined via initial condition at $x = 0$.

Chapter 2

(Left) Marchenko Integral Equation

In this chapter we develop a generalized approach for the Darboux transformation via the (left) Marchenko integral equation. We first present some preliminary results that are needed later on and then we analyze the resolvent kernel $r(x;y,z)$ of the (left) Marchenko integral equation. In Theorem 2.1.2 we prove that $r(x;y,z)$ can be expressed explicitly in terms of the solution $\alpha(x,y)$ to the (left) Marchenko integral equation (2.0.9). We then give the formulas for the Darboux transformation at the potential level and wave function level.

Let us consider the integral equation

$$\alpha(x,y)+\beta(x,y)+\int_x^\infty dz\,\alpha(x,z)\,\omega(z,y)=0, \qquad y>x, \tag{2.0.1}$$

where $\alpha(x,y)$ is the unknown term, $\beta(x,y)$ is the nonhomogeneous term, and $\omega(z,y)$ is the integral kernel. We assume that the equation given in (2.0.1) is uniquely solvable in $\mathcal{H}_2^{N\times N}$, namely, we get a unique $\alpha(x,y)$ when $\beta(x,y)$ and $\omega(z,y)$ are given to us. Here, $\mathcal{H}_p^{M\times N}$ is the complex Banach space of $M\times N$ matrix-valued measurable functions $F:(x,+\infty)\to\mathbb{C}^{M\times N}$ such that the matrix norm $\|F(\cdot)\|$ belongs to $L^p(x,+\infty)$ for $1\le p\le+\infty$. Note that the kernel $\omega(z,y)$ does not depend on the parameter $x\in R$ and it satisfies

$$\sup_{y>x}\int_x^\infty dz\,(\|\omega(z,y)\|+\|\omega(y,z)\|)<+\infty, \tag{2.0.2}$$

where $\|\cdot\|$ denotes any $N\times N$- matrix norm. Let us write (2.0.1) in the operator form

$$\alpha+\beta+\alpha\,\Omega=0, \tag{2.0.3}$$

where the operator Ω acts from the right. We suppose that, for each fixed $x \in R$, the operator $(I + \Omega)$ is invertible on $\mathcal{H}_1^{N\times N}$ and $\mathcal{H}_2^{N\times N}$. Note that I denotes the identity operator. We use $(I + R)$ to denote the corresponding resolvent operator, where

$$I + R = (I + \Omega)^{-1}, \qquad R := (I + \Omega)^{-1} - I. \tag{2.0.4}$$

By solving (2.0.3) for α we get

$$\alpha = -\beta\,(I + \Omega)^{-1}. \tag{2.0.5}$$

Then by substituting (2.0.4) into (2.0.5), we obtain

$$\alpha = -\beta(I + R), \tag{2.0.6}$$

or explicitly

$$\alpha(x, y) = -\beta(x, y) - \int_x^\infty dz\,\beta(x, z)\,r(x; z, y), \tag{2.0.7}$$

where $r(x; z, y)$ is the integral kernel of the resolvent operator R. Let us write (2.0.1) in the form

$$\alpha + \omega + \alpha\,\Omega = 0, \tag{2.0.8}$$

where the nonhomogeneous term and the integral kernel are related to each other. Then (2.0.8) can explicitly be written as

$$\alpha(x, y) + \omega(x, y) + \int_x^\infty dz\,\alpha(x, z)\,\omega(z, y) = 0, \qquad y > x, \tag{2.0.9}$$

which is usually called [3, 4, 6, 10, 13, 14, 18] the (left) Marchenko integral equation. We make the assumption that the (left) Marchenko integral equation is uniquely solvable in $\mathcal{H}_2^{N\times N}$, namely, we assume that there is a unique $\alpha(x, y)$ as a solution to (2.0.9) when $\omega(x, y)$ is given. We construct the resolvent kernel $r(x; z, y)$ appearing in (2.0.7) in terms of the unique $\alpha(x, y)$. Since $\alpha(x, y)$ is uniquely determined, we will then obtain $r(x; z, y)$ uniquely. By solving (2.0.8) with help of (2.0.4) we get

$$\alpha = -\omega(I + R). \tag{2.0.10}$$

The unique solvability of (2.0.9) in $\mathcal{H}_1^{N\times N}$ and the condition in (2.0.2) imply that

$$\sup_{y>x} \int_x^\infty dz\,(\|\alpha(z,y)\| + \|\alpha(y,z)\|) < +\infty. \tag{2.0.11}$$

We consider (2.0.8) when the integral operator Ω appearing in (2.0.3) is $N\times N$ matrix-valued and J-selfadjoint in the sense that

$$\Omega = J\,\Omega^\ddagger J, \qquad \omega(y,z) = J\,\left[\omega(z,y)\right]^\dagger J, \tag{2.0.12}$$

where the single dagger denotes the matrix adjoint (complex conjugate and matrix transpose) and the double dagger denotes not just only the matrix adjoint, but also switching the two arguments in the kernel of the operator, namely,

$$T^\ddagger(x,y) := T^\dagger(y,x), \tag{2.0.13}$$

for any operator T with the corresponding kernel $T(x,y)$. For any two integral operators A and B, we then get

$$\begin{aligned}(AB)^\ddagger(y,z) &= (AB)^\dagger(z,y)\\ &= \left[\int ds\, A(z,s)\,B(s,y)\right]^\dagger,\end{aligned}$$

or equivalently

$$(AB)^\ddagger(y,z) = \int ds\, B^\dagger(s,y)\,A^\dagger(z,s). \tag{2.0.14}$$

By using the property given in (2.0.13), we can rewrite (2.0.14) as

$$(AB)^\ddagger(y,z) = \int ds\, B^\ddagger(y,s)\,A^\ddagger(s,z), \tag{2.0.15}$$

which yields

$$(AB)^\ddagger(y,z) = (B^\ddagger A^\ddagger)(y,z). \tag{2.0.16}$$

Note also that J is an $N\times N$ selfadjoint involution, i.e.

$$J = J^\dagger = J^{-1}. \tag{2.0.17}$$

For instance, J may be assumed in the form of

$$J := \begin{bmatrix} I_j & 0 \\ 0 & -I_{N-j} \end{bmatrix},$$

where I_j is the $j \times j$ identity matrix for some $1 \le j \le N$.

Associated with the unperturbed problem $\mathcal{L}\,\Psi = \lambda\,\Psi$, we have [3, 4, 6, 10, 13, 14, 18] the fundamental integral equation given in (2.0.8), where α is related to the Fourier transform of Ψ and ω is related to the Fourier transform of the scattering data $S(\lambda)$ for the operator $\mathcal{L}$.

Associated with the perturbed problem $\tilde{\mathcal{L}}\,\tilde{\Psi} = \lambda\,\tilde{\Psi}$, we have the fundamental integral equation

$$\tilde{\alpha} + \tilde{\omega} + \tilde{\alpha}\,\tilde{\Omega} = 0, \tag{2.0.18}$$

or explicitly

$$\tilde{\alpha}(x,y) + \tilde{\omega}(x,y) + \int_x^\infty dz\,\tilde{\alpha}(x,z)\,\tilde{\omega}(z,y) = 0, \qquad x < y. \tag{2.0.19}$$

In the analysis of Darboux transformations, the perturbation will correspond to the case where the integral operators $\tilde{\Omega}$ and Ω appearing in (2.0.18) and (2.0.8), respectively, differ by a finite-rank operator and we denote that difference operator by FG, i.e.

$$\tilde{\Omega} = \Omega + FG, \qquad \tilde{\omega}(x,y) = \omega(x,y) + f(x)\,g(y). \tag{2.0.20}$$

Furthermore, note that we cannot in general expect F and G to commute, and hence in general $fg \neq gf$.

2.1 Construction of the Resolvent Kernel

In this section, we analyze the resolvent kernel $r(x;y,z)$ appearing in (2.0.7) and then we show that $r(x;y,z)$ can be expressed explicitly in terms of the solution

$\alpha(x,y)$ to (2.0.9).

Proposition 2.1.1 *Assume that* (2.0.3) *is uniquely solvable in* $\mathcal{H}_2^{N\times N}$ *and that* Ω *satisfies* (2.0.12). *Then, the operator* R *given in* (2.0.4) *and the corresponding kernel* $r(x;y,z)$ *appearing in* (2.0.7) *satisfy*

$$R = J\,R^{\ddagger}J, \qquad r(x;y,z) = J\,[r(x;z,y)]^{\dagger}\,J, \tag{2.1.1}$$

where J *is the involution matrix appearing in* (2.0.17).

Proof. From (2.0.4) we know that $(I+R) = (I+\Omega)^{-1}$ and thus

$$(I+\Omega)(I+R) = I = (I+R)(I+\Omega). \tag{2.1.2}$$

Then we get

$$R + \Omega + R\,\Omega = 0, \tag{2.1.3}$$

and

$$R + \Omega + \Omega\,R = 0. \tag{2.1.4}$$

By applying on (2.1.3) the double dagger transformation defined in (2.0.17), i.e. by taking the adjoint and switching the arguments, and further applying J on both sides, we obtain

$$JR^{\ddagger}J + J\Omega^{\ddagger}J + (J\Omega^{\ddagger}J)(JR^{\ddagger}J) = 0, \tag{2.1.5}$$

equivalently, by using (2.0.12) we get

$$JR^{\ddagger}J + \Omega + \Omega\,(JR^{\ddagger}J) = 0. \tag{2.1.6}$$

Since (2.0.3) is assumed to be uniquely solvable in $\mathcal{H}_2^{N\times N}$, by comparing (2.1.4) and (2.1.6) we get $R = JR^{\ddagger}J$.

Even though it is clear that $R = JR^{\ddagger}J$ implies $r(x;y,z) = J\,[r(x;z,y)]^{\dagger}\,J$, for clarity, let us also prove the same result by directly working with the kernel $r(x;y,z)$

for $y < z$ and $z < y$. Let us first consider (2.1.4) for $y < z$, which is explicitly written as

$$r(x; y, z) + \omega(y, z) + \int_x^\infty ds\, \omega(y, s)\, r(x; s, z) = 0, \qquad y < z. \tag{2.1.7}$$

By taking the adjoint of (2.1.7) and then multiplying the resulting equation by J on the left and on the right, we obtain

$$J\, [r(x; y, z)]^\dagger\, J + J\, [\omega(y, z)]^\dagger\, J + \int_x^\infty ds\, J\, [\omega(y, s)\, r(x; s, z)]^\dagger\, J, \qquad y < z, \tag{2.1.8}$$

or equivalently

$$J\, [r(x; y, z)]^\dagger\, J + J\, [\omega(y, z)]^\dagger\, J + \int_x^\infty ds\, J\, [r(x; s, z)]^\dagger\, J\, J\, [\omega(y, s)]^\dagger\, J, \qquad y < z. \tag{2.1.9}$$

Using (2.0.12) in (2.1.9) we get

$$J\, [r(x; y, z)]^\dagger\, J + \omega(z, y) + \int_x^\infty ds\, J\, [r(x; s, z)]^\dagger\, J\, \omega(s, y), \qquad y < z. \tag{2.1.10}$$

By interchanging y and z in (2.1.10), we obtain the equivalent expression

$$J\, [r(x; z, y)]^\dagger\, J + \omega(y, z) + \int_x^\infty ds\, J\, [r(x; s, y)]^\dagger\, J\, \omega(s, z), \qquad z < y. \tag{2.1.11}$$

By comparing (2.1.7) and (2.1.11) and by using the uniqueness of the solution to (2.1.7), we see that $r(x; y, z) = J\, [r(x; z, y)]^\dagger\, J$ and we have

$$r(x; y, z) + \omega(y, z) + \int_x^\infty ds\, r(x; y, s)\, \omega(s, z), \qquad z < y. \tag{2.1.12}$$

Similarly, by considering (2.1.4) for $z < y$, we obtain

$$r(x; y, z) + \omega(y, z) + \int_x^\infty ds\, r(x; y, s)\, \omega(s, z), \qquad y < z. \tag{2.1.13}$$

Thus, we have shown that $r(x; y, z) = J\, [r(x; z, y)]^\dagger\, J$ for $y < z$ and $z < y$, which completes the proof. ∎

We will give the first key result in the next theorem. This result is stated in [2] and a proof is given in [2] when $x < y < z$, and it is stated in [2] that the proof is similarly obtained for $x < z < y$. We provide the full proof by showing all details.

Theorem 2.1.2 *Assume that* (2.0.3) *is uniquely solvable in* $\mathcal{H}_2^{N\times N}$ *and that* Ω *satisfies* (2.0.12). *Then, the corresponding kernel* $r(x; y, z)$ *appearing in* (2.0.7) *can be expressed explicitly in terms of the solution* $\alpha(x, y)$ *to* (2.0.9) *as*

$$r(x; y, z) = \begin{cases} \alpha(y, z) + \int_x^y ds\, J\, [\alpha(s, y)]^\dagger J\, \alpha(s, z), & x < y < z, \\ J\, [\alpha(z, y)]^\dagger J + \int_x^z ds\, J\, [\alpha(s, y)]^\dagger J\, \alpha(s, z), & x < z < y, \end{cases} \tag{2.1.14}$$

where J *is the involution matrix appearing in* (2.0.12).

Proof. Since (2.0.3) is assumed to be uniquely solvable in $\mathcal{H}_2^{N\times N}$, so is (2.0.8) and hence the solution R to (2.1.3) is unique. Thus, it suffices to prove that the quantity defined in (2.1.14) satisfies (2.1.3), i.e. the quantity in (2.1.14) satisfies the integral equation

$$r(x; y, z) + \omega(y, z) + \int_x^\infty ds\, r(x; y, s)\, \omega(s, z) = 0, \qquad x < \min\{y, z\}. \tag{2.1.15}$$

We will give the proof for both cases of $y < z$ and $z < y$.

Case 1: $x < y < z$

Let us use $\int_x^\infty = \int_x^y + \int_y^\infty$ in the integral appearing in (2.1.15). Then the left-hand side in (2.1.15) becomes

$$r(x; y, z) + \omega(y, z) + \int_x^y ds\, r(x; y, s)\, \omega(s, z) + \int_y^\infty ds\, r(x; y, s)\, \omega(s, z). \tag{2.1.16}$$

By using the first line of (2.1.14) in the integral $\int_y^\infty$ and the second line of (2.1.14) in the integral $\int_x^y$ in (2.1.16), we obtain as the left-hand side of (2.1.15)

$$\alpha(y,z)+\int_x^y ds\, J\,[\alpha(s,y)]^\dagger\, J\,\alpha(s,z)+\omega(y,z)$$
$$+\int_y^\infty ds\left[\alpha(y,s)+\int_x^y dt\, J\,[\alpha(t,y)]^\dagger\, J\,\alpha(t,s)\right]\omega(s,z)$$
$$+\int_x^y ds\left[J\,[\alpha(s,y)]^\dagger\, J+\int_x^s dt\, J\,[\alpha(t,y)]^\dagger\, J\,\alpha(t,s)\right]\omega(s,z).$$

After the distribution, the left-hand side of (2.1.15) then becomes

$$\alpha(y,z)+\int_x^y ds\, J\,[\alpha(s,y)]^\dagger\, J\,\alpha(s,z)+\omega(y,z)$$
$$+\int_y^\infty ds\,\alpha(y,s)\,\omega(s,z)+\int_y^\infty ds\int_x^y dt\, J\,[\alpha(t,y)]^\dagger\, J\,\alpha(t,s)\,\omega(s,z)$$
$$+\int_x^y ds\, J\,[\alpha(s,y)]^\dagger\, J\,\omega(s,z)+\int_x^y ds\int_x^s dt\, J\,[\alpha(t,y)]^\dagger\, J\,\alpha(t,s)\,\omega(s,z).$$

Let us now define a_1, a_2, and a_3 as

$$a_1:=\alpha(y,z)+\omega(y,z)+\int_y^\infty ds\,\alpha(y,s)\,\omega(s,z),$$
$$a_2:=\int_x^y dt\, J\,[\alpha(t,y)]^\dagger\, J\,\alpha(t,z)+\int_x^y dt\, J\,[\alpha(t,y)]^\dagger\, J\,\omega(t,z),$$
$$a_3:=\int_y^\infty ds\int_x^y dt\, J\,[\alpha(t,y)]^\dagger\, J\,\alpha(t,s)\,\omega(s,z)$$
$$+\int_x^y ds\int_x^s dt\, J\,[\alpha(t,y)]^\dagger\, J\,\alpha(t,s)\,\omega(s,z).$$

We have $a_1=0$ because of the (left) Marchenko integral equation given in (2.0.9). The orders of the two iterated integrals in a_3 can be changed to $\int_x^y dt\int_y^\infty ds$ and $\int_x^y dt\int_t^y ds$, respectively. Using $\int_t^y+\int_y^\infty=\int_t^\infty$, the quantity a_3 becomes

$$a_3=\int_x^y dt\int_t^\infty ds\, J\,[\alpha(t,y)]^\dagger\, J\,\alpha(t,s)\,\omega(s,z).$$

Then we get

$$a_2+a_3=\int_x^y dt\, J\,[\alpha(t,y)]^\dagger\, J\left[\alpha(t,z)+\omega(t,z)+\int_t^\infty ds\,\alpha(t,s)\,\omega(s,z)\right]. \tag{2.1.17}$$

Since $z > t$, the quantity inside the brackets in (2.1.17) vanishes due to (2.0.9). Thus, we obtain $a_2 + a_3 = 0$, and hence $a_1 + a_2 + a_3 = 0$ for the case of $x < y < z$.

Case 2: $x < z < y$

Our goal is to show that the expression given in (2.1.14) satisfies (2.1.15) in the case $x < z < y$. A direct proof in the case $x < z < y$ does not seem to be feasible as it was done in the case $x < y < z$. Therefore, we will proceed as follows. Since any compact operator can be approximated by a sequence of finite-rank operators, we will approximate $\omega(y, z)$ in (2.1.15) by the sequence $\xi_n(y)\,\eta_n(z)$. This will allow us to solve (2.1.15) explicitly and then to get a solution. We will call that solution as $r_n(x; y, z)$. After approximation, we will obtain (2.1.14) where $\alpha(y, z)$ is replaced by $\alpha_n(y, z)$ which is the solution to the (left) Marchenko integral equation given in (2.1.18) when $\omega(y, z)$ is replaced with $\xi_n(y)\,\eta_n(z)$. Finally, by taking the limit when $n \to +\infty$, we will obtain $r(x; y, z)$ given in (2.1.14).

Consider the (left) Marchenko integral equation

$$\alpha(y, z) + \omega(y, z) + \int_y^\infty ds\, \alpha(y, s)\, \omega(s, z) = 0, \qquad z > y, \tag{2.1.18}$$

which obtained by replacing x and y in (2.0.9) by y and z, respectively. Suppose

$$\omega(y, z) = \xi(y)\, \eta(z), \tag{2.1.19}$$

i.e. suppose that the integral kernel in (2.1.14) is separable. We first would like to evaluate $\alpha(y, z)$ appearing in (2.1.14) in terms of the kernel parts ξ and η. Isolating $\alpha(y, z)$ in (2.1.18), we get

$$\alpha(y, z) = -\omega(y, z) - \int_y^\infty ds\, \alpha(y, s)\, \omega(s, z). \tag{2.1.20}$$

Substituting (2.1.19) into (2.1.20) we obtain

$$\alpha(y, z) = -\xi(y)\, \eta(z) - \int_y^\infty ds\, \alpha(y, s)\, \xi(s)\, \eta(z), \tag{2.1.21}$$

or equivalently

$$\alpha(y,z) = -\left[\xi(y) + \int_y^\infty ds\, \alpha(y,s)\, \xi(s)\right] \eta(z). \tag{2.1.22}$$

Define $\tau(y)$ as

$$\tau(y) := \xi(y) + \int_y^\infty ds\, \alpha(y,s)\, \xi(s). \tag{2.1.23}$$

Then we get

$$\alpha(y,z) = -\tau(y)\, \eta(z), \tag{2.1.24}$$

By replacing $\alpha(y,z)$ appearing in the integrand in (2.1.23) with (2.1.24) we obtain

$$\tau(y) = \xi(y) - \int_y^\infty ds\, \tau(y)\, \eta(s)\, \xi(s), \tag{2.1.25}$$

which yields

$$\tau(y) \left[I + \int_y^\infty ds\, \eta(s)\, \xi(s)\right] = \xi(y), \tag{2.1.26}$$

where I is the identity matrix. Thus we obtain

$$\tau(y) = \xi(y) \left[I + \int_y^\infty ds\, \eta(s)\, \xi(s)\right]^{-1}, \tag{2.1.27}$$

provided the matrix inverse exists. Consequently,

$$\alpha(y,z) = -\xi(y) \left[I + \int_y^\infty ds\, \eta(s)\, \xi(s)\right]^{-1} \eta(z). \tag{2.1.28}$$

Finally, define

$$\Upsilon(y) := I + \int_y^\infty ds\, \eta(s)\, \xi(s). \tag{2.1.29}$$

Hence, by substituting (2.1.29) into (2.1.28) we obtain

$$\alpha(y,z) = -\xi(y) \Upsilon(y)^{-1} \eta(z). \tag{2.1.30}$$

We would now like to evaluate $J\,[\alpha(z,y)]^\dagger\, J$ in terms of ξ and η. For this, let us consider (2.1.18). By taking the adjoint of (2.1.18) and then multiplying the resulting equation by J on the left and on the right, we get

$$J\,[\alpha(y,z)]^\dagger\, J + J\,[\omega(y,z)]^\dagger\, J + \int_y^\infty ds\, J\,[\alpha(y,s)\,\omega(s,z)]^\dagger\, J = 0, \tag{2.1.31}$$

or equivalently

$$J\,[\alpha(y,z)]^\dagger\,J + J\,[\omega(y,z)]^\dagger\,J + \int_y^\infty ds\,J\,[\omega(s,z)]^\dagger\,J\,J\,[\alpha(y,s)]^\dagger\,J = 0. \tag{2.1.32}$$

Using (2.0.12) in (2.1.32) we get

$$J\,[\alpha(y,z)]^\dagger\,J + \omega(z,y) + \int_y^\infty ds\,\omega(z,s)\,J\,[\alpha(y,s)]^\dagger\,J = 0. \tag{2.1.33}$$

By substituting (2.1.19) into (2.1.33) we obtain

$$J\,[\alpha(y,z)]^\dagger\,J + \xi(z)\eta(y) + \int_y^\infty ds\,\xi(z)\eta(s)\,J\,[\alpha(y,s)]^\dagger\,J = 0. \tag{2.1.34}$$

Let us set

$$J\,[\alpha(y,z)]^\dagger\,J = \xi(z)K(y), \tag{2.1.35}$$

with $K(y)$ to be determined. Substituting (2.1.35) into (2.1.34) and then equating the right coefficients of $\xi(z)$ from both sides of (2.1.34), we get

$$K(y) + \eta(y) + \int_y^\infty ds\,\eta(s)\,\xi(s)\,K(y) = 0. \tag{2.1.36}$$

By solving (2.1.36) for $K(y)$ we obtain

$$K(y) = -\left[I + \int_y^\infty ds\,\eta(s)\,\xi(s)\right]^{-1}\eta(y). \tag{2.1.37}$$

Using (2.1.29) in (2.1.37) we get

$$K(y) = -\Upsilon(y)^{-1}\,\eta(y). \tag{2.1.38}$$

By substituting (2.1.38) into (2.1.35) we obtain

$$J\,[\alpha(y,z)]^\dagger\,J = -\xi(z)\,\Upsilon(y)^{-1}\,\eta(y). \tag{2.1.39}$$

Hence, by interchanging y and z in (2.1.39) we obtain

$$J\,[\alpha(z,y)]^\dagger\,J = -\xi(y)\,\Upsilon(z)^{-1}\,\eta(z). \tag{2.1.40}$$

Now, let us solve the integral equation

$$r(x;y,z)+\omega(y,z)+\int_x^\infty ds\, r(x;y,s)\,\omega(s,z)=0. \tag{2.1.41}$$

Set

$$r(y,z):=H(y)\,\eta(z), \tag{2.1.42}$$

with $H(y)$ to be determined. Substituting (2.1.19) and (2.1.42) into (2.1.41) and then equating the left coefficients of $\eta(z)$ from both sides of (2.1.41), we get

$$H(y)+\xi(y)+\int_x^\infty ds\, H(y)\,\eta(s)\,\xi(s)=0,$$

which yields

$$H(y)\left[I+\int_x^\infty ds\,\eta(s)\,\xi(s)\right]=-\xi(y). \tag{2.1.43}$$

Then we get

$$H(y)=-\xi(y)\left[I+\int_x^\infty ds\,\eta(s)\,\xi(s)\right]^{-1}. \tag{2.1.44}$$

As a result

$$r(x;y,z)=-\xi(y)\left[I+\int_x^\infty ds\,\eta(s)\,\xi(s)\right]^{-1}\eta(z). \tag{2.1.45}$$

From (2.1.29) we see that the matrix in the brackets in (2.1.45) is equal to $\Upsilon(x)$. Thus we can write (2.1.45) as

$$r(x;y,z)=-\xi(y)\Upsilon(x)^{-1}\eta(z). \tag{2.1.46}$$

Then for $x<y<z$, with help of (2.1.30) and (2.1.40) we compute

$$\begin{aligned}
\alpha(y,z)&+\int_x^y ds\, J\,[\alpha(s,y)]^\dagger\, J\,\alpha(s,z)\\
&=-\xi(y)\,\Upsilon(y)^{-1}\,\eta(z)+\int_x^y ds\,\xi(y)\,\Upsilon(s)^{-1}\,\eta(s)\,\xi(s)\,\Upsilon(s)^{-1}\,\eta(z)\\
&=-\xi(y)\left[\Upsilon(y)^{-1}-\int_x^y ds\,\Upsilon(s)^{-1}\,\eta(s)\,\xi(s)\,\Upsilon(s)^{-1}\right]\eta(z)\\
&=-\xi(y)\left[\Upsilon(y)^{-1}-\int_x^y ds\left(\frac{d}{ds}\left[I+\int_s^\infty dt\,\eta(t)\,\xi(t)\right]^{-1}\right)\right]\eta(z).
\end{aligned} \tag{2.1.47}$$

We can write (2.1.47) as

$$\begin{aligned}\alpha(y,z)+\int_x^y ds\, J\,[\alpha(s,y)]^\dagger\, J\,\alpha(s,z) &= -\xi(y)\left\{\Upsilon(y)^{-1}-\left[\Upsilon(y)^{-1}-\Upsilon(x)^{-1}\right]\right\}\eta(z)\\ &= -\xi(y)\,\Upsilon(x)^{-1}\eta(z),\end{aligned}$$

which is equal to $r(x;y,z)$ because of (2.1.46). Similarly, for $x<z<y$ we compute

$$\begin{aligned}J\,[\alpha(z,y)]^\dagger\, J+\int_x^z ds\, J\,[\alpha(s,y)]^\dagger\, J\,\alpha(s,z) &= -\xi(y)\,\Upsilon(z)^{-1}\,\eta(z)+\int_x^z ds\,\xi(y)\,\Upsilon(s)^{-1}\,\eta(s)\,\xi(s)\,\Upsilon(s)^{-1}\,\eta(z)\\ &= -\xi(y)\left[\Upsilon(z)^{-1}-\int_x^z ds\,\Upsilon(s)^{-1}\,\eta(s)\,\xi(s)\,\Upsilon(s)^{-1}\right]\eta(z)\\ &= -\xi(y)\left[\Upsilon(z)^{-1}-\int_x^z ds\left(\frac{d}{ds}\left[I+\int_s^\infty dt\,\eta(t)\,\xi(t)\right]^{-1}\right)\right]\eta(z)\\ &= -\xi(y)\left\{\Upsilon(z)^{-1}-\left[\Upsilon(z)^{-1}-\Upsilon(x)^{-1}\right]\right\}\eta(z)\\ &= -\xi(y)\,\Upsilon(x)^{-1}\,\eta(z),\end{aligned}$$

which is equal to $r(x;y,z)$ because of (2.1.46). Thus we have shown that

$$r(x;y,z)=\begin{cases}\alpha(y,z)+\displaystyle\int_x^y ds\, J\,[\alpha(s,y)]^\dagger\, J\,\alpha(s,z), & x<y<z,\\ J\,[\alpha(z,y)]^\dagger\, J+\displaystyle\int_x^z ds\, J\,[\alpha(s,y)]^\dagger\, J\,\alpha(s,z), & x<z<y,\end{cases}\tag{2.1.48}$$

where we have obtained $\alpha(y,z)$ and $[\alpha(z,y)]^\dagger$ in terms of the kernel parts ξ and η. The above calculations could have been done on the following four spaces:

a. $L^1((x,+\infty)\to\mathbb{C}^N)$: $\xi\in L^1$ and $\eta\in L^\infty$.

b. $L^p((x,+\infty)\to\mathbb{C}^N)$: $\xi\in L^p$ and $\eta\in L^{p/(p-1)}$.

c. $L^\infty((x,+\infty)\to\mathbb{C}^N)$: $\xi\in L^\infty$ and $\eta\in L^1$.

d. $L^2((x,+\infty)\to\mathbb{C}^N)$: $\xi\in L^2$ and $\eta\in L^2$.

In all of these cases, the integrals $\int_x^\infty ds\, \eta(s)\, \xi(s)$ are absolutely convergent.

As a summary, in the case $x < z < y$, we can approximate the kernel of the Marchenko integral equation in (2.1.18), namely, $\omega(y,z)$ by the sequence of separable kernels $\xi_n(y)\,\eta_n(z)$, where the approximation is understood in the sense

$$\lim_{n\to+\infty} \operatorname*{ess\,sup}_{z>x} \int_x^\infty dy\, \|\omega(y,z) - \xi_n(y)\,\eta_n(z)\| = 0, \tag{2.1.49}$$

i.e., let us assume that

$$\lim_{n\to+\infty} \|\Omega - \Omega_n\|_{L^1\to L^1} = 0. \tag{2.1.50}$$

Since we then also have

$$\lim_{n\to+\infty} \|(I+\Omega)^{-1} - (I+\Omega_n)^{-1}\|_{L^1\to L^1} = 0, \tag{2.1.51}$$

we get for the resolvent kernels

$$\lim_{n\to+\infty} \operatorname*{ess\,sup}_{z>x} \int_x^\infty dy\, \|r(x;y,z) - r_n(x;y,z)\| = 0. \tag{2.1.52}$$

Recall that $\|\cdot\|$ is any $N\times N$-matrix norm. By using the results in the degenerate case, we get

$$r_n(x;y,z) = \begin{cases} \alpha_n(y,z) + \displaystyle\int_x^y ds\, J\, [\alpha_n(s,y)]^\dagger\, J\, \alpha_n(s,z), & x<y<z, \\ J\, [\alpha_n(z,y)]^\dagger\, J + \displaystyle\int_x^z ds\, J\, [\alpha_n(s,y)]^\dagger\, J\, \alpha_n(s,z), & x<z<y. \end{cases}$$

By taking the limit as $n\to+\infty$, we obtain

$$r(x;y,z) = \begin{cases} \alpha(y,z) + \displaystyle\int_x^y ds\, J\, [\alpha(s,y)]^\dagger\, J\, \alpha(s,z), & x<y<z, \\ J\, [\alpha(z,y)]^\dagger\, J + \displaystyle\int_x^z ds\, J\, [\alpha(s,y)]^\dagger\, J\, \alpha(s,z), & x<z<y, \end{cases}$$

which completes the proof. ∎

2.2 Darboux Transformation at the Potential Level

Now, we show that the solution $\tilde{\alpha}(x,y)$ to (2.0.19) can be expressed explicitly in terms of $\alpha(x,y)$, $f(x)$, and $g(y)$ appearing in (2.0.9) and (2.0.20), respectively. Then, we will obtain the formula for the Darboux transformation at the potential level.

Let us define the intermediate quantities $n(x)$ and $q(y)$ as

$$n(x) := f(x) + \int_x^\infty dz\, \alpha(x,z)\, f(z), \quad q(y) := g(y) + \int_y^\infty dz\, g(z)\, J\, [\alpha(y,z)]^\dagger J, \tag{2.2.1}$$

where J is the involution matrix appearing in (2.0.12).

Theorem 2.2.1 *Consider the perturbed operator* $\tilde{\Omega}$ *of* (2.0.18) *and the unperturbed operator* Ω *of* (2.0.8)*. Let* F*,* G*,* f*, and* g *be the quantities appearing in* (2.0.20)*. When* $\tilde{\Omega} - \Omega$ *is the finite-rank perturbation* FG *given in* (2.0.20)*, we can transform* (2.0.18) *into an integral equation that has a degenerate kernel and hence obtain* $\tilde{\alpha}$ *explicitly by linear algebraic methods.*

Proof. By substituting (2.0.20) into (2.0.18) we obtain

$$\omega + fg + \tilde{\alpha}(I + \Omega + FG) = 0, \tag{2.2.2}$$

which yields

$$\tilde{\alpha}(I + \Omega + FG) = -\omega - fg. \tag{2.2.3}$$

By applying on (2.2.3) from the right with the resolvent kernel operator $(I + R)$ appearing in (2.0.4) we get

$$\tilde{\alpha}\,[(I+\Omega)(I+R) + FG\,(I+R)] = -\omega(I+R) - fg\,(I+R). \tag{2.2.4}$$

Because of (2.0.4) we have

$$(I+\Omega)(I+R) = I. \tag{2.2.5}$$

Furthermore, from (2.0.10) we see that the first term on the right hand side in (2.2.4) is equal to α. Hence, by using $\alpha = -\omega(I + R)$ and with help of (2.2.5), we can write (2.2.4) in the form

$$\tilde{\alpha}\,[I + FG\,(I + R)] = \alpha - fg\,(I + R). \tag{2.2.6}$$

Now, let us define the operator $\tilde{G}$ as

$$\tilde{G} := G\,(I + R), \qquad \tilde{g}(x, y) := g(y) + \int_x^\infty dz\, g(z)\, r(x; z, y), \tag{2.2.7}$$

where $r(x; y, z)$ is the kernel given in (2.1.14). Since $f(y)\,\tilde{g}(x, z)$ is the kernel of $F\tilde{G}$, (2.2.6) can be written as

$$\tilde{\alpha}(I + F\,\tilde{G}) = \alpha - f\,\tilde{g}. \tag{2.2.8}$$

Let us now solve (2.2.6) for $\tilde{\alpha}$. We would like to have a solution in the form

$$\tilde{\alpha}(x, y) = \alpha(x, y) + p(x)\,\tilde{g}(x, y), \tag{2.2.9}$$

with $p(x)$ to be determined. Substituting (2.2.9) into (2.2.8), we get

$$\alpha + p\,\tilde{g} + p\,\tilde{g}\,F\tilde{G} + \alpha\,F\,\tilde{G} = \alpha - f\tilde{g}. \tag{2.2.10}$$

After cancelling α from boths side of (2.2.10), we obtain

$$(\alpha F + p + p\tilde{g}F + f)\tilde{G} = 0, \tag{2.2.11}$$

which yields

$$p(I + \tilde{g}F) = -(f + \alpha F), \tag{2.2.12}$$

or explicitly written as

$$p(x) = -n(x)\left[I + \int_x^\infty ds\,\tilde{g}(x, s)\,f(s)\right]^{-1}. \tag{2.2.13}$$

Substituting (2.2.13) into (2.2.9) we obtain

$$\tilde{\alpha}(x, y) = \alpha(x, y) - n(x)\left[I + \int_x^\infty ds\,\tilde{g}(x, s)\,f(s)\right]^{-1}\tilde{g}(x, y). \tag{2.2.14}$$

which completes the proof. ∎

Next, we show that $\tilde{g}(x,y)$ defined in (2.2.7) can be expressed explicitly in terms of $\alpha(x,y)$ and $g(y)$ appearing in (2.0.8) and (2.0.20), respectively.

Proposition 2.2.2 *The quantity* $\tilde{g}(x,y)$ *defined in* (2.2.7) *can be expressed explicitly in terms of the solution* $\alpha(x,y)$ *to* (2.0.8) *and the quantities* $f(x)$ *and* $g(y)$ *appearing in* (2.0.20) *as*

$$\tilde{g}(x,y) = q(y) + \int_x^y ds\, q(s)\, \alpha(s,y), \tag{2.2.15}$$

where q(y) is the quantity defined in (2.2.1).

Proof. Let us consider (2.2.7). Using $\int_x^\infty = \int_x^y + \int_y^\infty$ in (2.2.7), we write the second equation in (2.2.7) as

$$\tilde{g}(x,y) = g(y) + \int_x^y ds\, g(s)\, r(x;s,y) + \int_y^\infty ds\, g(s)\, r(x;s,y). \tag{2.2.16}$$

Substituting the first line of (2.1.14) in the integral $\int_x^y$ of (2.2.16) and the second line in the integral $\int_y^\infty$ of(2.2.16), we get

$$\begin{aligned}\tilde{g}(x,y) = g(y) &+ \int_x^y ds\, g(s)\, \left[\alpha(s,y) + \int_x^s dt\, J\, [\alpha(t,s)]^\dagger\, J\, \alpha(t,y)\right] \\ &+ \int_y^\infty ds\, g(s) \left[J\, [\alpha(y,s)]^\dagger\, J + \int_x^y dt\, J\, [\alpha(t,s)]^\dagger\, J\, \alpha(t,y)\right].\end{aligned}$$

After the distribution, we obtain

$$\begin{aligned}\tilde{g}(x,y) = g(y) &+ \int_y^\infty ds\, g(s)\, J\, [\alpha(y,s)]^\dagger\, J + \int_x^y ds\, g(s)\, \alpha(s,y) \\ &+ \left(\int_x^y ds \int_x^s dt + \int_y^\infty ds \int_x^y dt\right) g(s)\, J\, [\alpha(t,s)]^\dagger\, J\, \alpha(t,y).\end{aligned} \tag{2.2.17}$$

The orders of the two iterated integrals in (2.2.17) can be changed to $\int_x^y dt \int_t^y ds$ and $\int_x^y dt \int_y^\infty ds$, respectively. Using $\int_t^y + \int_y^\infty = \int_t^\infty$, we get

$$\begin{aligned}\tilde{g}(x,y) = g(y) &+ \int_y^\infty ds\, g(s)\, J\, [\alpha(y,s)]^\dagger\, J + \int_x^y ds\, g(s)\, \alpha(s,y) \\ &+ \int_x^y dt \int_t^\infty ds\, g(s)\, J\, [\alpha(t,s)]^\dagger\, J\, \alpha(t,y).\end{aligned}$$

We can replace $g(y) + \int_y^\infty ds\, g(s)\, J\, [\alpha(y,s)]^\dagger\, J$ with $q(y)$ because of (2.2.1). Then we get

$$\tilde{g}(x,y) = q(y) + \int_x^y ds\, g(s)\, \alpha(s,y) + \int_x^y dt \int_t^\infty ds\, g(s)\, J\, [\alpha(t,s)]^\dagger\, J\, \alpha(t,y). \quad (2.2.18)$$

We now interchange s and t in the iterated integral in (2.2.18), and thus obtain

$$\tilde{g}(x,y) = q(y) + \int_x^y ds\, g(s)\, \alpha(s,y) + \int_x^y ds \int_s^\infty dt\, g(t)\, J\, [\alpha(s,t)]^\dagger\, J\, \alpha(s,y), \quad (2.2.19)$$

or equivalently

$$\tilde{g}(x,y) = q(y) + \int_x^y ds \left[g(s) + \int_s^\infty dt\, g(t)\, J\, [\alpha(s,t)]^\dagger\, J \right] \alpha(s,y). \quad (2.2.20)$$

By replacing $g(s) + \int_s^\infty dt\, g(t)\, J\, [\alpha(s,t)]^\dagger\, J$ with $q(s)$ because of (2.2.1), from (2.2.20) we get (2.2.15). ∎

Note that $\tilde{g}(x,x) = q(x)$. Let us define the matrix $\Gamma(x)$ as

$$\Gamma(x) := I + \int_x^\infty ds\, \tilde{g}(x,s)\, f(s). \quad (2.2.21)$$

Proposition 2.2.3 *The quantity* $\Gamma(x)$ *defined in* (2.2.21) *can be expressed explicitly in terms of the solution* $\alpha(x,y)$ *to* (2.0.8) *and the quantities* $f(x)$ *and* $g(y)$ *appearing in* (2.0.20) *as*

$$\Gamma(x) = I + \int_x^\infty ds\, q(s)\, n(s), \quad (2.2.22)$$

where we have defined $n(x)$ *and* $q(x)$ *in* (2.2.1).

Proof. By substituting (2.2.15) into (2.2.21), then we obtain

$$\Gamma(x) = I + \int_x^\infty ds \left[q(s) + \int_x^s dt\, q(t)\, \alpha(t,s) \right] f(s),$$

or equivalently

$$\Gamma(x) = I + \int_x^\infty ds\, q(s)\, f(s) + \int_x^\infty ds \int_x^s dt\, q(t)\, \alpha(t,s)\, f(s). \quad (2.2.23)$$

The order of the integration in (2.2.23) can be changed to $\int_x^\infty dt \int_t^\infty ds$, and interchanging s and t in the iterated integral we get

$$\Gamma(x) = I + \int_x^\infty ds\, q(s)\, f(s) + \int_x^\infty ds \int_s^\infty dt\, q(s)\, \alpha(s,t)\, f(t), \qquad (2.2.24)$$

or equivalently

$$\Gamma(x) = I + \int_x^\infty ds\, q(s) \left[f(s) + \int_s^\infty dt\, q(s)\, \alpha(s,t)\, f(t) \right]. \qquad (2.2.25)$$

By replacing $f(s) + \int_s^\infty dt\, q(s)\, \alpha(s,t)\, f(t)$ with $n(s)$ because of (2.2.1), we obtain

$$\Gamma(x) = I + \int_x^\infty ds\, q(s)\, n(s). \qquad (2.2.26)$$

Thus, the proof is complete. ∎

The next theorem describes the formula for the Darboux transformation at the potential level.

Theorem 2.2.4 *Let* α *and* $\tilde{\alpha}$ *be the solutions to the integral equations* (2.0.8) *and* (2.0.18)*, respectively, and let* $n(x)$, $\Gamma(x)$, *and* $\tilde{g}(x,y)$ *be the quantities given in* (2.2.1)*,* (2.2.22)*, and* (2.2.15)*, respectively. Then,* $\tilde{\alpha}(x,y) - \alpha(x,y)$ *can be written in terms of* $\alpha(x,y)$*,* $f(x)$*, and* $g(y)$ *as*

$$\tilde{\alpha}(x,y) - \alpha(x,y) = -n(x)\, \Gamma(x)^{-1}\, \tilde{g}(x,y). \qquad (2.2.27)$$

Hence, the Darboux transformation at the potential level is obtained from

$$\tilde{\alpha}(x,x) - \alpha(x,x) = -n(x)\, \Gamma(x)^{-1}\, q(x). \qquad (2.2.28)$$

Proof. By substituting (2.2.21) into (2.2.14) we obtain (2.2.27). Using $\tilde{g}(x,x) = q(x)$ in (2.2.27) we obtain (2.2.28). ∎

2.3 Darboux Transformation at the Wave Function Level

We would now like to obtain the Darboux transformation at the wave function level.

The Fourier transform of the $N \times N$ matrix-valued quantity $\alpha(x, y)$ in (2.0.9) is related to the wave function $\Psi(\lambda, x)$ appearing in the unperturbed problem $\mathcal{L}\,\Psi = \lambda\,\Psi$. The relationship is given by

$$\Psi(\lambda, x) := e^{-\mathrm{i}\lambda J x} + \int_x^\infty dy\, \alpha(x, y)\, e^{-\mathrm{i}\lambda J y}, \tag{2.3.1}$$

where J is the involution matrix appearing in (2.0.12). Using the inverse Fourier transform on (2.3.1) we obtain

$$\alpha(x, y) = \frac{1}{2\pi} \int_{-\infty}^\infty d\lambda\, \left[\Psi(\lambda, x) - e^{-\mathrm{i}\lambda J x}\right] e^{\mathrm{i}\lambda J y}. \tag{2.3.2}$$

Similar to (2.3.1) and (2.3.2), we have

$$\tilde\Psi(\lambda, x) := e^{-\mathrm{i}\lambda J x} + \int_x^\infty dy\, \tilde\alpha(x, y)\, e^{-\mathrm{i}\lambda J y}, \tag{2.3.3}$$

$$\tilde\alpha(x, y) = \frac{1}{2\pi} \int_{-\infty}^\infty d\lambda\, \left[\tilde\Psi(\lambda, x) - e^{-\mathrm{i}\lambda J x}\right] e^{\mathrm{i}\lambda J y}. \tag{2.3.4}$$

Now define $\gamma(\lambda, x)$ as

$$\gamma(\lambda, x) := \int_x^\infty dy\, \tilde g(x, y)\, e^{-\mathrm{i}\lambda J y}. \tag{2.3.5}$$

Using (2.2.15) in (2.3.5) we get

$$\gamma(\lambda, x) = \int_x^\infty dy\, \left[q(y) + \int_x^y ds\, q(s)\, \alpha(s, y)\right] e^{-\mathrm{i}\lambda J y}, \tag{2.3.6}$$

or equivalently

$$\gamma(\lambda, x) = \int_x^\infty ds\, q(s) e^{-\mathrm{i}\lambda J s} + \int_x^\infty dy \int_x^y ds\, q(s)\, \alpha(s, y)\, e^{-\mathrm{i}\lambda J y}, \tag{2.3.7}$$

where we have replaced the dummy variable y with s in the first integral in (2.3.7). The orders of the iterated integral in (2.3.7) can be changed to $\int_x^\infty ds \int_s^\infty dy$, and we obtain

$$\gamma(\lambda, x) = \int_x^\infty ds\, q(s) e^{-\mathrm{i}\lambda J s} + \int_x^\infty ds \int_s^\infty dy\, q(s)\, \alpha(s, y)\, e^{-\mathrm{i}\lambda J y}, \tag{2.3.8}$$

or equivalently

$$\gamma(\lambda, x) = \int_x^\infty ds\, q(s) \left[e^{-\mathrm{i}\lambda J s} + \int_s^\infty dy\, \alpha(s,y)\, e^{-\mathrm{i}\lambda J y} \right]. \tag{2.3.9}$$

By using (2.3.1) in (2.3.9) we get

$$\gamma(\lambda, x) = \int_x^\infty ds\, q(s)\, \Psi(\lambda, s). \tag{2.3.10}$$

The next theorem describes the formula for the Darboux transformation at the wave function level.

Theorem 2.3.1 *Let* α *and* $\tilde{\alpha}$ *be the solutions to the integral equations* (2.0.8) *and* (2.0.18), *respectively, and let* $n(x)$, $\Gamma(x)$, *and* $\gamma(\lambda, x)$ *be the quantities given in* (2.2.1), (2.2.22), *and* (2.3.10), *respectively. Then,* $\tilde{\Psi}(\lambda, x) - \Psi(\lambda, x)$ *can be written in terms of* $\alpha(x, y)$, $f(x)$ *and* $g(y)$ *as*

$$\tilde{\Psi}(\lambda, y) - \Psi(\lambda, y) = -n(x)\, \Gamma(x)^{-1}\, \gamma(\lambda, x). \tag{2.3.11}$$

Proof. By subtracting (2.3.3) from (2.3.1) we obtain

$$\tilde{\Psi}(\lambda, x) - \Psi(\lambda, x) = \int_x^\infty dy\, [\tilde{\alpha}(x,y) - \alpha(x,y)]\, e^{-\mathrm{i}\lambda J y}. \tag{2.3.12}$$

By substituting (2.2.27) into (2.3.12) we get

$$\tilde{\Psi}(\lambda, x) - \Psi(\lambda, x) = \int_x^\infty dy\, \left[-n(x)\, \Gamma(x)^{-1}\, \tilde{g}(x,y) \right] e^{-\mathrm{i}\lambda J y}, \tag{2.3.13}$$

which yields

$$\tilde{\Psi}(\lambda, x) - \Psi(\lambda, x) = -n(x)\, \Gamma(x)^{-1} \int_x^\infty dy\, \tilde{g}(x,y)\, e^{-\mathrm{i}\lambda J y}. \tag{2.3.14}$$

Using (2.3.5) in (2.3.14) we get

$$\tilde{\Psi}(\lambda, y) - \Psi(\lambda, y) = -n(x)\, \Gamma(x)^{-1}\, \gamma(\lambda, x). \tag{2.3.15}$$

Thus, the proof is complete. ∎

2.4 Darboux Transformation via a Constant Matrix Triplet

In this section we show that the results given in Theorem (2.2.4) and Theorem (2.3.1) provide a unified approach to derive Darboux transformations.

Suppose we add a discrete eigenvalue λ_j with multiplicity n_j to the existing spectrum. Then, associated with the eigenvalue λ_j, there are n_j parameters $c_{j0}, \dots, c_{j(n_j-1)}$, usually known as norming constants [5]. Consequently, for each discrete eigenvalue λ_j added to the spectrum, there will be an n_j-parameter family of potentials $\tilde{u}(x)$ where the norming constants act as the parameters. In case several discrete eigenvalues $\lambda_1, \dots, \lambda_N$ are added all at once, it is convenient to use a square matrix A whose eigenvalues are related to λ_j for $j = 1, \dots, N$ in a simple manner. It is also convenient to use a matrix C whose entries are related to the norming constants c_{js} for $j = 1, \dots, N$ and $s = 0, 1, \dots, n_j - 1$.

The matrices $f(x)$ and $g(y)$ appearing in (2.0.20) can be written in the form

$$f(x) = \begin{bmatrix} 0 & B^\dagger e^{-A^\dagger x} \\ C\, e^{-Ax} & 0 \end{bmatrix}, \qquad g(y) = \begin{bmatrix} e^{-Ay} B & 0 \\ 0 & -e^{-A^\dagger y} C^\dagger \end{bmatrix}, \tag{2.4.1}$$

where A is a constant square matrix with all eigenvalues having positive real parts, and B and C are constant matrices of appropriate sizes so that the matrix product $f(x)\, g(y)$ is well defined and given by

$$f(x)\, g(y) = \begin{bmatrix} 0 & -B^\dagger e^{-A^\dagger (x+y)} C^\dagger \\ C\, e^{-A(x+y)} B & 0 \end{bmatrix}. \tag{2.4.2}$$

For $f(x)$ and $g(y)$ given in (2.4.1), let us evaluate $\tilde{g}(x, y)$ given in (2.2.15) and the intermediate quantities $n(x)$ and $q(x)$ given in (2.2.1) explicitly in terms of the wavefunction $\Psi(\lambda, x)$ evaluated at the eigenvalues of A. By taking the matrix adjoint of $\alpha(x, y)$ appearing in (2.3.2) we obtain

$$J\, [\alpha(x, y)]^\dagger\, J = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda J y} \left[J\, [\Psi(\lambda, x)]^\dagger\, J - e^{i\lambda J x} \right]. \tag{2.4.3}$$

By substituting (2.3.2) in the first formula in (2.2.1) we get

$$n(x) = f(x) + \int_x^\infty dy \, \frac{1}{2\pi} \int_{-\infty}^\infty d\lambda \, \left[\Psi(\lambda, x) - e^{-\mathrm{i}\lambda J x} \right] \, e^{\mathrm{i}\lambda J y} f(y), \tag{2.4.4}$$

or equivalently

$$n(x) = f(x) + \int_x^\infty dy \left[\frac{1}{2\pi} \int_{-\infty}^\infty d\lambda \, \Psi(\lambda, x) e^{\mathrm{i}\lambda J y} - \frac{1}{2\pi} \int_{-\infty}^\infty d\lambda \, e^{\mathrm{i}\lambda J (y-x)} \right] f(y). \tag{2.4.5}$$

Using the fact that

$$\frac{1}{2\pi} \int_{-\infty}^\infty ds \, e^{\pm \mathrm{i} a s} = \delta(a), \tag{2.4.6}$$

where δ is the Dirac delta distribution, from (2.4.5) we obtain

$$n(x) = f(x) + \int_x^\infty dy \left[\left[\frac{1}{2\pi} \int_{-\infty}^\infty d\lambda \, \Psi(\lambda, x) e^{\mathrm{i}\lambda J y} \right] - \delta(x - z) \right] f(y), \tag{2.4.7}$$

or equivalently

$$n(x) = f(x) + \frac{1}{2\pi} \int_x^\infty dy \int_{-\infty}^\infty d\lambda \, \Psi(\lambda, x) e^{\mathrm{i}\lambda J y} f(y) - \int_{-\infty}^\infty d\lambda \, \delta(x - z) f(y). \tag{2.4.8}$$

By using (2.4.6) in (2.4.8), we get

$$n(x) = f(x) + \frac{1}{2\pi} \left[\int_x^\infty dy \int_{-\infty}^\infty d\lambda \, \Psi(\lambda, x) e^{\mathrm{i}\lambda J y} f(y) \right] - f(x). \tag{2.4.9}$$

After simplification in (2.4.9), we obtain

$$n(x) = \frac{1}{2\pi} \int_{-\infty}^\infty d\lambda \, \Psi(\lambda, x) \int_x^\infty dy \, e^{\mathrm{i}\lambda J y} \, f(y). \tag{2.4.10}$$

Using in (2.4.10) the value of $f(x)$ given in (2.4.1), we get

$$n(x) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^\infty d\lambda \, \Psi(\lambda, x) \, e^{\mathrm{i}\lambda J x} \, \mathcal{N}(\lambda, x), \tag{2.4.11}$$

where we have defined

$$\mathcal{N}(\lambda, x) := \begin{bmatrix} 0 & -B^\dagger (\lambda I + \mathrm{i} A^\dagger)^{-1} e^{-A^\dagger x} \\ C(\lambda I - \mathrm{i} A)^{-1} e^{-A x} & 0 \end{bmatrix}. \tag{2.4.12}$$

Similarly, by using (2.4.3) in the second formula in (2.2.1), we evaluate $q(x)$ as

$$q(x) = g(x) + \int_x^\infty dy\, g(x) \left[\frac{1}{2\pi} \int_{-\infty}^\infty d\lambda\, e^{-i\lambda J y} \left[J\,[\Psi(\lambda,x)]^\dagger\, J - e^{i\lambda J x} \right] \right], \tag{2.4.13}$$

or equivalently

$$q(x) = g(x) + \int_x^\infty dy\, g(x) \left[\frac{1}{2\pi} \int_{-\infty}^\infty d\lambda\, e^{-i\lambda J y} J\,[\Psi(\lambda,x)]^\dagger\, J - \frac{1}{2\pi} \int_{-\infty}^\infty d\lambda\, e^{-i\lambda J y}\, e^{i\lambda J x} \right], \tag{2.4.14}$$

By using (2.4.6) in (2.4.14) and by proceeding as in (2.4.5)-(2.4.10), we get

$$q(x) = \frac{1}{2\pi} \int_{-\infty}^\infty d\lambda \int_x^\infty dy\, g(y)\, e^{-i\lambda J y}\, J\, [\Psi(\lambda,x)]^\dagger\, J. \tag{2.4.15}$$

Using in (2.4.15) the expression for $g(y)$ given in (2.4.1), we obtain

$$q(x) = \frac{1}{2\pi i} \int_{-\infty}^\infty d\lambda\, \Theta(\lambda,x)\, e^{-i\lambda J x}\, J\, [\Psi(\lambda,x)]^\dagger\, J, \tag{2.4.16}$$

where we have defined

$$\Theta(\lambda,x) := \begin{bmatrix} e^{-Ax}(\lambda I - iA)^{-1}B & 0 \\ 0 & e^{-A^\dagger x}(\lambda I + iA^\dagger)^{-1}C^\dagger \end{bmatrix}. \tag{2.4.17}$$

We still need to evaluate $\tilde{g}(x,y)$ in terms of the wave function $\Psi(\lambda,x)$. For this purpose, we use (2.3.2) in (2.2.15), (2.3.1), (2.4.6), and (2.4.16), and thus we obtain

$$\tilde{g}(x,y) = \frac{1}{2\pi} \int_{-\infty}^\infty d\mu \int_x^\infty dz\, q(z)\, \Psi(\mu,z)\, e^{i\mu J y}. \tag{2.4.18}$$

Using (2.4.16) in (2.4.18) we get

$$\tilde{g}(x,y) = \int_{-\infty}^\infty d\lambda \int_{-\infty}^\infty d\mu \int_x^\infty dz\, E(\lambda,\mu,z)\, e^{i\mu J y}, \tag{2.4.19}$$

where we have defined

$$E(\lambda,\mu,x) := \frac{1}{4\pi^2 i}\, \Theta(\lambda,x)\, e^{-i\lambda J x}\, J\, [\Psi(\lambda,x)]^\dagger\, J\, \Psi(\mu,x). \tag{2.4.20}$$

Using the expression for $n(x)$ and $q(x)$ given in (2.4.11) and (2.4.16), respectively, with the help of (2.4.20) we evaluate $\Gamma(x)$ given in (2.2.22) as

$$\Gamma(x) = I - \mathrm{i} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\mu \int_{x}^{\infty} dy \, E(\lambda, \mu, y) \, e^{\mathrm{i}\mu J y} \, \mathcal{N}(\mu, y). \tag{2.4.21}$$

Finally, by substituting (2.4.16) in (2.3.10) and using (2.4.20), we obtain

$$\gamma(\lambda, x) = 2\pi \int_{x}^{\infty} ds \int_{-\infty}^{\infty} d\lambda \, E(\lambda, \lambda, s). \tag{2.4.22}$$

Chapter 3

(Right) Marchenko Integral Equation

In this chapter we develop a generalized approach for the Darboux transformation via the (right) Marchenko integral equation. We first present some preliminary results that are needed later on and then we analyze the resolvent kernel $r(x;z,y)$ of the (right) Marchenko integral equation. In Theorem 3.1.2 we prove that $r(x;z,y)$ can be expressed explicitly in terms of the solution $\alpha(x,y)$ to the (right) Marchenko integral equation (3.0.9). We then give the formulas for the Darboux transformation at the potential level and wave function level.

The results in this chapter are somewhat similar to the results in Chapter 2, but for the benefit of the reader, we provide all the details. Furthermore, for the reader to see the similarity, we use similar notations for the corresponding quantities.

Let us consider the integral equation

$$\alpha(x,y)+\beta(x,y)+\int_{-\infty}^{x} dz\,\alpha(x,z)\,\omega(z,y)=0, \qquad x>y, \tag{3.0.1}$$

where $\alpha(x,y)$ is the unknown term, $\beta(x,y)$ is the nonhomogeneous term, and $\omega(z,y)$ is the integral kernel. We assume that the equation given in (3.0.1) is uniquely solvable in $\mathcal{H}_2^{N\times N}$, namely, we get a unique $\alpha(x,y)$ when $\beta(x,y)$ and $\omega(z,y)$ are given to us. Recall that $\mathcal{H}_p^{M\times N}$ is the complex Banach space of $M\times N$ matrix-valued measurable functions $F:(x,+\infty)\to\mathbb{C}^{M\times N}$ such that the matrix norm $\|F(\cdot)\|$ belongs to $L^p(x,+\infty)$ for $1\le p\le+\infty$. Note that the kernel $\omega(z,y)$ does not depend on the parameter $x\in R$ and it satisfies

$$\sup_{x>y}\int_{-\infty}^{x} dz\,(\|\omega(z,y)\|+\|\omega(y,z)\|)<+\infty. \tag{3.0.2}$$

Recall that $\|\cdot\|$ denotes any $N \times N$- matrix norm. Let us write (3.0.1) in the operator form

$$\alpha + \beta + \alpha\,\Omega = 0, \tag{3.0.3}$$

where the operator Ω acts from the right. We suppose that, for each fixed $x \in R$, the operator $(I + \Omega)$ is invertible on $\mathcal{H}_1^{N\times N}$ and $\mathcal{H}_2^{N\times N}$. Note that I denotes the identity operator. We use $(I + R)$ to denote the corresponding resolvent operator, where

$$I + R = (I + \Omega)^{-1}, \qquad R := (I + \Omega)^{-1} - I. \tag{3.0.4}$$

By solving (3.0.3) for α we get

$$\alpha = -\beta\,(I + \Omega)^{-1}. \tag{3.0.5}$$

Using (3.0.4) in (3.0.5), we obtain

$$\alpha = -\beta(I + R), \tag{3.0.6}$$

or explicitly

$$\alpha(x, y) = -\beta(x, y) - \int_{-\infty}^{x} dz\,\beta(x, z)\,r(x; z, y), \tag{3.0.7}$$

where $r(x; z, y)$ is the integral kernel of the resolvent operator R. Let us write (3.0.1) in the form

$$\alpha + \omega + \alpha\,\Omega = 0, \tag{3.0.8}$$

where the nonhomogeneous term and the integral kernel are related to each other. Then (3.0.8) can explicitly be written as

$$\alpha(x, y) + \omega(x, y) + \int_{-\infty}^{x} dz\,\alpha(x, z)\,\omega(z, y) = 0, \qquad x > y, \tag{3.0.9}$$

which is usually called [3, 4, 6, 10, 13, 14, 18] the (right) Marchenko integral equation. We make the assumption that the (right) Marchenko integral equation is uniquely solvable in $\mathcal{H}_2^{N\times N}$, namely, we assume that there is a unique $\alpha(x,y)$ as a solution to (3.0.9) when $\omega(x,y)$ is given. We construct the resolvent kernel $r(x;z,y)$ appearing in (3.0.7) in terms of the unique $\alpha(x,y)$ then the resolvent kernel will also be unique. Since $\alpha(x,y)$ is uniquely determined, we will then obtain $r(x;z,y)$ uniquely. By solving (3.0.8) with help of (3.0.4) we get

$$\alpha = -\omega(I+R). \tag{3.0.10}$$

The unique solvability of (3.0.9) in $\mathcal{H}_1^{N\times N}$ and the condition in (3.0.2) imply that

$$\sup_{x>y} \int_{-\infty}^{x} dz\, (\|\alpha(z,y)\| + \|\alpha(y,z)\|) < +\infty. \tag{3.0.11}$$

We consider (3.0.8) when the integral operator Ω appearing in (3.0.3) is $N \times N$ matrix-valued and J-selfadjoint in the sense that

$$\Omega = J\,\Omega^{\ddagger} J, \qquad \omega(y,z) = J\; [\omega(z,y)]^{\dagger}\, J. \tag{3.0.12}$$

Recall that the dagger denotes the matrix adjoint (complex conjugate and matrix transpose) and the double dagger denotes not just only the matrix adjoint, but also switching the two arguments of the kernel in the operator, namely, for any two integral operators A and B

$$(AB)^{\ddagger}(y,z) = (B^{\ddagger}A^{\ddagger})(y,z), \tag{3.0.13}$$

which is already stated in (2.0.16). Note also that J is an $N\times N$ selfadjoint involution, i.e.

$$J = J^{\dagger} = J^{-1}. \tag{3.0.14}$$

Associated with the unperturbed problem $\mathcal{L}\,\Psi = \lambda\,\Psi$, we have [3, 4, 6, 10, 13, 14, 18] the fundamental integral equation given in (3.0.8), where α is related to the Fourier transform of Ψ and ω is related to the Fourier transform of the scattering data $S(\lambda)$ for the operator $\mathcal{L}$.

Associated with the perturbed problem $\tilde{\mathcal{L}}\,\tilde{\Psi} = \lambda\,\tilde{\Psi}$, we have the fundamental integral equation

$$\tilde{\alpha} + \tilde{\omega} + \tilde{\alpha}\,\tilde{\Omega} = 0, \tag{3.0.15}$$

or explicitly

$$\tilde{\alpha}(x,y) + \tilde{\omega}(x,y) + \int_{-\infty}^{x} dz\,\tilde{\alpha}(x,z)\,\tilde{\omega}(z,y) = 0, \qquad y < x. \tag{3.0.16}$$

As we indicated in Chapter 2, in the analysis of Darboux transformations, the perturbation will correspond to the case where the integral operators $\tilde{\Omega}$ and Ω appearing in (3.0.15) and (3.0.8), respectively, differ by a finite-rank operator and we denote that difference operator by FG, i.e.

$$\tilde{\Omega} = \Omega + FG, \qquad \tilde{\omega}(x,y) = \omega(x,y) + f(x)\,g(y). \tag{3.0.17}$$

Recall that we cannot in general expect F and G to commute, and hence in general $f\,g \neq g\,f$.

3.1 Construction of the Resolvent Kernel

In this section, we analyze the resolvent kernel $r(x;z,y)$ appearing in (3.0.7) and then we show that $r(x;z,y)$ can be expressed explicitly in terms of the solution $\alpha(x,y)$ to (3.0.9).

Proposition 3.1.1 *Assume that* (3.0.3) *is uniquely solvable in* $\mathcal{H}_2^{N\times N}$ *and that* Ω *satisfies* (3.0.12). *Then, the operator* R *given in* (3.0.4) *and the corresponding kernel* $r(x;z,y)$ *appearing in* (3.0.7) *satisfy*

$$R = J\,R^{\ddagger} J, \qquad r(x;z,y) = J\,\left[r(x;y,z)\right]^{\dagger} J, \tag{3.1.1}$$

where J *is the involution matrix appearing in* (3.0.14).

Proof. From (3.0.4) we know that $(I+R)=(I+\Omega)^{-1}$ and thus

$$(I+\Omega)(I+R)=I=(I+R)(I+\Omega). \tag{3.1.2}$$

Then we get

$$R+\Omega+R\,\Omega=0, \tag{3.1.3}$$

and

$$R+\Omega+\Omega\,R=0. \tag{3.1.4}$$

By applying on (3.1.3) the double dagger transformation defined in (3.0.14), i.e. by taking the adjoint and switching the arguments, and further applying J on both sides, we obtain

$$JR^{\ddagger}J+J\Omega^{\ddagger}J+(J\Omega^{\ddagger}J)(JR^{\ddagger}J)=0, \tag{3.1.5}$$

equivalently, by using (3.0.12) we get

$$JR^{\ddagger}J+\Omega+\Omega\,(JR^{\ddagger}J)=0. \tag{3.1.6}$$

Since (3.0.3) is assumed to be uniquely solvable in $\mathcal{H}_2^{N\times N}$, by comparing (3.1.4) and (3.1.6) we get $R=JR^{\ddagger}J$.

Even though it is clear that $R=JR^{\ddagger}J$ implies $r(x;y,z)=J\,[r(x;z,y)]^{\dagger}\,J$, for clarity, let us also prove the same result by directly working with the kernel $r(x;z,y)$ for $y<z$ and $z<y$. Let us first consider (3.1.4) for $z<y$, which is explicitly written as

$$r(x;y,z)+\omega(y,z)+\int_{-\infty}^{x} ds\,\omega(y,s)\,r(x;s,z)=0, \qquad z<y. \tag{3.1.7}$$

By taking the adjoint of (3.1.7) and then multiplying the resulting equation by J on the left and on the right, we obtain

$$J\,[r(x;y,z)]^{\dagger}\,J+J\,[\omega(y,z)]^{\dagger}\,J+\int_{-\infty}^{x} ds\,J\,[\omega(y,s)\,r(x;s,z)]^{\dagger}\,J, \qquad z<y, \tag{3.1.8}$$

or equivalently

$$J\ [r(x;y,z)]^\dagger\, J + J\ [\omega(y,z)]^\dagger\, J + \int_{-\infty}^{x} ds\, J\ [r(x;s,z)]^\dagger\, J\, J\, [\omega(y,s)]^\dagger\, J, \qquad z<y. \tag{3.1.9}$$

Using (3.0.12) in (3.1.9) we get

$$J\ [r(x;y,z)]^\dagger\, J + \omega(z,y) + \int_{-\infty}^{x} ds\, J\ [r(x;s,z)]^\dagger\, J\, \omega(s,y), \qquad z<y. \tag{3.1.10}$$

By interchanging y and z in (3.1.10), we obtain the equivalent expression

$$J\ [r(x;z,y)]^\dagger\, J + \omega(y,z) + \int_{-\infty}^{x} ds\, J\ [r(x;s,y)]^\dagger\, J\, \omega(s,z), \qquad y<z. \tag{3.1.11}$$

By comparing (3.1.7) and (3.1.11) and by using the uniqueness of the solution to (3.1.7), we see that $r(x;y,z) = J\ [r(x;z,y)]^\dagger\, J$ and we have

$$r(x;y,z) + \omega(y,z) + \int_{-\infty}^{x} ds\, r(x;y,s)\, \omega(s,z), \qquad y<z. \tag{3.1.12}$$

Similarly, by considering (3.1.4) for $y<z$, we obtain

$$r(x;y,z) + \omega(y,z) + \int_{-\infty}^{x} ds\, r(x;y,s)\, \omega(s,z), \qquad z<y. \tag{3.1.13}$$

Thus, we have shown that $r(x;y,z) = J\ [r(x;z,y)]^\dagger\, J$ for $y<z$ and $z<y$ which completes the proof. ∎

Theorem 3.1.2 *Assume that* (3.0.3) *is uniquely solvable in* $\mathcal{H}_2^{N\times N}$ *and that* Ω *satisfies* (3.0.12)*. Then, the corresponding kernel* $r(x;z,y)$ *appearing in* (3.0.7) *can be expressed explicitly in terms of the solution* $\alpha(x,y)$ *to* (3.0.9) *as*

$$r(x;z,y) = \begin{cases} \alpha(z,y) + \displaystyle\int_{z}^{x} ds\, J\ [\alpha(s,z)]^\dagger\, J\, \alpha(s,y), & y<z<x, \\ J\ [\alpha(y,z)]^\dagger\, J + \displaystyle\int_{y}^{x} ds\, J\ [\alpha(s,z)]^\dagger\, J\, \alpha(s,y), & z<y<x, \end{cases} \tag{3.1.14}$$

where J *is the involution matrix appearing in* (3.0.12).

Proof. Since (3.0.3) is assumed to be uniquely solvable in $\mathcal{H}_2^{N\times N}$, so is (3.0.8) and hence the solution R to (3.1.3) is unique. Thus, it suffices to prove that the quantity defined in (3.1.14) satisfies (3.1.3), i.e. the quantity in (3.1.14) satisfies the integral equations

$$r(x;z,y)+\omega(z,y)+\int_{-\infty}^{x} ds\, r(x;z,s)\,\omega(s,y)=0, \qquad \max\{y,z\}<x. \tag{3.1.15}$$

We will give the proof for both cases of $y<z$ and $z<y$.

Case 1: $y<z<x$

Let us use $\int_{-\infty}^{x}=\int_{-\infty}^{z}+\int_{z}^{x}$ in the integral appearing in (3.1.15). Then the left-hand side in (3.1.15) becomes

$$r(x;z,y)+\omega(z,y)+\int_{-\infty}^{z} ds\, r(x;z,s)\,\omega(s,y)+\int_{z}^{x} ds\, r(x;z,s)\,\omega(s,y). \tag{3.1.16}$$

By using the first line of (3.1.14) in the integral $\int_{-\infty}^{z}$ and the second line of (3.1.14) in the integral $\int_{z}^{x}$ in (3.1.16), we obtain as the left-hand side of (3.1.15)

$$\begin{aligned}\alpha(z,y)&+\int_{z}^{x} ds\, J\,[\alpha(s,z)]^{\dagger}\,J\,\alpha(s,y)+\omega(z,y)\\&+\int_{-\infty}^{z} ds\left[\alpha(z,s)+\int_{z}^{x} dt\, J\,[\alpha(t,z)]^{\dagger}\,J\,\alpha(t,s)\right]\omega(s,y)\\&+\int_{z}^{x} ds\left[J\,[\alpha(s,z)]^{\dagger}\,J+\int_{s}^{x} dt\, J\,[\alpha(t,z)]^{\dagger}\,J\,\alpha(t,s)\right]\omega(s,y).\end{aligned}$$

After the distribution, the left-hand side of (3.1.15) then becomes

$$\begin{aligned}\alpha(z,y)&+\int_{z}^{x} ds\, J\,[\alpha(s,z)]^{\dagger}\,J\,\alpha(s,y)+\omega(z,y)\\&+\int_{-\infty}^{z} ds\,\alpha(z,s)\,\omega(s,y)+\int_{-\infty}^{z} ds\int_{z}^{x} dt\, J\,[\alpha(t,z)]^{\dagger}\,J\,\alpha(t,s)\,\omega(s,y)\\&+\int_{z}^{x} ds\, J\,[\alpha(s,z)]^{\dagger}\,J\,\omega(s,y)+\int_{z}^{x} ds\int_{s}^{x} dt\, J\,[\alpha(t,z)]^{\dagger}\,J\,\alpha(t,s)\,\omega(s,y).\end{aligned}$$

Let us now define b_1, b_2, and b_3 as

$$\begin{aligned}
b_1 &:= \alpha(z,y) + \omega(z,y) + \int_{-\infty}^{z} ds\, \alpha(z,s)\, \omega(s,y), \\
b_2 &:= \int_z^x dt\, J\, [\alpha(t,z)]^\dagger\, J\, \alpha(t,y) + \int_z^x dt\, J\, [\alpha(t,z)]^\dagger\, J\, \omega(t,y), \\
b_3 &:= \int_{-\infty}^{z} ds \int_z^x dt\, J\, [\alpha(t,z)]^\dagger\, J\, \alpha(t,s)\, \omega(s,y) \\
&\quad + \int_z^x ds \int_s^x dt\, J\, [\alpha(t,z)]^\dagger\, J\, \alpha(t,s)\, \omega(s,y).
\end{aligned}$$

We have $b_1 = 0$ because of the (right) Marchenko integral equation given in (3.0.9). The orders of the two iterated integrals in b_3 can be changed to $\int_z^x dt \int_{-\infty}^z ds$ and $\int_z^x dt \int_z^t ds$, respectively. Using $\int_{-\infty}^z + \int_z^t = \int_{-\infty}^t$, the quantity b_3 becomes

$$b_3 = \int_z^x dt \int_{-\infty}^{t} ds\, J\, [\alpha(t,z)]^\dagger\, J\, \alpha(t,s)\, \omega(s,y).$$

Then we get

$$b_2 + b_3 = \int_z^x dt\, J\, [\alpha(t,z)]^\dagger\, J \left[\alpha(t,y) + \omega(t,y) + \int_{-\infty}^{t} ds\, \alpha(t,s)\, \omega(s,y) \right]. \quad (3.1.17)$$

Since $t > y$, the quantity inside the brackets in (3.1.17) vanishes due to (3.0.9). Thus, we obtain $b_2 + b_3 = 0$, and hence $b_1 + b_2 + b_3 = 0$ for the case of $y < z < x$.

Case 2: $z < y < x$

Our goal is to show that the expression given in (3.1.14) satisfies (3.1.15) in the case $z < y < x$. A direct proof in the case $z < y < x$ does not seem to be feasible as it was done in the case $y < z < x$. Therefore, we will proceed as follows. Since any compact operator can be approximated by a sequence of finite-rank operators, we will approximate $\omega(z,y)$ in (3.1.15) by the sequence $\xi_n(z)\, \eta_n(y)$. This will allow us to solve (3.1.15) explicitly and then to get a solution. We will call that solution as $r_n(x;z,y)$. Thus we will obtain (3.1.14) where $\alpha(z,y)$ is replaced by $\alpha_n(z,y)$ which is the solution to the (right) Marchenko integral equation given in (3.1.18) when $\omega(z,y)$ is replaced with $\xi_n(z)\,\eta_n(y)$. Finally, by taking the limit when $n \to +\infty$, we will obtain $r(x; z, y)$ given in (3.1.14)

Consider the (right) Marchenko integral equation

$$\alpha(z, y) + \omega(z, y) + \int_{-\infty}^{z} ds\, \alpha(z, s)\, \omega(s, y) = 0, \qquad z > y, \tag{3.1.18}$$

which obtained by replacing x and y in (3.0.9) by z and y, respectively. Suppose

$$\omega(z, y) = \xi(z)\, \eta(y), \tag{3.1.19}$$

i.e. suppose that the integral kernel in (3.1.14) is separable. We first would like to evaluate $\alpha(z, y)$ appearing in (3.1.14) in terms of the kernel parts ξ and η. Isolating $\alpha(z, y)$ in (3.1.18), we get

$$\alpha(z, y) = -\omega(z, y) - \int_{-\infty}^{z} ds\, \alpha(z, s)\, \omega(s, y). \tag{3.1.20}$$

Substituting (3.1.19) into (3.1.20) we obtain

$$\alpha(z, y) = -\xi(z)\, \eta(y) - \int_{-\infty}^{z} ds\, \alpha(z, s)\, \xi(s)\, \eta(y), \tag{3.1.21}$$

or equivalently

$$\alpha(z, y) = -\left[\xi(z) + \int_{-\infty}^{z} ds\, \alpha(z, s)\, \xi(s)\right] \eta(y). \tag{3.1.22}$$

Define $\tau(z)$ as

$$\tau(z) := \xi(z) + \int_{-\infty}^{z} ds\, \alpha(z, s)\, \xi(s). \tag{3.1.23}$$

Then we get

$$\alpha(z, y) = -\tau(z)\, \eta(y), \tag{3.1.24}$$

By replacing $\alpha(z, y)$ appearing in the integrand in (3.1.23) with (3.1.24) we obtain

$$\tau(z) = \xi(z) - \int_{-\infty}^{z} ds\, \tau(z)\, \eta(s)\, \xi(s), \tag{3.1.25}$$

which yields

$$\tau(z)\left[I+\int_{-\infty}^{z} ds\,\eta(s)\,\xi(s)\right]=\xi(z), \tag{3.1.26}$$

where I is the identity matrix. Thus we obtain

$$\tau(z)=\xi(z)\left[I+\int_{-\infty}^{z} ds\,\eta(s)\,\xi(s)\right]^{-1}, \tag{3.1.27}$$

provided the matrix inverse exists. Consequently,

$$\alpha(z,y)=-\xi(z)\left[I+\int_{-\infty}^{z} ds\,\eta(s)\,\xi(s)\right]^{-1}\eta(y). \tag{3.1.28}$$

Finally, define

$$\Upsilon(z):=I+\int_{-\infty}^{z} ds\,\eta(s)\,\xi(s). \tag{3.1.29}$$

By substituting (3.1.29) into (3.1.28) we obtain

$$\alpha(z,y)=-\xi(z)\Upsilon(z)^{-1}\eta(y). \tag{3.1.30}$$

We would now like to evaluate $J\,[\alpha(y,z)]^\dagger\,J$ in terms of ξ and η. For this, let us consider (3.1.18). By taking the adjoint of (3.1.18) and then multiplying the resulting equation by J on the left and on the right, we get

$$J\,[\alpha(z,y)]^\dagger\,J+J\,[\omega(z,y)]^\dagger\,J+\int_{-\infty}^{z} ds\,J\,[\alpha(z,s)\,\omega(s,y)]^\dagger\,J=0, \tag{3.1.31}$$

or equivalently

$$J\,[\alpha(z,y)]^\dagger\,J+J\,[\omega(z,y)]^\dagger\,J+\int_{-\infty}^{z} ds\,J\,[\omega(s,y)]^\dagger\,J\,J\,[\alpha(z,s)]^\dagger\,J=0. \tag{3.1.32}$$

Using (3.0.12) in (3.1.32) we get

$$J\,[\alpha(z,y)]^\dagger\,J+\omega(y,z)+\int_{-\infty}^{z} ds\,\omega(y,s)\,J\,[\alpha(z,s)]^\dagger\,J=0. \tag{3.1.33}$$

By substituting (3.1.19) into (3.1.33) we obtain

$$J\,[\alpha(z,y)]^\dagger\,J+\xi(y)\eta(z)+\int_{-\infty}^{z} ds\,\xi(y)\eta(s)\,J\,[\alpha(z,s)]^\dagger\,J=0. \tag{3.1.34}$$

Let us set

$$J\,[\alpha(z,y)]^\dagger\, J := \xi(y)K(z), \tag{3.1.35}$$

with $K(z)$ to be determined. Substituting (3.1.35) into (3.1.34) and then equating the right coefficients of $\xi(y)$ from both sides of (3.1.34), we get

$$K(z) + \eta(z) + \int_{-\infty}^{z} ds\, \eta(s)\, \xi(s)\, K(z) = 0. \tag{3.1.36}$$

By solving (3.1.36) for $K(z)$ we obtain

$$K(z) = -\left[I + \int_{-\infty}^{z} ds\, \eta(s)\, \xi(s)\right]^{-1} \eta(z). \tag{3.1.37}$$

Using (3.1.29) in (3.1.37) we get

$$K(z) = -\Upsilon(z)^{-1}\, \eta(z). \tag{3.1.38}$$

By substituting (3.1.38) into (3.1.35) we obtain

$$J\,[\alpha(z,y)]^\dagger\, J = -\xi(y)\, \Upsilon(z)^{-1}\, \eta(z). \tag{3.1.39}$$

Hence, by interchanging y and z in (3.1.39) we obtain

$$J\,[\alpha(y,z)]^\dagger\, J = -\xi(z)\, \Upsilon(y)^{-1}\, \eta(y). \tag{3.1.40}$$

Now, let us solve the integral equation

$$r(x;z,y) + \omega(z,y) + \int_{-\infty}^{x} ds\, r(x;z,s)\, \omega(s,y) = 0. \tag{3.1.41}$$

Define

$$r(z,y) = H(z)\, \eta(y), \tag{3.1.42}$$

with $H(z)$ to be determined. Substituting (3.1.19) and (3.1.42) into (3.1.41) and then equating the left coefficients of $\eta(y)$ from both sides of (3.1.41), we get

$$H(z) + \xi(z) + \int_{-\infty}^{x} ds\, H(z)\, \eta(s)\, \xi(s) = 0,$$

which yields

$$H(z)\left[I+\int_{-\infty}^{x} ds\,\eta(s)\,\xi(s)\right]=-\xi(z). \tag{3.1.43}$$

Then we get

$$H(z)=-\xi(z)\left[I+\int_{-\infty}^{x} ds\,\eta(s)\,\xi(s)\right]^{-1}. \tag{3.1.44}$$

As a result

$$r(x;z,y)=-\xi(z)\left[I+\int_{-\infty}^{x} ds\,\eta(s)\,\xi(s)\right]^{-1}\eta(y). \tag{3.1.45}$$

From (3.1.29) we see that the matrix in the brackets in (3.1.45) is equal to $\Upsilon(x)$. Thus we can write (3.1.45) as

$$r(x;z,y)=-\xi(z)\Upsilon(x)^{-1}\eta(y). \tag{3.1.46}$$

Then for $x<y<z$, with help of (3.1.30) and (3.1.40) we compute

$$\begin{aligned}
&\alpha(z,y)+\int_{z}^{x} ds\,J\,[\alpha(s,z)]^{\dagger}\,J\,\alpha(s,y)\\
&\quad=-\xi(z)\,\Upsilon(z)^{-1}\,\eta(y)+\int_{z}^{x} ds\,\xi(z)\,\Upsilon(s)^{-1}\,\eta(s)\,\xi(s)\,\Upsilon(s)^{-1}\,\eta(y)\\
&\quad=-\xi(z)\left[\Upsilon(z)^{-1}-\int_{z}^{x} ds\,\Upsilon(s)^{-1}\,\eta(s)\,\xi(s)\,\Upsilon(s)^{-1}\right]\eta(y)\\
&\quad=-\xi(z)\left[\Upsilon(z)^{-1}-\int_{z}^{x} ds\left(-\frac{d}{ds}\left[I+\int_{-\infty}^{s} dt\,\eta(t)\,\xi(t)\right]^{-1}\right)\right]\eta(y)\\
&\quad=-\xi(z)\left\{\Upsilon(z)^{-1}-\left[\Upsilon(z)^{-1}-\Upsilon(x)^{-1}\right]\right\}\eta(y)\\
&\quad=-\xi(z)\,\Upsilon(x)^{-1}\eta(y),
\end{aligned}$$

which is equal to $r(x;y,z)$ because of (3.1.46). Similarly, for $x<z<y$ we compute

$$\begin{aligned}
&J\,[\alpha(y,z)]^{\dagger}\,J+\int_{y}^{x} ds\,J\,[\alpha(s,z)]^{\dagger}\,J\,\alpha(s,y)\\
&\quad=-\xi(z)\,\Upsilon(y)^{-1}\,\eta(y)+\int_{y}^{x} ds\,\xi(z)\,\Upsilon(s)^{-1}\,\eta(s)\,\xi(s)\,\Upsilon(s)^{-1}\,\eta(y)\\
&\quad=-\xi(z)\left[\Upsilon(y)^{-1}-\int_{y}^{x} ds\,\Upsilon(s)^{-1}\,\eta(s)\,\xi(s)\,\Upsilon(s)^{-1}\right]\eta(y).
\end{aligned} \tag{3.1.47}$$

We can write (3.1.47) as

$$
\begin{aligned}
J\,[\alpha(y,z)]^\dagger\,J + \int_y^x ds\, J\,[\alpha(s,z)]^\dagger\,J\,\alpha(s,y)& \\
&= -\xi(z)\left[\Upsilon(y)^{-1} - \int_y^x ds \left(-\frac{d}{ds}\left[I + \int_{-\infty}^s dt\,\eta(t)\,\xi(t)\right]^{-1}\right)\right]\eta(y) \\
&= -\xi(z)\left\{\Upsilon(y)^{-1} - \left[\Upsilon(y)^{-1} - \Upsilon(x)^{-1}\right]\right\}\eta(y) \\
&= -\xi(z)\,\Upsilon(x)^{-1}\,\eta(y),
\end{aligned}
$$

which is equal to $r(x;y,z)$ because of (2.1.46). Thus we have shown that

$$
r(x;z,y) = \begin{cases} \alpha(z,y) + \displaystyle\int_z^x ds\, J\,[\alpha(s,z)]^\dagger\,J\,\alpha(s,y), & y<z<x, \\ J\,[\alpha(y,z)]^\dagger\,J + \displaystyle\int_y^x ds\, J\,[\alpha(s,z)]^\dagger\,J\,\alpha(s,y), & z<y<x, \end{cases} \tag{3.1.48}
$$

where we have obtained $\alpha(z,y)$ and $[\alpha(y,z)]^\dagger$ in terms of the kernel parts ξ and η.

As a summary, in the case $z<y<x$, we can approximate the kernel of the Marchenko integral equation in (3.1.18), namely, $\omega(z,y)$ by the sequence of separable kernels $\xi_n(z)\,\eta_n(y)$, where the approximation is understood in the sense

$$
\lim_{n\to+\infty}\ \operatorname*{ess\,sup}_{y<x} \int_{-\infty}^x dz\, \|\omega(z,y) - \xi_n(z)\,\eta_n(y)\| = 0, \tag{3.1.49}
$$

i.e., let us assume that

$$
\lim_{n\to+\infty} \|\Omega - \Omega_n\|_{L^1\to L^1} = 0. \tag{3.1.50}
$$

Since we then also have

$$
\lim_{n\to+\infty} \|(I+\Omega)^{-1} - (I+\Omega_n)^{-1}\|_{L^1\to L^1} = 0, \tag{3.1.51}
$$

we get for the resolvent kernels

$$
\lim_{n\to+\infty}\ \operatorname*{ess\,sup}_{y<x} \int_{-\infty}^x dz\, \|r(x;z,y) - r_n(x;z,y)\| = 0. \tag{3.1.52}
$$

Recall that $\|\cdot\|$ is any $N \times N$-matrix norm. By using the results in the degenerate case, we get

$$r_n(x;z,y) = \begin{cases} \alpha_n(z,y) + \displaystyle\int_z^x ds\, J\, [\alpha_n(s,z)]^\dagger\, J\, \alpha_n(s,y), & y < z < x, \\ J\, [\alpha_n(y,z)]^\dagger\, J + \displaystyle\int_y^x ds\, J\, [\alpha_n(s,z)]^\dagger\, J\, \alpha_n(s,y), & z < y < x. \end{cases}$$

By taking the limit as $n \to +\infty$, we obtain

$$r(x;z,y) = \begin{cases} \alpha(z,y) + \displaystyle\int_z^x ds\, J\, [\alpha(s,z)]^\dagger\, J\, \alpha(s,y), & y < z < x, \\ J\, [\alpha(y,z)]^\dagger\, J + \displaystyle\int_y^x ds\, J\, [\alpha(s,z)]^\dagger\, J\, \alpha(s,y), & z < y < x. \end{cases}$$

which completes the proof. ∎

3.2 Darboux Transformation at the Potential Level

Now, we show that the solution $\tilde{\alpha}(x,y)$ to (3.0.16) can be expressed explicitly in terms of $\alpha(x,y)$, $f(x)$, and $g(y)$ appearing in (3.0.9) and (3.0.17), respectively. Then, we will obtain the formula for the Darboux transformation at the potential level.

Let us define the intermediate quantities $n(x)$ and $q(y)$ as

$$n(x) := f(x) + \int_{-\infty}^x dz\, \alpha(x,z)\, f(z), \quad q(y) := g(y) + \int_{-\infty}^y dz\, g(z)\, J\, [\alpha(y,z)]^\dagger\, J, \tag{3.2.1}$$

where J is the involution matrix appearing in (3.0.12).

Theorem 3.2.1 *Consider the perturbed operator $\tilde{\Omega}$ of* (3.0.15) *and the unperturbed operator Ω of* (3.0.8). *Let F, G, f, and g be the quantities appearing in* (3.0.17). *When $\tilde{\Omega} - \Omega$ is the finite-rank perturbation FG given in* (3.0.17), *we can transform* (3.0.15) *into an integral equation that has a degenerate kernel and hence obtain $\tilde{\alpha}$ explicitly by linear algebraic methods.*

Proof. By substituting (3.0.17) into (3.0.15) we obtain

$$\omega + f\, g + \tilde{\alpha}(I + \Omega + F\, G) = 0, \tag{3.2.2}$$

which yields

$$\tilde{\alpha}(I + \Omega + F\,G) = -\omega - f\,g. \tag{3.2.3}$$

By applying on (3.2.3) from the right with the resolvent kernel operator $(I + R)$ appearing in (3.0.4) we get

$$\tilde{\alpha}\,[(I + \Omega)(I + R) + F\,G\,(I + R)] = -\omega(I + R) - f\,g\,(I + R). \tag{3.2.4}$$

Because of (3.0.4) we have

$$(I + \Omega)(I + R) = I. \tag{3.2.5}$$

Furthermore, from (3.0.10) we see that the first term on the right hand side in (3.2.4) is equal to α. Hence, by using $\alpha = -\omega(I + R)$ and with help of (3.2.5), we can write (3.2.4) in the form

$$\tilde{\alpha}\,[I + FG\,(I + R)] = \alpha - fg\,(I + R). \tag{3.2.6}$$

Now, let us define the operator $\tilde{G}$ as

$$\tilde{G} := G\,(I + R), \qquad \tilde{g}(x, y) := g(y) + \int_{-\infty}^{x} dz\, g(z)\, r(x; z, y), \tag{3.2.7}$$

where $r(x; z, y)$ is the kernel given in (3.1.14). Since $f(y)\,\tilde{g}(x, z)$ is the kernel of $F\tilde{G}$, (3.2.6) can be written as

$$\tilde{\alpha}(I + F\,\tilde{G}) = \alpha - f\tilde{g}. \tag{3.2.8}$$

Let us now solve (3.2.6) for $\tilde{\alpha}$. We would like to have a solution in the form

$$\tilde{\alpha}(x, y) = \alpha(x, y) + p(x)\,\tilde{g}(x, y), \tag{3.2.9}$$

with $p(x)$ to be determined. Substituting (3.2.9) into (3.2.8), we get

$$\alpha + p\,\tilde{g} + p\,\tilde{g}\,F\,\tilde{G} + \alpha\,F\,\tilde{G} = \alpha - f\tilde{g}. \tag{3.2.10}$$

After cancelling α from boths side of (3.2.10), we obtain

$$(\alpha\,F + p + p\,\tilde{g}\,F + f)\,\tilde{G} = 0, \tag{3.2.11}$$

which yields

$$p(I+\tilde{g}\,F)=-(f+\alpha\,F), \tag{3.2.12}$$

or explicitly written as

$$p(x)=-n(x)\left[I+\int_{-\infty}^{x}ds\,\tilde{g}(x,s)\,f(s)\right]^{-1}. \tag{3.2.13}$$

Substituting (3.2.13) into (3.2.9) we obtain

$$\tilde{\alpha}(x,y)=\alpha(x,y)-n(x)\left[I+\int_{-\infty}^{x}ds\,\tilde{g}(x,s)\,f(s)\right]^{-1}\tilde{g}(x,y). \tag{3.2.14}$$

which completes the proof. ∎

Next, we show that $\tilde{g}(x,y)$ defined in (3.2.7) can be expressed explicitly in terms of $\alpha(x,y)$ and $g(y)$ appearing in (3.0.8) and (3.0.17), respectively.

Propostion 3.2.2 *The quantity* $\tilde{g}(x,y)$ *defined in* (3.2.7) *can be expressed explicitly in terms of the solution* $\alpha(x,y)$ *to* (3.0.8) *and the quantities* $f(x)$ *and* $g(y)$ *appearing in* (3.0.17) *as*

$$\tilde{g}(x,y)=q(y)+\int_{y}^{x}ds\,q(s)\,\alpha(s,y), \tag{3.2.15}$$

where q(y) is the quantity defined in (3.2.1).

Proof. Let us consider (3.2.7). Using $\int_{-\infty}^{x}=\int_{-\infty}^{y}+\int_{y}^{x}$ in (3.2.7), we write the second equation in (3.2.7) as

$$\tilde{g}(x,y)=g(y)+\int_{-\infty}^{y}ds\,g(s)\,r(s,y)+\int_{y}^{x}ds\,g(s)\,r(s,y). \tag{3.2.16}$$

Substituting the first line of (3.1.14) in the integral $\int_{y}^{x}$ of (3.2.16) and the second line in the integral $\int_{-\infty}^{y}$ of (3.2.16), we get

$$\begin{aligned}\tilde{g}(x,y)=g(y)&+\int_{-\infty}^{y}ds\,g(s)\,\left[J\,[\alpha(y,s)]^{\dagger}\,J+\int_{y}^{x}dt\,J\,[\alpha(t,s)]^{\dagger}\,J\,\alpha(t,y)\right]\\&+\int_{y}^{x}ds\,g(s)\left[\alpha(s,y)+\int_{s}^{x}dt\,J\,[\alpha(t,s)]^{\dagger}\,J\,\alpha(t,y)\right].\end{aligned}$$

After the distribution, we obtain

$$\tilde{g}(x,y) = g(y) + \int_{-\infty}^{y} ds\, g(s)\, J\, [\alpha(y,s)]^{\dagger}\, J + \int_{-\infty}^{y} ds \int_{y}^{x} dt\, g(s)\, J\, [\alpha(t,s)]^{\dagger}\, J\, \alpha(t,y)$$
$$+ \int_{y}^{x} ds\, g(s)\, \alpha(s,y) + \int_{y}^{x} ds \int_{s}^{x} dt\, g(s)\, J\, [\alpha(t,s)]^{\dagger}\, J\, \alpha(t,y). \tag{3.2.17}$$

The orders of the two iterated integrals in (3.2.17) can be changed to $\int_y^x dt \int_{-\infty}^y ds$ and $\int_y^x dt \int_y^t ds$, respectively. Using $\int_{-\infty}^y + \int_y^t = \int_{-\infty}^t$, we get

$$\tilde{g}(x,y) = g(y) + \int_{-\infty}^{y} ds\, g(s)\, J\, [\alpha(y,s)]^{\dagger}\, J + \int_{y}^{x} ds\, g(s)\, \alpha(s,y)$$
$$+ \int_{y}^{x} dt \int_{-\infty}^{t} ds\, g(s)\, J\, [\alpha(t,s)]^{\dagger}\, J\, \alpha(t,y).$$

We can replace $g(y) + \int_{-\infty}^y ds\, g(s)\, J\, [\alpha(y,s)]^\dagger\, J$ with $q(y)$ because of (3.2.1). Then we get

$$\tilde{g}(x,y) = q(y) + \int_{y}^{x} ds\, g(s)\, \alpha(s,y) + \int_{y}^{x} dt \int_{-\infty}^{t} ds\, g(s)\, J\, [\alpha(t,s)]^{\dagger}\, J\, \alpha(t,y). \tag{3.2.18}$$

We now interchange s and t in the iterated integral in (3.2.18), and thus obtain

$$\tilde{g}(x,y) = q(y) + \int_{y}^{x} ds\, g(s)\, \alpha(s,y) + \int_{y}^{x} ds \int_{-\infty}^{s} dt\, g(t)\, J\, [\alpha(s,t)]^{\dagger}\, J\, \alpha(s,y), \tag{3.2.19}$$

or equivalently

$$\tilde{g}(x,y) = q(y) + \int_{y}^{x} ds \left[g(s) + \int_{-\infty}^{s} dt\, g(t)\, J\, [\alpha(s,t)]^{\dagger}\, J \right] \alpha(s,y). \tag{3.2.20}$$

By replacing $g(s) + \int_{-\infty}^s dt\, g(t)\, J\, [\alpha(s,t)]^\dagger\, J$ with $q(s)$ because of (3.2.1), from (3.2.20) we get (3.2.15). ∎

Note that $\tilde{g}(x,x) = q(x)$. Let us define the matrix $\Gamma(x)$ as

$$\Gamma(x) := I + \int_{-\infty}^{x} ds\, \tilde{g}(x,s)\, f(s). \tag{3.2.21}$$

Proposition 3.2.3 *The quantity* $\Gamma(x)$ *defined in* (3.2.21) *can be expressed explicitly in terms of the solution* $\alpha(x,y)$ *to* (3.0.8) *and the quantities* $f(x)$ *and* $g(y)$ *appearing in* (3.0.17) *as*

$$\Gamma(x) = I + \int_{-\infty}^{x} ds\, q(s)\, n(s), \tag{3.2.22}$$

where we have defined $n(x)$ *and* $q(x)$ *in* (3.2.1).

Proof. By substituting (3.2.15) into (3.2.21), then we obtain

$$\Gamma(x) = I + \int_{-\infty}^{x} ds \left[q(s) + \int_{s}^{x} dt\, q(t)\, \alpha(t,s) \right] f(s),$$

or equivalently

$$\Gamma(x) = I + \int_{-\infty}^{x} ds\, q(s)\, f(s) + \int_{-\infty}^{x} ds \int_{s}^{x} dt\, q(t)\, \alpha(t,s)\, f(s). \tag{3.2.23}$$

The order of the integration in (3.2.23) can be changed to $\int_{-\infty}^{x} dt \int_{-\infty}^{t} ds$, and interchanging s and t in the iterated integral we get

$$\Gamma(x) = I + \int_{-\infty}^{x} ds\, q(s)\, f(s) + \int_{-\infty}^{x} ds \int_{-\infty}^{s} dt\, q(s)\, \alpha(s,t)\, f(t), \tag{3.2.24}$$

or equivalently

$$\Gamma(x) = I + \int_{-\infty}^{x} ds\, q(s) \left[f(s) + \int_{-\infty}^{s} dt\, q(s)\, \alpha(s,t)\, f(t) \right]. \tag{3.2.25}$$

By replacing $f(s) + \int_{-\infty}^{s} dt\, q(s)\, \alpha(s,t)\, f(t)$ with $n(s)$ because of (3.2.1), we obtain

$$\Gamma(x) = I + \int_{-\infty}^{x} ds\, q(s)\, n(s). \tag{3.2.26}$$

Thus, the proof is complete. ∎

The next theorem describes the formula for the Darboux transformation at the potential level.

Theorem 3.2.4 *Let* α *and* $\tilde{\alpha}$ *be the solutions to the integral equations* (3.0.8) *and* (3.0.15), *respectively, and let* $n(x)$, $\Gamma(x)$, *and* $\tilde{g}(x,y)$ *be the quantities given in* (3.2.1), *(3.2.22), and (3.2.15), respectively. Then, $\tilde{\alpha}(x,y)-\alpha(x,y)$ can be written in terms of $\alpha(x,y)$, $f(x)$, and $g(y)$ as*

$$\tilde{\alpha}(x,y)-\alpha(x,y)=-n(x)\,\Gamma(x)^{-1}\,\tilde{g}(x,y). \tag{3.2.27}$$

Hence, the Darboux transformation at the potential level is obtained from

$$\tilde{\alpha}(x,x)-\alpha(x,x)=-n(x)\,\Gamma(x)^{-1}\,q(x). \tag{3.2.28}$$

Proof. By substituting (3.2.21) into (3.2.14) we obtain (3.2.27). Using $\tilde{g}(x,x)=q(x)$ in (3.2.27) we obtain (3.2.28).

3.3 Darboux Transformation at the Wave Function Level

We would now like to obtain the Darboux transformation at the wave function level.

The Fourier transform of the $N\times N$ matrix-valued quantity $\alpha(x,y)$ in (3.0.9) is related to the wave function $\Psi(\lambda,x)$ appearing in the unperturbed problem $\mathcal{L}\,\Psi=\lambda\,\Psi$. The relationship is given by

$$\Psi(\lambda,x):=e^{-\mathrm{i}\lambda Jx}+\int_{-\infty}^{x}dy\,\alpha(x,y)\,e^{-\mathrm{i}\lambda Jy}. \tag{3.3.1}$$

Recall that J is the involution matrix appearing in (3.0.12). Using the inverse Fourier transform on (3.3.1) we obtain

$$\alpha(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\,\left[\Psi(\lambda,x)-e^{-\mathrm{i}\lambda Jx}\right]\,e^{\mathrm{i}\lambda Jy}. \tag{3.3.2}$$

Similar to (3.3.1) and (3.3.2), we have

$$\tilde{\Psi}(\lambda,x):=e^{-\mathrm{i}\lambda Jx}+\int_{-\infty}^{x}dy\,\tilde{\alpha}(x,y)\,e^{-\mathrm{i}\lambda Jy}, \tag{3.3.3}$$

$$\tilde{\alpha}(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\,\left[\tilde{\Psi}(\lambda,x)-e^{-\mathrm{i}\lambda Jx}\right]\,e^{\mathrm{i}\lambda Jy}. \tag{3.3.4}$$

Now define $\gamma(\lambda, x)$ as

$$\gamma(\lambda, x) := \int_{-\infty}^{x} dy\, \tilde{g}(x, y)\, e^{-\mathrm{i}\lambda J y}. \tag{3.3.5}$$

Using (3.2.15) in (3.3.5) we get

$$\gamma(\lambda, x) = \int_{-\infty}^{x} dy\, \left[q(y) + \int_{y}^{x} ds\, q(s)\, \alpha(s, y) \right] e^{-\mathrm{i}\lambda J y}, \tag{3.3.6}$$

or equivalently

$$\gamma(\lambda, x) = \int_{-\infty}^{x} ds\, q(s) e^{-\mathrm{i}\lambda J s} + \int_{-\infty}^{x} dy \int_{y}^{x} ds\, q(s)\, \alpha(s, y)\, e^{-\mathrm{i}\lambda J y}, \tag{3.3.7}$$

where we have replaced the dummy variable y with s in the first integral in (3.3.7). The orders of the iterated integral in (3.3.7) can be changed to $\int_{-\infty}^{x} ds \int_{-\infty}^{s} dy$, and we obtain

$$\gamma(\lambda, x) = \int_{-\infty}^{x} ds\, q(s) e^{-\mathrm{i}\lambda J s} + \int_{-\infty}^{x} ds \int_{-\infty}^{s} dy\, q(s)\, \alpha(s, y)\, e^{-\mathrm{i}\lambda J y}, \tag{3.3.8}$$

or equivalently

$$\gamma(\lambda, x) = \int_{-\infty}^{x} ds\, q(s) \left[e^{-\mathrm{i}\lambda J s} + \int_{-\infty}^{s} dy\, \alpha(s, y)\, e^{-\mathrm{i}\lambda J y} \right]. \tag{3.3.9}$$

By using (3.3.1) in (3.3.9) we get

$$\gamma(\lambda, x) = \int_{-\infty}^{x} ds\, q(s)\, \Psi(\lambda, s). \tag{3.3.10}$$

The next theorem describes the formula for the Darboux transformation at the wave function level.

Theorem 3.3.1 *Let α and $\tilde{\alpha}$ be the solutions to the integral equations* (3.0.8) *and* (3.0.15)*, respectively, and let $n(x)$, $\Gamma(x)$, and $\gamma(\lambda, x)$ be the quantities given in* (3.2.1), (3.2.22)*, and* (3.3.10)*, respectively. Then, $\tilde{\Psi}(\lambda, x) - \Psi(\lambda, x)$ can be written in terms of $\alpha(x, y)$, $f(x)$, and $g(y)$ as*

$$\tilde{\Psi}(\lambda, y) - \Psi(\lambda, y) = -n(x)\, \Gamma(x)^{-1}\, \gamma(\lambda, x). \tag{3.3.11}$$

Proof. By subtracting (3.3.3) from (3.3.1) we obtain

$$\tilde{\Psi}(\lambda, x) - \Psi(\lambda, x) = \int_{-\infty}^{x} dy \, [\tilde{\alpha}(x, y) - \alpha(x, y)] \, e^{-\mathrm{i}\lambda J y}. \tag{3.3.12}$$

By substituting (3.2.27) into (3.3.12) we get

$$\tilde{\Psi}(\lambda, x) - \Psi(\lambda, x) = \int_{-\infty}^{x} dy \left[-n(x) \, \Gamma(x)^{-1} \, \tilde{g}(x, y)\right] e^{-\mathrm{i}\lambda J y}, \tag{3.3.13}$$

which yields

$$\tilde{\Psi}(\lambda, x) - \Psi(\lambda, x) = -n(x) \, \Gamma(x)^{-1} \int_{-\infty}^{x} dy \, \tilde{g}(x, y) \, e^{-\mathrm{i}\lambda J y}. \tag{3.3.14}$$

Using (3.3.5) in (3.3.14) we get

$$\tilde{\Psi}(\lambda, y) - \Psi(\lambda, y) = -n(x) \, \Gamma(x)^{-1} \, \gamma(\lambda, x). \tag{3.3.15}$$

Thus, the proof is complete. ∎

3.4 Darboux Transformation via a Constant Matrix Triplet

In this section we show that the results given in Theorem (3.2.4) and Theorem (3.3.1) provide a unified approach to derive Darboux transformations.

Suppose we add a discrete eigenvalue λ_j with multiplicity n_j to the existing spectrum. Then, associated with the eigenvalue λ_j, there are n_j parameters $c_{j0}, \ldots, c_{j(n_j-1)}$, usually known as norming constants [5]. Consequently, for each discrete eigenvalue λ_j added to the spectrum, there will be an n_j-parameter family of potentials $\tilde{u}(x)$ where the norming constants act as the parameters. In case several discrete eigenvalues $\lambda_1, \ldots, \lambda_N$ are added all at once, it is convenient to use a square matrix A whose eigenvalues are related to λ_j for $j = 1, \ldots, N$ in a simple manner. It is also convenient to use a matrix C whose entries are related to the norming constants c_{js} for $j = 1, \ldots, N$ and $s = 0, 1, \ldots, n_j - 1$.

The matrices $f(x)$ and $g(y)$ appearing in (3.0.17) can be written in the form

$$f(x) = \begin{bmatrix} 0 & B^\dagger e^{-A^\dagger x} \\ C\, e^{-Ax} & 0 \end{bmatrix}, \qquad g(y) = \begin{bmatrix} e^{-Ay} B & 0 \\ 0 & -e^{-A^\dagger y} C^\dagger \end{bmatrix}, \tag{3.4.1}$$

where A is a constant square matrix with all eigenvalues having positive real parts, and B and C are constant matrices of appropriate sizes so that the matrix product $f(x)\, g(y)$ is well defined and given by

$$f(x)\, g(y) = \begin{bmatrix} 0 & -B^\dagger e^{-A^\dagger(x+y)} C^\dagger \\ C\, e^{-A(x+y)} B & 0 \end{bmatrix}. \tag{3.4.2}$$

For $f(x)$ and $g(y)$ given in (3.4.1), let us evaluate $\tilde{g}(x,y)$ given in (3.2.15) and the intermediate quantities $n(x)$ and $q(x)$ given in (3.2.1) explicitly in terms of the wave-function $\Psi(\lambda,x)$ evaluated at the eigenvalues of A. By taking the matrix adjoint of $\alpha(x,y)$ appearing in (3.3.2) we obtain

$$J\, [\alpha(x,y)]^\dagger\, J = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda\, e^{-\mathrm{i}\lambda J y} \left[J\, [\Psi(\lambda,x)]^\dagger\, J - e^{\mathrm{i}\lambda J x} \right]. \tag{3.4.3}$$

By substituting (3.3.2) in the first formula in (3.2.1) we get

$$n(x) = f(x) + \frac{1}{2\pi} \int_{-\infty}^{x} dy \int_{-\infty}^{\infty} d\lambda\, \left[\Psi(\lambda,x) - e^{-\mathrm{i}\lambda J x} \right]\, e^{\mathrm{i}\lambda J y} f(y), \tag{3.4.4}$$

or equivalently

$$n(x) = f(x) + \int_{-\infty}^{x} dy \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda\, \Psi(\lambda,x) e^{\mathrm{i}\lambda J y} - \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda\, e^{\mathrm{i}\lambda J (y-x)} \right] f(y). \tag{3.4.5}$$

Using the fact that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} ds\, e^{\pm \mathrm{i} a s} = \delta(a), \tag{3.4.6}$$

where δ is the Dirac delta distribution, from (3.4.5) we obtain

$$n(x) = f(x) + \int_{-\infty}^{x} dy \left[\left[\frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda\, \Psi(\lambda,x) e^{\mathrm{i}\lambda J y} \right] - \delta(x-z) \right] f(y), \tag{3.4.7}$$

or equivalently

$$n(x) = f(x) + \frac{1}{2\pi} \int_{-\infty}^{x} dy \int_{-\infty}^{\infty} d\lambda \, \Psi(\lambda, x) e^{\mathrm{i}\lambda J y} f(y) - \int_{-\infty}^{\infty} d\lambda \, \delta(x - z) f(y). \quad (3.4.8)$$

By using (3.4.6) in (3.4.8), we get

$$n(x) = f(x) + \frac{1}{2\pi} \left[\int_{-\infty}^{x} dy \int_{-\infty}^{\infty} d\lambda \, \Psi(\lambda, x) e^{\mathrm{i}\lambda J y} f(y) \right] - f(x). \quad (3.4.9)$$

After simplification in (3.4.9), we obtain

$$n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \, \Psi(\lambda, x) \int_{-\infty}^{x} dy \, e^{\mathrm{i}\lambda J y} \, f(y). \quad (3.4.10)$$

Using in (3.4.10) the value of $f(x)$ given in (3.4.1), we get

$$n(x) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} d\lambda \, \Psi(\lambda, x) \, e^{\mathrm{i}\lambda J x} \, \mathcal{N}(\lambda, x), \quad (3.4.11)$$

where we have defined

$$\mathcal{N}(\lambda, x) := \begin{bmatrix} 0 & -B^\dagger (\lambda I + \mathrm{i}A^\dagger)^{-1} e^{-A^\dagger x} \\ C(\lambda I - \mathrm{i}A)^{-1} e^{-Ax} & 0 \end{bmatrix}. \quad (3.4.12)$$

Similarly, by using (3.4.3) in the second formula in (3.2.1), we evaluate $q(x)$ as

$$q(x) = g(x) + \int_{-\infty}^{x} dy \, g(x) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \, e^{-\mathrm{i}\lambda J y} \left[J \left[\Psi(\lambda, x) \right]^\dagger J - e^{\mathrm{i}\lambda J x} \right] \right], \quad (3.4.13)$$

or equivalently

$$q(x) = g(x) + \int_{-\infty}^{x} dy \, g(x) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \, e^{-\mathrm{i}\lambda J y} J \left[\Psi(\lambda, x) \right]^\dagger J - \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \, e^{-\mathrm{i}\lambda J y} \, e^{\mathrm{i}\lambda J x} \right], \quad (3.4.14)$$

By using (3.4.6) in (3.4.14) and by proceeding as in (3.4.5)-(3.4.10), we get

$$q(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{x} dy \, g(y) \, e^{-\mathrm{i}\lambda J y} \, J \, \left[\Psi(\lambda, x) \right]^\dagger J. \quad (3.4.15)$$

Using in (3.4.15) the expression for $g(y)$ given in (3.4.1), we obtain

$$q(x) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} d\lambda \, \Theta(\lambda, x) \, e^{-\mathrm{i}\lambda J x} \, J \, \left[\Psi(\lambda, x) \right]^\dagger J, \quad (3.4.16)$$

where we have defined

$$\Theta(\lambda, x) := \begin{bmatrix} e^{-Ax}(\lambda I - \mathrm{i}A)^{-1}B & 0 \\ 0 & e^{-A^\dagger x}(\lambda I + \mathrm{i}A^\dagger)^{-1}C^\dagger \end{bmatrix}. \tag{3.4.17}$$

We still need to evaluate $\tilde{g}(x, y)$ in terms of the wave function $\Psi(\lambda, x)$. For this purpose, we use (3.3.2) in (3.2.15), (3.3.1), (3.4.6), and (3.4.16), and thus we obtain

$$\tilde{g}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mu \int_{-\infty}^{x} dz\, q(z)\, \Psi(\mu, z)\, e^{\mathrm{i}\mu J y}. \tag{3.4.18}$$

Using (3.4.16) in (3.4.18) we get

$$\tilde{g}(x, y) = \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\mu \int_{-\infty}^{x} dz\, E(\lambda, \mu, z)\, e^{\mathrm{i}\mu J y}, \tag{3.4.19}$$

where we have defined

$$E(\lambda, \mu, x) := \frac{1}{4\pi^2 \mathrm{i}}\, \Theta(\lambda, x)\, e^{-\mathrm{i}\lambda J x}\, J\, [\Psi(\lambda, x)]^\dagger\, J\, \Psi(\mu, x). \tag{3.4.20}$$

Using the expression for $n(x)$ and $q(x)$ given in (3.4.11) and (3.4.16), respectively, with the help of (3.4.20) we evaluate $\Gamma(x)$ given in (3.2.22) as

$$\Gamma(x) = I - \mathrm{i} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\mu \int_{-\infty}^{x} dy\, E(\lambda, \mu, y)\, e^{\mathrm{i}\mu J y}\, \mathcal{N}(\mu, y). \tag{3.4.21}$$

Finally, by substituting (3.4.16) in (3.3.10) and using (3.4.20), we obtain

$$\gamma(\lambda, x) = 2\pi \int_{-\infty}^{x} ds \int_{-\infty}^{\infty} d\lambda\, E(\lambda, \lambda, s). \tag{3.4.22}$$

Chapter 4

Gel'fand-Levitan Equation

In this chapter we develop a generalized approach for the Darboux transformation via the Gel'fand-Levitan equation. We first present some preliminary results that are needed later on and then we analyze the resolvent kernel $r(x; z, y)$ of the Gel'fand-Levitan equation. In Theorem 4.1.2 we prove that $r(x; z, y)$ can be expressed explicitly in terms of the solution $\alpha(x, y)$ to the Gel'fand-Levitan equation (4.0.9). We then give the formulas for the Darboux transformation at the potential level and wave function level.

The results in this chapter are somewhat similar to the results in Chapter 2, but for the benefit of the reader, we provide all the details. Furthermore, for the reader to see the similarity, we use similar notations for the corresponding quantities.

Let us consider the integral equation

$$\alpha(x,y) + \beta(x,y) + \int_0^x dz\, \alpha(x,z)\, \omega(z,y) = 0, \qquad 0 < y < x, \tag{4.0.1}$$

where $\alpha(x, y)$ is the unknown term, $\beta(x, y)$ is the nonhomogeneous term, and $\omega(z, y)$ is the integral kernel. We assume that the equation given in (4.0.1) is uniquely solvable in $\mathcal{H}_2^{N\times N}$, namely, we get a unique $\alpha(x, y)$ when $\beta(x, y)$ and $\omega(z, y)$ are given to us. Recall that $\mathcal{H}_p^{M\times N}$ is the complex Banach space of $M \times N$ matrix-valued measurable functions $F : (x, +\infty) \to \mathbb{C}^{M\times N}$ such that the matrix norm $\|F(\cdot)\|$ belongs to $L^p(x, +\infty)$ for $1 \le p \le +\infty$. Note that the kernel $\omega(z, y)$ does not depend on the parameter $x \in R$ and it satisfies

$$\sup_{x>y} \int_0^x dz\, (\|\omega(z,y)\| + \|\omega(y,z)\|) < +\infty. \tag{4.0.2}$$

Recall that $\|\cdot\|$ denotes any $N\times N$- matrix norm. Let us write (4.0.1) in the operator form

$$\alpha+\beta+\alpha\,\Omega=0, \tag{4.0.3}$$

where the operator Ω acts from the right. We suppose that, for each fixed $x\in R$, the operator $(I+\Omega)$ is invertible on $\mathcal{H}_1^{N\times N}$ and $\mathcal{H}_2^{N\times N}$. Note that I denotes the identity operator. We use $(I+R)$ to denote the corresponding resolvent operator, where

$$I+R=(I+\Omega)^{-1},\qquad R:=(I+\Omega)^{-1}-I. \tag{4.0.4}$$

By solving (4.0.3) for α we get

$$\alpha=-\beta\,(I+\Omega)^{-1}. \tag{4.0.5}$$

Using (4.0.4) in (4.0.5), we obtain

$$\alpha=-\beta(I+R), \tag{4.0.6}$$

or explicitly

$$\alpha(x,y)=-\beta(x,y)-\int_0^x dz\,\beta(x,z)\,r(x;z,y), \tag{4.0.7}$$

where $r(x;z,y)$ is the integral kernel of the resolvent operator R. Let us write (4.0.1) in the form

$$\alpha+\omega+\alpha\,\Omega=0, \tag{4.0.8}$$

where the nonhomogeneous term and the integral kernel are related to each other. Then (4.0.8) can explicitly be written as

$$\alpha(x,y)+\omega(x,y)+\int_0^x dz\,\alpha(x,z)\,\omega(z,y)=0,\qquad 0<y<x, \tag{4.0.9}$$

which is usually called [3, 4, 6, 10, 13, 14, 18] the Gel'fand-Levitan equation. We make the assumption that the Gel'fand-Levitan equation is uniquely solvable in $\mathcal{H}_2^{N\times N}$, namely, we assume that there is a unique $\alpha(x,y)$ as a solution to (4.0.9) when $\omega(x,y)$ is given. We construct the resolvent kernel $r(x;z,y)$ appearing in (4.0.7) in terms of the unique $\alpha(x,y)$ then the resolvent kernel will also be unique. Since $\alpha(x,y)$ is uniquely determined, we will then obtain $r(x;z,y)$ uniquely. By solving (4.0.8) with help of (4.0.4) we get

$$\alpha = -\omega(I+R). \tag{4.0.10}$$

The unique solvability of (4.0.9) in $\mathcal{H}_1^{N\times N}$ and the condition in (4.0.2) imply that

$$\sup_{x>y}\int_0^x dz\,(\|\alpha(z,y)\|+\|\alpha(y,z)\|)<+\infty. \tag{4.0.11}$$

We consider (4.0.8) when the integral operator Ω appearing in (4.0.3) is $N\times N$ matrix-valued and J-selfadjoint in the sense that

$$\Omega = J\,\Omega^{\ddagger}J, \qquad \omega(y,z)=J\,[\omega(z,y)]^{\dagger}\,J. \tag{4.0.12}$$

Recall that the single dagger denotes the matrix adjoint (complex conjugate and matrix transpose) and the double dagger denotes not just only the matrix adjoint, but also switching the two arguments in the kernel of the operator, namely, for any two integral operators A and B

$$(A\,B)^{\ddagger}(y,z)=(B^{\ddagger}\,A^{\ddagger})(y,z), \tag{4.0.13}$$

which is already stated in (2.0.16). Note also that J is an $N\times N$ selfadjoint involution, i.e.

$$J=J^{\dagger}=J^{-1}. \tag{4.0.14}$$

Associated with the unperturbed problem $\mathcal{L}\,\Psi=\lambda\,\Psi$, we have [3, 4, 6, 10, 13, 14, 18] the fundamental integral equation given in (4.0.8), where α is related to the Fourier transform of Ψ and ω is related to the Fourier transform of the scattering data $S(\lambda)$ for the operator $\mathcal{L}$.

Associated with the perturbed problem $\tilde{\mathcal{L}}\,\tilde{\Psi} = \lambda\,\tilde{\Psi}$, we have the fundamental integral equation

$$\tilde{\alpha} + \tilde{\omega} + \tilde{\alpha}\,\tilde{\Omega} = 0, \tag{4.0.15}$$

or explicitly

$$\tilde{\alpha}(x,y) + \tilde{\omega}(x,y) + \int_0^x dz\,\tilde{\alpha}(x,z)\,\tilde{\omega}(z,y) = 0, \quad y < x. \tag{4.0.16}$$

As we indicated in Chapter 2, in the analysis of Darboux transformations, the perturbation will correspond to the case where the integral operators $\tilde{\Omega}$ and Ω appearing in (4.0.15) and (4.0.8), respectively, differ by a finite-rank operator and we denote that difference operator by FG, i.e.

$$\tilde{\Omega} = \Omega + FG, \qquad \tilde{\omega}(x,y) = \omega(x,y) + f(x)\,g(y). \tag{4.0.17}$$

Recall that we cannot in general expect F and G to commute, and hence in general $f\,g \neq g\,f$.

4.1 Construction of the Resolvent Kernel

In this section, we analyze the resolvent kernel $r(x;z,y)$ appearing in (4.0.7) and then we show that $r(x;z,y)$ can be expressed explicitly in terms of the solution $\alpha(x,y)$ to (4.0.9).

Proposition 4.1.1 *Assume that* (4.0.3) *is uniquely solvable in* $\mathcal{H}_2^{N\times N}$ *and that* Ω *satisfies* (4.0.12)*. Then, the operator* R *given in* (4.0.4) *and the corresponding kernel* $r(x;z,y)$ *appearing in* (4.0.7) *satisfy*

$$R = J\,R^{\ddagger}J, \qquad r(x;z,y) = J\,\left[r(x;y,z)\right]^{\dagger}J, \tag{4.1.1}$$

where J *is the involution matrix appearing in* (4.0.14).

Proof. From (4.0.4) we know that $(I+R)=(I+\Omega)^{-1}$ and thus

$$(I+\Omega)(I+R)=I=(I+R)(I+\Omega). \tag{4.1.2}$$

Then we get

$$R+\Omega+R\,\Omega=0, \tag{4.1.3}$$

and

$$R+\Omega+\Omega\,R=0. \tag{4.1.4}$$

By applying on (4.1.3) the double dagger transformation defined in (4.0.14), i.e. by taking the adjoint and switching the arguments, and further applying J on both sides, we obtain

$$JR^{\ddagger}J+J\Omega^{\ddagger}J+(J\Omega^{\ddagger}J)(JR^{\ddagger}J)=0, \tag{4.1.5}$$

equivalently, by using (4.0.12) we get

$$JR^{\ddagger}J+\Omega+\Omega\,(JR^{\ddagger}J)=0. \tag{4.1.6}$$

Since (4.0.3) is assumed to be uniquely solvable in $\mathcal{H}_2^{N\times N}$, by comparing (4.1.4) and (4.1.6) we get $R=JR^{\ddagger}J$.

Even though it is clear that $R=JR^{\ddagger}J$ implies $r(x;y,z)=J\,[r(x;z,y)]^{\dagger}\,J$, for clarity, let us also prove the same result by directly working with the kernel $r(x;z,y)$ for $y<z$ and $z<y$. Let us first consider (4.1.4) for $z<y$, which is explicitly written as

$$r(x;y,z)+\omega(y,z)+\int_0^x ds\,\omega(y,s)\,r(x;s,z)=0, \qquad z<y. \tag{4.1.7}$$

By taking the adjoint of (4.1.7) and then multiplying the resulting equation by J on the left and on the right, we obtain

$$J\,[r(x;y,z)]^{\dagger}\,J+J\,[\omega(y,z)]^{\dagger}\,J+\int_0^x ds\,J\,[\omega(y,s)\,r(x;s,z)]^{\dagger}\,J, \qquad z<y, \tag{4.1.8}$$

or equivalently

$$J\,[r(x;y,z)]^\dagger\,J + J\,[\omega(y,z)]^\dagger\,J + \int_0^x ds\,J\,[r(x;s,z)]^\dagger\,J\,J\,[\omega(y,s)]^\dagger\,J, \qquad z<y. \tag{4.1.9}$$

Using (4.0.12) in (4.1.9) we get

$$J\,[r(x;y,z)]^\dagger\,J + \omega(z,y) + \int_0^x ds\,J\,[r(x;s,z)]^\dagger\,J\,\omega(s,y), \qquad z<y. \tag{4.1.10}$$

By interchanging y and z in (4.1.10), we obtain the equivalent expression

$$J\,[r(x;z,y)]^\dagger\,J + \omega(y,z) + \int_0^x ds\,J\,[r(x;s,y)]^\dagger\,J\,\omega(s,z), \qquad y<z. \tag{4.1.11}$$

By comparing (4.1.7) and (4.1.11) and by using the uniqueness of the solution to (4.1.7), we see that $r(x;y,z) = J\,[r(x;z,y)]^\dagger\,J$ and we have

$$r(x;y,z) + \omega(y,z) + \int_0^x ds\,r(x;y,s)\,\omega(s,z), \qquad y<z. \tag{4.1.12}$$

Similarly, by considering (4.1.4) for $y<z$, we obtain

$$r(x;y,z) + \omega(y,z) + \int_0^x ds\,r(x;y,s)\,\omega(s,z), \qquad z<y. \tag{4.1.13}$$

Thus, we have shown that $r(x;y,z) = J\,[r(x;z,y)]^\dagger\,J$ for $y<z$ and $z<y$ which completes the proof. ∎

Theorem 4.1.2 *Assume that* (4.0.3) *is uniquely solvable in* $\mathcal{H}_2^{N\times N}$ *and that* Ω *satisfies* (4.0.12). *Then, the corresponding kernel* $r(x;z,y)$ *appearing in* (4.0.7) *can be expressed explicitly in terms of the solution* $\alpha(x,y)$ *to* (4.0.9) *as*

$$r(x;z,y) = \begin{cases} \alpha(z,y) + \int_z^x ds\,J\,[\alpha(s,z)]^\dagger\,J\,\alpha(s,y), & 0<y<z<x, \\ J\,[\alpha(y,z)]^\dagger\,J + \int_y^x ds\,J\,[\alpha(s,z)]^\dagger\,J\,\alpha(s,y), & 0<z<y<x, \end{cases} \tag{4.1.14}$$

where J *is the involution matrix appearing in* (4.0.12).

Proof. Since (4.0.3) is assumed to be uniquely solvable in $\mathcal{H}_2^{N\times N}$, so is (4.0.8) and hence the solution R to (4.1.3) is unique. Thus, it suffices to prove that the quantity defined in (4.1.14) satisfies (4.1.3), i.e. the quantity in (4.1.14) satisfies the integral equations

$$r(x;z,y)+\omega(z,y)+\int_0^x ds\, r(x;z,s)\,\omega(s,y)=0,\quad 0<\max\{y,z\}<x. \tag{4.1.15}$$

We will give the proof for both cases of $y<z$ and $z<y$.

Case 1: $0<y<z<x$

Let us use $\int_0^x=\int_0^z+\int_z^x$ in the integral appearing in (4.1.15). Then the left-hand side in (4.1.15) becomes

$$r(x;z,y)+\omega(z,y)+\int_0^z ds\, r(x;z,s)\,\omega(s,y)+\int_z^x ds\, r(x;z,s)\,\omega(s,y). \tag{4.1.16}$$

By using the first line of (4.1.14) in the integral $\int_0^z$ and the second line of (4.1.14) in the integral $\int_z^x$ in (4.1.16), we obtain as the left-hand side of (4.1.15)

$$\begin{aligned}\alpha(z,y)&+\int_z^x ds\, J\,[\alpha(s,z)]^\dagger J\,\alpha(s,y)+\omega(z,y)\\&+\int_0^z ds\left[\alpha(z,s)+\int_z^x dt\, J\,[\alpha(t,z)]^\dagger J\,\alpha(t,s)\right]\omega(s,y)\\&+\int_z^x ds\left[J\,[\alpha(s,z)]^\dagger J+\int_s^x dt\, J\,[\alpha(t,z)]^\dagger J\,\alpha(t,s)\right]\omega(s,y).\end{aligned}$$

After the distribution, the left-hand side of (4.1.15) then becomes

$$\begin{aligned}\alpha(z,y)&+\int_z^x ds\, J\,[\alpha(s,z)]^\dagger J\,\alpha(s,y)+\omega(z,y)\\&+\int_0^z ds\,\alpha(z,s)\,\omega(s,y)+\int_0^z ds\int_z^x dt\, J\,[\alpha(t,z)]^\dagger J\,\alpha(t,s)\,\omega(s,y)\\&+\int_z^x ds\, J\,[\alpha(s,z)]^\dagger J\,\omega(s,y)+\int_z^x ds\int_s^x dt\, J\,[\alpha(t,z)]^\dagger J\,\alpha(t,s)\,\omega(s,y).\end{aligned}$$

Let us now define c_1, c_2, and c_3 as

$$\begin{aligned}
c_1 &:= \alpha(z,y) + \omega(z,y) + \int_0^z ds\, \alpha(z,s)\, \omega(s,y), \\
c_2 &:= \int_z^x dt\, J\, [\alpha(t,z)]^\dagger\, J\, \alpha(t,y) + \int_z^x dt\, J\, [\alpha(t,z)]^\dagger\, J\, \omega(t,y), \\
c_3 &:= \int_0^z ds \int_z^x dt\, J\, [\alpha(t,z)]^\dagger\, J\, \alpha(t,s)\, \omega(s,y) \\
&\quad + \int_z^x ds \int_s^x dt\, J\, [\alpha(t,z)]^\dagger\, J\, \alpha(t,s)\, \omega(s,y).
\end{aligned}$$

We have $c_1 = 0$ because of the Gel'fand-Levitan equation given in (4.0.9). The orders of the two iterated integrals in c_3 can be changed to $\int_z^x dt \int_0^z ds$ and $\int_z^x dt \int_z^t ds$, respectively. Using $\int_0^z + \int_z^t = \int_0^t$, the quantity c_3 becomes

$$c_3 = \int_z^x dt \int_0^t ds\, J\, [\alpha(t,z)]^\dagger\, J\, \alpha(t,s)\, \omega(s,y).$$

Then we get

$$c_2 + c_3 = \int_z^x dt\, J\, [\alpha(t,z)]^\dagger\, J \left[\alpha(t,y) + \omega(t,y) + \int_0^t ds\, \alpha(t,s)\, \omega(s,y)\right]. \qquad (4.1.17)$$

Since $t > y$, the quantity inside the brackets in (4.1.17) vanishes due to (4.0.9). Thus, we obtain $c_2 + c_3 = 0$, and hence $c_1 + c_2 + c_3 = 0$ for the case of $y < z < x$.

Case 2: $0 < z < y < x$

Our goal is to show that the expression given in (4.1.14) satisfies (4.1.15) in the case $0 < z < y < x$. A direct proof in the case $0 < z < y < x$ does not seem to be feasible as it was done in the case $0 < y < z < x$. Therefore, we will proceed as follows. Since any compact operator can be approximated by a sequence of finite-rank operators, we will approximate $\omega(z,y)$ in (4.1.15) by the sequence $\xi_n(z)\, \eta_n(y)$. This will allow us to solve (4.1.15) explicitly and then to get a solution. We will call that solution as $r_n(x;z,y)$. Thus we will obtain (4.1.14) where $\alpha(z,y)$ is replaced by $\alpha_n(z,y)$ which is the solution to the Gel'fand-Levitan equation given in (4.1.18) when

$\omega(z,y)$ is replaced with $\xi_n(z)\,\eta_n(y)$. Finally, by taking the limit when $n \to +\infty$, we will obtain $r(x;z,y)$ given in (4.1.14)

Consider the Gel'fand-Levitan equation

$$\alpha(z,y) + \omega(z,y) + \int_0^z ds\,\alpha(z,s)\,\omega(s,y) = 0, \qquad 0 < y < z, \tag{4.1.18}$$

which obtained by replacing x and y in (4.0.9) by y and z, respectively. Suppose

$$\omega(z,y) = \xi(z)\,\eta(y), \tag{4.1.19}$$

i.e. suppose that the integral kernel in (4.1.14) is separable. We first would like to evaluate $\alpha(z,y)$ appearing in (4.1.14) in terms of the kernel parts ξ and η. Isolating $\alpha(z,y)$ in (4.1.18), we get

$$\alpha(z,y) = -\omega(z,y) - \int_0^z ds\,\alpha(z,s)\,\omega(s,y). \tag{4.1.20}$$

Substituting (4.1.19) into (4.1.20) we obtain

$$\alpha(z,y) = -\xi(z)\,\eta(y) - \int_0^z ds\,\alpha(z,s)\,\xi(s)\,\eta(y), \tag{4.1.21}$$

or equivalently

$$\alpha(z,y) = -\left[\xi(z) + \int_0^z ds\,\alpha(z,s)\,\xi(s)\right]\eta(y). \tag{4.1.22}$$

Define $\tau(z)$ as

$$\tau(z) := \xi(z) + \int_0^z ds\,\alpha(z,s)\,\xi(s). \tag{4.1.23}$$

Then we get

$$\alpha(z,y) = -\tau(z)\,\eta(y), \tag{4.1.24}$$

By replacing $\alpha(z,y)$ appearing in the integrand in (4.1.23) with (4.1.24) we obtain

$$\tau(z) = \xi(z) - \int_0^z ds\,\tau(z)\,\eta(s)\,\xi(s), \tag{4.1.25}$$

which yields

$$\tau(z)\left[I+\int_0^z ds\,\eta(s)\,\xi(s)\right]=\xi(z), \tag{4.1.26}$$

where I is the identity matrix. Thus we obtain

$$\tau(z)=\xi(z)\left[I+\int_0^z ds\,\eta(s)\,\xi(s)\right]^{-1}, \tag{4.1.27}$$

provided the matrix inverse exists. Consequently,

$$\alpha(z,y)=-\xi(z)\left[I+\int_0^z ds\,\eta(s)\,\xi(s)\right]^{-1}\eta(y). \tag{4.1.28}$$

Finally, define

$$\Upsilon(z):=I+\int_0^z ds\,\eta(s)\,\xi(s). \tag{4.1.29}$$

By substituting (4.1.29) into (4.1.28) we obtain

$$\alpha(z,y)=-\xi(z)\Upsilon(z)^{-1}\eta(y). \tag{4.1.30}$$

We would now like to evaluate $J\left[\alpha(y,z)\right]^\dagger J$ in terms of ξ and η. For this, let us consider (4.1.18). By taking the adjoint of (4.1.18) and then multiplying the resulting equation by J on the left and on the right, we get

$$J\left[\alpha(z,y)\right]^\dagger J+J\left[\omega(z,y)\right]^\dagger J+\int_0^z ds\,J\left[\alpha(z,s)\,\omega(s,y)\right]^\dagger J=0, \tag{4.1.31}$$

or equivalently

$$J\left[\alpha(z,y)\right]^\dagger J+J\left[\omega(z,y)\right]^\dagger J+\int_0^z ds\,J\left[\omega(s,y)\right]^\dagger J\,J\left[\alpha(z,s)\right]^\dagger J=0. \tag{4.1.32}$$

Using (4.0.12) in (4.1.32) we get

$$J\left[\alpha(z,y)\right]^\dagger J+\omega(y,z)+\int_0^z ds\,\omega(y,s)\,J\left[\alpha(z,s)\right]^\dagger J=0. \tag{4.1.33}$$

By substituting (4.1.19) into (4.1.33) we obtain

$$J\left[\alpha(z,y)\right]^\dagger J+\xi(y)\eta(z)+\int_0^z ds\,\xi(y)\eta(s)\,J\left[\alpha(z,s)\right]^\dagger J=0. \tag{4.1.34}$$

Let us set

$$J\left[\alpha(z,y)\right]^\dagger J := \xi(y)K(z), \tag{4.1.35}$$

with $K(z)$ to be determined. Substituting (4.1.35) into (4.1.34) and then equating the right coefficients of $\xi(y)$ from both sides of (4.1.34), we get

$$K(z) + \eta(z) + \int_0^z ds\, \eta(s)\, \xi(s)\, K(z) = 0. \tag{4.1.36}$$

By solving (4.1.36) for $K(z)$ we obtain

$$K(z) = -\left[I + \int_0^z ds\, \eta(s)\, \xi(s)\right]^{-1} \eta(z). \tag{4.1.37}$$

Using (4.1.29) in (4.1.37) we get

$$K(z) = -\Upsilon(z)^{-1}\, \eta(z). \tag{4.1.38}$$

By substituting (4.1.38) into (4.1.35) we obtain

$$J\left[\alpha(z,y)\right]^\dagger J = -\xi(y)\, \Upsilon(z)^{-1}\, \eta(z). \tag{4.1.39}$$

Hence, by interchanging y and z in (4.1.39) we obtain

$$J\left[\alpha(y,z)\right]^\dagger J = -\xi(z)\, \Upsilon(y)^{-1}\, \eta(y). \tag{4.1.40}$$

Now, let us solve the integral equation

$$r(x;z,y) + \omega(z,y) + \int_0^x ds\, r(x;z,s)\, \omega(s,y) = 0. \tag{4.1.41}$$

Define

$$r(z,y) = H(z)\, \eta(y), \tag{4.1.42}$$

with $H(z)$ to be determined. Substituting (4.1.19) and (4.1.42) into (4.1.41) and then equating the left coefficients of $\eta(y)$ from both sides of (4.1.41), we get

$$H(z) + \xi(z) + \int_0^x ds\, H(z)\, \eta(s)\, \xi(s) = 0,$$

which yields

$$H(z)\left[I+\int_0^x ds\,\eta(s)\,\xi(s)\right]=-\xi(z). \tag{4.1.43}$$

Then we get

$$H(z)=-\xi(z)\left[I+\int_0^x ds\,\eta(s)\,\xi(s)\right]^{-1}. \tag{4.1.44}$$

As a result

$$r(x;z,y)=-\xi(z)\left[I+\int_0^x ds\,\eta(s)\,\xi(s)\right]^{-1}\eta(y). \tag{4.1.45}$$

From (4.1.29) we see that the matrix in the brackets in (4.1.45) is equal to $\Upsilon(x)$. Thus we can write (4.1.45) as

$$r(x;z,y)=-\xi(z)\Upsilon(x)^{-1}\eta(y). \tag{4.1.46}$$

Then for $x<y<z$, with help of (4.1.30) and (4.1.40) we compute

$$\begin{aligned}
&\alpha(z,y)+\int_z^x ds\,J\,[\alpha(s,z)]^\dagger\,J\,\alpha(s,y)\\
&\quad=-\xi(z)\,\Upsilon(z)^{-1}\,\eta(y)+\int_z^x ds\,\xi(z)\,\Upsilon(s)^{-1}\,\eta(s)\,\omega_1(s)\,\Upsilon(s)^{-1}\,\eta(y)\\
&\quad=-\xi(z)\left[\Upsilon(z)^{-1}-\int_z^x ds\,\Upsilon(s)^{-1}\,\eta(s)\,\xi(s)\,\Upsilon(s)^{-1}\right]\eta(y)\\
&\quad=-\xi(z)\left[\Upsilon(z)^{-1}-\int_z^x ds\left(-\frac{d}{ds}\left[I+\int_0^s dt\,\eta(t)\,\xi(t)\right]^{-1}\right)\right]\eta(y)\\
&\quad=-\xi(z)\left\{\Upsilon(z)^{-1}-\left[\Upsilon(z)^{-1}-\Upsilon(x)^{-1}\right]\right\}\eta(y)\\
&\quad=-\xi(z)\,\Upsilon(x)^{-1}\eta(y),
\end{aligned}$$

which is equal to $r(x;y,z)$ because of (4.1.46). Similarly, for $x<z<y$ we compute

$$\begin{aligned}
&J\,[\alpha(y,z)]^\dagger\,J+\int_y^x ds\,J\,[\alpha(s,z)]^\dagger\,J\,\alpha(s,y)\\
&\quad=-\xi(z)\,\Upsilon(y)^{-1}\,\eta(y)+\int_y^x ds\,\xi(z)\,\Upsilon(s)^{-1}\,\eta(s)\,\xi(s)\,\Upsilon(s)^{-1}\,\eta(y)\\
&\quad=-\xi(z)\left[\Upsilon(y)^{-1}-\int_y^x ds\,\Upsilon(s)^{-1}\,\eta(s)\,\xi(s)\,\Upsilon(s)^{-1}\right]\eta(y).
\end{aligned} \tag{4.1.47}$$

We can write (4.1.47) as

$$
\begin{aligned}
J\,[\alpha(y,z)]^\dagger\,J+\int_y^x ds\,J\,[\alpha(s,z)]^\dagger\,J\,\alpha(s,y)
&= -\xi(z)\left[\Upsilon(y)^{-1}-\int_y^x ds\left(-\frac{d}{ds}\left[I+\int_0^s dt\,\eta(t)\,\xi(t)\right]^{-1}\right)\right]\eta(y)\\
&= -\xi(z)\left\{\Upsilon(y)^{-1}-\left[\Upsilon(y)^{-1}-\Upsilon(x)^{-1}\right]\right\}\eta(y)\\
&= -\xi(z)\,\Upsilon(x)^{-1}\,\eta(y),
\end{aligned}
$$

which is equal to $r(x;y,z)$ because of (2.1.46). Thus we have shown that

$$
r(x;z,y)=\begin{cases}\alpha(z,y)+\displaystyle\int_z^x ds\,J\,[\alpha(s,z)]^\dagger\,J\,\alpha(s,y), & 0<y<z<x,\\ J\,[\alpha(y,z)]^\dagger\,J+\displaystyle\int_y^x ds\,J\,[\alpha(s,z)]^\dagger\,J\,\alpha(s,y), & 0<z<y<x,\end{cases} \tag{4.1.48}
$$

where we have obtained $\alpha(z,y)$ and $[\alpha(y,z)]^\dagger$ in terms of the kernel parts ξ and η.

As a summary, in the case $z<y<x$, we can approximate the kernel of the Marchenko integral equation in (4.1.18), namely, $\omega(z,y)$ by the sequence of separable kernels $\xi_n(z)\,\eta_n(y)$, where the approximation is understood in the sense

$$
\lim_{n\to+\infty}\ \operatorname*{ess\,sup}_{y<x}\int_0^x dz\,\|\omega(z,y)-\xi_n(z)\,\eta_n(y)\|=0, \tag{4.1.49}
$$

i.e., let us assume that

$$
\lim_{n\to+\infty}\|\Omega-\Omega_n\|_{L^1\to L^1}=0. \tag{4.1.50}
$$

Since we then also have

$$
\lim_{n\to+\infty}\|(I+\Omega)^{-1}-(I+\Omega_n)^{-1}\|_{L^1\to L^1}=0, \tag{4.1.51}
$$

we get for the resolvent kernels

$$
\lim_{n\to+\infty}\ \operatorname*{ess\,sup}_{y<x}\int_0^x dz\,\|r(x;z,y)-r_n(x;z,y)\|=0. \tag{4.1.52}
$$

Recall that $\|\cdot\|$ is any $N \times N$-matrix norm. By using the results in the degenerate case, we get

$$r_n(x;z,y) = \begin{cases} \alpha_n(z,y) + \displaystyle\int_z^x ds\, J\, [\alpha_n(s,z)]^\dagger\, J\, \alpha_n(s,y), & 0 < y < z < x, \\ J\, [\alpha_n(y,z)]^\dagger\, J + \displaystyle\int_y^x ds\, J\, [\alpha_n(s,z)]^\dagger\, J\, \alpha_n(s,y), & 0 < z < y < x. \end{cases}$$

By taking the limit as $n \to +\infty$, we obtain

$$r(x;z,y) = \begin{cases} \alpha(z,y) + \displaystyle\int_z^x ds\, J\, [\alpha(s,z)]^\dagger\, J\, \alpha(s,y), & 0 < y < z < x, \\ J\, [\alpha(y,z)]^\dagger\, J + \displaystyle\int_y^x ds\, J\, [\alpha(s,z)]^\dagger\, J\, \alpha(s,y), & 0 < z < y < x. \end{cases}$$

which completes the proof. ∎

4.2 Darboux Transformation at the Potential Level

Now, we show that the solution $\tilde{\alpha}(x,y)$ to (4.0.16) can be expressed explicitly in terms of $\alpha(x,y)$, $f(x)$, and $g(y)$ appearing in (4.0.9) and (4.0.17), respectively. Then, we will obtain the formula for the Darboux transformation at the potential level.

Let us define the intermediate quantities $n(x)$ and $q(y)$ as

$$n(x) := f(x) + \int_0^x dz\, \alpha(x,z)\, f(z), \quad q(y) := g(y) + \int_0^y dz\, g(z)\, J\, [\alpha(y,z)]^\dagger\, J, \quad (4.2.1)$$

where J is the involution matrix appearing in (4.0.12).

Theorem 4.2.1 *Consider the perturbed operator* $\tilde{\Omega}$ *of* (4.0.15) *and the unperturbed operator* Ω *of* (4.0.8). *Let* F, G, f, *and* g *be the quantities appearing in* (4.0.17). *When* $\tilde{\Omega} - \Omega$ *is the finite-rank perturbation* FG *given in* (4.0.17), *we can transform* (4.0.15) *into an integral equation that has a degenerate kernel and hence obtain* $\tilde{\alpha}$ *explicitly by linear algebraic methods.*

Proof. By substituting (4.0.17) into (4.0.15) we obtain

$$\omega + f\, g + \tilde{\alpha}(I + \Omega + F\, G) = 0, \qquad (4.2.2)$$

which yields

$$\tilde{\alpha}(I+\Omega+F\,G)=-\omega-f\,g. \tag{4.2.3}$$

By applying on (4.2.3) from the right with the resolvent kernel operator $(I+R)$ appearing in (4.0.4) we get

$$\tilde{\alpha}\,[(I+\Omega)(I+R)+F\,G\,(I+R)]=-\omega(I+R)-f\,g\,(I+R). \tag{4.2.4}$$

Because of (4.0.4) we have

$$(I+\Omega)(I+R)=I. \tag{4.2.5}$$

Furthermore, from (4.0.10) we see that the first term on the right hand side in (4.2.4) is equal to α. Hence, by using $\alpha=-\omega(I+R)$ and with help of (4.2.5), we can write (4.2.4) in the form

$$\tilde{\alpha}\,[I+FG\,(I+R)]=\alpha-fg\,(I+R). \tag{4.2.6}$$

Now, let us define the operator $\tilde{G}$ as

$$\tilde{G}:=G\,(I+R),\qquad \tilde{g}(x,y):=g(y)+\int_0^x dz\,g(z)\,r(x;z,y), \tag{4.2.7}$$

where $r(x;z,y)$ is the kernel given in (4.1.14). Since $f(y)\,\tilde{g}(x,z)$ is the kernel of $F\tilde{G}$, (4.2.6) can be written as

$$\tilde{\alpha}(I+F\,\tilde{G})=\alpha-f\tilde{g}. \tag{4.2.8}$$

Let us now solve (4.2.6) for $\tilde{\alpha}$. We would like to have a solution in the form

$$\tilde{\alpha}(x,y)=\alpha(x,y)+p(x)\,\tilde{g}(x,y), \tag{4.2.9}$$

with $p(x)$ to be determined. Substituting (4.2.9) into (4.2.8), we get

$$\alpha+p\,\tilde{g}+p\,\tilde{g}\,F\,\tilde{G}+\alpha\,F\,\tilde{G}=\alpha-f\,\tilde{g}. \tag{4.2.10}$$

After cancelling α from boths side of (4.2.10), we obtain

$$(\alpha\,F+p+p\,\tilde{g}\,F+f)\,\tilde{G}=0, \tag{4.2.11}$$

which yields

$$p(I+\tilde{g}\,F)=-(f+\alpha\,F), \tag{4.2.12}$$

or explicitly written as

$$p(x)=-n(x)\left[I+\int_0^x ds\,\tilde{g}(x,s)\,f(s)\right]^{-1}. \tag{4.2.13}$$

Substituting (4.2.13) into (4.2.9) we obtain

$$\tilde{\alpha}(x,y)=\alpha(x,y)-n(x)\left[I+\int_0^x ds\,\tilde{g}(x,s)\,f(s)\right]^{-1}\tilde{g}(x,y). \tag{4.2.14}$$

which completes the proof. ∎

Next, we show that $\tilde{g}(x,y)$ defined in (4.2.7) can be expressed explicitly in terms of $\alpha(x,y)$ and $g(y)$ appearing in (4.0.8) and (4.0.17), respectively.

Proposition 4.2.2 *The quantity* $\tilde{g}(x,y)$ *defined in* (4.2.7) *can be expressed explicitly in terms of the solution* $\alpha(x,y)$ *to* (4.0.8) *and the quantities* $f(x)$ *and* $g(y)$ *appearing in* (4.0.17) *as*

$$\tilde{g}(x,y)=q(y)+\int_y^x ds\,q(s)\,\alpha(s,y), \tag{4.2.15}$$

where q(y) is the quantity defined in (4.2.1).

Proof. Let us consider (4.2.7). Using $\int_0^x=\int_0^y+\int_y^x$ in (4.2.7), we write the second equation in (4.2.7) as

$$\tilde{g}(x,y)=g(y)+\int_0^y ds\,g(s)\,r(x;s,y)+\int_y^x ds\,g(s)\,r(x;s,y). \tag{4.2.16}$$

Substituting the first line of (4.1.14) in the integral $\int_y^x$ of (4.2.16) and the second line in the integral $\int_0^y$ of (4.2.16), we get

$$\begin{aligned}\tilde{g}(x,y)=g(y)&+\int_y^x ds\,g(s)\,\left[\alpha(s,y)+\int_s^x dt\,J\,[\alpha(t,s)]^\dagger\,J\,\alpha(t,y)\right]\\&+\int_0^y ds\,g(s)\left[J\,[\alpha(y,s)]^\dagger\,J+\int_y^x dt\,J\,[\alpha(t,s)]^\dagger\,J\,\alpha(t,y)\right].\end{aligned}$$

After the distribution, we obtain

$$\begin{aligned}\tilde{g}(x,y) = g(y) &+ \int_0^y ds\, g(s)\, J\, [\alpha(y,s)]^\dagger\, J + \int_y^x ds\, g(s)\, \alpha(s,y) \\ &+ \left(\int_y^x ds \int_s^x dt + \int_0^y ds \int_y^x dt \right) g(s)\, J\, [\alpha(t,s)]^\dagger\, J\, \alpha(t,y). \quad (4.2.17)\end{aligned}$$

The orders of the two iterated integrals in (4.2.17) can be changed to $\int_y^x dt \int_y^t ds$ and $\int_y^x dt \int_0^y ds$, respectively. Using $\int_0^y + \int_y^t = \int_0^t$, we get

$$\begin{aligned}\tilde{g}(x,y) = g(y) &+ \int_0^y ds\, g(s)\, J\, [\alpha(y,s)]^\dagger\, J + \int_y^x ds\, g(s)\, \alpha(s,y) \\ &+ \int_y^x dt \int_0^t ds\, g(s)\, J\, [\alpha(t,s)]^\dagger\, J\, \alpha(t,y).\end{aligned}$$

We can replace $g(y) + \int_0^y ds\, g(s)\, J\, [\alpha(y,s)]^\dagger\, J$ with $q(y)$ because of (4.2.1). Then we get

$$\tilde{g}(x,y) = q(y) + \int_y^x ds\, g(s)\, \alpha(s,y) + \int_y^x dt \int_0^t ds\, g(s)\, J\, [\alpha(t,s)]^\dagger\, J\, \alpha(t,y). \quad (4.2.18)$$

We now interchange s and t in the iterated integral in (4.2.18), and thus obtain

$$\tilde{g}(x,y) = q(y) + \int_y^x ds\, g(s)\, \alpha(s,y) + \int_y^x ds \int_0^s dt\, g(t)\, J\, [\alpha(s,t)]^\dagger\, J\, \alpha(s,y), \quad (4.2.19)$$

or equivalently

$$\tilde{g}(x,y) = q(y) + \int_y^x ds \left[g(s) + \int_0^s dt\, g(t)\, J\, [\alpha(s,t)]^\dagger\, J \right] \alpha(s,y). \quad (4.2.20)$$

By replacing $g(s) + \int_0^s dt\, g(t)\, J\, [\alpha(s,t)]^\dagger\, J$ with $q(s)$ because of (4.2.1), from (4.2.20) we get (4.2.15). ▮

Note that $\tilde{g}(x,x) = q(x)$. Let us define the matrix $\Gamma(x)$ as

$$\Gamma(x) := I + \int_0^x ds\, \tilde{g}(x,s)\, f(s). \quad (4.2.21)$$

Proposition 4.2.3 *The quantity* $\Gamma(x)$ *defined in* (4.2.21) *can be expressed explicitly in terms of the solution* $\alpha(x,y)$ *to* (4.0.8) *and the quantities* $f(x)$ *and* $g(y)$ *appearing in* (4.0.17) *as*

$$\Gamma(x) = I + \int_0^x ds\, q(s)\, n(s), \quad (4.2.22)$$

where we have defined $n(x)$ *and* $q(x)$ *in* (3.2.1).

Proof. By substituting (4.2.15) into (4.2.21), then we obtain

$$\Gamma(x) = I + \int_0^x ds \left[q(s) + \int_s^x dt\, q(t)\, \alpha(t,s) \right] f(s),$$

or equivalently

$$\Gamma(x) = I + \int_0^x ds\, q(s)\, f(s) + \int_0^x ds \int_s^x dt\, q(t)\, \alpha(t,s)\, f(s). \tag{4.2.23}$$

The order of the integration in (4.2.23) can be changed to $\int_0^x dt \int_0^t ds$, and interchanging s and t in the iterated integral we get

$$\Gamma(x) = I + \int_0^x ds\, q(s)\, f(s) + \int_0^x ds \int_0^s dt\, q(s)\, \alpha(s,t)\, f(t), \tag{4.2.24}$$

or equivalently

$$\Gamma(x) = I + \int_0^x ds\, q(s) \left[f(s) + \int_0^s dt\, q(s)\, \alpha(s,t)\, f(t) \right]. \tag{4.2.25}$$

By replacing $f(s) + \int_0^s dt\, q(s)\, \alpha(s,t)\, f(t)$ with $n(s)$ because of (4.2.1), we obtain

$$\Gamma(x) = I + \int_0^x ds\, q(s)\, n(s). \tag{4.2.26}$$

Thus, the proof is complete. ∎

The next theorem describes the formula for the Darboux transformation at the potential level.

Theorem 4.2.4 *Let* α *and* $\tilde{\alpha}$ *be the solutions to the integral equations* (4.0.8) *and* (4.0.15), *respectively, and let* $n(x)$, $\Gamma(x)$, *and* $\tilde{g}(x,y)$ *be the quantities given in* (4.2.1), (4.2.22), *and* (4.2.15), *respectively. Then,* $\tilde{\alpha}(x,y) - \alpha(x,y)$ *can be written in terms of* $\alpha(x,y)$, $f(x)$, *and* $g(y)$ *as*

$$\tilde{\alpha}(x,y) - \alpha(x,y) = -n(x)\, \Gamma(x)^{-1}\, \tilde{g}(x,y). \tag{4.2.27}$$

Hence, the Darboux transformation at the potential level is obtained from

$$\tilde{\alpha}(x,x) - \alpha(x,x) = -n(x)\,\Gamma(x)^{-1}\,q(x). \tag{4.2.28}$$

Proof. By substituting (4.2.21) into (4.2.14) we obtain (4.2.27). Using $\tilde{g}(x,x) = q(x)$ in (4.2.27) we obtain (4.2.28). ∎

4.3 Darboux Transformation at the Wave Function Level

We would now like to obtain the Darboux transformation at the wave function level.

The Fourier transform of the $N \times N$ matrix-valued quantity $\alpha(x,y)$ in (4.0.9) is relatedto the wave function $\Psi(\lambda,x)$ appearing in the unperturbed problem $\mathcal{L}\,\Psi = \lambda\,\Psi$. The relationship is given by

$$\Psi(\lambda,x) := e^{-\mathrm{i}\lambda Jx} + \int_0^x dy\,\alpha(x,y)\,e^{-\mathrm{i}\lambda Jy}. \tag{4.3.1}$$

Recall that J is the involution matrix appearing in (4.0.12). Using the inverse Fourier transform on (4.3.1) we obtain

$$\alpha(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\,\left[\Psi(\lambda,x) - e^{-\mathrm{i}\lambda Jx}\right]\,e^{\mathrm{i}\lambda Jy}. \tag{4.3.2}$$

Similar to (4.3.1) and (4.3.2), we have

$$\tilde{\Psi}(\lambda,x) := e^{-\mathrm{i}\lambda Jx} + \int_0^x dy\,\tilde{\alpha}(x,y)\,e^{-\mathrm{i}\lambda Jy}, \tag{4.3.3}$$

$$\tilde{\alpha}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\,\left[\tilde{\Psi}(\lambda,x) - e^{-\mathrm{i}\lambda Jx}\right]\,e^{\mathrm{i}\lambda Jy}. \tag{4.3.4}$$

Now define $\gamma(\lambda,x)$ as

$$\gamma(\lambda,x) := \int_0^x dy\,\tilde{g}(x,y)\,e^{-\mathrm{i}\lambda Jy}. \tag{4.3.5}$$

Using (4.2.15) in (4.3.5) we get

$$\gamma(\lambda,x) = \int_0^x dy\,\left[q(y) + \int_y^x ds\,q(s)\,\alpha(s,y)\right]e^{-\mathrm{i}\lambda Jy}, \tag{4.3.6}$$

or equivalently

$$\gamma(\lambda,x)=\int_0^x ds\,q(s)e^{-\mathrm{i}\lambda Js}+\int_0^x dy\int_y^x ds\,q(s)\,\alpha(s,y)\,e^{-\mathrm{i}\lambda Jy},\tag{4.3.7}$$

where we have replaced the dummy variable y with s in the first integral in (4.3.7). The orders of the iterated integral in (4.3.7) can be changed to $\int_0^x ds \int_0^s dy$, and we obtain

$$\gamma(\lambda,x)=\int_0^x ds\,q(s)e^{-\mathrm{i}\lambda Js}+\int_0^x ds\int_0^s dy\,q(s)\,\alpha(s,y)\,e^{-\mathrm{i}\lambda Jy},\tag{4.3.8}$$

or equivalently

$$\gamma(\lambda,x)=\int_0^x ds\,q(s)\left[e^{-\mathrm{i}\lambda Js}+\int_0^s dy\,\alpha(s,y)\,e^{-\mathrm{i}\lambda Jy}\right].\tag{4.3.9}$$

By using (4.3.1) in (4.3.9) we get

$$\gamma(\lambda,x)=\int_0^x ds\,q(s)\,\Psi(\lambda,s).\tag{4.3.10}$$

The next theorem describes the formula for the Darboux transformation at the wave function level.

Theorem 4.3.1 *Let α and $\tilde{\alpha}$ be the solutions to the integral equations* (4.0.8) *and* (4.0.15)*, respectively, and let $n(x)$, $\Gamma(x)$, and $\gamma(\lambda,x)$ be the quantities given in* (4.2.1), (4.2.22)*, and* (4.3.10)*, respectively. Then, $\tilde{\Psi}(\lambda,x)-\Psi(\lambda,x)$ can be written in terms of $\alpha(x,y)$, $f(x)$, and $g(y)$ as*

$$\tilde{\Psi}(\lambda,y)-\Psi(\lambda,y)=-n(x)\,\Gamma(x)^{-1}\,\gamma(\lambda,x).\tag{4.3.11}$$

Proof. By subtracting (4.3.3) from (4.3.1) we obtain

$$\tilde{\Psi}(\lambda,x)-\Psi(\lambda,x)=\int_0^x dy\,[\tilde{\alpha}(x,y)-\alpha(x,y)]\,e^{-\mathrm{i}\lambda Jy}.\tag{4.3.12}$$

By substituting (4.2.27) into (4.3.12) we get

$$\tilde{\Psi}(\lambda,x)-\Psi(\lambda,x)=\int_0^x dy\,\left[-n(x)\,\Gamma(x)^{-1}\,\tilde{g}(x,y)\right]e^{-\mathrm{i}\lambda Jy},\tag{4.3.13}$$

which yields

$$\tilde{\Psi}(\lambda, x) - \Psi(\lambda, x) = -n(x)\,\Gamma(x)^{-1} \int_0^x dy\, \tilde{g}(x, y)\, e^{-i\lambda J y}. \tag{4.3.14}$$

Using (4.3.5) in (4.3.14) we get

$$\tilde{\Psi}(\lambda, y) - \Psi(\lambda, y) = -n(x)\,\Gamma(x)^{-1}\,\gamma(\lambda, x). \tag{4.3.15}$$

Thus, the proof is complete. ▮

4.4 Darboux Transformation via a Constant Matrix Triplet

In this section we show that the results given in Theorem (4.2.4) and Theorem (4.3.1) provide a unified approach to derive Darboux transformations.

Suppose we add a discrete eigenvalue λ_j with multiplicity n_j to the existing spectrum. Then, associated with the eigenvalue λ_j, there are n_j parameters $c_{j0}, \ldots, c_{j(n_j-1)}$, usually known as norming constants [5]. Consequently, for each discrete eigenvalue λ_j added to the spectrum, there will be an n_j-parameter family of potentials $\tilde{u}(x)$ where the norming constants act as the parameters. In case several discrete eigenvalues $\lambda_1, \ldots, \lambda_N$ are added all at once, it is convenient to use a square matrix A whose eigenvalues are related to λ_j for $j = 1, \ldots, N$ in a simple manner. It is also convenient to use a matrix C whose entries are related to the norming constants c_{js} for $j = 1, \ldots, N$ and $s = 0, 1, \ldots, n_j - 1$.

The matrices $f(x)$ and $g(y)$ appearing in (4.0.17) can be written in the form

$$f(x) = \begin{bmatrix} 0 & B^\dagger e^{-A^\dagger x} \\ C\, e^{-Ax} & 0 \end{bmatrix}, \qquad g(y) = \begin{bmatrix} e^{-Ay}\, B & 0 \\ 0 & -e^{-A^\dagger y}\, C^\dagger \end{bmatrix}, \tag{4.4.1}$$

where A is a constant square matrix, and B and C are constant matrices of appropriate sizes so that the matrix product $f(x)\,g(y)$ is well defined and given by

$$f(x)\,g(y)=\begin{bmatrix}0 & -B^\dagger\,e^{-A^\dagger(x+y)}C^\dagger\\ C\,e^{-A(x+y)}\,B & 0\end{bmatrix}. \tag{4.4.2}$$

For $f(x)$ and $g(y)$ given in (4.4.1), let us evaluate $\tilde{g}(x,y)$ given in (4.2.15) and the intermediate quantities $n(x)$ and $q(x)$ given in (4.2.1) explicitly in terms of the wavefunction $\Psi(\lambda,x)$ evaluated at the eigenvalues of A. By taking the matrix adjoint of $\alpha(x,y)$ appearing in (4.3.2) we obtain

$$J\,[\alpha(x,y)]^\dagger\,J=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\,e^{-\mathrm{i}\lambda Jy}\left[J\,[\Psi(\lambda,x)]^\dagger\,J-e^{\mathrm{i}\lambda Jx}\right]. \tag{4.4.3}$$

By substituting (4.3.2) in the first formula in (4.2.1) we get

$$n(x)=f(x)+\int_0^x dy\,\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\,\left[\Psi(\lambda,x)-e^{-\mathrm{i}\lambda Jx}\right]\,e^{\mathrm{i}\lambda Jy}f(y), \tag{4.4.4}$$

or equivalently

$$n(x)=f(x)+\int_0^x dy\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\,\Psi(\lambda,x)e^{\mathrm{i}\lambda Jy}-\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\,e^{\mathrm{i}\lambda J(y-x)}\right]f(y). \tag{4.4.5}$$

Using the fact that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}ds\,e^{\pm\mathrm{i}as}=\delta(a), \tag{4.4.6}$$

where δ is the Dirac delta distribution, from (4.4.5) we obtain

$$n(x)=f(x)+\int_0^x dy\left[\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\,\Psi(\lambda,x)e^{\mathrm{i}\lambda Jy}\right]-\delta(x-z)\right]f(y), \tag{4.4.7}$$

or equivalently

$$n(x)=f(x)+\frac{1}{2\pi}\int_0^x dy\int_{-\infty}^{\infty}d\lambda\,\Psi(\lambda,x)e^{\mathrm{i}\lambda Jy}f(y)-\int_{-\infty}^{\infty}d\lambda\,\delta(x-z)f(y). \tag{4.4.8}$$

By using (4.4.6) in (4.4.8), we get

$$n(x)=f(x)+\frac{1}{2\pi}\left[\int_0^x dy\int_{-\infty}^{\infty}d\lambda\,\Psi(\lambda,x)e^{\mathrm{i}\lambda Jy}f(y)\right]-f(x). \tag{4.4.9}$$

After simplification in (4.4.9), we obtain

$$n(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\,\Psi(\lambda,x)\int_0^x dy\,e^{\mathrm{i}\lambda Jy}\,f(y). \tag{4.4.10}$$

Using in (4.4.10) the value of $f(x)$ given in (4.4.1), we get

$$n(x)=\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}d\lambda\,\Psi(\lambda,x)\,e^{\mathrm{i}\lambda Jx}\,\mathcal{N}(\lambda,x), \tag{4.4.11}$$

where we have defined

$$\mathcal{N}(\lambda,x):=\begin{bmatrix}0 & -B^\dagger(\lambda I+\mathrm{i}A^\dagger)^{-1}e^{-A^\dagger x}\\ C(\lambda I-\mathrm{i}A)^{-1}e^{-Ax} & 0\end{bmatrix}. \tag{4.4.12}$$

Similarly, by using (4.4.3) in the second formula in (4.2.1), we evaluate $q(x)$ as

$$q(x)=g(x)+\int_0^x dy\,g(x)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\,e^{-\mathrm{i}\lambda Jy}\left[J\,[\Psi(\lambda,x)]^\dagger\,J-e^{\mathrm{i}\lambda Jx}\right]\right], \tag{4.4.13}$$

or equivalently

$$q(x)=g(x)+\int_0^x dy\,g(x)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\,e^{-\mathrm{i}\lambda Jy}J\,[\Psi(\lambda,x)]^\dagger\,J-\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\,e^{-\mathrm{i}\lambda Jy}\,e^{\mathrm{i}\lambda Jx}\right], \tag{4.4.14}$$

By using (4.4.6) in (4.4.14) and by proceeding as in (4.4.5)-(4.4.10), we get

$$q(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\int_0^x dy\,g(y)\,e^{-\mathrm{i}\lambda Jy}\,J\,\left[\Psi(\lambda,x)\right]^\dagger J. \tag{4.4.15}$$

Using in (4.4.15) the expression for $g(y)$ given in (4.4.1), we obtain

$$q(x)=\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}d\lambda\,\Theta(\lambda,x)\,e^{-\mathrm{i}\lambda Jx}\,J\,\left[\Psi(\lambda,x)\right]^\dagger J, \tag{4.4.16}$$

where we have defined

$$\Theta(\lambda,x):=\begin{bmatrix}e^{-Ax}(\lambda I-\mathrm{i}A)^{-1}B & 0\\ 0 & e^{-A^\dagger x}(\lambda I+\mathrm{i}A^\dagger)^{-1}C^\dagger\end{bmatrix}. \tag{4.4.17}$$

We still need to evaluate $\tilde{g}(x,y)$ in terms of the wave function $\Psi(\lambda,x)$. For this purpose, we use (4.3.2) in (4.2.15), (4.3.1), (4.4.6), and (4.4.16), and thus we obtain

$$\tilde{g}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\mu \int_0^x dz\, q(z)\, \Psi(\mu,z)\, e^{\mathrm{i}\mu J y}. \tag{4.4.18}$$

Using (4.4.16) in (4.4.18) we get

$$\tilde{g}(x,y) = \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\mu \int_0^x dz\, E(\lambda,\mu,z)\, e^{\mathrm{i}\mu J y}, \tag{4.4.19}$$

where we have defined

$$E(\lambda,\mu,x) := \frac{1}{4\pi^2\mathrm{i}}\,\Theta(\lambda,x)\, e^{-\mathrm{i}\lambda J x}\, J\, [\Psi(\lambda,x)]^\dagger\, J\, \Psi(\mu,x). \tag{4.4.20}$$

Using the expression for $n(x)$ and $q(x)$ given in (4.4.11) and (4.4.16), respectively, with the help of (4.4.20) we evaluate $\Gamma(x)$ given in (4.2.22) as

$$\Gamma(x) = I - \mathrm{i}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\mu \int_0^x dy\, E(\lambda,\mu,y)\, e^{\mathrm{i}\mu J y}\, \mathcal{N}(\mu,y). \tag{4.4.21}$$

Finally, by substituting (4.4.16) in (4.3.10) and using (4.4.20), we obtain

$$\gamma(\lambda,x) = 2\pi \int_0^x ds \int_{-\infty}^{\infty} d\lambda\, E(\lambda,\lambda,s). \tag{4.4.22}$$

Chapter 5

Examples

In this chapter we consider the Schrödinger equation on the full line

$$-\frac{d^2}{dx^2}\Psi(k,x)+u(x)\Psi(k,x)=k^2\,\Psi(k,x),\qquad x\in(-\infty,+\infty),\tag{5.0.1}$$

where k^2 is related to the spectral parameter λ as $\lambda := k^2$. The corresponding (left) Marchenko integral equation for (5.0.1) is given by

$$\alpha(y,z)+\omega(y,z)+\int_y^\infty ds\,\alpha(y,s)\,\omega(s,z)=0,\quad y<z,\tag{5.0.2}$$

where $\alpha(x,y)$ is related to $\Psi(k,x)$ as

$$\alpha(x,y)=\frac{1}{2\pi}\int_{-\infty}^\infty dk\left[\Psi(k,x)-e^{\mathrm{i}kx}\right]e^{-\mathrm{i}ky},\tag{5.0.3}$$

Note that (5.0.3) is similar to (2.3.2), but in (5.0.3) k is used instead of λ and -1 is used instead of J. Under the appropriate conditions on the potential $u(x)$, it is known [4, 6] that $\alpha(y,z)$ vanishes for $y>z$. For example, if $u(x)$ is real valued and

$$\int_{-\infty}^\infty dx\,(1+|x|)\,|u(x)|<+\infty\tag{5.0.4}$$

then $\alpha(y,z)=0$ for $y>z$. We first obtain $\alpha(y,z)$ by solving (5.0.2), and then we show that $r(x;y,z)$ appearing in (2.1.15) can explicitly be written in terms of $\alpha(y,z)$.

5.1 An Example for the Evaluation of the Resolvent Kernel

Example 5.1.1 Suppose that

$$\omega(y,z)=c_1^2\,e^{-k_1(y+z)},\tag{5.1.1}$$

where c_1 and k_1 are some positive constants. The substitution of (5.1.1) in (5.0.2) suggests that we write $\alpha(y,z)$ as

$$\alpha(y,z) := h(y)\, e^{-k_1 z}, \tag{5.1.2}$$

where $h(y)$ is to be determined. Substituting (5.1.1) and (5.1.2) into (5.0.2) and then cancelling $e^{-k_1 z}$ from both sides of the resulting equation, we get

$$h(y) + c_1^2\, e^{-k_1 y} + \int_y^\infty ds\, h(y)\, e^{-2k_1 s}\, c_1^2 = 0. \tag{5.1.3}$$

By solving (5.1.3) for $h(y)$ we obtain

$$h(y) = \frac{-c_1^2\, e^{-k_1 y}}{1 + \dfrac{c_1^2}{2k_1}\, e^{-2k_1 y}}. \tag{5.1.4}$$

Substituting (5.1.4) into (5.1.2) we get

$$\alpha(y,z) = \begin{cases} \dfrac{-c_1^2\, e^{-k_1 y}\, e^{-k_1 z}}{1 + \dfrac{c_1^2}{2k_1}\, e^{-2k_1 y}}, & y < z, \\ 0, & z < y. \end{cases} \tag{5.1.5}$$

Let us now solve (2.1.15) to obtain $r(x;y,z)$. Substituting (5.1.1) into (2.1.15) we get

$$r(x;y,z) + c_1^2\, e^{-k_1 y}\, e^{-k_1 z} + \int_x^\infty ds\, r(x;y,s)\, c_1^2\, e^{-k_1 s}\, e^{-k_1 z} = 0, \tag{5.1.6}$$

corresponding to (2.1.3). Let us write

$$r(x;y,z) := p(x;y)\, e^{-k_1 z}, \tag{5.1.7}$$

where $p(x;y)$ is to be determined. Substituting (5.1.7) into (5.1.6) and then cancelling $e^{-k_1 z}$ from both sides of the resulting equation, we get

$$p(x;y) + c_1^2\, e^{-k_1 y} + p(x;y) \int_x^\infty ds\, e^{-k_1 s}\, c_1^2\, e^{-k_1 s} = 0. \tag{5.1.8}$$

By solving (5.1.8) for $p(x;y)$ we obtain

$$p(x;y) = \frac{-c_1^2\, e^{-k_1 y}}{1 + \dfrac{c_1^2}{2k_1}\, e^{-2k_1 x}}. \tag{5.1.9}$$

Hence, using (5.1.9) in (5.1.7) we have

$$r(x;y,z) = \frac{-c_1^2\, e^{-k_1 y}\, e^{-k_1 z}}{1 + \dfrac{c_1^2}{2k_1}\, e^{-2k_1 x}}. \tag{5.1.10}$$

Notice that the expression $r(x;y,z)$ given in (5.1.10) is a solution to (2.1.15) both when $x < y < z$ and $x < z < y$.

Having solved (2.1.15) and obtained the solution $r(x;y,z)$ given in (5.0.2), we now solve the integral equation corresponding to (2.1.4), namely

$$r(x;y,z) + \omega(y,z) + \int_x^\infty ds\, \omega(y,s)\, r(x;s,z) = 0, \tag{5.1.11}$$

which is obtained from (2.1.15) by interchanging the orders of $\omega(y,s)$ and $r(x;s,z)$ in the integrand. Substituting (5.1.1) into (5.1.11) we get

$$r(x;y,z) + c_1^2\, e^{-k_1 y}\, e^{-k_1 z} + \int_x^\infty ds\, c_1^2\, e^{-k_1 y}\, e^{-k_1 s}\, r(x;s,z) = 0. \tag{5.1.12}$$

Let us write

$$r(x;y,z) := e^{-k_1 y}\, m(x;z), \tag{5.1.13}$$

where $m(x;z)$ is to be determined. Substituting (5.1.13) into (5.1.12) and then cancelling $e^{-k_1 y}$ from both sides of the resulting equation, we get

$$m(x;z) + c_1^2\, e^{-k_1 z} + \left[\int_x^\infty ds\, c_1^2\, e^{-2k_1 s}\, e^{-k_1 s}\right] m(z) = 0. \tag{5.1.14}$$

By solving (5.1.14) for $m(z)$ we obtain

$$m(x;z) = \frac{-c_1^2\, e^{-k_1 z}}{1 + \dfrac{c_1^2}{2k_1}\, e^{-2k_1 x}}. \tag{5.1.15}$$

Hence, using (5.1.15) in (5.1.13) we have

$$r(x;y,z) = \frac{-c_1^2\, e^{-k_1 y}\, e^{-k_1 z}}{1 + \dfrac{c_1^2}{2k_1}\, e^{-2k_1 x}}. \tag{5.1.16}$$

Notice that the expression given in (5.1.16) is valid both for $x < y < z$ and $x < z < y$. Also note that, from the equivalence of (5.1.10) and (5.1.16), we conclude that the solutions to (5.1.6) and (5.1.12) are given by the same $r(x;y,z)$.

We have obtained the solutions to (2.1.15) and (5.1.10) directly. Now we would like to verify that the expression given in (2.1.14) agrees with the solutions given in (5.1.10) and (5.1.16), respectively.

As seen from (5.1.5), $\alpha(y,z)$ is real valued and a scalar quantity, and thus

$$\alpha(y,z)^\dagger = \alpha(y,z). \tag{5.1.17}$$

Furthermore, there is no loss of generality choosing $J = -1$ in the scalar case, where J is the involution matrix appearing in (2.0.17). Thus the integral expression in the first line of (2.1.14) is evaluated as follows.

The integral term in the first line of (2.1.14) is evaluated by using (5.1.5) there. With the help of (5.1.17) and then using (5.1.5), we obtain

$$\begin{aligned}
\int_x^y ds\, J\,\alpha(s,y)^\dagger\, J\,\alpha(s,z) &= \int_x^y ds\, \alpha(s,y)\,\alpha(s,z) \\
&= \int_x^y ds \left[\frac{c_1^4\, e^{-k_1 s}\, e^{-k_1 y}}{1 + \dfrac{c_1^2}{2k_1}\, e^{-2k_1 s}}\right] \left[\frac{e^{-k_1 s}\, e^{-k_1 z}}{1 + \dfrac{c_1^2}{2k_1}\, e^{-2k_1 s}}\right] \\
&= c_1^4 \left[\int_x^y ds\, \frac{e^{-2k_1 s}}{\left(1 + \dfrac{c_1^2}{2k_1}\, e^{-2k_1 s}\right)^2}\right] e^{-k_1 y}\, e^{-k_1 z} \\
&= c_1^4 \left[\frac{1}{c_1^2}\, \frac{1}{1 + \dfrac{c_1^2}{2k_1}\, e^{-2k_1 s}}\right|_x^y \; e^{-k_1 y}\, e^{-k_1 z}
\end{aligned}$$

which yields

$$\int_x^y ds\, J\,\alpha(s,y)^\dagger\, J\,\alpha(s,z) = c_1^2 \left[\frac{1}{1+\dfrac{c_1^2}{2k_1}\, e^{-2k_1 y}} - \frac{1}{1+\dfrac{c_1^2}{2k_1}\, e^{-2k_1 x}} \right] e^{-k_1 y}\, e^{-k_1 z}. \tag{5.1.18}$$

Next, using (5.1.5) and (5.1.18) in the first line of (2.1.14), we obtain $r(x;y,z)$ when $z > y$ as

$$\begin{aligned} r(x;y,z) &= \alpha(y,z) + \int_x^y ds\, J\,\alpha(s,y)^\dagger\, J\,\alpha(s,z) \\ &= \frac{-c_1^2\, e^{-k_1 y}\, e^{-k_1 z}}{1+\dfrac{c_1^2}{2k_1}\, e^{-2k_1 y}} + c_1^2 \left[\frac{1}{1+\dfrac{c_1^2}{2k_1}\, e^{-2k_1 y}} - \frac{1}{1+\dfrac{c_1^2}{2k_1}\, e^{-2k_1 x}} \right] e^{-k_1 y}\, e^{-k_1 z} \\ &= \frac{-c_1^2\, e^{-k_1 y}\, e^{-k_1 z}}{1+\dfrac{c_1^2}{2k_1}\, e^{-2k_1 x}}, \end{aligned}$$

which agrees with (5.1.10) and (5.1.16). Similarly, using (5.1.5) and (5.1.18) in the second line of (2.1.14), we obtain $r(x;y,z)$ when $y > z$ as

$$\begin{aligned} r(x;y,z) &= J\,\alpha(z,y)^\dagger\, J + \int_x^z ds\, J\,\alpha(s,y)^\dagger\, J\,\alpha(s,z) \\ &= \frac{-c_1^2\, e^{-k_1 z}\, e^{-k_1 y}}{1+\dfrac{c_1^2}{2k_1}\, e^{-2k_1 z}} + c_1^2 \left[\left. \frac{1}{1+\dfrac{c_1^2}{2k_1}\, e^{-2k_1 s}} \right|_x^z \right] e^{-k_1 y}\, e^{-k_1 z} \\ &= \frac{-c_1^2\, e^{-k_1 z}\, e^{-k_1 y}}{1+\dfrac{c_1^2}{2k_1}\, e^{-2k_1 z}} + c_1^2 \left[\frac{1}{1+\dfrac{c_1^2}{2k_1}\, e^{-2k_1 z}} - \frac{1}{1+\dfrac{c_1^2}{2k_1}\, e^{-2k_1 x}} \right] e^{-k_1 y}\, e^{-k_1 z} \\ &= \frac{-c_1^2\, e^{-k_1 y}\, e^{-k_1 z}}{1+\dfrac{c_1^2}{2k_1}\, e^{-2k_1 x}}, \end{aligned}$$

which agrees with (5.1.10) and (5.1.16).

Thus, in Example 5.1.1 we have verified that the expression $r(x;y,z)$ given in (2.1.14) is a solution to both $R+\Omega+R\,\Omega=0$ and $R+\Omega+\Omega\,R=0$, which are given in (2.1.3) and (2.1.4), respectively.

In Example 5.1.1, $\omega(y,z)$ given in (5.1.1) satisfies the additional symmetry property

$$\omega(y,z) = \omega(z,y). \tag{5.1.19}$$

This has resulted in the additional symmetry

$$r(x;y,z) = r(x;z,y), \tag{5.1.20}$$

which is apparent from (5.1.10) and (5.1.16). The additional symmetry condition given in (5.1.19) corresponds to a special case of (2.0.12).

In the next example, we will evaluate the resolvent kernel $r(x;y,z)$ corresponding to a Marchenko integral kernel $\omega(y,z)$ not satisfying the additional symmetry in (5.1.19) but still satisfying (2.0.12). We will then see that we no longer have the additional symmetry given in (5.1.20).

5.2 Another Example for the Evaluation of the Resolvent Kernel

Example 5.2.1 Suppose that

$$\omega(y,z) = c\, e^{-\mathrm{i}k_1(y-z)-\epsilon(y+z)}, \tag{5.2.1}$$

where c, k_1, and ϵ are some positive constants. Let us write the solution $\alpha(y,z)$ to (5.0.2) as

$$\alpha(y,z) := h_2(y)\, e^{\mathrm{i}k_1 z-\epsilon z}, \tag{5.2.2}$$

where $h_2(y)$ is to be determined. Substituting (5.2.1) and (5.2.2) into (5.0.2) we obtain

$$h_2(y)\, e^{\mathrm{i}k_1 z-\epsilon z} + c\, e^{-\mathrm{i}k_1 y-\epsilon y}\, e^{\mathrm{i}k_1 z-\epsilon z} + \int_y^\infty ds\, h_2(y)\, e^{\mathrm{i}k_1 s-\epsilon s} c\, e^{-\mathrm{i}k_1 s-\epsilon s} e^{\mathrm{i}k_1 z-\epsilon z} = 0. \tag{5.2.3}$$

By cancelling $e^{\mathrm{i}k_1 z-\epsilon z}$ from both sides of (5.2.3) and simplifying the integrand in (5.2.3) we get

$$h_2(y) + c\, e^{-\mathrm{i}k_1 y-\epsilon y} + \int_y^\infty ds\, h_2(y)\, c\, e^{-2\epsilon s} = 0. \tag{5.2.4}$$

By solving (5.2.4) for $h_2(y)$ we obtain

$$h_2(y) = \frac{-c\, e^{-\mathrm{i}k_1 y - \epsilon y}}{1 + \dfrac{c}{2\epsilon}\, e^{-2\epsilon y}}. \tag{5.2.5}$$

Substituting (5.2.5) into (5.2.2) we get

$$\alpha(y, z) = \frac{-c\, e^{-\mathrm{i}k_1 y - \epsilon y}\, e^{-\mathrm{i}k_1 z - \epsilon z}}{1 + \dfrac{c}{2\epsilon}\, e^{-2\epsilon y}}. \tag{5.2.6}$$

Let us now solve (2.1.15) to obtain $r(x; y, z)$ corresponding to (5.2.1). Substituting (5.2.1) into (2.1.15) we get

$$r(x; y, z) + c\, e^{-\mathrm{i}k_1(y-z) - \epsilon(y+z)} + \int_x^\infty ds\, r(x; y, s)\, c\, e^{-\mathrm{i}k_1(s-z) - \epsilon(s+z)} = 0. \tag{5.2.7}$$

Let us write

$$r(x; y, z) := p_2(x; y)\, e^{\mathrm{i}k_1 z - \epsilon z}, \tag{5.2.8}$$

where $p_2(x; y)$ is to be determined. Substituting (5.2.8) into (5.2.7) and then cancelling $e^{\mathrm{i}k_1 z - \epsilon z}$ from both sides of (5.2.7), we get

$$p_2(x; y) + c\, e^{-\mathrm{i}k_1 y - \epsilon y} + \int_x^\infty ds\, p_2(x; y)\, c\, e^{-2\epsilon s} = 0. \tag{5.2.9}$$

By solving (5.2.9) for $p_2(x; y)$ we obtain

$$p_2(x; y) = \frac{-c\, e^{-\mathrm{i}k_1 y - \epsilon y}}{1 + \dfrac{c}{2\epsilon}\, e^{-2\epsilon x}}. \tag{5.2.10}$$

Hence, using (5.2.10) in (5.2.8) we have

$$r(x; y, z) = \frac{-c\, e^{-\mathrm{i}k_1 y - \epsilon y} e^{\mathrm{i}k_1 z - \epsilon z}}{1 + \dfrac{c}{2\epsilon}\, e^{-2\epsilon x}}. \tag{5.2.11}$$

Notice that the expression $r(x; y, z)$ given in (5.2.11) is a solution to (2.1.15) both when $x < y < z$ and $x < z < y$.

Having solved (2.1.15), we now solve the integral equation given in (5.1.11) for $r(x;y,z)$. Substituting (5.2.1) into (5.1.11) we get

$$r(x;y,z) + c\,e^{-\mathrm{i}k_1(y-z)-\epsilon(y+z)} + \int_x^\infty ds\, c\, e^{-\mathrm{i}k_1(y-s)-\epsilon(y+s)}\, r(x;s,z) = 0. \qquad (5.2.12)$$

Let us write

$$r(x;y,z) := e^{-\mathrm{i}k_1 y-\epsilon y}\, m_2(x;z)\,, \qquad (5.2.13)$$

where $m_2(x;z)$ is to be determined. Substituting (5.2.13) into (5.2.12) and then cancelling $e^{-\mathrm{i}k_1 y-\epsilon y}$ from both sides of (5.2.7), we get

$$m_2(x;z) + c\,e^{\mathrm{i}k_1 z-\epsilon z} + \int_x^\infty ds\, c\, e^{-2\epsilon s}\, m_2(x;z) = 0. \qquad (5.2.14)$$

By solving (5.2.14) for $m_2(x;z)$ we obtain

$$m_2(x;z) = \frac{-c\, e^{\mathrm{i}k_1 z-\epsilon z}}{1+\dfrac{c}{2\epsilon}\, e^{-2\epsilon x}}. \qquad (5.2.15)$$

Hence, using (5.2.15) in (5.2.13) we have

$$r(x;y,z) = \frac{-e^{-\mathrm{i}k_1 y-\epsilon y}\, c\, e^{\mathrm{i}k_1 z-\epsilon z}}{1+\dfrac{c}{2\epsilon}\, e^{-2\epsilon x}}. \qquad (5.2.16)$$

Notice that the expression given in (5.2.11) and (5.2.16) is valid both for $x < y < z$ and $x < z < y$.

By comparing Example 5.1.1 and Example 5.2.1, we conclude that the additional symmetry $\omega(y,z) = \omega(z,y)$ given in (5.1.19) implies that $r(x;y,z) = r(x;z,y)$ given in (5.1.20). On the other hand, $\omega(y,z)$ in Example 5.2.1 does not satisfy (5.1.19) and the corresponding $r(x;y,z)$ does not satisfy (5.1.20). In Example 5.2.1, $\omega(y,z)$ and $r(x;y,z)$ still satisfy the property (2.0.12) and (2.1.1), respectively.

5.3 An Example for the Evaluation of the Darboux Transformation

Example 5.3.1 Consider the Schrödinger equation given in (5.0.1) on the full line when the potential $u(x)$ is real valued, satisfies (5.0.4) and does not have a bound state at

$k = \mathrm{i}\kappa$, with κ being a positive constant. If we add one bound state at $\lambda = -\kappa^2$ with norming constant c, then the perturbed problem will be

$$-\frac{d^2}{dx^2}\tilde{\Psi}(k,x) + \tilde{u}(x)\,\tilde{\Psi}(k,x) = k^2\,\tilde{\Psi}(k,x). \tag{5.3.1}$$

The goal is to find the Darboux transformation at the potential level and at the wave function level via the (left) Marchenko integral equation when the unperturbed potential $u(x)$ is zero. For this, let us first evaluate the intermediate quantities $n(x)$ and $q(y)$. In this case $f(x)$ and $g(y)$ appearing in (2.0.20) are given by

$$f(x) = c\,e^{-\kappa\,x}, \qquad g(y) = c\,e^{-\kappa\,y}. \tag{5.3.2}$$

The wave function $\Psi(k,x)$ in this case is given by

$$\Psi(k,x) = e^{\mathrm{i}kx}. \tag{5.3.3}$$

As seen from (5.0.3), we then get

$$\alpha(x,y) = 0. \tag{5.3.4}$$

By using (5.3.2) into (2.2.1), we evaluate the corresponding quantities $n(x)$ and $q(y)$ as

$$n(x) = c\,e^{-\kappa\,x}, \tag{5.3.5}$$

$$q(y) = c\,e^{-\kappa\,y}, \tag{5.3.6}$$

where the involution J appearing in (2.0.17) has been chosen as -1. Next, using (5.3.4), (5.3.5), and (5.3.6) in (2.2.15) and (2.2.22), we evaluate $\tilde{g}(x,y)$ and $\Gamma(x)$, respectively, as

$$\tilde{g}(x,y) = c\,e^{-\kappa\,y}, \tag{5.3.7}$$

$$\Gamma(x) = 1 + \int_x^\infty ds\,c\,e^{-\kappa\,s}\,c\,e^{-\kappa\,s}. \tag{5.3.8}$$

We can explicitly evaluate $\Gamma(x)$ from (5.3.8) as

$$\Gamma(x) = 1 + \frac{c^2}{2\kappa} e^{-2\kappa\, x}. \tag{5.3.9}$$

We would now like to evaluate $\tilde{\alpha}(x,y) - \alpha(x,y)$. By substituting (5.3.5), (5.3.7), and (5.3.9) into (2.2.27), we get

$$\tilde{\alpha}(x,y) - \alpha(x,y) = -c\, e^{-\kappa\, x} \left[1 + \frac{c^2}{2\kappa} e^{-2\kappa\, x}\right]^{-1} c\, e^{-\kappa\, y}. \tag{5.3.10}$$

Since $\alpha(x,y) = 0$ as indicated in (5.3.4), we obtain from (5.3.10)

$$\tilde{\alpha}(x,y) = -\frac{c^2\, e^{-\kappa\,(x+y)}}{1 + \frac{c^2}{2\kappa} e^{-2\kappa\, x}}. \tag{5.3.11}$$

Furthermore, by replacing y by x in (5.3.11) we get

$$\tilde{\alpha}(x,x) = -\frac{c^2\, e^{-2\kappa\, x}}{1 + \frac{c^2}{2\kappa} e^{-2\kappa\, x}}. \tag{5.3.12}$$

The potential $\tilde{u}(x)$ is recovered as [4, 6, 13, 14]

$$\tilde{u}(x) = -2 \frac{d}{dx}\, \tilde{\alpha}(x,x). \tag{5.3.13}$$

By substituting (5.3.12) into (5.3.13) we get

$$\tilde{u}(x) = 2 \frac{d}{dx} \left[\frac{c^2\, e^{-2\kappa\, x}}{1 + \frac{c^2}{2\kappa} e^{-2\kappa\, x}}\right], \tag{5.3.14}$$

which yields

$$\tilde{u}(x) = 2c^2 \frac{-1}{\left(e^{2\kappa x} + \frac{c^2}{2\kappa}\right)^2}\, 2\kappa\, e^{2\kappa x}, \tag{5.3.15}$$

or equivalently

$$\tilde{u}(x) = \frac{-4\, c^2\, \kappa}{\left(e^{\kappa x} + \frac{c^2}{2\kappa} e^{-\kappa x}\right)^2}. \tag{5.3.16}$$

By replacing $\frac{c^2}{2\kappa}$ in (5.3.16) with $e^{2\ln\sqrt{c^2/(2\kappa)}}$ we get

$$\begin{aligned}\tilde{u}(x) &= \frac{-4\,c^2\,\kappa}{\left(e^{\kappa x} + e^{2\ln\sqrt{c^2/(2\kappa)}}e^{-\kappa x}\right)^2} \\ &= \frac{-4\,c^2\,\kappa}{\left[e^{\ln\sqrt{c^2/(2\kappa)}}\left(e^{\kappa x-\ln\sqrt{c^2/(2\kappa)}} + e^{-\kappa x+\ln\sqrt{c^2/(2\kappa)}}\right)\right]^2} \\ &= \frac{-4\,c^2\,\kappa}{\dfrac{c^2}{2\kappa}\left[e^{\kappa x-\ln\sqrt{c^2/(2\kappa)}} + e^{-\kappa x+\ln\sqrt{c^2/(2\kappa)}}\right]^2},\end{aligned}$$

which yields

$$\tilde{u}(x) = \frac{-8\,\kappa^2}{\left[e^{\kappa x-\ln\sqrt{c^2/(2\kappa)}} + e^{-\kappa x+\ln\sqrt{c^2/(2\kappa)}}\right]^2}. \tag{5.3.17}$$

We can write (5.3.17) in the form

$$\tilde{u}(x) = \frac{-8\,\kappa^2}{\left[2\cosh\left(\kappa x - \ln\sqrt{c^2/(2\kappa)}\right)\right]^2}, \tag{5.3.18}$$

or equivalently

$$\tilde{u}(x) = \frac{-8\,\kappa^2}{4\cosh^2\left(\kappa x - \ln\sqrt{c^2/(2\kappa)}\right)}. \tag{5.3.19}$$

Hence, the Darboux transformation at the potential level is given as

$$\tilde{u}(x) = -2\kappa^2 \mathrm{sech}^2\left(\kappa x - \ln\sqrt{c^2/(2\kappa)}\right). \tag{5.3.20}$$

Let us now find the Darboux transformation at the wave function level. For this, let us first evaluate the quantity $\gamma(k,x)$ by using the formula given in (2.3.10). The wave function appearing in (2.3.1) corresponds to the Jost solution from the left satisfying $\Psi(k,x) = e^{\mathrm{i}kx}\,[1+o(1)]$ as $x \to +\infty$, and in fact since $u(x) \equiv 0$, we have $\Psi(k,x) = e^{\mathrm{i}kx}$, as stated in (5.3.3). Thus we obtain from (2.3.10)

$$\gamma(k,x) = \int_x^\infty ds\, c\, e^{-\kappa s}\, e^{\mathrm{i}ks}, \tag{5.3.21}$$

which yields,

$$\gamma(k,x) = \frac{\mathrm{i}c\, e^{(\mathrm{i}k-\kappa)x}}{k+\mathrm{i}\kappa}. \tag{5.3.22}$$

By substituting (5.3.2), (5.3.9), and (5.3.22) into (2.3.11), we obtain

$$\tilde{\Psi}(k,x) - \Psi(k,x) = -c\, e^{-\kappa\, x} \left[1 + \frac{c^2}{2\kappa} e^{-2\kappa\, x}\right]^{-1} \frac{\mathrm{i}c\, e^{(\mathrm{i}k-\kappa)x}}{k+\mathrm{i}\kappa}, \tag{5.3.23}$$

which yields, as a result of (5.3.3),

$$\tilde{\Psi}(k,x) - e^{\mathrm{i}kx} = -\frac{\mathrm{i}c^2\, e^{-2\kappa\, x} e^{\mathrm{i}kx}}{\left(1 + \frac{c^2}{2\kappa} e^{-2\kappa\, x}\right)(k+\mathrm{i}\kappa)}. \tag{5.3.24}$$

Hence, the Darboux transformation at the wave function level is given as

$$\tilde{\Psi}(k,x) = e^{\mathrm{i}kx} \left[1 - \frac{\mathrm{i}c^2\, e^{-2\kappa\, x}}{\left(1 + \frac{c^2}{2\kappa} e^{-2\kappa\, x}\right)(k+\mathrm{i}\kappa)}\right]. \tag{5.3.25}$$

As a summary, in this example the Darboux transformation at the potential level is given by

$$u(x) = 0 \quad \longmapsto \quad \tilde{u}(x) = -2\kappa^2 \mathrm{sech}^2 \left(\kappa x - \ln \sqrt{c^2/(2\kappa)}\right), \tag{5.3.26}$$

and the the Darboux transformation at the wave fuction level is given by

$$\Psi(k,x) = e^{\mathrm{i}kx} \longmapsto \tilde{\Psi}(k,x) = e^{\mathrm{i}kx} \left[1 - \frac{\mathrm{i}c^2 e^{-2\kappa x}}{(k+\mathrm{i}\kappa)\left(1 + \frac{c^2}{2\kappa} e^{-2\kappa x}\right)}\right]. \tag{5.3.27}$$

Chapter 6

Conclusion

In this thesis we consider Darboux transformations for linear differential operators denoted by $\mathcal{L}$ acting on some suitable function spaces. The spectrum of the operator $\mathcal{L}$ usually consists of two parts: the discrete spectrum and the continuous spectrum. The Darboux transformation determines how the (generalized) eigenfunctions change when a finite number of discrete eigenvalues are added or subtracted from the spectrum of $\mathcal{L}$ without changing the continuous spectrum.

The Darboux transformation has two parts, the first part is at the potential level and the second part is at the wave function level. It provides the perturbed potential and wave function in terms of the unperturbed quantities when a finite number of bound states are added or subtracted. At the potential level, the Darboux transformation consists of determining the perturbed potential $\tilde{u}(x)$ in terms of the unperturbed potential $u(x)$ and of the quantities evaluated at the discrete λ-eigenvalues appearing in the perturbation. At the wave function level, the Darboux transformation consists of determining the perturbed wave function $\tilde{\Psi}(\lambda, x)$ in terms of the unperturbed wave function $\Psi(\lambda, x)$ and of the quantities evaluated at the discrete λ-eigenvalues appearing in the perturbation.

In our thesis we provide a method to derive Darboux transformations for a wide variety of spectral problems for differential equations. This is done with the help of a fundamental integral equation, related to the so-called Marchenko integral equations or Gel'fand-Levitan integral equation.

Our approach allows us to obtain a Darboux transformation for any wave function, whereas in other approaches a Darboux transformation is given only for some special wave function. Our method is not specific to a particular differential equation. It is a "unified approach," applicable to a large class of differential equations.

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