TENSOR PRODUCTS OF A FINITE-DIMENSIONAL REPRESENTATION AND AN INFINITE-DIMENSIONAL REPRESENTATION

by

FELICIA DEWANAGA

Presented to the Faculty of the Graduate School of

The University of Texas at Arlington in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE

THE UNIVERSITY OF TEXAS AT ARLINGTON

May 2016

ACKNOWLEDGEMENTS

I would like to express my deep gratitiude to Dr. Dimitar Grancharov for his superlative guidance, inspiration, support, patient and encouragement which have been instrumental toward my graduate research success. I would like to thank Dr. Michaela Vancliff and Dr. Ruth Gornet for serving on my dissertation committee.

I would like to also thank my family for their endless love and support. Great thanks to my friends, especially Imelda Trejo, without whom I would not have made it this far.

Thank you to my love, Gustavo Puerto, who has been very supportive and caring during my graduate studies.

April 25, 2016

ABSTRACT

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FELICIA DEWANAGA, M.Sc.

The University of Texas at Arlington, 2016

Supervising Professor: Dimitar Grantcharov

In this project, we explicitly find the decomposition of the tensor product of a Verma module $Z(\lambda)$ and the standard module $\mathbb{C}^n$ of the Lie algebras $\mathfrak{sl}(n)$, $n = 2, 3$. The result provides an explicit description of the translation functor introduced by Bernstein and Gelfand.

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CHAPTER 1

INTRODUCTION

Lie algebras arise naturally as vector spaces of linear transformations. Their representations are studied by both mathematicians and theoretical physicists. The finite-dimensional representations of simple finite-dimensional Lie algebras were classified by H. Weyl in the early 20th century. One remarkable theorem of H. Weyl states that every finite-dimensional representation is a direct sum of simple representations. Another important discovery made by H. Weyl was an explicit way to define these representations by using tensor products of the natural and the dual to the natural representations.

The study of the infinite-dimensional representations of Lie algebras emerged in the 1960's as an effort to address problems motivated by theoretical physics. Unfortunately, there is no analog of Weyl's theorem for the infinite-dimensional representations; namely, there are representations that do not decompose as direct sums of irreducible ones. Among the infinite-dimensional representations, there is one class of special interest: representations with highest weight vectors, called also vacuum vectors.

Tensor products of finite- and infinite-dimensional representations were extensively studied in the last 40 years. Fundamental results have been discovered by J.

Bernstein and S. Gelfand [1], J. Jantzen [3], and G. Zuckerman [4]. In this thesis, we reprove some of the results using explicit computations for lower rank Lie algebras.

The content of this project is as follows. In Chapter 2, we list some basic definitions and results for Lie algebras and their representations. In Chapter 3, we prove explicitly some standard results for tensor products of finite-dimensional representations of $\mathfrak{sl}(n)$ and $\mathfrak{gl}(n)$. In the last chapter we prove our main results: we find an explicit decomposition of the tensor product of the Verma module $Z(\lambda)$ and the standard module $\mathbb{C}^n$ of the Lie algebras $\mathfrak{sl}(n)$ for $n = 2$ and $n = 3$.

CHAPTER 2

BACKGROUND MATERIAL

2.1 Definition and examples of Lie Algebras

Definition 2.1.1. A *Lie algebra* is a vector space L over a field $\mathtt{F}$, together with a binary operation $[.,.] : L \times L \to L$, called the Lie bracket or commutator, which satisfies the following axioms:

$$(i) \quad [ax+by, z] = a[x,z] + b[y,z]$$

$$[z, ax+by] = a[z,x] + b[z,y] \qquad \text{(bilinearity)} \quad (2.1)$$

$$(ii) \quad [x,y] = -[y,x] \qquad \text{(skew-symmetry)} \quad (2.2)$$

$$(iii) \quad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \qquad \textit{(Jacobi identity)} \quad (2.3)$$

for all $x, y, z \in L$.

A subspace K of L is a *Lie subalgebra* K of L if $[x,y] \in K$, if $x, y \in K$.

2.1.1 Examples

Let V be a vector space over $\mathtt{F}$. The ring of homomorphism $V \to V$ is denoted by $\mathrm{End}(V)$. When $\mathrm{End}(V)$ is given a new operation defined as $[x,y] = xy - yx$, called the bracket of x and y, $\mathrm{End}(V)$ constitute a Lie algebra over $\mathtt{F}$: the brackets and linear combinations are again linear transformations from V to V and the conditions

2.1 to 2.3 are satisfied. When End(V) has the bracket operation, we write $\mathfrak{gl}(V)$ for End(V) to distinguish this algebraic structure from the old associative one.

If a basis of V is fixed then $\mathfrak{gl}(V)$ is isomorphic as a vector space to the space of all $n \times n$ matrices over $\mathtt{F}$ whenever V has dimension n. We denote this Lie algebra consisting square n-dimensional matrices by $\mathfrak{gl}(n, F)$, or $\mathfrak{gl}(n)$ if the $\mathtt{F}$ is clear in the context.

Some further example is the set of endomorphisms of V (of finite dimension n) with trace zero, and we denote it by $\mathfrak{sl}(n, F)$ (which is isomorphic to $\mathfrak{sl}(V)$). It is a fact that $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$ and $\mathrm{Tr}(x + y) = \mathrm{Tr}(x) + \mathrm{Tr}(y)$; therefore, $\mathfrak{sl}(n)$ is a Lie subalgebra of $\mathfrak{gl}(n)$.

From now on we fix our ground field to be $\mathtt{F} = \mathbb{C}$. It will be convenient to have at our disposal the following notations: The symbol E_{ij} will denote the matrix with 1 in the ith row and jth column, and 0 elsewhere. We let the symbol e_i denote the single column matrix with 1 in the ith row and 0 elsewhere, and the symbol f_i represent the map $f_i : \mathbb{C}^n \to \mathbb{C}$ such that $f_i(e_j) = \delta_{ij}$.

2.2 Cartan subalgebra

Definition 2.2.1. A *Cartan subalgebra* $\mathfrak{h}$ of a Lie algebra L is a subalgebra of L which is nilpotent and self-normalizing.

From now on we fix the following Cartan subalgebras of $\mathfrak{gl}(n)$ and $\mathfrak{sl}(n)$. The Cartan subalgebra $\mathfrak{h}_{\mathfrak{gl}(n)}$ will be the subalgebra of $\mathfrak{gl}(n)$ consisting of all diagonal matrices, $\{h = b_{11}E_{11} + ... + b_{nn}E_{nn} \mid b_{ii} \in \mathbb{C}\}$. The Cartan subalgebra $\mathfrak{h}_{\mathfrak{sl}(n)}$ will be the subalgebra of $\mathfrak{sl}(n)$ consisting of all diagonal matrices in $\mathfrak{sl}(n)$, i.e. $\mathfrak{h}_{\mathfrak{sl}(n)} = \mathfrak{h}_{\mathfrak{gl}(n)} \cap \mathfrak{sl}(n)$. We also fix the bases of $\mathfrak{h}_{\mathfrak{gl}(n)}$ and $\mathfrak{h}_{\mathfrak{sl}(n)}$ to be $\{E_{11}, ..., E_{nn}\}$ and $\{E_{11} - E_{22}, ..., E_{n-1,n-1} - E_{nn}\}$, respectively.

We define ε_i as the element of $\mathfrak{h}^*$ for which $\varepsilon_i(E_{jj}) = \delta_{ij}$. Let L_α denote the space $\{x \in L | [hx] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$, where $\alpha \in \mathfrak{h}^*$. The set of all nonzero $\alpha \in \mathfrak{h}^*$ for which $L_\alpha \neq 0$ is denoted by Δ and is called *the root system of* L. The elements of Δ are called the *roots of* L *relative to* $\mathfrak{h}$. We have a decomposition of $L = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} L_\alpha$, and we call this the *root space decomposition.* The set of roots of $\mathfrak{sl}(n)$ is $\Delta(\mathfrak{sl}(n), \mathfrak{h}_{\mathfrak{sl}(n)}) = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\}$. The root system of $\mathfrak{gl}(n)$ is the same: $\Delta(\mathfrak{gl}(n), \mathfrak{h}_{\mathfrak{gl}(n)}) = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\}$. The root decomposition of $\mathfrak{sl}(n)$ (respectively $\mathfrak{gl}(n)$) is $\mathfrak{h}_{\mathfrak{sl}(n)} \oplus \bigoplus_{i \neq j} \mathrm{Span}_{\mathbb{C}}\{E_{ij}\}$ (respectively, $\mathfrak{h}_{\mathfrak{gl}(n)} \oplus \bigoplus_{i \neq j} \mathrm{Span}_{\mathbb{C}}\{E_{ij}\}$). Let $\Pi = \{\varepsilon_1 - \varepsilon_2, ..., \varepsilon_{n-1} - \varepsilon_n\}$. We call Π the *standard base* of Δ. We have that for every $\beta \in \Delta$, there are unique integers k_α such that $\beta = \sum k_\alpha \alpha \, (\alpha \in \Pi)$. If all $k_\alpha \geq 0$ (respectively, all $k_\alpha \leq 0$), we call β *positive root* relative to Π (respectively, *negative root*) and write $\beta \succ 0$ (respectively $\prec 0$). We let $\mathfrak{n}$ (respectively, $\mathfrak{n}^-$) to be the Lie subalgebra of $\mathfrak{sl}(n)$ and $\mathfrak{gl}(n)$ spanned by E_{ij}, $i < j$ (respectively, E_{ij}, $i > j$). Namely, $\mathfrak{n} = \bigoplus_{\alpha \succ 0} L_\alpha$ and $\mathfrak{n}^- = \bigoplus_{\alpha \prec 0} L_\alpha$.

2.3 Dual to the Cartan subalgebras of $\mathfrak{gl}(n)$ and $\mathfrak{sl}(n)$

There is a natural surjective homomorphism $\gamma : \mathfrak{h}^*_{\mathfrak{gl}(n)} \to \mathfrak{h}^*_{\mathfrak{sl}(n)}$ with kernel $\ker \gamma = \langle \varepsilon_1 + \ldots + \varepsilon_n \rangle$. In what follows we will identify $\mathfrak{h}^*_{\mathfrak{sl}(n)}$ with the space

$$H = \left\{ \sum_{i=1}^{n} a_i \varepsilon_i \mid \sum_{i=1}^{n} a_i = 0, a_i \in \mathbb{C} \right\}$$

through the isomorphism $\mathfrak{h}^*_{\mathfrak{gl}(n)} / \ker \gamma \to H$ defined by

$$\gamma \left(\sum_{i=1}^{n} a_i \varepsilon_i + \langle \varepsilon_1 + \ldots + \varepsilon_n \rangle \right) = \sum_{i=1}^{n} a_i \varepsilon_i - \frac{\sum_{i=1}^{n} a_i}{n} (\varepsilon_1 + \ldots + \varepsilon_n).$$

2.4 Representations

Definition 2.4.1. A *representation* of a Lie algebra L is a Lie algebra homomorphism $\phi : L \to \mathfrak{gl}(V)$. We denote this representation by (ρ, V), or simply by V.

We shall see that there is an equivalency between representations and modules. Recall the definition of an L-module.

Definition 2.4.2. A vector space V is a module over a Lie algebra L: if V is endowed with an operation $L \times V \to V$, $(x, v) \mapsto x \cdot v$, that satisfies the following:

$$(i) \quad (ax + by) \cdot v = a(x \cdot v) + b(y \cdot v), \tag{2.4}$$

$$(ii) \quad x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w), \tag{2.5}$$

$$(iii) \quad [x, y] \cdot v = x \cdot y \cdot v - y \cdot x \cdot v, \tag{2.6}$$

for all $x, y \in L; v, w \in V; a, b \in \mathbb{C}$.

We will often write xv for $x \cdot v$. A representation $\phi : L \to \mathfrak{gl}(V)$ is viewed as an L-module through the action $x.v = \phi(x)(v)$. Conversely, given an L-module(V), we can define a representation $\phi : L \to \mathfrak{gl}(V)$ by the action we just defined.

If V is an L-module, then we can make the dual vector space V^* an L-module if we define $(x.f)(v) = -f(x.v)$ for $f \in V^*$, $v \in V$, $x \in L$. Axioms 2.4 and 2.5 are quite easy to check. To check Axiom 2.6, we have

$$\begin{aligned} ([x,y] \cdot f)(v) &= -f([x,y] \cdot v) \\ &= -f(x \cdot y \cdot v - y \cdot x \cdot v) \\ &= -f(x \cdot y \cdot v) + f(y \cdot x \cdot v) \\ &= (x \cdot f)(y \cdot v) - (y \cdot f)(x \cdot v) \\ &= -(y \cdot x \cdot f)(v) + (x \cdot y \cdot f)(v) \\ &= ((x \cdot y - y \cdot x) \cdot f)(v) \end{aligned}$$

We call V^* *the dual module* of V.

2.4.1 Tensor products of representations

Let L be a Lie algebra over $\mathbb{C}$. If V, W are L-modules, let $V \otimes W$ be the tensor product of the vector spaces V and W over $\mathbb{C}$. In order to endow $V \otimes W$ with a module structure, we define the action of L on the generators $v \otimes w$ as follows: $x \cdot (v \otimes w) = x \cdot v \otimes w + v \otimes x \cdot w$, $x \in L$. It is straightforward to check the conditions

2.4 and 2.5 to construct the tensor product $V \otimes W$ as an L-module. It follows to verify Axiom 2.6:

$$\begin{aligned}
[x,y]\cdot(v\otimes w) &= [x,y]\cdot v\otimes w + v\otimes[x,y]\cdot w \\
&= (x\cdot y\cdot v - y\cdot x\cdot v)\otimes w + v\otimes(x\cdot y\cdot w - y\cdot x\cdot w) \\
&= (x\cdot y\cdot v\otimes w + v\otimes x\cdot y\cdot w) - (y\cdot x\cdot v\otimes w + v\otimes y\cdot x\cdot w) \\
&= (x\cdot y - y\cdot x)\cdot(v\otimes w)
\end{aligned}$$

2.4.2 Symmetric and exterior representations

Let $\sigma : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n, \sum v_i \otimes w_i \mapsto \sum w_i \otimes v_i$. Set $S^2(\mathbb{C}^n) = \{u \in \mathbb{C}^n \otimes \mathbb{C}^n | \sigma(u) = u\}$ and $\Lambda^2(\mathbb{C}^n) = \{u \in \mathbb{C}^n \otimes \mathbb{C}^n | \sigma(u) = -u\}$. We have the following lemma,

Lemma 2.4.3. *$S^2(\mathbb{C}^n) = \mathrm{Span}_{\mathbb{C}}\{v\otimes w + w\otimes v | v, w \in \mathbb{C}^n\}$ and $\Lambda^2(\mathbb{C}^n) = \mathrm{Span}_{\mathbb{C}}\{v\otimes w - w\otimes v | v, w \in \mathbb{C}^n\}$. Moreover, $S^2(\mathbb{C}^n)$ and $\Lambda^2(\mathbb{C}^n)$ are $\mathfrak{gl}(n)$-representations (and, hence, $\mathfrak{sl}(n)$-representations).*

Proof. From the definitions of $S^2(\mathbb{C}^n)$ and $\Lambda^2(\mathbb{C}^n)$, we have that $\mathrm{Span}_{\mathbb{C}}\{v\otimes w + w\otimes v | v, w \in \mathbb{C}^n\} \subset S^2(\mathbb{C}^n)$ and $\mathrm{Span}_{\mathbb{C}}\{v\otimes w - w\otimes v | v, w \in \mathbb{C}^n\} \subset \Lambda^2(\mathbb{C}^n)$. The reverse inclusions follow by a standard reasoning.

□

We call $S^2(\mathbb{C}^n)$ the *second symmetric representation* and $\Lambda^2(\mathbb{C}^n)$ *second exterior representation* of $\mathfrak{gl}(n)$ (and of $\mathfrak{sl}(n)$).

2.5 Weights and maximal vectors of representations

Let V be an arbitrary L-module and let $\mathfrak{h}$ be a Cartan subalgebra of L. Let V be a weight module of L, i.e.

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda,$$

where $V_\lambda = \{v \in V | h \cdot v = \lambda(h)v, \text{ for every } h \in \mathfrak{h}\}$. We call λ a *weight* of V if $V_\lambda \neq 0$. Also, V_λ is called a *weight space* of weight λ. A *maximal vector* (or a *highest weight vector*) of weight λ in an L-module V is a nonzero vector $v \in V$ annihilated by all $L_\alpha, \alpha \succ 0$. The weight of a maximal vector is called the *highest weight* of V.

2.5.1 Finite-dimensional representations of $\mathfrak{sl}(2)$

In this subsection, let the Lie algebra L denote $\mathfrak{sl}(2)$ whose basis is $\{x, y, h\}$ with $[x, y] = h$,$[h, y] = -2y$, $[h, x] = 2x$. In matrix form, $x = E_{12},\ y = E_{21},\ h = E_{11} - E_{22}$. The fixed Cartan subalgebra is $\mathfrak{h} = \mathrm{Span}_{\mathbb{C}}\{h\}$. We identify $\mathfrak{h}^*$ with $\mathbb{C}$ through the bijection $c(\varepsilon_1 - \varepsilon_2) \mapsto 2c$, equivalently, $\gamma(c\varepsilon_1) \mapsto c$.

Assume now that V is an irreducible finite-dimensional L-module. Let $v_0 \in V_\lambda$ be a maximal vector; set $v_{-1} = 0,\ v_i = (1/i!)y^i,\ v_0, i \in \mathbb{Z}_{\geq 0}$. There are equations relating the elements of the basis of $\mathfrak{sl}(2)$ with the vectors v_i. The proof of the following lemma can be found in [2].

Lemma 2.5.1.

$$h \cdot v_i = (\lambda - 2i)v_i,$$

$$y \cdot v_i = (i+1)v_{i+1},$$

$$x \cdot v_i = (\lambda - i + 1)v_{i-1}, i \geq 0.$$

Theorem 2.5.2. *For arbitrary $m \geq 0$, the formulas in Lemma* 2.5.1 *define an irreducible L-module of L on an $m+1$-dimensional vector space over $\mathbb{C}$ with basis $(v_0, v_1, ..., v_m)$, and we call this L-module $V(m)$.*

Note that due to the identification $\mathfrak{h}^* \to \mathbb{C}$, $V(m) = V(\gamma(m\varepsilon_1))$.

2.5.2 Finite-dimensional representations of $\mathfrak{sl}(n)$

Consider now for an arbitrary semisimple Lie algebra L over $\mathbb{C}$. If V is a finite dimensional L-module, then V is a weight module.

Proposition 2.5.3. $\mathbb{C}^n = V(\gamma(\varepsilon_1))$ *and* $(\mathbb{C}^n)^* = V(\gamma(-\varepsilon_n))$

Proof. Let first us consider $\mathbb{C}^n$ as a $\mathfrak{gl}(n)$-module. Since $E_{1j}e_j = e_1 \neq 0$ for $j > 1$, e_1 is the only maximal vector up to a nonzero scalar multiple. For obtaining the weight of e_1, we need to find λ such that $h \cdot v = \lambda(h)v$ for all $h \in \mathfrak{h}_{\mathfrak{gl}(n)}$. Let $h = b_{11}E_{11} + ... + b_{nn}E_{nn}$. Then

$$(b_{11}E_{11} + ... + b_{nn}E_{nn})e_1 = \begin{pmatrix} b_{11} & 0 & \dots & \dots & 0 \\ 0 & b_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & b_{nn} \end{pmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} b_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = b_{11}e_1.$$

We have ε_1 as the $\mathfrak{gl}(n)$-weight of e_1. Now let us consider $\mathbb{C}^n$ as an $\mathfrak{sl}(n)$-module. Then $\varepsilon_1 \notin \{a_1\varepsilon_1 + \cdots + a_n\varepsilon_n | \sum a_i = 0\}$, the Cartan subalgebra of $\mathfrak{sl}(n)$. The $\mathfrak{sl}(n)$-weight of e_1 is $\gamma(\varepsilon_1)$. By Theorem 2.5.2, e_1 generates an irreducible L-module $V(\gamma(\varepsilon_1))$. Therefore, $\mathbb{C}^n = V(\gamma(\varepsilon_1))$.

Let us consider now $(\mathbb{C}^n)^*$ as a $\mathfrak{gl}(n)$-module. We have $(\mathbb{C}^n)^* = \mathbb{C}f_1 \oplus \ldots \oplus \mathbb{C}f_n$. We use the action $(E_{kl}f_i)e_j = -f_i(E_{kl}e_j)$ as discussed in Section 2.4. Then it is an easy calculation to get

$$(E_{kl}f_i)e_j = -1 \text{ for } 0 < k < l \leq n,\ i = k,\ j = l,$$

$$(E_{kl}f_i)e_j = 0 \text{ for } i \neq k,\ j \neq i,\ 0 < k < l \leq n.$$

Therefore, $E_{kl}f_i(e_j) = 0$ if $i = n,\ k < l \leq n,$ and $j, l \in [1, n]$, which leads us to f_n as the only maximal vector up to a nonzero scalar multiple. Now we need to find its highest weight. By similar reasoning done for $\mathbb{C}^n$, we have $hf_n = -b_{nn}f_n$, and so $-\varepsilon_n$ is the $\mathfrak{gl}(n)$-weight of f_n. We also have $\gamma(-\varepsilon_n)$ as the $\mathfrak{sl}(n)$-weight of f_n. By Theorem 2.5.2, f_n generates an irreducible L-module $V(\gamma(-\varepsilon_n))$. Therefore, $\mathbb{C}^n = V(\gamma(-\varepsilon_n))$. $\square$

Example 2.5.4. Note that $\gamma(\varepsilon_1) = \gamma(\varepsilon_2) = (\varepsilon_1 - \varepsilon_2)/2$. Thus $\mathbb{C}^2 \cong (\mathbb{C}^2)^*$ as $\mathfrak{sl}(2)$-modules.

CHAPTER 3

TENSOR PRODUCTS OF FINITE-DIMENSIONAL REPRESENTATIONS

In this chapter, we establish some explicit results on the decomposition of the tensor products of finite-dimensional representations of $\mathfrak{gl}(n)$ and $\mathfrak{sl}(n)$. That is, for this chapter, $L = \mathfrak{gl}(n)$ or $L = \mathfrak{sl}(n)$ and we consider $\mathbb{C}^n$ and the dual vector space $(\mathbb{C}^n)^*$ as L-modules; then we use tensor products to obtain new representations. In particular, we express $\mathbb{C}^n \otimes \mathbb{C}^n$ and $\mathbb{C}^n \otimes (\mathbb{C}^n)^*$ as direct sums of some important L-modules as shown in Theorem 3.2.1 and 3.2.3.

3.1 Dual to the natural representations of $\mathfrak{gl}(n)$ and $\mathfrak{sl}(n)$

We first discuss when the natural representation V of $\mathfrak{gl}(n)$ and $\mathfrak{sl}(n)$ is isomorphic to its dual V^*. We note that that $V \simeq V^*$ for the Lie algebras $\mathfrak{sp}(2n)$ and $\mathfrak{so}(n)$.

Lemma 3.1.1. *Let $L = \mathfrak{sl}(n)$ and $V = \mathbb{C}^n$. Then $V \simeq V^*$ if and only if $n = 2$.*

Proof. Recall that by Proposition 2.5.3, $V \simeq V(\gamma(\varepsilon_1))$ and $V^* \simeq V(\gamma(-\varepsilon_n))$. Therefore $V \simeq V^*$ if and only if $\gamma(\varepsilon_1) = \gamma(-\varepsilon_n)$. One can easily check that $\gamma(\varepsilon_1 + \varepsilon_n) = 0$ if and only if $n = 2$ (see Example 2.5.4). $\square$

Using reasoning similar to the one used in the proof of Proposition 2.5.3, we can establish the following more general result.

Theorem 3.1.2. *If λ is a dominant integral and if $\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + ... + \lambda_n\varepsilon_n$, then $(V(\lambda))^* = V(-\lambda^T)$, where $\lambda^T = \lambda_n\varepsilon_1 + \lambda_{n-1}\varepsilon_2 + ... + \lambda_1\varepsilon_n$.*

3.2 Tensor products of finite-dimensional representations of $\mathfrak{sl}(n)$

Theorem 3.2.1. *We have the following identity of $\mathfrak{sl}(n)$-modules:*

$$\mathbb{C}^n \otimes \mathbb{C}^n = S^2(\mathbb{C}^n) \oplus \Lambda^2(\mathbb{C}^n).$$

Before we prove Theorem 3.2.1, let us present $S^2(\mathbb{C}^n)$ and $\Lambda^2(\mathbb{C}^n)$ as highest weight modules and let us find their highest weights. We need following proposition,

Proposition 3.2.2. *$S^2(\mathbb{C}^n) = V(\gamma(2\varepsilon_1))$ and $\Lambda^2(\mathbb{C}^n) = V(\gamma(\varepsilon_1 + \varepsilon_2))$.*

Proof. To find a maximal vector $v \in S^2(\mathbb{C}^n)$, we need to find v such that $E_{12}v = E_{23}v = \ldots = E_{n-1,n}v = 0$. It is easy to verify that $E_{12}e_1 \otimes e_1 = E_{23}e_1 \otimes e_1 = ... = E_{n-1,n}e_1 \otimes e_1 = 0$. Therefore, $v = e_1 \otimes e_1$ is a maximal vector. We now show that all maximal vectors are scalar multiple of $e_1 \otimes e_1$.

By Lemma 2.4.3, it follows that $v = \sum a_{ij}\frac{e_i\otimes e_j + e_j\otimes e_i}{2}$. Therefore, it is straightforward to verify that $E_{12}(\sum a_{ij}\frac{e_i\otimes e_j+e_j\otimes e_i}{2}) = \ldots = E_{n-1,n}(\sum a_{ij}\frac{e_i\otimes e_j+e_j\otimes e_i}{2}) = 0$ implies that all a_{ij}'s are equal to 0 except for a_{11} (for more explicitness, see 3.1 below for similar work). Thus, up to a nonzero scalar multiple, $e_1 \otimes e_1$ is the only maximal vector in $S^2(\mathbb{C}^n)$. Since $h(e_1 \otimes e_1) = 2b_{11}(e_1 \otimes e_1)$, the highest weight of the $\mathfrak{sl}(n)$-module $S^2(\mathbb{C}^n)$ is $\gamma(2\varepsilon_1)$. Thus $S^2(\mathbb{C}^n) = V(\gamma(2\varepsilon_1))$.

By Lemma 2.4.3, v is of a form $\sum a_{ij}\frac{e_i\otimes e_j - e_j\otimes e_i}{2}$. It is straightforward to check that $E_{12}(\sum a_{ij}\frac{e_i\otimes e_j-e_j\otimes e_i}{2}) = ... = E_{n-1,n}(\sum a_{ij}\frac{e_i\otimes e_j-e_j\otimes e_i}{2}) = 0$ implies that all a_{ij}'s are equal to 0 except for a_{12}. Thus $v = \frac{e_1 \otimes e_2 - e_2 \otimes e_1}{2}$ is the only maximal vector for $\lambda^2(\mathbb{C}^n)$ up to a scalar multiple. Then $h(\frac{e_1 \otimes e_2 - e_2 \otimes e_1}{2}) = b_{11} + b_{22}$. Therefore, $\Lambda^2(\mathbb{C}^n) = V(\gamma(\varepsilon_1 + \varepsilon_2))$. □

We are now ready to prove Theorem 3.2.1.

Proof. Let $(e_1, e_2, ..., e_n)$ be a basis of $\mathbb{C}^n$. Thus the basis of $\mathbb{C}^n \otimes \mathbb{C}^n$ consists of n^2 vectors $e_i \otimes e_j$. Let w be an arbitrary vector in $\mathbb{C}^n \otimes \mathbb{C}^n$. Then w is a vector of a form $\sum a_{ij} e_i \otimes e_j$. To find a maximal vector w of $\mathbb{C}^n \otimes \mathbb{C}^n$, we need $E_{12}w = E_{23}w = ... = E_{n-1,n}w = 0$ to hold. We will use a system with n^2 unknowns as follows:

$$\begin{aligned}
0 &= E_{12}w \\
&= E_{12}(\sum a_{ij} e_i \otimes e_j) \\
&= E_{12}(a_{12}e_1 \otimes e_2 + a_{22}e_2 \otimes e_2 + \ldots + a_{n2}e_n \otimes e_2 \\
&\qquad + a_{21}e_2 \otimes e_1 + a_{23}e_2 \otimes e_3 + \ldots + a_{2n}e_2 \otimes e_n) \\
&= a_{12}e_1 \otimes e_1 + a_{22}e_1 \otimes e_2 + a_{22}e_2 \otimes e_1 + a_{32}e_3 \otimes e_1 + \ldots + a_{n2}e_n \otimes e_1 \\
&\qquad + a_{21}e_1 \otimes e_1 + a_{23}e_1 \otimes e_3 + \ldots + a_{2n}e_1 \otimes e_n
\end{aligned} \tag{3.1}$$

Since $e_k \otimes e_l$ and $e_r \otimes e_s$ are linearly independent whenever $k \neq r$ or $l \neq s$, we have $a_{12} + a_{21} = 0, a_{i2} = a_{2i} = 0$, for $n \geq i \geq 2$.

$$\begin{aligned}
0 &= E_{23}w \\
&= E_{23}(\sum a_{ij} e_i \otimes e_j) = E_{23}(a_{13}e_1 \otimes e_3 + a_{23}e_2 \otimes e_3 + ... + a_{n3}e_n \otimes e_3 \\
&\qquad + a_{31}e_3 \otimes e_1 + a_{32}e_3 \otimes e_2 + ... + a_{3n}e_3 \otimes e_n) \\
&= a_{13}e_1 \otimes e_2 + a_{23}e_2 \otimes e_2 + a_{33}e_2 \otimes e_3 + a_{33}e_3 \otimes e_2 + ... + a_{n3}e_n \otimes e_3 \\
&\qquad + a_{31}e_2 \otimes e_1 + a_{32}e_2 \otimes e_2 + a_{34}e_2 \otimes e_4 + ... + a_{3n}e_2 \otimes e_n.
\end{aligned}$$

By linear independence, $a_{i3} = a_{3i} = 0$, for $n \geq i \geq 3$ and $i = 1$.

Repeating the same steps for $E_{34}w = 0, ..., E_{n-1,n}w = 0$ as above, we get that the only nonzero a_{ij}'s are a_{11}, a_{12}, and a_{21} with $a_{12} + a_{21} = 0$. By substituting all the values of the a_{ij}'s in $\sum a_{ij} e_i \otimes e_j = w$, this means that

$$w = a_{11}e_1 \otimes e_1 + a_{12}e_1 \otimes e_2 - a_{21}e_2 \otimes e_1 = a_{11}e_1 \otimes e_1 + 2a_{12}\frac{e_1 \otimes e_2 + e_2 \otimes e_1}{2}.$$

But we have seen $e_1 \otimes e_1$ and $\frac{e_1 \otimes e_2 + e_2 \otimes e_1}{2}$ are also maximal vectors in $S^2(\mathbb{C}^n)$ and in $\Lambda^2(\mathbb{C}^n)$, respectively. So we conclude that $\mathbb{C}^n \otimes \mathbb{C}^n$ is the direct sum of the two irreducible $\mathfrak{sl}(n)$-modules $S^2(\mathbb{C}^n)$ and $\Lambda^2(\mathbb{C}^n)$. $\square$

Theorem 3.2.3. *We have the following isomorphism of $\mathfrak{sl}(n)$-modules and $\mathfrak{gl}(n)$-modules:*

$$\mathbb{C}^n \otimes (\mathbb{C}^n)^* \simeq \mathfrak{gl}(n).$$

Hence, $\mathbb{C}^n \otimes (\mathbb{C}^n)^* = \mathfrak{sl}(n) \oplus \mathbb{C}$ *as* $\mathfrak{sl}(n)$*-modules.*

Proof. Let $(e_1, e_2, ..., e_n)$ be a basis of $\mathbb{C}^n$, and $(f_1, f_2, ..., f_n)$ be a basis of $(\mathbb{C}^n)^*$. Thus the basis of $\mathbb{C}^n \otimes (\mathbb{C}^n)^*$ consists of n^2 vectors $e_i \otimes f_j$. If w is an arbitrary vector in $\mathbb{C}^n \otimes (\mathbb{C}^n)^*$, then by similar work in the Equation 3.1 above in this proof, $a_{ij} = 0$ for all (i, j) except for $(i, j) = (1, n), (1, 1), (2, 2), ..., (n, n)$ where $(1, 1) = (2, 2) = ... = (n, n)$. Then $w = a_{1n}e_1 \otimes f_n + a_{11}e_1 \otimes f_1 + a_{22}e_2 \otimes f_2 + ... + a_{nn}e_n \otimes f_n = a_{1n}e_1 \otimes f_n + a_{11}(e_1 \otimes f_1 + .. + e_n \otimes f_n)$.

Thus maximal vectors in $\mathbb{C}^n \otimes (\mathbb{C}^n)^*$ are $a_{in}e_1 \otimes f_n$ and $a_{11}(e_1 \otimes f_1 + .. + e_n \otimes f_n)$. To find the highest weight, we let $h = b_{11}E_{11} + ... + b_{nn}E_{nn}$. Then

$$h(a_{in}e_1 \otimes f_n) = a_{in}he_1 \otimes f_n + a_{in}e_1 \otimes hf_n = a_{in}b_{11}e_1 \otimes f_n + a_{in}e_1 \otimes b_{nn}f_n = (b_{11} - b_{nn})a_{in}e_1 \otimes f_n.$$

Thus the weight of $a_{in}e_1 \otimes f_n$ is $\varepsilon_1 - \varepsilon_n$. With similar reasoning, the weight of $a_{11}(e_1 \otimes f_1 + .. + e_n \otimes f_n)$ is 0.

Thus $\mathbb{C}^n \otimes (\mathbb{C}^n)^* \simeq V(\varepsilon_1 - \varepsilon_n) \oplus V(0)$ as $\mathfrak{sl}(n)$-modules. In fact, $\mathfrak{sl}(n) = V(\varepsilon_1 - \varepsilon_n)$, and, obviously, $V(0) = \mathbb{C}$. □

CHAPTER 4

TENSOR PRODUCTS OF A FINITE-DIMENSIONAL REPRESENTATION AND AN INFINITE-DIMENSIONAL REPRESENTAION OF $\mathfrak{sl}(n)$

4.1 The case of $\mathfrak{sl}(2)$

Given an arbitrary scalar $\lambda \in \mathbb{C}$, let $Z(\lambda)$ be a vector space over $\mathbb{C}$ with a countably infinite basis $(v_0, v_1, v_2, \ldots)$. Using Lemma 2.5.1, we define an action of $L = \mathfrak{sl}(2)$ on $Z(\lambda)$. Then as one easily checks, $Z(\lambda)$ becomes an (infinite-dimensional) $\mathfrak{sl}(2)$-module. This module is called the *Verma module* of highest weight λ. In this section, we study the tensor product $Z(\lambda) \otimes \mathbb{C}^2$. We know that this tensor product of a finite-dimensional representation and an infinite-dimensional representation of $\mathfrak{sl}(2)$ is an $\mathfrak{sl}(2)$-module of the underlying vector spaces, as described in Chapter 2. We will see that the decomposition of $Z(\lambda) \otimes \mathbb{C}^2$ is much more complicated than the one of $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes (\mathbb{C}^2)^*$ obtained in the previous chapter.

We use the same methods as before – we find the maximal vectors of the infinite-dimensional representation $Z(\lambda) \otimes \mathbb{C}^2$ of $\mathfrak{sl}(2)$. We first note that $Z(\lambda) \otimes \mathbb{C}^2$ has a basis consisting of the vectors $v_i \otimes e_j$, $i \geq 0$, $j = 1, 2$.

Lemma 4.1.1. *The module $Z(\lambda) \otimes \mathbb{C}^2$ has maximal vectors of weight $\lambda + 1$ and $\lambda - 1$. Moreover, if $M_1 = \mathrm{Span}_{\mathbb{C}}\{E_{21}^k(v_0 \otimes e_1) | k \in \mathbb{Z}_{\geq 0}\}$ and $M_2 = \mathrm{Span}_{\mathbb{C}}\{E_{21}^k(v_1 \otimes e_1 - \lambda v_0 \otimes$*

$e_2)|k \in \mathbb{Z}_{\geq 0}\}$, then M_1 and M_2 are submodules of $Z(\lambda) \otimes \mathbb{C}^2$ of highest weights $\lambda + 1$ and $\lambda - 1$, respectively.

Proof. Let $v = \sum a_{ij} v_i \otimes e_j$ (finite sum) be an arbitrary vector in $Z(\lambda) \otimes \mathbb{C}^2$. Recall that $E_{12} = x$, and that v is a maximal vector of $Z(\lambda) \otimes \mathbb{C}^2$ if $x \cdot v = 0$. Therefore:

$$\begin{aligned}
0 &= E_{12} \sum a_{ij} v_i \otimes e_j \\
&= \sum a_{ij} x \cdot v_i \otimes e_j + \sum a_{ij} v_i \otimes x \cdot e_j \\
&= \sum a_{ij} (\lambda - i + 1) v_{i-1} \otimes e_j + \sum a_{i2} v_i \otimes e_1 \\
&= a_{11}(\lambda - 1 + 1) v_0 \otimes e_1 + a_{21}(\lambda - 2 + 1) v_1 \otimes e_1 + \cdots + a_{k1}(\lambda - k + 1) v_{k-1} \otimes e_1 \\
&\quad + a_{12}(\lambda - 1 + 1) v_0 \otimes e_2 + a_{22}(\lambda - 2 + 1) v_1 \otimes e_2 + \cdots + a_{k2}(\lambda - k + 1) v_{k-1} \otimes e_2 \\
&\quad + a_{02} v_0 \otimes e_1 + a_{12} v_1 \otimes e_1 + a_{22} v_2 \otimes e_1 + \ldots + a_{k2} v_k \otimes e_1.
\end{aligned}$$

Then by the linear independence of the vectors $v_i \otimes e_j$, we see that

$$\begin{array}{rl}
-a_{11}(\lambda) = a_{02} & \qquad a_{12}(\lambda) = 0 \\
-a_{21}(\lambda) = a_{12} & \qquad a_{22}(\lambda - 1) = 0 \\
-a_{31}(\lambda) = a_{22} & \qquad a_{32}(\lambda - 2) = 0 \\
\vdots & \qquad \vdots \\
-a_{k1}(\lambda - (k-1)) = a_{k2} & \qquad a_{k2}(\lambda - (k-1)) = 0.
\end{array}$$

We proceed with two cases depending on λ.

Case 1: If $\lambda \notin \{0, 1, 2, 3, \ldots\}$

$$a_{12} = a_{22} = a_{32} = \ldots = a_{k2} = 0$$

$$\Rightarrow \quad a_{21} = a_{31} = a_{41} = \ldots = a_{k1} = 0.$$

This gives us that $a_{ij} = 0$, $\forall (i, j)$, except for $(i, j) = (0, 1), (1, 1), (0, 2)$ such that $-a_{11}(\lambda) = a_{02}$.

Substituting all the a_{ij}'s, we have

$$\begin{aligned} v &= a_{01} v_0 \otimes e_1 + a_{11} v_1 \otimes e_1 + a_{02} v_0 \otimes e_2 \\ &= a_{01} v_0 \otimes e_1 + a_{11}(v_1 \otimes e_1 - \lambda v_0 \otimes e_2). \end{aligned}$$

Thus, $a_{01} v_0 \otimes e_1$ and $a_{11}(v_1 \otimes e_1 - \lambda v_0 \otimes e_2)$ are maximal vectors of $Z(\lambda) \otimes \mathbb{C}^2$. More precisely, the space of maximal vectors is $\mathrm{Span}_{\mathbb{C}}\{v_1 \otimes e_1 - \lambda v_0 \otimes e_2, v_0 \otimes e_1\}$. Then we find the highest weights:

Let $h = E_{11} - E_{22}$. Since $h \cdot (v_0 \otimes e_1) = h \cdot v_0 \otimes e_1 + v_0 \otimes h \cdot e_1 = (\lambda + 1) v_0 \otimes e_1$, the $\mathfrak{sl}(2)$ weight of $v_0 \otimes e_1$ is $\lambda + 1$. With similar reasoning, we have the weight of $v_1 \otimes e_1 - \lambda v_0 \otimes e_2$ to be $\lambda - 1$. By Theorem 2.5.2, there exists an $\mathfrak{sl}(2)$-submodule of $Z(\lambda) \otimes \mathbb{C}^2$ spanned by $\{E_{21}^k (v_0 \otimes e_1) | k \in \mathbb{Z}_{\geq 0}\}$ with highest weight $\lambda + 1$, and an $\mathfrak{sl}(2)$-submodule spanned by $\{E_{21}^k (v_1 \otimes e_1 - \lambda v_0 \otimes e_2) | k \in \mathbb{Z}_{\geq 0}\}$ with highest weight $\lambda - 1$. The first module is M_1, while the second one is M_2.

Case 2: If $\lambda \in \mathbb{Z}_{\geq 0}$, then the system of unknown a_{ij}'s in Lemma 4.1.1 gives us:

For $\lambda = 0$

$$a_{21} = a_{12},$$

$$a_{02} = a_{22} \quad = a_{32} = \ldots = a_{k2} = 0,$$

$$a_{31} = a_{41} \quad = a_{51} = \ldots = a_{k1} = 0.$$

For $\lambda = 1$

$$-a_{11} = a_{02},$$

$$a_{31} = a_{22},$$

$$a_{12} = a_{32} \quad = a_{42} = \ldots = a_{k2} = 0,$$

$$a_{41} = a_{51} \quad = a_{61} = \ldots = a_{k1} = 0.$$

For $\lambda = 2$

$$-2a_{11} = a_{02},$$

$$a_{41} = a_{32},$$

$$a_{12} = a_{22} \quad = a_{42} = a_{52} = \ldots = a_{k2} = 0,$$

$$a_{21} = a_{51} \quad = a_{61} = a_{71} = \ldots = a_{k1} = 0.$$

Repeating these steps, we see that for $\lambda \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} a_{02} &= -i a_{11} \\ a_{2+\lambda,1} &= a_{1+\lambda,2} \\ a_{i2} &= \quad 0 \qquad \forall i,\ \lambda + 1 < i \leq \lambda. \end{aligned}$$

Therefore, $a_{ij} = 0$, $\forall (i,j)$ except $a_{01}, a_{02}, a_{11}, a_{2+\lambda,1}, a_{1+\lambda,2}, a_{1+\lambda,1}$, where $a_{02} = -\lambda a_{11}, a_{2+\lambda,1} = a_{1+\lambda,2}$. Therefore, $v = a_{01} v_0 \otimes e_1 + a_{11}(v_1 \otimes e_1 - \lambda v_0 \otimes e_2) + a_{2+\lambda,1}(v_{2+\lambda} \otimes e_1 + v_{1+\lambda} \otimes e_2) + a_{1+\lambda,1}(v_{1+\lambda} \otimes e_1)$, and the maximal vector space is the span of the set $\{v_1 \otimes e_1 - \lambda v_0 \otimes e_2, v_0 \otimes e_1, v_{2+\lambda} \otimes e_1 + v_{1+\lambda} \otimes e_2, v_{1+\lambda} \otimes e_1\}$. However, the vector $v_{2+\lambda} \otimes e_1 + v_{1+\lambda} \otimes e_2$ is generated by $v_0 \otimes e_1$, while $v_{1+\lambda} \otimes e_1$ is generated by $v_1 \otimes e_1 - \lambda v_0 \otimes e_2$. Since these maximal vectors are the same ones in Case 1, we conclude that M_1 and M_2 are submodules of $Z(\lambda) \otimes \mathbb{C}^2$ with their corresponding highest weights as in Case 1. However, these submodules are definitely not irreducible, since each has two linearly independent maximal vectors. $\square$

Theorem 4.1.2. *If $\lambda \neq -1$, then $Z(\lambda) \otimes \mathbb{C}^2 \simeq Z(\lambda+1) \oplus Z(\lambda-1)$.*

Proof. By Lemma 4.1.1, the tensor product $Z(\lambda) \otimes \mathbb{C}^2$ contains $\mathfrak{sl}(2)$-submodules M_1 and M_2 of highest weights $\lambda+1$ and $\lambda-1$, respectively. To prove the isomorphism in the statement of the theorem, we will check that $M_1 \simeq Z(\lambda+1)$ and $M_2 \simeq Z(\lambda-1)$. Then we will check whether $Z(\lambda) \otimes \mathbb{C}^2 = Z(\lambda+1) \oplus Z(\lambda-1)$ as vector spaces, by verifying that $Z(\lambda) \otimes \mathbb{C}^2 = Z(\lambda+1) + Z(\lambda-1)$ and that $Z(\lambda+1)$ and $Z(\lambda-1)$ have trivial intersection.

In $M_1 = \mathrm{Span}_{\mathbb{C}}\{w_k = E_{21}^k(v_0 \otimes e_1) \mid k \in \mathbb{Z}_{\geq 0}\}$, repeated application of the formula $w_n = E_{21}^n(v_0 \otimes e_1)$, $\forall n \in \{0, 1, \ldots, k\}$ and induction on k, we observe that

$$w_k = k!(v_k \otimes e_1 + v_{k-1} \otimes e_2). \tag{4.1}$$

Similarly, with $M_2 = \mathrm{Span}_{\mathbb{C}}\{u_k = E_{21}^k(v_1 \otimes e_1 - \lambda v_0 \otimes e_2) \mid k \in \mathbb{Z}_{\geq 0}\}$, we observe that,

$$u_k = (k+1)! v_{k+1} \otimes e_1 + k!(k-\lambda) v_k \otimes e_2. \tag{4.2}$$

Now we need to verify that the action of $\mathfrak{sl}(2)$ on M_1 is given by the formulas in Lemma 2.5.1 to prove that $M_1 \simeq Z(\lambda+1)$. Let $h = E_{11} - E_{22}, x = E_{12}$, and $y = E_{21}$. Note that

$$\begin{aligned} h \cdot w_k &= k!(h \cdot v_k \otimes e_1 + v_k \otimes h \cdot e_1 + h \cdot v_{k-1} \otimes e_2 + v_{k-1} \otimes h \cdot e_2 \\ &= k!((\lambda - 2k) v_k \otimes e_1 + v_k \otimes e_1 + (\lambda - 2k + 2) v_{k-1} \otimes e_2 - v_{k-1} \otimes e_2) \\ &= ((\lambda+1) - 2k) w_k. \end{aligned}$$

Similarly,

$$\begin{aligned} y \cdot w_k &= k!(y \cdot v_k \otimes e_1 + v_k \otimes y \cdot e_1 + y \cdot v_{k-1} \otimes e_2 + v_{k-1} \otimes y \cdot e_2 \\ &= k!((k+1) v_{k+1} \otimes e_1 + v_k \otimes e_2 + k v_k \otimes e_2) \\ &= k!(k+1)(v_{k+1} \otimes e_1 + v_k \otimes e_2) \\ &= w_{k+1}, \end{aligned}$$

and also,

$$\begin{aligned} x \cdot w_k &= k!(x \cdot v_k \otimes e_1 + v_k \otimes x \cdot e_1 + x \cdot v_{k-1} \otimes e_2 + v_{k-1} \otimes x \cdot e_2 \\ &= k!((\lambda - k + 1) v_{k-1} \otimes e_1 + (\lambda - k + 2) v_{k-2} \otimes e_2 + v_{k-1} \otimes e_1) \\ &= k!(\lambda - k + 2)(v_{k-1} \otimes e_1 + v_{k-2} \otimes e_2) \\ &= k((\lambda+1) - k + 1) w_{k-1}. \end{aligned}$$

Then, we see that $\phi : M_1 \to Z(\lambda+1)$ is an isomorphism of $\mathfrak{sl}(2)$-modules, where $\phi : w_k \mapsto k!v_k$. The proof is as follows:

$$\begin{aligned} y \cdot \phi(w_k) &= y \cdot (k!v_k) \\ &= k!y \cdot (v_k) \\ &= k!(k+1)v_{k+1} \\ &= (k+1)!v_{k+1} \\ &= \phi(w_{k+1}) \\ &= \phi(y \cdot w_k). \end{aligned}$$

Analogously

$$\begin{aligned} x \cdot \phi(w_k) &= x \cdot (k!v_k) \\ &= k!x \cdot (v_k) \\ &= (\lambda + 1 - k + 1)k!v_{k-1} \\ &= k(\lambda - k + 2)(k-1)!v_{k-1} \\ &= k(\lambda - k + 2)\phi(w_{k-1}) \\ &= \phi(k(\lambda - k + 2)w_{k-1}) \\ &= \phi(x \cdot w_k) \end{aligned}$$

The relation $h \cdot \phi(w_k) = \phi(h \cdot w_k)$ is easy to show. It is also easy to show that ϕ is linear and bijective. Therefore, ϕ is an isomorphism of $\mathfrak{sl}(2)$-modules. So we have $M_1 \simeq Z(\lambda+1)$. By similar reasoning, we can prove that $M_2 \simeq Z(\lambda - 1)$.

Now we want to show that $Z(\lambda)\otimes\mathbb{C}^2 = Z(\lambda+1)\oplus Z(\lambda-1)$ as vector spaces.

To show the sum of vector space: Let $v = \sum a_{ij}v_i\otimes e_j \in Z(\lambda)\otimes\mathbb{C}^2$. Then expanding the series, we have

$$\begin{aligned} v &= \sum a_{ij}v_i\otimes e_j \\ &= a_{01}v_0\otimes e_1 + a_{11}v_1\otimes e_1 + a_{21}v_2\otimes e_1 + \cdots + a_{k1}v_k\otimes e_1 \\ &\quad + a_{02}v_0\otimes e_2 + a_{12}v_1\otimes e_2 + a_{22}v_2\otimes e_2 + \cdots + a_{k2}v_k\otimes e_2 \\ &= \sum b_i(i!(v_i\otimes e_1 + v_{i-1}\otimes e_2)) + \sum c_i((i+1)!v_{i+1}\otimes e_1 + i!(i-\lambda)v_i\otimes e_2) \\ &= w+u, \end{aligned}$$

where w is in $M_1 = \mathrm{Span}_{\mathbb{C}}\{k!(v_k\otimes e_1 + v_{k-1}\otimes e_2) \mid k\in\mathbb{Z}_{\geq 0}\}$ and u is in $M_2 = \mathrm{Span}_{\mathbb{C}}\{(k+1)!v_{k+1}\otimes e_1 + k!(k-\lambda)v_k\otimes e_2 \mid k\in\mathbb{Z}_{\geq 0}\}$. To show that $Z(\lambda+1)\cap Z(\lambda-1) = 0$, it is sufficient to show that they intersect trivially on each weight space. Hence we need to check if $w_{i+1} = (i+1)!(v_{i+1}\otimes e_1 + v_1\otimes e_2)$ and $u_i = (i+1)!v_{i+1}\otimes e_1 + i!(i-\lambda)v_1\otimes e_2$ are linearly independent. Suppose that $w_{i+1} = cu_i$. Without loss of generality, let $c = 1$. Then it implies that

$$(i+1)! = i!\,(i-\lambda) \qquad \Leftrightarrow \qquad i+1 = i-\lambda \qquad \Leftrightarrow \qquad \lambda = -1.$$

In other words, $w_{i+1} = u_i$ if and only if $\lambda = -1$. Thus, the result holds. □

Theorem 4.1.3. *Let λ be an $\mathfrak{sl}(2)$-weight.*

(a) $Z(\lambda)$ is simple if and only if $\lambda \notin \mathbb{Z}_{\geq 0}$.

(b) If $\lambda\in\mathbb{Z}_{\geq 0}$, there is a nonsplit exact sequence of $\mathfrak{sl}(2)$-modules

$$0 \to Z(\lambda-2) \xrightarrow{\psi} Z(\lambda) \xrightarrow{\phi} V(\lambda) \to 0$$

Proof. The proof of this theorem is standard. See for example §7 of [2]. In the case $\lambda = 0$, the theorem can be proven using tensor products, see Remark 4.1.6. $\square$

4.1.1 The module $Z(-1) \otimes \mathbb{C}^2$

We have seen earlier in this section that in order to show that $Z(\lambda) \otimes \mathbb{C}^2$ is a direct sum of $Z(\lambda+1)$ and $Z(\lambda-1)$, the later vector spaces M_1 and M_2 must intersect trivially, and the latter fails when $\lambda \neq -1$. It remains to describe $Z(\lambda) \otimes \mathbb{C}^2$ when $\lambda = -1$. For the rest of this subsection, we will assume that $\lambda = -1$, and later show that the $\mathfrak{sl}(2)$-module $Z(-1) \otimes \mathbb{C}^2$ does not split, namely $Z(-1) \otimes \mathbb{C}^2$ does not contain nonzero submodules A and B such that $Z(-1) \otimes \mathbb{C}^2 = A \oplus B$.

Theorem 4.1.4. *There is a short exact sequence of $\mathfrak{sl}(2)$-modules*

$$0 \to M_1 \to Z(-1) \otimes \mathbb{C}^2 \to M_1' \to 0,$$

where $M_1 \simeq Z(0)$ and M_1' is isomorphic to the module M_2 defined in Lemma 4.1.1. In particular, $M_1' \simeq Z(-2)$.

Proof. Let $Z(-1) \otimes \mathbb{C}^2 / M_1 = M_1'$ be a $\mathrm{Span}_{\mathbb{C}}\{v_i \otimes e_1 + M_1, v_i \otimes e_2 + M_1 \mid i \in \mathbb{Z}_{\geq 0}\}$, where, as usual, the v_i's and the e_i's are bases of $Z(-1)$ and $\mathbb{C}^2$, respectively. We claim that the set $\{v_0 \otimes e_2 + M_1, v_1 \otimes e_2 + M_1, \ldots\}$ is a basis of M_1'. Proof of the claim: We know from Equation (4.1), $M_1 = \mathrm{Span}_{\mathbb{C}}\{k!(v_k \otimes e_1 + v_{k-1} \otimes e_2) \mid k \in \mathbb{Z}_{\geq 0}\}$.

Therefore, in M_1'

$$\begin{aligned} & k!(v_k \otimes e_1 + v_{k-1} \otimes e_2 + M_1 = M_1 = 0 + M_1 \\ \Rightarrow \quad & k!(v_k \otimes e_1 + v_{k-1} \otimes e_2 \in M_1 \\ \Rightarrow \quad & v_k \otimes e_1 + M_1 = -v_{k-1} \otimes e_2 + M_1 \\ \Rightarrow \quad & M_1' = \mathrm{Span}_{\mathbb{C}}\{v_k \otimes e_2 + M_1 \mid k \in \mathbb{Z}_{\geq 0}\}. \end{aligned}$$

Assume now that:

$$a_0(v_0 \otimes e_2 + M_1) + a_1(v_1 \otimes e_2 + M_1) + \ldots + a_k(v_k \otimes e_2 + M_1) = 0.$$

Therefore: $\quad a_0 v_0 \otimes e_2 + a_1 v_1 \otimes e_2 + \ldots + a_k v_k \otimes e_2 \in M_1$

$$\begin{aligned} \Rightarrow \quad & a_0 v_0 \otimes e_2 + a_1 v_1 \otimes e_2 + \ldots + a_k v_k \otimes e_2 = \\ & b_0(v_0 \otimes e_1) + b_1(v_1 \otimes e_1 + v_0 \otimes e_2) + \ldots + b_{k+1}(k+1)!(v_{k+1} \otimes e_1 + v_k \otimes e_2). \end{aligned}$$

By the linear independence of the set of all vectors $v_i \otimes e_j$, $i \geq 0$, $j = 1, 2$, in $Z(-1) \otimes \mathbb{C}^2$, we then have

$$\begin{aligned} & b_0 = b_1 = \ldots = b_{k+1} = 0 \\ & a_0 - b_1 = 0 \\ & \qquad \vdots \\ & a_k - b_{k+1} = 0 \\ \Rightarrow \quad & a_0 = \ldots = a_k = 0. \end{aligned}$$

It implies $\{v_k \otimes e_2 + M_1 \mid k \in \mathbb{Z}_{\geq 0}\}$ is linearly independent. The claim is proved.

Now we want to prove that $Z(-1) \otimes \mathbb{C}^2/Z(0)$ is isomorphic to $Z(-2)$. Since $M_1 \simeq Z(\lambda+1) = Z(0)$ and $M_2 \simeq Z(\lambda-1) = Z(-2)$, as proven in Theorem 4.1.2, it is sufficient to show that $M_1' \simeq M_2$, where M_2 is $\mathrm{Span}_{\mathbb{C}}\{(k+1)!(v_{k+1} \otimes e_1 + v_k \otimes e_2 \mid k \in \mathbb{Z}_{\geq 0}\}$ from Equation (4.2).

Let $\phi : M_2 \to M_1'$ be a map via $(i+1)(v_{i+1} \otimes e_1 + v_i \otimes e_2) \mapsto v_i \otimes e_2 + M_1$. It is not difficult to check that ϕ is linear, bijective and satisfy the definition of homomorphism of L-modules. The result holds. □

Corollary 4.1.5. *Let M_1, M_2, M_1' be the modules defined in Lemma 4.1.1 and Theorem 4.1.4.*

(a) We have the following nonsplit exact sequence of $\mathfrak{sl}(2)$-modules:

$$0 \to M_1 \xrightarrow{\psi} Z(-1) \otimes \mathbb{C}^2 \xrightarrow{\phi} M_1' \to 0.$$

(b) We have the following nonsplit exact sequence of $\mathfrak{sl}(2)$-modules:

$$0 \to M_2 \xrightarrow{\psi} M_1 \xrightarrow{\phi} M_2' \to 0$$

where $M_2' = \mathrm{Span}_{\mathbb{C}}\{w_0 + M_2\}$.

Proof. (a) We have that $\{v_i \otimes e_1; v_i \otimes e_2 | i \in \mathbb{Z}_{\geq 0}\}$ is a basis of $Z(-1) \otimes \mathbb{C}^2$ and from the proof of Theorem 4.1.4, $\{v_i \otimes e_2 + M_1 | i \in \mathbb{Z}_{\geq 0}\}$ is a basis of M_1'. Let $\{u_0, u_1, \ldots\}$ be a basis of M_1, ψ denote the map $u_i \mapsto v_i \otimes e_1$, and ϕ denote the map $v_i \otimes e_2 \mapsto v_i \otimes e_2 + M_1, v_i \otimes e_1 \mapsto 0$. Then the map ψ is injective and ϕ is surjective. The image of the map ψ equals the span of $\{v_i \otimes e_1 | i \in \mathbb{Z}_{\geq 0}\}$ which is also the kernel of ϕ. Thus, the sequence given in (a) is exact. It is nonsplit because there is no nonzero $\mathfrak{sl}(2)$-module homomorphism η of M_1' into $Z(-1) \otimes \mathbb{C}^2$ such that $\phi\eta = 0$. The latter follows from the fact that if such a nonzero homomorphism η exists, then η maps the highest weight vector $v_0 \otimes e_2 + M_1$ of M_1' to zero.

(b) Recall $M_1 = \mathrm{Span}_{\mathbb{C}}\{w_k = k!(v_k \otimes e_1 + v_{k-1} \otimes e_2) | k \in \mathbb{Z}_{\geq 0}\}$ and $M_2 = \mathrm{Span}_{\mathbb{C}}\{u_k = (k+1)! v_{k+1} \otimes e_1 + k!(k-\lambda) v_k \otimes e_2 | k \in \mathbb{Z}_{\geq 0}\}$ from Equations (4.1) and (4.2). In the proof of Theorem 4.1.2, we noted that $w_{i+1} = u_1$ for all $i \in \mathbb{Z}_{\geq 0}$ if and only if $\lambda = -1$. So $w_1 = u_0, w_2 = u_1$, and so on. Then $\mathrm{Span}_{\mathbb{C}}\{w_1, w_2, \ldots\} \subseteq \mathrm{Span}_{\mathbb{C}}\{w_0, w_1, \ldots\} \subseteq Z(-1) \otimes \mathbb{C}^2$.

It is easy to check that $\{w_k = k!(v_k \otimes e_1 + v_{k-1} \otimes e_2) | k \in \mathbb{Z}_{\geq 0}\}$ is linearly independent. So M_2 has a basis $\{w_1, w_2, \ldots\}$, while M_1 has a basis $\{w_0, w_1, \ldots\}$.

Let ψ denote the map $w_i \mapsto w_i$, and ϕ denote the natural projection $w_i \mapsto w_i + M_2'$. Clearly, ψ is injective and ϕ is surjective. The image of ψ is $M_2 \subseteq M_1$, and the kernel of ϕ is also M_2. Thus, we have the exact sequence. It is nonsplit because there is no nontrivial $\mathfrak{sl}(2)$-module homomorphism of M_2' into M_1. $\square$

Remark 4.1.6. We note that part (b) of the last corollary is in fact equivalent to Theorem 4.1.3 (b) for $\lambda = 0$. In this way, we have a tensor product proof of the case $\lambda = 0$ of that theorem.

Corollary 4.1.7. *For any $\lambda \in \mathbb{C}$, the module $Z(\lambda) \otimes \mathbb{C}^2$ has a filtration whose subquotients are isomorphic to $Z(\lambda - 1)$ and $Z(\lambda + 1)$.*

4.2 The case of $\mathfrak{sl}(3)$ and $\mathfrak{gl}(3)$

In this section, we work with $L = \mathfrak{sl}(3)$. The results automatically transfer to the Lie algebra $\mathfrak{gl}(3)$. We fix the basis of $\mathfrak{h}_{\mathfrak{sl}(3)}$ to be $h_1 = E_{11} - E_{22}, h_2 = E_{22} - E_{33}$. The basis of $\mathfrak{sl}(3)$ is h_1, h_2, together with all E_{ij} $(i \neq j)$.

Let $\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \lambda_3\varepsilon_3$ be an element of $\mathfrak{h}^*_{\mathfrak{sl}(3)}$. In particular, $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ are such that $\sum \lambda_i = 0$. The Verma module $Z(\lambda)$ is defined as the tensor product $U(\mathfrak{sl}(3)) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n})} \mathbb{C}_\lambda$, where $U(\mathfrak{a})$ denotes the universal enveloping algebra of $\mathfrak{a}$, and $\mathbb{C}_\lambda$ is the one dimensional $(\mathfrak{h} \oplus \mathfrak{n})$-module $\mathbb{C}v$ defined by $(h + n)v = \lambda(h)v$.

The Verma module $Z(\lambda)$ has a standard basis: $Z(\lambda) = \mathrm{Span}_{\mathbb{C}}\{E_{21}^k E_{32}^l E_{31}^m v | k, l, m \in \mathbb{Z}_{\geq 0}\}$. By definition:

$$h_1 \cdot v = (\lambda_1 - \lambda_2)v, \quad h_2 \cdot v = (\lambda_2 - \lambda_3)v, \quad E_{12} \cdot v = E_{13} \cdot v = E_{23} \cdot v = 0. \tag{4.3}$$

Lemma 4.2.1. *The module $Z(\lambda) \otimes \mathbb{C}^3$ has maximal vectors of $\mathfrak{gl}(3)$ weight $\gamma(\lambda + \varepsilon_1)$, $\gamma(\lambda + \varepsilon_2)$, and $\gamma(\lambda + \varepsilon_3)$.*

Proof. We look for maximal vectors $\sum_i u_i \otimes e_i$ such that u_i is a linear combination of $E_{21}^k E_{32}^l E_{31}^m v$ for small k, l, m. We first notice that $v \otimes e_1$ is a maximal vector of $Z(\lambda) \otimes \mathbb{C}^3$ because $E_{12}v = E_{13}v = E_{23}v = 0$ and $E_{12}e_1 = E_{13}e_1 = E_{23}e_1 = 0$

forces $v \otimes e_1$ to be a maximal vector. We next look at maximal vectors of the form $Av \otimes e_2 + BE_{21}v \otimes e_1$.

$$\begin{aligned}
E_{12}(Av \otimes e_2 + BE_{21}v \otimes e_1) &= AE_{12}v \otimes e_2 + Av \otimes E_{12}e_2 + BE_{12}E_{21}v \otimes e_1 \\
&\quad +BE_{21}v \otimes E_{12}e_1 \\
&= Av \otimes e_1 + BE_{21}E_{12}v \otimes e_1 + B[E_{12}E_{21}]v \otimes e_1 \\
&= Av \otimes e_1 + B(\lambda_1 - \lambda_2)v \otimes e_1 = 0 \\
&\Rightarrow A + B(\lambda_1 - \lambda_2) = 0.
\end{aligned}$$

Also,

$$\begin{aligned}
E_{23}(Av \otimes e_2 + BE_{21}v \otimes e_1) &= AE_{23}v \otimes e_2 + Av \otimes E_{23}e_2 + BE_{23}E_{21}v \otimes e_1 \\
&\quad +BE_{21}v \otimes E_{23}e_1 \\
&= BE_{21}E_{23}v \otimes e_1 + B[E_{23}E_{21}]v \otimes e_1 = 0.
\end{aligned}$$

Case 1: If $\lambda_1 - \lambda_2 \neq 0$ and $B = -1$, then $A = \lambda_1 - \lambda_2$. Then $Av \otimes e_2 + BE_{21}v \otimes e_1 = (\lambda_1 - \lambda_2)v \otimes e_2 - E_{21}v \otimes e_1$ is a maximal vector.

Case 2: If $\lambda_1 - \lambda_2 = 0$ and $B = 1$, then $A = 0$. Then $Av \otimes e_2 + BE_{21}v \otimes e_1 = E_{21}v \otimes e_1$ is a maximal vector.

Now we find the weight of $(\lambda_1 - \lambda_2)v \otimes e_2 - E_{21}v \otimes e_1$,

$h_1((\lambda_1 - \lambda_2)v \otimes e_2 - E_{21}v \otimes e_1)$

$= (\lambda_1 - \lambda_2)h_1v \otimes e_2 + (\lambda_1 - \lambda_2)v \otimes h_1e_2 - h_1E_{21}v \otimes e_1 - E_{21}v \otimes h_1e_1$

$= (\lambda_1 - \lambda_2)^2v \otimes e_2 - (\lambda_1 - \lambda_2)v \otimes e_2 - E_{21}h_1v \otimes e_1 - [h_1E_{21}]v \otimes e_1 - E_{21}v \otimes e_1$

$= (\lambda_1 - \lambda_2)^2 v \otimes e_2 - (\lambda_1 - \lambda_2) v \otimes e_2 - (\lambda_1 - \lambda_2) E_{21} v \otimes e_1 - 2E_{21} v \otimes e_1 - E_{21} v \otimes e_1$

$= (\lambda_1 - \lambda_2 - 1)(\lambda_1 - \lambda_2) v \otimes e_2 - (\lambda_1 - \lambda_2 - 1) E_{21} v \otimes e_1$

$= (\lambda_1 - \lambda_2 - 1)((\lambda_1 - \lambda_2) v \otimes e_2 - E_{21} v \otimes e_1).$

Similarly, $h_2((\lambda_1 - \lambda_2) v \otimes e_2 - E_{21} v \otimes e_1) = (\lambda_2 - \lambda_3 + 1)((\lambda_1 - \lambda_2) v \otimes e_2 - E_{21} v \otimes e_1)$, $h_1(E_{21} v \otimes e_1) = (\lambda_1 - \lambda_2 - 1) E_{21} v \otimes e_1$, and $h_2(E_{21} v \otimes e_1) = (\lambda_2 - \lambda_3 + 1) E_{21} v \otimes e_1$.

Since the weight of $(\lambda_1 - \lambda_2) v \otimes e_2 - E_{21} v \otimes e_1$ is $(a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3)(\lambda_1 - \lambda_2) v \otimes e_2 - E_{21} v \otimes e_1)$ such that $\sum a_i = 0$, then we have

$$\begin{aligned} (a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3)h_1 &= \lambda_1 - \lambda_2 - 1, \\ (a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3)h_2 &= \lambda_2 - \lambda_3 + 1. \end{aligned}$$

This implies that

$$a_1 - a_2 = \lambda_1 - \lambda_2 - 1,$$

$$a_2 - a_3 = \lambda_2 - \lambda_3 + 1,$$

$$\text{where } a_1 + a_2 + a_3 = 0; \sum \lambda_i = 0.$$

We solve for a_i's: $a_1 = \lambda_1 - \frac{1}{3}, a_2 = \lambda_2 + \frac{2}{3}, a_3 = \lambda_3 - \frac{1}{3}$.

Recall $\gamma : \mathfrak{h}^*_{\mathfrak{gl}(3)} \to \mathfrak{h}^*_{\mathfrak{sl}(3)}$, where $\gamma : \lambda + (a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3) \mapsto (\lambda_1 + a_1 - \frac{|a|}{3})\varepsilon_1 + (\lambda_2 + a_2 - \frac{|a|}{3})\varepsilon_2 + (\lambda_3 + a_3 - \frac{|a|}{3}) a_3\varepsilon_3$, where $|a| = \sum_i a_i$. We easily compute that the $\mathfrak{sl}(3)$-weight of the maximal vector $E_{21} v \otimes e_1$ is $\gamma(\lambda + \varepsilon_2)$. Hence, the $\mathfrak{sl}(3)$-weight of $(\lambda_1 - \lambda_2) v \otimes e_2 - E_{21} v \otimes e_1$ is also $\gamma(\lambda + \varepsilon_2)$.

Now we look at maximal vector of the form $Av \otimes e_3 + BE_{32}v \otimes e_2 + CE_{31}v \otimes e_1 + DE_{21}E_{32}v \otimes e_1$. By similar calculation as in the previous maximal vector case, we have:

$$E_{12}(Av \otimes e_3 + BE_{32}v \otimes e_2 + CE_{31}v \otimes e_1 + DE_{21}E_{32}v \otimes e_1)$$

$$= BE_{32}v \otimes e_1 - CE_{32}v \otimes e_1 + D(\lambda_1 - \lambda_2)E_{32}v \otimes e_1 + DE_{32}v \otimes e_1$$

$$\Rightarrow B - C + D(\lambda_1 - \lambda_2 + 1) = 0.$$

Furthermore,

$$E_{23}(Av \otimes e_3 + BE_{32}v \otimes e_2 + CE_{31}v \otimes e_1 + DE_{21}E_{32}v \otimes e_1) = 0$$

$$\Rightarrow A + B(\lambda_2 - \lambda_3) = 0, C + D(\lambda_2 - \lambda_3) = 0$$

We let for simplicity $D = -1$. Then $C = \lambda_2 - \lambda_3$, $B = C - D(\lambda_1 - \lambda_2 + 1) = \lambda_2 - \lambda_3 + (\lambda_1 - \lambda_2 + 1) = \lambda_1 - \lambda_3 + 1$, and $A = -(\lambda_1 - \lambda_3 + 1)(\lambda_2 - \lambda_3)$. Thus, we obtain

$$u = -(\lambda_1 - \lambda_3 + 1)(\lambda_2 - \lambda_3)v \otimes e_3 + (\lambda_1 - \lambda_3 + 1)E_{32}v \otimes e_2 + (\lambda_2 - \lambda_3)E_{31}v \otimes e_1 - E_{21}E_{32}v \otimes e_1$$

is a maximal vector with $\mathfrak{sl}(3)$-highest weight $\gamma(\lambda + \varepsilon_3)$. □

Theorem 4.2.2. *Define*

$$\begin{aligned}
M_1 &= \text{Span}_{\mathbb{C}}\{E_{21}^k E_{32}^l E_{31}^m (v \otimes e_1) | k, l, m \in \mathbb{Z}_{\geq 0}\}, \\
M_2' &= \text{Span}_{\mathbb{C}}\{E_{21}^k E_{32}^l E_{31}^m ((\lambda_1 - \lambda_2)v \otimes e_2 - E_{21}v \otimes e_1) + M_1 | k, l, m \in \mathbb{Z}_{\geq 0}\}, \\
M_3' &= \text{Span}_{\mathbb{C}}\{E_{21}^k E_{32}^l E_{31}^m (-(\lambda_1 - \lambda_3 + 1)(\lambda_2 - \lambda_3)v \otimes e_3 + (\lambda_1 - \lambda_3 + 1)E_{32}v \otimes e_2 \\
&\quad +(\lambda_2 - \lambda_3)E_{31}v \otimes e_1 - E_{21}E_{32}v \otimes e_1) + M_1 | k, l, m \in \mathbb{Z}_{\geq 0}\}
\end{aligned}$$

1. *M_1 is a submodule of $Z(\lambda) \otimes \mathbb{C}^3$ and $M_1 \simeq Z(\gamma(\lambda + \varepsilon_1))$.*

2. *M_2' and M_3' are submodules of $M_1' = Z(\lambda) \otimes \mathbb{C}^3 / M_1$; and $M_2' \simeq Z(\gamma(\lambda + \varepsilon_2))$ and $M_3' \simeq Z(\gamma(\lambda + \varepsilon_3))$.*

3. *$M_1' = M_2' \oplus M_3'$.*

Proof. 1. By Lemma 4.2.1 and Theorem 2.5.2, there exists an $\mathfrak{sl}(3)$-submodule M_1 of $Z(\lambda) \otimes \mathbb{C}^3$ with highest weight $\gamma(\lambda + \varepsilon_1)$. To prove that $M_1 \simeq Z(\gamma(\lambda + \varepsilon_1))$, we first recall that the identities (4.3) hold for $v \otimes e_1$ with $\gamma(\lambda + \varepsilon_1) = (\lambda_1 + \frac{2}{3})\varepsilon_1 + (\lambda_2 - \frac{1}{3})\varepsilon_2 + (\lambda_3 - \frac{1}{3})\varepsilon_3$:

$$\begin{aligned}
h_1 \cdot (v \otimes e_1) &= h_1 \cdot v \otimes e_1 + v \otimes h_1 \cdot e_1 \\
&= (\lambda_1 - \lambda_2) v \otimes e_1 + v \otimes e_1 \\
&= (\lambda_1 - \lambda_2 + 1) v \otimes e_1 \\
&= (\lambda_1 + \frac{2}{3} - (\lambda_2 - \frac{1}{3})) v \otimes e_1, \\
h_2 \cdot (v \otimes e_1) &= h_2 \cdot v \otimes e_1 + v \otimes h_2 \cdot e_1 \\
&= (\lambda_2 - \lambda_3) v \otimes e_1 \\
&= (\lambda_2 - \lambda_3) v \otimes e_1, \\
&= (\lambda_2 - \frac{1}{3} - (\lambda_3 - \frac{1}{3})) v \otimes e_1.
\end{aligned}$$

The identities $E_{12} \cdot v \otimes e_1 = E_{12} \cdot v \otimes e_1 = E_{12} \cdot v \otimes e_1 = 0$ follow from the fact that $v \otimes e_1$ is a maximal vector. Then we look at the map

$$\Phi : M_1 \to Z(\gamma(\lambda + \varepsilon_1)),\ E_{21}^k E_{32}^l E_{31}^m (v \otimes e_1) \mapsto E_{21}^k E_{32}^l E_{31}^m v'$$

where v' is a vector of weight $\gamma(\lambda + \varepsilon_1)$. It is not difficult to check that Φ is an isomorphism of $\mathfrak{sl}(3)$-modules.

2. Since M_2 and M_3 are $\mathfrak{gl}(3)$-submodules and $\mathfrak{sl}(3)$-submodules of of $Z(\lambda) \otimes \mathbb{C}^3$, by Lemma 4.2.1 and Theorem 2.5.2, it is obvious that M_2' and M_3' are $\mathfrak{gl}(3)$-submodules and $\mathfrak{sl}(3)$-submodules of $M_1' = Z(\lambda) \otimes \mathbb{C}^3/M_1$. Similarly in part 1 of this proof, we prove that $M_2' \simeq Z(\gamma(\lambda+\varepsilon_2))$, by verifying that the identitites (4.3) hold for $((\lambda_1 - \lambda_2)v \otimes e_2 - E_{21}v \otimes e_1) + M_1$ with $\gamma(\lambda+\varepsilon_2) = (\lambda_1 - \frac{1}{3})\varepsilon_1 + (\lambda_2 + \frac{2}{3})\varepsilon_2 + (\lambda_3 - \frac{1}{3})\varepsilon_3$:

$$
\begin{aligned}
&h_1 \cdot (((\lambda_1 - \lambda_2)v \otimes e_2 - E_{21}v \otimes e_1) + M_1) = \\
&h_1 \cdot ((\lambda_1 - \lambda_2)v \otimes e_2 + M_1) - h_1 \cdot (E_{21}v \otimes e_1 + M_1) = \\
&((\lambda_1 - \lambda_2)h_1 \cdot v \otimes e_2 + M_1) + ((\lambda_1 - \lambda_2)v \otimes h_1 \cdot e_2 + M_1) \\
&\qquad -(h_1 \cdot (E_{21}v) \otimes e_1 + M_1) - (E_{21}v \otimes h_1 \cdot e_1 + M_1) = \\
&((\lambda_1 - \lambda_2)h_1 \cdot v \otimes e_2 + M_1) + ((\lambda_1 - \lambda_2)v \otimes h_1 \cdot e_2 + M_1) \\
&\qquad -([h_1, E_{21}]v \otimes e_1 + M_1) - (E_{21}h_1 \cdot v \otimes e_1 + M_1) - (E_{21}v \otimes h_1 \cdot e_1 + M_1) = \\
&((\lambda_1 - \lambda_2)^2 v \otimes e_2 + M_1) - ((\lambda_1 - \lambda_2)v \otimes e_2 + M_1) \\
&\qquad -((\lambda_1 - \lambda_2)E_{21}v \otimes e_1 + M_1) - (E_{21}v \otimes e_1 + M_1) = \\
&(\lambda_1 - \lambda_2 - 1)(((\lambda_1 - \lambda_2)v \otimes e_2 - E_{21}v \otimes e_1) + M_1) = \\
&(\lambda_1 - \tfrac{1}{3} - (\lambda_2 + \tfrac{2}{3}))(((\lambda_1 - \lambda_2)v \otimes e_2 - E_{21}v \otimes e_1) + M_1);
\end{aligned}
$$

similarly,

$$
\begin{aligned}
&h_2 \cdot (((\lambda_1 - \lambda_2)v \otimes e_2 - E_{21}v \otimes e_1) + M_1) = \\
&(\lambda_2 + \tfrac{2}{3} - (\lambda_1 - \tfrac{1}{3}))(((\lambda_1 - \lambda_2)v \otimes e_2 - E_{21}v \otimes e_1) + M_1);
\end{aligned}
$$

$E_{12}\cdot(((\lambda_1-\lambda_2)v\otimes e_2-E_{21}v\otimes e_1)+M_1)$

$=E_{12}\cdot((\lambda_1-\lambda_2)v\otimes e_2+M_1)-E_{12}\cdot(E_{21}v\otimes e_1+M_1)$

$=((\lambda_1-\lambda_2)E_{12}\cdot v\otimes e_2+M_1)+((\lambda_1-\lambda_2)v\otimes E_{12}\cdot e_2+M_1)$

$-(E_{12}\cdot(E_{21}v)\otimes e_1+M_1)-(E_{21}v\otimes E_{12}\cdot e_1+M_1)$

$=((\lambda_1-\lambda_2)E_{12}\cdot v\otimes e_2+M_1)+((\lambda_1-\lambda_2)v\otimes E_{12}\cdot e_2+M_1)$

$-([E_{12},E_{21}]v\otimes e_1+M_1)-(E_{21}(E_{12}\cdot v)\otimes e_1+M_1)-(E_{21}v\otimes E_{12}\cdot e_1+M_1)$

$=((\lambda_1-\lambda_2)v\otimes e_1+M_1)-(h_1\cdot v\otimes e_1+M_1)$

$=((\lambda_1-\lambda_2)v\otimes e_1+M_1)-((\lambda_1-\lambda_2)v\otimes e_1+M_1)=0;$

and similarly,

$$\begin{aligned} E_{13}\cdot(((\lambda_1-\lambda_2)v\otimes e_2-E_{21}v\otimes e_1)+M_1) &= 0 \\ E_{23}\cdot(((\lambda_1-\lambda_2)v\otimes e_2-E_{21}v\otimes e_1)+M_1) &= 0; \end{aligned}$$

as needed. Then, like in the first case, we obtain $M_2'\simeq Z(\gamma(\lambda+\varepsilon_2))$. With the same reasoning, $M_3'\simeq Z(\gamma(\lambda+\varepsilon_3))$.

3. Let $t\in M_1'$ be an arbitrary vector. Then we can say that $t=\sum a_{k,l,m}E_{21}^kE_{32}^le_{31}^m(v\otimes e_1)+M_1, k,l,m\in\mathbb{Z}_{\geq 0}$. The expansion of the series $t=\sum a_{k,l,m}E_{21}^kE_{32}^lE_{31}^m(v\otimes e_1)$ is equal to the $\sum w=(\sum b_{k,l,m}E_{21}^kE_{32}^lE_{31}^m((\lambda_1-\lambda_2)v\otimes e_2-E_{21}v\otimes e_1)+M_1)$ and $u=(\sum c_{k,l,m}E_{21}^kE_{32}^lE_{31}^m(-(\lambda_1-\lambda_3+1)(\lambda_1-\lambda_2)v\otimes e_3+(\lambda_1-\lambda_3+1)E_{32}v\otimes e_2+(\lambda_2-\lambda_3)E_{31}v\otimes e_1-E_{21}e_{32}v\otimes e_1)+M_1, k,l,m\in\mathbb{Z}_{\geq 0}$. But $w\in M_2'$ and $u\in M_3'$. Thus

$M_1' = M_2' + M_3'$.

Now let w and u be arbitrary vectors in M_2' and M_3', respectively. Let

$$\begin{aligned}
& w && = u \\
\Rightarrow\quad & w - u && = (\sum b_{k,l,m} E_{21}^k E_{32}^l E_{31}^m ((\lambda_1 - \lambda_2) v \otimes e_2 - E_{21} v \otimes e_1) + M_1) \\
& && \quad -(\sum c_{k,l,m} E_{21}^k E_{32}^l E_{31}^m (-(\lambda_1 - \lambda_3 + 1)(\lambda_1 - \lambda_2) v \otimes e_3 \\
& && \quad +(\lambda_1 - \lambda_3 + 1) E_{32} v \otimes e_2 \\
& && \quad +(\lambda_2 - \lambda_3) E_{31} v \otimes e_1 - E_{21} e_{32} v \otimes e_1) + M_1 \qquad (4.4) \\
& && = 0 + M_1.
\end{aligned}$$

Using linear independence arguments, Equation (4.4) implies that $b_{k,l,m} = 0$ and $c_{k,l,m} = 0, \forall k, l, m \in \mathbb{Z}_{\geq 0}$. Then $0 = w = u \in M_2' \cap M_3'$, and M_2' intersects with M_3' trivially. Thus the result holds. $\square$

Corollary 4.2.3. *For any $\lambda \in \mathfrak{h}^*_{\mathfrak{sl}(3)}$, the module $Z(\lambda) \otimes \mathbb{C}^3$ has a filtration whose subquotients are isomorphic to $Z(\gamma(\lambda + \varepsilon_1)$, $Z(\gamma(\lambda + \varepsilon_2))$, and $Z(\gamma(\lambda + \varepsilon_3))$.*

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