Representations of the Extended Poincaré Superalgebras in Four Dimensions

by

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The only true voyage would be not to travel through a hundred different lands with the same pair of eyes, but to see the same land through a hundred different pairs of eyes.

Marcel Proust

One cannot be human by oneself. There is no selfhood where there is no community. We do not relate to others as the persons we are; we are who we are in relating to others.

James P. Carse

CHAPTER 0

Introduction

Emmy Noether's theorem proved that a conserved physical quantity can be described as a symmetry of nature. A couple of decades later, Eugene Wigner used the Poincaré group to induce representations from the fundamental internal space-time symmetries of (special) relativistic quantum particles [32] (in a flat Minkowski space-time). However, many people found this paper so inaccessible that Wigner's students spent considerable amount of time translating passages of this paper into more detailed and accessible papers [20] and books.

Since the 1930's, many mathmeticians have worked on the problem of *restricted representations* (or branching rules) of groups; we refer the reader to some relevant articles: [21], [22], and [3]. The theory of branching rules is used extensively in classical invariant theory; the most notable contributors to this field are, historically, H. Weyl, F. Murnaghan, and D. Littlewood. Thankfully, branching rules have become easier to work with due to modern techniques invented, most notably, by R. Brauer, R Howe [19], B. Kostant, J. Lepowsky [24], and P. Littelmann.

In 1975, R. Haag *et al.* [9] investigated the possible extensions of the symmetries of relativistic quantum particles. They showed that the only consistent (super)symmetric extensions to the standard model of physics are obtained by using *super charges* to generate the odd part of a Lie superalgebra whos even part is generated by the Poincaré group; this theory has become known as *supersymmetry.* In this paper, R. Haag *et al.* used a notation called *supermultiplets* to give the dimension of a representation and its multiplicity; this notation is described mathematically in chapter 5 of this thesis.

By 1980 other possible extensions to the standard model of physics, such as string theory, had been invented; many of these other theories use space-times with dimension not equal to four. S. Ferrara *et al.* began classifying the representations of these algebras for dimensions greater than four, and in 1986 Strathdee published considerable work [29] (with the aid of [33]) listing some representations for the Poincaré superalgebra in any finite dimension. Further, Strathdee began to restrict the representations to only the ones considered "acceptable" for physical theories. This work has been continued to date [11].

We found the work of S. Ferrara *et al.* [12] to be essential to our understanding extended supersymmetries. This paper was the most usable source, because it contained the most explicit mathematics and it avoided much of the conflicting, abused, or undefined notations found in much of the other literature. However, this paper was written using imprecise language meant for physicists, so it was far from trivial to understand the mathematical interperetation of this work. The results of [12] go a long way towards classifying the irreducible representations of the extended supersymmetries in four dimensions.

In this thesis, we provide a "translation" of the results in [12] and [29] (along with some other literature on the Extended Poincaré Superalgebras) into a rigorous mathematical setting, which makes the subject more accessible to a larger audience. Having a mathematical model allows us to give explicit results and detailed proofs. Further, this model allowes us to see beyond just the physical interpretation and it allows investigation by a purely mathematically adept audience.

Our work was motivated by a paper written in 2012 by M. Chaichian *et al,* which classified all of the unitary, irreducible representations of the extended Poincaré superalgebra in three dimensions. The three dimensional case is of interest to string theorists who work in 11 dimensions and reduce the dimensions to three by using the periodic nature of Clifford modules.

We consider only the four dimensional case, which is of interest to physicists working on quantum supergravity models without cosmological constant, and we provide explicit branching rules for the invariant subgroups corresponding to the most physically relevant symmetries of the irreducible representations of the Extended Poincaré Superalgebra in four dimensions. However, it is possible to further generalize this work into any finite dimension. Such work would classify all possible finitely extended supersymmetric models.

CHAPTER 1

Preliminary Definitions

In all subsequent chapters, we will denote the $n \times n$ identity matrix by I_n and we will denote the transpose of a matrix A by A^t.

In this chapter, we let the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Unless otherwise stated, all vector spaces, homomorphisms, and tensor products will be assumed to be over $\mathbb{F}$. We denote $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}_2$, and for $i \in \mathbb{Z}$, by $\bar{i}$ we denote $i + 2\mathbb{Z}$ in $\mathbb{Z}_2$.

1.1 Lie Groups

Definition 1.1.1. By $M_{\mathbb{F}}(n)$ we denote the set of all $n \times n$ matrices with entries in $\mathbb{F}$.

Remark 1.1.2. Note that $M_{\mathbb{F}}(n)$ is an associative algebra over $\mathbb{F}$ with identity I_n.

Definition 1.1.3. A *Lie group* is a group G which is also a (real or complex) differentiable manifold and for which the group multiplication map $G \times G \to G$, $(x, y) \mapsto xy$ and the inverse map $G \to G$, $x \mapsto x^{-1}$ are smooth maps of differentiable manifolds.

In this thesis we will work both with real and complex Lie groups. Also, we deal mostly with reductive algebraic groups, in particular, subgroups of the general linear group defined below.

Definition 1.1.4. If V is a vector space over $\mathbb{F}$, then the set of all invertible endomorphisms of V forms a group with binary operation composition. We denote this group by $GL_{\mathbb{F}}(V)$. In the case when $\mathbb{V} = \mathbb{F}^n$ we have that $GL_{\mathbb{F}}(V) \simeq GL_{\mathbb{F}}(n)$ where the latter is the *general linear group*:

$$GL_{\mathbb{F}}(n) = \{A \in M_{\mathbb{F}}(n) \mid AB = BA = I_n \text{ for some } B \in M_{\mathbb{F}}(n)\}.$$

In the case $\mathbb{F} = \mathbb{R}$ (respectively, $\mathbb{F} = \mathbb{C}$), $GL_{\mathbb{F}}(n)$ is a real (complex) Lie group, see for example [17].

Definition 1.1.5. The *special linear group* is

$$SL_{\mathbb{F}}(n) = \{A \in GL_{\mathbb{F}}(n) \mid \det(A) = 1\}.$$

Definition 1.1.6. The *real orthogonal group* is

$$O(n) = \left\{A \in GL_{\mathbb{R}}(n) \mid A^t A = I_n\right\},$$

where A^t denotes the transpose of the matrix A.

Remark 1.1.7. For $A \in O(n)$, $\det A = \pm 1$.

Definition 1.1.8. The *special orthogonal group* is

$$SO(n) = O(n) \cap SL_{\mathbb{R}}(n).$$

Definition 1.1.9. The *generalized orthogonal group* is

$$O(p,q) = \{A \in M_{\mathbb{R}}(n) \mid A^t I_{p,q} A = I_{p,q}\}$$

where $p, q \in \mathbb{Z}_{\geq 0}$, $p + q = n$, and

$$I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}.$$

Definition 1.1.10.

$$SO(p,q) = O(p,q) \cap SL_{\mathbb{R}}(p+q)$$

Remark 1.1.11.

- $O(n,0) \cong O(0,n) \cong O(n)$
- $O(p,q) \cong O(q,p)$

Definition 1.1.12. The *complex symplectic group* is

$$Sp_{\mathbb{C}}(n) = \left\{A \in GL_{\mathbb{C}}(2n) \mid A^T J_{2n} A = J_{2n}\right\},$$

where $J_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$.

Definition 1.1.13. The *unitary group* is

$$U(n) = \left\{A \in GL_{\mathbb{C}}(n) \mid \overline{A^t}A = I_n\right\},$$

where $\overline{A}$ denote the complex conjugate of the matrix A.

Remark 1.1.14. For $A \in U(n)$, $\det(A) = \pm 1$.

Definition 1.1.15. The *special unitary group* is

$$SU(n) = U(n) \cap SL_{\mathbb{C}}(n).$$

Definition 1.1.16. The *compact symplectic group* is

$$Sp(n) = Sp_{\mathbb{C}}(n) \cap U(2n).$$

Definition 1.1.17.

$$U(p,q) = \left\{A \in GL_{\mathbb{C}}(n) \mid \overline{A^t}I_{p,q}A = I_{p,q}\right\}.$$

1.2 Lie Algebras

Definition 1.2.1. A vector space V over a field $\mathbb{F}$ is called a *Lie algebra* if it has a binary operation $[\cdot,\cdot] : V \times V \to V$ called the *Lie bracket* which satisfies the following three axioms for $a, b \in \mathbb{F}$, and $x, y, z \in V$:

$$[ax + by, z] = a[x,z] + b[y,z] \quad \text{and}$$

$$[x, ay + bz] = a[x,y] + b[x,z] \quad \text{(bilinearity)}, \tag{1.1}$$

$$[x,y] = -[y,x] \quad \text{(skew-symmetry), and} \tag{1.2}$$

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \quad \text{(the Jacobi identity)}. \tag{1.3}$$

Definition 1.2.2. A subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called a *Lie subalgebra of* $\mathfrak{g}$, if $[x, y] \in \mathfrak{h}$ whenever $x, y \in \mathfrak{h}$.

Definition 1.2.3. A *Lie algebra homomorphism* is a vector space homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}$.

Definition 1.2.4. A *Lie algebra isomorphism* is a Lie algebra homomorphism that is also one-to-one and onto.

Definition 1.2.5. Let V be a vector space over $\mathbb{F}$. Then we denote by $\mathfrak{gl}_{\mathbb{F}}(V)$ the space of all endomorphisms on V along with the bracket operation $[S, T] = ST - TS$. We call $\mathfrak{gl}_{\mathbb{F}}(V)$ the *general linear Lie algebra of* V.

Remark 1.2.6. The proof that $\mathfrak{gl}_{\mathbb{F}}(V)$ is indeed a Lie algebra includes a straightforward calculation for (1.3). When $\dim_{\mathbb{F}} V = n$, this Lie algebra is isomorphic to the Lie algebra $\mathfrak{gl}(n)$ consisting of all $n \times n$ matrices with entries in $\mathbb{F}$, $M_n(\mathbb{F}) \cong \text{End}(V)$, and Lie bracket $[S, T] = ST - TS$.

Definition 1.2.7.

$$\mathfrak{sl}_{\mathbb{F}}(V) = \{A \in \mathfrak{gl}_{\mathbb{F}}(V) \mid \text{tr}(A) = 0\}$$

Definition 1.2.8. A *representation* of a Lie algebra $\mathfrak{g}$ (or equivalently, a $\mathfrak{g}$-*module*) is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}_{\mathbb{F}}(V)$. To identify the representation we will use the pair (ρ, V), or simply V when ρ is given in context.

Remark 1.2.9. Equivalently, V is a $\mathfrak{g}$-module if the map $\mathfrak{g} \times V \to V$, $(x, v) \mapsto x \cdot v = \rho(x)v$ satisfies the following axioms for $a, b \in \mathbb{F}$, $v, w \in V$, and $x, y \in \mathfrak{g}$:

$$(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v), \tag{1.4}$$

$$x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w), \text{ and} \tag{1.5}$$

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v). \tag{1.6}$$

From now on, $x \cdot v$ will be denoted by xv for $x \in \mathfrak{g}$ and $v \in V$.

Definition 1.2.10. Let (ρ, V) be a representation of $\mathfrak{g}$. Then a *subrepresentation of* (ρ, V) is a pair (ρ, W), where W is a subspace of V for which $(\rho(g))(w) \in W$ for any $g \in \mathfrak{g}$ and any $w \in W$.

Definition 1.2.11. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ and let (ρ, V) be a representation of $\mathfrak{g}$. The pair $(\rho|_{\mathfrak{h}}, V)$ is called *the restriction of the representation* (ρ, V) *on* $\mathfrak{h}$. Here $\rho|_{\mathfrak{h}}$ is the restriction map of ρ on $\mathfrak{h}$; that is, for any $h \in \mathfrak{h}$, $\rho|_{\mathfrak{h}}(h) = \rho(h)$.

Definition 1.2.12. Let $\mathfrak{g}$ be a Lie algebra and let $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ be homomorphisms of vector spaces. Then $\varphi_1 \oplus \varphi_2 : U_1 \oplus U_2 \to V_1 \oplus V_2$ is the homomorphism defined by $(\varphi_1 \oplus \varphi_2)(u_1, u_2) = (\varphi_1(u_1), \varphi_2(u_2))$. In the case when we consider $U_1 \oplus U_2$ as an inner direct sum, i.e. U_1 and U_2 are subspaces of a vector space U, we will write $(\varphi_1 \oplus \varphi_2)(u_1 + u_2) = \varphi_1(u_1) + \varphi(u_2)$.

Definition 1.2.13. Let $\mathfrak{g}$ be a Lie algebra and $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ be homomorphisms of $\mathfrak{g}$-modules U_1 and U_2. Then $\varphi_1 \oplus \varphi_2 : U_1 \oplus U_2 \to V_1 \oplus V_2$ is a homomorphism of $\mathfrak{g}$-modules, called the *direct sum of* φ_1 *and* φ_2.

Definition 1.2.14. Let W be a subrepresentation of V over a Lie algebra $\mathfrak{g}$. Then the quotient vector space V/W is a representation of $\mathfrak{g}$ defined by $x(v+W) = xv+W$; this representation is called a *quotient representation*.

Remark 1.2.15. Whenever $v_1 + W = v_2 + W$, we have $x(v_1 + W) = x(v_2 + W)$. Since $x(v_1 - v_2) \in W$, the above action of $\mathfrak{g}$ on V/W is well-defined.

1.3 Lie Superalgebras

Definition 1.3.1. A $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ over $\mathbb{F}$ is called a *vector superspace* with *even part* $V_{\bar{0}}$ and *odd part* $V_{\bar{1}}$.

We denote a vector superspace, over the field $\mathbb{F}$, with even part of dimension m and odd part of dimension n by $\mathbb{F}^{m|n}$.

For a vector v in $V_{\bar{i}}$, we call $\bar{i}$ *the parity of* v and denote it by $p(v)$.

Remark 1.3.2. When we write $p(v)$, we assume by default that $v \in V_{\bar{i}}$ for $\bar{i} \in \mathbb{Z}_2$.

Remark 1.3.3. Note that not every element in a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ has parity. For example, an element $v_0 + v_1$, where $v_0 \in V_{\bar{0}}$ and $v_1 \in V_{\bar{1}}$ are nonzero vectors, that does not have an explicitly defined parity. An element with well defined parity, i.e. in $V_{\bar{0}}$ or in $V_{\bar{1}}$, is called *homogeneous.* Although most of the vectors of a vector superspace are non-homogeneous, many definitions and results for superspaces can be stated for homogeneous vectors and then extended using bilinearity.

Definition 1.3.4. For a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$, we denote by ΠV the vector superspace V with the parity reversed. We call ΠV the Π-*dual of* V.

Specifically, $\Pi V = V$ as a vector space such that $(\Pi V)_{\bar{0}} = V_{\bar{1}}$ and $(\Pi V)_{\bar{1}} = V_{\bar{0}}$.

Example 1.3.5. The vector space $V = \mathbb{C}^{m|n}$ has an even part $V_{\bar{0}} = \mathbb{C}^m$ and an odd part $V_{\bar{1}} = \mathbb{C}^n$. Further, $\Pi\mathbb{C}^{m|n} = \mathbb{C}^{n|m}$.

Definition 1.3.6. A vector subspace V of a vector superspace W is called a *vector subsuperspace* whenever $V_{\bar{i}} \subset W_{\bar{i}}$

Definition 1.3.7. Let $U = U_{\bar{0}} \oplus U_{\bar{1}}$ and $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be vector subsuperspaces of a vector superspace $W = W_{\bar{0}} \oplus W_{\bar{1}}$. Then we say that W is the *direct sum of* U *and* V, $W = U \oplus V$, exactly when $W_{\bar{i}} = U_{\bar{i}} \oplus V_{\bar{i}}$ for $\bar{i} \in \mathbb{Z}_2$

Definition 1.3.8. We say that a map $\varphi : V_{\bar{0}} \oplus V_{\bar{1}} \to W_{\bar{0}} \oplus W_{\bar{1}}$ is *parity invariant* whenever $\varphi(v_{\bar{i}}) \in W_{\bar{i}}$ where $v_{\bar{i}} \in V_{\bar{i}}$ and $\bar{i} \in \mathbb{Z}_2$.

Definition 1.3.9. A *vector superspace homomorphism* is a vector space homomorphism which is parity invariant.

Definition 1.3.10. A *Lie superalgebra* $\mathfrak{g}$ is a vector superspace over $\mathbb{F}$, $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie superbracket*, such that the superbracket obeys the following four axioms for $a, b \in \mathbb{F}$ and $x, y, z \in \mathfrak{g}$:

$$p([x,y]) = p(x) + p(y) \qquad \text{(parity invariance)}, \tag{1.7}$$

$$[ax+by, z] = a[x,z] + b[y,z] \qquad \text{and}$$

$$[x, ay+bz] = a[x,y] + b[x,z] \qquad \text{(bilinearity)}, \tag{1.8}$$

$$[x,y] = -(-1)^{p(y)p(x)}[y,x] \qquad \text{(super skew-symmetry), and} \tag{1.9}$$

$$(-1)^{p(x)p(z)}[x,[y,z]] + (-1)^{p(y)p(x)}[y,[z,x]] + (-1)^{p(z)p(y)}[z,[x,y]] = 0$$

$$\text{(the super Jacobi identity)}. \tag{1.10}$$

Definition 1.3.11. A vector subsuperspace $\mathfrak{h}$ of a Lie superalgebra $\mathfrak{g}$ is called a *Lie subsuperalgebra of* $\mathfrak{g}$ if $\mathfrak{h}$ is also a Lie superalgebra. Equivalently, a subsuperspace $\mathfrak{h}$ of $\mathfrak{g}$ is a Lie subsuperalgebra of $\mathfrak{g}$ if $[a,b] \in \mathfrak{h}$ whenever $a, b \in \mathfrak{h}$.

Definition 1.3.12. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace. Then the space $\mathfrak{gl}(V)$ of endomorphisms of V is a superspace with even part

$$\mathfrak{gl}(V)_{\bar{0}} = \{T \in \mathfrak{gl}(V) \mid T(V_{\bar{0}}) \subset V_{\bar{0}},\ T(V_{\bar{1}}) \subset V_{\bar{1}}\}$$

and odd part

$$\mathfrak{gl}(V)_{\bar{1}} = \{T \in \mathfrak{gl}(V) \mid T(V_{\bar{0}}) \subset V_{\bar{1}},\ T(V_{\bar{1}}) \subset V_{\bar{0}}\}.$$

The superspace $\mathfrak{gl}(V)$ together with the superbracket
$[S,T] = ST - (-1)^{p(T)p(S)}TS$ is a Lie superalgebra, called the *general linear Lie superalgebra of* V.

Remark 1.3.13. The maps S and T above are vector space homomorphisms but not necessarily vector superspace homomorphisms.

Remark 1.3.14. Let $V = \mathbb{C}^{m|n}$

Denote by $\mathfrak{gl}(m|n)$ the Lie superalgebra consisting of $(m+n)\times(m+n)$ matrices $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, where $A \in M_m(\mathbb{C})$, $D \in M_n(\mathbb{C})$, B is an $m \times n$ complex matrix, and C is an $n \times m$ complex matrix. The even part $\mathfrak{gl}(m|n)_{\bar{0}}$ consists of all matrices of the form $\left[\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right]$ while the odd part $\mathfrak{gl}(m|n)_{\bar{1}}$ consists of the matrices of the form $\left[\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right]$. The Lie superbracket of $\mathfrak{gl}(m|n)$ is defined by $[X,Y] = XY - (-1)^{p(X)p(Y)}YX$. In particular, $\mathfrak{gl}(m|n)$ has a basis consisting of all elementary matrices and $\dim_{\mathbb{C}}\mathfrak{gl}(m|n) = (m+n)^2$. After fixing a basis for $V = \mathbb{C}^{m|n}$, we can show that $\mathfrak{gl}(V) \cong \mathfrak{gl}(m|n)$. The proof is similar to the proof that $\mathfrak{gl}(\mathbb{C}^k) \cong \mathfrak{gl}(k)$.

Definition 1.3.15. A *Lie superalgebra isomorphism* is a Lie superalgebra homomorphism that is also one-to-one and onto.

Definition 1.3.16. A *representation of the Lie superalgebra* $\mathfrak{g}$ (or a $\mathfrak{g}$-module) is a Lie superalgebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, where V is a vector superspace. This is denoted by (ρ, V)

Definition 1.3.17. For a Lie superalgebra $\mathfrak{g}$, a vector superspace V is a $\mathfrak{g}$-*module* if the operation $\mathfrak{g} \times V \to V$ such that $(g, v) \to gv$ satisfies the following axioms for $a, b \in \mathbb{F}$, $v, w \in V$, and $x, y \in \mathfrak{g}$:

$$\mathfrak{g}_{\bar{i}}V_{\bar{j}} \subset V_{\bar{i}+\bar{j}} \tag{1.11}$$

$$(ax + by)v = a(xv) + b(yv), \tag{1.12}$$

$$x(av + bw) = a(xv) + b(xw), \text{ and} \tag{1.13}$$

$$[x, y]v = x(yv) - y(xv). \tag{1.14}$$

Remark 1.3.18. Every representation (ρ, V) of a Lie superalgebra is a $\mathfrak{g}$-module through the map $(g, v) \mapsto \rho(g)v$ for each $g \in \mathfrak{g}$ and $v \in V$.

Conversely, we have that every $\mathfrak{g}$-module W is a representation of $\mathfrak{g}$ by defining $\rho(g)(v) = gv$.

Remark 1.3.19. If V is a representation of $\mathfrak{g}$, then, as we will check, ΠV is also a representation of $\mathfrak{g}$.

For equation (1.11), $\mathfrak{g}_{\bar{i}}(\Pi V)_{\bar{j}} = \mathfrak{g}_{\bar{i}} V_{\bar{j}+\bar{1}} \subset V_{\bar{i}+\bar{j}+\bar{1}} = (\Pi V)_{\bar{i}+\bar{j}}$. Showing that (1.12)-(1.14) hold is straightforward.

Definition 1.3.20. Let V be a representation of a Lie superalgebra $\mathfrak{g}$. A subsuperspace W of V is a *subrepresentation of* V, if it is a representation of $\mathfrak{g}$ itself. Equivalently, W is a subrepresentation, if $gw \in W$ for any $g \in \mathfrak{g}$ and $w \in W$.

Definition 1.3.21. Let $\mathfrak{g}$ be a Lie superalgebra and let $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ be homomorphisms of vector superspaces U_1 and U_2. Then $\varphi_1 \oplus \varphi_2 : U_1 \oplus U_2 \to V_1 \oplus V_2$ is the homomorphism defined similarly to definition 1.2.12.

Definition 1.3.22. Let $\mathfrak{g}$ be a Lie superalgebra and $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ be vector superspace homomorphisms of $\mathfrak{g}$-modules U_1 and U_2. Then $\varphi_1 \oplus \varphi_2 : U_1 \oplus U_2 \to V_1 \oplus V_2$ is a homomorphism of $\mathfrak{g}$-modules, called the *direct sum of* φ_1 *and* φ_2.

CHAPTER 2

Invariant Theory

In this chapter, we present results from invariant theory that are needed for the results in chapter 5. All of the content of this chapter comes from Goodman and Wallach [15], to which the reader is encouraged to refer for more details.

We will assume that the ground field is $\mathbb{C}$ unless otherwise noted, and we refer the reader to Appendix C for background material for associative algebras and group algebras.

2.1 Duality at the Level of Associative Algebras

Recall that for any vector space V, $\mathrm{End}(V)$ is an associative algebra with unity I_V, the identity map on V.

Definition 2.1.1. For any subset $U \subset \mathrm{End}(V)$, let

$$\mathrm{Comm}(U) = \{T \in \mathrm{End}(V) | TS = ST \text{ for any } S \in U\}$$

denote the *commutant of* U *in* $\mathrm{End}(V)$.

Remark 2.1.2. $\mathrm{Comm}(U)$ is an associative subalgebra of $\mathrm{End}(V)$.

Theorem 2.1.3 (Double Commutant)**.** Suppose $\mathcal{A} \subset \mathrm{End}(V)$ is an associative algebra with unity. Then let $\mathcal{B} = \mathrm{Comm}(\mathcal{A})$.
If V is a completely reducible $\mathcal{A}$-module, then $\mathrm{Comm}(\mathcal{B}) = \mathcal{A}$.

Recall that a representation of G is a group homomorphism $\rho : G \to GL_{\mathbb{C}}(V)$, where V is a complex vector space. From now until the end of the chapter we fix $G \subset GL_{\mathbb{C}}(V)$ to be a reductive linear algebraic group.

Definition 2.1.4. By $[G]$ we denote the set of equivalence classes of irreducible representations of G. On the other hand, $\widehat{G}$ will stand for the subset of $[G]$ of equivalence classes of finite-dimensional irreducible representations of G. The corresponding sets of equivalence classes of representations of an associative algebra $\mathcal{A}$, and a Lie algebra $\mathfrak{g}$ will be denoted by $[\mathcal{A}]$, $\widehat{\mathcal{A}}$, $[\mathfrak{g}]$, $\widehat{\mathfrak{g}}$, respectively.

If A is G or $\mathfrak{g}$, then every representation of A will be considered as a pair (ρ, V), where V is a complex vector space and ρ is the corresponding homomorphism (of groups or of Lie algebras). We will write $(\rho^\lambda, F^\lambda)$ for a representative of the class λ, for each λ in $[G]$ or $[\mathfrak{g}]$.

Definition 2.1.5. By $\mathcal{A}(G)$ (or, by $\mathbb{C}[G]$) we denote the group algebra associated with the group G.

Remark 2.1.6. Every G-module is considered as an $\mathcal{A}(G)$-module and vice-versa, see Example 2, §4.1.1, [15]

2.2 Duality at the Level of Groups

The following theorem is a corollary of Proposition 4.1.12 in [15]. The original statement is stronger as it holds for locally completely reducible G-modules. We recall that G is a linear reductive group.

Theorem 2.2.1. Let (ρ, W) be a finite-dimensional representation of G. Then

$$W \cong \sum_{\lambda \in \widehat{G}} \operatorname{Hom}(F^\lambda, W) \otimes F^\lambda \tag{2.1}$$

as a G-module.

Definition 2.2.2. We write $\operatorname{Spec}(\rho)$ for the set of representation types λ that occur in the decomposition (2.1) of (ρ, W):

$$W \cong \sum_{\lambda \in \operatorname{Spec}(\rho)} \operatorname{Hom}(F^\lambda, W) \otimes F^\lambda.$$

We call $\mathrm{Hom}(F^\lambda, W)$ the *multiplicity spaces* in this decomposition.

Definition 2.2.3. Let $\mathcal{R} \subset \mathrm{End}(W)$ be a subalgebra such that

1. $\mathcal{R}$ acts irreducibly on W.
2. If $g \in G$ and $T \in \mathcal{R}$, then $(g, T) \mapsto \rho(g)T\rho(g^{-1}) \in \mathcal{R}$ defines an action of G on $\mathcal{R}$.

Then we denote by

$$\mathcal{R}^G = \{T \in \mathcal{R} | \rho(g)T = T\rho(g) \text{ for all } g \in G\}$$

the *commutant of* $\rho(G)$ *in* $\mathcal{R}$.

By $\mathrm{End}_G(W) = \mathrm{Comm}(\rho(G))$ we denote the commutant of $\rho(G)$ in $\mathrm{End}(W)$.

Remark 2.2.4. Since elements of $\mathcal{R}^G$ commute with elements from $\mathcal{A}(G)$, we may define an $\mathcal{R}^G \otimes \mathcal{A}(G)$-module structure on W. Alternatively, we may consider W as an $(\mathcal{R}^G, \mathcal{A}(G))$-bimodule.

Let $E^\lambda = \mathrm{Hom}_G(F^\lambda, L)$ for $\lambda \in \widehat{G}$. Then E^λ is an $\mathcal{R}^G$-module satisfying

$$Tu(\pi^\lambda(g)v) = T\rho(g)u(v) = \rho(g)(Tu(v)),$$

where $u \in E^\lambda$, $v \in F^\lambda$, $T \in \mathcal{R}^G$, and $g \in G$.

As a corollary we obtain the following result (see (4.14) in [15]).

Theorem 2.2.5. As an $\mathcal{R}^G \otimes \mathcal{A}(G)$-module, the space W decomposes as

$$W \cong \bigoplus_{\lambda \in \mathrm{Spec}(\rho)} E^\lambda \boxtimes F^\lambda, \tag{2.2}$$

where $E \boxtimes F$ stands for the outer (external) tensor product of the $\mathcal{R}^G$-module E and of the $\mathcal{A}(G)$-module F.

Theorem 2.2.6 (Duality)**.** Each multiplicity space E^λ is an irreducible $\mathcal{R}^G$-module. Further, if $\lambda, \mu \in \mathrm{Spec}(\rho)$ and $E^\lambda \cong E^\mu$ as an $\mathcal{R}^G$-module, then $\lambda = \mu$.

Theorem 2.2.7 (Duality Correspondence)**.** Let σ be the representation of $\mathcal{R}^G$ on W and let Spec(σ) denote the set of equivalence classes of the irreducible representation $\{E^\lambda\}$ of the algebra $\mathcal{R}^G$ occurring in W. Then the following hold.

- The representation (σ, W) is a dirrect sum of irreducible R^G-modules, and each irreducible submodule E^λ occurs with finite mulitplicity, $\dim(F^\lambda)$.
- The map $F^\lambda \to E^\lambda$ is a bijection between Spec(ρ) and Spec(σ).

2.3 Duality at the Level of Lie Algebras

Let $\mathfrak{g}$ be a semisimple Lie algebra. Then we fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a choice of positive roots of $\mathfrak{h}$, and let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the associated triangular decomposition of $\mathfrak{g}$.

Definition 2.3.1. By P_{++} we denote the set of dominant integral weights of $\mathfrak{g}$.

Remark 2.3.2. There is a bijection between P_{++} and $\widehat{\mathfrak{g}}$; the map from P_{++} to $\widehat{\mathfrak{g}}$ is given by $\mu \mapsto (\pi^\mu, V^\mu)$, where V^μ is a simple highest weight module of $\mathfrak{g}$ with highest weight μ (see for example Theorem 3.2.5 in [15]).

Definition 2.3.3. Let V be a finite-dimensional $\mathfrak{g}$-module.

- For any $\mu \in \mathfrak{h}^*$, $V_\mu = \{v \in V \mid hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\}$ denotes the μ-weight space of V. Since V is finite-dimensional,

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu.$$

  We call this decomposition *the weight space decomposition of* V.
- We set $V^{\mathfrak{n}^+} = \{v \in V | Xv = 0 \text{ for all } X \in \mathfrak{n}^+\}$.

Definition 2.3.4. Let V be a finite-dimensional representation of $\mathfrak{g}$. Then we call $P(V) = \{\mu \in P_{++}(\mathfrak{g}) | V_\mu^{\mathfrak{n}^+} \neq 0\}$ the set of highest weights of V.

Remark 2.3.5. If $T \in \mathrm{End}_{\mathfrak{g}}(V)$, then T preserves $V^{\mathfrak{n}^+}$ and the weight space decomposition $V^{\mathfrak{n}^+} = \bigoplus_{\mu \in P(V)} V_\mu^{\mathfrak{n}^+}$.

For each $\mu \in P(V)$ we fix an irreducible representation (π^μ, V^μ) with highest weight μ. We will consider V^μ both as a module over $\mathfrak{g}$ and over the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$.

Theorem 2.3.6. For $T \in \mathrm{End}_{\mathfrak{g}}(V)$, the restriction map $\phi : T \to T|_{V^{\mathfrak{n}^+}}$ gives an algebra isomorphism,

$$\mathrm{End}_{\mathfrak{g}}(V) \cong \bigoplus_{\mu \in P(V)} \mathrm{End}(V_\mu^{\mathfrak{n}^+}).$$

- For every $\mu \in P(V)$, the space $V_\mu^{\mathfrak{n}^+}$ is an irreducible module for $\mathrm{End}_{\mathfrak{g}}(V)$.
- Further, distinct values of μ give the inequivalent modules for $\mathrm{End}_{\mathfrak{g}}(V)$.
- Under the action of $\mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}_{\mathfrak{g}}(V)$, V has the canonical decomposition

$$V \cong \bigoplus_{\mu \in P(V)} V^\mu \boxtimes V_\mu^{\mathfrak{n}^+}. \tag{2.3}$$

Remark 2.3.7. The decomposition (2.3) is often considered as a decomposition under the "joint action" of $\mathfrak{g}$ and $\mathrm{End}_{\mathfrak{g}}(V)$.

2.4 $Sp(2N)$ action on $\bigwedge \mathbb{C}^{2N}$

We have taken the following example directly from §5.5 of Goodman and Wallach.

For the rest of this section $G = Sp(2N)$. We fix $V = \mathbb{C}^{2N}$ and Ω to be a non-degenerate skew-symmetric bilinear form on V, we fix $\{e_i\}$ to be a basis of V, and we fix $\{\varphi^i\}$ to be the Ω-dual basis of V, i.e. such that $\Omega(e_i, \varphi^j) = \delta_{ij}$.

Definition 2.4.1. For $x \in V$ and $x^* \in V^*$ we define the *exterior product* operator $\varepsilon(x) : \bigwedge^k \mathbb{C}^{2N} \to \bigwedge^{k+1} \mathbb{C}^{2\ell}$, by $\varepsilon(x)(v_1 \wedge ... \wedge v_k) = x \wedge v_1 \wedge ... \wedge v_k$, and the *interior product* operator on $\bigwedge V$, $\iota : \bigwedge^k \mathbb{C}^{2N} \to \bigwedge^{k-1} \mathbb{C}^{2N}$, by $\iota(v^*)(v_1 \wedge \cdots \wedge v_k) = \sum_{j=1}^k (-1)^{j-1} \langle v^*, v_j \rangle v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_k$.

Remark 2.4.2. We have the following relations:

$$\{\varepsilon(x), \varepsilon(y)\} = 0,$$
$$\{\iota(x^*), \iota(y^*)\} = 0,$$
$$\{\varepsilon(x), \iota(x^*)\} = \Omega(x^*, x) Id_{\bigwedge^k \mathbb{C}^{2\ell}}.$$

Definition 2.4.3. By $E = \sum_{i=1}^{2N} \varepsilon(e_i)\iota(\varphi^i)$ we denote the *skew-symmetric Euler operator* on $\bigwedge V$.

We also set $\theta = \sum_{i=1}^{2N} e_i \otimes \varphi^i$.

Remark 2.4.4. For $u \in \bigwedge^k V$, $Eu = ku$.

Definition 2.4.5. Let $Y = \frac{1}{2}\varepsilon(\theta)$, $X = -Y^*$, and $H = NId - E$.

Lemma 2.4.6. The following identities hold in $\mathrm{End}(\bigwedge \mathbb{C}^{2N})$.

$$[E, X] = -2X,\ [E, Y] = 2Y,\ [Y, X] = E - NId$$

In particular, $\mathfrak{g}' = \mathrm{Span}\{X, E - N\mathrm{Id}, Y\}$ is isomorphic to $\mathfrak{sl}_{\mathbb{C}}(2)$.

Theorem 2.4.7. The commutant $\mathrm{Comm}(Sp(\mathbb{C}^{2N}))$ of the action of $Sp(\mathbb{C}^{2N})$ on $\mathrm{End}(\bigwedge \mathbb{C}^{2N})$ is generated by $X, Y, E - N\mathrm{Id}$, i.e. it is isomorphic to $\mathcal{U}(\mathfrak{g}')$.

Definition 2.4.8. We call a k-vector $u \in \bigwedge^k \mathbb{C}^{2N}$ Ω-*harmonic* when $Xu = 0$.

We denote the k-homogeneous space of Ω-harmonic elements in $\bigwedge^k \mathbb{C}^{2N}$ by $\mathcal{H}(\bigwedge^k \mathbb{C}^{2N})$, i.e.

$$\mathcal{H}\left(\bigwedge^k \mathbb{C}^{2N}\right) = \left\{u \in \bigwedge^k \mathbb{C}^{2N} | Xu = 0\right\}.$$

We denote the space of all harmonic elements in $\bigwedge \mathbb{C}^{2N}$ by $\mathcal{H}(\bigwedge \mathbb{C}^{2N})$.

Recall that $G = Sp(V, \Omega) = Sp(2N)$, $\mathfrak{g}' \cong \mathfrak{sl}_{\mathbb{C}}(2)$, and that $F^{(k)}$ stands for the irreducible $\mathfrak{g}'$-module of dimension $k+1$. We shorten $\mathcal{H}(\bigwedge^k \mathbb{C}^{2N})$ to $\mathcal{H}^k$.

Theorem 2.4.9. (i) If $p > N$, then $\mathcal{H}^p = 0$. If $0 \leq k \leq N$, then the space $\mathcal{H}^k$ is an irreducible representation of G which is isomorphic to the k-th fundamental representation, i.e. its highest weight is the k-th fundamental weight ϖ_k.

(ii) As $(\mathbb{C}[G], \mathcal{U}(\mathfrak{g}'))$-bimodules,

$$\bigwedge \mathbb{C}^{2N} \cong \bigoplus_{k=0}^{N} \mathcal{H}^k \boxtimes F^{(N-k)}.$$

In the following chapters, we use the term $(G, \mathfrak{g}')$-modules instead of $(\mathbb{C}[G], U(\mathfrak{g}')$-bimodules.

Corollary 2.4.10. As G-modules,

$$\bigwedge^k \mathbb{C}^{2N} = \bigoplus_{p=0}^{\lfloor k/2 \rfloor} \theta^p \wedge \mathcal{H}\left(\bigwedge^{k-2p} \mathbb{C}^{2N}, \Omega\right).$$

CHAPTER 3

Quantum Field Theory

One of the most celebrated achievements of Minkowski is that he placed time on an equal footing with the three spatial dimensions: $x_\eta = (ct, \vec{x})^t = (ct, x_1, x_2, x_3)^t = (x_0, x_1, x_2, x_3)^t$, where c is the speed of light in a vacuum. The new terminology (Minkowski) *space-time* emphasizes this union and it reinforces the need to express physical objects in four dimensions – rather than just in three. This mathematical structure is a natural environment for Einstein's special relativity.

Formally, we consider the vector x_η to represent the four-component vector with any fixed basis $\{e_i | i = 0, 1, 2, 3\}$: $x_\eta = x_0 + x_1 + x_2 + x_3 = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3$ – not a particular (coordinate) component, say, a_2. Specifically, the index η corresponds to the basis which is used for the corresponding vector space; we refer the reader to appendix B for more details. Mathematically, Minkowski embedded the three-dimensional vector kinematics into a psuedo-Riemannian structure which physicists call *Minkowski space*, in which operators act on the configuration and momentum spaces. Because the momentum space describes the potential and kinetic energy of the system, the momentum space gives a more illuminating picture of what is happening physically.

We consider a four-momentum to be the classical three dimensional physical presentation of momentum, $\vec{p}$, concatinated by the particle's rest energy: $p_\alpha = (Mc^2, \vec{p}c)^t$. Given the vector space isomorphism $\mathbb{R}^{1,3} \cong \mathbb{R}^4$, the Euclidean metric of the four-momentum in Minkowski space represents the square of the particle's energy at rest plus the square of the particle's kinetic energy; this is given by Einstein's famous equation

$$E^2 = (Mc^2)^2 + (\vec{p}c)^2,$$

where $\vec{p}$ is the three-dimensional momentum. From here on we set the physical constants to unity ($c = 1 = \hbar$) and we consider the energy of a particle to be positive, while the energy of its anti-partle is negative.

If $E^2 > 0$, then it is useful to "boost" our reference frame to that of the particle; mathematically, we change the basis of our coordinate system until the particle appears to be at rest. Then $E^2 = M^2$ and $\vec{p} = 0$, which fixes a canonical choice of presentation for $p_\alpha = (M, 0, 0, 0)^t$ and which motivates the terminalogy *a massive particle at rest.*

If $E^2 = 0$, then we have two cases: the zero momentum case and the massless (light-like) case. In the zero momentum case, $p_\alpha = (0, 0, 0, 0)^t$. In the massless case, the square of the particle's energy equals the square of the particle's momentum: $E^2 - \vec{p} \cdot \vec{p} = M^2 = 0$. In other words, all of the particle's energy is in its (3-dimensional) momentum. Hence, the canonical choice of presentation $p_\alpha = (\omega, \omega, 0, 0)^t$.

To recapitulate the above, the four-momentum, p_α, of a particle is of one of the three standard forms:

(i) $p_\alpha = (0, 0, 0, 0)^t$ (zero momentum),

(ii) $p_\alpha = (\omega, \omega, 0, 0)^t$ (massless), or

(iii) $p_\alpha = (M, 0, 0, 0)^t$ (massive).

In this chapter, we use tensor notation by default; we refer the reader to Appendix B for more details. We have adapted much of this chapter from [26]

3.1 Minkowski Space

Let $\mathbb{R}^{1,3}$ denote the four-dimensional Minkowski space with signature $(+1, -1, -1, -1)$.

- For every four-component (covariant) vector $x_\alpha \in \mathbb{R}^{1,3}$, there exists a four-component dual (contravariant) vector $x^\alpha \in (\mathbb{R}^{1,3})^*$ such that $x^\alpha = \sum_{\beta=0}^{3} g^{\alpha\beta} x_\beta$. The indices α correspond to a fixed (dual) basis $\{\varphi^0, \varphi^1, \varphi^2, \varphi^3\}$ of $(\mathbb{R}^{1,3})^*$ – rather than the basis elements of $\mathbb{R}^{1,3}$.
- We consider the indices of $g^{\alpha\beta}$ to each correlate with the dual basis of $\mathbb{R}^{1,3}$, $(\mathbb{R}^{1,3})^*$. For instance, if we wish to consider the matrix of $g^{\alpha\beta}$, then we write $(g^{\alpha\beta})^\rho_\eta$, where α and β still correspond to basis elements of the dual space, $(\mathbb{R}^{1,3})^*$ and where the indices ρ and η correspond to row or columns of the matrix. To be more explicit, we need to establish a convention; let x_ρ represent a column vector. Then ρ above labels the rows of the matrix, $(g^{\alpha\beta})^\rho_\eta$, while η labels the columns of the matrix.
- If e is a basis element and φ is a dual basis element, then $e\varphi = \varphi(e)$; if x_α is a column vector (so that x^α is the dual of x_α), then $x_\alpha y^\alpha = \sum_{\alpha=0}^{3} y^\alpha x_\alpha = \sum_{\alpha=0}^{3} y_\alpha x_\alpha$.
- From here on, we will use Einstein's notation; namely, that repeated upper and lower indices imply a partial evaluation (or a contraction); in example, $\sum_{\alpha=0}^{3} x^\alpha y_\alpha$ will be denoted by $x^\alpha y_\alpha$. The convention is that greek letters are summed over $0, 1, 2,$ and 3 while latin letters are summed over $1, 2,$ and 3; however, we will need to depart from this convention in the following chapters, which deal extensively with equations of tensors and spinors together.

As we will see, the *natural pairing*, $x^\mu x_\mu$, in Minkowski space is a defining invariant of the Lorentz group.

3.2 Lorentz Algebra

We denote the six (anti-symmetric) generators of the *Lorentz algebra* by J_α^β, where $J_\alpha^\beta = -J_\beta^\alpha$.

As a matrix, we have $(J_\alpha^\beta)_\eta^\zeta = \sqrt{-1}(g_\alpha^\zeta \delta_\eta^\beta - g_\zeta^\beta \delta_\alpha^\zeta)$, where ζ and η label the rows and columns, respectively, and α and β label the co/contra-variant basis elements, respectively.

Then

$$[J_\alpha^\beta, J_\eta^\zeta] = \sqrt{-1}(g_\alpha^\zeta J_\eta^\beta - g^{\beta\zeta} J_{\alpha\eta} - g_{\alpha\eta} J^{\beta\zeta}). \tag{3.1}$$

This is the Lie algebra $\mathfrak{so}(1,3)$, which we call the *Lorentz algebra.*

Sometimes J_α^β is written more explicitely as two types of spatial vectors, namely, $K^i = J_0^i = -J_i^0$ and $J_k^j = \epsilon^{ijk} J^i$, where $i, j, k = 1, 2$, or 3. Explicitely, we can write

$$J^1 = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{-1} \\ 0 & 0 & \sqrt{-1} & 0 \end{array}\right], \quad K^1 = \left[\begin{array}{c|ccc} 0 & -1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$J^2 = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \sqrt{-1} \\ 0 & 0 & 0 & 0 \\ 0 & -\sqrt{-1} & 0 & 0 \end{array}\right], \quad K^2 = \left[\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$J^3 = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad K^3 = \left[\begin{array}{c|ccc} 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right].$$

The subalgebra of $\mathfrak{so}(1,3)$ which leaves the time component invariant is $\mathfrak{so}(3) \cong \mathfrak{su}(2)$, which generates the group of rotations in the three spatial planes (x, y), (x, z), and (y, z). The other three generators, K^i, generate the three hyperbolic rotations, in the (t, x_1), (t, x_2), or (t, x_3) planes, which we call *boosts*.

In vector notation, the relation for $\mathfrak{so}(1,3)$ given in equation (3.1) becomes

$$[J_i, J_j] = \sqrt{-1}\epsilon^{ijk} J_k, \tag{3.2}$$

$$[J_i, K_j] = \sqrt{-1}\epsilon^{ijk} K_k, \text{ and} \tag{3.3}$$

$$[K_i, K_j] = -\sqrt{-1}\epsilon^{ijk} J_k. \tag{3.4}$$

We set out to decouple this algebra into two separate (commuting) algebras, so we let $J_L^i = \frac{1}{2}(J^i - \sqrt{-1}K^i)$ and $J_R^i = \frac{1}{2}(J^i + \sqrt{-1}K^i)$. Then we have

$$\left[J_L^i, J_L^j\right] = \sqrt{-1}\epsilon^{ijk} J_L^k,\ \left[J_R^i, J_R^j\right] = \sqrt{-1}\epsilon^{ijk} J_R^k,\ \text{ and } [J_L^i, J_R^j] = 0.$$

Thus, $\mathfrak{so}(1,3) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ as Lie algebras, and, in this context, we may describe the physical states as two angular momenta (j, j'), corresponding to J_L^i and J_R^i.

We fix the *Pauli matrices* to be

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ \text{and } \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where $[\sigma_i, \sigma_j] = 2\sqrt{-1}\epsilon_{ijk}\sigma_k$ and $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$. In addition to the three Pauli matrices above, we introduce a fourth matrix, $\sigma_0 = I_2$. The four Pauli matrices form a basis of *the space of* 2×2 *Hermitian matrices*; we denote this space by $\mathbb{H}(2) \subset M_{\mathbb{C}}(2)$. We use the fact that $(\sigma_0)^2 = I_2$ and $(\sigma_i)^2 = -I_2$ to motivate our choice of signature for the Minkowski space, which is isomorphic to the group of quaternions with unit Euclidean length.

We say that a square matrix is *Hermitian* when it is invariant under the action of the adjoint operator, $A = \overline{A^t}$, and we say that a square matrix is *skew-Hermitian* when it changes sign under the action of the adjoint operator: $A = -\overline{A^t}$. The Pauli matrices are Hermitian while $\sqrt{-1}\sigma_i$ are skew-Hermitian. This motivates our use of the factor of $\sqrt{-1}$ above.

In the semi-spinor representation of the Lorentz algebra, we use the Pauli matrices as a basis for the Minkowski space, $\mathbb{R}^{1,3}$. Explicitely, we use the injective vector space homomorphism $\mathbb{R}^{1,3} \to \mathbb{H}(2)$ such that $e_\eta \mapsto \sigma_\eta$. Then we have

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_0 + x_3 & x_1 - \sqrt{-1}x_2 \\ x_1 + \sqrt{-1}x_2 & x_0 - x_3 \end{bmatrix}.$$

The natural pairing in the Minkowski space, which is ivariant under this homomorphism, becomes the determinant of the 2×2 matrix presentation of the vector in Hermitian space.

The following calculations show that σ_2 plays a special role:

$$\sigma_2\sigma_1\sigma_2 = -\sigma_1 = -\overline{\sigma_1},$$

$$\sigma_2\sigma_2\sigma_2 = \sigma_2 = -\overline{\sigma_2}, \text{ and}$$

$$\sigma_2\sigma_3\sigma_2 = -\sigma_3 = -\overline{\sigma_3}.$$

In terms of our results below, we have $\overline{\sigma_2}^t\sigma_2 = I_2$ and $\overline{\sigma_2}^t\sigma_\eta\sigma_2 = -\overline{\sigma_\eta}$.

3.3 Semi-Spinor Representations

We now investigate the representations of $SO(3)$ and $SU(2)$ by investigating their generating Lie algebra $[J^i, J^j] = \sqrt{-1}\epsilon^{ijk}J^k$ (presented in vector notation) acting on $\mathbb{R}^3$. We begin by defining the orthogonal transformation $\Lambda \in O(3)$ where $\Lambda : V \to V$ such that

$$x_i' = \Lambda_i^j x_j. \tag{3.5}$$

Then we move from acting on $\mathbb{R}^3$ to acting on $\mathbb{C}^2$ by changing our basis as described above. When viewed as an embedding of $\mathbb{R}^3$ in $\mathbb{R}^{1,3}$, $\sigma_i = \Lambda_i^\alpha \sigma_\alpha$, and we still have $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ and $[\sigma_\alpha, \sigma_\beta] = 2\sqrt{-1}\epsilon_{\alpha\beta k}\sigma_k$. If the matrix of the homomorphism given by equation (3.5) is in $SO(3)$, then Λ_i^j is an algebra automorphism and there is an invertible homomorphism ρ such that $A = \rho\Lambda\rho^{-1} \in SL_{\mathbb{C}}(2)$. The map ρ such that $\Lambda \mapsto \rho(\Lambda)$ is called the *semi- (or half-) spinor representation of* $SO(3)$ *(or* $SU(2)$*) in* $GL_{\mathbb{R}}(3)$ *(or* $GL_{\mathbb{C}}(2)$*)*. For more information on this homomorphism, we refer the reader to [10].

Remark 3.3.1. It is interesting that, under conjugation by $\zeta = -\sqrt{-1}\sigma_2$, the Hermitian matrices, $\mathbb{H}(2)$, generate $SL_{\mathbb{C}}(2)$: for any $N \in SL_{\mathbb{C}}(2)$, there is some $M \in \mathbb{H}(2)$ such that

$$N = \zeta M \zeta^{-1}.$$

Because an inner automorphism determines the conjugating matrix modulo the kernel, ρ is only unique up to a non-zero constant multiple $\theta \in \mathbb{C}^\times$; the spinor representation is the homomorphism $SO(3) \to SL_{\mathbb{C}}(2)/\{\pm I_2\}$ which is double-valued:

$$SO(3) \cong SU(2)/\{\pm I_2\}.$$

3.4 Lorentz Group

Given the Lorentz algebra above, we construct the corresponding Lie group called the Lorentz group. A general element of the Lorentz algebra is of the form $\theta_i J^i \pm \omega_i\sqrt{-1}K^i$ (in vector notation with Einstein summation notation), so a general element of the Lorentz group is of the form $\Lambda = exp(\theta_i J^i \pm \sqrt{-1}\omega_i K^i)$. More invariantly, the *Lorentz group* is

$$O(1,3) = \{\Lambda_\alpha^\beta \mid g^{\alpha\beta}x_\alpha x_\beta = g^{\alpha\beta}(\Lambda_\alpha^\zeta x_\zeta)(\Lambda_\beta^\eta x_\eta)\}.$$

It can be shown that $\det(\Lambda_\alpha^\beta) = \pm 1$. The transformations such that $\det(\Lambda_\alpha^\beta) = +1$ form a subgroup of $O(1,3)$ which is called the *proper Lorentz transformations*; this group is denoted by $SO(1,3)$. The proper Lorentz group has two disconnected components, which either preserve the direction of time or reverse it: $\Lambda_0^0 \geq 1$ or $\Lambda_0^0 \leq -1$, respectively. These transformations are called *orthochronous* or *non-orthochronous*, respectively.

We use representations from the only connected component which contains the identity element (the proper orthochronous elements), because it is the only connected component which forms a subgroup of the Lorentz group. We then extend the representations of the restricted Lorentz group to representations of the entire Lorentz group by acting on the restricted representation by the operations P, T, and PT defined below.

The transformations such that $\det(\Lambda_\alpha^\beta) = -1$ can occur in three different ways which generate the three remaining disconnected components from the component containing the identity element:

- $P(t,x,y,z) = (t,-x,-y,-z)$ is called the *spatial parity operator*,
- $T(t,x,y,z) = (-t,x,y,z)$ is called the time reversal operation, and
- Transformations such that $(t,x,y,z) \mapsto (t,-x,y,z)$ or $(t,x,-y,z)$ or $(t,x,y,-z)$ are reflections about a single spatial axis.

3.5 Spinor Representations

Recalling the semi- (or half-) spinor representation theory of $SU(2)$ and that $\mathfrak{so}(1,3) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, we can write $\mathbb{C}^4$ as $\mathbb{C}^2 \oplus \mathbb{C}^2$, which indicates that $\mathbb{C}^4$ is the sum of two representations of $SU(2)$ (each of which is conjugate to the other vector space). Now we give the four-component spinors to be

$$\psi = \begin{bmatrix} \xi \\ \eta \end{bmatrix},$$

where $\psi \in \mathbb{C}^4$, $\eta \in \mathbb{C}^2$, and $\xi \in \overline{\mathbb{C}^2}$. The spatial parity operator defined above is such that $\xi \mapsto \eta$ and $\eta \mapsto \xi$.

3.6 Clifford Algebras Associated with Minkowski Space

Given that the σ's give representations of $\mathfrak{so}(3)$ and since $\mathfrak{so}(1,3) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, we define γ's in block diagonal form, such a basis is called a *chiral basis*.

Definition 3.6.1. We define the four γ-*matrices* to be

$$\gamma_0 = \begin{bmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{bmatrix} \text{ and } \gamma_k = \begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix},$$

where k =1, 2, or 3.

Then the γ's obey the relation

$$\{\gamma_\eta, \gamma_\zeta\} = 2g_{\eta\zeta}. \tag{3.6}$$

In other words, $\{\gamma_0,\ \gamma_1,\ \gamma_2,\ \gamma_3\}$ generate a Clifford algebra whose bilinear form is determined by $(g_{\alpha\beta})$. For more details, see Appendix D. We now recall the following standard fact (for example see lemma 15.6 in [27]).

Lemma 3.6.2. The associative unital algebra generated by $\{\gamma_0,\ \gamma_1,\ \gamma_2,\ \gamma_3\}$ has a basis

$$I, \gamma_\eta, \gamma_\eta\gamma_\zeta\ (\eta < \zeta), \gamma_\eta\gamma_\zeta\gamma_\mu\ (\eta < \zeta < \mu), \gamma_0\gamma_1\gamma_2\gamma_3.$$

Hence this algebra is isomorphic to the matrix algebra $M_{\mathbb{C}}(4)$.

(For a Geometric Algebra interpretation of the above, we refer the reader to [18].)

Now we turn our attention to the spinor representations of the the Lorentz group. Consider the Minkowski space $\mathbb{R}^{1,3}$ and an arbitrary element $\Lambda \in O(1,3)$. Then

$$\gamma'_\eta = \Lambda_\eta^\zeta \gamma_\zeta,$$

satisfy the identities $\{\gamma'_\eta, \gamma'_\zeta\} = 2g_{\eta\zeta}$. Therefore the maps $1 \mapsto 1, \gamma_\eta \mapsto \gamma'_\eta$ define an automorphism $A(\Lambda)$ of $M_{\mathbb{C}}(4)$. Since every automorphism of $M_{\mathbb{C}}(4)$ is inner, we have that there is some $\rho = \rho(\Lambda)$ in $GL_{\mathbb{C}}(4)$ such that $A(\Lambda)(X) = \rho X \rho^{-1}$. The map $\Lambda \mapsto \rho(\Lambda)$ is called the *spinor representation of the group* $O(1,3)$ *in the group* $GL_{\mathbb{C}}(4)$. We note that $\rho(\Lambda) \in SL_{\mathbb{C}}(4)$ is double-valued and defined up to a multiplicative complex constant.

The 4-dimensional complex space, $\mathbb{C}^4$, with the above spinor representation, $\Lambda \mapsto \rho(\Lambda)$, is called the *space of four-component spinors*. The elements of this space are (column vectors and are) called *spinors*.

As a generalization of the conjugate action of σ_2 above, we introduce the *charge conjugation matrix* (in the chiral basis),

$$C = \sqrt{-1}\gamma_2,$$

which satisfies $\overline{C}^t C = I_4$ and $\overline{C}^t \gamma_\mu C = -(\overline{\gamma_\mu})$.

The four γ's above generate a Clifford algebra on four generators:

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} I_4$$

such that $\gamma_\mu \gamma_\nu = -\gamma_\nu, \gamma_\mu$ whenever $\mu \neq \nu$. This Clifford algebra was discovered by P.A.M. Dirac while investigating the Klein-Gordon equation, which is the wave equation for massive relativistic quantum particles with spin 1/2. The relations for the Clifford algebra turned out to be necessary and sufficient conditions for the Klein-Gordon equation,

$$-\left(g_{\alpha\beta}\frac{\partial}{\partial\gamma_\alpha}\frac{\partial}{\partial\gamma_\beta}+M^2\right)\psi=0,$$

to be expressed as the product of two first order operators,

$$\left(\sqrt{-1}\gamma_\alpha\frac{\partial}{\partial\gamma_\alpha}+M\right)\left(\sqrt{-1}\gamma_\beta\frac{\partial}{\partial\gamma_\beta}-M\right)\psi=0.$$

The *Dirac equation* (for the four component spinor ψ) is

$$\left(\sqrt{-1}\gamma_\beta\frac{\partial}{\partial\gamma_\beta}-M\right)\psi=0, \tag{3.7}$$

while the *conjugate Dirac equation* is

$$\left(\sqrt{-1}\gamma_\alpha\frac{\partial}{\partial\gamma_\alpha}+m\right)\overline{\psi}=0,$$

where $\overline{\psi}=\psi^*\gamma_0$ is the *Dirac conjugate*. For more information on Clifford algebras in relativistic electrodynamics, we refer the reader to [16], [23], and [30].

We define a fifth γ matrix:

$$\gamma_5=\sqrt{-1}\gamma_0\gamma_1\gamma_2\gamma_3.$$

The label 5 for this element comes from physicists who originally worked over Euclidean space where the vector components are labelled by 1,2,3, and 4 – rather than in Minkowski space where the vector components are labelled by 0,1,2, and 3.

We call the operator $\pi_\pm=\frac{1}{2}(I_4\pm\gamma_5)$ the *Lorentz invariant projection operator*; we can use $\pi_\pm$ to separate the four-component chiral spinors into two two-component spinors,

$$\psi_\pm=\pi_\pm\psi$$

where ψ satisfies the Dirac equation.

3.7 Poincaré Algebra and Group

In full generalization of the relativity described by the Euclidean group, Weyl introduced the generators of space-time translations, $P^\alpha = \sqrt{-1}\frac{\partial}{\partial x_\alpha}$, as a symmetry of the special theory of relativity. In this way, a generic translation operator in the Poincaré group is of the form $\exp(-\sqrt{-1}P^\alpha a_\alpha)$, where a_α is a fixed four-component position vector and P^α is the four component energy-momentum operator associated to the vector x_α which is being translated. Further,

$$[P^\alpha, P^\beta] = 0.$$

In tensor notation, we have the relation

$$[P^\alpha, J^\zeta_\eta] = \sqrt{-1}(g^{\alpha\zeta}P_\eta - g^\alpha_\eta P^\zeta).$$

In explicit terms, the *Poincaré algebra* is generated by J^α_β and P^α, subject to the relations

$$[P^\alpha, P^\beta] = 0, \tag{3.8}$$

$$[J^\alpha_\beta, P^\zeta] = \sqrt{-1}\left(g^\zeta_\beta P^\alpha - g^{\alpha\zeta}P_\beta\right), \quad \text{and} \tag{3.9}$$

$$[J^\alpha_\beta, J^{\zeta\eta}] = \sqrt{-1}\left(g^\zeta_\beta J^\alpha_\eta + g^\alpha_\eta J^\zeta_\beta - g^{\alpha\zeta}J_{\beta\eta} - g_{\beta\eta}J^{\alpha\zeta}\right). \tag{3.10}$$

In analogy with the Lorentz algebra, the Poincaré algebra can be expressed in the less compact but more illucidating notation:

$$[J^i, J^j] = \sqrt{-1}\epsilon^{ijk}J^k,\ [J^i, K^j] = \sqrt{-1}\epsilon^{ijk}K^k,\ [J^i, P^j] = \sqrt{-1}\epsilon^{ijk}P^k,$$
$$[K^i, K^j] = -\sqrt{-1}\epsilon^{ijk}J^k,\ [P^i, P^j] = 0,\ [K^i, P^j] = \sqrt{-1}P^0\delta^{ij}I_3,$$
$$[J^i, P^0] = 0,\ [P^i, P^0] = 0,\ [K^i, P^0] = \sqrt{-1}P^i.$$

We define the Poincaré group as the semi-direct product $O(1,3) \ltimes \mathbb{R}^{1,3}$ acting on $\mathbb{R}^{1,3}$ such that

$$(\Lambda, a) : x_\alpha \mapsto \Lambda^\beta_\alpha x_\beta + a_\alpha,$$

where $\Lambda = \Lambda_\alpha^\beta \in O(1,3)$ and $a_\alpha \in \mathbb{R}^{1,3}$ are fixed and for any $x_\alpha \in \mathbb{R}^{1,3}$. This action leaves the natural pairing in Minkowski space, $x^\beta x_\beta = g^{\alpha\beta} x_\alpha x_\beta$, invariant, which is an alternate definition for the group.

We extend representations from the connected component of the Poincaré group that contains the identity element, so we define the notation

$$L = \{\Lambda \in SO(1,3) \mid (\Lambda_\alpha^\beta)_0^0 > 0\}.$$

Then, for fixed P^α, we call

$$L_p = \{\Lambda \in L \mid L(P) = P\}$$

the *little subgroup* or the *little group* of the Poincaré group on four dimensions.

We note that $L_p \cong SO(3)$ for massive representations and $L_p \cong SO(2)$ for massless representations, see for example the proof of Theorem 6 in [5].

Given a fixed four-momentum, p_α, we let L_p denote the subgroup of L which leaves p_α invariant. Then $\mathfrak{l}_p = \mathrm{Lie}(L_p)$ is a subalgebra of $\mathfrak{g}$, and the Harish-Chandra pair $(L_p \ltimes \mathbb{R}^{1,3}, \mathfrak{g}_{\bar{0}})$ is a subsupergroup of the extended Poincaré supergroup, which is considered in the next chapter.

CHAPTER 4

Mathematical Model of the Extended Poincaré Superalgebras

We are ready to define our main object of study - the finitely Extended Poincaré superalgebra. We use the definition according to Ferrara *et al*; however, other equivalent definitions exist. We highly encourage the reader to refer to the previous chapter and the appendices for more details.

Definition 4.0.1. The *Extended Poincaré superalgebra of N supercharges* on four dimensions is the Lie superalgebra, EPS(N), whose even part, EPS$(N)_{\bar{0}}$, is generated by $J_{\alpha,\beta}$, P_α, U^{ij}, V^{ij}, and whose odd part, EPS$(N)_{\bar{1}}$, is generated by Q^i_α, $\alpha, \beta = 1, 2$, $i, j = 1, 2, \ldots, N$, subject to the relations

$$[J^\alpha_\beta, P^\zeta] = \sqrt{-1}\left(g^\zeta_\beta P^\alpha - g^{\alpha\zeta} P_\beta\right), \tag{4.1}$$

$$[J^\alpha_\beta, J^\zeta_\eta] = \sqrt{-1}\left(g^\zeta_\beta J^\alpha_\eta + g^\alpha_\eta J^\zeta_\beta - g^{\alpha\zeta} J_{\beta\eta} - g_{\beta\eta} J^{\alpha\zeta}\right), \tag{4.2}$$

$$\{Q^i_\alpha, Q^j_\beta\} = (\gamma^\zeta C)_{\alpha\beta} P_\zeta \delta^{ij} + C_{\alpha\beta} U^{ij} + (\gamma_5 C)_{\alpha\beta} V^{ij}, \tag{4.3}$$

$$[Q^i_\alpha, J^{\zeta\eta}] = \sqrt{-1}(\Sigma_{\zeta\eta})^\rho_\alpha Q^i_\rho, \tag{4.4}$$

$$[P^\alpha, P^\beta] = 0, \tag{4.5}$$

$$[Q^i_\alpha, P^\zeta] = 0, \tag{4.6}$$

$$[P^\alpha, U^{ij}] = [P^\alpha, V^{ij}] = 0, \tag{4.7}$$

$$[J^{\alpha\beta}, U^{ij}] = [J^{\alpha\beta}, V^{ij}] = 0, \tag{4.8}$$

$$[U^{ij}, Q^k_\alpha] = [V^{ij}, Q^k_\alpha] = 0, \quad \text{and} \tag{4.9}$$

$$[U^{ij}, U^{\ell m}] = [V^{ij}, V^{\ell m}] = [V^{ij}, U^{\ell m}] = 0, \tag{4.10}$$

where $\rho = 1, 2$, $\zeta, \eta = 0, 1, 2, 3$, U^{ij} and V^{ij} are Hermitian operators, and $\Sigma_{\alpha\beta} = \frac{1}{2}[\gamma_\alpha, \gamma_\beta]$. The Q^i_α's are spinors which we call *supercharges* (or Fermi charges), and we call U^{ij} and V^{ij} *central charges*. We remind the reader of one of the properties of spinors, namely $(CQ^*)^i_\alpha = Q^i_\alpha$.

Remark 4.0.2. Because the probablities described by the Schrödinger equation are invariant under unitary transformations, unitary representations are the most interesting representations for quantum mechanics. Recall that $\rho : G \to GL(n)$ is a *unitary representation* of the group G if the image $\rho(G)$ is a subgroup of $U(n)$. We similarly define unitary representations of Lie (super)algebras and associative algebras. From now on, whenever we write A^* for an element A in a group or a Lie superalgebra, we mean the complex conjugate of $\rho(A)$ in a fixed representation ρ.

Along these lines, in addition to the $2N$ supercharges Q^i_α, there are $2N$ conjugates, Q^{*i}_α. As a result, there are $4N$ supercharges; this will be discussed below in more detail.

At the level of categories, we may regard a Lie supergroup as a super Harish-Chandra pair $(G_{\bar{0}}, \mathfrak{g})$, where $G_{\bar{0}} = L_p \ltimes \mathbb{R}^{1,3}$ is the little Poincaré group and $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \mathrm{Lie}(G_{\bar{0}}) \oplus \mathfrak{g}_{\bar{1}}$ is a Lie superalgebra which is a $G_{\bar{0}}$-module such that the $\mathrm{Lie}(G_{\bar{0}})$-action is the differential of the $G_{\bar{0}}$-action on $\mathfrak{g}$. We adopt the supergroup terminology of [8] to which we refer the reader for more details. We refer the reader to [13] for a deep mathematical article about the relevant superspaces.

We may present the extended Poincaré superalgebra with fewer generators while being more explicit. For this purpose, we define

$$Z^{jk} = -V^{jk} + \sqrt{-1}U^{jk},$$

and, in analogy with the previous chapter, we introduce the following notation for our (two-component) supercharges:

$$Q^i_\alpha = (Q_\alpha)^i$$
$$(Q^\alpha)^i = \left(\epsilon^{\alpha\beta} Q^*_\beta\right)^i = \left(\sqrt{-1}\,(\sigma_2)^{\alpha\beta}\, Q^*_\beta\right)^i,$$

where $i, j = 1, \ldots N$, $\alpha, \beta = 1, 2$; we have chosen to suppress the traditional Van der Waerden notation for spinors. In our notation, the four-component spinors are such that

$$Q_\alpha = \begin{pmatrix} Q_\beta \\ \epsilon^{\beta\eta} Q^*_\eta \end{pmatrix},$$

where $\alpha = 1, \ldots, 4$ and $\beta, \eta = 1, 2$.

Now, in four-component notation, equation (4.3) becomes

$$\{Q^i_\alpha, Q^{*j}_\beta\} = (\sigma_\mu)_{\alpha\beta} P^\mu \delta^{ij} \tag{4.11}$$

$$\{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta} Z^{ij} \tag{4.12}$$

$$\{Q^{*i}_\alpha, Q^{*j}_\beta\} = \epsilon_{\alpha\beta} Z^{ij}, \tag{4.13}$$

where $\alpha, \beta = 1, 2$ and $i, j = 1, \ldots, N$.

We want to simplify these relations; for this reason we apply unitary transformations,

$$Q^i_\alpha \mapsto \widetilde{Q}^i_\alpha = \sum_{j=1}^{N} X^{ij} Q^j_\alpha \tag{4.14}$$

$$Q^{*i\alpha} \mapsto \widetilde{Q}^{*i\alpha} = \sum_{j=1}^{N} X^{*ij} Q^{*j\alpha}, \tag{4.15}$$

such that the matrix of Z^{ij} transforms to a block diagonal matrix. Note that the equations (4.11)-(4.13) do not change the rest frame under any unitary transformation – specifically for $X^{ij} \in U(N)$.

Hence, for massive transformations, $\{\widetilde{Q}^i_\alpha, \widetilde{Q}^{*j\beta}\} = \delta^\beta_\alpha \delta^{ij} P^0 I_2$. However, the Z^{ij}'s change according to the transformation

$$Z^{ij} \mapsto (XZX^T)^{ij} = \widetilde{Z}^{ij}, \tag{4.16}$$

where Z^{ij} is in block digonal form.

1. If N is even, then $\widetilde{Z} = \sqrt{-1}\sigma_2 \otimes \widehat{Z}_{N/2}$, where $\widehat{Z}_{N/2} = \mathrm{diag}(z_1, \ldots, z_{N/2})$,
2. If N is odd, then
$$\widetilde{Z} = \left[\begin{array}{c|c} \sqrt{-1}\sigma_2 \otimes \widehat{Z}_{(N-1)/2} & 0 \\ \hline 0 & 0 \end{array}\right],$$
where $\widehat{Z}_{(N-1)/2} = \mathrm{diag}(z_1, \ldots, z_{(N-1)/2})$ and where $z_1, z_2, \ldots, z_{\lfloor N/2 \rfloor}$ are the eigenvalues of the anti-symmetric matrix Z^{ij} (and $\lfloor N/2 \rfloor$ is the floor function of $N/2$); we choose the z_i's to be non-negative. Please see Appendix B for more details about the tensor product of matrices.

Now we have the $2N$ (two-component) supercharges and their $2N$ conjugates. Wanting to eventually multiplex these into four-component spinors, we introduce a new notation for the index $i = 1, \ldots, N$ by splitting the single index into two separate indices $a = 1, 2$ and $m = 1, \ldots, \lfloor N/2 \rfloor$. Keeping with the psuedo-Van der Waerden notation for spinors used above, the positions of a and α, as indices, will indicate the spatial parity (left or right"handedness") of the spinors.

From now on we will consider all elements of $EPS(N)$ as endomorphisms of a fixed representation of $EPS(N)$. Namely, if $\rho : EPS(N) \to \mathrm{End}\, V$ is a Lie superalgebra homomorphism, then we identify $EPS(N)$ with $\rho(EPS(N))$. We also consider representations of $EPS(N)$ on which $\widetilde{Z}^{ij}$ act by constant multiplication. Similarly

$P^\zeta v = p^\zeta v$ for $\zeta = 0, 1, 2, 3$. Furthermore, recall that there are only three types of representations:

- *Massive representations.*

  These are $EPS(N)$-representations for which $(p^0, p^1, p^2, p^3) = (M, 0, 0, 0)$ with $M > 0$.

- *Massless (light-like) representations.*

  These are $EPS(N)$-representations for which $(p^0, p^1, p^2, p^3) = (\omega, \omega, 0, 0)$ with $\omega > 0$.

- *Zero momentum representations.*

  These are $EPS(N)$-representations for which $(p^0, p^1, p^2, p^3) = (0, 0, 0, 0)$.

4.1 Massive Representations of EPS(N) with Even N

We now focus on the first case, i.e. on the massive representations. We also assume, for simplicity, that N is even. The case where N is odd is similar and is discussed below.

Then we have the following relations after applying the transformation (4.16):

$$\{\widetilde{Q}^{am}_\alpha, \widetilde{Q}^{bn}_\beta\} = \epsilon_{\alpha\beta}\epsilon^{ab}\delta^{mn}z_n \tag{4.17}$$

$$\{\widetilde{Q}^{am}_\alpha, \widetilde{Q}^{*\beta n}_b\} = \epsilon^\beta_\alpha \epsilon^a_b \delta^{mn} M. \tag{4.18}$$

where $a, b, \alpha, \beta = 1, 2$ and $m, n = 1, 2, \ldots, N/2$. We apply one more transformation in order to obtain generators and relations in convenient terms. Then we define

$$S^{am}_{\alpha(1)} = \frac{\widetilde{Q}^{am}_\alpha + \widetilde{Q}^{*\alpha m}_a}{\sqrt{2}} \text{ and } S^{am}_{\alpha(2)} = (-1)^{a+1}\frac{\widetilde{Q}^{am}_\alpha - \widetilde{Q}^{*\alpha m}_a}{\sqrt{2}}, \tag{4.19}$$

where $a, \alpha = 1, 2$ and $m = 1, 2, \ldots, N/2$ such that $S^{am}_{\alpha(i)} = \epsilon^{ab}\epsilon_{\alpha\beta}S^{*bm}_{\beta(i)}$. If N is even, then equations (4.17)-(4.18) become

$$\{S^{m}_{\alpha(i)}, S^{*n}_{\beta(j)}\} = \delta_{\alpha\beta}\delta^{mn}\delta_{ij}(M - (-1)^{j} z_n) \tag{4.20}$$

$$\{S^{m}_{\alpha(i)}, S^{n}_{\beta(j)}\} = 0 \tag{4.21}$$

$$\{S^{*m}_{\alpha(i)}, S^{*n}_{\beta(j)}\} = 0, \tag{4.22}$$

where $\alpha, \beta, i, j = 1, 2$ and $m, n = 1, 2, \ldots, N/2$.

Remark 4.1.1. Note that this notation does not require us to keep track of upper verse lower indices – only repeated indices.

From the above, we have the *positivity bound* $0 \leq z_n \leq M$ for $n = 1, 2, \ldots, \lfloor N/2 \rfloor$. A direct consequence of this bound is that all z_n's vanish when $M = 0$.

Proposition 4.1.2. The extended Poincaré superalgebra action on a massive representation has the following relations

$$\begin{aligned}
[J^{\zeta}_{\eta}, P^{\chi}] &= \sqrt{-1}\left(g^{\chi}_{\eta} P^{\zeta} - g^{\zeta\chi} P_{\eta}\right) \\
[J^{\zeta}_{\eta}, J^{\chi}_{\xi}] &= \sqrt{-1}\left(g^{\chi}_{\eta} J^{\zeta}_{\xi} + g^{\zeta}_{\xi} J^{\chi}_{\eta} - g^{\zeta\chi} J_{\eta\xi} - g_{\eta\xi} J^{\zeta\chi}\right) \\
\{S^{m}_{\alpha(i)}, S^{*n}_{\beta(j)}\} &= \delta_{\alpha\beta}\delta_{ij}\delta^{mn}\left(M - (-1)^{j} z_n\right) \\
[S^{m}_{\alpha(i)}, J_{\zeta\eta}] &= (\Theta_{\zeta\eta})^{\rho}_{\alpha} S^{m}_{\rho i} \\
[P^{\zeta}, P^{\eta}] &= 0 \\
[S^{m}_{\alpha(i)}, P_{\zeta}] &= 0 \\
[S^{m}_{\alpha(i)}, S^{n}_{\beta(j)}] &= [S^{*m}_{\alpha(i)}, S^{*n}_{\beta(j)}] = 0 \\
[\widetilde{Z}^{ij}, P^{\zeta}] &= [\widetilde{Z}^{ij}, J^{\zeta\eta}] = [S^{m}_{\alpha(i)}, \widetilde{Z}^{ij}] = 0,
\end{aligned}$$

where $i, j, \alpha, \beta = 1, 2$, $m, n = 1, \ldots, N/2$, $\eta, \zeta = 1, 2, 3, 4$, and Θ can be found from equations (4.4), (4.14), and (4.19). Recall that P_{ζ}, $J_{\zeta\eta}$ and $\widetilde{Z}^{ij}$ generate $\mathrm{EPS}(N)_{\bar{0}}$, while the S^{ma}_{α} generate $\mathrm{EPS}(N)_{\bar{1}}$.

Definition 4.1.3. We denote by $\mathbf{Q}$ the space spanned by Q^i_α and Q^{*i}_α, and we let $\mathrm{Cliff}(\mathbf{Q})$ denote the Clifford algebra generated by $\mathbf{Q}$ subject to the equations (4.17), (4.18), and (4.23)-(4.25).

Every representation of $EPS(N)$ is a representation of the Clifford algebra $\mathrm{Cliff}(\mathbf{Q})$ generated by Q^i_α. Conversely, we are interested in representations of $EPS(N)$ that come from representations of $\mathrm{Cliff}(\mathbf{Q})$. Namely, for every $\mathrm{Cliff}(\mathbf{Q})$-representation V_0, we can consider the induced representation V through the embedding $\mathrm{Cliff}(\mathbf{Q}) \subset EPS(N)$; see for example equation (3.1) in [6]. We note that this reference uses analytic induction.

We are interested in the most physically relevant restrictions of the simple spin module $\mathbb{S} = \bigwedge \mathbb{C}^{2N}$ of $\mathrm{Cliff}(\mathbf{Q})$. More precisely, we look at the following restrictions.

(i) Restriction to the group $U(2N)$. Here we use the fact that $\mathbb{S}$ is a $GL_{\mathbb{C}}(2N)$-module.

(ii) Restriction to the group $SU(2) \times Sp(2N)$. Here we use the fact that $\mathbb{S}$ is an $(\mathfrak{sl}(2), Sp(2n))$-module (see example 2.4 in Chapter 2).

(iii) Restriction to the group $U(1) \times SU(N)$. Here we use the fact that $\mathbb{S}$ is a $(\mathfrak{gl}(1), SU(N))$-module.

(iv) Restriction to the group $Sp(2q_1) \times \cdots Sp(2q_n) \times U(N - 2\sum_{i=1}^n q_i)$. This is the case when

$$
\begin{aligned}
&z_1 = \cdots = z_{q_1} > z_{q_1+1} = \cdots = \\
&\quad = z_{q_1+q_2} > \cdots > z_{q-q_n} = \cdots = z_q > z_{q+1} = \cdots = z_{N/2} = 0,
\end{aligned}
$$

where $q = \sum_{i=1}^n q_i$. We will not deal with this case in this thesis, but we plan to generalize our results to account for this case in the future.

4.2 Massive Representations of EPS(N) with Odd N

Now we consider the case when N is odd. We still have the relations (4.20)-(4.22) with $m, n = 1, 2, \ldots, (N-1)/2$, but we need the following relations for the one remaining super charge, Q^N_α:

$$\{Q^N_\alpha, Q^{*N}_\beta\} = \delta_{\alpha\beta} M, \tag{4.23}$$

$$\{Q^N_\alpha, Q^N_\beta\} = 0, \text{ and} \tag{4.24}$$

$$\{Q^N_\alpha, S^m_{\beta(i)}\} = \{Q^N_\alpha, S^{*m}_{\beta(i)}\} = 0. \tag{4.25}$$

Remark 4.2.1.

- For even N, $\mathbf{Q}$ is spanned by $\widetilde{Q}^{am}_\alpha$ and $\widetilde{Q}^{*\alpha m}_a$ or by $S^{am}_{\alpha(i)}$.
- For odd N, $\mathbf{Q}$ is spanned by Q^N_α, $\widetilde{Q}^{am}_\alpha$ and $\widetilde{Q}^{*\alpha m}_a$ (or by Q^N_α and $S^{am}_{\alpha(i)}$).
- Cliff($\mathbf{Q}$) is realized as a Clifford algebra by the relations given by equations (4.20)-(4.22) and by (4.23)-(4.25).

4.3 Massless Representations of EPS(N)

From equations (4.1)-(4.10), we get the following relations for the two-component spinors:

$$\{Q^i_\alpha, Q^{*j\beta}\} = 2(\sigma_\mu)^\beta_\alpha P^\mu \delta^{ij},$$

$$\{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta} Z^{ij}, \text{ and}$$

$$\{Q^{*i\alpha}, Q^{*j\beta}\} = \epsilon^{\alpha\beta} Z^{ij}.$$

After some calculations with $P^\alpha = (\omega, \omega, 0, 0)^t$, we have $\{Q^i_1, Q^{*j1}\} = 0$ and $\{Q^i_2, Q^{*j2}\} = 2\omega\delta^{ij}$. Further, from [6] we see that $Z^{ij} = 0$; this result can also be found in reference [1], which also provides a brief modern treatment of extended supersymmetry.

4.4 General Massive Multiplets in Extended Supersymmetry

In this section we collect the main results from [12]. Our goal in the thesis is to generalize these results for the restrictions (i)-(iv) listed above; we focus on (ii) and (iii). Note that in this section we will follow the notation of [12], while, in the next chapter, we will use the notation introduced earlier (following mostly [15]). The reader should keep in mind that the representations listed below are identified with their dimensions or with their highest weights. The correspondence $V \to \dim V$ is one-to-one for all representations considered below, but certainly fails to be injective in general.

We have the following motivation to consider the above restrictions. When considering the supercharges as a vector, the set of equations described by equation (4.3) define a Clifford algebra with the invariant group $SO(4N)$.

Case (ii): Restriction to $SU(2) \times Sp(2N)$.

We wish to decompose $SO(4N)$ into the invariant $SU(2) \times Sp(2N)$. The lowest dimensional supermultiplet is obtained from the *Clifford vacuum* $\mid 0\rangle$, defined by

$$Q^i_\alpha \mid 0\rangle = 0$$

for all α and i. By applying the creation operators Q^{*i}_α we generate 2^{2N} states, which form a basis for the spinorial representation $\mathbb{S}$ of $SO(4N)$:

$$\mid 0\rangle,\ Q^{*i_1}_{\alpha_1} \mid 0\rangle,\ Q^{*i_1}_{\alpha_1} Q^{*i_2}_{\alpha_2} \mid 0\rangle,\ \ldots,\ Q^{*i_1}_{\alpha_1} \ldots Q^{*i_N}_{\alpha_N} \mid 0\rangle.$$

$SO(4N)$ branches into two irreducible representations corresponding to Bosons and Fermions; each of the irreducible subrepresentations have dimension 2^{2N-1}. The following is the restriction of $\mathbb{S}$ as a representation of $SU(2) \times Sp(2N)$ as follows

$$\mathbb{S} = (N+1, [2N]_0) + (N, [2N]_1) + \cdots + (N+1-k, [2N]_k) + \cdots + (1, [2N]_N), \quad (4.26)$$

where the first label is the dimension of the $SU(2)$ multiplet (the J spin is $(N-k)/2$) and where $[2N]_k = \binom{2N}{k} - \binom{2N}{k-2}$ is the dimension of the totally antisymmetric traceless representation of $Sp(2N)$. The formula for the dimensionality can be found in [15]; however, an alternative approach to derive this formula can be found in [4]. To compare with the results from the literature, we refer the reader to equation (9) in [12] and to table 4 with $D = 4$ in [29]. By the above identity we obtain a classification of the states of given intrinsic $SU(2)$ spin.

Case (iii): Restriction to $U(1) \times SU(N)$.

Using the decompostion above, we can further restrict the massive supermultiplet into massless (or light-like) representations. Here we will use the inclusions

$$U(1) \times SU(N) \subset SU(2) \times Sp(2N) \subset SO(4N)$$

where $U(1)$ is generated by the $SU(2)$ spin projection, λ, and the $2N$ representation of $Sp(2N)$ decomposes under $U(N)$ via the map $U(N) \to Sp(2N)$ such that $A \mapsto (A, \overline{A})$, or in the notation of invariant theory, $\mathbb{C}^{2N} \downarrow_{U(N)} = \mathbb{C}^N \oplus \overline{\mathbb{C}^N}$. Physically, we may interpret this restriction as a massive particle being expressed as the combination of massless particles.

Our massive supermultiplet branches into 2^N massless supermultiplets, each of dimension 2^N, and we have

$$\mathbb{S} = \left\{\frac{N}{2}, [N]_0\right\} + \left\{\frac{N-1}{2}, [N]_1\right\} + \dots \left\{\frac{N-k}{2}, [N]_k\right\} + \dots + \{0, [N]_N\},$$

where the braces denote a massless supermultiplet specified by a state of maximal spin projection, λ_{max}, belonging to the antisymmetric representation $[N]_k$ of $SU(N)$, namely

$$\{\lambda_{max}, [N]_k\} = \left(\lambda_{max}, [N]_k \otimes [\bar{N}]_0\right) + \left(\lambda_{max} - \frac{1}{2}, [N]_k \otimes [\bar{N}]_1\right) + \dots + \\ + \left(\lambda_{max} - \frac{\ell}{2}, [N]_k \otimes [\bar{N}]_\ell\right) + \dots + \left(\lambda_{max} - \frac{N}{2}, [N]_k \otimes [\bar{N}]_N\right).$$

Example 4.4.1. For $N = 4$,

$$\mathbb{S} = \{4,1\} + \{3,4\} + \{2,6\} + \{1,4\} + \{0,1\},$$

which has dimension 2^8. Explicitly, we have the following decomposition:

$$\begin{aligned}
\mathbb{S} = {} & (4, [4]_0 \otimes [\bar{4}]_0) + (7/2, [4]_0 \otimes [\bar{4}]_1) + \\
& + (3, [4]_0 \otimes [\bar{4}]_2) + (3, [4]_1 \otimes [\bar{4}]_0) + \\
& + (5/2, [4]_0 \otimes [\bar{4}]_3) + (5/2, [4]_1 \otimes [\bar{4}]_1) + \\
& + (2, [4]_0 \otimes [\bar{4}]_4) + (2, [4]_1 \otimes [\bar{4}]_2) + \\
& + (2, [4]_1 \otimes [\bar{4}]_2) + (2, [4]_2 \otimes [\bar{4}]_0) + \\
& + (3/2, [4]_1 \otimes [\bar{4}]_3) + (3/2, [4]_2 \otimes [\bar{4}]_1) + \\
& + (1, [4]_1 \otimes [\bar{4}]_4) + (1, [4]_2 \otimes [\bar{4}]_2) + \\
& + (1, [4]_3 \otimes [\bar{4}]_0) + (1/2, [4]_2 \otimes [\bar{4}]_3) + \\
& + (1/2, [4]_3 \otimes [\bar{4}]_1) + (0, [4]_2 \otimes [\bar{4}]_4) + \\
& + (0, [4]_3 \otimes [\bar{4}]_2) + (0, [4]_4 \otimes [\bar{4}]_0) + \\
& + (-1/2, [4]_3 \otimes [\bar{4}]_3) + (-1/2, [4]_4 \otimes [\bar{4}]_1) + \\
& + (-1, [4]_3 \otimes [\bar{4}]_4) + (-1, [4]_4 \otimes [\bar{4}]_2) + \\
& + (-3/2, [4]_4 \otimes [\bar{4}]_3) + (-2, [4]_4 \otimes [\bar{4}]_4).
\end{aligned}$$

We leave checking that the dimensions work out to the reader.

CHAPTER 5

Representation Theoretic Results

We denote by $\mathbb{S} = \mathbb{S}(z_1, ..., z_n, M)$ the massive spinor representation of the Clifford subalgebra $\mathrm{Cliff}(\mathbf{Q})$ of $ESP(N)$. Recall that $\mathrm{Cliff}(\mathbf{Q})$ is generated by Q^i_α and $Q^{*i\alpha}$. We denote by $\mathbb{S}(z_1, ..., z_n, M)$ the corresponding (induced) massive representation of $ESP(N)$. Note that $\mathbb{S}(z_1, ..., z_n, M)$ depends on the n central parameters $z_1, z_2, ..., z_n$, and the mass M, where $n = \lfloor N/2 \rfloor$. For simplicity, we will assume, unless otherwise stated, that N is even.

Definition 5.0.1. If $\lambda \in \mathbb{R}$, then $\mathbb{C}_\lambda$ denotes the $U(1)$-module with underlying space $\mathbb{C}$ and action defined by $\exp(\sqrt{-1}\varphi) \cdot z = \exp(\sqrt{-1}\lambda\varphi)z$.

In all results below, the restrictions $V \downarrow_{G\times\mathfrak{g}}$ for a Lie group G and a Lie algebra $\mathfrak{g}$ should be understood as restrictions to the tensor product of the corresponding associative algebras, i.e. as $V \downarrow_{\mathcal{A}(G)\otimes U(\mathfrak{g})}$, and we will also use the symbol $\boxtimes$ for the outer tensor products.

5.1 Induced Representations

E. Wigner induced representations of the Poincaré group analytically using the "Mackey machine" [25]; the following is a brief summary of this process which has been adapted from [31]. We will then present the appropriate version of this process for the subsupergroup that we are working with in this thesis, which has been adapted from [12]. There is also an algebraic version of induced representations, which might be more consistent with the language of this thesis; we refer the reader to [7] for details on algebraic induction.

5.1.1 Induction from a Closed Subgroup

We let G be a separable Lie group and let K be a closed subgroup of G, and we assume the existance of an invariant measure $d\mu$ on G/K. Then for a given unitary representation, π_0, of K we define the Hilbert space of functions, on which π_0 acts, by

$$V = \left\{ \varphi : G \to \mathfrak{X} \mid \varphi(gk^{-1}) = \pi_0(k)\varphi(g),\ \int_{G/K} ||\varphi(g)||^2 d\mu(\overline{g}) < \infty \right\},$$

where the first condition defines the action of π_0 and the second condition is the norm on V with $\mathfrak{X}$ a complex Hilbert space.

Claim 5.1.1. We can define a left-action on V by G such that $[\pi(h)\varphi](g) = \varphi(h^{-1}g)$, for any $h, g \in G$.

Proof. We need only show that if $\varphi \in V$, then $\pi(h)\varphi \in V$.

$$\begin{aligned}
[\pi(h)\varphi](gk^{-1}) &= \pi(h)\varphi(gk^{-1}) \\
&= \varphi(h^{-1}gk^{-1}) \\
&= \pi_0(k)\varphi(h^{-1}g) \\
&= \pi_0(k)\pi(h)\varphi(g) \\
&= \pi_0(k)[\pi(h)\varphi](g)
\end{aligned}$$

□

Now we construct a map $s : G/K \to G$ such that $s(gK) \in gK$ and we define a Borel measurable set $S = s(G/K)$. Then for any $g \in G$, we have a unique form of g, $g = sk^{-1}$ for some $k \in K$; here we have denoted $s(gK)$ by s. Further, we define the measure $d\mu(s) = d\mu(\overline{g})$. Now we construct a restriction of φ such that $\zeta(s) = \varphi(s)$ and $\varphi(g) = \varphi(sk^{-1}) = \pi_0(k)\varphi(s) = \pi_0(k)\zeta(s)$. Thus, $V = L^2(S, d\mu(s); \mathfrak{X})$ and any $\zeta \in V$ is called a *Wigner state*. The action of G on V is $\pi^W(h)\zeta(s) = \pi(h)\varphi(s) = \varphi(h^{-1}s)$.

Definition 5.1.2. We call the representation $h \mapsto \pi(h)$ of V the *representation of G induced by the representation of π_0 of K.*

5.1.2 Inductions from $\mathbf{Q}$ and L_p to EPS(N)

Following [12], we denote the Lorentz group by L and we let π_0 be an irreducible unitary representation of the subsupergroup $(L_p \ltimes \mathbb{R}^{1,3}, \mathfrak{g}_0)$ acting on the complex Hilbert superspace $V_0(p, z, h)$, where $z = (z_k)$ is an eigenvalue of Z^{ij} and h is some label of L_p. We construct the Hilbert superspace $V(p, z, h)$ of $V_0(p, z, h)$-valued L^2 functions on L with the following property: for any $\phi \in V(p, z)$ and any $g \in L$, $\phi(gh) = \pi_0(h^{-1})\phi(g)$.

Then the action of the extended Poincaré supergroup on $V(p, z, h)$ is defined by

$$\begin{aligned}
(h\phi)(g) &= \phi(h^{-1}g) \\
(a\phi)(g) &= \exp(\sqrt{-1}\mathrm{Ad}_g(p)_\eta a^\eta)\phi(g) \\
(Q^i_\alpha\phi)(g) &= \pi_0(\mathrm{Ad}_{g^{-1}}(Q^i_\alpha))\phi(g) \\
Z^{ij}\phi(g) &= z^k\phi(g),
\end{aligned}$$

where $h \in L$, $a \in \mathbb{R}^{1,3}$, and for any $g \in L$. This leads to an induction type of functor from the category of representations of the subsupergroup $(L_p \ltimes \mathbb{R}^{1,3}, \mathfrak{g}_0)$ to the category of representations of EPS(N). We denote this functor by $\mathrm{Ind}_{L_p}^{EPS}$.

For our consideration we will need another induction functor, namely the one from the category of representations of $\mathbf{Q}$ to the category of representations of EPS(N). Let us denote this functor by $\mathrm{Ind}_{\mathbf{Q}}^{EPS}$, or simply by Ind. For detailed study of the two indiction functors $\mathrm{Ind}_{L_p}^{EPS}$ and $\mathrm{Ind}_{\mathbf{Q}}^{EPS}$ we refer the reader to §4 of [5] and §3, §5 of [28].

In particular we have this important property: every simple (unitary) representation of EPS(N) is isomorphic to a quotient of the induced representation Ind(S) of a (unitary) Cliff($\mathbf{Q}$)-representation S. For this reason, for the remainder of this chapter we focus on the simple representations of Cliff($\mathbf{Q}$) and their decompositions as G-representations for some special Lie groups G.

5.2 Restriction to $U(1) \times U(2N)$

In order to prove some of the results we will need the definition of the Euler operator E on the exterior algebra $\bigwedge V$ of the finite-dimensional vector space $V = \mathbb{C}^{2N}$. Fix a basis $\{e_1, ..., e_n\}$ of V, and a dual basis $\{\varphi_1, ..., \varphi_n\}$ of the dual vector space V^*; namely, $\varphi_i(e_j) = \delta_{ij}$. For vectors $v \in V$ and $v^* \in V^*$ introduce the exterior product operator $\epsilon(v)$ and the interior product operator $\iota(v^*)$ on $\bigwedge V$ as follows

$$\begin{aligned} \epsilon(v)(v_1 \wedge ... \wedge v_k) &= v \wedge v_1 \wedge ... \wedge v_k, \\ \iota(v^*)(v_1 \wedge ... \wedge v_k) &= \sum_{i=1}^{k} (-1)^i v^*(v_i) v_1 \wedge ... \wedge \widehat{v_i} \wedge ... \wedge v_k \end{aligned}$$

where $\widehat{v_i}$ denotes the omition of v_i. In what follows we set $\chi = \frac{1}{2}(N\mathrm{Id} - E)$ and call it the helicity operator

Definition 5.2.1. The skew-symmetric Euler operator on $\bigwedge V$ is defined as

$$E = \sum_{i=1}^{n} \epsilon(e_i)\iota(e_i^*).$$

Theorem 5.2.2. The restriction of $\mathbb{S}$ to $U(1) \times U(2N)$ is given by the formula

$$\mathbb{S} \downarrow_{U(1)\times U(2N)} = \bigoplus_{k=0}^{2N} \mathbb{C}_k \boxtimes \bigwedge^k \mathbb{C}^{2N}.$$

Note that the action of the helicity operator χ on $\mathbb{C}_k$ is given by multiplication by $\frac{N-k}{2}$

Proof. From Corollary 5.5.3 in [15] we have

$$\mathbb{S}\downarrow_{GL(2N)}=\bigoplus_{k=0}^{2N}\bigwedge^{k}\mathbb{C}^{2N}.$$

Now we use the fact that the commutant of $GL(n)$ in End($\mathbb{S}$) is the algebra generated by the Euler operator E on $\bigwedge\mathbb{C}^{2N}$. Now using Theorem 2.2.5 we complete the proof. □

5.3 Restriction to $SU(2)\times Sp(2N)$

Recall from §2.4 that $\mathcal{H}^k=\mathcal{H}\left(\bigwedge^k\mathbb{C}^{2N}\right)$ denotes the k-harmonic subspace of $\bigwedge\mathbb{C}^{2N}$. The space $\mathcal{H}^k$ is an irreducible representation of $Sp(2N)$, which is isomorphic to the k-th fundamental representation. Finally, we recall that $F^{(N-k)}$ denotes the finite dimensional $\mathfrak{sl}_{\mathbb{C}}(2)$-representation with highest weight $N-k$ and dimension $N-k+1$. Considering that the representations of $SU(2)$ and $\mathfrak{sl}_{\mathbb{C}}(2)$ are in 1-1 correspondence, by Theorem 2.4.9, we have the following theorem.

Theorem 5.3.1. The restriction of $\mathbb{S}$ to $SU(2)\times Sp(2N)$ is given by the formula

$$\mathbb{S}\downarrow_{SU(2)\times Sp(2N)}=\bigoplus_{k=0}^{N}F^{(N-k)}\boxtimes\mathcal{H}^k.$$

Note that the above theorem provides a rigorous representation theory interpretation of the identity (4.26) from [12] as explained in the next section.

5.4 Restriction to $U(1)\times SU(N)$

We first note that to obtain the restriction of $\mathbb{S}$ to $U(1)\times SU(N)$ is a branching rule problem for $Sp(2N)\downarrow_{U(N)}$. This branching problem in general is difficult, but for the representations we consider, it is relatively easy. To obtain the needed branching rule we will directly look at the branching $U(2N)\downarrow_{U(N)}$. We first note the obvious fact.

Lemma 5.4.1. The standard $U(2N)$-representation, $\mathbb{C}^{2N}$, as a $U(N)\times U(N)$-module decomposes as

$$\mathbb{C}^{2N}\downarrow_{U(N)\times U(N)}=\mathbb{C}^N\boxtimes\mathbb{C}^N.$$

Consider now the embedding $U(N)\to U(N)\times U(N)$ defined by $A\mapsto(A,\overline{A})$. Then, using this embedding, we have $\mathbb{C}^{2N}\downarrow_{U(N)}=\mathbb{C}^N\oplus\overline{\mathbb{C}^N}$.

Theorem 5.4.2. The restriction of $\mathbb{S}$ to $U(1)\times SU(N)$ is given by the formula

$$\mathbb{S}\downarrow_{U(1)\times SU(N)}\cong\bigoplus_{k=0}^{2N}\bigoplus_{p=0}^{k}\mathbb{C}_k\boxtimes\left(\bigwedge^{p}\mathbb{C}^N\otimes\bigwedge^{k-p}\overline{\mathbb{C}^N}\right).$$

Note that $\chi|_{\mathbb{C}_k}=\left(\frac{N-k}{2}\right)\mathrm{Id}$.

Proof. From Theorem 5.2.2 we see that to find $\mathbb{S}\downarrow_{U(1)\times SU(N)}$ it is enough to find $\left(\bigwedge^k\mathbb{C}^{2N}\right)\downarrow_{U(N)}$. But

$$\left(\bigwedge^{k}\mathbb{C}^{2N}\right)\downarrow_{U(N)}=\left(\bigwedge^{k}(\mathbb{C}^N\oplus\overline{\mathbb{C}^N})\right)\downarrow_{U(N)}=\bigoplus_{p=0}^{k}\left(\bigwedge^{p}\mathbb{C}^N\otimes\bigwedge^{k-p}\overline{\mathbb{C}^N}\right).$$

Now combining the last identity with Theorem 5.2.2 we obtain the desired result. □

5.5 Interpretation of [12] in Representation Theoretic Terms

In this section we describe the main results in [12] (and in [29]) in rigorous mathematical language.

5.5.1 Massive Supermultiplets

We start with the equation

$$2^{2N}=(N+1,[2N]_0)+(N,[2N]_1)+\cdots+(N+1-k,[2N]_k)+\cdots+(1,[2N]_N),$$

see (4.26), which classifies the intrinsic states of given intrinsic $SU(2)$ J-spin. The left hand side of the identity is the unique, up to isomorphism, simple module $\mathbb{S}$ of

Cliff($\mathbf{Q}$), and the right hand side coincides with the right hand side of the isomorphism in Theorem 5.3.1. In other words, $(N+1-k, [2N]_k)$ corresponds to the outer tensor product $F^{(N-k)} \boxtimes \mathcal{H}^k$. Hence, the above equation corresponds to the canonical decomposition of $\bigwedge \mathbb{C}^{2N}$ as a $(G, \mathrm{Comm}(G))$-module, where $G = GL(2N)$ and $\mathrm{Comm}(G)$ is the commutant of G in $\mathrm{End}(\bigwedge \mathbb{C}^{2N})$.

5.5.2 Massless Supermultiplets

We now look at the equation

$$2^{2N} = \left\{\frac{N}{2}, [N]_0\right\} + \left\{\frac{N-1}{2}, [N]_1\right\} + \ldots \left\{\frac{N-k}{2}, [N]_k\right\} + \cdots + \{0, [N]_N\},$$

where the braces denote a massless supermultiplet specified by a state of maximal spin projection, λ_{max}, belonging to the antisymmetric representation $[N]_k$ of $SU(N)$, namely,

$$\begin{aligned}\{\lambda_{max}, [N]_k\} = \left(\lambda_{max}, [N]_k \otimes [\bar{N}]_0\right) + \left(\lambda_{max} - \frac{1}{2}, [N]_k \otimes [\bar{N}]_1\right) + \cdots + \\ + \left(\lambda_{max} - \frac{\ell}{2}, [N]_k \otimes [\bar{N}]_\ell\right) + \cdots + \left(\lambda_{max} - \frac{N}{2}, [N]_k \otimes [\bar{N}]_N\right).\end{aligned}$$

The J-spin projection $\frac{N-k}{2}$ is given by the action of $\frac{1}{2}(N\mathrm{Id} - E)$, where E is the Euler operator, see Definition 2.4.3 and the discussion thereafter. The second component of $\left\{\frac{N-k}{2}, [N]_k\right\}$ corresponds to the representation $\bigwedge^k \mathbb{C}^{2N}$ of $U(N)$ (equivalently, of $SU(N)$). By Theorem 5.4.2, we can write the following physics-mathematics correspondences:

$$\begin{aligned}\left\{\frac{N-k}{2}, [N]_k\right\} \quad &\leftrightarrow \quad \bigoplus_{i=k}^{N+k} \mathbb{C}_i \boxtimes \left(\bigwedge^k \mathbb{C}^N \otimes \bigwedge^{i-k} \overline{\mathbb{C}^N}\right), \\ \left(\frac{N-k}{2} - \frac{\ell}{2}, [N]_k \otimes [\bar{N}]_\ell\right) \quad &\leftrightarrow \quad \mathbb{C}_{k+\ell} \boxtimes \left(\bigwedge^k \mathbb{C}^N \otimes \bigwedge^\ell \overline{\mathbb{C}^N}\right).\end{aligned}$$

5.5.3 Examples

Example 5.5.1. Let $N = 4$ and $q = 2$ (i.e. $0 < z_1 = z_2 = M$). Then $G = SU(2) \times Sp(4)$. By the above, $\mathbb{S}$ has dimension 2^8 and

$$\mathbb{S} \downarrow_{SU(4)} = 2^4 \cdot 2^4 = 2^4 \left([4]_0 + \cdots + [4]_4\right).$$

$$2^4 \downarrow_{SU(2)\times Sp(4)} = (3, [4]_0) + (2, [4]_1) + (1, [4]_2) \tag{5.1}$$

$$= (3, 1) + (2, 4) + (1, 5) \tag{5.2}$$

Example 5.5.2. Let $N = 4$ and $q = 1$ (i.e. $0 < z_2 < z_1 = M$). Then $G = Sp(2) \times Sp(2) \cong SU(2) \times SU(2)$. Considering the branching $U(4) \downarrow_{SU(2)\times SU(2)}$, we get

$$[4]_0 \downarrow_{SU(2)\times SU(2)} = (1, 1) \tag{5.3}$$

$$[4]_1 \downarrow_{SU(2)\times SU(2)} = (2, 1) + (1, 2) \tag{5.4}$$

$$[4]_2 \downarrow_{SU(2)\times SU(2)} = 2(1, 1) + (2, 2). \tag{5.5}$$

So, $2^4 = (3, 1, 1) + (2, 2, 1) + (2, 1, 2) + (1, 1, 1) + (1, 2, 2)$.

Below we provide a table which organizes the combinations of N and q which have representations with $J_{\max}$ up to 2:

Dim	$J_{\max}$	N	q	N	q	N	q	N	q	N	q	N	q	N	q	N	q
2^{16}	4															8	0
2^{14}	7/2													7	0	8	1
2^{12}	3											6	0	7	1	8	2
2^{10}	5/2									5	0	6	1	7	2	8	3
2^{8}	2							4	0	5	1	6	2	7	3	8	4
2^{6}	3/2					3	0	4	1	5	2	6	3				
2^{4}	1			2	0	3	1	4	2								
2^{2}	1/2	1	0	2	1												

.

Appendix A

Finite-Dimensional Irreducible Representations of $\mathfrak{sl}_{\mathbb{C}}(2)$

In this appendix, we list some standard facts from the representation theory of $\mathfrak{g} = \mathfrak{sl}_{\mathbb{C}}(2)$, and we fix the base field to be $\mathbb{C}$. For the reader's convenice some proofs are included.

A.1 Explicit Construction

The matrices

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \text{ and } h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

form a basis for $\mathfrak{g}$ and satisfy the commutation relations

$$[h, x] = 2x, \ [h, y] = -2y, \text{ and } [x, y] = h.$$

Let $\mathfrak{h}$ be the subalgebra of diagonal matrices of $\mathfrak{g}$, i.e. $\mathfrak{h} = \mathbb{C}h$. Let $\alpha \in \mathfrak{h}^*$ be defined by $\alpha(h) = 2$. We identify $\mathfrak{h}^*$ with $\mathbb{C}$ through the map $c\alpha \mapsto 2c$.

Lemma A.1.1. Let V be a $\mathfrak{g}$-module and let $v_0 \in V$ be such that $xv_0 = 0$ and $hv_0 = \lambda v_0$ for some $\lambda \in \mathbb{C}$. Set $v_j = y^j v_0$ for $j \in \mathbb{Z}_{>0}$ and $v_j = 0$ for $j < 0$. Then $yv_j = v_{j+1}$, $hv_j = (\lambda - 2j)v_j$, and $xv_j = j(\lambda - j + 1)v_{j-1}$ for $j \in \mathbb{Z}_{>0}$.

Let V be a finite-dimensional $\mathfrak{g}$-module. Then we decompose V into generalized eigenspaces for the action of h:

$$V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda,$$

where $V_\lambda = \bigcup_{k \geq 1} \operatorname{Ker}(h - \lambda Id)^k|_V$ is the λ-weight space of V.

Definition A.1.2. If $V_\lambda \neq 0$, then λ is called a *weight* of V with *weight space* V_λ.

In particular, $v \in V_\lambda$ then $(h-\lambda)^k v = 0$ for some $k \geq 1$. As linear transformations on V,

$$x(h-\lambda) = (h-\lambda-2)x \text{ and } y(h-\lambda) = (h-\lambda+2)x.$$

Hence, $(h-\lambda-2)^k xv = x(h-\lambda)^k v = 0$ and $(h-\lambda+2)^k yv = y(h-\lambda)^k v = 0$. Thus,

$$xV_\lambda \subset V_{\lambda+2} \text{ and } yV_\lambda \subset V_{\lambda-2}$$

for all $\lambda \in \mathbb{C}$.

Lemma A.1.3. Suppose V is a finite-dimensional $\mathfrak{g}$-module and $0 \neq v_0 \in V$ satisfies $hv_0 = \lambda v_0$ and $xv_0 = 0$. Let k be the smallest non-negative integer such that $y^k v_0 \neq 0$ and $y^{k+1} v_0 = 0$. Then $\lambda = k$ and the space $W = \mathrm{Span}_{\mathbb{C}}\{v_0,\ yv_0, \ldots,\ y^k v_0\}$ is a $(k+1)$-dimensional $\mathfrak{g}$-module.

We can provide a specific action of $\mathfrak{g}$ on the subspace W, from the previous lemma, in matrix form as follows. For $k \in \mathbb{Z}_{>0}$, we define the $(k+1) \times (k+1)$ matrices

$$X_k = \begin{bmatrix} 0 & k & 0 & 0 & \ldots & 0 \\ 0 & 0 & 2(k-1) & 0 & \ldots & 0 \\ 0 & 0 & 0 & 3(k-2) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & k \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix},\ Y_k = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}, \text{ and}$$

$H_k = \mathrm{Diag}[k, k-2, \ldots, 2-k, -k]$, which satisfy

$$[H_k, X_k] = 2X_k,\ [H_k, Y_k] = -2Y_k,\ \text{and } [X_k, Y_k] = H_k.$$

Proposition A.1.4. Let $k \geq 0$ be an integer. Then the representation $(\rho_k, F^{(k)})$ of $\mathfrak{g}$ on $\mathbb{C}^{k+1}$ defined by

$$\rho_k(x) = X_k,\ \rho_k(h) = H_k,\ \text{and } \rho_k(y) = Y_k$$

is irreducible.

Corollary A.1.5. The weights of a finite-dimensional $\mathfrak{g}$-module V are integers.

Appendix B

Tensor Analysis

In this chapter we mainly follow [2]. Let the ground field be $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

B.1 Tensor Spaces

Definition B.1.1. Let V and W be vector spaces and $f : V \to W$. Then we call f a *homomorphism*, if for all $v_1, v_2 \in V$ and $a \in \mathbb{F}$,

- $f(v_1 + v_2) = f(v_1) + f(v_2)$ and
- $f(av_1) = af(v_1)$.

Definition B.1.2. The set of homomorphisms from $V \to W$ forms a vector space, which we denote by $L(V, W)$.

Remark B.1.3. The sum of homomorphisms is $(f + g)(v) = f(v) + g(v)$, and the scalar product of $a \in \mathbb{F}$ and $f \in L(V, W)$ is $(af)(v) = a(f(v))$ for any $v \in V$.

Definition B.1.4. We call $L(V, \mathbb{R})$ the *dual space of* V, and we denote it by V^*.

Remark B.1.5. For each basis $\{e_i\}$ of V there is a unique basis $\{\epsilon^i\}$ of V^* such that $\epsilon^i e_j = \delta^i_j$, where

$$\delta^i_j = \begin{cases} 1 & , \ i = j \\ 0 & , \ \text{otherwise} \end{cases} .$$

.

Definition B.1.6. The linear functionals $\epsilon^i : V \to \mathbb{R}$ are called the *dual basis* to the basis $\{e_i\}$.

Remark B.1.7. When $\dim V < \infty$, $(V^*)^* \cong V$ as vector spaces.

Definition B.1.8. If $\{e_i \mid i = 1, \ldots, d\}$ is a basis of V $(v = \sum_{i=1}^{d} a^i e_i)$ and $\{\epsilon^i \mid i = 1, \ldots, d\}$ is the corresponding dual basis $(\tau = \sum_{i=1}^{d} b_i \epsilon^i)$, then the *natural pairing* on V is

$$\begin{aligned}\langle v, \tau \rangle &= \sum_{i=1}^{d} b_i \epsilon^i \left(\sum_{j=1}^{d} a^j e_j \right) \\ &= \sum_{i=1}^{d} \sum_{j=1}^{d} b_i a^j (\epsilon^i e_j) \\ &= \sum_{i=1}^{d} \sum_{j=1}^{d} b_i a^j \delta^i_j \\ &= \sum_{i=1}^{d} b_i a^i.\end{aligned}$$

Definition B.1.9. Let U and V be vector spaces. Then the *tensor product* of U and V is a vector space $U \otimes V$ together with a bilinear map from $U \times V \to U \otimes V$ such that $\tau : (u, v) \mapsto u \otimes v$ and satisfying the *universal mapping property*:

Given any vector space W and bilinear map $\beta : U \times V \to W$, there exists a unique linear map $B : U \otimes V \to W$ such that $\beta = B \circ \tau$

$$\begin{array}{ccc} U \times V & \xrightarrow{\tau} & U \otimes V \\ & \searrow^{\beta} & \downarrow{\scriptstyle B} \\ & & W \end{array} \quad . \tag{B.1}$$

Remark B.1.10. Let U, V, and W be three vector spaces and let $\tau : (U \otimes V) \times W \to U \otimes (V \otimes W)$ such that $\tau(u \otimes v, w) = u \otimes (v \otimes w)$. Then the the universal mapping property is satisfied. Further, $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$. In this way, we can iteratively take tensor products of vector spaces.

Definition B.1.11. Let $V^{\otimes k}$ be the k-fold tensor product of V with itself, for $k = 0, 1, \ldots,$ where $V^{\otimes 0} = \mathbb{F}$. Then we define the *tensor algebra* as

$$\mathcal{T}(V) = \bigoplus_{k \geq 0} V^{\otimes k}.$$

Multiplication respects the grading: $V^{\otimes k} \times V^{\otimes m} \to V^{\otimes(k+m)}$ such that $(x_1 \otimes \cdots \otimes x_k, y_1 \otimes \cdots \otimes y_m) \mapsto x_1 \otimes \cdots \otimes x_k \otimes y_1 \otimes \cdots \otimes y_m$ for $x_i, y_j \in V$.

B.2 Homomorphisms on Tensor Spaces

Definition B.2.1.

- We call the real-valued multilinear functions on $V_1^* \times \cdots \times V_n^* \times V_{n+1} \times V_{n+\ell}$ *tensors over* $V_1 \otimes \cdots \otimes V_n \otimes V_{n+1}^* \otimes V_{n+\ell}^*$, and we call the vector spaces they form *tensor spaces*.
- The number n, in the space given above, is called the *contravariant degree* of the tensor space, and
- The number ℓ, in the space given above, is called the *covariant degree*.
- We call the *degree* of a tensor the ordered pair with content contravariant degree then covariant degree.

Example B.2.2. The degree of the example above is (n, ℓ).

Definition B.2.3.

- By convention, we consider the trivial case of a tensor with degree $(0, 0)$ to be a scalar, $\mathcal{T}_0^0 = \mathbb{F}$.
- Two more trival cases are tensors with degree $(1, 0)$ or $(0, 1)$. We call a tensor with degree $(1, 0)$ a *contravariant vector*, and we call a tensor of degree $(0, 1)$ a *covariant vector*.
- We call a tensor with degree $(r, 0)$ a *contravariant tensor*, and we call a tensor with degree $(0, s)$ a *covariant tensor*, for $r, s > 1$.

Remark B.2.4. Note that $\mathcal{T}(V \oplus V^*) = \bigoplus_{k,\ell} \mathcal{T}_k^\ell$.

Remark B.2.5. For $\dim_{\mathbb{F}}(V) < \infty$, $A \in \mathcal{T}_1^1$, and for a fixed $\tau \in V^*$, $A(\tau, v)$ is a linear function of $v \in V$, so we denote by $B : V^* \to V^*$ the function defined by $A(\tau, v) = \langle v, B\tau \rangle$.

Hence, for each tensor of type $\mathcal{T}_1^1$ we get an epimorphism on V^*. Conversely, if B is an epimorphism, then we define a tensor $A \in \mathcal{T}_1^1$ by $A(\tau, v) = \langle v, B\tau \rangle$.

Remark B.2.6. For $A \in \mathcal{T}_1^1$, the action of B on V^* (or V with $\dim(V) = \dim(V^*) = d$) can be viewed as a *partial evaluation*: for $\tau = \sum_{i=1}^d b_i \epsilon^i \in V^*$,

$$\begin{aligned}
B\tau &= \left(\sum_{i=1}^d \sum_{j=1}^d B_j^i e_i \otimes \epsilon^j \right) \tau \\
&= \sum_{i=1}^d \sum_{j=1}^d \sum_{k=1}^d B_j^i b_k (e_i \otimes \epsilon^j) \epsilon^k \\
&= \sum_{i=1}^d \sum_{j=1}^d \sum_{k=1}^d B_j^i b_k \epsilon^j \langle e_i, \epsilon^k \rangle \\
&= \sum_{i=1}^d \sum_{j=1}^d \sum_{k=1}^d B_j^i b_k \epsilon^j \delta_i^k \\
&= \sum_{i=1}^d \sum_{j=1}^d B_j^i b_i \epsilon^j
\end{aligned}$$

Example B.2.7. Let V and W be vector spaces with basis $\{e_i \mid i = 1, \ldots, D\}$ and $\{E_i \mid i = 1, \ldots, d\}$, respectively. Then for $v \in V$ and $w \in W$ the tensor product of two vectors, $v \otimes w$, can be represented by a matrix. We begin with the general form of the tensor product of two vectors.

$$\begin{aligned}
v \otimes w &= \left(\sum_{i=1}^D a_i e_i \right) \otimes \left(\sum_{j=1}^d b_j E_j \right) \\
&= \sum_{i=1}^D a_i e_i \otimes \sum_{j=1}^d b_j E_j \\
&= \sum_{i=1}^D \sum_{j=1}^d a_i b_j (e_i \otimes E_j) \\
&= \sum_{i=1}^D a_i \sum_{j=1}^d b_j (e_i \otimes E_j)
\end{aligned}$$

At this point we must realize the size of our matrix. Our column vectors above have size $D \times 1$ and $d \times 1$, respectively. Recall that the matrix product is defined for $(1 \times d) \times (D \times 1) = 1 \times 1$ or $(1 \times D) \times (d \times 1) = 1 \times 1$ if, and only if, $D = d$.

However, the tensor product of our two vectors can be defined as $(d \times 1) \times (1 \times D) = d \times D$ or as $(D \times 1) \times (1 \times d) = D \times d$ – with no restrictions on D or d.

Specifically, we consider $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$ with $v = e_1 + 2e_2 \in V$ and $w = 3E_1 + 2E_2 + E_3$. Then choosing the 2×3 matrix, we let $e_i \otimes E_j$ represent an elementary matrix which has its non-zero entry in the $(i,j)^{\text{th}}$ position. Then we have:

$$
\begin{aligned}
v \otimes w &= (e_1 + 2e_2) \otimes (3E_1 + 2E_2 + E_3) \\
&= e_1 \otimes (3E_1 + 2E_2 + E_3) + 2e_2 \otimes (3E_1 + 2E_2 + E_3) \\
&= e_1 \otimes 3E_1 + e_1 \otimes 2E_2 + e_1 \otimes E_3 + 2e_2 \otimes 3E_1 + 2e_2 \otimes 2E_2 + 2e_2 \otimes E_3 \\
&= 3(e_1 \otimes E_1) + 2(e_1 \otimes E_2) + (e_1 \otimes E_3) + 6(e_2 \otimes E_1) + 4(e_2 \otimes E_2) + 2(e_2 \otimes E_3) \\
&= 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
&+ 6\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \end{pmatrix}.
\end{aligned}
$$

In summary, $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \end{pmatrix}$.

We may extend this concept to the tensor product of more than two vectors. Then we should be careful about which two vectors spaces will form our matrix and which vectors are not explicitely being used, say, in a partial evaluation. To this end, we make clever use of index notation in practice.

Definition B.2.8. If T is a tangent space at a point q on a d-dimensional manifold and if the bases are obtained as coordinate vector fields with respect to two systems of coordinates (x^i) and (y^i) at q, then $e_i = \frac{\partial}{\partial x^i}$, $\epsilon^i = dx^i$, $f_i = \frac{\partial}{\partial y^i}$, $\phi^i = dy^i$, $a_i^j = \frac{\partial x^j}{\partial y^i}$, and $b_j^i = \frac{\partial y^i}{\partial x^j}$ all evaluated at q, for $i, j \in \{1, \ldots, d\}$. Further, $A_{jk}^{y,i} = \sum_{i=1}^d \sum_{j=1}^d \sum_{k=1}^d A_{np}^{x,q} \frac{\partial y^i}{\partial x^q} \frac{\partial x^n}{\partial y^j} \frac{\partial x^p}{\partial y^k}$.

Definition B.2.9. We say a tensor is *symmetric in the* p^{th} *and* q^{th} *contravariant indices*, if the components with respect to every basis are unchanged when these indices are interchanged.

Definition B.2.10. We say a tensor is *symmetric in the* p^{th} *and* q^{th} *variables* (of the same type), if its values as a multilinear function are unchanged when these variables are interchanged.

Definition B.2.11. We call a tensor *contravariant (covariant) symmetric*, if it is symmetric in every pair of contravariant (covariant) indices.

Remark B.2.12. By convention, we take tensors of degree $(0, 0)$, $(0, 1)$, or $(1, 0)$ to be symmetric.

Definition B.2.13. We call a tensor *skew symmetric in the* p^{th} *and* q^{th} *contravariant indices*, if the components of the tensor, with respect to every basis, are changed in sign when these indices are interchanged.

Definition B.2.14. We call a tensor *skew symmetric in the* p^{th} *and* q^{th} *variables* (of the same type), if $A = 0$, as a multilinear function, when these variables are the same (regardless of basis).

Remark B.2.15. An example of a skew symmetric tensor (over three indeces) is the *Levi-Civita tensor*:

$$\epsilon^{ijk} = \begin{cases} 1 & , \ ijk \text{ is an even permutation of } 1,2,3 \\ 0 & , \ i=j \text{ or } j=k \text{ or } i=k \\ -1 & , \ ijk \text{ is an odd permutation of } 1,2,3 \end{cases}.$$

Definition B.2.16. The multiplication of skew symmetric tensors is called the *exterior product*, and the resulting algebra is called the *geometric exterior algebra.*

This product is denoted by $\wedge$.

This symbol is also used to denote the space of skew symmetric tensors of type $(r, 0)$, $\bigwedge^r V$ or the tensor space of type $(0, s)$ which is denoted by $\bigwedge^s V^*$.

Theorem B.2.17. The dimension of $\bigwedge^r V$ is $\binom{d}{r}$, where $d = \dim(V)$.

Theorem B.2.18 (Cartan)**.** Let $\{e_i\}$, for $i = 1, \dots, d$, be a basis of V, and let $v_i \in V$, for $i = 1, \dots, p$, such that $\sum_{i=1}^{p} e_i \wedge v_i = 0$. Then there exists scalars, A_{ij}, such that $v_i = \sum_{j=1}^{p} A_{ij} e_j$ and $A_{ij} = A_{ji}$.

Definition B.2.19. A tensor $A \in \bigwedge^p V$ is called *decomposable* if there exist $v_1, \dots, v_p \in V$ such that $A = v_1 \wedge \cdots \wedge v_p$.

Otherwise, we call A *indecomposable.*

Remark B.2.20. If $A \in \bigwedge^2 V$, then A is decomposable if, and only if, $A \wedge A = 0$, or equivalently, for all i, j_1, j_2, and j_3,

$$A^{ij_1} A^{j_2 j_3} + A^{ij_3} A^{j_1 j_2} = A^{ij_2} A^{j_1 j_3}.$$

Appendix C

Representations of Associative Algebras

In this chapter, we take the ground field to be $\mathbb{C}$, and we let $\mathcal{A}$ and $\mathcal{B}$ be an associative algebra over $\mathbb{C}$ with unity $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$, respecively. Then a family, $\mathcal{F}$, of linear subspaces $\mathbb{C}I_{\mathcal{A}} = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n \subset \ldots$ such that

$$\mathcal{A}_j \cdot \mathcal{A}_k \subset \mathcal{A}_{j+k}, \text{ and } \bigcup_{j \geq 0} \mathcal{A}_j = \mathcal{A}$$

is called a *filtration* on $\mathcal{A}$, and we say $\mathcal{B}$ is *graded* when $\mathcal{B}$ has a family, $\mathcal{G}$, of subspaces $\mathbb{C}1 = \mathcal{B}^0, \mathcal{B}^1, \ldots, \mathcal{B}^n, \ldots$ such that

$$\mathcal{B}^j \cdot \mathcal{B}^k \subset \mathcal{B}^{j+k} \text{ and } \bigoplus_{j \geq 0} \mathcal{B}^j = \mathcal{B}.$$

C.1 Tensor Algebra

Let V be a vector space over $\mathbb{C}$. Then the tensor algebra generated by V, $\mathcal{T}(V)$, has the following universal mapping property:

Suppose $\mathcal{A}$ is any associatgive algebra over $\mathbb{C}$ and $\varphi : V \to \mathcal{A}$ is any homomorphism of vector spaces. Then φ extends uniquely to a homomorphism $\tilde{\varphi} : \mathcal{T}(V) \to \mathcal{A}$ by the formula

$$\tilde{\varphi}(x_1 \otimes \cdots \otimes x_k) = \varphi(x_1) \ldots \varphi(x_k)$$

for $x_i \in V$. Since $\mathcal{A}$ is associative, $\tilde{\varphi}$ is an algebra homomorphism. Hence,

$$\begin{array}{ccc} V & \xrightarrow{\iota} & \mathcal{T}(V) \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \tilde{\varphi}} \\ & & \mathcal{A} \end{array} , \tag{C.1}$$

where ι is the natural inclusion map.

Definition C.1.1. The spaces $\{V^{\otimes k}\}_{k\geq 0}$ define the *standard grading.*

C.2 Symmetric Algebra

For any associative algebra $\mathcal{A}$, the symmetric algebra, $\mathcal{S}(V)$, is universal for homomorphisms $\varphi : V \to \mathcal{A}$ that satisfy

$$\varphi(x)\varphi(y) = \varphi(y)\varphi(x),$$

where $x, y \in V$. Hence, given any linear map $\varphi : V \to \mathcal{A}$ as above, there is a unique algebra homomorphism $\hat{\varphi} : \mathcal{S}(V) \to \mathcal{A}$ such that we get the commuting diagram

$$\begin{array}{ccc} V & \xrightarrow{\gamma} & \mathcal{S}(V) \\ & \searrow^{\varphi} & \downarrow{\scriptstyle\hat{\varphi}} \\ & & \mathcal{A} \end{array}, \tag{C.2}$$

Specifically, we construct $\mathcal{S}(V)$ as the quotient of $\mathcal{T}(V)$ modulo the two-sided ideal $\mathcal{I} = \langle x \otimes y - y \otimes x \rangle$ for $x, y \in V$

$$\mathcal{S}(V) = \frac{\mathcal{T}(V)}{\mathcal{I}}$$

with $\gamma : \mathcal{T}(V) \to \mathcal{T}(V)/\mathcal{I}$ being the quotient map.

C.3 Exterior Algebra

The *exterior algebra*, $\bigwedge V$, for a vector space V is the associative algebra generated by V that is universal relative to a homomorphism ψ from V to an associative algebras $\mathcal{A}$ such that

$$\psi(x)\psi(y) = -\psi(y)\psi(x),$$

where $x, y \in V$.

Given any homomorphism $\psi : V \to \mathcal{A}$ as above, there exists a unique algebra homomorphism $\check{\psi} : S(V) \to \mathcal{A}$ such that we have the commuting diagram

$$\begin{array}{ccc} V & \xrightarrow{\ \delta\ } & \bigwedge(V) \\ & \overset{\psi}{\searrow} & \Big\downarrow \check{\psi} \\ & & \mathcal{A} \end{array} \quad . \tag{C.3}$$

Specifically, we construct $\bigwedge V$ as the quotient of $\mathcal{T}(V)$ modulo the two sided ideal $\mathcal{J} = \langle x \otimes y + y \otimes x \rangle$, for $x, y \in V$ and δ the canonical quotient map:

$$\bigwedge V = \frac{\mathcal{T}(V)}{\mathcal{J}}.$$

C.4 Enveloping Algebra

Definition C.4.1. If $\mathcal{A}$ is an associative algebra, then, using the multiplication in $\mathcal{A}$, we define the *commutator* $[x, y] = xy - yx$.

Remark C.4.2.

- This product is anti-symmetric.
- Since A is associative, the commutator satisfies the Jacobi identity.

Definition C.4.3. We denote by $\mathcal{A}_{\text{Lie}}$ the Lie algebra with commutator whose underlying vector space is $\mathcal{A}$.

Remark C.4.4. Let $\mathcal{A}$ and $\mathcal{B}$ be associative algebras over $\mathbb{C}$ and let $\varphi : \mathcal{A} \to \mathcal{B}$ be an associative algebra homomorphism. Then

$$[\varphi(x), \varphi(y)] = \varphi([x, y])$$

for $x, y \in \mathcal{A}$. Thus, $\varphi : \mathcal{A}_{\text{Lie}} \to \mathcal{B}_{Lie}$ is a Lie algebra homomorphism.

Let $\varphi : \mathfrak{g} \to \mathfrak{G}$ be Lie algebra homomorphism whose image generates $\mathfrak{G}$ as an associative algebra with unity over $\mathbb{C}$. Then we define the *universal enveloping algebra* of $\mathfrak{g}$ to be the pair $(\mathfrak{G}, \varphi)$ satisfying the following *universal mapping property*:

Given any associative algebra $\mathcal{A}$ over $\mathbb{C}$ and a Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathcal{A}_{\text{Lie}}$, there exists an associative algebra homomorphism $\Psi : \mathfrak{G} \to \mathcal{A}$ such that

$$\psi(x) = \Psi(\varphi(x))$$

for $x \in \mathfrak{g}$

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi} & \mathfrak{G} \\ & \searrow^{\psi} & \downarrow{\scriptstyle \Psi} \\ & & \mathcal{A} \end{array} \quad . \tag{C.4}$$

Remark C.4.5.
- The universal enveloping algebra exists and is unique.
- We denote the universal enveloping algebra on $\mathfrak{g}$ by $\mathcal{U}(\mathfrak{g})$.

C.5 Representations of Associative Algebras

Until the end of this appendix we fix $\mathcal{A}$ to be an associative algebra over $\mathbb{C}$ with unity.

Definition C.5.1. A *representation of* $\mathcal{A}$ is a pair (ρ, V), where V is a vector space over $\mathbb{C}$ and $\rho : \mathcal{A} \to \text{End}(V)$ is an associative algebra homomorphism.

If V, W are both $\mathcal{A}$-modules, then we make the vector space $V \oplus W$ into an $\mathcal{A}$-module by the action $a \cdot (v + w) = av + aw$.

Definition C.5.2. If $U \subset V$ is a linear subspace such that $\rho(a)U \subset U$ for all $a \in \mathcal{A}$, then we say that U is *invariant* under the representation.

In this case, we define the representations (ρ_U, U) and $(\rho_{V/U}, V/U)$ by the restriction of $\rho(\mathcal{A})$ to U and by the natural quotient action of $\rho(\mathcal{A})$ on V/U, respectively.

Definition C.5.3. A representation (ρ, V) is *irreducible* if the only invariant subspaces are $\{0\}$ and V.

Definition C.5.4. Let (ρ, V) and (τ, W) be representations of $\mathcal{A}$. Then we denote by $\text{Hom}_{\mathcal{A}}(V, W)$ the set of all $T \in \text{Hom}(V, W)$ such that $T\rho(a) = \tau(a)T$ for all $a \in \mathcal{A}$.

Such a map is called a *module homomorphism* or an *intertwining operator* between the two representations.

Lemma C.5.5 (Schur)**.** Let (ρ, V) and (τ, W) be irreducible representations of an associative algebra $\mathcal{A}$. Assume that V and W have countable dimension over $\mathbb{C}$. Then

$$\dim \mathrm{Hom}_{\mathcal{A}}(V, W) = \begin{cases} 1, & (\rho, V) \cong (\tau, W) \\ 0, & \text{otherwise} \end{cases}.$$

Definition C.5.6. A finite dimensional $\mathcal{A}$-module V is *completely reducible* if for every $\mathcal{A}$-invariant subspace $W \subset V$ there exists a complementary invariant subspace $U \subset V$ such that $V = W \oplus U$.

Lemma C.5.7. Let (ρ, V) be completely reducible and suppose $W \subset V$ is an invariant subspace. Set $\sigma(x) = \rho(x) \mid_W$ and $\pi(x)(v + W) = \rho(x)v + W$ for $x \in \mathcal{A}$ and $v \in V$. Then the representations (σ, W) and $(\pi, V/W)$ are completely reducible.

Proposition C.5.8. Let (ρ, V) be a finite dimensional representation of the associative algebra $\mathcal{A}$. The following are equivalent:

- (ρ, V) is completely reducible.
- $V = W_1 \oplus \cdots \oplus W_s$ with each W_i an irreducible $\mathcal{A}$-module.
- $V = V_1 + \cdots + V_d$ as a vector space, where each V_i is an irreducible $\mathcal{A}$-submodule.

Further, if V satisfies these conditions and if all the V_i in the last statement are equivalent to a single irreducible $\mathcal{A}$-module W, then every $\mathcal{A}$-submodule of V is isomorphic to a direct sum of copies of W.

Corollary C.5.9. Suppose (ρ, V) and (σ, W) are completely reducible representations of $\mathcal{A}$. Then $(\rho \oplus \sigma, V \oplus W)$ is a completely reducible representation.

Definition C.5.10. Let $[\mathcal{A}]$ denote the set of all equivalence classes of irreducible $\mathcal{A}$-modules. On the other hand, let $\widehat{\mathcal{A}}$ denote the set of all equivalence classes of finite-dimensional irreducible $\mathcal{A}$-modules.

Definition C.5.11. Suppose that V is an $\mathcal{A}$-module. For each $\lambda \in [\mathcal{A}]$ we define the λ-*isotypic* subspace

$$V(\lambda) = \sum_{U \subset V, \widehat{U} = \lambda} U.$$

Fix a module F^λ in the class λ for each $\lambda \in \widehat{\mathcal{A}}$. Then there is a tautological homomorphism $S_\lambda : \mathrm{Hom}_{\mathcal{A}}(F^\lambda, V) \otimes F^\lambda \to V$, $S_\lambda(u \otimes w) = u(w)$. Make $\mathrm{Hom}_{\mathcal{A}}(F^\lambda, V) \otimes F^\lambda$ into an $\mathcal{A}$-module with action $x \cdot (u \otimes w) = u \otimes (xw)$ for $x \in \mathcal{A}$. Then S_λ is an $\mathcal{A}$-intertwining map. If $0 \neq u \in \mathrm{Hom}_{\mathcal{A}}(F^\lambda, V)$, then Schur's Lemma implies that $u(F^\lambda)$ is an irreducible $\mathcal{A}$-submodule of V isomorphic to F^λ. Hence,

$$S_\lambda(\mathrm{Hom}_{\mathcal{A}}(F^\lambda, V) \otimes F^\lambda) \subset V_{(\lambda)}$$

for every $\lambda \in \widehat{\mathcal{A}}$.

Appendix D

Spin Representations

Let V be a finite dimensional complex vector space with a symmetric bilinear (not necessarily non-degenerate) form β.

D.1 Clifford Algebras

For detailed information on Clifford algebras and groups, we refer the reader to [14].

Definition D.1.1. A *Clifford algebra* for (V, β) is an associative algebra, $\mathrm{Cliff}(V, \beta)$, with unity, I_V, over $\mathbb{C}$ and a homomorphism $\gamma : V \to \mathrm{Cliff}(V, \beta)$ satisfying the following properties:

- $\{\gamma(x), \gamma(y)\} = \beta(x, y) I_V$ for $x, y \in V$, where $\{a, b\} = ab + ba$ is the anticommutator of a, b.
- $\gamma(V)$ generates $\mathrm{Cliff}(V, \beta)$ as an algebra.
- Given any complex associative algebra $\mathcal{A}$ with unity $I_{\mathcal{A}}$ and a homomorphism

$$\varphi : V \to \mathcal{A} \text{ such that } \{\varphi(x), \varphi(y)\} = \beta(x, y) I_{\mathcal{A}}, \tag{D.1}$$

there exists an associative algebra homomorphism $\tilde{\varphi} : \mathrm{Cliff}(V, \beta) \to \mathcal{A}$ such that $\varphi = \tilde{\varphi} \circ \gamma$:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & \mathcal{A} \\ & \searrow^{\gamma} & \uparrow{\scriptstyle \tilde{\varphi}} \\ & & \mathrm{Cliff}(V, \beta) \end{array} \quad . \tag{D.2}$$

To show the existence of a Clifford algebra, we begin with the tensor algebra $\mathcal{T}(V)$ and let $\mathcal{J}(V, \beta)$ be the two sided ideal of $\mathcal{T}(V)$ given by $\langle x \otimes y + y \otimes x - \beta(x, y) I_V \rangle$

for $x, y \in V$. Then define $\mathrm{Cliff}(V, \beta) = \frac{\mathcal{T}(V)}{\mathcal{J}(V,\beta)}$ and let $\gamma : V \to \mathrm{Cliff}(V, \beta)$ be the natural quotient map coming from the embedding $V \hookrightarrow \mathcal{T}(V)$.

Definition D.1.2. Let $\mathrm{Cliff}_k(V, \beta)$ be the span of 1 and the operators $\gamma(a_1) \dots \gamma(a_p)$ for $a_i \in V$ and $p \leq k$.

The subspaces $\mathrm{Cliff}_k(V, \beta)$, for $k = 0, 1, \dots,$ provides a natural filtration of the Clifford algebra:

$$\mathrm{Cliff}_k(V, \beta) \cdot \mathrm{Cliff}_m(V, \beta) \subset \mathrm{Cliff}_{k+m}(V, \beta).$$

Let $\{v_i \mid i = 1, \dots, n\}$ be a basis for V. Since $\{\gamma(v_i), \gamma(v_j)\} = \beta(v_i, v_j)I_V$, we get that $\mathrm{Cliff}_k(V, \beta)$ is tt spanned by 1 and the products $\gamma(v_{i_1}) \dots \gamma(v_{i_p})$, where $1 \leq i_1 < i_2 < \dots < i_p \leq n$ and $p \leq k$. Further, $\mathrm{Cliff}(V, \beta) = \mathrm{Cliff}_n(V, \beta)$, so $\dim \mathrm{Cliff}(V, \beta) \leq 2^n$.

Let the homomorphism $V \to \mathrm{Cliff}(V, \beta)$ be such that $v \mapsto -\gamma(v)$ satisfies equation (D.1), so this homomorphism extends to an algebra homomorphism α : $\mathrm{Cliff}(V, \beta) \to \mathrm{Cliff}(V, \beta)$ such that $\alpha(\gamma(v_1) \dots \gamma(v_k)) = (-1)^k \gamma(v_1) \dots \gamma(v_k)$. Since $\alpha^2(v) = v$ for all $v \in V$, α is an automorphism, which we call the *main involution of* $\mathrm{Cliff}(V, \beta)$. Further, there is a decomposition

$$\mathrm{Cliff}(V, \beta) = \mathrm{Cliff}^+(V, \beta) \oplus \mathrm{Cliff}^-(V, \beta),$$

where $\mathrm{Cliff}^+(V, \beta)$ is spanned by products of an even number of elements of V and $\mathrm{Cliff}^-(V, \beta)$ is spanned by products of an odd number of elements of V. Finally, α acts by ± 1 on $\mathrm{Cliff}^\pm(V, \beta)$.

Let V be a finite dimensional complex vector space with non-degenerate, symmetric bilinear form Ω.

Definition D.1.3. Let S be a complex vector space and let $\gamma : V \to \mathrm{End}(S)$ be a vector space homomorphism. Then we call (S, Ω) a *space of spinors* for (V, Ω) if

- $\{\gamma(x), \gamma(y)\} = \Omega(x, y)I_V$ for all $x, y \in V$.

- The only subspaces of S that are invariant under $\gamma(V)$ are 0 and S.

If (S,Ω) is a space of spinors for (V,β), then the map γ extends to an irreducible representation $\tilde{\gamma}:\mathrm{Cliff}(V,\Omega)\to\mathrm{End}(S)$. Conversely, every irreducible representation of $\mathrm{Cliff}(V,\Omega)$ arises this way.

Remark D.1.4. The following are standard and can be found in [15].

Theorem D.1.5. If $\dim V = 2\ell$ is even, then, up to isomorphism, there is exactly one space of spinors for (V,Ω), and it has dimension 2^ℓ.

Proposition D.1.6. Suppose $\dim V = n$ is even. Let (S,γ) be a space of spinors for (V,β). Then $(\mathrm{End}S,\gamma)$ is a Clifford algebra for (V,Ω). Thus, $\mathrm{Cliff}(V,\Omega)$ is a simple algebra of dimension 2^n.

The map $\gamma: V\to\mathrm{Cliff}(V,\Omega)$ is injective, and for any basis $\{v_1,\dots,v_n\}$ of V the set of all ordered products $\gamma(v_{i_1})\dots\gamma(v_{i_p})$, where $1\le i_1<\dots<i_p\le n$, is a basis for $\mathrm{Cliff}(V,\Omega)$.

Theorem D.1.7. If $\dim V = 2\ell+1$ is odd, then there are exactly two nonisomorphic spaces of spinors for (V,Ω), and each space has dimension 2^ℓ.

Proposition D.1.8. Suppose $\dim V = 2\ell+1$ is odd. Let (S,γ_+) and (S,γ_-) be the two inequivalent spaces of spinors for (V,Ω), and let $\gamma: V\to \mathrm{End}S\oplus\mathrm{End}S$ be defined by $\gamma(v)=\gamma_+(v)\oplus\gamma_-(v)$. Then $(\mathrm{End}S\oplus\mathrm{End}S,\gamma)$ is a Clifford algebra for (V,Ω). Thus, $\mathrm{Cliff}(V,\Omega)$ is a semisimple algebra and is the sum of two simple ideals of dimension 2^{n-1}.

The map $\gamma: V\to\mathrm{Cliff}(V,\Omega)$ is injective. For any basis $\{v_1,\dots,v_n\}$ of V the set of all ordered products $\gamma(v_{i_1})\dots\gamma(v_{i_p})$, where $1\le i_1<\dots<i_p\le n$, is a basis for $\mathrm{Cliff}(V,\Omega)$.

Definition D.1.9. Given $a,b\in V$ we define $R_{a,b}\in\mathrm{End}(V)$ by $R_{a,b}v=\Omega(b,v)a-\Omega(a,v)b$.

Since $\Omega(R_{a,b}x, y) = \Omega(b, x)\Omega(a, y) - \Omega(a, x)\Omega(b, y) = -\Omega(x, R_{a,b}y)$, we have $R_{a,b} \in \mathfrak{so}(V, \Omega)$.

Lemma D.1.10. The homomorphisms $R_{a,b}$ span $\mathfrak{so}(V, \Omega)$, for all $a, b \in V$.

Lemma D.1.11. Define a homomorphism $\varphi : \mathfrak{so}(V) \to \mathrm{Cliff}_2(V, \Omega)$ by $\varphi(R_{a,b}) = (1/2)[\gamma(a), \gamma(b)]$ for $a, b \in V$. Then φ is an injective Lie algebra homomorphism, and $[\varphi(X), \gamma(v)] = \gamma(Xv)$ for $X \in \mathfrak{so}(V, \Omega)$ and $v \in V$.

Definition D.1.12. We denote by $C^\bullet(W)$ the sum $\bigoplus_{p=0}^{\dim W} C^p(W)$, where $C^p(W)$ is the space of p-multilinear functions on W that are skew-symmetric.

Assume that $\dim V$ is even and fix a decomposition $V = W \oplus W^*$ with W and W^* maximal Ω-isotropic subspaces. Let $(C^\bullet(W), \gamma)$ be a fixed choice for the space of spinors. Then we define the *even* and *odd spin spaces* by $C^+(W) = \bigoplus_{p \text{ even}} C^p(W)$ and $C^-(W) = \bigoplus_{p \text{ odd}} C^p(W)$, respectively. Then $\gamma(v) : C^\pm(W) \to C^\mp(W)$ for $v \in V$, so the action of $\gamma(V)$ interchanges the even and odd spin spaces. Denote by $\tilde{\gamma}$ the extension of γ to a representation of $\mathrm{Cliff}(V, \Omega)$ on $C^\bullet(W)$.

Let $\varphi : \mathfrak{so}(V, \Omega) \to \mathrm{Cliff}(V, \Omega)$ be the Lie algebra homomorphism in the previous lemma. Set $\pi(X) = \tilde{\gamma}(\varphi(X))$ for $X \in \mathfrak{so}(V, \Omega)$. Since $\varphi(X)$ is an even element in the Clifford algebra and since γ interchanges the spin spaces, $\pi(X)$ preserves the even and odd subspaces $C^\pm(W)$.

Remark D.1.13. The labeling $\pm$ depends on a particular choice of the space of spinors.

Definition D.1.14. Let

$$\pi^\pm(X) = \pi(X) \mid_{C^\pm(W)} .$$

Then we call $\pi^\pm$ the *semi-spin (or half-spin) representations of* $\mathfrak{so}(V, \Omega)$.

Proposition D.1.15. For $\dim V = 2\ell$, the half-spin representations $\pi^{\pm}$ of $\mathfrak{so}(V, \Omega)$ are irreducible with highest weights $\varpi_{\pm} = (\varepsilon_1 + \cdots + \varepsilon_{\ell-1} \pm \varepsilon_\ell)/2$. The weights are

$$(\pm\varepsilon_1 \pm \cdots \pm \varepsilon_\ell)/2,$$

where an even number of minus signs for π^+ and an odd number of minus signs for π^- and each has multiplicity one.

Please refer to [15] pages 92 and 140 for more details about the roots of $\mathfrak{so}(V)$.

For $\dim V = 2\ell+1$. We fix a decomposition $V = W \oplus \mathbb{C}e_0 \oplus W^*$ with W and W^* maximal Ω-isotropic subspaces. Then we take the space of spinors $(C^\bullet(W), \gamma_+)$, and we define a representation of $\mathfrak{so}(V, \Omega)$ on $C^\bullet(W)$ by $\pi = \widetilde{\gamma}_+ \circ \varphi$, where $\varphi : \mathfrak{so}(V, \Omega) \to \mathrm{Cliff}(V, \Omega)$ is the homomorphism as in above lemma and $\widetilde{\gamma}_+$ is the canonical extension of γ_+ to a representation of $\mathrm{Cliff}(V, \Omega)$ on $C^\bullet(W)$. We call π the *spin representation* of $\mathfrak{so}(V, \Omega)$.

Proposition D.1.16. For $\dim V = 2\ell + 1$, the spin represntation of $\mathfrak{so}(V, \Omega)$ is irreducible and has highest weight $\varpi_\ell = (\varepsilon_1 + \cdots + \varepsilon_{\ell-1} + \varepsilon_\ell)/2$. The weights are $(\pm\varepsilon_1 \pm \cdots \pm \varepsilon_\ell)/2$, and each weight has multiplicity one.

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