On the Quantum Spaces of Some Quadratic Regular Algebras of Global Dimension Four

by

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Out of the night that covers me,
Black as the pit from pole to pole,
I thank whatever gods may be
For my unconquerable soul.

In the fell clutch of circumstance
I have not winced or cried aloud.
Under the bludgeonings of chance
My head is bloody, but unbowed.

Beyond this place of wrath and tears
Looms but the Horror of the shade,
And yet the menace of the years
Finds, and shall find me unafraid.

It matters not how strait the gate,
How charged with punishments the scroll,
I am the master of my fate,
I am the captain of my soul.

– William Ernest Henley, Invictus

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Abstract

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A quantum $\mathbb{P}^3$ is a noncommutative analogue of a polynomial ring on four variables, and, herein, it is taken to be a regular algebra of global dimension four. It is well known that if a generic quadratic quantum $\mathbb{P}^3$ exists, then it has a point scheme consisting of exactly twenty distinct points and a one-dimensional line scheme.

In this thesis, we compute the line scheme of a family of algebras whose generic member is a candidate for a generic quadratic quantum $\mathbb{P}^3$. We find that, as a closed subscheme of $\mathbb{P}^5$, the line scheme of the generic member is the union of seven curves; namely, a nonplanar elliptic curve in a $\mathbb{P}^3$, four planar elliptic curves and two nonsingular conics.

Afterward, we compute the point scheme and line scheme of several (nongeneric) quadratic quantum $\mathbb{P}^3$'s related to the Lie algebra $\mathfrak{sl}(2, \Bbbk)$. In doing so, we identify some notable features of the algebras, such as the existence of an element that plays the role of a Casimir element of the underlying Lie-type algebra.

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Chapter 1

Introduction

Algebraic geometry has long been a tool in the study of commutative algebras. In the 1980's, a movement began that had the goal of extending the study of algebraic geometry to noncommutative algebras. This movement has grown significantly since 1990; this is mainly due to the work of Artin, Tate and Van den Bergh that introduced a method of encoding the multiplication of a noncommutative algebra using geometry.

In [3], Artin, Tate and Van den Bergh defined the notion of a point module and a line module. These modules have the property that there are points and lines in projective space associated to them. In lieu of associating geometry directly to a noncommutative algebra, Artin, Tate and Van den Bergh associated the geometry to these modules.

The collection of all points associated to the point modules of an algebra is known as the point scheme. In [3], Artin, Tate and Van den Bergh gave a method for computing the point scheme of an algebra; one could then determine the point modules from these points. They did not, however, parametrize the line modules by a scheme. This was accomplished in 2002 by Shelton and Vancliff in [29, 30] for certain kinds of algebras.

Artin, Tate and Van den Bergh's geometry has been exceptionally useful in the study of Artin-Schelter regular algebras (also called AS-regular algebras). These algebras are considered to be noncommutative analogues of polynomial rings owing to the fact that the Gorenstein condition required in the definition of Artin-Schelter regularity is a symmetry condition that replaces the symmetry condition of commutativity of polynomial rings. Shelton and Vancliff's construction of the line scheme gives us a method of parametrizing the line modules by a scheme for certain Artin-Schelter regular algebras of global dimension four; these algebras are quadratic domains, have four generators, six defining relations, and the same Hilbert series as that of the polynomial ring on four variables.

Quadratic Artin-Schelter regular algebras of global dimension four have become known as quadratic quantum $\mathbb{P}^3$'s. Artin was the first to introduce the terminology "quantum $\mathbb{P}^2$" in [1] to refer to an AS-regular algebra of global dimension three. The name came about from the increased number of such algebras emerging from the field of quantum mechanics at the time.

Our first consideration in this thesis will be a family of algebras defined by Cassidy and Vancliff in [5] whose generic member is a candidate for a generic quadratic quantum $\mathbb{P}^3$. In the mid-1990's, Van den Bergh proved that if a generic quadratic quantum $\mathbb{P}^3$ exists, then its point scheme consists of twenty distinct points (counted with multiplicity) and has a one-parameter family of line modules (cf. [34]); in the language of Shelton and Vancliff in [29, 30], a generic quadratic quantum $\mathbb{P}^3$ has a one-dimensional line scheme.

Many algebras with a point scheme consisting of twenty distinct points are known; likewise, many algebras with a one-dimensional line scheme are known. However, it was not until 2001 that an algebra with both these properties was discovered; Shelton and Tingey defined such an algebra in [28]. Unfortunately, Shelton and Tingey found this algebra with the aid of a computer-algebra program and trial-and-error; it was the only known example for nearly a decade.

In 2010, Cassidy and Vancliff defined a new type of algebra known as a graded skew Clifford algebra. In [5], they gave examples of several families of regular graded skew Clifford algebras; the first family is the one we consider in Chapter 3 of this thesis. The generic member of this family has a point scheme consisting of twenty distinct points and a one-dimensional line scheme; furthermore, the algebra given by Shelton and Tingey in [28] is a member of this family. However, the methods used by Cassidy and Vancliff computed only the dimension of the line scheme, not the line scheme itself.

In Chapter 3 of this thesis, we compute the line scheme of this family of algebras and, for a generic member, find it to be the union of a nonplanar elliptic curve in a $\mathbb{P}^3$, four planar elliptic curves and two nonsingular conics. We will also describe the lines in $\mathbb{P}^3$ determined by the line scheme and describe some distinguished properties of the algebras highlighted by this geometry. Further analysis of the line scheme has led to the conjecture that the algebras in this family are not truly generic quadratic quantum $\mathbb{P}^3$'s. The analysis did however give a candidate for the line scheme of a generic quadratic quantum $\mathbb{P}^3$ (or perhaps a class of generic quadratic quantum $\mathbb{P}^3$'s); namely the union of two nonplanar elliptic curves in a $\mathbb{P}^3$ and four planar elliptic curves.

Our second consideration in this thesis concerns certain algebras related to the Lie algebra $\mathfrak{sl}(2, \Bbbk)$, where $\Bbbk$ is an algebraically closed field of characteristic zero. These algebras include the Lie superalgebra $\mathfrak{sl}(1|1)$, a color Lie algebra obtained via a cocycle twist of $\mathfrak{sl}(2, \Bbbk)$ and a quantum analogue of the universal enveloping algebra of $\mathfrak{sl}(2, \Bbbk)$, denoted $\mathcal{U}_q(\mathfrak{sl}(2, \Bbbk))$. Each of these algebras, and $\mathfrak{sl}(2, \Bbbk)$ itself, appear in quantum mechanics in some fashion.

In order to analyze these algebras in Chapter 4 by using Artin, Tate and Van den Bergh's geometry, we first pass to the universal enveloping algebra of the algebra (this step does not apply to $\mathcal{U}_q(\mathfrak{sl}(2, \Bbbk))$). This process gives an associative $\Bbbk$-algebra on three generators. However, these algebras are not graded; hence, they are not quadratic quantum $\mathbb{P}^3$'s. We instead associate Artin, Tate and Van den Bergh's geometry to a graded algebra that maps onto the ungraded universal enveloping algebra (respectively quantized universal enveloping algebra); we then show that this graded algebra is a quadratic quantum $\mathbb{P}^3$ by showing that it is either an Ore extension, or a normal extension, of an AS-regular algebra of global dimension three.

The geometry we use is able to identify distinguished elements of the quadratic quantum $\mathbb{P}^3$'s we consider (and therefore of the (respectively quantized) universal enveloping algebra). In particular, an element analogous to a Casimir element is identified in $\mathcal{U}(\mathfrak{sl}(1|1))$ and $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{k}))$. Furthermore, in the case of $\mathcal{U}(\mathfrak{sl}(1|1))$, the geometry motivated work which led to the realization of the associated quadratic quantum $\mathbb{P}^3$ being a twist by an automorphism of the coordinate ring of quantum 2×2 matrices.

Chapter 2

Preliminary Concepts

Herein, we will assume that $\Bbbk$ is an algebraically closed field; additional assumptions on $\Bbbk$ will be imposed in Chapters 3 and 4. We will begin by examining some basic definitions regarding abstract algebra and define what is meant by a regular algebra. We will then examine some concepts from algebraic geometry as well as Artin, Tate and Van den Bergh's construction for using algebraic geometry with noncommutative algebras. We will conclude with a brief explanation of graded skew Clifford algebras and Lie-type algebras.

Let $B \subset A$ indicate that the set B is a subset of the set A, where possibly $A = B$. We denote the set of positive integers by $\mathbb{N}$. Also, let $\Bbbk^{m\times n}$ denote the set of all $m \times n$ matrices with entries in $\Bbbk$ and $\mathbb{M}_n(\Bbbk)$ denote the set of all $n \times n$ matrices with entries in $\Bbbk$. If $M \in \Bbbk^{m\times n}$, then we write M_{ij} to denote the ijth entry of M. For a subset A of a field or vector space, we write $A^\times$ for the nonzero elements of A.

2.1 Abstract Algebra

The definitions, results and examples in this section can be found in books such as [10], [14], [18], and [26].

2.1.1 Algebras and Modules

Definition 2.1.1.1. Associative $\Bbbk$-Algebra

An associative $\Bbbk$-algebra A is a vector space over $\Bbbk$ and a ring such that $\Bbbk$ is contained in the center of A, and $\alpha(ab) = (\alpha a)b = a(\alpha b)$, for all $\alpha \in \mathbb{k}$ and $a, b \in A$. If $ab = ba$ for all $a, b \in A$, then we call A a commutative algebra.

Example 2.1.1.2.

1. Let $A = \mathbb{k}[x_1, ..., x_n]$ be the collection of all polynomials in the variables $x_1, ..., x_n$ with coefficients in $\mathbb{k}$. As a vector space, $\dim(A) = \infty$, and A is a commutative algebra under standard polynomial multiplication.
2. If $A = \mathbb{M}_n(\mathbb{k})$, then A is an n^2-dimensional vector space and is a noncommutative algebra under standard matrix multiplication.
3. The free $\mathbb{k}$-algebra on generators $x_1, ..., x_n$, denoted $\mathbb{k}\langle x_1, ..., x_n\rangle$, is the $\mathbb{k}$-algebra whose vector-space basis consists of all words in $x_1, ..., x_n$, including the empty word. Addition and scalar multiplication is defined in the standard way, but the multiplication of two basis elements is done via concatenation. Multiplication is then extended to the entire algebra using distribution.

Any $\mathbb{k}$-algebra on n generators can be viewed as a quotient of the free $\mathbb{k}$-algebra on n generators.

Example 2.1.1.3.

1. Defining relations of the polynomial ring $\mathbb{k}[x_1, ..., x_n]$ can be chosen to be of the form $x_i x_j = x_j x_i$ for $i, j = 1, ..., n$. Thus, $\mathbb{k}[x_1, ..., x_n] = \mathbb{k}\langle x_1, ..., x_n\rangle / I$, where $I = \langle x_i x_j - x_j x_i : i, j = 1, ..., n\rangle$.
2. Consider $A = \mathbb{M}_2(\mathbb{k})$. A vector space basis for A is $\{E_{ij} : i, j = 1, 2\}$, where E_{ij} is the 2×2 matrix with a 1 in the ijth entry and 0 elsewhere. For such matrices, $E_{ij}E_{kl} = \delta_{jk}E_{il}$, where δ_{jk} is the kronecker-delta. So, we may express A as
$$\frac{\mathbb{k}\langle E_{11}, E_{12}, E_{21}, E_{22}\rangle}{\langle E_{ij}E_{kl} - \delta_{jk}E_{il},\ E_{11} + E_{22} - 1 : i, j = 1, 2\rangle}.$$

Notice that $E_{11} = E_{12}E_{21}$ and $E_{22} = E_{21}E_{12}$. So, E_{11} and E_{22} are redundant as generators and we obtain the isomorphic algebra

$$\frac{\Bbbk\langle E_{12}, E_{21}\rangle}{\langle\ E_{12}^2,\ E_{21}^2,\ E_{12}E_{21} + E_{21}E_{12} - 1\ \rangle}.$$

Definition 2.1.1.4. Positively Graded, Connected $\Bbbk$-Algebra

A $\Bbbk$-algebra A is called positively graded if $A = \bigoplus_{i=0}^{\infty} A_i$, where A_i is a subspace of A for all i and $A_iA_j \subset A_{i+j}$. We call the elements of $A_i^{\times}$ the homogeneous elements of degree i. We denote the degree of an element $x \in A$ by $\deg(x)$. We say that A is connected if $A_0 = \Bbbk$.

Example 2.1.1.5. The polynomial ring $A = \Bbbk[x_1, ..., x_n]$ is a positively graded, connected $\Bbbk$-algebra with $A_i = \bigoplus_{i_1+\cdots+i_n=i} \Bbbk x_1^{i_1} \cdots x_n^{i_n}$, for all i.

If we view an algebra A as a quotient of the free algebra, then A is graded if and only if the defining relations of A are homogeneous.

Example 2.1.1.6.

1. If $\deg(x) = \deg(y) = 1$, then $\Bbbk[x, y] = \Bbbk\langle x, y\rangle/\langle yx - xy\rangle$ is graded.
2. If $\deg(x) = \deg(y) = 1$, then $A = \Bbbk\langle x, y\rangle/\langle x - y^2\rangle$ is not graded. However, if we change the grading of the generators of A so that $\deg(x) = 2$ and $\deg(y) = 1$, then A is graded.
3. If $\deg(x) = \deg(y) = \deg(z) = 1$, then $A = \Bbbk\langle x, y, z\rangle/\langle xy - yx, xz + zx + z^2, yzy - y^3\rangle$ is graded.

Definition 2.1.1.7. Quadratic Algebra

We call a graded $\Bbbk$-algebra A quadratic if it is generated by degree-one elements and each of its defining relations is homogeneous of degree-two.

Example 2.1.1.8.

1. If $\deg(x) = \deg(y) = 1$, then $\Bbbk[x, y] = \Bbbk\langle x, y\rangle/\langle yx - xy\rangle$ is quadratic.
2. If $\deg(x) = \deg(y) = 1$, then $A = \Bbbk\langle x, y\rangle/\langle x - y^2\rangle$ is not quadratic since its defining relation is not homogeneous of degree-two. If we change the grading of the generators of A so that $\deg(x) = 2$ and $\deg(y) = 1$, then A is still not quadratic since one of its generators is not of degree-one.
3. If $\deg(x) = \deg(y) = \deg(z) = 1$, then $A = \Bbbk\langle x, y, z\rangle/\langle xy - yx, xz + zx + z^2, yzy - y^3\rangle$ is not quadratic since one of its defining relations is homogeneous of degree-three.

Definition 2.1.1.9. Left A-Module

Let A be a $\Bbbk$-algebra. A left A-module is an abelian group, $(M, +)$, and an action of A on M such that for all $a, b \in A$ and $m, n \in M$:

(i) $(a + b)m = am + bm$,

(ii) $(ab)m = a(bm)$,

(iii) $a(m + n) = am + an$, and

(iv) $1m = m$, where 1 is the unity element in A.

A right A-module is defined in a similar manner.

Example 2.1.1.10.

1. If $A = \Bbbk$, then any $\Bbbk$-vector space is an A-module.
2. If $A = \Bbbk[x, y]$, then $M = \Bbbk[X]$ is an A-module under the action defined by $xf = Xf$ and $yf = 0$, for all $f \in M$.

Definition 2.1.1.11. Cyclic Module

If A is a $\Bbbk$-algebra and M is a left A-module, then we call M cyclic if there exists an $m \in M$ such that $M = Am$. In this case, $M \cong A/\mathrm{Ann}(m)$, where $\mathrm{Ann}(m) = \{a \in A : am = 0\}$ is called the (left) annihilator of m.

Definition 2.1.1.12. Graded A-Module

If $A = \bigoplus_{i=0}^{\infty} A_i$ is a graded $\Bbbk$-algebra and M is a left A-module, then M is called a graded module if $M = \bigoplus_{j=-\infty}^{\infty} M_j$, where M_j is a subspace of M, for all j, and $A_i M_j \subset M_{i+j}$, for all i, j.

Example 2.1.1.13. If $A = \Bbbk[x, y]$ and $M = \Bbbk[X]$ as in Example 2.1.1.10, then M is cyclic since $M = A \cdot 1$. Also, $\operatorname{Ann}(1) = \langle y \rangle \subset A$ and $M = \Bbbk[X] \cong \Bbbk[x, y]/\langle y \rangle$. If we define $\deg(x) = \deg(y) = \deg(X) = 1$, then M becomes a graded A-module.

For graded modules, we often want a convenient way to summarize the dimension of each homogenous degree-i subspace of the module. For this purpose, one tool that we use is the Hilbert series.

Definition 2.1.1.14. Hilbert Series

Let A be a graded $\Bbbk$-algebra and $M = \bigoplus_{i=n}^{\infty} M_i$ be a graded A-module. The Hilbert series of M is $H_M(t) = \sum_{i=n}^{\infty} \dim_{\Bbbk}(M_i) t^i$.

Note that we are also allowed to discuss the Hilbert series of a graded $\Bbbk$-algebra, A, since A is a module over itself.

Example 2.1.1.15.

1. If $A = \Bbbk[x_1, ..., x_n] = \bigoplus_{i=0}^{\infty} A_i$ where $A_i = \bigoplus_{i_1+\cdots+i_n=i} \Bbbk x_1^{i_1} \cdots x_n^{i_n}$ for all i, then a basis for A_i is $\mathfrak{B}_i = \{x_1^{i_1} \cdots x_n^{i_n} : i_1 + \cdots + i_n = i\}$. So, a basis for A is $\mathfrak{B} = \bigcup_{i=0}^{\infty} \mathfrak{B}_i$, and $\dim_{\Bbbk}(A_i) = \binom{i+n-1}{i}$. Thus, $H_A(t) = \sum_{i=0}^{\infty} \dim_{\Bbbk}(A_i) t^i = \sum_{i=0}^{\infty} \binom{i+n-1}{i} t^i = \frac{1}{(1-t)^n}$.
2. If $A = \Bbbk[x_1, x_2, x_3, x_4]$ and $M = A/\langle x_1, x_2 - x_3, x_4 \rangle$, then M is a graded A-module with action $x_1 f = 0, x_2 f = \overline{x_2} f, x_3 f = \overline{x_2} f, x_4 f = 0$ for all $f \in M$, where $\overline{x_2}$ is the image of x_2 in M. Also, $M \cong \Bbbk[x_2]$, as a vector space, so $\dim_{\Bbbk}(M_i) = 1$, for all i, and $H_M(t) = \sum_{i=0}^{\infty} t^i = \frac{1}{1-t}$.

2.1.2 Tensor Products

Given two vector spaces M and N, the direct sum $M \oplus N$ gives us a way to "add" two vector spaces such that $\dim_{\Bbbk}(M \oplus N) = \dim_{\Bbbk}(M) + \dim_{\Bbbk}(N)$. A method of "multiplying" two vector spaces, which we will denote by $M \otimes N$, is a way to combine modules in such that $\dim_{\Bbbk}(M \otimes N) = \dim_{\Bbbk}(M)\dim_{\Bbbk}(N)$ holds.

Definition 2.1.2.1. Tensor Product of Vector Spaces
Let M and N be $\Bbbk$-vector spaces. The vector space generated by the Cartesian product of M and N is

$$\mathfrak{F}(M \times N) = \left\{ \sum_{i=1}^{r} q_i(m_i, n_i) : r \in \mathbb{N}, q_i \in \Bbbk, m_i \in M, n_i \in N, \text{ for all } i \right\}.$$

Let K be the subspace of $\mathfrak{F}(M \times N)$ generated by the elements

$$(m_1 + m_2, n) - (m_1, n) - (m_2, n), \qquad (m, n_1 + n_2) - (m, n_1) - (m, n_2),$$
$$(qm, n) - q(m, n), \qquad (m, qn) - q(m, n),$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $q \in \Bbbk$. The tensor product of M and N is $M \otimes N = \mathfrak{F}(M \times N)/K$. We denote $(m, n) + K \in M \otimes N$ by $m \otimes n$.

Under this construction, $M \otimes N$ has the following properties:

(i) $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$,

(ii) $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$, and

(iii) $q(m \otimes n) = (qm) \otimes n = m \otimes (qn) = (m \otimes n)q$.

The reader should note that not every element of $M \otimes N$ is of the form $m \otimes n$; however, if $\mathfrak{B}_1 = \{m_1, ..., m_r\}$ is a basis for M and $\mathfrak{B}_2 = \{n_1, ..., n_s\}$ is a basis for N, then $\mathfrak{B} = \{m_i \otimes n_j : 1 \leq i \leq r, 1 \leq j \leq s\}$ is a basis for $M \otimes N$. Hence, $\dim_{\Bbbk}(M \otimes N) = \dim_{\Bbbk}(M)\dim_{\Bbbk}(N)$ as desired.

Proposition 2.1.2.2. *If $M, N, Q, M_1, M_2, ..., N_1, N_2, ...$ are vector spaces, then:*

(i) $(M \otimes N) \otimes Q \cong M \otimes (N \otimes Q)$,

(ii) $\left(\bigoplus_{i=1}^{\infty} M_i\right) \otimes N \cong \bigoplus_{i=1}^{\infty} (M_i \otimes N)$, *and*

(iii) $M \otimes \left(\bigoplus_{i=1}^{\infty} N_i\right) \cong \bigoplus_{i=1}^{\infty} (M \otimes N_i)$.

These properties tell us that the tensor product is associative and distributes across direct sums. Using the tensor product, we can now define a $\Bbbk$-algebra from a vector space V.

Definition 2.1.2.3. Tensor Algebra
Let V be a $\Bbbk$-vector space. Define $T^k(V) = \bigotimes_{i=1}^{k} V$ and $T(V) = \bigoplus_{k=0}^{\infty} T^k(V)$. We call $T(V)$ the tensor algebra on V.

Addition and scalar multiplication are defined on $T(V)$ using the addition and scalar multiplication on the tensor product. Multiplication is defined via concatenation; that is, if $u = u_1 \otimes \cdots \otimes u_n$ and $w = w_1 \otimes \cdots \otimes w_m$, then

$$uw = u_1 \otimes \cdots \otimes u_n \otimes w_1 \otimes \cdots \otimes w_m.$$

This, together with Proposition 2.1.2.2, verifies that $T(V)$ is a $\Bbbk$-algebra. Furthermore, $T(V)$ is a graded $\Bbbk$-algebra with $(T(V))_k = T^k(V)$ and, if $V = \bigoplus_{i=1}^{N} \Bbbk v_i$, then $T(V) \cong \Bbbk\langle v_1, ..., v_N\rangle$.

Example 2.1.2.4. If $A = \Bbbk[x, y] \cong \Bbbk\langle x, y\rangle/\langle xy - yx\rangle$ and $V = \Bbbk x \oplus \Bbbk y$, then $A \cong T(V)/\langle x \otimes y - y \otimes x\rangle$.

Let V be a $\Bbbk$-vector space with basis $\mathfrak{B} = \{v_1, ..., v_n\}$. The dual space to V is $V^* = \{f : V \to \Bbbk \mid f \text{ is } \Bbbk\text{-linear}\}$, which has basis $\mathfrak{B}^* = \{z_1, ..., z_n\}$, where $z_i(v_j) = \delta_{ij}$, the kronecker-delta. Since V^* is also a $\Bbbk$-vector space, we may form the vector space $V^* \otimes V^*$ as described above. This vector space has a natural action on

$V \otimes V$ defined via $(z_i \otimes z_j)(v_k \otimes v_l) = z_i(v_k) \cdot z_j(v_l)$. Given this action and a subspace W of $V \otimes V$, we define a subspace of $V^* \otimes V^*$ by $W^\perp = \{f \in V^* \otimes V^* : f(w) = 0, \text{ for all } w \in W\}$. With this in mind, we now define the Koszul dual of a quadratic algebra.

Definition 2.1.2.5. Koszul Dual
Let A be a finitely-generated quadratic $\Bbbk$-algebra. It follows that $A \cong T(V)/\langle W \rangle$, where V is a finite-dimensional vector space and W is a subspace of $V \otimes V$. The Koszul dual of A is the $\Bbbk$-algebra $A^! = T(V^*)/\langle W^\perp \rangle$.

For any finite-dimensional vector space U and subspace S of U, we have that $\dim(U) = \dim(S) + \dim(S^\perp)$. Therefore, if we assume that $\dim(V) = n < \infty$ and $\dim(W) = m$, then $\dim(W^\perp) = \dim(V \otimes V) - \dim(W) = n^2 - m$. So, in order to determine the Koszul dual of a finitely-generated quadratic $\Bbbk$-algebra, we need only find $n^2 - m$ linearly independent elements of $T^2(V^*)$ that vanish on the generators of W.

Example 2.1.2.6. Let $A = \Bbbk[x_1, x_2] \cong \Bbbk\langle x_1, x_2 \rangle / \langle x_1x_2 - x_2x_1 \rangle \cong T(\Bbbk x_1 \oplus \Bbbk x_2)/\langle x_1 \otimes x_2 - x_2 \otimes x_1 \rangle$. Here $\dim(V) = 2$ and $\dim(W) = 1$, so $\dim(W^\perp) = 2^2 - 1 = 3$. Hence, we seek three linearly independent elements of $T^2(\Bbbk z_1 \oplus \Bbbk z_2)$ that vanish on $x_1 \otimes x_2 - x_2 \otimes x_1$, where $\{z_1, z_2\}$ is the dual basis to $\{x_1, x_2\}$. In particular,

$$
\begin{aligned}
(z_1 \otimes z_1)(x_1 \otimes x_2 - x_2 \otimes x_1) &= z_1(x_1)z_1(x_2) - z_1(x_2)z_1(x_1) = 0, \\
(z_2 \otimes z_2)(x_1 \otimes x_2 - x_2 \otimes x_1) &= z_2(x_1)z_2(x_2) - z_2(x_2)z_2(x_1) = 0, \\
(z_1 \otimes z_2 + z_2 \otimes z_1)(x_1 \otimes x_2 - x_2 \otimes x_1) &= z_1(x_1)z_2(x_2) - z_1(x_2)z_2(x_1) \\
&\quad + z_2(x_1)z_1(x_2) - z_2(x_2)z_1(x_1) = 0.
\end{aligned}
$$

So, the Koszul dual of A is

$$A^! = \frac{T(\Bbbk z_1 \oplus \Bbbk z_2)}{\langle\ z_1 \otimes z_1,\ z_2 \otimes z_2,\ z_1 \otimes z_2 + z_2 \otimes z_1\ \rangle} \cong \frac{\Bbbk\langle z_1, z_2\rangle}{\langle\ z_1^2,\ z_2^2,\ z_1 z_2 + z_2 z_1\ \rangle},$$

which is the exterior algebra on two generators.

2.1.3 Regularity

All of the graded algebras we will consider in Chapters 3 and 4 are known as regular algebras. In noncommutative algebra, one would like to have an analogue of objects and concepts from commutative algebra. In particular, polynomial rings are central objects in commutative algebra so it would be desirable to have a noncommutative analogue; many believe that regular algebras are the "correct" analogue.

Definition 2.1.3.1. Global Dimension

Let A be a $\Bbbk$-algebra.

(i) An A-module P is called projective if, given any two A-modules M and N, an A-module epimorphism $g : M \to N$ and an A-module homomorphism $f : P \to N$, there exists an A-module homomorphism $h : P \to M$ such that $gh = f$; that is, the diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "f"] & \\ M \arrow[r, two heads, "g"] & N \arrow[r] & 0 \end{tikzcd}$$

of A-module homomorphisms commutes.

(ii) Let M be an A-module, $P_0, P_1, ...$ be projective A-modules and $d_0, d_1, ...$ be A-module homomorphisms such that $d_i : P_i \to P_{i-1}$, for $i \geq 1$, and $d_0 : P_0 \to M$. We call the sequence

$$\cdots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

a projective resolution of the module M if $\mathrm{Im}(d_k) = \ker(d_{k-1})$, for all k.

(iii) The projective dimension of an A-module M, denoted $\text{pdim}(M)$, is the infimum of the length of all projective resolutions of M, not including d_0 or the trivial maps at the beginning or end of the resolutions.

(iv) The left (respectively right) global dimension of A is the supremum of the projective dimensions of all left (respectively right) A-modules. If A is positively graded and connected, then the global dimension of A, denoted $\text{gldim}(A)$, is $\text{gldim}(A) = \text{pdim}({}_A\Bbbk) = \text{pdim}(\Bbbk_A)$, where ${}_A\Bbbk$ (respectively $\Bbbk_A$) is the trivial left (respectively right) A-module (cf. [3]).

Definition 2.1.3.2. Polynomial Growth

Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a positively graded, connected $\Bbbk$-algebra. We say that A has polynomial growth if there exist positive $a, b \in \mathbb{R}$ such that $\dim_{\Bbbk}(A_n) \leq an^b$, for all n.

Definition 2.1.3.3. Gorenstein Condition [2]

We say that a positively graded, connected $\Bbbk$-algebra A of finite global dimension satisfies the Gorenstein condition if

(i) the projective modules in a minimal projective resolution, $\mathbb{X}$, of ${}_A\Bbbk$ are finitely-generated A-modules, and

(ii) the sequence obtained by applying the functor $\text{Hom}_A(\cdot, A)$ to the modules in $\mathbb{X}$ is a projective resolution of a graded right A-module isomorphic to the right trivial module $\Bbbk_A$.

Definition 2.1.3.4. Artin-Schelter Regular Algebras [2]

Suppose that $A = \bigoplus_{i=0}^{\infty} A_i$ is a positively graded connected $\Bbbk$-algebra generated by A_1. We say that A is Artin-Schelter (AS) regular of global dimension d if

(i) $\text{gldim}(A) = d < \infty$,

(ii) A has polynomial growth, and

(iii) A satisfies the Gorenstein condition.

The Gorenstein condition imposes a symmetry condition on A. So, it acts as an analogue of the commutative property inherent to polynomial rings. This is one of the motivating reasons for claiming that regular algebras are the “correct” noncommutative analogue of polynomial rings. In fact, polynomial rings satisfy these conditions.

Example 2.1.3.5. If $A = \Bbbk[x_1, x_2]$, then

$$0 \longrightarrow A \xrightarrow{d_2} A^2 \xrightarrow{d_1} A \xrightarrow{d_0} {}_A\Bbbk \longrightarrow 0$$

is a minimal projective resolution of ${}_A\Bbbk$, where

$$d_1(a_1, a_2) = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad d_2(a) = a \begin{bmatrix} x_2 & -x_1 \end{bmatrix},$$

$a_1, a_2, a \in A$ and d_0 is the canonical map; so $\operatorname{gldim}(A) = 2$. Applying $\operatorname{Hom}_A(\cdot, A)$ to this resolution yields a projective resolution of $\Bbbk_A$ where the maps are left multiplication by the matrices in the original resolution. So, A is Gorenstein and A has polynomial growth since $\dim_{\Bbbk}(A_n) \leq n + 1 \leq 2n$, for all $n \geq 1$. Therefore, A is a regular algebra of global dimension two.

In this thesis, we will use the terminology “quadratic quantum $\mathbb{P}^3$” to refer to an AS-regular algebra of global dimension four. A second type of regularity that we will briefly use is known as Auslander regularity.

Definition 2.1.3.6. Auslander Regular Algebra (cf. [23])

Let A be a noetherian $\Bbbk$-algebra.

(i) An A-module M satisfies the Auslander-condition if, for all $q \geq 0$,

$$q \leq \inf\left\{i : \operatorname{Ext}^i_A(N, A) \neq 0\right\},$$

for all A submodules N of $\operatorname{Ext}^q(M, A)$.

(ii) The algebra A is said to be Auslander regular of global dimension d if $\mathrm{gldim}(A) = d < \infty$ and every finitely generated A-module satisifes the Auslander-condition.

By [23] and [24, Proposition 1.3], if a quadratic $\Bbbk$-algebra A is Auslander regular and has polynomial growth, then it is Artin-Schelter regular.

2.1.4 Ore Extensions and Twists by Automorphisms

In this subsection, we will look at two methods of obtaining new algebras from existing ones. The first, known as an Ore extension, allows one to append a new generator to an algebra and obtain an algebra of higher global dimension. The second, twisting by an automorphism, creates a new graded algebra from a known graded algebra that will have the same vector space structure and quantum space (which will be defined in Section 2.3).

Definition 2.1.4.1. Ore Extension [14]
Let A be a $\Bbbk$-algebra.

(i) Let φ be an endomorphism of A. We call a linear map $\delta : A \to A$ a left φ-derivation on A if $\delta(ab) = \varphi(a)\delta(b) + \delta(a)b$, for all $a, b \in A$.

(ii) Let φ be an endomorphism of A and δ be a left φ-derivation on A. We shall write $B = A[x; \varphi, \delta]$ provided that

  (a) B is a $\Bbbk$-algebra, containing A as a subalgebra,

  (b) x is an element of B,

  (c) B is a free left A-module with basis $\{1, x, x^2, x^3, ...\}$, and

  (d) $xa = \varphi(a)x + \delta(a)$, for all $a \in A$.

  Such an algebra is called an Ore extension of A.

Example 2.1.4.2. Let $B = \mathbb{k}[x,y] = \mathbb{k}\langle x,y\rangle/\langle xy - yx\rangle$. Define linear maps $\varphi : \mathbb{k}\langle x,y\rangle \to \mathbb{k}\langle x,y\rangle$ and $\delta : \mathbb{k}\langle x,y\rangle \to \mathbb{k}\langle x,y\rangle$ as follows:

$$\varphi(x) = x, \qquad \varphi(y) = qy, \qquad \delta(x) = 0, \qquad \delta(y) = q^{-1}y^2,$$

where $q \in \mathbb{k}^\times$. We must first show that φ and δ descend to maps on B; we show that φ and δ map $\langle xy - yx\rangle$ to itself. Since B is a polynomial ring, φ naturally descends to an endomorphism of B. We must show that δ descends to a left φ-derivation on B:

$$\begin{aligned}\delta(xy - yx) = \delta(xy) - \delta(yx) &= \varphi(x)\delta(y) + \delta(x)y - \varphi(y)\delta(x) - \delta(y)x\\ &= q^{-1}xy^2 + 0 - 0 - q^{-1}y^2x \in \langle xy - yx\rangle,\end{aligned}$$

and it follows that $\delta(\langle xy - yx\rangle) \subset \langle xy - yx\rangle$. Thus, δ is a left φ-derivation on B. Therefore, the Ore extension $A = B[z; \varphi, \delta]$ is well defined. Explicitly, we have that

$$A = \frac{\mathbb{k}\langle x,y,z\rangle}{\langle xy - yx, xz - zx, zy - qyz - q^{-1}y^2\rangle}.$$

We now turn our attention to twisting an algebra by an automorphism.

Definition 2.1.4.3. Twist by an Automorphism [4]
Let A be a graded $\mathbb{k}$-algebra and $\tau \in \mathrm{Aut}(A)$ with $\tau(A_i) = A_i$, for all i. We define the twist of A by τ as the $\mathbb{k}$-algebra A^τ where:

- $A^\tau \cong A$ as a $\mathbb{k}$-vector space, and
- the multiplication in A^τ, denoted $\star$, is defined by $\overline{a} \star \overline{b} = a\tau^d(b)$, where $a \in A_d$, $b \in A_i$, and $\overline{a}$, $\overline{b}$ are the elements in A^τ corresponding to a and b, respectively.

Example 2.1.4.4. Let $A = \Bbbk\langle x, y\rangle/\langle yx - qxy\rangle$, where $q \in \Bbbk^{\times}$ and A is graded in the standard way. Define an automorphism $\tau : A \to A$ by $\tau(x) = x$ and $\tau(y) = qy$. It follows that

$$\overline{y} \star \overline{x} = y\tau(x) = yx = qxy = x(qy) = x\tau(y) = \overline{x} \star \overline{y},$$

so, $A^{\tau} \cong \Bbbk[\overline{x}, \overline{y}]$.

By [4, Corollary 8.5], the quantum space (defined in Section 2.3) of an algebra is invariant under twisting. So, if two $\Bbbk$-algebras have the same quantum space, then it is possible that the algebras are isomorphic to, or twists of, one another. This will motivate some of our work in Chapter 4.

2.2 Projective Algebraic Geometry

We now continue our preliminary chapter with some discussion of basic ideas from algebraic geometry. We will focus mainly on the definition of projective space. The definitions, results and examples in this section can be found in books such as [9], [11], [15], and [16].

Definition 2.2.0.1. Projective n-Space
Define an equivalence relation on $\Bbbk^{n+1} \setminus \{0\}$ by $(\alpha_0, ..., \alpha_n) \sim (\beta_0, ..., \beta_n)$ if and only if there exists $\lambda \in \Bbbk^{\times}$ such that $\beta_i = \lambda\alpha_i$, for all i. Projective n-space is $\mathbb{P}^n = (\Bbbk^{n+1} \setminus \{0\})/ \sim$.

Definition 2.2.0.2. Projective Variety
If $f_1, ..., f_m \in \Bbbk[x_0, ..., x_n]$ are homogeneous, then the projective algebraic variety determined by $f_1, ..., f_m$ is $\mathcal{V}(f_1, ..., f_m) = \{p \in \mathbb{P}^n : f_i(p) = 0, \text{ for all } i\}$.

Example 2.2.0.3. If $V = \mathcal{V}(x-y) \subset \mathbb{P}^2$, then $(\alpha, \alpha, \beta) \in V$ for all $\alpha, \beta \in \Bbbk$, not both zero. So, if $\alpha = 0$, then we obtain the point $(0,0,1) \in \mathbb{P}^2$. If $\alpha \neq 0$, we may take $\alpha = 1$ and obtain the line of points $\{(1,1,\beta) : \beta \in \Bbbk\}$. Thus,

$$V = \{(1,1,\beta) : \beta \in \Bbbk\} \cup \{(0,0,1)\}.$$

It should be noted that only the zero locus of homogeneous polynomials is well defined in projective space. Furthermore, it will be useful in our analysis to look only at irreducible varieties.

It is also possible to obtain a projective variety from an affine variety. If $V \subset \Bbbk^n$ is an affine variety, then the projective closure in $\mathbb{P}^n$ of V is denoted $\mathbb{P}(V)$ and is the smallest projective variety in $\mathbb{P}^n$ that contains V.

We will also be discussing projective schemes in this thesis. The official definition of a scheme is attributed to Groethendiek; details can be found in [15, 16]. Herein, it is enough to consider a projective scheme as a projective variety that encodes the multiplicity of the points in the scheme.

Example 2.2.0.4. If $V = \mathcal{V}(x)$ and $W = \mathcal{V}(x^2)$ in $\mathbb{P}^2$, then as projective varieties, $V = W = \{(0,1)\}$. However, $V \neq W$ as projective schemes as we consider V to be a point and W a double point (or a point with multiplicity 2).

2.3 Artin, Tate and Van den Bergh's Geometry

In this section we define the geometry developed by Artin, Tate and Van den Bergh in [3] and discuss how Shelton and Vancliff added to the field with their work in [29, 30].

Definition 2.3.0.1. Point Module [3]

Let $A = \bigoplus_{i=0}^{\infty} A_i$, with $A_0 = \Bbbk$, be an associative graded $\Bbbk$-algebra, generated by A_1, with $\dim(A_1) = n < \infty$. A graded right A-module $M = \bigoplus_{j=0}^{\infty} M_j$ is called a point module if M is cyclic, generated by M_0 and $\dim_{\Bbbk}(M_j) = 1$, for all j.

To every point module M over A, one can associate a point in $\mathbb{P}^{n-1}$ as follows. Assume that $M = \bigoplus_{j=0}^{\infty} \Bbbk m_j$, where $m_j \in M_j^{\times}$. Since M is graded, $m_0 a = \alpha_a m_1$, where $\alpha_a \in \Bbbk$, for all $a \in A_1$; since M is cyclic, there exists $a \in A_1$ such that $\alpha_a \neq 0$. Define a $\Bbbk$-linear epimorphism $\varphi : A_1 \to \Bbbk$ by $\varphi(a) = \alpha_a$. If $U = \ker(\varphi)$, then $\Bbbk \cong A_1/U$ and so $\dim_{\Bbbk}(U) = n - 1$. The annihilator of U, denoted $U^{\perp}$, in A_1^* is one-dimensional; thus, $\mathbb{P}(U^{\perp})$ is zero-dimensional and is, hence, a point in $\mathbb{P}(A_1^*)$.

Example 2.3.0.2. If $A = \Bbbk[x_1, x_2, x_3, x_4]$ and $M = A/(x_1 A + (x_2 - x_3)A + x_4 A)$, then M is a point module of A with associated point $(0, 1, 1, 0) \in \mathbb{P}^3$.

The collection of all such points, counted with multiplicity, that can be associated to the point modules of a quadratic $\Bbbk$-algebra A is called the point scheme of A [3]. Following the method in [3], under certain hypotheses one can compute the point scheme by first writing the defining relations of a quadratic algebra A with generators $x_1, ..., x_n$, as a matrix equation of the form $Nx = 0$, were $x = (x_1, ..., x_n)^T$. The point scheme of A can be identified with the zero locus in $\mathbb{P}(A_1^*)$ of the maximal minors of N.

Definition 2.3.0.3. Line Module [3]
Let $A = \bigoplus_{i=0}^{\infty} A_i$, with $A_0 = \Bbbk$, be an associative graded $\Bbbk$-algebra, generated by A_1, with $\dim(A_1) = n < \infty$. A graded right A-module $L = \bigoplus_{j=0}^{\infty} L_j$ is called a line module if L is cyclic, generated by L_0 and $\dim_{\Bbbk}(L_j) = j + 1$, for all j.

Similar to a point module, to every line module one can associate a line in $\mathbb{P}^{n-1}$. Since L is cyclic, A maps onto L in a natural way. Also, because both A and L are graded, $A_1 \twoheadrightarrow L_1$. The kernel, U, of this map has dimension $n-2$; so, $\mathbb{P}(U^\perp) \subset \mathbb{P}(A_1^*)$ is one-dimensional and is, hence, a line in $\mathbb{P}(A_1^*)$.

Example 2.3.0.4. If $A = \Bbbk[x_1, x_2, x_3, x_4]$ and $L = A/(x_1A + (x_2 - x_3)A)$, then L is a line module of A with associated line $\{(0, 1, 1, \alpha) \in \mathbb{P}^3 : \alpha \in \mathbb{P}^1\}$.

By [29], the collection of all such lines that can be associated to the line modules of a quadratic quantum $\mathbb{P}^3$, A, is called the line scheme, $\mathfrak{L}$, of A; $\mathfrak{L}$ can be realized as a subscheme of $\mathbb{P}^5$. The method of computing the line scheme of a quadratic quantum $\mathbb{P}^3$, A, is given in [30]. We outline the procedure here.

1. Compute the Koszul dual of A to obtain a quadratic $\Bbbk$-algebra on the dual generators $z_1, \ldots, z_4$ with 10 defining relations.
2. Rewrite the defining relations as $\tilde{M}z = 0$ where $\tilde{M}$ is a 10×4 matrix and $z = (z_1 \cdots z_4)^T$.
3. Produce a 10×8 matrix from $\tilde{M}$ by concatenating two 10×4 matrices; the first is obtained by replacing every z_i in $\tilde{M}$ with $u_i \in \Bbbk$ and the second by using $v_i \in \Bbbk$.
4. Each of the 45 8×8 minors of this matrix is a bihomogeneous polynomial of bi-degree $(4, 4)$ in the u_i and v_i and so each minor is a linear combination of products of polynomials of the form $N_{ij} = u_iv_j - u_jv_i$, for $1 \leq i < j \leq 4$.
5. Apply the map
$$\begin{aligned} N_{12} &\mapsto M_{34}, & N_{13} &\mapsto -M_{24}, & N_{14} &\mapsto M_{23}, \\ N_{23} &\mapsto M_{14}, & N_{24} &\mapsto -M_{13}, & N_{34} &\mapsto M_{12}, \end{aligned}$$
to the polynomials to yield quartic polynomials in the six Plücker coordinates, M_{ij}, for $1 \leq i < j \leq 4$.
6. The line scheme of A may be realized as the scheme of zeros of these 45 polynomials and the Plücker polynomial $P = M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}$.

Note that the scheme of all lines in $\mathbb{P}^3$ is parametrized by $\mathcal{V}(P) \subset \mathbb{P}^5$. This is known as the Plücker embedding. The task of recovering the lines in $\mathbb{P}^3$ that correspond to line modules from the line scheme is done for specific algebras in Section 3.3.1 and Chapter 4. For more information on Plücker coordinates, the reader is referred to [15].

In this thesis, given a quadratic quantum $\mathbb{P}^3$, A, we use the term "quantum space" to refer to the collection of all point modules and line modules of A.

2.4 Graded Skew Clifford Algebras

The family of algebras that we will consider in Chapter 3 was defined in [5]. This family is an example of a type of algebra that Cassidy and Vancliff defined in [5] that generalizes the notion of a graded Clifford algebra [20]. Before defining this type of algebra, we must generalize the notion of a symmetric matrix.

Definition 2.4.0.1. μ-Symmetric Matrix [5]
Let $\mu_{ij} \in \mathbb{k}^\times$, where $1 \leq i, j \leq n$, such that $\mu_{ij}\mu_{ji} = 1$ for all $i \neq j$. We write $\mu = (\mu_{ij}) \in \mathbb{M}_n(\mathbb{k})$. A matrix $M \in \mathbb{M}_n(\mathbb{k})$ is called μ-symmetric if $M_{ij} = \mu_{ij} M_{ji}$ for all i, j.

In this thesis, we will assume that $\mu_{ii} = 1$ for all i.

Example 2.4.0.2.

1. If $\mu \in \mathbb{M}_n(\mathbb{k})$ with $\mu_{ij} = 1$ for all i, j, then μ-symmetric matrices are precisely symmetric matrices.
2. If $\mu = \begin{bmatrix} 1 & 1/3 & -i \\ 3 & 1 & i \\ i & -i & 1 \end{bmatrix}$, then $M = \begin{bmatrix} 5 & 1 & 0 \\ 3 & 8 & 8 \\ 0 & -8i & i \end{bmatrix}$ is a μ-symmetric matrix.

We will also need the definition of a normalizing sequence in order to generalize the concept of a graded Clifford algebra.

Definition 2.4.0.3. Normalizing Sequence [26, §4.1.13]
For a $\Bbbk$-algebra A, we call $\{b_1, ..., b_m\} \subset A$ a normalizing sequence in A if $\langle b_1, ..., b_m\rangle \neq A$, b_1 is normal in A (i.e., $Ab_1 = b_1A$) and the image of b_{k+1} is normal in $A/\langle b_1, ..., b_k\rangle$, for all k.

Example 2.4.0.4. Let $A = \Bbbk\langle x_1, x_2\rangle/\langle x_1x_2 - x_2x_1 - x_1^2\rangle$. Notice that x_2 is not normal in A but $x_1x_2 = (x_2 + x_1)x_1$, so x_1 is normal in A. Also, $A/\langle x_1\rangle \cong \Bbbk[x_2]$ and so x_2 is normal in $A/\langle x_1\rangle$. Thus, $\{x_1, x_2\}$ is a normalizing sequence of A.

With these concepts, we may now give the definition of a graded skew Clifford algebra.

Definition 2.4.0.5. Graded Skew Clifford Algebra [5]
Let $\mu \in \mathbb{M}_n(\Bbbk)$ be as above and $M_1, ..., M_n$ be μ-symmetric matrices. A graded skew Clifford algebra $A = A(\mu, M_1, ..., M_n)$ associated to μ and $M_1, ..., M_n$ is a graded $\Bbbk$-algebra on degree-one generators $x_1, ..., x_n$ and on degree-two generators $y_1, ..., y_n$ with defining relations given by the following:

(a) $x_ix_j + \mu_{ij}x_jx_i = \sum_{k=1}^{n}(M_k)_{ij}y_k$ for all $i, j = 1, ..., n$, and
(b) any additional relations needed to guarantee the existence of a normalizing sequence that spans $\Bbbk y_1 + \cdots + \Bbbk y_n$.

Like a graded Clifford algebra, we may associate some geometry to a graded skew Clifford algebra through the quadratic forms related to the defining matrices. This geometry will be in the spirit of Artin, Tate and Van den Bergh's geometry.

To each μ-symmetric matrix M_k, we can associate a noncommutative quadratic form $q_k \in S = \Bbbk\langle z_1, ..., z_n\rangle/\langle z_j z_i - \mu_{ij} z_i z_j : 1 \leq i, j \leq n\rangle$ via

$$q_k = [z_1 \cdots z_n]\ M_k\ [z_1 \cdots z_n]^T.$$

The collection of all such quadratic forms is called a quadric system. We say that a quadric system Q is right base-point free if $S/\langle Q\rangle$ has no point modules or fat point modules [5].

The family of graded skew Clifford algebras we consider will consist of regular algebras. In [5], Cassidy and Vancliff gave some equivalent conditions for a graded skew Clifford algebra to be regular. We now list those below.

Theorem 2.4.0.6. *[5] A graded skew Clifford algebra $A = A(\mu, M_1, ..., M_n)$ is a quadratic Auslander regular algebra of global dimension n that satisifies the Cohen-Macaulay property with Hilbert series $1/(1-t)^n$ if and only if the associated quadric system $\{q_1, ..., q_n\}$ is normalizing and base-point free; in this case, A is Artin-Schelter regular, a noetherian domain and unique up to isomorphism.*

The $\Bbbk$-algebras we consider in Chapter 3 were constructed in [5] to be AS-regular. Details of the construction of these $\Bbbk$-algebras may be found in [5].

2.5 Lie-Type Algebras

We finish our preliminary material with discussions on Lie-type algebras. We will first consider the traditional Lie algebra, then consider the Lie superalgebra and conclude with the definition of a color Lie algebra.

2.5.1 Lie Algebras and Universal Enveloping Algebras

Definition 2.5.1.1. Lie algebra (cf. [17])

Assume that $\mathrm{char}(\mathbb{k}) \neq 2$. A $\mathbb{k}$-vector space $\mathfrak{g}$ with an operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted $(x, y) \mapsto [x, y]$ and called the bracket of x and y, is called a Lie algebra over $\mathbb{k}$ if:

(i) the bracket operation is bilinear,

(ii) $[x, y] = -[y, x]$, for all $x, y \in \mathfrak{g}$, and

(iii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, for all $x, y, z \in \mathfrak{g}$.

The final condition in the above definition is known as the Jacobi identity and generalizes the notion of associativity from an associative $\mathbb{k}$-algebra.

Example 2.5.1.2.

(a) If $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{k})$ is the vector space of all 2×2 matrices with entries in $\mathbb{k}$, then $\mathfrak{g}$ becomes a Lie algebra with bracket defined by $[X, Y] = XY - YX$, for all $X, Y \in \mathfrak{gl}(2, \mathbb{k})$. If we take $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ as the standard basis of $\mathfrak{gl}(2, \mathbb{k})$, then the bracket is defined by $[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$, where δ_{jk} is the kronecker-delta.

(b) If $\mathfrak{h} = \mathbb{k}x \oplus \mathbb{k}y \oplus \mathbb{k}z$, then $\mathfrak{h}$ becomes a Lie algebra under the bracket $[x, y] = z$ and $[x, z] = 0 = [y, z]$; $\mathfrak{h}$ is known as the Heisenberg Lie algebra.

(c) Define $\mathfrak{sl}(2, \mathbb{k}) = \{M \in \mathfrak{gl}(2, \mathbb{k}) : \mathrm{tr}(M) = 0\}$, where $\mathrm{tr}(M)$ is the trace of M. Every element of $\mathfrak{sl}(2, \mathbb{k})$ is of the form

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix},$$

where $a, b, c \in \mathbb{k}$; hence, $\mathfrak{sl}(2, \mathbb{k})$ is a three-dimensional vector space with basis elements

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The vector space $\mathfrak{sl}(2,\Bbbk)$ becomes a Lie algebra under the commutator bracket induced by $\mathfrak{gl}(2,\Bbbk)$. Using the basis $\{e,f,h\}$, the Lie bracket on $\mathfrak{sl}(2,\Bbbk)$ is defined by

$$[e,f]=h, \qquad [h,e]=2e, \qquad [h,f]=-2f.$$

The geometric constructions of Artin, Tate and Van den Bergh require an associative algebra. So in order to associate geometry to a Lie algebra, we will associate the geometry instead to its universal enveloping algebra.

Definition 2.5.1.3. Universal Enveloping Algebra of a Lie Algebra (cf. [17])
If $\mathfrak{g}$ is a finite-dimensional Lie algebra with basis $\{x_1,...,x_n\}$, then the universal enveloping algebra of $\mathfrak{g}$ is the associative $\Bbbk$-algebra defined by

$$\mathcal{U}(\mathfrak{g})=\frac{\Bbbk\langle x_1,...,x_n\rangle}{\langle x_ix_j-x_jx_i-[x_i,x_j]:i,j=1,...,n\rangle}.$$

This definition of $\mathcal{U}(\mathfrak{g})$ is sufficient for the purposes of this thesis. A more general definition that does not rely on the selection of a basis of $\mathfrak{g}$ is given in [17].

Example 2.5.1.4. If $\mathfrak{h}$ is the Heisenberg Lie algebra, then

$$\mathcal{U}(\mathfrak{h})=\frac{\Bbbk\langle x,y,z\rangle}{\langle xy-yx-z,xz-zx,yz-zy\rangle}.$$

Consideration of $\mathcal{U}(\mathfrak{g})$ when studying the modules of $\mathfrak{g}$ is quite natural as the category of modules of $\mathcal{U}(\mathfrak{g})$ is equivalent to the category of modules of $\mathfrak{g}$ [17].

The universal enveloping algebra of certain Lie algebras contain a distinguished element known as the Casimir element. For details on the construction of this element, the reader is again referred to [17]. The distinguishing feature of this element is that it is central in $\mathcal{U}(\mathfrak{g})$ and, in the case of $\mathcal{U}(\mathfrak{sl}(2,\Bbbk))$, it generates the center of $\mathcal{U}(\mathfrak{sl}(2,\Bbbk))$.

Example 2.5.1.5. Let $\mathfrak{so}(3, \Bbbk)$ be the Lie-algebra with basis $\{E_{32} - E_{23}, E_{13} - E_{31}, E_{21} - E_{12}\}$ and bracket defined by $[X, Y] = XY - YX$, for all $X, Y \in \mathfrak{so}(3, \Bbbk)$. Under the identification $x_1 = E_{32} - E_{23}, x_2 = E_{13} - E_{31}$, and $x_3 = E_{21} - E_{12}$, the bracket is given by

$$[x_1, x_2] = x_3, \qquad [x_3, x_1] = x_2, \qquad [x_2, x_3] = x_1.$$

The universal enveloping algebra of $\mathfrak{so}(3, \Bbbk)$ is

$$\mathcal{U}(\mathfrak{so}(3, \Bbbk)) = \frac{\Bbbk\langle x_1, x_2, x_3\rangle}{\langle x_1x_2 - x_2x_1 - x_3, x_3x_1 - x_1x_3 - x_2, x_2x_3 - x_3x_2 - x_1\rangle},$$

and the Casimir element is $\omega = x_1^2 + x_2^2 + x_3^2$. Notice that ω is central in $\mathcal{U}(\mathfrak{so}(3, \Bbbk))$.

2.5.2 Lie Superalgebras and Universal Enveloping Algebras

Definition 2.5.2.1. Superspace (cf. [7])

A superspace is a $\mathbb{Z}_2$-graded $\Bbbk$-vector space; that is, a superspace is a $\Bbbk$-vector space V with subspaces V_0 and V_1 such that $V = V_0 \oplus V_1$. We call V_0 the even part of V and V_1 the odd part of V.

Example 2.5.2.2. The $\Bbbk$-vector space

$$\Bbbk^{m|n} = \{(\alpha_1, \alpha_2, ..., \alpha_m; \beta_1, \beta_2, ..., \beta_n) : \alpha_i, \beta_j \in \Bbbk, \text{ for all } i, j\}$$

is a superspace with

$$V_0 = \{(\alpha_1, \alpha_2, ..., \alpha_m; 0, 0, ..., 0) : \alpha_i \in \Bbbk, \text{ for all } i\}$$

and

$$V_1 = \{(0, 0, ..., 0; \beta_1, \beta_2, ..., \beta_n) : \beta_j \in \Bbbk, \text{ for all } j\}.$$

Given a superspace V and $x \in V$, if $x \in V_i$, then we say that $|x| = i$ is the parity of x. If $|x| = 0$, then we call x even; if $|x| = 1$, then we call x odd. If x has parity, we also say that x is homogeneous.

Definition 2.5.2.3. Lie Superalgebra (cf. [7])

Let $\mathrm{char}(\Bbbk) = 0$. A superspace $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with an operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted $(x, y) \mapsto [x, y]$ and called the superbracket of x and y, is called a Lie superalgebra over $\Bbbk$ if:

(i) $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j \pmod 2}$, for all i, j,

(ii) the bracket operation is bilinear,

(iii) $[x, y] = -(-1)^{|x||y|}[y, x]$, for all homogeneous $x, y \in \mathfrak{g}$, and

(iv) $(-1)^{|x||z|}[x, [y, z]]+(-1)^{|x||y|}[y, [z, x]]+(-1)^{|y||z|}[z, [x, y]] = 0$, for all homogeneous $x, y, z \in \mathfrak{g}$.

The last condition above is known as the super-Jacobi identity. Note that the above also implies that $\mathfrak{g}_0$ has a Lie algebra structure with the induced bracket.

Example 2.5.2.4. If $\mathfrak{gl}(m|n)$ denotes the $\Bbbk$-vector space of $(m + n) \times (m + n)$ block matrices of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A \in \Bbbk^{m\times m}$, $B \in \Bbbk^{m\times n}$, $C \in \Bbbk^{n\times m}$ and $D \in \Bbbk^{n\times n}$, then $\mathfrak{gl}(m|n)$ has a superspace structure with

$$\mathfrak{g}_0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} : A \in \Bbbk^{m\times m}, D \in \Bbbk^{n\times n} \right\}, \ \mathfrak{g}_1 = \left\{ \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} : B \in \Bbbk^{m\times n}, C \in \Bbbk^{n\times m} \right\}.$$

Furthermore, $\mathfrak{gl}(m|n)$ is a Lie superalgebra with bracket $[X, Y] = XY - (-1)^{|X||Y|}Y X$, for all homogeneous $X, Y \in \mathfrak{gl}(m|n)$.

The concepts of a universal enveloping algebra and Casimir element generalize from Lie algebras to Lie superalgebras.

Definition 2.5.2.5. Universal Enveloping Algebra of a Lie Superalgebra (cf. [7])

If $\mathfrak{g}$ is a finite-dimensional Lie superalgebra with basis $\{x_1, ..., x_n\}$ of homogeneous elements, then the universal enveloping algebra of $\mathfrak{g}$ is the associative $\Bbbk$-algebra defined by

$$\mathcal{U}(\mathfrak{g}) = \frac{\Bbbk\langle x_1, ..., x_n\rangle}{\langle x_i x_j - (-1)^{|x_i||x_j|} x_j x_i - [x_i, x_j] : i, j = 1, ..., n\rangle}.$$

The Casimir element of a Lie superalgebra supercommutes in the algebra; that is, if ω is the Casimir element and $x \in \mathcal{U}(\mathfrak{g})$ has parity, then $x\omega = (-1)^{|x|}\omega x$.

2.5.3 Color Lie Algebras and Universal Enveloping Algebras

Before defining a color Lie algebra, we must first define a bicharacter map. If G is an abelian group, then $\epsilon : G \times G \to \Bbbk^{\times}$ is an antisymmetric bicharacter map if:

(a) $\epsilon(g, h)\epsilon(h, g) = 1$, for all $g, h \in G$,

(b) $\epsilon(g, hk) = \epsilon(g, h)\epsilon(g, k)$, for all $g, h, k \in G$, and

(c) $\epsilon(gh, k) = \epsilon(g, k)\epsilon(h, k)$, for all $g, h, k \in G$.

Definition 2.5.3.1. Color Lie Algebra [27]

Let ϵ be a bicharacter map on G and assume that $\mathrm{char}(\Bbbk) = 0$. By a color Lie algebra $\mathfrak{g}$, we mean $\mathfrak{g} = \bigoplus_{g\in G} \mathfrak{g}_g$ is a G-graded space over $\Bbbk$, equipped with a bilinear multiplication $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that:

(i) $[\mathfrak{g}_g, \mathfrak{g}_h] \subset \mathfrak{g}_{gh}$, where $g, h \in G$,

(ii) $[x, y] = -\epsilon(g, h)[y, x]$, where $x \in \mathfrak{g}_g$, $y \in \mathfrak{g}_h$, and $g, h \in G$, and

(iii) $\epsilon(k, g)[x, [y, z]] + \epsilon(g, h)[y, [z, x]] + \epsilon(h, k)[z, [x, y]] = 0$ where $x \in \mathfrak{g}_g$, $y \in \mathfrak{g}_h$, $z \in \mathfrak{g}_k$ and $g, h, k \in G$.

This last condition is known as the color-Jacobi identity. Note that if we take G to be the field $\mathbb{Z}_2 = \{0, 1\}$ and take $\epsilon(g, h) = (-1)^{gh}$, where $g, h \in \mathbb{Z}_2$, then we may realize Lie superalgebras as a subclass of color Lie algebras. As such, the definition of the universal enveloping algebra of a color Lie algebra is a natural generalization of that of the universal enveloping algebra of a Lie superalgebra.

Definition 2.5.3.2. Universal Enveloping Algebra of a Color Lie Algebra (cf. [27]) If $\mathfrak{g}$ is a finite-dimensional G-graded color Lie algebra over $\mathbb{k}$ with basis $\{x_1, ..., x_n\}$ of homogeneous elements, then the universal enveloping algebra of $\mathfrak{g}$ is the associative $\mathbb{k}$-algebra defined by

$$\mathcal{U}(\mathfrak{g}) = \frac{\mathbb{k}\langle x_1, ..., x_n\rangle}{\langle x_i x_j - \epsilon(g,h) x_j x_i - [x_i, x_j] : x_i \in \mathfrak{g}_g, x_j \in \mathfrak{g}_h, g, h \in G, i, j = 1, ..., n\rangle}.$$

Chapter 3

A Family of Quadratic Quantum $\mathbb{P}^3$'s

In this chapter we will examine a family of algebras whose generic member is a candidate for a generic quadratic quantum $\mathbb{P}^3$. This family was originally defined in [5]; it was constructed in such a way that the generic member has a point scheme consisting of twenty distinct points and a one-dimensional line scheme. Prior to Cassidy and Vancliff constructing this family, the only known example of such an algebra was the algebra found by Shelton and Tingey in [28]. However, techniques used in [28] and [5] determined only the dimension of the line schemes of these algebras and not the line scheme itself.

3.1 The Family of Algebras $\mathcal{A}(\gamma)$

Definition 3.1.0.1. The Family of Algebras $\mathcal{A}(\gamma)$ [5]
Let $\gamma \in \Bbbk^{\times}$ and write $\mathcal{A}(\gamma)$ for the $\Bbbk$-algebra on generators x_1, x_2, x_3, x_4 with defining relations:

$$x_4x_1 = ix_1x_4, \qquad x_3^2 = x_1^2, \qquad x_3x_1 = x_1x_3 - x_2^2,$$

$$x_3x_2 = ix_2x_3, \qquad x_4^2 = x_2^2, \qquad x_4x_2 = x_2x_4 - \gamma x_1^2,$$

where $i^2 = -1$.

By construction of $\mathcal{A}(\gamma)$ in [5], $\mathcal{A}(\gamma)$ is a regular noetherian domain of global dimension four with Hilbert series the same as that of the polynomial ring on four variables. The special member $\mathcal{A}(1)$ is the algebra introduced in [28]. It should be noted that the polynomials given in [28] that define the point scheme of $\mathcal{A}(\gamma)$ have some sign errors. Moreover, $\mathcal{A}(1)$ was studied in [13] in the context of finding the scheme of lines associated to each point of the point scheme; the entire line scheme was not analyzed. Thank you to B. Shelton and M. Vancliff for providing their notes on a potential approach toward computing the line scheme of the algebra defined in [28].

It is useful to observe that $\mathcal{A}(\gamma) \cong \mathcal{A}(-\gamma)$, for all $\gamma \in \Bbbk^\times$, under the map that sends $x_2 \mapsto -x_2$ and $x_k \mapsto x_k$, for all $k \neq 2$. In fact, there exist two antiautomorphisms of $\mathcal{A}(\gamma)$ defined by $\psi_1 : x_1 \leftrightarrow x_3$, $x_2 \leftrightarrow x_4$, and $\psi_2 : x_2 \leftrightarrow \lambda x_3$, $x_4 \leftrightarrow \lambda x_1$, where $\lambda \in \Bbbk^\times$ and $\lambda^4 = \gamma$.

3.2 The Quantum Space of $\mathcal{A}(\gamma)$

In this section we will compute both the point scheme and line scheme of $\mathcal{A}(\gamma)$. The method follows that of [3] and [29, 30]. We will assume that $\mathrm{char}(\Bbbk) \neq 2$ in this section. Let $e_1, ..., e_4$ denote the four elementary points in $\mathbb{P}^3$; that is, $e_i = \mathcal{V}(x_j, x_k, x_l)$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

3.2.1 The Point Scheme of $\mathcal{A}(\gamma)$

Theorem 3.2.1.1.

(a) *For every $\gamma \in \Bbbk^\times$, the point scheme of $\mathcal{A}(\gamma)$ is $\mathfrak{p}(\gamma) = \{e_1, e_2, e_3, e_4\} \cup \mathcal{Z}_\gamma$, where $\mathcal{Z}_\gamma = \mathcal{V}(\ x_4^8 - 4x_4^4 + \gamma^2,\ x_3^2 - ix_3x_4^2 - 1,\ \gamma x_2 - 2ix_4^3 + x_3x_4^5\ )$. We call the points belonging to $\mathcal{Z}_\gamma$ the generic points of $\mathfrak{p}(\gamma)$.*

(b) *If $\gamma^2 \neq 4$, then $\mathfrak{p}(\gamma)$ has twenty distinct points.*

(c) *If* $\gamma^2 = 4$, *then* $\mathfrak{p}(\gamma)$ *has exactly twelve distinct points; the eight closed points of* $\mathcal{Z}_\gamma$ *have multiplicity two and the elementary points* e_1, e_2, e_3, e_4 *each have multiplicity one.*

Proof. Following [3], we write the defining relations of $\mathcal{A}(\gamma)$ in the form $Mx = 0$, where M is a 6×4 matrix and x is a column vector given by $x^T = (x_1, x_2, x_3, x_4)$. Thus, we may take M to be the matrix

$$M = \begin{bmatrix} x_4 & 0 & 0 & -ix_1 \\ 0 & x_3 & -ix_2 & 0 \\ x_1 & 0 & -x_3 & 0 \\ 0 & x_2 & 0 & -x_4 \\ x_3 & x_2 & -x_1 & 0 \\ \gamma x_1 & x_4 & 0 & -x_2 \end{bmatrix}.$$

The point scheme of $\mathcal{A}(\gamma)$ can be identified with the zero locus in $\mathbb{P}(\mathcal{A}(\gamma)_1^*)$ of the 4×4 minors of M. The polynomials that define $\mathfrak{p}(\gamma)$ are listed in Appendix 5.1.1.

If $x_1 = 0$, then computing a Gröbner basis using the polynomials in Appendix 5.1.1 yields the polynomials:

$$x_3x_4^3, \quad x_3^2x_4^2, \quad x_3^3x_4, \quad x_2x_4^3, \quad x_2x_3x_4^2, \quad x_2x_3^2x_4,$$

$$x_2x_3^3, \quad x_2^2x_4^2, \quad x_2^2x_3x_4, \quad x_2^2x_3^2, \quad x_2^3x_4, \quad x_2^3x_3.$$

An easy computation shows that these polynomials vanish precisely if the x_i pairwise vanish, for $i = 2, 3, 4$; that is, $e_2, e_3, e_4 \in \mathfrak{p}(\gamma)$. If $x_1 \neq 0$, we may take $x_1 = 1$. A Gröbner basis computation yields the polynomials

$$ix_4(\gamma^2 - 4x_4^4 + x_4^8), \quad ix_3(\gamma^2 - 4x_4^4 + x_4^8), \quad x_3^2 - ix_3x_4^2 - 1, \quad \gamma x_2 - 2ix_4^3 + x_3x_4^5.$$

If, in addition, $x_4 = 0$, we obtain that $e_1 \in \mathfrak{p}(\gamma)$. Otherwise, we see that

$$\mathcal{Z}_\gamma = \mathcal{V}(\ \gamma^2 - 4x_4^4 + x_4^8,\ x_3^2 - ix_3x_4^2 - 1,\ \gamma x_2 - 2ix_4^3 + x_3x_4^5\ )$$

gives the remaining points of the point scheme.

Since $x_4^8 - 4x_4^4 + \gamma^2 = 0$ if and only if $(x_4^4 - 2)^2 = 4 - \gamma^2$, we see that $x_4^8 - 4x_4^4 + \gamma^2$ has eight distinct zeros if and only if $\gamma^2 \neq 4$; if $\gamma^2 = 4$ then $x_4^8 - 4x_4^4 + \gamma^2$ has exactly four distinct zeros, each of multiplicity two. Given a zero, x_4, of $x_4^8 - 4x_4^4 + \gamma^2$, the equation $x_3^2 - ix_3x_4^2 - 1 = 0$ has a unique solution for x_3 if and only if $x_4^4 = 4$, which is not a solution of $x_4^8 - 4x_4^4 + \gamma^2 = 0$ since $\gamma \neq 0$; hence, there are two roots of $x_3^2 - ix_3x_4^2 - 1$, each of multiplicity one. Finally, given zeros x_3 and x_4 of $x_4^8 - 4x_4^4 + \gamma^2$ and $x_3^2 - ix_3x_4^2 - 1$, the equation $\gamma x_2 - 2ix_4^3 + x_3x_4^5 = 0$ has a unique solution.

If the point scheme of a quadratic algebra with four generators and six defining relations is finite, then it consists of twenty points counted with multiplicity (cf. [34]). Therefore, (b) and (c) are proved. $\square$

Corollary 3.2.1.2. *Let $A = A(\gamma)$ and $V = A_1$.*

(a) The points in $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$ on which the defining relations of A vanish are of the form (e_1, e_2), (e_2, e_1), $(e_3, e_4), (e_4, e_3)$ and

$$\left((1, \alpha_2, \alpha_3, \alpha_4), (1, i\alpha_2\alpha_3^{-2}, \alpha_3^{-1}, -i\alpha_4)\right),$$

where $(1, \alpha_2, \alpha_3, \alpha_4) \in \mathcal{Z}_\gamma$ and $i^2 = -1$.

(b) For all $\gamma \in \Bbbk^\times$, there exists an automorphism $\sigma : \mathfrak{p}(\gamma) \to \mathfrak{p}(\gamma)$ which, on closed points, is defined by: $e_1 \leftrightarrow e_2$, $e_3 \leftrightarrow e_4$, and

$$\sigma(1, \alpha_2, \alpha_3, \alpha_4) = (1, i\alpha_2\alpha_3^{-2}, \alpha_3^{-1}, -i\alpha_4),$$

for all $(1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Z}_\gamma$. Hence, on the closed points of $\mathfrak{p}(\gamma)$, σ has two orbits of length two and n orbits of length four, where $n = 4$ if $|\mathcal{Z}_\gamma| = 16$ and $n = 2$ if $|\mathcal{Z}_\gamma| = 8$.

Proof. Part (a) is easily reached by computation. The existence of the map in (b) follows from (a) and [22, Theorem 4.1.3]. The size of the orbits may be verified by computation. □

3.2.2 The Line Scheme of $\mathcal{A}(\gamma)$

For this section, we assume that $\mathrm{char}(\Bbbk) = 0$.

In [30], a method was given for computing the line scheme of any quadratic algebra on four generators that is a domain and has Hilbert series the same as that of the polynomial ring on four variables. In this subsection, we summarize that method while applying it to $\mathcal{A}(\gamma)$; further details may be found in [30].

The first step in the process is to compute the Koszul dual of $\mathcal{A}(\gamma)$. Let $\{z_1, z_2, z_3, z_4\}$ be the basis of V^* dual to $\{x_1, x_2, x_3, x_4\}$. The Koszul dual of $\mathcal{A}(\gamma)$ is the $\Bbbk$-algebra $\mathcal{A}(\gamma)^!$ on generators z_1, z_2, z_3, z_4 with defining relations:

$$\begin{aligned} z_1z_2 &= 0, & z_2z_1 &= 0, \\ z_3z_4 &= 0, & z_4z_3 &= 0, \\ iz_4z_1 &= -z_1z_4, & iz_3z_2 &= -z_2z_3, \\ z_3z_1 &= -z_1z_3, & z_4z_2 &= -z_2z_4, \\ z_1^2 + z_3^2 &= -\gamma z_2z_4, & z_2^2 + z_4^2 &= -z_1z_3. \end{aligned}$$

One then rewrites these relations in the form of a matrix equation similar to that used in Section 3.2.1; in this case, however, it yields the equation $\hat{M}z = 0$, where $z^T = (z_1, \ldots, z_4)$ and $\hat{M}$ is a 10×4 matrix whose entries are linear forms in the z_i.

One then produces a 10×8 matrix from $\hat{M}$ by concatenating two 10×4 matrices, the first of which is obtained from $\hat{M}$ by replacing every z_i in $\hat{M}$ by $u_i \in \Bbbk$, and the second is obtained from $\hat{M}$ by replacing every z_i in $\hat{M}$ by $v_i \in \Bbbk$, where

$(u_1, \dots, u_4), (v_1, \dots, v_4) \in \mathbb{P}^3$. For $\mathcal{A}(\gamma)$, this process yields the following 10×8 matrix:

$$\mathcal{M}(\gamma) = \begin{bmatrix} 0 & u_1 & 0 & 0 & 0 & v_1 & 0 & 0 \\ u_2 & 0 & 0 & 0 & v_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_3 & 0 & 0 & 0 & v_3 \\ 0 & 0 & u_4 & 0 & 0 & 0 & v_4 & 0 \\ u_3 & 0 & u_1 & 0 & v_3 & 0 & v_1 & 0 \\ 0 & u_4 & 0 & u_2 & 0 & v_4 & 0 & v_2 \\ -u_4 & 0 & 0 & iu_1 & -v_4 & 0 & 0 & iv_1 \\ 0 & -u_3 & iu_2 & 0 & 0 & -v_3 & iv_2 & 0 \\ u_1 & 0 & u_3 & \gamma u_2 & v_1 & 0 & v_3 & \gamma v_2 \\ 0 & u_2 & u_1 & u_4 & 0 & v_2 & v_1 & v_4 \end{bmatrix}.$$

Each of the forty-five 8×8 minors of $\mathcal{M}(\gamma)$ is a bihomogeneous polynomial of bidegree $(4, 4)$ in the u_i and v_i, and so each such minor is a linear combination of products of polynomials of the form $N_{ij} = u_i v_j - u_j v_i$, where $1 \leq i < j \leq 4$. Hence, $\mathcal{M}(\gamma)$ yields forty-five quartic polynomials in the six variables N_{ij}. Following [30], one then applies the map:

$$N_{12} \mapsto M_{34}, \quad N_{13} \mapsto -M_{24}, \quad N_{14} \mapsto M_{23},$$
$$N_{23} \mapsto M_{14}, \quad N_{24} \mapsto -M_{13}, \quad N_{34} \mapsto M_{12},$$

to the polynomials, which yields forty-five quartic polynomials in the Plücker coordinates M_{ij} on $\mathbb{P}^5$.

The line scheme $\mathfrak{L}(\gamma)$ of $\mathcal{A}(\gamma)$ may be realized in $\mathbb{P}^5$ as the scheme of zeros of these forty-five polynomials in the M_{ij} together with the Plücker polynomial $P = M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}$. For $\mathcal{A}(\gamma)$, these polynomials were found by using Wolfram's Mathematica and are listed in Appendix 5.1.2.

In the remainder of this section, we compute and describe $\mathfrak{L}(\gamma)$ as a subscheme of $\mathbb{P}^5$. The lines in $\mathbb{P}(V^*)$ that correspond to the points of $\mathfrak{L}(\gamma)$ are described in Section 3.3.

3.2.3 Computing the Closed Points of the Line Scheme of $\mathcal{A}(\gamma)$

Our procedure in this subsection focuses on finding the closed points of the line scheme $\mathfrak{L}(\gamma)$ of $\mathcal{A}(\gamma)$; in the next subsection, we will prove that $\mathfrak{L}(\gamma)$ is reduced and so is given by its closed points. We denote the variety of closed points of $\mathfrak{L}(\gamma)$ by $\mathfrak{L}'(\gamma)$.

Subtracting the polynomials 5.1.2.19 and 5.1.2.20 produces $M_{14}M_{23}M_{24}^2$. If $M_{14} = M_{23} = M_{24} = 0$, then $M_{12} = 0 = M_{34}$, so there is a unique solution in this case. This leaves six cases to consider:

$$\begin{array}{rlrl}
\text{(I)} & M_{14}M_{23} \neq 0,\ M_{24} = 0, & \text{(IV)} & M_{23} \neq 0,\ M_{14} = 0 = M_{24}, \\
\text{(II)} & M_{23}M_{24} \neq 0,\ M_{14} = 0, & \text{(V)} & M_{14} \neq 0,\ M_{23} = 0 = M_{24}, \\
\text{(III)} & M_{14}M_{24} \neq 0,\ M_{23} = 0, & \text{(VI)} & M_{24} \neq 0,\ M_{14} = 0 = M_{23}.
\end{array}$$

We will outline the analysis for (I), (II), (IV) and (VI); the other cases follow from these four cases by using the map ψ_1 defined in Section 3.1. In applying the map ψ_1, the reader should recall that $M_{ji} = -M_{ij}$ for all $i \neq j$.

Case (I): $M_{14}M_{23} \neq 0$ and $M_{24} = 0$.

With the assumption that $M_{24} = 0$, a computation of a Gröbner basis yields several polynomials, one of which is $M_{13}^2 M_{14}M_{23}$. Hence, $M_{13} = 0$, and another computation of a Gröbner basis yields several polynomials, two of which are:

$$M_{14}M_{23} + M_{12}M_{34},$$
$$M_{34}^4 - M_{14}^2M_{34}^2 - M_{23}^2M_{34}^2 + \gamma M_{14}M_{23}M_{34}^2 + M_{14}^2M_{23}^2,$$

so that, in particular, $M_{12}M_{34} \neq 0$. Using the first polynomial to substitute for $M_{14}M_{23}$, and using the assumption that $M_{34} \neq 0$, we find that the second polynomial vanishes if and only if $M_{12}^2 + M_{34}^2 + \gamma M_{14}M_{23} - M_{14}^2 - M_{23}^2 = 0$. Another computation of a Gröbner basis yields only these polynomials, so that this case provides the component

$$\mathfrak{L}_1 = \mathcal{V}(\ M_{13},\ M_{24}, M_{14}M_{23} + M_{12}M_{34},\ M_{12}^2 + M_{34}^2 + \gamma M_{14}M_{23} - M_{14}^2 - M_{23}^2\ ).$$

In Theorem 3.2.3.1, we will prove that $\mathfrak{L}_1$ is irreducible if and only if $\gamma^2 \neq 16$. Here we show that if $\gamma^2 = 16$, then $\mathfrak{L}_1$ is the union of two nonsingular conics. Since $\mathcal{A}(4) \cong \mathcal{A}(-4)$, it suffices to consider $\gamma = 4$. In fact, let $\alpha \in \Bbbk$ and let

$$Q = M_{12}^2 + M_{34}^2 + \gamma M_{14}M_{23} - M_{14}^2 - M_{23}^2 + 2\alpha(M_{14}M_{23} + M_{12}M_{34}),$$

and associate to Q the symmetric matrix

$$\begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & -1 & \alpha + \frac{\gamma}{2} & 0 \\ 0 & \alpha + \frac{\gamma}{2} & -1 & 0 \\ \alpha & 0 & 0 & 1 \end{bmatrix},$$

which has rank at most two if and only if Q factors. This happens if and only if $(\gamma, \alpha) = (\pm 4, \mp 1)$. It follows that if $\gamma = 4$, then

$$Q = (M_{12} - M_{34} + M_{14} - M_{23})(M_{12} - M_{34} - M_{14} + M_{23}),$$

and $\mathfrak{L}_1 = \mathfrak{L}_{1a} \cup \mathfrak{L}_{1b}$, where

$$\mathfrak{L}_{1a} = \mathcal{V}(\ M_{13},\ M_{24},\ M_{14}M_{23} + M_{12}M_{34},\ M_{12} + M_{14} - M_{23} - M_{34}\ ),$$
$$\mathfrak{L}_{1b} = \mathcal{V}(\ M_{13},\ M_{24},\ M_{14}M_{23} + M_{12}M_{34},\ M_{12} - M_{14} + M_{23} - M_{34}\ ),$$

and each of $\mathfrak{L}_{1a}$ and $\mathfrak{L}_{1b}$ is a nonsingular conic, since using the last polynomial in each case to substitute for M_{12} in $M_{14}M_{23} + M_{12}M_{34}$ yields a rank-3 quadratic form in each case. Moreover, $\mathfrak{L}_{1b}$ is ψ_1 applied to $\mathfrak{L}_{1a}$.

Case (II): $M_{23}M_{24} \neq 0$ and $M_{14} = 0$.

With the assumption that $M_{14} = 0$, a computation of a Gröbner basis yields several polynomials, two of which are $M_{13}M_{23}M_{24}^2$ and $M_{23}M_{24}M_{34}^2$. Hence, $M_{13} = M_{34} = 0$. With these additional criteria, another computation of a Gröbner basis yields exactly three polynomials: $M_{12}f$, $M_{23}f$, $M_{24}f$, where $f = M_{12}^3 - M_{12}M_{23}^2 - iM_{23}M_{24}^2$. Thus, $f = 0$. It follows that this case yields the irreducible component

$$\mathfrak{L}_2 = \mathcal{V}(\ M_{13}, M_{14},\ M_{34},\ M_{12}^3 - M_{12}M_{23}^2 - iM_{23}M_{24}^2\ )$$

of $\mathfrak{L}'(\gamma)$.

Case (III): $M_{14}M_{24} \neq 0$ and $M_{23} = 0$.

This case is computed by applying ψ_1 to case (II), giving

$$\mathfrak{L}_3 = \mathcal{V}(\ M_{12},\ M_{13},\ M_{23}, M_{34}^3 - M_{14}^2M_{34} + iM_{14}M_{24}^2\ ).$$

Case (IV): $M_{23} \neq 0$ and $M_{14} = 0 = M_{24}$.

If, additionally, $M_{12} \neq 0$, then $M_{13} = 0$ and $M_{i4} = 0$ for all $i = 1, 2, 3$. It follows that $M_{12}^2 = M_{23}^2$, and so these assumptions yield a subvariety of $\mathfrak{L}_2$. Hence, we may assume that $M_{12} = 0$. It follows that this case yields the irreducible component

$$\mathfrak{L}_4 = \mathcal{V}(\ M_{12},\ M_{14},\ M_{24},\ M_{23}^2M_{34} + i\gamma M_{13}^2M_{23} - M_{34}^3\ )$$

of $\mathfrak{L}'(\gamma)$, so $\mathfrak{L}_4$ is ψ_2 applied to $\mathfrak{L}_2$.

Case (V): $M_{14} \neq 0$ and $M_{23} = 0 = M_{24}$.

This case is computed by applying ψ_1 to case (IV), giving the irreducible component

$$\mathfrak{L}_5 = \mathcal{V}(\ M_{23},\ M_{24},\ M_{34},\ M_{12}M_{14}^2 - i\gamma M_{13}^2M_{14} - M_{12}^3\ )$$

of $\mathfrak{L}'(\gamma)$, which is also ψ_2 applied to $\mathfrak{L}_3$.

Case (VI): $M_{24} \neq 0$ and $M_{14} = 0 = M_{23}$.

Using $M_{14} = 0 = M_{23}$, a computation of a Gröbner basis yields several polynomials, one of which is $M_{12}M_{34} - M_{13}M_{24}$ whereas the others are multiples of $M_{12}^2 + M_{34}^2$. In particular, two of those polynomials are: $M_{12}M_{24}(M_{12}^2 + M_{34}^2)$ and $M_{34}^2(M_{12}^2 + M_{34}^2)$. It follows that $M_{12}^2 + M_{34}^2 = 0$, so that this case yields the component $\mathfrak{L}_6 = \mathfrak{L}_{6a} \cup \mathfrak{L}_{6b}$ of $\mathfrak{L}'(\gamma)$, where

$$\mathfrak{L}_{6a} = \mathcal{V}(\ M_{14},\ M_{23},\ M_{12}M_{34} - M_{13}M_{24},\ M_{12} + iM_{34}\ ),$$
$$\mathfrak{L}_{6b} = \mathcal{V}(\ M_{14},\ M_{23},\ M_{12}M_{34} - M_{13}M_{24},\ M_{12} - iM_{34}\ ),$$

and each of $\mathfrak{L}_{6a}$ and $\mathfrak{L}_{6b}$ is a nonsingular conic, since using $M_{12} \pm iM_{34}$ to substitute for M_{12} in $M_{12}M_{34} - M_{13}M_{24}$ yields a rank-3 quadratic form in each case. Moreover, $\mathfrak{L}_{6b}$ is ψ_1 applied to $\mathfrak{L}_{6a}$.

Having completed this analysis, we can see that the point

$$\mathcal{V}(\ M_{12},\ M_{14},\ M_{23},\ M_{24},\ M_{34}\ ),$$

that was found earlier, is contained in $\mathfrak{L}_4 \cap \mathfrak{L}_5 \cap \mathfrak{L}_6$. We summarize the above work in the next result.

Theorem 3.2.3.1. *Let $\mathfrak{L}'(\gamma)$ denote the reduced variety of the line scheme $\mathfrak{L}(\gamma)$ of $\mathcal{A}(\gamma)$. If $\gamma^2 \neq 16$, then $\mathfrak{L}'(\gamma)$ is the union, in $\mathbb{P}^5$, of the following seven irreducible components:*

(I) $\mathfrak{L}_1 = \mathcal{V}(\ M_{13},\ M_{24},\ M_{14}M_{23} + M_{12}M_{34},\ M_{12}^2 + M_{34}^2 + \gamma M_{14}M_{23} - M_{14}^2 - M_{23}^2\ )$, *which is a nonplanar elliptic curve in a $\mathbb{P}^3$.*

(II) $\mathfrak{L}_2 = \mathcal{V}(\ M_{13},\ M_{14},\ M_{34},\ M_{12}^3 - M_{12}M_{23}^2 - iM_{23}M_{24}^2\ )$, *which is a planar elliptic curve.*

(III) $\mathfrak{L}_3 = \mathcal{V}(\ M_{12},\ M_{13},\ M_{23},\ M_{34}^3 - M_{14}^2M_{34} + iM_{14}M_{24}^2\ )$, *which is a planar elliptic curve.*

(IV) $\mathfrak{L}_4 = \mathcal{V}(\ M_{12},\ M_{14},\ \ M_{24},\ M_{23}^2M_{34} + i\gamma M_{13}^2M_{23} - M_{34}^3\ )$, *which is a planar elliptic curve.*

(V) $\mathfrak{L}_5 = \mathcal{V}(\ M_{23},\ M_{24},\ M_{34},\ M_{12}M_{14}^2 - i\gamma M_{13}^2 M_{14} - M_{12}^3\ )$, *which is a planar elliptic curve.*

(VIa) $\mathfrak{L}_{6a} = \mathcal{V}(\ M_{14},\ M_{23},\ M_{12}M_{34} - M_{13}M_{24},\ M_{12} + iM_{34}\ )$, *which is a nonsingular conic.*

(VIb) $\mathfrak{L}_{6b} = \mathcal{V}(\ M_{14},\ M_{23},\ M_{12}M_{34} - M_{13}M_{24},\ M_{12} - iM_{34}\ )$, *which is a nonsingular conic.*

If $\gamma = 4$, then $\mathfrak{L}'(\gamma)$ is the union, in $\mathbb{P}^5$, of eight irreducible components, six of which are $\mathfrak{L}_2$, $\mathfrak{L}_3$, $\mathfrak{L}_4$, $\mathfrak{L}_5$, $\mathfrak{L}_{6a}$, $\mathfrak{L}_{6b}$ (as above) and two of which are

$$\mathfrak{L}_{1a} = \mathcal{V}(\ M_{13},\ M_{24},\ M_{14}M_{23} + M_{12}M_{34},\ M_{12} + M_{14} - M_{23} - M_{34}\ ),$$
$$\mathfrak{L}_{1b} = \mathcal{V}(\ M_{13},\ M_{24},\ M_{14}M_{23} + M_{12}M_{34},\ M_{12} - M_{14} + M_{23} - M_{34}\ ),$$

which are nonsingular conics.

Proof. The polynomials were found in the preceding work, as was the geometric description for $\mathfrak{L}_{1a}$, $\mathfrak{L}_{1b}$, $\mathfrak{L}_{6a}$ and $\mathfrak{L}_{6b}$, so here we discuss only the geometric description of the other components.

(I) Write $q_1 = M_{14}M_{23} + M_{12}M_{34}$ and $q_2 = M_{12}^2 + M_{34}^2 + \gamma M_{14}M_{23} - M_{14}^2 - M_{23}^2$ viewed in $\mathbb{k}[M_{12}, M_{14}, M_{23}, M_{34}]$. Since

$$q_2 = M_{12}^2 - (\gamma/2)M_{12}M_{34} + M_{34}^2 - \left(M_{14}^2 - (\gamma/2)M_{14}M_{23} + M_{23}^2\right)$$

modulo q_1, and since $\mathrm{char}(\mathbb{k}) \neq 2$, we may take the Jacobian matrix of this system of two polynomials to be the 2×4 matrix

$$\begin{bmatrix} M_{34} & M_{23} & M_{14} & M_{12} \\ 2M_{12} - (\gamma/2)M_{34} & -(2M_{14} - (\gamma/2)M_{23}) & -(2M_{23} - (\gamma/2)M_{14}) & 2M_{34} - (\gamma/2)M_{12} \end{bmatrix}.$$

Assuming that all the 2×2 minors are zero, we find that $M_{34}^2 = M_{12}^2$ (from columns one and four) and $M_{23}^2 = M_{14}^2$ (from columns two and three). Substituting these relations into the minor obtained from the last two columns yields that either

$(\gamma \pm 4)M_{12}M_{14} = 0$ or $\gamma M_{12}M_{14} = 0$, so $M_{12}M_{14} = 0$ (since $\gamma(\gamma^2 - 16) \neq 0$). Substitution into q_1 implies that there is no solution, and so the Jacobian matrix has rank two at all points of $\mathcal{V}(q_1, q_2)$. It follows that $\mathcal{V}(q_1, q_2)$, viewed as a subvariety of $\mathbb{P}^3 = \mathcal{V}(M_{13}, M_{24})$, is reduced, and so $\mathfrak{L}_1$ is reduced. Following the method of the proof of [32, Proposition 2.5], if $\mathcal{V}(q_1, q_2)$ is not irreducible, then there exists a point in the intersection of two of its irreducible components, and so the Jacobian matrix has rank at most one at that point, which is a contradiction. Hence, $\mathcal{V}(q_1, q_2)$ is irreducible, and thus nonsingular since it is reduced. Moreover, its genus is $4 - 2 - 2 + 1 = 1$. It follows that $\mathcal{V}(q_1, q_2)$ is an elliptic curve, and the same is true of $\mathfrak{L}_1$.

(II) Viewing $h = M_{12}^3 - M_{12}M_{23}^2 - iM_{23}M_{24}^2$ as a polynomial in $\Bbbk[M_{12},\ M_{23},\ M_{24}]$, the Jacobian matrix of h is a 1×3 matrix that has rank one at all points of $\mathcal{V}(h)$ (since $\mathrm{char}(\Bbbk) \neq 2$), so $\mathcal{V}(h)$ is nonsingular in $\mathbb{P}^2 = \mathcal{V}(M_{13},\ M_{14},\ M_{34})$.

(III), (IV), (V) These cases follow from (II) by applying ψ_1 or ψ_2 as appropriate.

□

3.2.4 Description of the Line Scheme of $\mathcal{A}(\gamma)$

In this subsection, we prove that the line scheme $\mathfrak{L}(\gamma)$ of $\mathcal{A}(\gamma)$ is reduced and so is given by $\mathfrak{L}'(\gamma)$ described in Theorem 3.2.3.1.

Lemma 3.2.4.1. *For all $\gamma \in \Bbbk^\times$, the irreducible components of $\mathfrak{L}(\gamma)$ have dimension one; in particular, $\mathfrak{L}(\gamma)$ has no embedded points.*

Proof. By [5], $\mathcal{A}(\gamma)$ is a regular noetherian domain that is Auslander-regular and satisfies the Cohen Macaulay property and has Hilbert series the same as that of the polynomial ring on four variables. Hence, by [29, Remark 2.10], we may apply [29, Corollary 2.6] to $\mathcal{A}(\gamma)$, which gives us that the irreducible components of $\mathfrak{L}(\gamma)$ have dimension at least one. However, by Theorem 3.2.3.1, they have dimension at most one, so equality follows. Let X_1 denote the 11-dimensional subscheme of $\mathbb{P}(V \otimes V)$ consisting of the elements of rank at most two, and, for all $\gamma \in \Bbbk^\times$, let X_2 denote the 5-dimensional linear subscheme of $\mathbb{P}(V \otimes V)$ given by the span of the defining relations of $\mathcal{A}(\gamma)$. By [29, Lemma 2.5], $\mathfrak{L}(\gamma) \cong X_1 \cap X_2$ for all $\gamma \in \Bbbk^\times$. Since X_i is a Cohen Macaulay scheme for $i = 1, 2$, and since $\dim(X_1 \cap X_2) = 1$, the proof of [29, Theorem 4.3] (together with Macaulay's Unmixedness Theorem) rules out the possibility of embedded components. □

Theorem 3.2.4.2. *For all $\gamma \in \Bbbk^\times$, the line scheme $\mathfrak{L}(\gamma)$ is a reduced scheme of degree twenty.*

Proof. Let X_1 and X_2 be as in the proof of Lemma 3.2.4.1, and let $X = X_1 \cap X_2$. Since $\deg(X_1) = 20$ by [15, Example 19.10], Bézout's Theorem for Cohen Macaulay schemes ([11, Theorem III-78]) implies that $\deg(X) = 20$. However, since $\mathfrak{L}(\gamma) \cong X$ by [29, Lemma 2.5], the reduced scheme X' of X is isomorphic to $\mathfrak{L}'(\gamma)$. Since the degrees of the irreducible components of $\mathfrak{L}'(\gamma)$ in Theorem 3.2.3.1 are as small as possible, $\deg(X') \geq 4 + 12 + 4 = 20$; that is, $20 = \deg(X) \geq \deg(X') \geq 20$, giving $\deg(X) = \deg(X')$. As X has no embedded points by Lemma 3.2.4.1, it follows that $X = X'$, so X is a reduced scheme. Thus, $\mathfrak{L}(\gamma)$ is reduced and has degree twenty since $\deg(\mathfrak{L}'(\gamma)) = 20$. □

We now offer two alternative proofs for the statement that $\mathfrak{L}(\gamma)$ is reduced. The first proof follows the same general format as the one above. The distinction is that in the above proof, the computation of the degree of X' uses the degree of curves in $\mathfrak{L}'(\gamma)$. The new proof computes curves in X' directly and makes use of their degrees.

Proof. Let X, X_1 and X_2 be as in the proofs of Lemma 3.2.4.1 and Theorem 3.2.4.2. Since $\deg(X_1) = 20$ by [15, Example 19.10], Bézout's Theorem for Cohen Macaulay schemes ([11, Theorem III-78]) implies that $\deg(X) = 20$. The polynomials that define X, called the Van den Bergh polynomials, are given in Appendix 5.1.4. For more information on the construction of these polynomials, the reader is referred to [29, 30].

We now compute X', the variety of closed points of X. Computing a Gröbner basis for the polynomials in Appendix 5.1.4 yields several polynomials, one of which is $y_2y_5y_6$. We make use of a symmetric argument to that in Section 3.2.3. If $\gamma^2 \neq 16$, then this computation yields the following irreducible components:

- $\mathfrak{X}_1 = \mathcal{V}(\ y_1,\ y_2,\ y_3^2 + \gamma y_3y_6 - y_5^2,\ y_4^2 - y_4y_5 - y_6^2\ )$, which is a nonplanar elliptic curve in a $\mathbb{P}^3$,
- $\mathfrak{X}_2 = \mathcal{V}(\ y_1,\ y_5,\ y_3 + \gamma y_6,\ y_2^2y_4 - i\gamma y_4^2y_6 + iy_6^3\ )$, which is a planar elliptic curve,
- $\mathfrak{X}_3 = \mathcal{V}(\ y_1,\ y_4,\ y_6,\ y_2^2y_3 + iy_3^2y_5 - iy_5^3\ )$, which is a planar elliptic curve,
- $\mathfrak{X}_4 = \mathcal{V}(\ y_2,\ y_6,\ y_4 + y_5,\ y_1^2y_3 - iy_3^2y_5 + iy_5^3\ )$, which is a planar elliptic curve,
- $\mathfrak{X}_5 = \mathcal{V}(\ y_1,\ y_2,\ y_5,\ y_1^2y_4 + i\gamma y_4^2y_6 + i\gamma y_6^3)$, which is a planar elliptic curve,
- $\mathfrak{X}_{6a} = \mathcal{V}(\ y_5,\ y_6,\ y_2^2 + iy_3y_4,\ y_2 - y_1)$, which is a nonsingular conic,
- $\mathfrak{X}_{6b} = \mathcal{V}(\ y_5,\ y_6,\ y_2^2 + iy_3y_4,\ y_2 + y_1)$, which is a nonsingular conic.

If $\gamma = 4$, then X' is determined by $\mathfrak{X}_2, \mathfrak{X}_3, \mathfrak{X}_4, \mathfrak{X}_5, \mathfrak{X}_{6a}, \mathfrak{X}_{6b}$, and two nonsingular conics:

$$\mathfrak{X}_{1a} = \mathcal{V}(\ y_1,\ y_2,\ y_3 - 2y_4 - y_5 + 2y_6,\ y_4^2 - y_4y_5 - y_6^2\ ),\ \text{and}$$

$$\mathfrak{X}_{1b} = \mathcal{V}(\ y_1,\ y_2,\ y_3 + 2y_4 + y_5 + 2y_6,\ y_4^2 - y_4y_5 - y_6^2\ ).$$

Verification that these components are elliptic curves and conics is done following the method in Section 3.2.3.

Having found these components, we see that, for all $\gamma \in \Bbbk^\times$, $\deg(X') = 4 + 12 + 4 = 20$. Therefore, $20 = \deg(X) \geq \deg(X') = 20$ which implies that $X = X'$

and the scheme is reduced, since X has no embedded points by Lemma 3.2.4.1. Since $\mathfrak{L}(\gamma) \cong X$, $\mathfrak{L}(\gamma)$ is also reduced. $\square$

We now give a second alternative proof to Theorem 3.2.4.2. This method makes use of the coordinate ring of $\mathfrak{L}(\gamma)$ and computing the dimension of the local rings associated to $\mathfrak{L}(\gamma)$. For this proof, we assume that $\mathrm{char}(\Bbbk) \neq 2$.

Proof. Suppose $\mathrm{char}(\Bbbk) \neq 2$. Consider the coordinate ring of $\mathfrak{L}(\gamma)$ defined as $R = \Bbbk[M_{12}, ..., M_{34}]/I$, where I is the ideal generated by the polynomials in Appendix 5.1.2. Let $f = M_{12} + M_{13} + iM_{14} + iM_{23} + M_{24}$. Since the intersection of $\mathfrak{L}(\gamma)$ with $\mathcal{V}(f)$ consists of finitely many points, we may use f to compute the degree of $\mathfrak{L}(\gamma)$; note that none of the intersection points are intersection points of the components of $\mathfrak{L}(\gamma)$. The points of intersection of $\mathfrak{L}(\gamma)$ and $\mathcal{V}(f)$ are given by the following:

(a) $\mathcal{V}(\ M_{12},\ M_{13},\ M_{23},\ M_{34} - 1,\ M_{14} - iM_{24},\ M_{24}^3 - M_{24}^2 - 1\ )$,

(b) $\mathcal{V}(\ M_{13},\ M_{24},\ M_{23} - 1,\ M_{12} + iM_{14} + i,\ i(M_{14} + 1)M_{34} - M_{14},\ 2M_{14}^4 + (6-\gamma)M_{14}^3 + (9 - 2\gamma)M_{14}^2 + (6 - \gamma)M_{14} + 2\ )$,

(c) $\mathcal{V}(\ M_{13},\ M_{14},\ M_{34},\ M_{23} - 1,\ M_{12} + M_{24} + i,\ M_{24}^3 + 4iM_{24}^2 - 4M_{24} - 2i\ )$,

(d) $\mathcal{V}(\ M_{14},\ M_{23},\ M_{13} - 1,\ M_{12} + M_{24} + 1,\ (M_{24} + 1)M_{34} + M_{24},\ (M_{24} + 1)^4 + M_{24}^2\ )$,

(e) $\mathcal{V}(\ M_{12},\ M_{14},\ M_{24},\ M_{13} - 1,\ M_{23} - i,\ M_{34}^3 + M_{34} + \gamma\ )$, and

(f) $\mathcal{V}(\ M_{23},\ M_{24},\ M_{34},\ M_{13} - 1,\ M_{12} + 1 + iM_{14}, 2M_{14}^3 - 4iM_{14}^2 + (\gamma - 3)M_{14} + i\ )$.

The method of computation for these points of intersection is the same as the method in the proof of Theorem 3.2.3.1 and Theorem 3.2.1; however, in this computation, we did not make use of Gröbner bases; this allows us to assume only that $\mathrm{char}(\Bbbk) \neq 2$ for this proof, instead of $\mathrm{char}(\Bbbk) = 0$ as before. For a generic $\Bbbk$, there will be exactly twenty intersection points. The reader should note that cases

(a) and (b) were also described explicitly in [8]. We will use the same method of computation for all cases.

Let J denote the ideal of $\Bbbk[M_{12}, ..., M_{34}]$ that is generated by the polynomials in Appendix 5.1.2 and f. Also, let m_{ij} (respectively, $\overline{J}$) represent the image of M_{ij} (respectively, J) in the localized ring in each case.

(a) In this case, it is straightforward to see that M_{14}, M_{24}, M_{34} are each nonzero. Setting $M_{34} = 1$ in J yields several polynomials, some of which are:

$$-m_{14}(-m_{13}m_{14} + m_{13}^2 m_{23} + im_{13}m_{14}m_{23} + im_{13}m_{23}^2 + m_{13}m_{23}m_{24} + im_{14}m_{23}m_{24}),$$
$$m_{14}(-im_{13}m_{14} + im_{13}^2 m_{23} - m_{13}m_{14}m_{23} - m_{13}m_{23}^2 + im_{13}m_{23}m_{24} + m_{14}m_{23}m_{24}),$$
$$m_{23}m_{24} - m_{14}^2 m_{23}m_{24} - m_{13}m_{14}m_{24}^2 + \gamma im_{13}m_{14}m_{23},$$
$$-im_{13}m_{14}m_{24} + m_{14}^2 m_{24} + im_{23}m_{24} + m_{14}m_{23}m_{24} - im_{14}m_{24}^2 + \gamma m_{13}m_{14}m_{23}.$$

Since m_{14} is nonzero in this case, we can invert it and obtain that the following polynomials belong to $\overline{J}$:

$$-m_{13}m_{14} + m_{13}^2 m_{23} + im_{13}m_{14}m_{23} + im_{13}m_{23}^2 + m_{13}m_{23}m_{24} + im_{14}m_{23}m_{24},$$
$$-im_{13}m_{14} + im_{13}^2 m_{23} - m_{13}m_{14}m_{23} - m_{13}m_{23}^2 + im_{13}m_{23}m_{24} + m_{14}m_{23}m_{24}.$$

The polynomial $2im_{14}m_{23}m_{24}$ is a linear combination of these polynomials. Since $\mathrm{char}(\Bbbk) \neq 2$ and m_{14} and m_{24} are nonzero, we can invert them and obtain that $m_{23} \in \overline{J}$. Now, using the third polynomial above, we see that $m_{13} \in \overline{J}$. Finally, the fourth polynomial then implies that $m_{14} - im_{24} \in \overline{J}$. This allows the remaining generators of $\overline{J}$ to be written as multiples of the generator $m_{24}^3 - m_{24}^2 - 1$. Therefore, the localized ring associated to these intersection points is isomorphic to a polynomial ring in one variable x with exactly one relation: $x^3 - x^2 - 1 = 0$; thus, the ring has dimension three.

(b) In this case, we can see that M_{14}, M_{23}, M_{34} are nonzero. Setting $M_{13} = 1$ in the ideal yields several polynomials, some of which are:

$$m_{14}(-m_{13} + im_{13}^2 - m_{13}m_{14} + im_{13}m_{24} + m_{14}m_{24} - im_{13}m_{14}m_{34}),$$
$$-m_{14}(im_{13} + m_{13}^2 + im_{13}m_{14} + m_{13}m_{24} + im_{14}m_{24} - m_{13}m_{14}m_{34}),$$
$$m_{14}(m_{14}m_{24} - im_{13}m_{14}m_{24} + m_{14}^2m_{24} - im_{14}m_{24}^2 + im_{24}m_{34} + \gamma m_{13}m_{14}).$$

As before, since m_{14} is nonzero, we can invert it. So, the first and second polynomials tell us that $m_{24} \in \overline{J}$ and this implies, together with the third polynomial, that $m_{13} \in \overline{J}$. This now allows all the remaining generators of $\overline{J}$ to be written as multiples of the generator

$$2m_{14}^4 + (6-\gamma)m_{14}^3 + (9-2\gamma)m_{14}^2 + (6-\gamma)m_{14} + 2.$$

Therefore, the localized ring associated to these intersection points is isomorphic to a polynomial ring in one variable with exactly one relation of degree four (as $\mathrm{char}(\Bbbk) \neq 2$), and so has dimension four.

(c) In this case, M_{12}, M_{23}, M_{24} are nonzero. Setting $M_{23} = 1$ yields several polynomials, some of which are:

$$(i + m_{13} + im_{14} + m_{24})(m_{13} - im_{13}^2 + m_{13}m_{14} - im_{13}m_{24} + m_{14}m_{24} + im_{13}m_{14}m_{34}),$$
$$(i + m_{13} + im_{14} + m_{24})(-m_{13} + im_{13}^2 - m_{13}m_{14} + im_{13}m_{24} + m_{14}m_{24} - im_{13}m_{14}m_{34}),$$
$$-m_{14}m_{24} + im_{13}m_{14}m_{24} - m_{14}^2m_{24} + im_{14}m_{24}^2 - im_{24}m_{34} - \gamma m_{13}m_{14},$$
$$m_{14} - m_{13}m_{24} - im_{34} - m_{13}m_{34} - im_{14}m_{34} - m_{24}m_{34}.$$

Note that the image of M_{12} in $\overline{J}$ is $-m_{13} - im_{14} - i - m_{24}$. Since the image of M_{12} is nonzero in $\overline{J}$, we can invert it. So, using the first and second polynomials, and the fact that m_{24} is nonzero in $\overline{J}$, we obtain that $m_{14} \in \overline{J}$. This, combined with the third polynomial, implies that $m_{34} \in \overline{J}$ and this, together with the fourth polynomial, implies that $m_{13} \in \overline{J}$. This now allows all the remaining generators of $\overline{J}$ to be written as a multiple of the generator

$$-2 + 4im_{24} + 4m_{24}^2 - im_{24}^3.$$

Therefore, the local ring associated to these intersection points is isomorphic to a polynomial ring in one variable with exactly one relation of degree three, and so has dimension three.

(d) In this case, $M_{12}, M_{13}, M_{24}, M_{34}$ are nonzero and $M_{24} \neq -1$. Setting $M_{13} = 1$ yields several polynomials, some of which are:

$$m_{24}(m_{23} + im_{14}m_{23} + im_{23}^2 + m_{23}m_{24} - im_{14}m_{23}m_{24} + m_{14}m_{34}),$$
$$-m_{34}(-im_{23} + m_{14}m_{23} + m_{23}^2 - im_{23}m_{24} - m_{14}m_{23}m_{24} + im_{14}m_{34}),$$
$$m_{14}m_{23}^2 + m_{23}m_{24} + im_{14}m_{23}m_{24}m_{34} - m_{14}m_{34}^2.$$

We can invert m_{24} and $-m_{34}$ in the first and second polynomials and take a linear combination of the resulting polynomials to obtain $2im_{14}m_{34} \in \overline{J}$ and we again invert m_{34} to obtain that $m_{14} \in \overline{J}$. This, together with the third polynomial, implies that $m_{23} \in \overline{J}$. This allows the remaining generators to be written as multiples of the generator

$$1 + 4m_{24} + 7m_{24}^2 + 4m_{24}^3 + m_{24}^4.$$

Therefore, the local ring associated to these intersection points is isomorphic to a polynomial ring in one variable with exactly one relation of degree four, and so has dimension four.

(e) In this case, M_{13}, M_{23}, M_{34} are nonzero. Taking $M_{13} = 1$ yields several polynomials, some of which are:

$$-2m_{14}m_{23}m_{24},$$
$$-m_{14}^2m_{23}m_{24} - m_{14}m_{24}^2 + m_{23}m_{24}m_{34}^2 + \gamma im_{14}m_{23}m_{34},$$
$$m_{34}(-im_{14}m_{24} + m_{14}^2m_{24} + m_{14}m_{23}m_{24} - im_{14}m_{24}^2 + im_{23}m_{24}m_{34} + \gamma m_{14}m_{23}),$$
$$im_{23} - m_{14}m_{23} - m_{23}^2 + im_{23}m_{24} + m_{14}m_{23}m_{24} - im_{14}m_{34}.$$

We may invert m_{23} in the first polynomial to obtain that $m_{14}m_{24} \in \overline{J}$. This fact, together with linear combinations of the next two polynomials, and the fact that m_{23} and m_{34} are invertible, implies that $m_{14}, m_{24} \in \overline{J}$. Finally, the last polynomial now tells us that $m_{23} - i \in \overline{J}$. It follows that the remaining generators of $\overline{J}$ can be written as multiples of the generator $-m_{34} - m_{34}^3 - \gamma$. Therefore, the local ring associated to these intersection points is isomorphic to a polynomial ring in one variable with exactly one relation of degree three, and so has dimension three.

(f) In this case, M_{12}, M_{13}, M_{14} are nonzero. Taking $M_{13} = 1$ yields several polynomials, some of which are:

$$-2m_{14}m_{23}m_{24},$$
$$m_{12}(m_{14}m_{24} + im_{14}^2m_{24} + im_{14}m_{23}m_{24} + m_{14}m_{24}^2 - m_{23}m_{24}m_{34} - \gamma im_{14}m_{23}),$$
$$m_{12}(-m_{14}m_{24} - im_{14}^2m_{24} - im_{14}m_{23}m_{24} - m_{14}m_{24}^2 + m_{23}m_{24}m_{34} - \gamma im_{14}m_{23}),$$
$$\begin{gathered} m_{23} + 2im_{14}m_{23} - 2m_{14}^2m_{23} + 2im_{23}^2 - 2m_{14}m_{23}^2 - m_{23}^3 - m_{14}m_{24} + 2m_{23}m_{24} + \\ + im_{14}m_{23}m_{24} + m_{14}^2m_{23}m_{24} + 2im_{23}^2m_{24} + m_{14}m_{23}^2m_{24} + m_{23}m_{24}^2 - im_{14}m_{23}m_{24}^2, \end{gathered}$$
$$im_{23} - m_{14}m_{23} - m_{23}^2 + im_{23}m_{24} + m_{14}m_{23}m_{24} - im_{14}m_{34}.$$

Since m_{14} is nonzero, we can invert it, and so the first polynomial tells us that $m_{23}m_{24} \in \overline{J}$. We may also invert m_{12}; so this, together with m_{14} being nonzero and using the second and third polynomials, tells us that $m_{23} \in \overline{J}$. All this, together with the fourth polynomial, implies that $m_{24} \in \overline{J}$ and finally the fifth polynomial then implies that $m_{34} \in \overline{J}$. These facts allow us to write the remaining generators as multiples of the generator $-1 - 3im_{14} + 4m_{14}^2 + 2im_{14}^3 + i\gamma m_{14}$. Therefore, the local ring associated to these intersection points is isomorphic to a polynomial ring in one variable with exactly one relation of degree three, and so has dimension three.

So, from the work above, we see that the degree of $\mathfrak{L}(\gamma)$ is 3+4+3+4+3+3=20. From our work in Section 3.2.3, we know that $\mathfrak{L}'(\gamma)$ has degree twenty. Therefore, $\mathfrak{L}(\gamma) = \mathfrak{L}'(\gamma)$ (as $\mathfrak{L}(\gamma)$ has no embedded points by Lemma 3.2.4.1).

$\square$

3.3 The Lines in $\mathbb{P}^3$ Parametrized by the Line Scheme of $\mathcal{A}(\gamma)$

In this section, we describe the lines in $\mathbb{P}(V^*)$ that are parametrized by the line scheme $\mathfrak{L}(\gamma)$ of $\mathcal{A}(\gamma)$. We also describe, in Theorem 3.3.3.1, the lines that pass through any given point of the point scheme; in particular, if p is one of the generic points of the point scheme (that is, $p \in \mathcal{Z}_\gamma$), then there are exactly six distinct lines of the line scheme that pass through p. Since we will use results from Section 3.2.3, we resume the assumption that $\mathrm{char}(\Bbbk) = 0$.

3.3.1 The Lines in $\mathbb{P}^3$

In this subsection, we find the lines in $\mathbb{P}(V^*)$ that are parametrized by the line scheme. We first recall how the Plücker coordinates $M_{12}, \ldots, M_{34}$ relate to lines in $\mathbb{P}^3$; details may be found in [9, §8.6]. Any line ℓ in $\mathbb{P}^3$ is uniquely determined by any two distinct points $a = (a_1, \ldots, a_4) \in \ell$ and $b = (b_1, \ldots, b_4) \in \ell$, and may be represented by a 2×4 matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

that has rank two; in particular, the points on ℓ are represented in homogeneous coordinates by linear combinations of the rows of this matrix. In general, there are infinitely many such matrices that may be associated to any line ℓ in $\mathbb{P}^3$, and they are all related to each other by applying row operations.

The Plücker coordinate M_{ij} is evaluated on this matrix as the minor $a_i b_j - a_j b_i$ for all $i \neq j$, and the Plücker polynomial $P = M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}$ vanishes on this matrix.

Since $\dim(V) = 4$, we identify $\mathbb{P}(V^*)$ with $\mathbb{P}^3$. By Theorem 3.2.4.2, $\mathfrak{L}(\gamma)$ is given by Theorem 3.2.3.1. We continue to use the notation e_j introduced in Section 3.2.1.

(I) In this case, $\gamma^2 \neq 16$ and the component is $\mathfrak{L}_1$, which is a nonplanar elliptic curve in a $\mathbb{P}^3$ (contained in $\mathbb{P}^5$), where

$$\mathfrak{L}_1 = \mathcal{V}(\ M_{13},\ M_{24},\ M_{14}M_{23} + M_{12}M_{34},\ M_{12}^2 + M_{34}^2 + \gamma M_{14}M_{23} - M_{14}^2 - M_{23}^2\ ).$$

It follows that any line ℓ in $\mathbb{P}(V^*)$ given by $\mathfrak{L}_1$ is represented by a 2×4 matrix of the form:

$$(*) \qquad \begin{bmatrix} a_1 & 0 & a_3 & 0 \\ 0 & b_2 & 0 & b_4 \end{bmatrix},$$

where $a_j,\ b_j \in \Bbbk$ for all j and $a_1^2b_2^2 + a_3^2b_4^2 - \gamma a_1b_2a_3b_4 - a_1^2b_4^2 - b_2^2a_3^2 = 0$. In particular, if $p \in \ell$, then $p = (\lambda_1 a_1, \lambda_2 b_2, \lambda_1 a_3, \lambda_2 b_4)$, for some $(\lambda_1, \lambda_2) \in \mathbb{P}^1$, such that $a_1^2b_2^2 + a_3^2b_4^2 - \gamma a_1b_2a_3b_4 - a_1^2b_4^2 - b_2^2a_3^2 = 0$. It is easily verified that p lies on the quartic surface

$$\mathcal{V}(\ x_1^2x_2^2 + x_3^2x_4^2 - \gamma x_1x_2x_3x_4 - x_1^2x_4^2 - x_2^2x_3^2\ )$$

in $\mathbb{P}(V^*)$ for all $(\lambda_1, \lambda_2) \in \mathbb{P}^1$. Hence, the lines parametrized by $\mathfrak{L}_1$ all lie on this quartic surface in $\mathbb{P}(V^*)$ and are given by:

$$\mathcal{V}(\ x_3,\ x_2 \pm x_4\ ), \quad \mathcal{V}(\ x_4,\ x_1 \pm x_3\ ), \quad \text{and} \quad \mathcal{V}(\ x_1 - \alpha x_3,\ x_2 - \beta x_4\ )$$

for all $\alpha, \beta \in \Bbbk$ such that $(\alpha^2 - 1)(\beta^2 - 1) = \gamma\alpha\beta$. The case $\gamma = 4$ is discussed below in cases **(Ia)** and **(Ib)**.

(II) In this case, the component is $\mathfrak{L}_2$, which is a planar elliptic curve, where

$$\mathfrak{L}_2 = \mathcal{V}(\ M_{13},\ M_{14},\ M_{34},\ M_{12}^3 - M_{12}M_{23}^2 - iM_{23}M_{24}^2\ ),$$

so any line in $\mathbb{P}(V^*)$ given by $\mathfrak{L}_2$ is represented by a 2×4 matrix of the form:

$$\begin{bmatrix} a_1 & 0 & a_3 & a_4 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

such that $a_1^3 - a_1a_3^2 + ia_3a_4^2 = 0$. It follows that $\mathfrak{L}_2$ parametrizes those lines in $\mathbb{P}(V^*)$ that pass through e_2 and meet the planar curve $\mathcal{V}(x_2, x_1^3 - x_1x_3^2 + ix_3x_4^2)$; this planar curve is a (nonsingular) elliptic curve since $\mathrm{char}(\Bbbk) = 0$.

(III) In this case, the component is $\mathfrak{L}_3$, which may be obtained as ψ_1 applied to $\mathfrak{L}_2$. Hence, $\mathfrak{L}_3$ parametrizes those lines in $\mathbb{P}(V^*)$ that pass through e_4 and meet the planar elliptic curve $\mathcal{V}(x_4, x_3^3 - x_1^2x_3 + ix_1x_2^2)$.

(IV) In this case, the component is $\mathfrak{L}_4$, which may be obtained as ψ_2 applied to $\mathfrak{L}_2$. Hence, $\mathfrak{L}_4$ parametrizes those lines in $\mathbb{P}(V^*)$ that pass through e_3 and meet the planar elliptic curve $\mathcal{V}(x_3, x_4^3 - x_2^2x_4 + i\gamma x_1^2x_2)$.

(V) In this case, the component is $\mathfrak{L}_5$, which may be obtained as ψ_1 applied to $\mathfrak{L}_4$. Hence, $\mathfrak{L}_5$ parametrizes those lines in $\mathbb{P}(V^*)$ that pass through e_1 and meet the planar elliptic curve $\mathcal{V}(x_1, x_2^3 - x_2x_4^2 + i\gamma x_3^2x_4)$.

(VI) In this case, the component is $\mathfrak{L}_6 = \mathfrak{L}_{6a} \cup \mathfrak{L}_{6b}$, where

$$\mathfrak{L}_{6a} = \mathcal{V}(\ M_{14},\ M_{23},\ M_{12}M_{34} - M_{13}M_{24},\ M_{12} + iM_{34}\ ),$$
$$\mathfrak{L}_{6b} = \mathcal{V}(\ M_{14},\ M_{23},\ M_{12}M_{34} - M_{13}M_{24},\ M_{12} - iM_{34}\ ),$$

which are nonsingular conics. Following the argument from case (I), any line in $\mathbb{P}(V^*)$ given by $\mathfrak{L}_{6a}$ is represented by a 2×4 matrix of the form:

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ \alpha a_1 & \beta a_2 & \beta a_3 & \alpha a_4 \end{bmatrix},$$

such that $\alpha, \beta,\ a_j \in \Bbbk$ for all j, $a_1a_2 = ia_3a_4$ and $\alpha \neq \beta$. A calculation similar to that used in (I) verifies that every point of the line lies on the quadric $\mathcal{V}(x_1x_2 - ix_3x_4)$. It follows that $\mathfrak{L}_{6a}$ parametrizes one of the rulings of the nonsingular quadric $\mathcal{V}(x_1x_2 - ix_3x_4)$; namely, the ruling that consists of the lines $\mathcal{V}(\delta x_1 - \epsilon x_4, \delta x_3 + i\epsilon x_2)$ for all

$(\delta, \epsilon) \in \mathbb{P}^1$. Since $\mathfrak{L}_{6b}$ may be obtained by applying ψ_1 to $\mathfrak{L}_{6a}$, we find $\mathfrak{L}_{6b}$ parametrizes one of the rulings of the nonsingular quadric $\mathcal{V}(x_3x_4 - ix_1x_2)$; namely, the ruling that consists of the lines $\mathcal{V}(\delta x_3 - \epsilon x_2, \delta x_1 + i\epsilon x_4)$ for all $(\delta, \epsilon) \in \mathbb{P}^1$.

(Ia) and **(Ib)** In this case, $\gamma = 4$ and the component is $\mathfrak{L}_1 = \mathfrak{L}_{1a} \cup \mathfrak{L}_{1b}$, where

$$\mathfrak{L}_{1a} = \mathcal{V}(\ M_{13},\ M_{24},\ M_{14}M_{23} + M_{12}M_{34},\ M_{12} + M_{14} - M_{23} - M_{34}\ ),$$
$$\mathfrak{L}_{1b} = \mathcal{V}(\ M_{13},\ M_{24},\ M_{14}M_{23} + M_{12}M_{34},\ M_{12} - M_{14} + M_{23} - M_{34}\ ),$$

which are nonsingular conics. Following the argument from case (I), any line in $\mathbb{P}(V^*)$ given by $\mathfrak{L}_{1a}$ is represented by a 2×4 matrix of the form $(*)$ such that $a_1b_2 + a_1b_4 + b_2a_3 = a_3b_4$. A calculation similar to that used in (I) verifies that every point of the line lies on the nonsingular quadric

$$Q_a = \mathcal{V}(\ x_1x_2 + x_1x_4 + x_2x_3 - x_3x_4\ )$$

in $\mathbb{P}(V^*)$. Hence, the lines parametrized by $\mathfrak{L}_{1a}$ all lie on Q_a and are:

$$\mathcal{V}(\ x_3,\ x_2 + x_4\ ) \quad \text{and} \quad \mathcal{V}(\ x_1 - \alpha x_3,\ (\alpha+1)x_2 + (\alpha-1)x_4\ )$$

for all $\alpha \in \Bbbk$, which yields one of the rulings on the quadric Q_a. Applying ψ_1 to these lines, it follows that the lines parametrized by $\mathfrak{L}_{1b}$ are:

$$\mathcal{V}(\ x_1,\ x_2 + x_4\ ) \quad \text{and} \quad \mathcal{V}(\ x_3 - \alpha x_1,\ (\alpha-1)x_2 + (\alpha+1)x_4\ )$$

for all $\alpha \in \Bbbk$, which yields one of the rulings on the nonsingular quadric

$$Q_b = \mathcal{V}(\ x_3x_4 + x_2x_3 + x_1x_4 - x_1x_2\ ).$$

3.3.2 The Intersection Points of the Line Scheme of $\mathcal{A}(\gamma)$

The intersections of the irreducible components of $\mathfrak{L}(\gamma)$ are straightforward to compute and are listed in Appendix 5.1.3.

For $i = 1, ..., 6$, let $E_i \in \mathbb{P}^5$ denote the point with the ith coordinate nonzero and all other coordinates equal zero. If $\gamma \in \Bbbk$ is generic, then the distinct intersection points of the components of $\mathfrak{L}(\gamma)$ are E_2, E_3, E_4, E_5, $E_1 \pm E_4$, $E_3 \pm E_6$, $E_4 \pm E_6$, $E_1 \pm E_3$.

Since $A(\gamma)$ is a graded skew Clifford algebra, $\mathcal{A}(\gamma)$ contains a normalizing sequence of four linearly independent homogeneous degree-two elements. One such normalizing sequence is $\{x_2^2, x_1^2, x_3x_4 + x_4x_3, x_1x_2 + x_2x_1\}$. We conjecture that the intersection points of the components of $\mathfrak{L}(\gamma)$ correspond to right ideals of $\mathcal{A}(\gamma)$ that have a "large intersection" with the normalizing sequence. We explore this idea below.

Denote $A = \mathcal{A}(\gamma)$. Using Section 3.3.1, we obtain the following correspondences between the intersection points of $\mathfrak{L}(\gamma)$ and right ideals of A:

(i) $E_2 \leftrightsquigarrow x_2A + x_4A$,

(ii) $E_3 \leftrightsquigarrow x_2A + x_3A$,

(iii) $E_4 \leftrightsquigarrow x_1A + x_4A$,

(iv) $E_5 \leftrightsquigarrow x_1A + x_3A$,

(v) $E_1 \pm E_4 \leftrightsquigarrow x_4A + (x_1 \mp x_3)A$,

(vi) $E_3 \pm E_6 \leftrightsquigarrow x_2A + (x_1 \pm x_3)A$,

(vii) $E_4 \pm E_6 \leftrightsquigarrow x_1A + (x_2 \mp x_4)A$,

(viii) $E_1 \pm E_3 \leftrightsquigarrow x_3A + (x_2 \pm x_4)A$.

Below we express each of the degree-two subspaces of the ideals above as a span of basis elements of $\mathcal{A}(\gamma)$. This is easily verified computationally.

$$(x_2A + x_4A)_2 = \Bbbk x_4x_3 \oplus \Bbbk x_2x_4 \oplus \Bbbk x_2x_3 \oplus \Bbbk x_2^2 \oplus \Bbbk x_2x_1 \oplus \Bbbk x_1x_4 \oplus \Bbbk x_1^2,$$

$$(x_2A + x_3A)_2 = \Bbbk x_3x_4 \oplus \Bbbk x_2x_4 \oplus \Bbbk x_2x_3 \oplus \Bbbk x_2^2 \oplus \Bbbk x_2x_1 \oplus \Bbbk x_1x_3 \oplus \Bbbk x_1^2,$$

$$(x_1A + x_4A)_2 = \Bbbk x_4x_3 \oplus \Bbbk x_2x_4 \oplus \Bbbk x_2^2 \oplus \Bbbk x_1x_4 \oplus \Bbbk x_1x_3 \oplus \Bbbk x_1x_2 \oplus \Bbbk x_1^2,$$

$$(x_1A + x_3A)_2 = \Bbbk x_3x_4 \oplus \Bbbk x_2x_3 \oplus \Bbbk x_2^2 \oplus \Bbbk x_1x_4 \oplus \Bbbk x_1x_3 \oplus \Bbbk x_1x_2 \oplus \Bbbk x_1^2,$$

$$\begin{aligned}(x_4A + (x_1 \pm x_3)A)_2 = {} & \Bbbk x_1x_4 \oplus \Bbbk x_4x_2 \oplus \Bbbk x_4x_3 \oplus \Bbbk x_2^2 \oplus \Bbbk(x_1^2 \pm x_1x_3) \\ & \oplus \Bbbk(x_1x_2 \pm ix_2x_3) \oplus \Bbbk x_3x_4,\end{aligned}$$

$$
\begin{aligned}
(x_2A+(x_1\pm x_3)A)_2 &= \Bbbk x_2x_1 \oplus \Bbbk x_2^2 \oplus \Bbbk x_2x_3 \oplus \Bbbk x_2x_4 \oplus \Bbbk x_1x_2 \\
&\quad \oplus \Bbbk(x_1x_3 \pm x_3^2) \oplus \Bbbk(x_1x_4 \pm x_3x_4), \\
(x_1A+(x_2\pm x_4)A)_2 &= \Bbbk x_1^2 \oplus \Bbbk x_1x_2 \oplus \Bbbk x_1x_3 \oplus \Bbbk x_1x_4 \oplus \Bbbk x_2x_1 \\
&\quad \oplus \Bbbk(x_2^2 \pm x_2x_4) \oplus \Bbbk(x_2x_3 \pm x_4x_3), \\
(x_3A+(x_2\pm x_4)A)_2 &= \Bbbk x_3x_1 \oplus \Bbbk x_2x_3 \oplus \Bbbk x_1^2 \oplus \Bbbk x_3x_4 \oplus \Bbbk(x_2x_1 \pm x_4x_1) \\
&\quad \oplus \Bbbk(x_2^2 \pm x_2x_4) \oplus \Bbbk x_4x_3.
\end{aligned}
$$

One can easily see that the intersection of any of these ideals with the normalizing sequence above is of cardinality 2. Furthermore, when checking the ideals corresponding to other points of the line scheme that are not intersection points, the intersection is either of cardinality 0 or 1.

3.3.3 The Lines of $\mathfrak{L}(\gamma)$ that Contain Points of $\mathfrak{p}(\gamma)$

In this subsection, we compute how many lines in $\mathbb{P}(V^*)$ that are parametrized by $\mathfrak{L}(\gamma)$ contain a given point of $\mathfrak{p}(\gamma)$. By [29, Remark 3.2], if the number of lines is finite, then it is six, counting multiplicity; hence, the generic case is considered to be six distinct lines. The reader should note that a result similar to Theorem 3.3.3.1 is given in [13, Theorem IV.2.5] for the algebra $\mathcal{A}(1)$, but that result is false as stated (perhaps as a consequence of the sign errors in the third relation of (3) on Page 797 of [28]).

Theorem 3.3.3.1. *Suppose $\gamma \in \Bbbk^\times$, and let $\mathcal{Z}_\gamma$ be as in Theorem 3.2.1.1.*

(a) *For any $j \in \{1, \ldots, 4\}$, e_j lies on infinitely many lines that are parametrized by $\mathfrak{L}(\gamma)$.*
(b) *Each point of $\mathcal{Z}_\gamma$ lies on exactly six distinct lines of those parametrized by $\mathfrak{L}(\gamma)$.*

Proof. Since (a) follows from (II)-(V) in Section 3.3.1, we focus on (b). Let $p = (1, \alpha_2, \alpha_3, \alpha_4) \in \mathcal{Z}_\gamma$. It follows that $\alpha_j \neq 0$ for all j. Suppose that $\gamma^2 \neq 16$.

Let $\alpha = 1/\alpha_3$ and $\beta = \alpha_2/\alpha_4$, so $(\alpha^2 - 1)(\beta^2 - 1) = \gamma\alpha\beta$, by 5.1.1.15 in Appendix 5.1.1. Hence, $p \in \mathcal{V}(x_1 - \alpha x_3, x_2 - \beta x_4)$, which is a line that corresponds to an element of $\mathfrak{L}_1$. Clearly, no other line given by $\mathfrak{L}_1$ contains p.

Let $r_2 = (1, 0, \alpha_3, \alpha_4)$ and let ℓ_2 denote the line through e_2 and r_2. By 5.1.1.9, we have $1 - \alpha_3^2 + i\alpha_3\alpha_4^2 = 0$, so $r_2 \in \mathcal{V}(x_2, x_1^3 - x_1x_3^2 + ix_3x_4^2)$. Thus, ℓ_2 corresponds to an element of $\mathfrak{L}_2$, and $p \in \ell_2$. Conversely, let $r_2' = (b_1,\ 0,\ b_3,\ b_4) \in \mathcal{V}(x_2, x_1^3 - x_1x_3^2 + ix_3x_4^2)$. If p lies on the line through r_2' and e_2, then there exists $(\lambda_1, \lambda_2) \in \mathbb{P}^1$ such that $p = (\lambda_1 b_1, \lambda_2, \lambda_1 b_3, \lambda_1 b_4)$. Thus, $\lambda_1 b_1 \neq 0$ and $\alpha_i = b_i/b_1$ for $i = 3, 4$. Hence, $r_2' = (b_1, 0, b_1\alpha_3, b_1\alpha_4) = (1, 0, \alpha_3, \alpha_4) = r_2$. It follows that no other line given by $\mathfrak{L}_2$ contains p.

Let $r_4 = (1, \alpha_2, \alpha_3, 0)$ and let ℓ_4 denote the line through e_4 and r_4. By 5.1.1.2, we have $\alpha_3^3 - \alpha_3 + i\alpha_2^2 = 0$, so $r_4 \in \mathcal{V}(x_4, x_3^3 - x_1^2x_3 + ix_1x_2^2)$. Thus, ℓ_4 corresponds to an element of $\mathfrak{L}_3$, and $p \in \ell_4$. An argument similar to that of $\mathfrak{L}_2$ proves that no other line given by $\mathfrak{L}_3$ contains p.

Let $r_3 = (1, \alpha_2, 0, \alpha_4)$ and let ℓ_3 denote the line through e_3 and r_3. By 5.1.1.5, we have $\alpha_4^3 - \alpha_2^2\alpha_4 + i\gamma\alpha_2 = 0$, so $r_3 \in \mathcal{V}(x_3, x_4^3 - x_2^2x_4 + i\gamma x_1^2x_2)$. Thus, ℓ_3 corresponds to an element of $\mathfrak{L}_4$, and $p \in \ell_3$. An argument similar to that of $\mathfrak{L}_2$ proves that no other line given by $\mathfrak{L}_4$ contains p.

Let $r_1 = (0, \alpha_2, \alpha_3, \alpha_4)$ and let ℓ_4 denote the line through e_1 and r_1. By 5.1.1.8, we have $\alpha_2^3 - \alpha_2\alpha_4^2 + i\gamma\alpha_3^2\alpha_4 = 0$, so $r_1 \in \mathcal{V}(x_1, x_2^3 - x_2x_4^2 + i\gamma x_3^2x_4)$. Thus, ℓ_4 corresponds to an element of $\mathfrak{L}_5$, and $p \in \ell_4$. An argument similar to that of $\mathfrak{L}_2$ proves that no other line given by $\mathfrak{L}_5$ contains p.

By 5.1.1.1, we have $\alpha_2 = \pm i\alpha_3\alpha_4$, so either $p \in \mathcal{V}(x_1x_2 - ix_3x_4)$ or $p \in \mathcal{V}(ix_1x_2 - x_3x_4)$ (but not both, since $\alpha_3\alpha_4 \neq 0$). In the first case, $p \in \mathcal{V}(\alpha_4x_1 - x_4, \alpha_4x_3 + ix_2)$

and, in the second, $p \in \mathcal{V}(\alpha_4 x_1 - x_4, i\alpha_4 x_3 + x_2)$. These lines correspond to elements of $\mathfrak{L}_{6a}$ and $\mathfrak{L}_{6b}$ respectively. Since each quadric has only two rulings, and since each irreducible component of $\mathfrak{L}_6$ parametrizes only one of the rulings in each case, no other line given by $\mathfrak{L}_6$ contains p.

If, instead, $\gamma = 4$, the only adjustment to the above reasoning is in the case of the lines parametrized by $\mathfrak{L}_1$. Since $\gamma = 4$, the polynomial 5.1.1.15 factors, so

$$(\dagger) \qquad (\alpha_2 + \alpha_4 + \alpha_2\alpha_3 - \alpha_3\alpha_4)(\alpha_2 - \alpha_4 - \alpha_2\alpha_3 - \alpha_3\alpha_4) = 0,$$

that is,

$$\big((1 + \alpha_3)\alpha_2 + (1 - \alpha_3)\alpha_4\big)\big((1 - \alpha_3)\alpha_2 - (1 + \alpha_3)\alpha_4\big) = 0,$$

which provides exactly two lines (of those parametrized by $\mathfrak{L}_1$) that could contain p. These lines are

$$\mathcal{V}(\ x_1 - (1/\alpha_3)x_3,\ \ ((1/\alpha_3) + 1)x_2 + (1/\alpha_3) - 1)x_4\ )$$

and

$$\mathcal{V}(\ x_3 - \alpha_3 x_1,\ \ (\alpha_3 - 1)x_2 + (\alpha_3 + 1)x_4\ ),$$

which correspond to elements of $\mathfrak{L}_{1a}$ and $\mathfrak{L}_{1b}$ respectively. If the first factor of (†) is zero, then p belongs to the first line, whereas if the second factor of (†) is zero, then p belongs to the second line. If both factors of (†) are zero, then $\alpha_2 = \alpha_3\alpha_4$, which forces $\alpha_3\alpha_4 = 0$, by 5.1.1.1, and this contradicts $p \in \mathcal{Z}_\gamma$. It follows that p belongs to exactly one line of those parametrized by $\mathfrak{L}_1$.

For all $\gamma \in \Bbbk^\times$, it is a straightforward calculation to show that the six lines found above are distinct. $\square$

Considering Theorems 3.2.3.1, 3.2.4.2 and 3.3.3.1 in the case where $\gamma^2 \neq 16$, we arrive at the following conjecture.

Conjecture 3.3.3.2. *The line scheme of the most generic quadratic quantum* $\mathbb{P}^3$ *is isomorphic to the union of two spatial elliptic curves and four planar elliptic curves. (Here,* spatial elliptic curve *means a nonplanar elliptic curve that is contained in a subscheme of* $\mathbb{P}^5$ *that is isomorphic to* $\mathbb{P}^3$*.)*

Chapter 4

Different Flavors of $\mathfrak{sl}(2,\mathbb{k})$

In this chapter we examine the quantum spaces of several quadratic quantum $\mathbb{P}^3$s that can be traced back, in some fashion, to the Lie algebra $\mathfrak{sl}(2,\mathbb{k})$. The first algebra discussed is $\mathfrak{sl}(2,\mathbb{k})$ itself; we will summarize some known results on its quantum space. The quantum spaces of the remaining algebras will then be discussed in the same fashion. We assume that $\mathrm{char}(\mathbb{k}) = 0$ in this chapter.

4.1 The Lie Algebra $\mathfrak{sl}(2,\mathbb{k})$

Let $\mathfrak{sl}(2,\mathbb{k})$ be as defined in Section 2.5.1. In order to associate any geometry to $\mathfrak{sl}(2,\mathbb{k})$, we first pass to its universal enveloping algebra, $\mathcal{U}(\mathfrak{sl}(2,\mathbb{k}))$, defined as

$$\mathcal{U}(\mathfrak{sl}(2,\mathbb{k})) = \frac{\mathbb{k}\langle e,f,h\rangle}{\langle\ he-eh-2e,\ hf-fh+2f,\ ef-fe-h\ \rangle}.$$

Note that the Casimir element in $\mathcal{U}(\mathfrak{sl}(2,\mathbb{k}))$ is $\Omega' = h^2 - 2h + 4ef$.

However, we are unable to associate Artin, Tate and Van den Bergh's geometry to $\mathcal{U}(\mathfrak{sl}(2,\mathbb{k}))$ directly since it is not graded. Therefore, we consider a graded $\mathbb{k}$-algebra obtained from $\mathcal{U}(\mathfrak{sl}(2,\mathbb{k}))$ by homogenizing using a central variable.

4.1.1 The Quadratic Quantum $\mathbb{P}^3$ Associated to $\mathcal{U}(\mathfrak{sl}(2,\mathbb{k}))$

Definition 4.1.1.1. The Quadratic Quantum $\mathbb{P}^3$ Associated to $\mathcal{U}(\mathfrak{sl}(2,\mathbb{k}))$

The quadratic quantum $\mathbb{P}^3$ associated to $\mathcal{U}(\mathfrak{sl}(2,\mathbb{k}))$ is the $\mathbb{k}$-algebra

$$\mathcal{H}(\mathfrak{sl}(2,\mathbb{k})) = \frac{\mathbb{k}\langle e,f,h,t\rangle}{\langle he-eh-2et, hf-fh+2ft, ef-fe-ht, te-et, tf-ft, th-ht\rangle}.$$

Theorem 4.1.1.2. *The* $\Bbbk$*-algebra* $\mathcal{H}(\mathfrak{sl}(2,\Bbbk))$ *is AS-regular; in fact,* $\mathcal{H}(\mathfrak{sl}(2,\Bbbk))$ *is an iterated Ore extension of* $\Bbbk[e,t]$.

Proof. Define $C = \Bbbk[e,t]$ and linear maps $\sigma_1 : \Bbbk\langle e,t\rangle \to \Bbbk\langle e,t\rangle$ and $\delta_1 : \Bbbk\langle e,t\rangle \to \Bbbk\langle e,t\rangle$ by

$$\sigma_1(e) = e, \qquad \sigma_1(t) = t, \qquad \delta_1(e) = 2et, \qquad \delta_1(t) = 0.$$

We must show that σ_1 and δ_1 descend to an automorphism of C and a left σ_1-derivation of C, respectively. Since C is a polynomial ring, σ_1 naturally descends to an automorphism of C. We see that δ_1 descends to a left σ_1-derivation of C since

$$\delta_1(et - te) = \sigma_1(e)\delta_1(t) + \delta_1(e)t - \sigma_1(t)\delta_1(e) - \delta_1(t)e = 2et^2 - 2tet;$$

it follows that $\delta_1(\langle et-te\rangle) \subset \langle et-te\rangle$. Therefore, $B = C[h;\sigma_1,\delta_1]$ is an Ore extension of C and

$$B = \frac{\Bbbk\langle e,h,t\rangle}{\langle\ he - eh - 2et,\ et - te,\ ht - th\ \rangle}.$$

Now, define linear maps $\sigma_2 : \Bbbk\langle e,h,t\rangle \to \Bbbk\langle e,h,t\rangle$ and $\delta_2 : \Bbbk\langle e,h,t\rangle \to \Bbbk\langle e,h,t\rangle$ by

$$\begin{aligned} &\sigma_2(e) = e, && \sigma_2(h) = h + 2t, && \sigma_2(t) = t, \\ &\delta_2(e) = -ht, && \delta_2(h) = 0, && \delta_2(t) = 0. \end{aligned}$$

We see that σ_2 and δ_2 descend to an automorphism on B and a left σ_2-derivation of B, respectively, since:

$$\begin{aligned} \sigma_2(et - te) &= et - te, \\ \sigma_2(he - eh - 2et) &= (h+2t)e - e(h+2t) - 2et = he - eh - 4et + 2te, \\ \sigma_2(ht - th) &= (h+2t)t - t(h+2t) = ht - th, \\ \delta_2(et - te) &= \sigma_2(e)\delta_2(t) + \delta_2(e)t - \sigma_2(t)\delta_2(e) - \delta_2(t)e = -ht^2 + tht, \end{aligned}$$

$$\delta_2(ht - th) = \sigma_2(h)\delta_2(t) + \delta_2(h)t - \sigma_2(t)\delta_2(h) - \delta_2(t)h = 0,$$

$$\begin{aligned}\delta_2(he - eh - 2et) &= \sigma_2(h)\delta_2(e) + \delta_2(h)e - \sigma_2(e)\delta_2(h) - \delta_2(e)h - 2\sigma_2(e)\delta_2(t) - 2\delta_2(e)t \\ &= -h^2t - 2tht + hth + 2ht^2;\end{aligned}$$

it follows that $\sigma_2(\langle he - eh - 2et, et - te, ht - th\rangle) \subset \langle he - eh - 2et, et - te, ht - th\rangle$ and $\delta_2(\langle he - eh - 2et, et - te, ht - th\rangle) \subset \langle he - eh - 2et, et - te, ht - th\rangle$. Hence, $A = B[f; \sigma_2, \delta_2] = \mathcal{H}(\mathfrak{sl}(2, \Bbbk))$ is an Ore extension of B. By [25], A is Auslander regular and Cohen Macaulay; by definition of Cohen Macaulay [23, Definition 5.8], A has polynomial growth and is, hence, AS-regular by [23]. $\square$

Corollary 4.1.1.3. *The* $\Bbbk$*-algebra* $\mathcal{H}(\mathfrak{sl}(2, \Bbbk))$ *is a quadratic quantum* $\mathbb{P}^3$*.*

Proof. Since $A = \mathcal{H}(\mathfrak{sl}(2, \Bbbk))$ is an iterated Ore extension of $\Bbbk[e, t]$, A is Auslander regular by [25]. By [23, Theorem 4.8], A is a domain and, thus, t is a normal regular element of A. If follows that A is a normal regular extension of $\Bbbk[e, f, h]$, in the language of [22], and so is AS-regular of global dimension four (cf. [22, Theorem 2.6, Corollary 2.7] and the paragraph after [22, Definition 3.1.1]). Hence, $\mathcal{H}(\mathfrak{sl}(2, \Bbbk))$ is a quadratic quantum $\mathbb{P}^3$. $\square$

4.1.2 The Quantum Space of $\mathcal{H}(\mathfrak{sl}(2, \Bbbk))$

In this section we discuss both the point scheme and the line scheme of $\mathcal{H}(\mathfrak{sl}(2, \Bbbk))$. We will then examine some properties of $\mathcal{U}(\mathfrak{sl}(2, \Bbbk))$ that can realized through the quantum space of $\mathcal{H}(\mathfrak{sl}(2, \Bbbk))$. For ease of notation, we will define $x_1 = e$, $x_2 = f$, $x_3 = h$, $x_4 = t$.

4.1.2.1 The Point Scheme of $\mathcal{H}(\mathfrak{sl}(2,\Bbbk))$

Theorem 4.1.2.1. *[21] The point scheme of* $\mathcal{H}(\mathfrak{sl}(2,\Bbbk))$ *is*

$$\mathfrak{p} = \mathcal{V}(x_4) \cup \mathcal{V}(x_1, x_2, x_3) \cup \mathcal{V}(x_4, 4x_1x_2 + x_3^2).$$

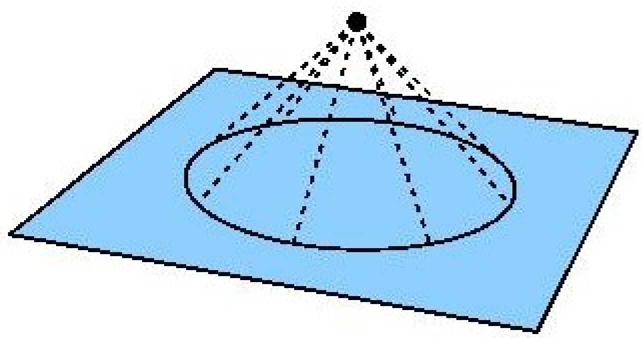

Figure 4.1: The Point Scheme of $\mathcal{H}(\mathfrak{sl}(2,\Bbbk))$

We state the following result as a corollary to the work in [21].

Corollary 4.1.2.2. *Let* $A = \mathcal{H}(\mathfrak{sl}(2,\Bbbk))$ *and* $V = A_1$*.*

(a) The closed points in $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$ *on which the defining relations of* $\mathcal{H}(\mathfrak{sl}(2,\Bbbk))$ *vanish are of the form* (p, p)*, where* $p \in \mathfrak{p}$*.*

(b) There exists an automorphism $\sigma : \mathfrak{p} \to \mathfrak{p}$ *which, on the closed points, is defined by* $\sigma(p) = p$*.*

Proof. Part (a) is easily computed by computation. The existence of the map in (b) follows from (a) and [22]. $\square$

The conic $\mathcal{V}(t, h^2 + 4ef) \subset \mathfrak{p}$ corresponds to a distinguished central element of $\mathcal{H}(\mathfrak{sl}(2,\Bbbk))$; namely, $\Omega = h^2 - 2ht + 4ef = h^2 + 2(ef + fe)$. The points on the conic are of the form $p(\alpha_1, \alpha_2) = (4\alpha_1^2, -\alpha_2^2, 4\alpha_1\alpha_2, 0)$ for $\alpha_1, \alpha_2 \in \Bbbk$. We associate Ω to points of this form since

$$\Omega(\ p(\alpha_1, \alpha_2)\ ,\ \sigma(p(\alpha_1, \alpha_2))\ ) = 0.$$

It is easily computed that Ω is central in $\mathcal{H}(\mathfrak{sl}(2, \Bbbk))$ and that the image of Ω in $\mathcal{U}(\mathfrak{sl}(2, \Bbbk)) \cong \mathcal{H}(\mathfrak{sl}(2, \Bbbk))/\langle t - 1\rangle$ is the Casimir element, $\Omega' = h^2 - 2h + 4ef$, which is the generator of the center of $\mathcal{U}(\mathfrak{sl}(2, \Bbbk))$.

4.1.2.2 The Line Scheme of $\mathcal{H}(\mathfrak{sl}(2, \Bbbk))$

Theorem 4.1.2.3. *[30] The line scheme, $\mathfrak{L}$, of $\mathcal{H}(\mathfrak{sl}(2, \Bbbk))$ consists of two components:*

(I) $\mathfrak{L}_1 = \mathcal{V}(M_{14}, M_{24}, M_{34})$ *counted with multiplicity 4, and*

(II) $\mathfrak{L}_2$, *counted with multiplicity one, which is given by the zero locus of the following polynomials:*

$$M_{12}^2 - M_{13}M_{23}, \quad 2M_{13}M_{24} - M_{12}M_{34}, \quad 2M_{12}M_{24} - M_{23}M_{34},$$
$$M_{12}M_{34} + 2M_{14}M_{23}, \quad M_{13}M_{34} + 2M_{12}M_{14}, \quad M_{34}^2 + 4M_{14}M_{24}.$$

The following corollary describes the lines in $\mathbb{P}^3$ that are parametrized by the line scheme. That is, it describes the lines in $\mathbb{P}^3$ that correspond to line modules of $\mathcal{H}(\mathfrak{sl}(2, \Bbbk))$. It should be noted that the corollary below was originally proved in [21] by using Borel subalgebras. However, it may also be proved using a technique similar to that in Section 3.3.1.

Corollary 4.1.2.4. *[21] The lines in $\mathbb{P}^3$ that are parametrized by the line scheme of $\mathcal{H}(\mathfrak{sl}(2, \Bbbk))$ are the lines in the pencil of quadrics $Q(\delta) = \mathcal{V}(x_3^2 + 4x_1x_2 - \delta^2 x_4^2)$, for all $\delta \in \mathbb{P}^1$.*

4.2 The Lie Superalgebra $\mathfrak{sl}(1|1)$

Consider the Lie superalgebra $\mathfrak{gl}(1|1)$ as in Example 2.5.2.4. The supertrace of a matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is $\mathrm{str}(M) = a - d$. Define $\mathfrak{sl}(1|1) = \{M \in \mathfrak{gl}(1|1) : \mathrm{str}(M) = 0\}$. Every element of $\mathfrak{sl}(1|1)$ is of the form

$$\begin{bmatrix} a & b \\ c & a \end{bmatrix},$$

for all $a, b, c \in \Bbbk$; hence, $\mathfrak{sl}(1|1)$ is a three-dimensional $\Bbbk$-vector space with basis elements

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The vector space $\mathfrak{sl}(1|1)$ becomes a Lie superalgebra under the supercommutator bracket induced by $\mathfrak{gl}(1|1)$. Using the basis $\{e, f, h\}$, the Lie superbracket on $\mathfrak{sl}(1|1)$ is defined by

$$[e, f] = h, \qquad [h, e] = 0 = [h, f], \qquad [e, e] = [f, f] = [h, h] = 0.$$

The universal enveloping algebra of $\mathfrak{sl}(1|1)$ is

$$\mathcal{U}(\mathfrak{sl}(1|1)) = \frac{\Bbbk\langle e, f, h\rangle}{\langle\ ef + fe - h,\ he - eh,\ hf - fh,\ e^2,\ f^2\ \rangle}.$$

Motivated by Le Bruyn and Smith's work in [21], and in order to obtain a graded algebra that maps onto $\mathcal{U}(\mathfrak{sl}(1|1))$ that has the potential to be a quadratic quantum $\mathbb{P}^3$, we construct the algebra given in the following section.

4.2.1 The Quadratic Quantum $\mathbb{P}^3$ Associated to $\mathcal{U}(\mathfrak{sl}(1|1))$

Definition 4.2.1.1. The Quadratic Quantum $\mathbb{P}^3$ Associated to $\mathcal{U}(\mathfrak{sl}(1|1))$

The quadratic quantum $\mathbb{P}^3$ associated to $\mathcal{U}(\mathfrak{sl}(1|1))$ is the algebra

$$\mathcal{H}(\mathfrak{sl}(1|1)) = \frac{\Bbbk\langle e, f, h, t\rangle}{\langle\ ef + fe - ht,\ he - eh,\ hf - fh,\ et - te,\ ft - tf,\ ht - th\ \rangle}.$$

Note that $\mathcal{U}(\mathfrak{sl}(1|1)) \cong \mathcal{H}(\mathfrak{sl}(1|1))/\langle t - 1, e^2, f^2\rangle$.

Theorem 4.2.1.2. *The* $\Bbbk$*-algebra* $\mathcal{H}(\mathfrak{sl}(1|1))$ *is AS-regular; in fact,* $\mathcal{H}(\mathfrak{sl}(1|1))$ *is an Ore extension of* $\Bbbk[e, h, t]$.

Proof. Define $B = \Bbbk[e, h, t]$ and linear maps $\sigma : \Bbbk\langle e, h, t\rangle \to \Bbbk\langle e, h, t\rangle$ and $\delta : \Bbbk\langle e, h, t\rangle \to \Bbbk\langle e, h, t\rangle$ by

$$\sigma(e) = -e, \qquad \sigma(h) = h, \qquad \sigma(t) = t, \qquad \delta(e) = ht, \qquad \delta(h) = 0, \qquad \delta(t) = 0.$$

Since B is a polynomial ring, σ naturally descends to an automorphism on B. We see that δ descends to a left σ-derivation of B since

$$\begin{aligned}
\delta(eh - he) &= \sigma(e)\delta(h) + \delta(e)h - \sigma(h)\delta(e) - \delta(h)e = hth - h^2t,\\
\delta(et - te) &= \sigma(e)\delta(t) + \delta(e)t - \sigma(t)\delta(e) - \delta(t)e = ht^2 - tht,\\
\delta(ht - th) &= 0,
\end{aligned}$$

and so $\delta(\langle et - te, eh - he, ht - th\rangle) \subset \langle et - te, eh - he, ht - th\rangle$. Hence, $A := B[f; \sigma, \delta]$ is an Ore extension where

$$A \cong \frac{\Bbbk\langle e, f, h, t\rangle}{\langle\ ef + fe - ht,\ he - eh,\ hf - fh,\ et - te,\ ft - tf,\ ht - th\ \rangle} = \mathcal{H}(\mathfrak{sl}(1|1)).$$

Thus, by [25], A is Auslander regular and Cohen Macaulay; by definition of Cohen Macaulay [23, Definition 5.8], A has polynomial growth and is, hence, AS-regular by [23]. $\square$

Corollary 4.2.1.3. *The $\Bbbk$-algebra $\mathcal{H}(\mathfrak{sl}(1|1))$ is a quadratic quantum $\mathbb{P}^3$.*

Proof. Since $A = \mathcal{H}(\mathfrak{sl}(1|1))$ is an Ore extension of $\Bbbk[e, h, t]$, A is Auslander regular by [25]. By [23, Theorem 4.8], A is a domain and, thus, t is a central regular element of A. It follows that A is a central regular extension of $\Bbbk[e, f, h]$, in the language of [22], and so is AS-regular of global dimension four (cf. [22, Theorem 2.6, Corollary 2.7] and the paragraph after [22, Definition 3.1.1]). Hence, $\mathcal{H}(\mathfrak{sl}(1|1))$ is a quadratic quantum $\mathbb{P}^3$. □

4.2.2 The Quantum Space of $\mathcal{H}(\mathfrak{sl}(1|1))$

We will now examine the point scheme and line scheme of $\mathcal{H}(\mathfrak{sl}(1|1))$. The process used to determine these is the same as that outlined in Section 3.2. Define $x_1 = e$, $x_2 = f$, $x_3 = h$, $x_4 = t$.

4.2.2.1 The Point Scheme $\mathcal{H}(\mathfrak{sl}(1|1))$

Theorem 4.2.2.1. *The point scheme of $\mathcal{H}(\mathfrak{sl}(1|1))$ is*

$$\mathfrak{p} = \mathcal{V}(x_3, x_4) \cup \mathcal{V}(x_3x_4 - 2x_1x_2),$$

that is, the union of a nonsingular quadric and a line in $\mathbb{P}^3$.

Proof. The polynomials that define $\mathfrak{p}$ are listed in Appendix 5.2.1. The zero locus of these polynomials is easily computed to be $\mathcal{V}(x_3, x_4) \cup \mathcal{V}(x_3x_4 - 2x_1x_2)$ using the logic in the proof of Theorem 3.2.1.1. It remains to show that the point scheme is given by its closed points.

The Jacobian matrix of the point scheme is $J(x_1, x_2, x_3, x_4)$, which is given in Appendix 5.2.2. If p is a multiple point contained in a d-dimensional, irreducible component of $\mathfrak{p}$, then the $(3 - d) \times (3 - d)$ minors of $J(p)$ vanish [15]. An easy computation shows that the only points where such minors vanish are the points $e_1 = (1, 0, 0, 0)$ and $e_2 = (0, 1, 0, 0)$ in $\mathcal{V}(x_3x_4 - 2x_1x_2)$.

By [9], the multiplicity of the point $p \in \mathfrak{p}$ is the vector-space dimension of $\mathcal{O}_{\mathfrak{p},p}$. Thus, by Bertini's Theorem, we may intersect the scheme with a generic, complementary-dimensional linear scheme that intersects $\mathfrak{p}$ at p and then compute the dimension of the local ring at p to determine the multiplicity.

The coordinate ring of $\mathfrak{p}$ is $\Bbbk[x_1, x_2, x_3, x_4]/I$, where I is the ideal generated by the polynomials in Appendix 5.2.1. Consider the projective line

$$L = \mathcal{V}(x_2 - x_3, x_4).$$

The coordinate ring of $\mathfrak{p} \cap L$ is $\Bbbk[x_1, x_2, x_3, x_4]/\langle x_4, x_2 - x_3, x_1x_3^3, x_1^2x_3^2\rangle$ which is isomorphic to $\Bbbk[x_1, x_2]/\langle x_1x_2^3, x_1^2x_2^2\rangle$. The points of intersection are e_1 and $(0, 1, 1, 0)$. In order to determine the multiplicity of e_1, we localize around e_1 and obtain $\Bbbk[x_2]/\langle x_2^2\rangle$, which is two-dimensional. Therefore, e_1 has multiplicity two, which implies that e_1 is a multiple point only as a consequence of it being an intersection point of two irreducible components. Also, because of the symmetry of x_1 and x_2 in $\mathcal{H}(\mathfrak{sl}(1|1))$, we can conclude that the same applies to e_2. Hence, $\mathfrak{p}$ is as proposed. □

Corollary 4.2.2.2. *Let* $A = \mathcal{H}(\mathfrak{sl}(1|1))$ *and* $V = A_1$.

(a) The points in $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$ *on which the defining relations of* $\mathcal{H}(\mathfrak{sl}(1|1))$ *vanish are of the form* (p, p)*, if* $p \in \mathcal{V}(x_3x_4 - 2x_1x_2)$*, and are of the form*

$$((\alpha_1, \alpha_2, 0, 0), (\alpha_1, -\alpha_2, 0, 0)),$$

if $(\alpha_1, \alpha_2, 0, 0) \in \mathcal{V}(x_3, x_4)$.

(b) There exists an automorphism $\sigma : \mathfrak{p} \to \mathfrak{p}$ *which, on the closed points, is defined by*

$$\sigma(p) = \sigma(p_1, p_2, p_3, p_4) = \begin{cases} (p_1, p_2, p_3, p_4), & p \in \mathcal{V}(x_3x_4 - 2x_1x_2) \\ (p_1, -p_2, 0, 0), & p \in \mathcal{V}(x_3, x_4) \end{cases}.$$

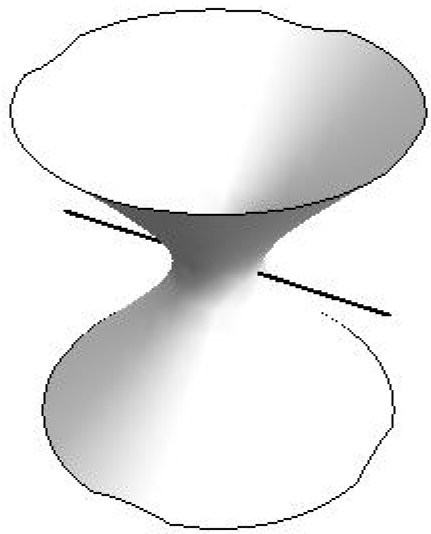

Figure 4.2: The Point Scheme of $\mathcal{H}(\mathfrak{sl}(1|1))$

Proof. Part (a) is easily computed by computation. The existence of the map in (b) follows from (a) and [22]. □

4.2.2.2 The Line Scheme of $\mathcal{H}(\mathfrak{sl}(1|1))$

In this section we compute the closed points of the lines scheme of $\mathcal{H}(\mathfrak{sl}(1|1))$, called the line variety, using the same process as in Section 3.2.2. In Section 4.2.3, we show that $\mathcal{H}(\mathfrak{sl}(1|1))$ is isomorphic to a twist of an algebra in the family discussed in [30, §3.1]; thus, the line scheme of $\mathcal{H}(\mathfrak{sl}(1|1))$ is a reduced scheme by the work in [30].

Theorem 4.2.2.3. *The line variety, $\mathfrak{L}$, of $\mathcal{H}(\mathfrak{sl}(1|1))$ has dimension three and is given by the irreducible components:*

(I) $\mathfrak{L}_1 = \mathcal{V}(\ M_{34},\ M_{13}M_{24} - M_{14}M_{23}\ )$,

(II) $\mathfrak{L}_2 = \mathcal{V}(\ M_{14},\ M_{23},\ M_{34}^2 + 2M_{13}M_{24},\ 2M_{12} + M_{34}\ )$,

(III) $\mathfrak{L}_3 = \mathcal{V}(\ M_{13},\ M_{24},\ M_{34}^2 + 2M_{14}M_{23},\ 2M_{12} - M_{34}\ )$.

Proof. The polynomials that define $\mathfrak{L}$ are given in Appendix 5.2.3. A Gröbner basis for these polynomials is given in Appendix 5.2.4. Polynomial 5.2.4.10 tells us that in order for these polynomials to vanish, either $M_{34} = 0$ or $2M_{13}M_{24}+2M_{14}M_{23}+M_{34}^2 = 0$. We will use the polynomials in Appendix 5.2.4 to analyze $\mathfrak{L}$.

If we assume that $M_{34} = 0$, computing a Gröbner basis yields that the polynomials that define the line scheme vanish only if $M_{13}M_{24} - M_{14}M_{23} = 0$. So this case yields the irreducible component $\mathfrak{L}_1$.

If $M_{34} \neq 0$, then we may assume $M_{34} = 1$, which implies that $2M_{13}M_{24} + 2M_{14}M_{23} + 1 = 0$. Computing a Gröbner basis with degree reverse-lexicographical ordering yields the polynomials:

$$M_{23}M_{24}, \qquad M_{14}M_{24}, \qquad M_{23}(2M_{14}M_{23}+1), \qquad M_{14}(2M_{14}M_{23}+1),$$
$$2M_{13}M_{24}+2M_{14}M_{23}+1, \qquad M_{13}M_{23}, \qquad M_{13}M_{14}, \qquad 2M_{12}+4M_{14}M_{23}+1.$$

To examine the zero locus of these polynomials further, we consider two subcases: $M_{14} = 0$ and $M_{14} \neq 0$.

If $M_{14} = 0$, then $M_{23} = 1 + 2M_{13}M_{24} = 1 + 2M_{12} = 0$. Since this was computed using a Gröbner basis with degree reverse-lexicographical ordering, we may rehomogenize these polynomials with respect to M_{34} to obtain the irreducible component $\mathfrak{L}_2$ [9].

If $M_{14} \neq 0$, then $M_{13} = M_{24} = 1 + 2M_{14}M_{23} = 2M_{12} - 1 = 0$. We then rehomogenize with respect to M_{34} to obtain the irreducible component $\mathfrak{L}_3$.

Therefore, the line variety is as proposed.

$\square$

The following corollary describes the lines in $\mathbb{P}^3$ that are parametrized by the line variety of $\mathcal{H}(\mathfrak{sl}(1|1))$. That is, it describes the lines in $\mathbb{P}^3$ that correspond to line modules of $\mathcal{H}(\mathfrak{sl}(1|1))$.

Corollary 4.2.2.4. The lines in $\mathbb{P}^3$ that are parametrized by the line variety of $\mathcal{H}(\mathfrak{sl}(1|1))$ are:

(i) those that intersect the line $\mathcal{V}(x_3, x_4)$ from the point scheme, and

(ii) those that belong to the rulings of the quadric $\mathcal{V}(2x_1x_2 - x_3x_4)$ from the point scheme.

Proof. Let $(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) \in \mathbb{P}^3$ be distinct points and let

$$\ell = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

represent the projective line between them.

(i) If ℓ is given by $\mathfrak{L}_1$, then $M_{34} = 0$ when evaluated on ℓ which implies that we may assume, for some $\alpha \in \mathbb{k}$, that

$$\ell = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & \alpha a_3 & \alpha a_4 \end{bmatrix}, \text{ or } \begin{bmatrix} a_1 & a_2 & a_3 & \alpha a_3 \\ b_1 & b_2 & b_3 & \alpha b_3 \end{bmatrix}.$$

Applying row operations, we find that we may assume that

$$\ell = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ c_1 & c_2 & 0 & 0 \end{bmatrix},$$

where $c_1, .., c_6 \in \mathbb{k}$. From this representation of ℓ, we can see that ℓ intersects $\mathcal{V}(x_3, x_4)$ if and only if ℓ is given by $\mathfrak{L}_1$.

(ii) If ℓ is given by $\mathfrak{L}_2$, then $M_{14} = M_{23} = 0$; an argument similar to that of (i) allows us to assume that

$$\ell = \begin{bmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_2 & a_3 & 0 \end{bmatrix},$$

for some $(a_1, a_4), (a_2, a_3) \in \mathbb{P}^1$. By requiring further that $M_{34}^2 + 2M_{13}M_{24} = 2M_{12} + M_{34} = 0$ when evaluated on ℓ, we see that $2a_1a_2 - a_3a_4 = 0$. So, ℓ passes through $(a_1, 0, 0, a_4)$ and $(0, a_2, a_3, 0)$, both of which lie on the quadric $\mathcal{V}(2x_1x_2 -$

x_3x_4); in fact, for any point $p = (a_1, \delta a_2, \delta a_3, a_4)$ belonging to ℓ, where $\delta \in \mathbb{P}^1$, we see that $p \in \mathcal{V}(2x_1x_2 - x_3x_4)$. Thus, ℓ belongs to one of the rulings of the quadric $\mathcal{V}(2x_1x_2 - x_3x_4)$. In particular, $\ell \in \{\mathcal{V}(\mu x_1 - x_4, 2x_2 - \mu x_3) : \mu \in \mathbb{P}^1\}$. By a symmetric argument, ℓ is given by $\mathfrak{L}_3$ if and only if ℓ belongs to the other ruling of $\mathcal{V}(2x_1x_2 - x_3x_4)$, namely, $\{\mathcal{V}(\mu x_1 - x_3, 2x_2 - \mu x_4) : \mu \in \mathbb{P}^1\}$.

□

4.2.3 Twisting $\mathcal{O}_q(\mathbb{M}_2)$ to $\mathcal{H}(\mathfrak{sl}(1|1))$

When computing the quantum space of $\mathcal{H}(\mathfrak{sl}(1|1))$, it was noticed that the quantum space was isomorphic to that of $\mathcal{O}_q(\mathbb{M}_2)$, the coordinate ring of quantum 2×2 matrices (see Definition 4.2.3.1). Since the quantum space is invariant (up to isomorphism) under twisting by an automorphism, this led to a conjecture that $\mathcal{H}(\mathfrak{sl}(1|1))$ is a twist by an automorphism of $\mathcal{O}_q(\mathbb{M}_2)$, for some $q \in \Bbbk^\times$.

Definition 4.2.3.1. The Coordinate Ring of Quantum 2×2 Matrices [12]
The coordinate ring of quantum 2×2 matrices is

$$\mathcal{O}_q(\mathbb{M}_2) = \frac{\Bbbk\langle a, b, c, d\rangle}{\langle ab - qba, cd - qdc, ac - qca, bd - qdb, bc - cb, ad - da - (q - q^{-1})bc\rangle},$$

where $q \in \Bbbk^\times, q^2 \neq 1$.

Lemma 4.2.3.2. *[33]*

(i) The point scheme of $\mathcal{O}_q(\mathbb{M}_2)$ is $\mathcal{V}(ad - bc) \cup \mathcal{V}(b, c)$.

(ii) The line scheme of $\mathcal{O}_q(\mathbb{M}_2)$ parametrizes all the lines in $\mathbb{P}^3$ that belong to $\mathcal{V}(ad-bc)$ and those lines that intersect $\mathcal{V}(b, c)$.

Theorem 4.2.3.3. *The algebra $\mathcal{H}(\mathfrak{sl}(1|1))$ is isomorphic to a twist by an automorphism (cf. Definition 2.1.4.3) of the algebra $\mathcal{O}_i(\mathbb{M}_2)$, where $i^2 = -1$.*

Proof. Let $A = \mathcal{O}_i(\mathbb{M}_2)$ and $\tau : A \to A$ be the automorphism defined by

$$\tau(a) = ia, \qquad \tau(b) = b, \qquad \tau(c) = c, \qquad \tau(d) = -id.$$

Also, let A^τ be the algebra obtained by twisting A by τ, with multiplication, $\star$, defined by $\overline{x} \star \overline{y} = x\tau(y)$, for all $\overline{x}, \overline{y} \in A_1^\tau$, where $\overline{x}, \overline{y}$ are the elements of A_1^τ corresponding to x and y in A_1. It follows that:

$$\begin{aligned}
\overline{a} \star \overline{b} - \overline{b} \star \overline{a} &= ab - iba = 0,\\
\overline{a} \star \overline{c} - \overline{c} \star \overline{a} &= ac - iac = 0,\\
\overline{b} \star \overline{c} - \overline{c} \star \overline{b} &= bc - cb = 0,\\
\overline{b} \star \overline{d} - \overline{d} \star \overline{b} &= -ibd - db = -i(bd - idb) = 0,\\
\overline{c} \star \overline{d} - \overline{d} \star \overline{c} &= -icd - dc = -i(cd - idc) = 0,\\
\overline{a} \star \overline{d} + \overline{d} \star \overline{a} - 2\overline{b} \star \overline{c} &= -iad + ida - 2bc = -i(ad - da - 2ibc) = 0.
\end{aligned}$$

Therefore,

$$A^\tau \cong \frac{\Bbbk\langle a, b, c, d\rangle}{\langle\ ab - ba,\ ac - ca,\ bc - cb,\ bd - db,\ cd - dc,\ ad + da - 2bc\ \rangle}.$$

This algebra is isomorphic to $\mathcal{H}(\mathfrak{sl}(1|1))$ under the isomorphism $\varphi : A^\tau \to \mathcal{H}(\mathfrak{sl}(1|1))$ defined by

$$\varphi(a) = e, \qquad \varphi(b) = \frac{h}{\sqrt{2}}, \qquad \varphi(c) = \frac{t}{\sqrt{2}}, \qquad \varphi(d) = f.$$

$\square$

This result leads to the identification of a distinguished supercommuting element of $\mathcal{H}(\mathfrak{sl}(1|1))$ (and therefore of $\mathcal{U}(\mathfrak{sl}(1|1))$). The element $ad - ibc \in \mathcal{O}_i(\mathbb{M}_2)$, called the quantum determinant, is central in $\mathcal{O}_i(\mathbb{M}_2)$; its image in $\mathcal{H}(\mathfrak{sl}(1|1))$ is

$i\left(ef - \frac{ht}{2}\right)$, which, by Corollary 4.2.2.2(a), corresponds to the quadric $\mathcal{V}(2ef - ht)$ in the point scheme of $\mathcal{H}(\mathfrak{sl}(1|1))$. We consider $2ef - ht = 2\left(ef - \frac{ht}{2}\right) \in \mathcal{H}(\mathfrak{sl}(1|1))$.

Like in $\mathfrak{sl}(1|1)$, we consider e and f to be odd elements of $\mathcal{H}(\mathfrak{sl}(1|1))$ and h to be an even element; we also take t to be an even element of $\mathcal{H}(\mathfrak{sl}(1|1))$. Thus,

$$\begin{aligned} e(2ef - ht) &= 2e(ht - fe) - eht = eht - 2efe = hte - 2efe = -(2ef - ht)e, \\ f(2ef - ht) &= 2(ht - ef)f - fht = htf - 2ef^2 = -(2ef - ht)f, \\ h(2ef - ht) &= (2ef - ht)h, \\ t(2ef - ht) &= (2ef - ht)t. \end{aligned}$$

So, the quantum space is indeed identifying a distinguished super-commuting element of $\mathcal{H}(\mathfrak{sl}(1|1))$. Note that the image of this element in $\mathcal{U}(\mathfrak{sl}(1|1))$ is $2ef - h = ef - fe$; this element also super-commutes within $\mathcal{U}(\mathfrak{sl}(1|1))$.

It is as if the element $2ef - h \in \mathcal{U}(\mathfrak{sl}(1|1))$ is playing the role of a Casimir element of a Lie superalgebra, but the Casimir element of $\mathcal{U}(\mathfrak{sl}(1|1))$ is not well-defined since the Killing Form on $\mathfrak{sl}(1|1)$ is degenerate. Hence, the geometry we associated to $\mathcal{H}(\mathfrak{sl}(1|1))$ was able identify a "generalized Casimir" element of $\mathcal{U}(\mathfrak{sl}(1|1))$.

4.3 The Color Lie Algebra $\mathfrak{sl}_k(2, \Bbbk)$

Let $\{e, f, h\}$ be the standard basis of $\mathfrak{sl}(2, \Bbbk)$ and define

$$a_1 = \frac{i}{2}(e - f), \qquad a_2 = -\frac{1}{2}(e + f), \qquad a_3 = \frac{i}{2}h,$$

so that the bracket on $\mathfrak{sl}(2, \Bbbk)$ is defined by

$$[a_1, a_2] = -a_3, \qquad [a_2, a_3] = a_1, \qquad [a_1, a_3] = -a_2.$$

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and define a G-grading on $\mathfrak{sl}(2, \Bbbk)$ by $\mathfrak{sl}(2, \Bbbk) = \bigoplus_{g \in G} X_g$, where

$$X_0 = \{0\}, \qquad X_{(1,0)} = \Bbbk a_1, \qquad X_{(0,1)} = \Bbbk a_2, \qquad X_{(1,1)} = \Bbbk a_3.$$

Define a bicharacter map $\epsilon : G \times G \to \mathbb{k}^{\times}$ by $\epsilon\left((\alpha_1, \alpha_2), (\beta_1, \beta_2)\right) = (-1)^{\alpha_1\beta_2 - \alpha_2\beta_1}$.

The color Lie algebra $\mathfrak{sl}_k(2, \mathbb{k})$, called Klein $\mathfrak{sl}(2, \mathbb{k})$, is the ϵ-Lie algebra with bracket

$$[a_1, a_2] = a_3, \qquad [a_3, a_1] = a_2, \qquad [a_2, a_3] = a_1.$$

For more details on the construction of $\mathfrak{sl}_k(2, \mathbb{k})$ from $\mathfrak{sl}(2, \mathbb{k})$, the reader is referred to [6].

The universal enveloping algebra of $\mathfrak{sl}_k(2, \mathbb{k})$ is

$$\mathcal{U}(\mathfrak{sl}_k(2, \mathbb{k})) = \frac{\mathbb{k}\langle a_1, a_2, a_3 \rangle}{\langle\ a_1a_2 + a_2a_1 - a_3,\ \ a_2a_3 + a_3a_2 - a_1,\ \ a_3a_1 + a_1a_3 - a_2\ \rangle}.$$

4.3.1 The Quadratic Quantum $\mathbb{P}^3$ Associated to $\mathcal{U}(\mathfrak{sl}_k(2, \mathbb{k}))$

Following the example of [21], we homogenize $\mathcal{U}(\mathfrak{sl}_k(2, \mathbb{k}))$ using a central variable.

Definition 4.3.1.1. The Quadratic Quantum $\mathbb{P}^3$ Associated to $\mathcal{U}(\mathfrak{sl}_k(2, \mathbb{k}))$
The quadratic quantum $\mathbb{P}^3$ associated to $\mathcal{U}(\mathfrak{sl}_k(2, \mathbb{k}))$ is the $\mathbb{k}$-algebra $\mathcal{H}(\mathfrak{sl}_k(2, \mathbb{k}))$ on generators a_1, a_2, a_3, a_4 with defining relations

$$a_1a_2 + a_2a_1 = a_3a_4, \qquad a_2a_3 + a_3a_2 = a_1a_4, \qquad a_3a_1 + a_1a_3 = a_2a_4,$$

$$a_1a_4 = a_4a_1, \qquad a_2a_4 = a_4a_2, \qquad a_3a_4 = a_4a_3.$$

Unlike $\mathcal{H}(\mathfrak{sl}(2, \mathbb{k}))$ and $\mathcal{H}(\mathfrak{sl}(1|1))$, $\mathcal{H}(\mathfrak{sl}_k(2, \mathbb{k}))$ does not appear to be an Ore extension of a polynomial ring. Instead, we make use of a result of Le Bruyn, Smith and Van den Bergh in order to prove regularity.

Theorem 4.3.1.2. *The algebra* $\mathcal{H}(\mathfrak{sl}_k(2, \mathbb{k}))$ *is a quadratic quantum* $\mathbb{P}^3$*.*

Proof. A computation using Bergman's Diamond Lemma shows that a basis for $\mathcal{U}(\mathfrak{sl}_k(2,\Bbbk))$ is $\mathfrak{B} = \{a_1^{i_1}a_2^{i_2}a_3^{i_3} : i_1, i_2, i_3 = 0, 1, 2, ...\}$. Let $D = \mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$. Note that $A = D/\langle a_4 \rangle$ is a skew-polynomial ring on three variables and therefore is an AS-regular algebra. We will prove that D is a central regular extension of A in the sense of [22].

Suppose $a_4 f = 0$ in D for some $f \in D$. We may assume $f = g_1 + g_2 + \cdots + g_m \in D$ is homogeneous, where each of the g_i are scalar multiples of monomials with the generators of D in increasing order. Let $\bar{f}$ denote the image of f in $D/\langle a_4 - 1 \rangle$. Since $a_4 f = 0$ in D, we have $\bar{f} = 0$ in $D/\langle a_4 - 1 \rangle \cong \mathcal{U}(\mathfrak{sl}_k(2,\Bbbk))$. Since the g_i are written with the generators in increasing order, $\bar{f} = 0$ must belong to the defining relations of $\mathcal{U}(\mathfrak{sl}_k(2,\Bbbk))$ and implies that an element of the form $a_1^{j_1}a_2^{j_2}a_3^{j_3}$, for some $j_1, j_2, j_3 = 0, 1, 2...$, is missing from $\mathfrak{B}$, which is a contradiction. Therefore such an f cannot exist and so a_4 is regular in D.

It follows that D is a central regular extension of A and by [22] is therefore an AS-regular algebra of global dimension four (cf. [22, Theorem 2.6, Corollary 2.7] and the paragraph after [22, Definition 3.1.1]). Moreover, A is a skew-polynomial ring, so it is Auslander regular. Hence, by [23, Section 5.10], D is also Auslander regular and satisfies the Cohen Macaulay property. □

Let S_3 be the symmetric group on $\{1, 2, 3\}$ and define $\rho = (1\ 2\ 3) \in S_3$. The map $\varphi : \mathcal{H}(\mathfrak{sl}_k(2,\Bbbk)) \to \mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$ defined by

$$\varphi(a_j) = a_{\rho(j)}, \qquad \varphi(a_4) = a_4,$$

for $j = 1, 2, 3$ is an automorphism of $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$. We will make use of both φ and φ^{-1} in our analysis of the quantum space of $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$.

4.3.2 The Quantum Space of $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$

To the compute the quantum space of $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$, we follow the same process as for $\mathcal{H}(\mathfrak{sl}(1|1))$.

4.3.2.1 The Point Scheme of $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$

Theorem 4.3.2.1. *The point scheme, $\mathfrak{p}$, of $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$ is the union of three lines and five points:*

(i) $\mathfrak{p}_1 = \mathcal{V}(a_1, a_4)$,

(ii) $\mathfrak{p}_2 = \mathcal{V}(a_2, a_4)$,

(iii) $\mathfrak{p}_3 = \mathcal{V}(a_3, a_4)$,

(iv) $\mathfrak{p}_4 = \mathcal{V}(a_1, a_2, a_3)$,

(v) $\mathfrak{p}_5 = \mathcal{V}(a_2 + a_1, a_3 + a_1, a_4 - 2a_1)$,

(vi) $\mathfrak{p}_6 = \mathcal{V}(a_2 + a_1, a_3 - a_1, a_4 + 2a_1)$,

(vii) $\mathfrak{p}_7 = \mathcal{V}(a_2 - a_1, a_3 - a_1, a_4 - 2a_1)$,

(viii) $\mathfrak{p}_8 = \mathcal{V}(a_2 - a_1, a_3 + a_1, a_4 + 2a_1)$,

where the points $e_1 \in \mathfrak{p}_2 \cap \mathfrak{p}_3$, $e_2 \in \mathfrak{p}_1 \cap \mathfrak{p}_3$, $e_3 \in \mathfrak{p}_1 \cap \mathfrak{p}_2$ are counted with multiplicity three and all other points are reduced.

Proof. The polynomials that define $\mathfrak{p}$ are listed in Appendix 5.3.1. A Gröbner basis for these polynomials is given in Appendix 5.3.2. The zero locus of these polynomials is easily computed to be $\bigcup_{i=1}^{8} \mathfrak{p}_i$ using the logic in the proof of Theorem 3.2.1.1. It remains to determine the multiplicity of the points in $\mathfrak{p}$.

The Jacobian matrix of the point scheme is $J(a_1, a_2, a_3, a_4)$, which is given is Appendix 5.3.3. We must examine the zero locus of the 2×2 minors to determine the multiplicity of points in $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$, and the zero locus of the 3×3 minors to determine the multiplicity of points in $\mathfrak{p}_4, ..., \mathfrak{p}_8$ [15]. An easy computation shows that the only points where the minors vanish are the points e_1, e_2, and e_3. So, e_1, e_2, e_3 are the only possible multiple points in the scheme.

The existence of the automorphism φ gives a symmetry between a_1, a_2 and a_3 in $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$. So, we need only compute the multiplicity of e_1; the symmetry gives that this will be the multiplicity of e_2 and e_3.

The coordinate ring of the point scheme is $\Bbbk[a_1, a_2, a_3, a_4]/I$, where I is the ideal generated by the polynomials in Appendix 5.3.2. Consider the projective plane

$$P = \mathcal{V}(a_2 - a_3 - a_4).$$

The coordinate ring of $\mathfrak{p} \cap P$ is the commutative algebra on $a_1, ..., a_4$ with defining relations

$$\begin{aligned} a_4^5 &= 0, & 2a_3a_4^3 + a_4^4 &= 0, \\ 4a_3^2a_4^2 - a_4^4 &= 0, & 8a_3^3a_4 + a_4^4 &= 0, \\ a_2 - a_3 - a_4 &= 0, & 2a_1a_4^3 + a_4^4 &= 0, \\ 4a_1a_3a_4^2 - a_4^4 &= 0, & 8a_1a_3^2a_4 + a_4^4 &= 0, \\ 16a_1a_3^3 - a_4^4 &= 0, & 4a_1^2a_4^2 - a_4^4 &= 0, \\ 8a_1^2a_3a_4 + a_4^4 &= 0, & 2a_1^2a_3^2 + a_1^3a_4 &= 0. \end{aligned}$$

The points of intersection of $\mathfrak{p}$ and P are e_1 and $(0, 1, 1, 0)$. In order to determine the multiplicity of e_1, we localize around e_1 and obtain a commutative algebra on generators a_2, a_3, a_4 with relations

$$a_4^2 = 0, \qquad a_3a_4 = 0, \qquad a_4 + 2a_3^2 = 0, \qquad a_2 - a_3 - a_4 = 0,$$

that is isomorphic to a polynomial ring on one variable x, with exactly one relation, x^3, and so has dimension three. Therefore, by Bertini's Theorem and [9], e_1 has multiplicity three; symmetry tells us that e_2 and e_3 also have multiplicity three.

□

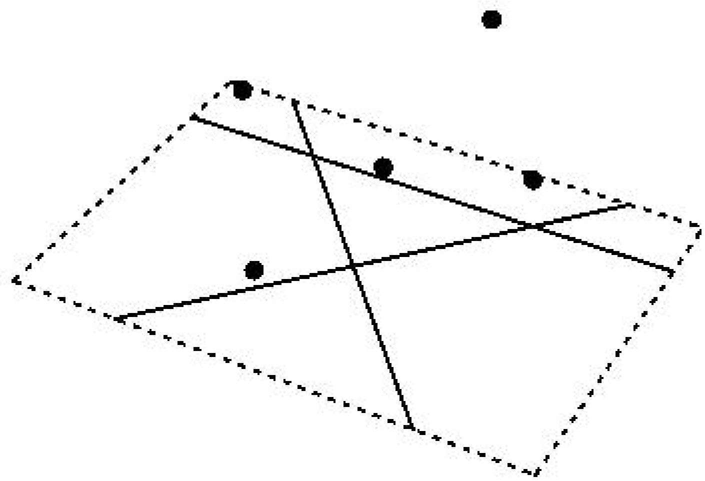

Figure 4.3: The Point Scheme of $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$

Corollary 4.3.2.2. *Let* $A = \mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$ *and* $V = A_1$.

(a) *The points in* $\mathbb{P}(V^*)\times\mathbb{P}(V^*)$ *on which the defining relations of* $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$ *vanish are of the form* (p,p), *if* $p \in \bigcup_{i=4}^{8}\mathfrak{p}_i$, *and are of the form*

$$\big(\,(\alpha_1,\alpha_2,\alpha_3,0)\,,\,((-1)^{\delta_{j2}}\alpha_1,(-1)^{\delta_{j3}}\alpha_2,(-1)^{\delta_{j1}}\alpha_3,0)\,\big)\,,$$

if $(\alpha_1,\alpha_2,\alpha_3,0)\in\mathfrak{p}_j$, *for* $j = 1,2,3$, *where* δ_{jk} *is the Kronecker-delta.*

(b) *There exists an automorphism* $\sigma : \mathfrak{p} \to \mathfrak{p}$ *which, on the closed points, is defined by*

$$\sigma(p) = \sigma(p_1,p_2,p_3,p_4) = \begin{cases} (p_1,p_2,p_3,p_4), & p\in\mathfrak{p}_j, j = 4,...,8 \\ ((-1)^{\delta_{j2}}p_1,(-1)^{\delta_{j3}}p_2,(-1)^{\delta_{j1}}p_3,0), & p\in\mathfrak{p}_j, j=1,2,3 \end{cases}.$$

Proof. Part (a) is easily computed by computation. The existence of the map in (b) follows from (a) and [22]. $\square$

4.3.2.2 The Line Scheme of $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$

We now discuss the line variety of $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$. That is, we examine the closed points of the line scheme.

Theorem 4.3.2.3. *The line variety, $\mathfrak{L}$, of $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$ has dimension two and is given by the union of the thirteen irreducible components:*

(I) $\mathfrak{L}_0 = \mathcal{V}(\ M_{14},\ M_{24},\ M_{34}\ ),$

(II) $\mathfrak{L}_1 = \mathcal{V}(\ M_{12},\ M_{34},\ M_{14} - M_{24},\ M_{13} - M_{23}\ ),$

(III) $\mathfrak{L}_2 = \mathcal{V}(\ M_{12},\ M_{34},\ M_{14} + M_{24},\ M_{13} + M_{23}\ ),$

(IV) $\mathfrak{L}_3 = \mathcal{V}(\ M_{34},\ M_{14} + 2M_{23},\ 2M_{13} + M_{24},\ 2M_{23} - M_{24}\ ),$

(V) $\mathfrak{L}_4 = \mathcal{V}(\ M_{34},\ M_{14} + 2M_{23},\ 2M_{13} + M_{24},\ 2M_{23} + M_{24}\ ),$

(VI) $\mathfrak{L}_5 = \mathcal{V}(\ M_{13},\ M_{24},\ M_{34} - M_{14},\ M_{12} + M_{23}\ ),$

(VII) $\mathfrak{L}_6 = \mathcal{V}(\ M_{13},\ M_{24},\ M_{34} + M_{14},\ M_{12} - M_{23}\ ),$

(VIII) $\mathfrak{L}_7 = \mathcal{V}(\ M_{24},\ M_{34} + 2M_{12},\ M_{14} - 2M_{23},\ 2M_{12} - M_{14}\ ),$

(IX) $\mathfrak{L}_8 = \mathcal{V}(\ M_{24},\ M_{34} + 2M_{12},\ M_{14} - 2M_{23},\ 2M_{12} + M_{14}\ ),$

(X) $\mathfrak{L}_9 = \mathcal{V}(\ M_{14},\ M_{23},\ M_{24} - M_{34},\ M_{13} - M_{12}\ ),$

(XI) $\mathfrak{L}_{10} = \mathcal{V}(\ M_{14},\ M_{23},\ M_{24} + M_{34},\ M_{13} + M_{12}\ ),$

(XII) $\mathfrak{L}_{11} = \mathcal{V}(\ M_{14},\ M_{24} - 2M_{13},\ M_{34} - 2M_{12},\ M_{34} + 2M_{13}\ ),$

(XIII) $\mathfrak{L}_{12} = \mathcal{V}(\ M_{14},\ M_{24} - 2M_{13},\ M_{34} - 2M_{12},\ M_{34} - 2M_{13}\ ).$

Proof. The polynomials that define $\mathfrak{L}$ are given in Appendix 5.3.4. A Gröbner basis for these polynomials is given in Appendix 5.3.5. Using Polynomial 5.3.5.3, we consider the following seven cases:

(i) $M_{14} = M_{24} = M_{34} = 0$,

(ii) $M_{34} = 0$, $M_{14}M_{24} \neq 0$,

(iii) $M_{24} = 0$, $M_{24}M_{34} \neq 0$,

(iv) $M_{14} = 0$, $M_{14}M_{34} \neq 0$,

(v) $M_{14} = M_{24} = 0$, $M_{34} \neq 0$,

(vi) $M_{14} = M_{34} = 0$, $M_{24} \neq 0$,

(vii) $M_{24} = M_{34} = 0$, $M_{14} \neq 0$.

We will analyze $\mathfrak{L}$ using the polynomials in Appendix 5.3.5.

(i) If $M_{14} = M_{24} = M_{34} = 0$, then all the polynomials vanish and we obtain the irreducible component $\mathfrak{L}_0$.

(ii) If $M_{34} = 0$ and $M_{14}M_{24} \neq 0$, then Polynomial 5.3.5.44 tells us that either $M_{12} = 0$ or $M_{14} + 2M_{23} = 0$.

- If $M_{12} = 0$, then computing a Gröbner basis yields the polynomials:

$$M_{24}^2(M_{14} - M_{24})(M_{14} + M_{24}), \qquad M_{23}M_{24}(M_{14} - M_{24})(M_{14} + M_{24}),$$

$$M_{23}^2(M_{14} - M_{24})(M_{14} + M_{24}), \qquad M_{14}M_{24}(M_{14} - M_{24})(M_{14} + M_{24}),$$

$$M_{14}M_{23}(M_{14}-M_{24})(M_{14}+M_{24}), \qquad (M_{14}-M_{24})(M_{14}+M_{24})\left(M_{14}^2 + M_{24}^2\right),$$

$$M_{13}M_{24} - M_{14}M_{23}, \qquad M_{14}M_{23}(M_{13}M_{14} - M_{23}M_{24}),$$

$$M_{13}M_{14}^3 - M_{23}M_{24}^3, \qquad (M_{13}M_{14} - M_{23}M_{24})(M_{13}M_{14} + M_{23}M_{24}).$$

  So, either $M_{14} - M_{24} = 0$ or $M_{14} + M_{24} = 0$. Computing another Gröbner basis with each of these polynomials yields the irreducible components $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively.

- If $M_{12} \neq 0$, then $M_{14} + 2M_{23} = 0$. Since $M_{14} \neq 0$, we know that $M_{23} \neq 0$. Computing a Gröbner basis yields the polynomials

$$M_{24}^2(2M_{23} - M_{24})(2M_{23} + M_{24}), \qquad M_{23}M_{24}(2M_{23} - M_{24})(2M_{23} + M_{24}),$$

$$(2M_{23} - M_{24})(2M_{23} + M_{24})\left(4M_{23}^2 + M_{24}^2\right), \qquad M_{13}M_{24} + 2M_{23}^2,$$

$$M_{23}\left(8M_{13}M_{23}^2 + M_{24}^3\right), \qquad \left(4M_{13}M_{23} - M_{24}^2\right)\left(4M_{13}M_{23} + M_{24}^2\right),$$

$$M_{12}M_{24}(2M_{23}-M_{24})(2M_{23}+M_{24}), \qquad M_{12}M_{23}(2M_{23}-M_{24})(2M_{23}+M_{24}),$$

$$M_{12}\left(8M_{13}M_{23}^2+M_{24}^3\right), \qquad M_{12}M_{23}(2M_{13}-M_{24})(2M_{13}+M_{24}),$$

$$M_{12}^2(2M_{23}-M_{24})(2M_{23}+M_{24}), \qquad M_{12}^2M_{23}(2M_{13}+M_{24}).$$

So either $2M_{23}-M_{24}=0$ or $2M_{23}+M_{24}=0$. Computing another Gröbner basis with each of these polynomials yields the irreducible components $\mathfrak{L}_3$ and $\mathfrak{L}_4$, respectively.

(iii) This case can be analyzed using the automorphism φ^{-1}. Applying this automorphism to each of the components $\mathfrak{L}_1$, $\mathfrak{L}_2$, $\mathfrak{L}_3$ and $\mathfrak{L}_4$ yields the components $\mathfrak{L}_5$, $\mathfrak{L}_6$, $\mathfrak{L}_7$ and $\mathfrak{L}_8$, respectively.

(iv) This case can be analyzed using the automorphism φ. Applying this automorphism to each of the components $\mathfrak{L}_1$, $\mathfrak{L}_2$, $\mathfrak{L}_3$ and $\mathfrak{L}_4$ yields the components $\mathfrak{L}_9$, $\mathfrak{L}_{10}$, $\mathfrak{L}_{11}$ and $\mathfrak{L}_{12}$, respectively.

(v) This case yields no solution. If $M_{14}=M_{24}=0$, then computing a Gröbner basis yields several polynomials, one of which is M_{34}. But this case assumes $M_{34}\neq 0$, and so the variety component is empty.

(vi) This case yields no solution. If $M_{14}=M_{34}=0$, then computing a Gröbner basis yields several polynomials, one of which is M_{24}. But this case assumes $M_{24}\neq 0$, and so the variety component is empty.

(vii) This case yields no solution. If $M_{24}=M_{34}=0$, then computing a Gröbner basis yields several polynomials, one of which is M_{14}. But this case assumes $M_{14}\neq 0$, and so the variety component is empty.

Therefore, the line variety is as proposed.

□

We will again make use of the automorphism φ when describing the lines in $\mathbb{P}^3$ that correspond to line modules of $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$. We will explicitly compute the lines given by $\mathfrak{L}_0$,...,$\mathfrak{L}_4$. The remaining lines are obtained in the same manner as $\mathfrak{L}_5$, ..., $\mathfrak{L}_{12}$ in the above proof.

Corollary 4.3.2.4. *The lines in* $\mathbb{P}^3$ *that are parametrized by the line variety of* $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$ *are:*

(i) *all lines in* $\mathcal{V}(a_4)$,

(ii) *all lines in* $\mathcal{V}(a_1 - a_2)$ *that pass through* $(1, 1, 0, 0)$,

(iii) *all lines in* $\mathcal{V}(a_1 + a_2)$ *that pass through* $(1, -1, 0, 0)$,

(iv) *all lines in* $\mathcal{V}(a_4 - 2a_3)$ *that pass through* $(1, -1, 0, 0)$,

(v) *all lines in* $\mathcal{V}(a_4 + 2a_3)$ *that pass through* $(1, 1, 0, 0)$,

(vi) *all lines in* $\mathcal{V}(a_1 - a_3)$ *that pass through* $(1, 0, 1, 0)$,

(vii) *all lines in* $\mathcal{V}(a_1 + a_3)$ *that pass through* $(1, 0, -1, 0)$,

(viii) *all lines in* $\mathcal{V}(a_4 - 2a_2)$ *that pass through* $(1, 0, -1, 0)$,

(ix) *all lines in* $\mathcal{V}(a_4 + 2a_2)$ *that pass through* $(1, 0, 1, 0)$,

(x) *all lines in* $\mathcal{V}(a_2 - a_3)$ *that pass through* $(0, 1, 1, 0)$,

(xi) *all lines in* $\mathcal{V}(a_2 + a_3)$ *that pass through* $(0, 1, -1, 0)$,

(xii) *all lines in* $\mathcal{V}(a_4 - 2a_1)$ *that pass through* $(0, 1, -1, 0)$, *and*

(xiii) *all lines in* $\mathcal{V}(a_4 + 2a_1)$ *that pass through* $(0, 1, 1, 0)$.

Proof. Let $(b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \in \mathbb{P}^3$ be distinct points and let

$$\ell = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}$$

represent the projective line through them.

If ℓ is given by $\mathfrak{L}_0$, then $M_{14} = M_{24} = M_{34} = 0$ when evaluated on ℓ which implies that we may assume that

$$\ell = \begin{bmatrix} b_1 & b_2 & b_3 & 0 \\ c_1 & c_2 & c_3 & 0 \end{bmatrix}.$$

From this representation of ℓ, we can see that every point of ℓ belongs to $\mathcal{V}(a_4)$. So, ℓ is given by $\mathfrak{L}_0$ if and only if ℓ belongs to $\mathcal{V}(a_4)$.

If ℓ is given by $\mathfrak{L}_1$, then $M_{12} = M_{34} = 0$; by an argument similar to the one above, we may assume that

$$\ell = \begin{bmatrix} d_1 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & d_4 \end{bmatrix},$$

where $(d_1, d_2), (d_3, d_4) \in \mathbb{P}^1$. By requiring further that $M_{14} - M_{24} = M_{13} - M_{23} = 0$ when evaluated on ℓ, we see that $(d_1 - d_2)d_3 = 0 = (d_1 - d_2)d_4$. So, $d_1 = d_2$, which implies that ℓ passes through $(1, 1, 0, 0)$ and belongs to the plane $\mathcal{V}(a_1 - a_2)$.

If ℓ is given by $\mathfrak{L}_2$, then me may assume that

$$\ell = \begin{bmatrix} d_1 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & d_4 \end{bmatrix},$$

as above. By requiring further that $M_{14}+M_{24} = M_{13}+M_{23} = 0$, we see that $d_2 = -d_1$. So ℓ passes through $(1, -1, 0, 0)$ and belongs to the plane $\mathcal{V}(a_1 + a_2)$.

If ℓ is given by $\mathfrak{L}_3$, then $M_{34} = 0$ implies that

$$\ell = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 \\ d_5 & d_6 & 0 & 0 \end{bmatrix},$$

where $d_1, ..., d_6 \in \Bbbk$. By requiring further that $M_{14} + 2M_{23} = 2M_{12} + M_{24} = 2M_{23} - M_{24} = 0$ when evaluated on ℓ, we see that

$$\begin{cases} d_4d_5 + 2d_3d_6 = 0 \\ 2d_3d_5 + d_4d_6 = 0 \\ (d_4 - 2d_3)d_6 = 0. \end{cases}$$

If $d_6 = 0$, then $d_3 = d_4 = 0$. Applying row operations on ℓ yields that

$$\ell = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

which is the line $\mathcal{V}(a_3, a_4)$. This line is also given by $\mathfrak{L}_0$.

If $d_6 \neq 0$, then $d_4 = 2d_3$ and we may take $d_6 = 1$. Thus, $d_3(d_5 + 1) = 0$ which implies that $d_3 = 0$ or $d_5 = -1$. If $d_3 = 0$, we again obtain the line $\mathcal{V}(a_3, a_4)$. Otherwise, using row operations, we may assume that

$$\ell = \begin{bmatrix} 0 & d_7 & d_3 & 2d_3 \\ 1 & -1 & 0 & 0 \end{bmatrix},$$

where $d_7 \in \Bbbk$. Thus, $\mathfrak{L}_3$ also gives all lines that pass through $(1, -1, 0, 0)$ and belongs to $\mathcal{V}(a_4 - 2a_3)$.

If ℓ is given by $\mathfrak{L}_4$, then

$$\ell = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 \\ d_5 & d_6 & 0 & 0 \end{bmatrix},$$

as above. By requiring further that $M_{14} + 2M_{23} = 2M_{12} + M_{24} = 2M_{23} + M_{24} = 0$ when evaluated on ℓ, we see that

$$\begin{cases} d_4 d_5 + 2d_3 d_6 = 0 \\ 2d_3 d_5 + d_4 d_6 = 0 \\ (d_4 + 2d_3)d_6 = 0. \end{cases}$$

So, as in the case of $\mathfrak{L}_3$, the component $\mathfrak{L}_4$ gives the line $\mathcal{V}(a_3, a_4)$ and lines, ℓ, of the form

$$\ell = \begin{bmatrix} 0 & d_7 & d_3 & -2d_3 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Thus, $\mathfrak{L}_4$ gives all lines that pass through $(1, 1, 0, 0)$ and belong to $\mathcal{V}(a_4 + 2a_3)$.

To analyze the lines given by the remaining components, one may now use φ and φ^{-1}. To obtain the lines given by $\mathfrak{L}_5, ..., \mathfrak{L}_8$, one may apply φ^{-1} to the lines described by $\mathfrak{L}_1, .., \mathfrak{L}_4$, respectively. To obtain the lines given by $\mathfrak{L}_9, ..., \mathfrak{L}_{12}$, one may apply φ to the lines described by $\mathfrak{L}_1, ..., \mathfrak{L}_4$, respectively. $\square$

We finish this section by remarking that, unlike $\mathcal{H}(\mathfrak{sl}(2,\Bbbk))$, $\mathcal{H}(\mathfrak{sl}(1|1))$ or $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ (which is discussed in the next section), the quantum space of $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$ does not contain a distinguished conic or quadric. In the case of the other algebras, certain conics and quadrics in the point schemes identified distinguished elements of the algebra, including an analogue of a Casimir element for each underlying Lie-type algebra. This suggests that $\mathfrak{sl}_k(2,\Bbbk)$ lacks such an element.

4.4 Quantum $\mathfrak{sl}(2,\Bbbk)$

The final algebra we will analyze is a quantum analogue of $\mathcal{U}(\mathfrak{sl}(2,\Bbbk))$, denoted $\mathcal{U}_q(\mathfrak{sl}(2,\Bbbk))$; it is a type of algebra known as a quantum group that plays a central role in mathematical physics.

Definition 4.4.0.1. Quantum $\mathfrak{sl}(2,\Bbbk)$ (cf. [31])
Quantum $\mathfrak{sl}(2,\Bbbk)$ is the $\Bbbk$-algebra

$$\mathcal{U}_q(\mathfrak{sl}(2,\Bbbk)) = \frac{\Bbbk\langle E, F, K, K^{-1}\rangle}{\left\langle\ KE - q^2EK,\ KF - q^{-2}FK,\ EF - FE - \frac{K^2-K^{-2}}{q^2-q^{-2}}\ \right\rangle},$$

where $q \in \Bbbk^\times$ and $q^4 \neq 1$.

It should be noted that this is not the current official version of a quantized $\mathfrak{sl}(2,\Bbbk)$; that version replaces the relation $EF - FE - \frac{K^2-K^{-2}}{q^2-q^{-2}} = 0$ with $EF - FE - \frac{K-K^{-1}}{q-q^{-1}} = 0$ in the defining relations (cf. [19]). We thank S. P. Smith of the University of Washington for bringing to our attention that if A denotes the graded algebra defined below in Definition 4.4.1.1 and if $\mathcal{O}_q$ denotes the current official version of quantized $\mathfrak{sl}(2,\Bbbk)$ (cf. [19]), then the ring of degree-zero elements in $A[(KT)^{-1}]$ is isomorphic to $\mathcal{O}_c$, where $c^2 = q$.

In order to recover $\mathcal{U}(\mathfrak{sl}(2,\Bbbk))$ from $\mathcal{U}_q(\mathfrak{sl}(2,\Bbbk))$, we make the change of variable $K = q^H$. We then do the following:

- Note that

$$\lim_{q\to 1}\frac{K^2-K^{-2}}{q^2-q^{-2}}=\lim_{q\to 1}\frac{q^{4H+2}-q^2}{q^{2H+4}-q^{2H}}=\lim_{q\to 1}\frac{(4H+2)q^{4H+1}-2q}{(2H+4)q^{2H+3}-2Hq^{2H-1}}=H.$$

So, in the limit, we obtain the relation $EF-FE-H=0$.

- Note that $\frac{d}{dq}(KE-q^2EK)=\frac{d}{dq}(q^HE-q^2Eq^H)=Hq^{H-1}E-2qEq^H-q^2EHq^{H-1}$. Taking $q=1$ yields the relation $HE-EH-2E=0$. A similar construction holds for the relation $KF-q^{-2}FK=0$ yielding $HF-FH+2F=0$.

4.4.1 The Quadratic Quantum $\mathbb{P}^3$ Associated to $\mathcal{U}_q(\mathfrak{sl}(2,\Bbbk))$

Definition 4.4.1.1. The Quadratic Quantum $\mathbb{P}^3$ Associated to $\mathcal{U}_q(\mathfrak{sl}(2,\Bbbk))$
The quadratic quantum $\mathbb{P}^3$ associated to $\mathcal{U}_q(\mathfrak{sl}(2,\Bbbk))$ is the $\Bbbk$-algebra $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ on generators E,F,H,T, with defining relations

$$KT=TK,\qquad KE=q^2EK,\qquad KF=q^{-2}FK,$$
$$ET=q^2TE,\qquad FT=q^{-2}TF,\qquad EF-FE=\frac{K^2-T^2}{q^2-q^{-2}},$$

where $q\in\Bbbk^\times$ and $q^4\neq 1$.

Note that $\mathcal{U}_q(\mathfrak{sl}(2,\Bbbk))\cong\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))/\langle KT-1\rangle$. The regularity of this algebra is readily seen as it is an Ore extension of a skew polynomial ring, as shown in the next result.

Theorem 4.4.1.2. *The $\Bbbk$-algebra $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ is AS-regular; in fact, $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ is an Ore extension of the skew polynomial ring*

$$B=\frac{\Bbbk\langle E,K,T\rangle}{\langle\ KT-TK,\ EK-q^{-2}KE,\ ET-q^2TE\ \rangle}.$$

Proof. Define linear maps $\sigma:\Bbbk\langle E,K,T\rangle\to\Bbbk\langle E,K,T\rangle$ and $\delta:\Bbbk\langle E,K,T\rangle\to\Bbbk\langle E,K,T\rangle$ by

$$\sigma(E)=E,\qquad \sigma(K)=q^2K,\qquad \sigma(T)=q^{-2}T,$$

$$\delta(E) = \frac{T^2 - K^2}{q^2 - q^{-2}}, \qquad \delta(K) = 0, \qquad \delta(T) = 0.$$

These maps descend to an automorphism of B and a left σ-derivation of B, respectively, since

$$\begin{aligned}
\sigma(KT - TK) &= KT - TK, \\
\sigma(EK - q^{-2}KE) &= q^2 EK - KE, \\
\sigma(ET - q^2 TE) &= q^{-2} ET - TE, \\
\delta(KT - TK) &= \sigma(K)\delta(T) + \delta(K)T - \sigma(T)\delta(K) - \delta(T)K = 0, \\
\delta(EK - q^{-2}KE) &= \sigma(E)\delta(K) + \delta(E)K - q^{-2}\left(\sigma(K)\delta(E) + \delta(K)E\right) \\
&= \left(\frac{T^2 - K^2}{q^2 - q^{-2}}\right) K - K \left(\frac{T^2 - K^2}{q^2 - q^{-2}}\right), \\
\delta(ET - q^2 TE) &= \sigma(E)\delta(T) + \delta(E)T - q^2\left(\sigma(T)\delta(E) + \delta(T)E\right) \\
&= \left(\frac{T^2 - K^2}{q^2 - q^{-2}}\right) T - T \left(\frac{T^2 - K^2}{q^2 - q^{-2}}\right);
\end{aligned}$$

it follows that $\sigma(\langle KT - TK, EK - q^{-2}KE, ET - q^2TE\rangle) \subset \langle KT - TK, EK - q^{-2}KE, ET - q^2TE\rangle$ and $\delta(\langle KT - TK, EK - q^{-2}KE, ET - q^2TE\rangle) \subset \langle KT - TK, EK - q^{-2}KE, ET - q^2TE\rangle$. So, $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk)) \cong B[F;\sigma,\delta]$ is an Ore extension of B. By [25], A is Auslander regular and Cohen Macaulay; by definition of Cohen Macaulay [23, Definition 5.8], A has polynomial growth and is, hence, AS-regular.

□

Corollary 4.4.1.3. *The* $\Bbbk$*-algebra* $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ *is a quadratic quantum* $\mathbb{P}^3$*.*

Proof. Since $A = \mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ is an Ore extension of B, A is Auslander regular by [25]. By [23, Theorem 4.8], A is a domain and, thus, T is a normal regular element of A. It follows that A is a normal regular extension of $A/\langle T\rangle$ (in the language of [22]), which is a skew polynomial ring, and so A is AS-regular of global dimension four (cf. [22, Theorem 2.6, Corollary 2.7] and the paragraph after [22, Definition 3.1.1]). Hence, $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ is a quadratic quantum $\mathbb{P}^3$. □

We will make use of an automorphism $\varphi : \mathcal{H}_q(\mathfrak{sl}(2,\Bbbk)) \to \mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ defined by:

$$\varphi(E) = F, \qquad \varphi(F) = E, \qquad \varphi(K) = T, \qquad \varphi(T) = K.$$

4.4.2 The Quantum Space of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$

Again, the quantum space of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ is computed as before. To ease notation, we define $x_1 = E$, $x_2 = F$, $x_3 = K$, and $x_4 = T$.

4.4.2.1 The Point Scheme of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$

Theorem 4.4.2.1. *For every $q \in \Bbbk^\times$ and $q^4 \neq 1$, the point scheme, $\mathfrak{p}(q)$, of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ is the union of a line, two conics and two points:*

(i) $\mathfrak{p}_1 = \mathcal{V}(x_3, x_4)$,

(ii) $\mathfrak{p}_2 = \mathcal{V}(x_3, q^4x_4^2 + (q^4-1)^2x_1x_2)$,

(iii) $\mathfrak{p}_3 = \mathcal{V}(x_4, q^4x_3^2 + (q^4-1)^2x_1x_2)$,

(iv) $\mathfrak{p}_4 = \mathcal{V}(x_1, x_2, x_3 + x_4)$, *and*

(v) $\mathfrak{p}_5 = \mathcal{V}(x_1, x_2, x_3 - x_4)$.

Proof. The polynomials that define $\mathfrak{p}(q)$ are given in Appendix 5.4.1. A Gröbner basis for these polynomials is given in Appendix 5.4.2. The zero locus of these polynomials are easily computed to be $\bigcup_{i=1}^5 \mathfrak{p}_i$ using the logic in Theorem 3.2.1.1.

The Jacobian matrix, $J_q(x_1, x_2, x_3, x_4)$, of $\mathfrak{p}(q)$ is given in Appendix 5.4.3; because of the size of the matrix, we present it in terms of its individual columns. We examine the zero locus of the 2×2 minors to determine the multiplicity of points in

$\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ and the zero locus of the 3×3 minors to determine the multiplicity of the points in $\mathfrak{p}_4$ and $\mathfrak{p}_5$ [15]. The only place where the minors vanish are $e_1, e_2 \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3$.

If we examine the multiplicity of e_1, we may make use of the automorphism φ to deduce that the multiplicity of e_2 is equal to that of e_1.

The coordinate ring of the point scheme is $\Bbbk[x_1, x_2, x_3, x_4]/I$, where I is the ideal generated by the polynomials in Appendix 5.4.2. Consider the projective plane

$$P = \mathcal{V}(x_2 - x_3 - x_4).$$

The coordinate ring of $\mathfrak{p}(q) \cap P$ is $\Bbbk[x_1, x_2, x_3, x_4]$ with defining relations

$$\begin{gathered} x_3^2x_4^2 + x_3x_4^3 = 0, \quad x_3^3x_4 - x_3x_4^3 = 0, \quad x_2 - x_3 - x_4 = 0, \\ x_1x_3x_4^2 = 0, \qquad x_1x_3^2x_4 = 0, \qquad x_1^2x_3x_4 = 0, \\ x_1x_4^3 - 2q^4x_1x_4^3 + q^8x_1x_4^3 + q^4x_3x_4^3 + q^4x_4^4 = 0, \quad x_1^2x_4^2 - 2q^4x_1^2x_4^2 + q^8x_1^2x_4^2 + q^4x_1x_4^3 = 0, \\ x_1x_3^3 - 2q^4x_1x_3^3 + q^8x_1x_3^3 + q^4x_3^4 + q^4x_3x_4^3 = 0, \quad x_1^2x_3^2 - 2q^4x_1^2x_3^2 + q^8x_1^2x_3^2 + q^4x_1x_3^3 = 0. \end{gathered}$$

The points of intersection of $\mathfrak{p}$ and P are

$$\left(-\tfrac{q^4}{(q^4-1)^2},\ 1,\ 0,\ 1\right), \quad \left(-\tfrac{q^4}{(q^4-1)^2},\ 1,\ 1,\ 0\right), \quad (\ 0,\ 0,\ 1,\ -1\ ), \quad (\ 1,\ 0,\ 0,\ 0\ );$$

by inverting $x_1 + \frac{q^4}{(q^4-1)^2}x_3 + \frac{q^4}{(q^4-1)^2}x_4$, we will determine the multiplicity of e_1. This yields a ring that is isomorphic to a commutative ring on generators x, y with relations $x^2 = 0$, $xy = 0$, $y^2 = 0$, which is three-dimensional. Thus, e_1 is a multiple point only as a consequence of it being an intersection point of three components; because of the automorphism φ, we can conclude the same applies to e_2.

Therefore, the point scheme is as proposed.

$\square$

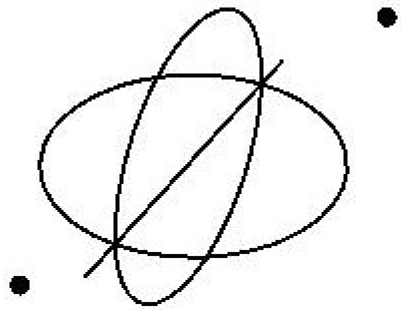

Figure 4.4: The Point Scheme of $\mathcal{H}_q(\mathfrak{sl}(2,\mathbb{k}))$

Corollary 4.4.2.2. *Let* $A = \mathcal{H}_q(\mathfrak{sl}(2,\mathbb{k}))$ *and* $V = A_1$.

(a) The points in $\mathbb{P}(V^*)\times\mathbb{P}(V^*)$ *on which the defining relations of* $\mathcal{H}_q(\mathfrak{sl}(2,\mathbb{k}))$ *vanish are of the form* (p,p), *if* $p \in \mathfrak{p}_1 \cup \mathfrak{p}_4 \cup \mathfrak{p}_5$, *and are of the form*

$$\Big((\alpha_1,\alpha_2,\alpha_3,\alpha_4)\,,\,(\alpha_1, q^{4(-1)^j}\alpha_2, q^{-2}\alpha_3, q^2\alpha_4) \Big)\,,$$

if $(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \in \mathfrak{p}_j$, *for* $j = 2,3$.

(b) There exists an automorphism $\sigma : \mathfrak{p} \to \mathfrak{p}$ *which, on the closed points, is defined by* $\sigma(p) = \sigma(p_1,p_2,p_3,p_4) = \begin{cases} (p_1,p_2,p_3,p_4), & p \in \mathfrak{p}_1 \cup \mathfrak{p}_4 \cup \mathfrak{p}_5 \\ (p_1, q^{4(-1)^j}p_2, q^{-2}p_3, q^2p_4), & p \in \mathfrak{p}_j,\ \textit{for } j = 2,3 \end{cases}.$

Proof. Part (a) is easily computed by computation. The existence of the map in (b) follows from (a) and [22]. $\square$

In the case of $\mathcal{H}(\mathfrak{sl}(2,\mathbb{k}))$ and $\mathcal{H}(\mathfrak{sl}(1|1))$, the embedded conic and the quadric in their respective point schemes corresponded to a Casimir element of the underlying Lie-type algebra. The same is true of $\mathcal{H}_q(\mathfrak{sl}(2,\mathbb{k}))$.

The quantum Casimir element of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{k}))$ is

$$\Omega_q = EF + FE + \left(\frac{q^4+1}{q^4-1}\right)\left(\frac{K^2+K^{-2}}{q^2-q^{-2}}\right).$$

The image of this element in $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ is

$$\Omega_q' = x_1x_2 + x_2x_1 + \left(\frac{q^4+1}{q^4-1}\right)\left(\frac{x_3^2+x_4^2}{q^2-q^{-2}}\right).$$

Let $p \in \mathfrak{p}_2 \setminus \mathcal{V}(x_1)$; we may express p in the form

$$p = \left((q^4-1)^2\alpha_1^2, -q^4\alpha_4^2, 0, (q^4-1)^2\alpha_1\alpha_4\right)$$

and

$$\sigma(p) = \left((q^4-1)^2\alpha_1^2, -q^8\alpha_4^2, 0, q^2(q^4-1)^2\alpha_1\alpha_4\right),$$

where $\alpha_1, \alpha_2, \alpha_4 \in \Bbbk$. An easy computation shows that

$$\Omega_q'(p,\sigma(p)) = -q^8(q^4-1)^2\alpha_1^2\alpha_4^2 - q^4(q^4-1)^2\alpha_1^2\alpha_4^2 + \left(\frac{q^4+1}{q^4-1}\right)\frac{q^2(q^4-1)^4\alpha_1^2\alpha_4^2}{q^2-q^{-2}} = 0.$$

Similarly, if $p \in \mathfrak{p}_3 \setminus \mathcal{V}(x_1)$, then

$$p = \left((q^4-1)^2\alpha_1^2, -q^4\alpha_3^2, (q^4-1)^2\alpha_1\alpha_3, 0\right),$$

$$\sigma(p) = \left((q^4-1)^2\alpha_1^2, -\alpha_3^2, q^{-2}(q^4-1)^2\alpha_1\alpha_3, 0\right),$$

and

$$\Omega_q'(p,\sigma(p)) = -(q^4-1)^2\alpha_1^2\alpha_3^2 - q^4(q^4-1)^2\alpha_1^2\alpha_3^2 + \left(\frac{q^4+1}{q^4-1}\right)\frac{q^{-2}(q^4-1)^4\alpha_1^2\alpha_3^2}{q^2-q^{-2}} = 0.$$

So the quantum Casimir element vanishes on the points in each conic; we may conclude that the geometry is identifying Ω_q' as a distinguished central element of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ and is, therefore, identifying Ω_q as a distinguished central element of $\mathcal{U}_q(\mathfrak{sl}(2,\Bbbk))$.

4.4.2.2 The Line Scheme of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$

Theorem 4.4.2.3. The line variety of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ is $\mathfrak{L}(q) = \bigcup_{i=1}^3 \mathfrak{L}_i$ where

(I) $\mathfrak{L}_1 = \mathcal{V}(\ M_{13},\ M_{23},\ M_{34}\ )$,

(II) $\mathfrak{L}_2 = \mathcal{V}(\ M_{14},\ M_{24},\ M_{34}\ )$, *and*

(III) $\mathfrak{L}_3$ *is given by the zero locus of the seven polynomials:*

$$q^4 M_{34}^2 + M_{14}M_{24}(-1+q^4)^2,$$
$$M_{12}M_{34} + M_{14}M_{23} - M_{13}M_{24},$$
$$q^4 M_{34}^2 + M_{13}M_{23}(-1+q^4)^2,$$
$$-M_{12}M_{24}M_{34} + M_{13}M_{24}^2 + M_{23}M_{34}^2 q^4,$$
$$(-M_{23}^2 + M_{24}^2)M_{34}q^4 + M_{12}M_{23}M_{24}(-1+q^4)^2,$$
$$-M_{12}M_{13}M_{34} + M_{13}^2 M_{24} + M_{14}M_{34}^2 q^4,$$
$$(M_{13} - M_{14})(M_{13} + M_{14})M_{34}q^4 + M_{12}M_{13}M_{14}(-1+q^4)^2.$$

Proof. The polynomials that define the line scheme are given in Appendix 5.4.4. A Gröbner basis for these polynomials is given in Appendix 5.4.5; we will use the polynomials in Appendix 5.4.5 to analyze $\mathfrak{L}(q)$.

Polynomial 5.4.5.1 implies that in order for all the polynomials to vanish, either $M_{34} = 0$ or $q^4 M_{34}^2 + (q^4-1)^2 M_{14}M_{24} = 0$.

(a) If $M_{34} = 0 = q^4 M_{34}^2 + (q^4-1)^2 M_{14}M_{24}$, then $M_{14} = 0$ or $M_{24} = 0$. If $M_{14} = 0 = M_{24}$, then all the polynomials vanish and we obtain the component $\mathfrak{L}_2$.

If $M_{14} = 0$ and $M_{24} \neq 0$, then Polynomial 5.4.5.42 implies that $M_{13} = 0$. A computation with a Gröbner basis yields the polynomials

$$M_{12}M_{23}^2 M_{24}, \qquad M_{12}M_{23}M_{24}^2, \qquad M_{12}^2 M_{23}M_{24}.$$

Since we are assuming $M_{24} \neq 0$, either $M_{12} = 0$ or $M_{23} = 0$. This yields the components $V_1 = \mathcal{V}(M_{12}, M_{13}, M_{14}, M_{34})$ and $V_2 = \mathcal{V}(M_{13}, M_{14}, M_{23}, M_{34})$.

If $M_{24} = 0$ and $M_{14} \neq 0$, then Polynomial 5.4.5.42 implies that $M_{23} = 0$. A computation with a Gröbner basis yields polynomials

$$M_{12}M_{13}M_{14}^2, \qquad M_{12}M_{13}^2M_{14}, \qquad M_{12}^2M_{13}M_{14}.$$

Since we are assuming $M_{14} \neq 0$, either $M_{12} = 0$ or $M_{13} = 0$. This yields the components $V_3 = \mathcal{V}(M_{12}, M_{23}, M_{24}, M_{34})$ and $V_4 = \mathcal{V}(M_{13}, M_{23}, M_{24}, M_{34})$.

(b) If $M_{34} = 0$ but $q^4M_{34}^2 + (q^4-1)^2M_{14}M_{24} \neq 0$, then $M_{14}M_{24} \neq 0$. Polynomial 5.4.5.30 implies that $M_{13} = 0$. A computation with a Gröbner basis yields the polynomials

$$M_{14}M_{23}, \qquad M_{12}M_{23}^2M_{24}, \qquad M_{12}M_{23}M_{24}^2, \qquad M_{12}^2M_{23}M_{24}.$$

Since we are assuming $M_{14}M_{24} \neq 0$, we see that $M_{23} = 0$ and all the polynomials vanish. So, we obtain the component $\mathfrak{L}_1$. Note that $V_2, V_4 \subset \mathfrak{L}_1$.

(c) If $M_{34} \neq 0$ but $q^4M_{34}^2 + (q^4-1)^2M_{14}M_{24} = 0$, then we may take $M_{34} = 1$. Computing a Gröbner basis with degree, reverse-lexicographic ordering yields the polynomials

$$\left(q^4-1\right)^2 M_{14}M_{24} + q^4, \qquad M_{12} - M_{13}M_{24} + M_{14}M_{23}, \qquad \left(q^4-1\right)^2 M_{13}M_{23} + q^4,$$

$$-M_{12}M_{24} + M_{13}M_{24}^2 + \frac{q^4M_{23}}{\left(q^4-1\right)^2}, \qquad -M_{12}M_{13} + M_{13}^2M_{24} + \frac{q^4M_{14}}{\left(q^4-1\right)^2},$$

$$-\frac{(1+q^4)\left(\left(q^4-1\right)^2 M_{12}M_{23}M_{24} + q^4\left(M_{24}^2 - M_{23}^2\right)\right)}{\left(q^4-1\right)^2},$$

$$\frac{\left(q^4-1\right)^2 M_{12}M_{13}M_{14} + q^4(M_{13}-M_{14})(M_{13}+M_{14})}{\left(q^4-1\right)^2}.$$

Multiplying these polynomials by powers of q^4-1 and rehomogenizing with respect to M_{34} shows that $\mathfrak{L}_3$ is as proposed.

□

The following corollary describes the lines in $\mathbb{P}^3$ that are parametrized by the line variety of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$. We thank S. P. Smith of the University of Washington for his suggestion to consider a pencil of quadrics in $\mathbb{P}^3$.

Corollary 4.4.2.4. *Let $\mathfrak{L}(q) = \bigcup_{i=1}^{3} \mathfrak{L}_i$ be the line variety of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ as above. Denote $x_1 = E$, $x_2 = F$, $x_3 = K$, and $x_4 = T$. The lines in $\mathbb{P}^3$ that correspond to line modules of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ are precisely those in the pencil of quadrics*

$$Q_q(\alpha,\beta) = \mathcal{V}\left(\ \alpha q^4(x_3^2 + x_4^2) + \alpha(q^4-1)^2 x_1x_2 + \beta q^4 x_3x_4\ \right),$$

where $(\alpha,\beta) \in \mathbb{P}^1$. More precisely,

(i) $\mathfrak{L}_1$ gives all lines in $\mathcal{V}(x_3)$,

(ii) $\mathfrak{L}_2$ gives all lines in $\mathcal{V}(x_4)$, and

(iii) $\mathfrak{L}_3$ gives the union of the following three families of lines:

(a) those in $\mathcal{V}(x_2)$ that pass through e_1,

(b) those in $\mathcal{V}(x_1)$ that pass through e_2, and

(c) those of the form $\mathcal{V}(x_1 - a_1x_3 - b_1x_4, x_2 - a_2x_3 - b_2x_4)$, where $a_1, a_2, b_1, b_2 \in \Bbbk$, $q^4 + (q^4-1)^2 a_1a_2 = 0$ and $a_1a_2 = b_1b_2$.

Proof. Let $(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) \in \mathbb{P}^3$ be distinct points and let

$$\ell = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

represent the projective line through them.

(i) If ℓ is given by $\mathfrak{L}_1$, then $M_{13} = M_{23} = M_{34} = 0$; an argument similar to (i) allows us to assume that

$$\ell = \begin{bmatrix} a_1 & a_2 & 0 & a_4 \\ b_1 & b_2 & 0 & b_4 \end{bmatrix}.$$

From this representation of ℓ, we can see that every point of ℓ belongs to $\mathcal{V}(x_3)$. So, ℓ belongs to $\mathcal{V}(x_3)$ if and only if ℓ is given by $\mathfrak{L}_1$.

(ii) Applying the automorphism φ to the lines described by $\mathfrak{L}_1$ gives that $\mathfrak{L}_2$ gives all lines in $\mathcal{V}(x_4)$.

(iii) Assume that ℓ is given by $\mathfrak{L}_3$.

If $M_{34} = 0$, then we may assume that

$$\ell = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ c_5 & c_6 & 0 & 0 \end{bmatrix},$$

where $c_1, ..., c_6 \in \Bbbk$. Requiring the polynomials defining $\mathfrak{L}_3$ to vanish yields the following system of equations:

$$\begin{aligned} c_4^2 c_5 c_6 &= 0, & c_3^2 c_5 c_6 &= 0, \\ c_3 c_4^2 c_5 c_6^2 &= 0, & c_3^2 c_4 c_5^2 c_6 &= 0, \\ c_3 c_4 c_6^2 (c_1 c_6 - c_2 c_5) &= 0, & c_3 c_4 c_5^2 (c_1 c_6 - c_2 c_5) &= 0. \end{aligned}$$

If $c_5 c_6 \neq 0$, then $c_3 = 0 = c_4$ and we again obtain $\mathcal{V}(x_3, x_4)$. If $c_5 c_6 = 0$, then $c_1 c_3 c_4 c_6^3 = 0 = c_2 c_3 c_4 c_5^3$. Since $(c_5, c_6, 0, 0) \in \mathbb{P}^3$, exactly one of c_5 and c_6 are zero. This implies that, if $c_5 = 0$, then $c_1 c_3 c_4 = 0$ and we make take $c_6 = 1$. Hence,

$$\ell = \begin{bmatrix} c_1 & 0 & c_3 & c_4 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The cases where $c_3 = 0$ or $c_4 = 0$ are described by $\mathfrak{L}_1$ and $\mathfrak{L}_2$. We need only discuss

$$\ell = \begin{bmatrix} 0 & 0 & c_3 & c_4 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

We see that ℓ belongs to $\mathcal{V}(x_1)$ and passes through e_2. By a similar argument, if $c_6 = 0$, then ℓ belongs to $\mathcal{V}(x_2)$ and passes through e_1.

If $M_{34} \neq 0$, we may take $M_{34} = 1$ and we may assume

$$\ell = \begin{bmatrix} a_1 & a_2 & 1 & 0 \\ b_1 & b_2 & 0 & 1 \end{bmatrix}.$$

By requiring that the polynomials that define $\mathfrak{L}_3$ vanish on ℓ, we see that $q^4 + (q^4-1)^2 a_1a_2 = 0 = q^4 + (q^4-1)^2 b_1b_2$, and hence, $a_1a_2 = b_1b_2$. Also, we see that any point on a line ℓ with this representation belongs to $\mathcal{V}(x_1 - a_1x_3 - b_1x_4, x_2 - a_2x_3 - b_2x_4)$.

The lines given by $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are precisely those in $Q_q(0,1)$. The lines given by $\mathfrak{L}_3$ of the form

$$\begin{bmatrix} 0 & 0 & c_3 & c_4 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & c_3 & c_4 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

for $c_3c_4 \neq 0$, belong to $Q_q\left(1, -\frac{c_3^2+c_4^2}{c_3c_4}\right)$. If $\ell = \mathcal{V}(x_1 - a_1x_3 - b_1x_4, x_2 - a_2x_3 - b_2x_4)$, where $a_1, a_2, b_1, b_2 \in \Bbbk$, $q^4 + (q^4-1)^2 a_1a_2 = 0$ and $a_1a_2 = b_1b_2$, then ℓ belongs to $Q_q\left(1, -\frac{(q^4-1)^2}{q^4}(a_1b_2 + a_2b_1)\right)$.

If $\alpha \neq 0$, then we may take $\alpha = 1$. If $\beta^2 \neq 4$, then $Q_q(1,\beta)$ has rulings

$$\left\{\mathcal{V}\left(\; x_1 - \mu q^4\,(x_3 - \delta_1 x_4)\;\;,\;\; \mu(q^4-1)^2 x_2 + x_3 - \delta_2 x_4\;\right) : \mu \in \mathbb{P}^1\right\}$$

and

$$\left\{\mathcal{V}\left(\; x_2 - \mu q^4\,(x_3 - \delta_1 x_4)\;\;,\;\; \mu(q^4-1)^2 x_1 + x_3 - \delta_2 x_4\;\right) : \mu \in \mathbb{P}^1\right\},$$

where δ_1 and δ_2 are distinct solutions of $\delta^2 + \beta\delta + 1 = 0$.

If $\beta^2 = 4$, then

$$Q_q(1,\beta) = \mathcal{V}\left(\; q^4\left(x_3 + \frac{\beta}{2}x_4\right)^2 + (q^4-1)^2 x_1x_2\;\right)$$

which is a rank-three quadric and so has only one ruling, namely

$$\left\{\mathcal{V}\left(\; x_1 - \mu q^4\left(x_3 + \frac{\beta}{2}x_4\right)\;\;,\;\; \mu(q^4-1)^2 x_2 + x_3 + \frac{\beta}{2}x_4\;\right) : \mu \in \mathbb{P}^1\right\}.$$

The lines in each of these rulings are given by $\mathfrak{L}_3$. Therefore, the lines corresponding to line modules of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$ are as proposed.

$\square$

Chapter 5

Appendix

In this Appendix, we list the polynomials that define the various schemes discussed throughout this thesis.

5.1 $\mathcal{A}(\gamma)$

5.1.1 The Polynomials Defining the Point Scheme of $\mathcal{A}(\gamma)$

5.1.1.1. $x_1^2x_2^2 + x_3^2x_4^2$,

5.1.1.2. $x_1\left(x_3^3 - x_1^2x_3 + ix_1x_2^2\right)$

5.1.1.3. $x_2\left(x_3^3 - x_1^2x_3 + ix_1x_2^2\right)$

5.1.1.4. $x_4\left(x_3^3 - x_1^2x_3 + ix_1x_2^2\right)$

5.1.1.5. $x_1\left(x_4^3 - x_2^2x_4 + i\gamma x_1^2x_2\right)$

5.1.1.6. $x_2\left(x_4^3 - x_2^2x_4 + i\gamma x_1^2x_2\right)$

5.1.1.7. $x_3\left(x_4^3 - x_2^2x_4 + i\gamma x_1^2x_2\right)$

5.1.1.8. $x_1\left(x_2^3 - x_2x_4^2 + i\gamma x_3^2x_4\right)$

5.1.1.9. $x_2\left(x_1^3 - x_1x_3^2 + ix_3x_4^2\right)$

5.1.1.10. $i\gamma x_1^2x_3^2 - x_1^2x_2x_4 - x_2x_3^2x_4$

5.1.1.11. $ix_2^2x_4^2 - x_1x_2^2x_3 - x_1x_3x_4^2$

5.1.1.12. $x_1^3x_4 + \gamma x_1^2x_2x_3 - x_1x_3^2x_4 + ix_2^2x_3x_4$

5.1.1.13. $x_2^3x_3 + \gamma x_1x_2^2x_4 - x_2x_3x_4^2 + i\gamma x_1^2x_3x_4$

5.1.1.14. $i\gamma x_1^3x_3 + \gamma x_1^2x_2^2 - 2x_1x_2x_3x_4 + ix_2^3x_4$

5.1.1.15. $x_1^2x_2^2 - x_2^2x_3^2 - \gamma x_1x_2x_3x_4 - x_1^2x_4^2 + x_3^2x_4^2$

5.1.2 The Polynomials Defining the Line Scheme of $\mathcal{A}(\gamma)$

5.1.2.1. $M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}$

5.1.2.2. $2M_{13}M_{14}M_{23}M_{24}$

5.1.2.3. $M_{12}(\gamma M_{13}M_{14}M_{23} + iM_{12}M_{14}M_{24} + iM_{23}M_{24}M_{34})$

5.1.2.4. $M_{12}(\gamma M_{13}M_{14}M_{23} - iM_{12}M_{14}M_{24} - iM_{23}M_{24}M_{34})$

5.1.2.5. $M_{13}(\gamma M_{13}M_{14}M_{23} + iM_{12}M_{14}M_{24} + iM_{23}M_{24}M_{34})$

5.1.2.6. $M_{13}(\gamma M_{13}M_{14}M_{23} - iM_{12}M_{14}M_{24} - iM_{23}M_{24}M_{34})$

5.1.2.7. $M_{13}(\gamma M_{13}M_{14}M_{23} + iM_{12}M_{14}M_{24} - iM_{23}M_{24}M_{34})$

5.1.2.8. $M_{14}(\gamma M_{13}M_{14}M_{23} + iM_{12}M_{14}M_{24} + iM_{23}M_{24}M_{34})$

5.1.2.9. $M_{23}(\gamma M_{13}M_{14}M_{23} + iM_{12}M_{14}M_{24} + iM_{23}M_{24}M_{34})$

5.1.2.10. $M_{23}(\gamma M_{13}M_{14}M_{23} - iM_{12}M_{14}M_{24} - iM_{23}M_{24}M_{34})$

5.1.2.11. $M_{24}(\gamma M_{13}M_{14}M_{23} + iM_{12}M_{14}M_{24} + iM_{23}M_{24}M_{34})$

5.1.2.12. $M_{34}(\gamma M_{13}M_{14}M_{23} + iM_{12}M_{14}M_{24} + iM_{23}M_{24}M_{34})$

5.1.2.13. $M_{12}(M_{12}M_{13}M_{23} + M_{13}M_{14}M_{34} + iM_{14}M_{23}M_{24})$

5.1.2.14. $M_{12}(M_{12}M_{13}M_{23} + M_{13}M_{14}M_{34} - iM_{14}M_{23}M_{24})$

5.1.2.15. $M_{13}(M_{12}M_{13}M_{23} + M_{13}M_{14}M_{34} + iM_{14}M_{23}M_{24})$

5.1.2.16. $M_{14}(M_{12}M_{13}M_{23} + M_{13}M_{14}M_{34} + iM_{14}M_{23}M_{24})$

5.1.2.17. $M_{14}(M_{12}M_{13}M_{23} + M_{13}M_{14}M_{34} - iM_{14}M_{23}M_{24})$

5.1.2.18. $M_{23}(M_{12}M_{13}M_{23} + M_{13}M_{14}M_{34} + iM_{14}M_{23}M_{24})$

5.1.2.19. $M_{24}(M_{12}M_{13}M_{23} + M_{13}M_{14}M_{34} + iM_{14}M_{23}M_{24})$

5.1.2.20. $M_{24}(M_{12}M_{13}M_{23} + M_{13}M_{14}M_{34} - iM_{14}M_{23}M_{24})$

5.1.2.21. $M_{24}(M_{12}M_{13}M_{23} - M_{13}M_{14}M_{34} + iM_{14}M_{23}M_{24})$

5.1.2.22. $M_{34}(M_{12}M_{13}M_{23} + M_{13}M_{14}M_{34} + iM_{14}M_{23}M_{24})$

5.1.2.23. $M_{13}^2M_{23}M_{24} + M_{13}M_{14}M_{23}^2 - M_{13}M_{14}M_{34}^2 + iM_{14}M_{23}M_{24}M_{34}$

5.1.2.24. $M_{12}^2M_{13}M_{23} + iM_{12}M_{14}M_{23}M_{24} - M_{13}^2M_{14}M_{24} - M_{13}M_{14}^2M_{23}$

5.1.2.25. $i\gamma M_{12}M_{13}M_{23}^2 - \gamma M_{14}M_{23}^2M_{24} - M_{12}M_{14}M_{24}M_{34} - M_{23}M_{24}M_{34}^2$

5.1.2.26. $i\gamma M_{13}M_{14}M_{23}M_{34} - M_{13}M_{14}M_{24}^2 - M_{14}^2M_{23}M_{24} + M_{23}M_{24}M_{34}^2$

5.1.2.27. $i\gamma M_{12}M_{13}M_{14}M_{23} - M_{12}^2M_{14}M_{24} + M_{13}M_{23}M_{24}^2 + M_{14}M_{23}^2M_{24}$

5.1.2.28. $\gamma M_{13}M_{14}^2M_{23} + M_{12}M_{13}M_{23}M_{34} + iM_{12}M_{14}^2M_{24} + M_{13}M_{14}M_{34}^2$

5.1.2.29. $\gamma M_{14}^2M_{23}^2 + M_{12}^2M_{14}M_{23} + M_{12}M_{14}^2M_{34} + M_{12}M_{23}^2M_{34} + M_{14}M_{23}M_{34}^2$

5.1.2.30. $-i\gamma M_{12}M_{13}^2M_{23} + \gamma M_{13}M_{14}M_{23}M_{24} + M_{12}^2M_{13}M_{24} + iM_{12}M_{14}M_{24}^2 + M_{13}M_{24}M_{34}^2$

5.1.2.31. $i\gamma M_{13}^3M_{14} + M_{12}^3M_{13} + iM_{12}^2M_{14}M_{24} - M_{12}M_{13}M_{14}^2 + M_{13}^2M_{24}M_{34}$

5.1.2.32. $\gamma M_{12}M_{13}M_{14}M_{23} + M_{12}^2M_{14}M_{24} - M_{12}M_{13}M_{23}^2 - M_{13}M_{14}M_{23}M_{34} - iM_{13}M_{23}M_{24}^2$

5.1.2.33. $i\gamma M_{13}^2M_{14}M_{34} + M_{12}^2M_{13}M_{24} + iM_{12}M_{14}M_{24}^2 - 2M_{13}M_{14}^2M_{24} + M_{13}M_{24}M_{34}^2$

5.1.2.34. $i\gamma M_{12}^2M_{13}M_{23} - \gamma M_{12}M_{14}M_{23}M_{24} - i\gamma M_{13}^2M_{14}M_{24} + M_{12}M_{14}^2M_{24} + M_{14}M_{23}M_{24}M_{34}$

5.1.2.35. $i\gamma M_{12}M_{13}^2M_{23} - M_{12}^2M_{13}M_{24} + 2M_{13}M_{23}^2M_{24} - M_{13}M_{24}M_{34}^2 + iM_{23}M_{24}^2M_{34}$

5.1.2.36. $i\gamma M_{12}^2M_{13}M_{23} - M_{12}^3M_{24} + M_{12}M_{23}^2M_{24} - M_{13}M_{24}^2M_{34} + iM_{23}M_{24}^3$

5.1.2.37. $\gamma M_{14}^2M_{23}M_{34} - M_{12}M_{14}^2M_{23} + M_{12}M_{23}M_{34}^2 - M_{14}^3M_{34} + iM_{14}^2M_{24}^2 + M_{14}M_{34}^3$

5.1.2.38. $i\gamma M_{13}^3M_{23} - \gamma M_{13}M_{14}M_{23}M_{34} - M_{12}M_{13}^2M_{24} - iM_{12}M_{14}M_{24}M_{34} + M_{13}M_{23}^2M_{34} - M_{13}M_{34}^3$

5.1.2.39. $\gamma M_{12}M_{14}^2M_{23} + i\gamma M_{13}^2M_{14}^2 + M_{12}^3M_{14} + M_{12}^2M_{23}M_{34} - M_{12}M_{14}^3 - M_{14}^2M_{23}M_{34}$

5.1.2.40. $i\gamma M_{13}^2M_{23}^2 - \gamma M_{14}M_{23}^2M_{34} + M_{12}M_{14}M_{23}^2 - M_{12}M_{14}M_{34}^2 + M_{23}^3M_{34} - M_{23}M_{34}^3$

5.1.2.41. $i\gamma M_{12}M_{14}M_{23}^2 + iM_{12}^3M_{23} + iM_{12}^2M_{14}M_{34} - iM_{12}M_{23}^3 - iM_{14}M_{23}^2M_{34} + M_{23}^2M_{24}^2$

5.1.2.42. $i\gamma M_{12}M_{13}M_{23}M_{34} - \gamma M_{14}M_{23}M_{24}M_{34} - M_{12}M_{13}M_{24}^2 + M_{14}^2M_{24}M_{34} - iM_{14}M_{24}^3 - M_{24}M_{34}^3$

5.1.2.43. $i\gamma M_{12}M_{14}M_{23}M_{34} - iM_{12}^2M_{14}M_{23} - iM_{12}M_{14}^2M_{34} - M_{12}M_{14}M_{24}^2 - iM_{12}M_{23}^2M_{34} - iM_{14}M_{23}M_{34}^2 + M_{23}M_{24}^2M_{34}$

5.1.2.44. $i\gamma M_{12}M_{13}^2M_{23} - \gamma M_{12}M_{14}M_{23}M_{34} - i\gamma M_{13}^2M_{14}M_{34} + M_{12}^2M_{14}M_{23} + M_{12}M_{14}^2M_{34} + M_{12}M_{23}^2M_{34} + M_{14}M_{23}M_{34}^2$

5.1.2.45. $\gamma M_{12}^2 M_{14} M_{23} + i\gamma M_{12} M_{13}^2 M_{14} + M_{12}^4 - M_{12}^2 M_{14}^2 - M_{12}^2 M_{23}^2 - iM_{12} M_{23} M_{24}^2 + M_{13}^2 M_{24}^2 + M_{14}^2 M_{23}^2$

5.1.2.46. $-i\gamma M_{13}^2 M_{23} M_{34} + \gamma M_{14} M_{23} M_{34}^2 + M_{13}^2 M_{24}^2 + M_{14}^2 M_{23}^2 - M_{14}^2 M_{34}^2 + iM_{14} M_{24}^2 M_{34} - M_{23}^2 M_{34}^2 + M_{34}^4$

5.1.3 The Intersection Points of the Line Scheme of $\mathcal{A}(\gamma)$

5.1.3.1. $\mathfrak{L}_1 \cap \mathfrak{L}_2 = \mathcal{V}(M_{34}, M_{24}, M_{14}, M_{13}, M_{12}^2 - M_{23}^2) = \{E_1 \pm E_4\}$

5.1.3.2. $\mathfrak{L}_1 \cap \mathfrak{L}_3 = \mathcal{V}(M_{24}, M_{23}, M_{14}^2 - M_{34}^2, M_{13}, M_{12}) = \{E_3 \pm E_6\}$

5.1.3.3. $\mathfrak{L}_1 \cap \mathfrak{L}_4 = \mathcal{V}(M_{24}, M_{23}^2 - M_{34}^2, M_{14}, M_{13}, M_{12}) = \{E_4 \pm E_6\}$

5.1.3.4. $\mathfrak{L}_1 \cap \mathfrak{L}_5 = \mathcal{V}(M_{34}, M_{24}, M_{23}, M_{13}, M_{12}^2 - M_{14}^2) = \{E_1 \pm E_3\}$

5.1.3.5. $\mathfrak{L}_1 \cap \mathfrak{L}_{6a} = \mathcal{V}(M_{34}^2, M_{24}, M_{23}, M_{14}, M_{13}, M_{12} + iM_{34}) = \emptyset$

5.1.3.6. $\mathfrak{L}_1 \cap \mathfrak{L}_{6b} = \mathcal{V}(M_{34}^2, M_{24}, M_{23}, M_{14}, M_{13}, M_{12} - iM_{34}) = \emptyset$

5.1.3.7. $\mathfrak{L}_2 \cap \mathfrak{L}_3 = \mathcal{V}(M_{34}, M_{23}, M_{14}, M_{13}, M_{12}) = \{E_5\}$

5.1.3.8. $\mathfrak{L}_2 \cap \mathfrak{L}_4 = \mathcal{V}(M_{34}, M_{24}, M_{14}, M_{13}, M_{12}) = \{E_4\}$

5.1.3.9. $\mathfrak{L}_2 \cap \mathfrak{L}_5 = \mathcal{V}(M_{34}, M_{24}, M_{23}, M_{14}, M_{13}, M_{12}^3) = \emptyset$

5.1.3.10. $\mathfrak{L}_2 \cap \mathfrak{L}_{6a} = \mathcal{V}(M_{34}, M_{23}, M_{14}, M_{13}, M_{12}) = \{E_5\}$

5.1.3.11. $\mathfrak{L}_2 \cap \mathfrak{L}_{6b} = \mathcal{V}(M_{34}, M_{23}, M_{14}, M_{13}, M_{12}) = \{E_5\}$

5.1.3.12. $\mathfrak{L}_3 \cap \mathfrak{L}_4 = \mathcal{V}(M_{34}^3, M_{24}, M_{23}, M_{14}, M_{13}, M_{12}) = \emptyset$

5.1.3.13. $\mathfrak{L}_3 \cap \mathfrak{L}_5 = \mathcal{V}(M_{34}, M_{24}, M_{23}, M_{13}, M_{12}) = \{E_3\}$

5.1.3.14. $\mathfrak{L}_3 \cap \mathfrak{L}_{6a} = \mathcal{V}(M_{34}, M_{23}, M_{14}, M_{13}, M_{12}) = \{E_5\}$

5.1.3.15. $\mathfrak{L}_3 \cap \mathfrak{L}_{6b} = \mathcal{V}(M_{34}, M_{23}, M_{14}, M_{13}, M_{12}) = \{E_5\}$

5.1.3.16. $\mathfrak{L}_4 \cap \mathfrak{L}_5 = \mathcal{V}(M_{34}, M_{24}, M_{23}, M_{14}, M_{12}) = \{E_2\}$

5.1.3.17. $\mathfrak{L}_4 \cap \mathfrak{L}_{6a} = \mathcal{V}(M_{34}, M_{24}, M_{23}, M_{14}, M_{12}) = \{E_2\}$

5.1.3.18. $\mathfrak{L}_4 \cap \mathfrak{L}_{6b} = \mathcal{V}(M_{34}, M_{24}, M_{23}, M_{14}, M_{12}) = \{E_2\}$

5.1.3.19. $\mathfrak{L}_5 \cap \mathfrak{L}_{6a} = \mathcal{V}(M_{34}, M_{24}, M_{23}, M_{14}, M_{12}) = \{E_2\}$

5.1.3.20. $\mathfrak{L}_5 \cap \mathfrak{L}_{6b} = \mathcal{V}(M_{34}, M_{24}, M_{23}, M_{14}, M_{12}) = \{E_2\}$

5.1.3.21. $\mathfrak{L}_{6a} \cap \mathfrak{L}_{6b} = \mathcal{V}(M_{34}, M_{23}, M_{14}, M_{13}M_{24}, M_{12}) = \{E_2, E_5\}$

5.1.3.22. $\mathfrak{L}_{1a} \cap \mathfrak{L}_2 = \mathcal{V}(M_{34}, M_{24}, M_{14}, M_{13}, M_{12} - M_{23}) = \{E_1 + E_4\}$

5.1.3.23. $\mathfrak{L}_{1b} \cap \mathfrak{L}_2 = \mathcal{V}(M_{34}, M_{24}, M_{14}, M_{13}, M_{12} + M_{23}) = \{E_1 - E_4\}$

5.1.3.24. $\mathfrak{L}_{1a} \cap \mathfrak{L}_3 = \mathcal{V}(M_{24}, M_{23}, M_{14} - M_{34}, M_{13}, M_{12}) = \{E_3 + E_6\}$

5.1.3.25. $\mathfrak{L}_{1b} \cap \mathfrak{L}_3 = \mathcal{V}(M_{24}, M_{23}, M_{14} + M_{34}, M_{13}, M_{12}) = \{E_3 - E_6\}$

5.1.3.26. $\mathfrak{L}_{1a} \cap \mathfrak{L}_4 = \mathcal{V}(M_{24}, M_{23} + M_{34}, M_{14}, M_{13}, M_{12}) = \{E_4 - E_6\}$

5.1.3.27. $\mathfrak{L}_{1b} \cap \mathfrak{L}_4 = \mathcal{V}(M_{24}, M_{23} - M_{34}, M_{14}, M_{13}, M_{12}) = \{E_4 + E_6\}$

5.1.3.28. $\mathfrak{L}_{1a} \cap \mathfrak{L}_5 = \mathcal{V}(M_{34}, M_{24}, M_{23}, M_{13}, M_{12} + M_{14}) = \{E_1 - E_3\}$

5.1.3.29. $\mathfrak{L}_{1b} \cap \mathfrak{L}_5 = \mathcal{V}(M_{34}, M_{24}, M_{23}, M_{13}, M_{12} - M_{14}) = \{E_1 + E_3\}$

5.1.3.30. $\mathfrak{L}_{1a} \cap \mathfrak{L}_{6a} = \mathcal{V}(M_{34}, M_{24}, M_{23}, M_{14}, M_{13}, M_{12}) = \emptyset$

5.1.3.31. $\mathfrak{L}_{1b} \cap \mathfrak{L}_{6a} = \mathcal{V}(M_{34}, M_{24}, M_{23}, M_{14}, M_{13}, M_{12}) = \emptyset$

5.1.3.32. $\mathfrak{L}_{1a} \cap \mathfrak{L}_{6b} = \mathcal{V}(M_{34}, M_{24}, M_{23}, M_{14}, M_{13}, M_{12}) = \emptyset$

5.1.3.33. $\mathfrak{L}_{1b} \cap \mathfrak{L}_{6b} = \mathcal{V}(M_{34}, M_{24}, M_{23}, M_{14}, M_{13}, M_{12}) = \emptyset$

5.1.3.34. $\mathfrak{L}_{1a} \cap \mathfrak{L}_{1b} = \mathcal{V}(M_{24}, M_{23}^2 + M_{34}^2, M_{14} - M_{23}, M_{13}, M_{12} - M_{34}) = \{E_1 \pm iE_3 \pm iE_4 + E_6\}$

5.1.4 The Van den Bergh Polynomials Defining $\mathfrak{L}(\gamma)$

5.1.4.1. $-i(y_2y_4y_5 - iy_1y_3y_6)$

5.1.4.2. $-y_2y_4y_5 - iy_1y_3y_6$

5.1.4.3. $-iy_1^2y_3 + y_3^2y_4 - y_4y_5^2 + \gamma y_3y_4y_6$

5.1.4.4. $i(y_1^2y_2 + iy_2y_3y_4 + i\gamma y_2y_4y_6 - y_1y_5y_6)$

5.1.4.5. $y_1^2y_2 + iy_2y_3y_4 + i\gamma y_2y_4y_6 + y_1y_5y_6$

5.1.4.6. $i(y_1y_2^2 + iy_1y_3y_4 + iy_1y_3y_5 - y_2y_5y_6)$

5.1.4.7. $y_1y_2^2 + iy_1y_3y_4 + iy_1y_3y_5 + y_2y_5y_6$

5.1.4.8. $iy_2^2y_3 - y_3^2y_4 - y_3^2y_5 + y_4y_5^2 + y_5^3 + i\gamma y_2^2y_6 - \gamma y_3y_4y_6 - \gamma y_3y_5y_6$

5.1.4.9. $y_1y_4y_5 + y_1y_5^2 + iy_2y_3y_6 + i\gamma y_2y_6^2$

5.1.4.10. $iy_1y_4y_5 + iy_1y_5^2 + y_2y_3y_6 + \gamma y_2y_6^2$

5.1.4.11. $-iy_2^2y_4 + y_3y_4^2 + y_3y_4y_5 - y_3y_6^2$

5.1.4.12. $y_1y_2y_5 - y_3^2y_6 + y_5^2y_6 - \gamma y_3y_6^2$

5.1.4.13. $y_1y_2y_5 + y_3^2y_6 - y_5^2y_6 + \gamma y_3y_6^2$

5.1.4.14. $y_4^2y_5 + y_4y_5^2 + y_1y_2y_6 - y_5y_6^2$

5.1.4.15. $-y_4^2y_5 - y_4y_5^2 + y_1y_2y_6 + y_5y_6^2$

5.1.4.16. $iy_1^2 - y_4 - y_3y_4^2 + iy_1^2y_5 - y_3y_4y_5 - \gamma y_4^2y_6 - \gamma y_4y_5y_6 + y_3y_6^2 + \gamma y_6^3$

5.2 $\mathcal{H}(\mathfrak{sl}(1|1))$

5.2.1 The Polynomials Defining the Point Scheme of $\mathcal{H}(\mathfrak{sl}(1|1))$

5.2.1.1. $x_4^2(2x_1x_2 - x_3x_4)$

5.2.1.2. $x_3x_4(2x_1x_2 - x_3x_4)$

5.2.1.3. $x_3^2(2x_1x_2 - x_3x_4)$

5.2.1.4. $x_2x_4(2x_1x_2 - x_3x_4)$

5.2.1.5. $x_2x_3(2x_1x_2 - x_3x_4)$

5.2.1.6. $x_1x_4(2x_1x_2 - x_3x_4)$

5.2.1.7. $x_1x_3(2x_1x_2 - x_3x_4)$

5.2.2 The Jacobian Matrix of the Point Scheme of $\mathcal{H}(\mathfrak{sl}(1|1))$

$$\begin{bmatrix} 2x_2x_4^2 & 2x_1x_4^2 & -x_4^3 & 4x_1x_2x_4 - 3x_3x_4^2 \\ 2x_2x_3x_4 & 2x_1x_3x_4 & 2x_1x_2x_4 - 2x_3x_4^2 & 2x_1x_2x_3 - 2x_3^2x_4 \\ 2x_2x_3^2 & 2x_1x_3^2 & 4x_1x_2x_3 - 3x_3^2x_4 & -x_3^3 \\ 2x_2^2x_4 & 4x_1x_2x_4 - x_3x_4^2 & -x_2x_4^2 & 2x_1x_2^2 - 2x_2x_3x_4 \\ 2x_2^2x_3 & 4x_1x_2x_3 - x_3^2x_4 & 2x_1x_2^2 - 2x_2x_3x_4 & -x_2x_3^2 \\ 4x_1x_2x_4 - x_3x_4^2 & 2x_1^2x_4 & -x_1x_4^2 & 2x_1^2x_2 - 2x_1x_3x_4 \\ 4x_1x_2x_3 - x_3^2x_4 & 2x_1^2x_3 & 2x_1^2x_2 - 2x_1x_3x_4 & -x_1x_3^2 \end{bmatrix}.$$

5.2.3 The Polynomials Defining the Line Scheme of $\mathcal{H}(\mathfrak{sl}(1|1))$

5.2.3.1. $-M_{13}M_{14}(M_{13}M_{24} - M_{14}M_{23})$

5.2.3.2. $(M_{14}M_{23} - M_{13}M_{24})\left(2M_{12}^2 + M_{13}M_{24} + M_{14}M_{23}\right)$

5.2.3.3. $M_{23}M_{24}(M_{14}M_{23} - M_{13}M_{24})$

5.2.3.4. $M_{12}M_{13}M_{14}M_{34}$

5.2.3.5. $M_{13}^2M_{14}M_{34}$

5.2.3.6. $-M_{13}M_{14}^2M_{34}$

5.2.3.7. $-2M_{12}M_{13}M_{23}M_{34}$

5.2.3.8. $-2M_{13}^2M_{23}M_{34}$

5.2.3.9. $-M_{13}M_{14}M_{23}M_{34}$

5.2.3.10. $-2M_{13}M_{23}^2M_{34}$

5.2.3.11. $-2M_{12}M_{14}M_{24}M_{34}$

5.2.3.12. $M_{13}M_{14}M_{24}M_{34}$

5.2.3.13. $-2M_{14}^2M_{24}M_{34}$

5.2.3.14. $M_{12}M_{23}M_{24}M_{34}$

5.2.3.15. $-M_{13}M_{23}M_{24}M_{34}$

5.2.3.16. $M_{13}M_{23}M_{24}M_{34}$

5.2.3.17. $-M_{14}M_{23}M_{24}M_{34}$

5.2.3.18. $M_{14}M_{23}M_{24}M_{34}$

5.2.3.19. $-M_{23}^2M_{24}M_{34}$

5.2.3.20. $-2M_{14}M_{24}^2M_{34}$

5.2.3.21. $M_{23}M_{24}^2M_{34}$

5.2.3.22. $-M_{13}M_{14}M_{34}^2$

5.2.3.23. $-M_{23}M_{24}M_{34}^2$

5.2.3.24. $M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}$

5.2.3.25. $-M_{14}\left(M_{12}^2M_{34} + M_{12}M_{13}M_{24} - M_{12}M_{14}M_{23} + M_{14}M_{23}M_{34}\right)$

5.2.3.26. $M_{23}\left(M_{12}^2M_{34} + M_{12}M_{13}M_{24} - M_{12}M_{14}M_{23} + M_{14}M_{23}M_{34}\right)$

5.2.3.27. $M_{13}\left(M_{12}^2M_{34} + M_{12}M_{13}M_{24} - M_{12}M_{14}M_{23} + M_{13}M_{24}M_{34}\right)$

5.2.3.28. $-M_{24}\left(M_{12}^2M_{34} + M_{12}M_{13}M_{24} - M_{12}M_{14}M_{23} + M_{13}M_{24}M_{34}\right)$

5.2.3.29. $M_{13}^2M_{34}(2M_{12} + M_{34})$

5.2.3.30. $-M_{14}^2M_{34}(2M_{12} - M_{34})$

5.2.3.31. $-M_{13}M_{23}\left(-M_{12}M_{34} + M_{13}M_{24} - M_{14}M_{23} + M_{34}^2\right)$

5.2.3.32. $-M_{13}M_{23}\left(-M_{12}M_{34} + M_{13}M_{24} - M_{14}M_{23} - M_{34}^2\right)$

5.2.3.33. $M_{14}M_{23}M_{34}(2M_{12} - M_{34})$

5.2.3.34. $M_{23}^2M_{34}(2M_{12} - M_{34})$

5.2.3.35. $M_{13}M_{24}M_{34}(2M_{12} + M_{34})$

5.2.3.36. $-M_{14}M_{24}\left(M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23} + M_{34}^2\right)$

5.2.3.37. $-M_{14}M_{24}\left(M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23} - M_{34}^2\right)$

5.2.3.38. $-M_{24}^2M_{34}(2M_{12} + M_{34})$

5.2.3.39. $M_{13}M_{34}\left(M_{12}M_{34} + M_{13}M_{24} + M_{14}M_{23} + M_{34}^2\right)$

5.2.3.40. $M_{14}M_{34}\left(-M_{12}M_{34} + M_{13}M_{24} + M_{14}M_{23} + M_{34}^2\right)$

5.2.3.41. $M_{23}M_{34}\left(-M_{12}M_{34} + M_{13}M_{24} + M_{14}M_{23} + M_{34}^2\right)$

5.2.3.42. $M_{24}M_{34}\left(M_{12}M_{34}+M_{13}M_{24}+M_{14}M_{23}+M_{34}^2\right)$

5.2.3.43. $M_{34}^2\left(2M_{13}M_{24}+2M_{14}M_{23}+M_{34}^2\right)$

5.2.4 A Gröbner Basis for the Line Scheme of $\mathcal{H}(\mathfrak{sl}(1|1))$

The following polynomials were found by computing a Gröbner basis using Wolfram's Mathematica and the polynomials given in Appendix 5.2.3.

5.2.4.1. $M_{23}M_{24}M_{34}^2$

5.2.4.2. $M_{23}M_{24}^2M_{34}$

5.2.4.3. $M_{23}^2M_{24}M_{34}$

5.2.4.4. $M_{14}M_{24}M_{34}^2$

5.2.4.5. $M_{14}M_{24}^2M_{34}$

5.2.4.6. $M_{14}M_{23}M_{24}M_{34}$

5.2.4.7. $M_{23}M_{34}\left(2M_{14}M_{23}+M_{34}^2\right)$

5.2.4.8. $M_{14}^2M_{24}M_{34}$

5.2.4.9. $M_{14}M_{34}\left(2M_{14}M_{23}+M_{34}^2\right)$

5.2.4.10. $M_{34}^2\left(2M_{13}M_{24}+2M_{14}M_{23}+M_{34}^2\right)$

5.2.4.11. $M_{24}M_{34}\left(2M_{13}M_{24}+M_{34}^2\right)$

5.2.4.12. $-M_{24}^2\left(-2M_{13}M_{24}+2M_{14}M_{23}-M_{34}^2\right)$

5.2.4.13. $M_{13}M_{23}M_{34}^2$

5.2.4.14. $M_{13}M_{23}M_{24}M_{34}$

5.2.4.15. $-M_{23}M_{24}(M_{14}M_{23}-M_{13}M_{24})$

5.2.4.16. $M_{13}M_{23}^2M_{34}$

5.2.4.17. $-M_{23}^2\left(-2M_{13}M_{24}+2M_{14}M_{23}+M_{34}^2\right)$

5.2.4.18. $M_{13}M_{14}M_{34}^2$

5.2.4.19. $M_{13}M_{14}M_{24}M_{34}$

5.2.4.20. $-M_{14}M_{24}(M_{14}M_{23}-M_{13}M_{24})$

5.2.4.21. $M_{13}M_{14}M_{23}M_{34}$

5.2.4.22. $-M_{14}M_{23}\left(-2M_{13}M_{24}+2M_{14}M_{23}+M_{34}^2\right)$

5.2.4.23. $M_{13}M_{14}^2M_{34}$

5.2.4.24. $-M_{14}^2\left(-2M_{13}M_{24}+2M_{14}M_{23}+M_{34}^2\right)$

5.2.4.25. $M_{13}M_{34}\left(2M_{13}M_{24}+M_{34}^2\right)$

5.2.4.26. $-\left(-2M_{13}M_{24}+2M_{14}M_{23}+M_{34}^2\right)\left(2M_{13}M_{24}+2M_{14}M_{23}+M_{34}^2\right)$

5.2.4.27. $M_{13}^2M_{23}M_{34}$

5.2.4.28. $M_{13}M_{23}(M_{13}M_{24}-M_{14}M_{23})$

5.2.4.29. $M_{13}^2M_{14}M_{34}$

5.2.4.30. $M_{13}M_{14}(M_{13}M_{24}-M_{14}M_{23})$

5.2.4.31. $M_{13}^2\left(2M_{13}M_{24}-2M_{14}M_{23}+M_{34}^2\right)$

5.2.4.32. $M_{12}M_{34}-M_{13}M_{24}+M_{14}M_{23}$

5.2.4.33. $-M_{24}\left(-4M_{12}M_{13}M_{24}+4M_{12}M_{14}M_{23}+M_{34}^3\right)$

5.2.4.34. $-M_{23}\left(-4M_{12}M_{13}M_{24}+4M_{12}M_{14}M_{23}+M_{34}^3\right)$

5.2.4.35. $-M_{14}\left(-4M_{12}M_{13}M_{24}+4M_{12}M_{14}M_{23}+M_{34}^3\right)$

5.2.4.36. $M_{13}\left(4M_{12}M_{13}M_{24}-4M_{12}M_{14}M_{23}-M_{34}^3\right)$

5.2.4.37. $8M_{12}^2M_{13}M_{24}-8M_{12}^2M_{14}M_{23}+4M_{14}M_{23}M_{34}^2+M_{34}^4$

5.3 $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$

5.3.1 The Polynomials Defining the Point Scheme of $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$

5.3.1.1. $a_2\left(a_4a_1^2-2a_2a_3a_1-a_2^2a_4+a_3^2a_4\right)$

5.3.1.2. $-a_1\left(a_4a_1^2+2a_2a_3a_1-a_2^2a_4-a_3^2a_4\right)$

5.3.1.3. $a_3\left(a_4a_1^2-2a_2a_3a_1+a_2^2a_4-a_3^2a_4\right)$

5.3.1.4. $a_1a_4(2a_1a_2-a_3a_4)$

5.3.1.5. $-(a_2 - a_3)(a_2 + a_3)a_4^2$

5.3.1.6. $-a_1a_4(2a_1a_3 - a_2a_4)$

5.3.1.7. $a_2a_4(2a_1a_2 - a_3a_4)$

5.3.1.8. $a_2a_4(2a_2a_3 - a_1a_4)$

5.3.1.9. $(a_1 - a_3)(a_1 + a_3)a_4^2$

5.3.1.10. $-a_4^2(2a_1a_2 - a_3a_4)$

5.3.1.11. $(a_1 - a_2)(a_1 + a_2)a_4^2$

5.3.1.12. $a_3a_4(2a_2a_3 - a_1a_4)$

5.3.1.13. $a_3a_4(2a_1a_3 - a_2a_4)$

5.3.1.14. $-a_4^2(2a_1a_3 - a_2a_4)$

5.3.1.15. $-a_4^2(2a_2a_3 - a_1a_4)$

5.3.2 A Gröbner Basis for the Point Scheme of $\mathcal{H}(\mathfrak{sl}_k(2, \Bbbk))$

The following polynomials were found by computing a Gröbner basis using Wolfram's Mathematica and the polynomials given in Appendix 5.3.1.

5.3.2.1. $a_3a_4^2(2a_3 - a_4)(2a_3 + a_4)$

5.3.2.2. $a_2a_4(2a_3 - a_4)(2a_3 + a_4)$

5.3.2.3. $a_4^2(a_2 - a_3)(a_2 + a_3)$

5.3.2.4. $a_3a_4(2a_2 - a_4)(2a_2 + a_4)$

5.3.2.5. $-a_4^2(2a_2a_3 - a_1a_4)$

5.3.2.6. $a_4^2(2a_1a_3 - a_2a_4)$

5.3.2.7. $a_3a_4(2a_1a_3 - a_2a_4)$

5.3.2.8. $a_4^2(2a_1a_2 - a_3a_4)$

5.3.2.9. $a_3\left(4a_1a_2a_3 + 2a_3^2a_4 - a_4^3\right)$

5.3.2.10. $a_2a_4(2a_1a_2 - a_3a_4)$

5.3.2.11. $a_2\left(4a_1a_2a_3+2a_2^2a_4-a_4^3\right)$

5.3.2.12. $a_4^2(a_1-a_3)(a_1+a_3)$

5.3.2.13. $a_3a_4(2a_1-a_4)(2a_1+a_4)$

5.3.2.14. $a_2a_4(2a_1-a_4)(2a_1+a_4)$

5.3.2.15. $a_1^3a_4+2a_1^2a_2a_3-a_2a_3a_4^2$

5.3.3 The Jacobian Matrix of the Point Scheme of $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$

The following matrix is determined by the polynomials in Appendix 5.3.2.

$$\begin{bmatrix}
0 & 0 & 3a_3^2a_4^2-a_4^4 & 2a_3^3a_4-4a_3a_4^3 \\
0 & 4a_3^2a_4-a_4^3 & 8a_2a_3a_4 & 4a_2a_3^2-3a_2a_4^2 \\
0 & 2a_2a_4^2 & -2a_3a_4^2 & 2a_2^2a_4-2a_3^2a_4 \\
0 & 8a_2a_3a_4 & 4a_2^2a_4-a_4^3 & 4a_2^2a_3-3a_3a_4^2 \\
a_4^3 & -2a_3a_4^2 & -2a_2a_4^2 & 3a_1a_4^2-4a_2a_3a_4 \\
2a_3a_4^2 & -a_4^3 & 2a_1a_4^2 & 4a_1a_3a_4-3a_2a_4^2 \\
2a_3^2a_4 & -a_3a_4^2 & 4a_1a_3a_4-a_2a_4^2 & 2a_1a_3^2-2a_2a_3a_4 \\
2a_2a_4^2 & 2a_1a_4^2 & -a_4^3 & 4a_1a_2a_4-3a_3a_4^2 \\
4a_2a_3^2 & 4a_1a_3^2 & 8a_1a_2a_3+6a_3^2a_4-a_4^3 & 2a_3^3-3a_3a_4^2 \\
2a_2^2a_4 & 4a_1a_2a_4-a_3a_4^2 & -a_2a_4^2 & 2a_1a_2^2-2a_2a_3a_4 \\
4a_2^2a_3 & 8a_1a_2a_3+6a_2^2a_4-a_4^3 & 4a_1a_2^2 & 2a_2^3-3a_2a_4^2 \\
2a_1a_4^2 & 0 & -2a_3a_4^2 & 2a_1^2a_4-2a_3^2a_4 \\
8a_1a_3a_4 & 0 & 4a_1^2a_4-a_4^3 & 4a_1^2a_3-3a_3a_4^2 \\
8a_1a_2a_4 & 4a_1^2a_4-a_4^3 & 0 & 4a_1^2a_2-3a_2a_4^2 \\
3a_1^2a_4+4a_1a_2a_3 & 2a_1^2a_3-a_3a_4^2 & 2a_1^2a_2-a_2a_4^2 & a_1^3-2a_2a_3a_4
\end{bmatrix}$$

5.3.4 The Polynomials Defining the Line Scheme of $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$

5.3.4.1. $-M_{14}\left(2M_{13}M_{12}^2+M_{24}M_{12}^2-M_{13}M_{14}^2+M_{13}M_{14}M_{23}-M_{13}^2M_{24}+M_{14}M_{23}M_{24}\right)$

5.3.4.2. $M_{12}^2M_{14}^2-M_{23}^2M_{14}^2+M_{13}M_{24}M_{14}^2-M_{23}M_{24}^2M_{14}-M_{12}^2M_{24}^2+M_{13}^2M_{24}^2$

5.3.4.3. $-M_{14}M_{24}(M_{13}M_{14}-M_{23}M_{24})$

5.3.4.4. $M_{24}\left(M_{14}M_{12}^2+2M_{23}M_{12}^2-M_{14}M_{23}^2-M_{23}M_{24}^2+M_{13}M_{14}M_{24}+M_{13}M_{23}M_{24}\right)$

5.3.4.5. $2M_{12}M_{14}M_{24}M_{34}$

5.3.4.6. $2M_{14}^2M_{24}M_{34}$

5.3.4.7. $2M_{14}M_{24}^2M_{34}$

5.3.4.8. $M_{14}M_{23} - M_{13}M_{24} + M_{12}M_{34}$

5.3.4.9. $-M_{12}M_{14}(2M_{12}M_{13} + M_{34}M_{13} + M_{12}M_{24})$

5.3.4.10. $M_{14}^2(2M_{12}M_{13} + M_{34}M_{13} + M_{12}M_{24})$

5.3.4.11. $M_{14}(2M_{12}M_{13}M_{23} - M_{12}M_{24}M_{23} - M_{13}M_{34}M_{23} - M_{12}M_{14}M_{24} + M_{13}M_{14}M_{34})$

5.3.4.12. $-M_{14}\left(M_{34}M_{12}^2 - 2M_{13}^2M_{12} + M_{14}^2M_{12} + M_{14}M_{23}M_{12} - M_{13}^2M_{34} + M_{14}M_{23}M_{34}\right)$

5.3.4.13. $M_{14}\left(M_{34}M_{12}^2 + M_{14}^2M_{12} + M_{14}M_{23}M_{12} + M_{13}M_{24}M_{12} + M_{14}M_{23}M_{34}\right)$

5.3.4.14. $-M_{14}M_{24}(2M_{12}M_{13} + M_{34}M_{13} + M_{12}M_{24})$

5.3.4.15. $M_{14}M_{24}(2M_{12}M_{13} - M_{34}M_{13} + M_{12}M_{24})$

5.3.4.16. $-M_{14}M_{34}M_{12}^2 - M_{23}M_{34}M_{12}^2 - M_{14}M_{23}^2M_{12} + M_{14}M_{24}^2M_{12} + M_{23}M_{24}^2M_{12}$
$- M_{13}M_{23}M_{24}M_{12} + M_{14}M_{23}^2M_{34} - M_{13}M_{14}M_{24}M_{34}$

5.3.4.17. $M_{12}M_{24}(M_{12}M_{14} + 2M_{12}M_{23} - M_{23}M_{34})$

5.3.4.18. $M_{12}(M_{13}M_{14} - 2M_{13}M_{23} + M_{23}M_{24})M_{34}$

5.3.4.19. $-M_{24}(M_{12}M_{13}M_{14} + M_{12}M_{24}M_{14} - M_{13}M_{34}M_{14} - 2M_{12}M_{13}M_{23} + M_{13}M_{23}M_{34})$

5.3.4.20. $M_{13}(M_{13}M_{14} - 2M_{13}M_{23} + M_{23}M_{24})M_{34}$

5.3.4.21. $M_{14}M_{24}(M_{12}M_{14} + 2M_{12}M_{23} - M_{23}M_{34})$

5.3.4.22. $-M_{13}M_{34}M_{12}^2 - M_{24}M_{34}M_{12}^2 - M_{13}M_{14}^2M_{12} + M_{13}M_{14}M_{23}M_{12} + M_{13}^2M_{24}M_{12} -$
$M_{14}^2M_{24}M_{12} + M_{13}^2M_{24}M_{34} - M_{14}M_{23}M_{24}M_{34}$

5.3.4.23. $M_{14}M_{24}(M_{12}M_{14} + 2M_{12}M_{23} + M_{23}M_{34})$

5.3.4.24. $-M_{14}(2M_{12}M_{13}M_{23} - M_{12}M_{24}M_{23} + M_{13}M_{34}M_{23} - M_{24}M_{34}M_{23} - M_{12}M_{14}M_{24})$

5.3.4.25. $M_{23}(M_{13}M_{14} - 2M_{13}M_{23} + M_{23}M_{24})M_{34}$

5.3.4.26. $M_{24}\left(M_{34}M_{12}^2 - M_{24}^2M_{12} - M_{14}M_{23}M_{12} - M_{13}M_{24}M_{12} + M_{13}M_{24}M_{34}\right)$

5.3.4.27. $M_{24}\left(M_{34}M_{12}^2 + 2M_{23}^2M_{12} - M_{24}^2M_{12} - M_{13}M_{24}M_{12} - M_{23}^2M_{34} + M_{13}M_{24}M_{34}\right)$

5.3.4.28. $M_{24}^2(M_{12}M_{14} + 2M_{12}M_{23} - M_{23}M_{34})$

5.3.4.29. $-M_{24}(M_{12}M_{13}M_{14} + M_{12}M_{24}M_{14} - 2M_{12}M_{13}M_{23} - M_{13}M_{23}M_{34} + M_{23}M_{24}M_{34})$

5.3.4.30. $-M_{14}M_{34}(2M_{12}M_{13}+M_{34}M_{13}-M_{12}M_{24})$

5.3.4.31. $M_{14}^2\left(M_{14}^2-M_{24}^2-M_{34}^2\right)$

5.3.4.32. $-M_{14}M_{24}M_{13}^2+M_{23}M_{24}M_{13}^2-M_{14}M_{23}^2M_{13}+M_{14}M_{34}^2M_{13}-M_{23}M_{34}^2M_{13}$
$+M_{12}M_{23}M_{34}M_{13}+M_{14}M_{23}^2M_{24}-M_{12}M_{14}M_{24}M_{34}$

5.3.4.33. $-M_{14}M_{34}(M_{12}M_{14}+M_{23}M_{34})$

5.3.4.34. $-M_{13}^2M_{14}^2+M_{23}^2M_{14}^2-M_{12}M_{34}M_{14}^2-M_{23}M_{34}^2M_{14}-M_{12}^2M_{34}^2+M_{13}^2M_{34}^2$

5.3.4.35. $-M_{24}M_{34}(M_{12}M_{24}-M_{13}M_{34})$

5.3.4.36. $-M_{13}^2M_{24}^2+M_{23}^2M_{24}^2-M_{12}M_{34}M_{24}^2+M_{13}M_{34}^2M_{24}+M_{12}^2M_{34}^2-M_{23}^2M_{34}^2$

5.3.4.37. $M_{14}M_{24}\left(M_{14}^2+M_{23}M_{14}-M_{24}^2-M_{34}^2-M_{13}M_{24}-M_{12}M_{34}\right)$

5.3.4.38. $M_{14}M_{24}\left(M_{14}^2+M_{23}M_{14}-M_{24}^2+M_{34}^2-M_{13}M_{24}-M_{12}M_{34}\right)$

5.3.4.39. $-M_{24}M_{34}(M_{12}M_{14}-2M_{12}M_{23}+M_{23}M_{34})$

5.3.4.40. $-M_{14}M_{24}M_{13}^2+M_{23}M_{24}M_{13}^2-M_{14}M_{23}^2M_{13}+M_{23}M_{34}^2M_{13}+M_{12}M_{23}M_{34}M_{13}-$
$M_{23}M_{24}M_{34}^2+M_{14}M_{23}^2M_{24}-M_{12}M_{14}M_{24}M_{34}$

5.3.4.41. $M_{24}^2\left(M_{14}^2-M_{24}^2+M_{34}^2\right)$

5.3.4.42. $-M_{34}\left(M_{24}M_{13}^2-M_{34}^2M_{13}+M_{14}M_{23}M_{13}-M_{12}M_{34}M_{13}+M_{12}M_{24}M_{34}\right)$

5.3.4.43. $-M_{14}M_{34}\left(M_{14}^2-M_{23}M_{14}+M_{24}^2-M_{34}^2-M_{13}M_{24}-M_{12}M_{34}\right)$

5.3.4.44. $-M_{34}\left(M_{14}M_{23}^2-M_{34}^2M_{23}+M_{13}M_{24}M_{23}+M_{12}M_{34}M_{23}-M_{12}M_{14}M_{34}\right)$

5.3.4.45. $-M_{24}M_{34}\left(M_{14}^2-M_{23}M_{14}+M_{24}^2-M_{34}^2-M_{13}M_{24}+M_{12}M_{34}\right)$

5.3.4.46. $-M_{34}^2\left(M_{14}^2+M_{24}^2-M_{34}^2\right)$

5.3.5 A Gröbner Basis for the Line Scheme of $\mathcal{H}(\mathfrak{sl}_k(2,\Bbbk))$

The following polynomials were found by computing a Gröbner basis using Wolfram's Mathematica and the polynomials given in Appendix 5.3.4.

5.3.5.1. $M_{24}(M_{24}-M_{34})M_{34}(M_{24}+M_{34})$

5.3.5.2. $M_{14}M_{24}M_{34}^2$

5.3.5.3. $M_{14}M_{24}^2M_{34}$

5.3.5.4. $M_{14}M_{23}M_{24}M_{34}$

5.3.5.5. $M_{34}^2\left(M_{14}^2+M_{24}^2-M_{34}^2\right)$

5.3.5.6. $M_{14}^2M_{24}M_{34}$

5.3.5.7. $M_{24}^2\left(M_{14}^2-M_{24}^2+M_{34}^2\right)$

5.3.5.8. $M_{23}M_{34}\left(M_{14}^2+M_{24}^2-M_{34}^2\right)$

5.3.5.9. $M_{23}M_{24}\left(M_{14}^2-M_{24}^2+M_{34}^2\right)$

5.3.5.10. $M_{14}(M_{14}-M_{34})M_{34}(M_{14}+M_{34})$

5.3.5.11. $M_{14}(M_{14}-M_{24})M_{24}(M_{14}+M_{24})$

5.3.5.12. $M_{14}M_{23}\left(M_{14}^2-M_{24}^2-M_{34}^2\right)$

5.3.5.13. $\left(M_{14}^2+M_{24}^2-M_{34}^2\right)\left(M_{14}^2-M_{24}^2+M_{34}^2\right)$

5.3.5.14. $-M_{24}\left(-M_{13}M_{24}^2+M_{14}M_{23}M_{24}+M_{13}M_{34}^2\right)$

5.3.5.15. $M_{23}(2M_{13}-M_{24})M_{24}M_{34}$

5.3.5.16. $-M_{23}M_{24}\left(M_{34}^2+2M_{14}M_{23}-2M_{13}M_{24}\right)$

5.3.5.17. $M_{34}\left(4M_{13}M_{23}^2-2M_{24}M_{23}^2+M_{13}M_{24}^2-M_{13}M_{34}^2\right)$

5.3.5.18. $(M_{13}M_{14}-2M_{13}M_{23}+M_{23}M_{24})M_{34}^2$

5.3.5.19. $M_{13}M_{14}M_{24}M_{34}$

5.3.5.20. $M_{24}\left(-M_{23}M_{24}^2+M_{13}M_{14}M_{24}+M_{23}M_{34}^2\right)$

5.3.5.21. $M_{13}M_{34}\left(M_{24}^2-M_{34}^2+2M_{14}M_{23}\right)$

5.3.5.22. $-M_{23}\left(M_{23}M_{14}^2-2M_{13}M_{24}M_{14}+M_{23}M_{24}^2-M_{23}M_{34}^2\right)$

5.3.5.23. $M_{13}M_{34}\left(M_{14}^2+M_{24}^2-M_{34}^2\right)$

5.3.5.24. $M_{14}M_{24}(M_{13}M_{14}-M_{23}M_{24})$

5.3.5.25. $M_{23}\left(2M_{13}M_{14}^2-2M_{23}M_{24}M_{14}-2M_{13}M_{34}^2+M_{24}M_{34}^2\right)$

5.3.5.26. $M_{13}M_{14}^3-M_{23}M_{24}^3-2M_{13}M_{23}M_{34}^2+2M_{23}M_{24}M_{34}^2$

5.3.5.27. $M_{23}\left(2M_{24}M_{13}^2+M_{34}^2M_{13}-2M_{14}M_{23}M_{13}-M_{24}M_{34}^2\right)$

5.3.5.28. $\left(2M_{14}M_{13}^2-4M_{23}M_{13}^2+M_{23}M_{24}^2\right)M_{34}$

5.3.5.29. $M_{14}(M_{13}-M_{23})(M_{13}+M_{23})M_{24}$

5.3.5.30. $(M_{13}-M_{23})(M_{13}+M_{23})\left(M_{14}^2+M_{24}^2-M_{34}^2\right)$

5.3.5.31. $M_{14}M_{23}-M_{13}M_{24}+M_{12}M_{34}$

5.3.5.32. $M_{24}\left(4M_{12}M_{23}^2-2M_{34}M_{23}^2-M_{12}M_{24}^2+M_{13}M_{24}M_{34}\right)$

5.3.5.33. $M_{24}^2(M_{12}M_{14}+2M_{12}M_{23}-M_{23}M_{34})$

5.3.5.34. $M_{24}\left(M_{12}M_{24}^2-M_{13}M_{34}M_{24}+2M_{12}M_{14}M_{23}\right)$

5.3.5.35. $M_{24}\left(M_{12}M_{14}^2-M_{12}M_{24}^2+M_{13}M_{24}M_{34}\right)$

5.3.5.36. $M_{23}\left(2M_{12}M_{14}^2+2M_{23}M_{34}M_{14}-2M_{12}M_{24}^2+M_{24}^2M_{34}\right)$

5.3.5.37. $M_{12}M_{14}^3+M_{23}M_{34}^3+2M_{12}M_{23}M_{24}^2-2M_{23}M_{24}^2M_{34}$

5.3.5.38. $M_{24}(2M_{13}+M_{24})(M_{12}M_{24}-M_{13}M_{34})$

5.3.5.39. $M_{23}M_{24}(2M_{12}M_{13}+M_{12}M_{24}-M_{24}M_{34})$

5.3.5.40. $M_{24}(2M_{12}M_{13}M_{14}-2M_{12}M_{23}M_{24}+M_{23}M_{24}M_{34})$

5.3.5.41. $-M_{12}M_{24}^3+M_{13}M_{34}^3+4M_{12}M_{13}M_{14}M_{23}$

5.3.5.42. $M_{12}M_{24}^3-2M_{13}M_{34}M_{24}^2+M_{13}M_{34}^3+2M_{12}M_{13}M_{14}^2$

5.3.5.43. $2M_{12}M_{14}M_{13}^2+2M_{23}M_{34}M_{13}^2+M_{12}M_{23}M_{24}^2-M_{23}M_{24}^2M_{34}$

5.3.5.44. $M_{24}\left(2M_{14}M_{12}^2+4M_{23}M_{12}^2-M_{23}M_{34}^2\right)$

5.3.5.45. $M_{12}^2M_{14}^2-M_{23}^2M_{14}^2-M_{12}^2M_{24}^2+M_{13}^2M_{24}^2$

5.3.5.46. $2M_{13}M_{14}M_{12}^2-2M_{23}M_{24}M_{12}^2-M_{13}M_{23}M_{34}^2+M_{23}M_{24}M_{34}^2$

5.4 $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$

5.4.1 The Polynomials Defining the Point Scheme of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$

5.4.1.1. $-(q-1)(q+1)\left(q^2+1\right)x_1x_3^2x_4$

5.4.1.2. $\frac{(q-1)(q+1)\left(q^2+1\right)x_2x_3^2x_4}{q^4}$

5.4.1.3. $-\frac{x_3^2\left(x_1x_2q^8+x_3^2q^4-x_4^2q^4-2x_1x_2q^4+x_1x_2\right)}{(q-1)q^2(q+1)(q^2+1)}$

5.4.1.4. $-\frac{(q-1)(q+1)\left(q^2+1\right)x_1x_3x_4^2}{q^2}$

5.4.1.5. $(q-1)(q+1)\left(q^2+1\right)x_1^2x_3x_4$

5.4.1.6. $\frac{x_3(x_3-x_4)x_4(x_3+x_4)}{(q-1)(q+1)(q^2+1)}$

5.4.1.7. $\frac{(q-1)(q+1)\left(q^2+1\right)x_2x_3x_4^2}{q^2}$

5.4.1.8. $-\frac{q^4x_3(x_3-x_4)x_4(x_3+x_4)}{(q-1)(q+1)(q^2+1)}$

5.4.1.9. $-\frac{(q-1)(q+1)\left(q^2+1\right)x_2^2x_3x_4}{q^4}$

5.4.1.10. $\frac{x_4^2\left(x_1x_2q^8-x_3^2q^4+x_4^2q^4-2x_1x_2q^4+x_1x_2\right)}{(q-1)q^2(q+1)(q^2+1)}$

5.4.1.11. $-\frac{(q-1)(q+1)\left(q^2+1\right)\left(q^4+1\right)x_1x_2x_3x_4}{q^4}$

5.4.1.12. $\frac{x_1x_3\left(-x_4^2q^8+x_1x_2q^8+x_3^2q^4-2x_1x_2q^4+x_1x_2\right)}{(q-1)q^2(q+1)(q^2+1)}$

5.4.1.13. $\frac{x_2x_3\left(x_1x_2q^8+x_3^2q^4-2x_1x_2q^4-x_4^2+x_1x_2\right)}{(q-1)q^2(q+1)(q^2+1)}$

5.4.1.14. $-\frac{x_1x_4\left(x_1x_2q^8+x_4^2q^4-2x_1x_2q^4-x_3^2+x_1x_2\right)}{(q-1)(q+1)(q^2+1)}$

5.4.1.15. $-\frac{x_2x_4\left(-x_3^2q^8+x_1x_2q^8+x_4^2q^4-2x_1x_2q^4+x_1x_2\right)}{(q-1)q^4(q+1)(q^2+1)}$

5.4.2 A Gröbner Basis for the Point Scheme of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$

The following polynomials were found by computing a Gröbner basis using Wolfram's Mathematica and the polynomials given in Appendix 5.4.1.

5.4.2.1. $x_3x_4(x_3-x_4)(x_3+x_4)$

5.4.2.2. $x_2x_3x_4^2$

5.4.2.3. $x_2x_3^2x_4$

5.4.2.4. $x_2^2x_3x_4$

5.4.2.5. $x_1x_3x_4^2$

5.4.2.6. $x_1x_3^2x_4$

5.4.2.7. $x_4^2\left(q^8x_1x_2-2q^4x_1x_2-q^4x_3^2+q^4x_4^2+x_1x_2\right)$

5.4.2.8. $\left(q^4+1\right)x_1x_2x_3x_4$

5.4.2.9. $x_3^2\left(q^8x_1x_2-2q^4x_1x_2+q^4x_3^2-q^4x_4^2+x_1x_2\right)$

5.4.2.10. $x_2x_4\left(q^8x_1x_2-2q^4x_1x_2+q^4x_4^2+x_1x_2\right)$

5.4.2.11. $x_2x_3\left(q^8x_1x_2-2q^4x_1x_2+q^4x_3^2+x_1x_2\right)$

5.4.2.12. $x_1^2x_3x_4$

5.4.2.13. $x_1x_4\left(q^8x_1x_2-2q^4x_1x_2+q^4x_4^2+x_1x_2\right)$

5.4.2.14. $x_1x_3\left(q^8x_1x_2-2q^4x_1x_2+q^4x_3^2+x_1x_2\right)$

5.4.3 The Jacobian Matrix of the Point Scheme of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$

The matrix comprised of the following columns, read from left to right, is determined by the polynomials in Appendix 5.4.2.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ x_3x_4^2 \\ x_3^2x_4 \\ (q-1)^2(q+1)^2\left(q^2+1\right)^2x_2x_4^2 \\ \left(q^4+1\right)x_2x_3x_4 \\ (q-1)^2(q+1)^2\left(q^2+1\right)^2x_2x_3^2 \\ (q-1)^2(q+1)^2\left(q^2+1\right)^2x_2^2x_4 \\ (q-1)^2(q+1)^2\left(q^2+1\right)^2x_2^2x_3 \\ 2x_1x_3x_4 \\ x_4\left(2q^8x_1x_2-4q^4x_1x_2+q^4x_4^2+2x_1x_2\right) \\ x_3\left(2q^8x_1x_2-4q^4x_1x_2+q^4x_3^2+2x_1x_2\right) \end{bmatrix}, \begin{bmatrix} 0 \\ x_3x_4^2 \\ x_3^2x_4 \\ 2x_2x_3x_4 \\ 0 \\ 0 \\ (q-1)^2(q+1)^2\left(q^2+1\right)^2x_1x_4^2 \\ \left(q^4+1\right)x_1x_3x_4 \\ (q-1)^2(q+1)^2\left(q^2+1\right)^2x_1x_3^2 \\ x_4\left(2q^8x_1x_2-4q^4x_1x_2+q^4x_4^2+2x_1x_2\right) \\ x_3\left(2q^8x_1x_2-4q^4x_1x_2+q^4x_3^2+2x_1x_2\right) \\ 0 \\ (q-1)^2(q+1)^2\left(q^2+1\right)^2x_1^2x_4 \\ (q-1)^2(q+1)^2\left(q^2+1\right)^2x_1^2x_3 \end{bmatrix},$$

$$\begin{bmatrix} x_4\left(3x_3^2-x_4^2\right) \\ x_2x_4^2 \\ 2x_2x_3x_4 \\ x_2^2x_4 \\ x_1x_4^2 \\ 2x_1x_3x_4 \\ -2q^4x_3x_4^2 \\ \left(q^4+1\right)x_1x_2x_4 \\ 2x_3\left(q^8x_1x_2-2q^4x_1x_2+2q^4x_3^2-q^4x_4^2+x_1x_2\right) \\ 0 \\ x_2\left(q^8x_1x_2-2q^4x_1x_2+3q^4x_3^2+x_1x_2\right) \\ x_1^2x_4 \\ 0 \\ x_1\left(q^8x_1x_2-2q^4x_1x_2+3q^4x_3^2+x_1x_2\right) \end{bmatrix}, \begin{bmatrix} x_3\left(x_3^2-3x_4^2\right) \\ 2x_2x_3x_4 \\ x_2x_3^2 \\ x_2^2x_3 \\ 2x_1x_3x_4 \\ x_1x_3^2 \\ 2x_4\left(q^8x_1x_2-2q^4x_1x_2-q^4x_3^2+2q^4x_4^2+x_1x_2\right) \\ \left(q^4+1\right)x_1x_2x_3 \\ -2q^4x_3^2x_4 \\ x_2\left(q^8x_1x_2-2q^4x_1x_2+3q^4x_4^2+x_1x_2\right) \\ 0 \\ x_1^2x_3 \\ x_1\left(q^8x_1x_2-2q^4x_1x_2+3q^4x_4^2+x_1x_2\right) \\ 0 \end{bmatrix}.$$

5.4.4 The Polynomials Defining the Line Scheme of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$

The following polynomials determine $\mathfrak{L}(q)$. Note that some of these are nonzero scalar multiples of those obtained using the process outlined in [30].

5.4.4.1. $M_{14}M_{23} - M_{13}M_{24} + M_{12}M_{34}$

5.4.4.2. $M_{13}\left(M_{12}M_{13}M_{14}q^8 - 2M_{12}M_{13}M_{14}q^4 + M_{13}^2M_{34}q^4 - M_{14}^2M_{34}q^4 + M_{12}M_{13}M_{14}\right)$

5.4.4.3. $M_{14}\left(M_{12}M_{23}M_{24}q^8 - 2M_{12}M_{23}M_{24}q^4 - M_{23}^2M_{34}q^4 + M_{24}^2M_{34}q^4 + M_{12}M_{23}M_{24}\right)$

5.4.4.4. $M_{24}\left(M_{12}M_{23}M_{24}q^8 - 2M_{12}M_{23}M_{24}q^4 - M_{23}^2M_{34}q^4 + M_{24}^2M_{34}q^4 + M_{12}M_{23}M_{24}\right)$

5.4.4.5. $M_{12}\left(M_{12}M_{23}M_{24}q^8 - 2M_{12}M_{23}M_{24}q^4 - M_{23}^2M_{34}q^4 + M_{24}^2M_{34}q^4 + M_{12}M_{23}M_{24}\right)$

5.4.4.6. $M_{34}\left(M_{12}M_{13}M_{14}q^8 - 2M_{12}M_{13}M_{14}q^4 + M_{13}^2M_{34}q^4 - M_{14}^2M_{34}q^4 + M_{12}M_{13}M_{14}\right)$

5.4.4.7. $(M_{13}M_{23} - M_{14}M_{24})M_{34}^2q^2$

5.4.4.8. $(M_{13}M_{23} - M_{14}M_{24})M_{34}^2q^2$

5.4.4.9. $M_{34}\left(M_{12}M_{23}M_{24}q^8 - 2M_{12}M_{23}M_{24}q^4 - M_{23}^2M_{34}q^4 + M_{24}^2M_{34}q^4 + M_{12}M_{23}M_{24}\right)$

5.4.4.10. $M_{12}M_{34}\left(M_{13}M_{23}q^8 + M_{34}^2q^4 - 2M_{13}M_{23}q^4 + M_{13}M_{23}\right)$

5.4.4.11. $M_{12}M_{34}\left(M_{14}M_{24}q^8 + M_{34}^2q^4 - 2M_{14}M_{24}q^4 + M_{14}M_{24}\right)$

5.4.4.12. $M_{13}M_{34}\left(M_{13}M_{23}q^8 + M_{34}^2q^4 - 2M_{13}M_{23}q^4 + M_{13}M_{23}\right)$

5.4.4.13. $M_{34}\left(-M_{13}M_{14}M_{24}q^8 - M_{13}M_{34}^2q^4 + M_{14}^2M_{23}q^4 + M_{13}M_{14}M_{24}q^4\right.$
$\left.+M_{12}M_{14}M_{34}q^4 - M_{14}^2M_{23} - M_{12}M_{14}M_{34}\right)$

5.4.4.14. $M_{13}M_{34}\left(M_{14}M_{24}q^8 + M_{34}^2q^4 - 2M_{14}M_{24}q^4 + M_{14}M_{24}\right)$

5.4.4.15. $M_{14}M_{34}\left(M_{14}M_{24}q^8 + M_{34}^2q^4 - 2M_{14}M_{24}q^4 + M_{14}M_{24}\right)$

5.4.4.16. $M_{34}\left(M_{13}^2M_{24}q^8 - M_{12}M_{13}M_{34}q^8 + M_{14}M_{34}^2q^4 - M_{13}M_{14}M_{23}q^4 - M_{13}^2M_{24}q^4\right.$
$\left.+M_{12}M_{13}M_{34}q^4 + M_{13}M_{14}M_{23}\right)$

5.4.4.17. $M_{14}M_{34}\left(M_{13}M_{23}q^8 + M_{34}^2q^4 - 2M_{13}M_{23}q^4 + M_{13}M_{23}\right)$

5.4.4.18. $M_{23}M_{34}\left(M_{14}M_{24}q^8 + M_{34}^2q^4 - 2M_{14}M_{24}q^4 + M_{14}M_{24}\right)$

5.4.4.19. $M_{23}M_{34}\left(M_{13}M_{23}q^8 + M_{34}^2q^4 - 2M_{13}M_{23}q^4 + M_{13}M_{23}\right)$

5.4.4.20. $M_{34}\left(-M_{13}M_{24}^2q^8+M_{12}M_{24}M_{34}q^8+M_{13}M_{24}^2q^4-M_{23}M_{34}^2q^4+M_{14}M_{23}M_{24}q^4\right.$
$\left.-M_{12}M_{24}M_{34}q^4-M_{14}M_{23}M_{24}\right)$

5.4.4.21. $M_{24}M_{34}\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.4.22. $M_{34}\left(-M_{13}M_{23}M_{24}q^8+M_{14}M_{23}^2q^4-M_{24}M_{34}^2q^4+M_{13}M_{23}M_{24}q^4+M_{12}M_{23}M_{34}q^4\right.$
$\left.-M_{14}M_{23}^2-M_{12}M_{23}M_{34}\right)$

5.4.4.23. $M_{24}M_{34}\left(M_{14}M_{24}q^8+M_{34}^2q^4-2M_{14}M_{24}q^4+M_{14}M_{24}\right)$

5.4.4.24. $M_{34}^2\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.4.25. $M_{34}^2\left(M_{14}M_{24}q^8+M_{34}^2q^4-2M_{14}M_{24}q^4+M_{14}M_{24}\right)$

5.4.4.26. $M_{14}\left(M_{12}M_{13}M_{14}q^8-2M_{12}M_{13}M_{14}q^4+M_{13}^2M_{34}q^4-M_{14}^2M_{34}q^4+M_{12}M_{13}M_{14}\right)$

5.4.4.27. $M_{12}M_{13}M_{14}M_{23}q^8+M_{13}^2M_{23}M_{34}q^8-M_{13}M_{14}M_{24}M_{34}q^8+M_{12}M_{14}M_{34}^2q^4$
$-2M_{12}M_{13}M_{14}M_{23}q^4+M_{12}M_{13}M_{14}M_{23}$

5.4.4.28. $M_{12}M_{14}^2M_{23}q^8+M_{12}^2M_{14}M_{34}q^8-M_{13}M_{14}M_{23}M_{34}q^8+M_{14}^2M_{24}M_{34}q^8$
$+M_{12}M_{13}M_{34}^2q^4-M_{12}M_{14}^2M_{23}q^4-M_{12}M_{13}M_{14}M_{24}q^4-M_{12}^2M_{14}M_{34}q^4$
$+M_{12}M_{13}M_{14}M_{24}$

5.4.4.29. $M_{12}M_{14}M_{23}^2q^8+M_{13}M_{23}^2M_{34}q^8+M_{12}^2M_{23}M_{34}q^8-M_{14}M_{23}M_{24}M_{34}q^8$
$-M_{12}M_{14}M_{23}^2q^4+M_{12}M_{24}M_{34}^2q^4-M_{12}M_{13}M_{23}M_{24}q^4-M_{12}^2M_{23}M_{34}q^4$
$+M_{12}M_{13}M_{23}M_{24}$

5.4.4.30. $M_{13}\left(M_{12}M_{23}M_{24}q^8-2M_{12}M_{23}M_{24}q^4-M_{23}^2M_{34}q^4+M_{24}^2M_{34}q^4+M_{12}M_{23}M_{24}\right)$

5.4.4.31. $M_{23}\left(M_{12}M_{23}M_{24}q^8-2M_{12}M_{23}M_{24}q^4-M_{23}^2M_{34}q^4+M_{24}^2M_{34}q^4+M_{12}M_{23}M_{24}\right)$

5.4.4.32. $M_{12}M_{13}M_{14}M_{24}q^8+M_{12}M_{13}M_{34}^2q^4-2M_{12}M_{13}M_{14}M_{24}q^4+M_{12}M_{13}M_{14}M_{24}$
$+M_{13}M_{14}M_{23}M_{34}-M_{14}^2M_{24}M_{34}$

5.4.4.33. $M_{12}M_{13}M_{14}M_{23}q^8+M_{12}M_{14}M_{34}^2q^4-M_{12}M_{13}M_{14}M_{23}q^4-M_{12}M_{13}^2M_{24}q^4$
$+M_{12}^2M_{13}M_{34}q^4+M_{12}M_{13}^2M_{24}-M_{12}^2M_{13}M_{34}-M_{13}^2M_{23}M_{34}$
$+M_{13}M_{14}M_{24}M_{34}$

5.4.4.34. $M_{13}M_{14}\left(q^4+1\right)\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.4.35. $M_{13}M_{14}\left(q^4+1\right)\left(M_{14}M_{24}q^8+M_{34}^2q^4-2M_{14}M_{24}q^4+M_{14}M_{24}\right)$

5.4.4.36. $M_{13}M_{14}^2M_{23}q^{12} + M_{14}^2M_{34}^2q^8 - M_{13}M_{14}^2M_{23}q^8 + M_{12}M_{13}M_{14}M_{34}q^8 + M_{13}^2M_{34}^2q^4 - M_{13}^2M_{14}M_{24}q^4 - M_{12}M_{13}M_{14}M_{34}q^4 + M_{13}^2M_{14}M_{24}$

5.4.4.37. $M_{12}M_{14}M_{23}M_{24}q^8 - M_{12}M_{13}M_{24}^2q^4 + M_{12}M_{23}M_{34}^2q^4 - M_{12}M_{14}M_{23}M_{24}q^4 + M_{12}^2M_{24}M_{34}q^4 + M_{12}M_{13}M_{24}^2 - M_{14}M_{24}^2M_{34} - M_{12}^2M_{24}M_{34} + M_{13}M_{23}M_{24}M_{34}$

5.4.4.38. $M_{13}M_{14}M_{23}M_{24}q^{12} + M_{13}M_{23}M_{34}^2q^8 - M_{13}M_{14}M_{23}M_{24}q^8 + M_{14}M_{24}M_{34}^2q^4 - M_{13}M_{14}M_{23}M_{24}q^4 + M_{13}M_{14}M_{23}M_{24}$

5.4.4.39. $M_{14}M_{23}^2M_{24}q^{12} + M_{23}^2M_{34}^2q^8 - M_{14}M_{23}^2M_{24}q^8 + M_{12}M_{23}M_{24}M_{34}q^8 - M_{13}M_{23}M_{24}^2q^4 + M_{24}^2M_{34}^2q^4 - M_{12}M_{23}M_{24}M_{34}q^4 + M_{13}M_{23}M_{24}^2$

5.4.4.40. $-M_{13}^2M_{23}M_{24}q^{12} - M_{13}M_{24}M_{34}^2q^8 + M_{13}^2M_{23}M_{24}q^8 + M_{12}M_{13}M_{23}M_{34}q^8 + M_{13}M_{14}M_{23}^2q^4 - M_{14}M_{23}M_{34}^2q^4 - M_{12}M_{13}M_{23}M_{34}q^4 - M_{13}M_{14}M_{23}^2$

5.4.4.41. $-M_{13}M_{14}M_{24}^2q^{12} + M_{13}M_{14}M_{24}^2q^8 - M_{13}M_{24}M_{34}^2q^8 + M_{12}M_{14}M_{24}M_{34}q^8 - M_{14}M_{23}M_{34}^2q^4 + M_{14}^2M_{23}M_{24}q^4 - M_{12}M_{14}M_{24}M_{34}q^4 - M_{14}^2M_{23}M_{24}$

5.4.4.42. $M_{23}M_{24}\left(q^4+1\right)\left(M_{13}M_{23}q^8 + M_{34}^2q^4 - 2M_{13}M_{23}q^4 + M_{13}M_{23}\right)$

5.4.4.43. $M_{23}M_{24}\left(q^4+1\right)\left(M_{14}M_{24}q^8 + M_{34}^2q^4 - 2M_{14}M_{24}q^4 + M_{14}M_{24}\right)$

5.4.4.44. $M_{12}^2M_{13}M_{14}q^8 - M_{13}^2M_{14}M_{23}q^8 + M_{13}M_{14}^2M_{24}q^8 - 2M_{12}^2M_{13}M_{14}q^4 + M_{12}M_{13}^2M_{34}q^4 - M_{12}M_{14}^2M_{34}q^4 + M_{12}^2M_{13}M_{14} + M_{13}^2M_{14}M_{23} - M_{13}M_{14}^2M_{24}$

5.4.4.45. $\left(M_{13}M_{23} - M_{14}M_{24}\right)\left(-M_{14}M_{23}q^8 + M_{12}M_{34}q^4 + M_{13}M_{24}\right)$

5.4.5 A Gröbner Basis for the Line Scheme of $\mathcal{H}_q(\mathfrak{sl}(2,\Bbbk))$

The following polynomials were found by computing a Gröbner basis using Wolfram's Mathematica and the polynomials given in Appendix 5.4.4.

5.4.5.1. $M_{34}^2\left(M_{14}M_{24}q^8 + M_{34}^2q^4 - 2M_{14}M_{24}q^4 + M_{14}M_{24}\right)$

5.4.5.2. $M_{24}M_{34}\left(M_{14}M_{24}q^8 + M_{34}^2q^4 - 2M_{14}M_{24}q^4 + M_{14}M_{24}\right)$

5.4.5.3. $M_{23}M_{34}\left(M_{14}M_{24}q^8 + M_{34}^2q^4 - 2M_{14}M_{24}q^4 + M_{14}M_{24}\right)$

5.4.5.4. $M_{23}M_{24}\left(q^4+1\right)\left(M_{14}M_{24}q^8 + M_{34}^2q^4 - 2M_{14}M_{24}q^4 + M_{14}M_{24}\right)$

5.4.5.5. $M_{23}M_{24}\left(q^4+1\right)\left(M_{14}M_{24}q^8 + M_{34}^2q^4 - 2M_{14}M_{24}q^4 + M_{14}M_{24}\right)$

5.4.5.6. $M_{23}M_{24}\left(q^4+1\right)\left(M_{14}M_{24}q^8+M_{34}^2q^4-2M_{14}M_{24}q^4+M_{14}M_{24}\right)$

5.4.5.7. $M_{23}^2\left(q^4+1\right)\left(M_{14}M_{24}q^8+M_{34}^2q^4-2M_{14}M_{24}q^4+M_{14}M_{24}\right)$

5.4.5.8. $M_{23}^2\left(q^4+1\right)\left(M_{14}M_{24}q^8+M_{34}^2q^4-2M_{14}M_{24}q^4+M_{14}M_{24}\right)$

5.4.5.9. $M_{23}^2\left(q^4+1\right)\left(M_{14}M_{24}q^8+M_{34}^2q^4-2M_{14}M_{24}q^4+M_{14}M_{24}\right)$

5.4.5.10. $M_{14}M_{34}\left(M_{14}M_{24}q^8+M_{34}^2q^4-2M_{14}M_{24}q^4+M_{14}M_{24}\right)$

5.4.5.11. $M_{14}M_{23}\left(q^4+1\right)\left(M_{14}M_{24}q^8+M_{34}^2q^4-2M_{14}M_{24}q^4+M_{14}M_{24}\right)$

5.4.5.12. $M_{14}M_{23}\left(q^4+1\right)\left(M_{14}M_{24}q^8+M_{34}^2q^4-2M_{14}M_{24}q^4+M_{14}M_{24}\right)$

5.4.5.13. $M_{14}M_{23}\left(q^4+1\right)\left(M_{14}M_{24}q^8+M_{34}^2q^4-2M_{14}M_{24}q^4+M_{14}M_{24}\right)$

5.4.5.14. $M_{34}^2\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.5.15. $M_{24}M_{34}\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.5.16. $-M_{13}M_{23}M_{24}^2q^8+M_{14}M_{23}^2M_{24}q^8+2M_{13}M_{23}M_{24}^2q^4+M_{23}^2M_{34}^2q^4-M_{24}^2M_{34}^2q^4-2M_{14}M_{23}^2M_{24}q^4-M_{13}M_{23}M_{24}^2+M_{14}M_{23}^2M_{24}$

5.4.5.17. $M_{23}M_{34}\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.5.18. $M_{23}M_{24}\left(q^4+1\right)\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.5.19. $M_{23}M_{24}\left(q^4+1\right)\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.5.20. $M_{23}M_{24}\left(q^4+1\right)\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.5.21. $M_{13}M_{34}\left(M_{14}M_{24}q^8+M_{34}^2q^4-2M_{14}M_{24}q^4+M_{14}M_{24}\right)$

5.4.5.22. $\left(M_{14}M_{23}-M_{13}M_{24}\right)\left(M_{14}M_{24}q^8+M_{34}^2q^4-2M_{14}M_{24}q^4+M_{14}M_{24}\right)$

5.4.5.23. $M_{14}M_{34}\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.5.24. $\left(q^4+1\right)\left(M_{13}M_{14}M_{23}M_{24}q^{16}-4M_{13}M_{14}M_{23}M_{24}q^{12}-M_{34}^4q^8\right.$
$\left.+6M_{13}M_{14}M_{23}M_{24}q^8-4M_{13}M_{14}M_{23}M_{24}q^4+M_{13}M_{14}M_{23}M_{24}\right)$

5.4.5.25. $\left(q^4+1\right)\left(M_{13}M_{14}M_{23}M_{24}q^{16}-4M_{13}M_{14}M_{23}M_{24}q^{12}-M_{34}^4q^8\right.$
$\left.+6M_{13}M_{14}M_{23}M_{24}q^8-4M_{13}M_{14}M_{23}M_{24}q^4+M_{13}M_{14}M_{23}M_{24}\right)$

5.4.5.26. $\left(q^4+1\right)\left(M_{13}M_{14}M_{23}M_{24}q^{16}-4M_{13}M_{14}M_{23}M_{24}q^{12}-M_{34}^4q^8\right.$
$\left.+6M_{13}M_{14}M_{23}M_{24}q^8-4M_{13}M_{14}M_{23}M_{24}q^4+M_{13}M_{14}M_{23}M_{24}\right)$

5.4.5.27. $M_{14}M_{23}\left(q^4+1\right)\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.5.28. $M_{14}M_{23}\left(q^4+1\right)\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.5.29. $M_{14}M_{23}\left(q^4+1\right)\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.5.30. $M_{13}M_{14}\left(q^4+1\right)\left(M_{14}M_{24}q^8+M_{34}^2q^4-2M_{14}M_{24}q^4+M_{14}M_{24}\right)$

5.4.5.31. $M_{13}M_{14}\left(q^4+1\right)\left(M_{14}M_{24}q^8+M_{34}^2q^4-2M_{14}M_{24}q^4+M_{14}M_{24}\right)$

5.4.5.32. $M_{13}M_{14}\left(q^4+1\right)\left(M_{14}M_{24}q^8+M_{34}^2q^4-2M_{14}M_{24}q^4+M_{14}M_{24}\right)$

5.4.5.33. $M_{14}^2\left(q^4+1\right)\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.5.34. $M_{14}^2\left(q^4+1\right)\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.5.35. $M_{14}^2\left(q^4+1\right)\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.5.36. $M_{13}M_{34}\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.5.37. $(M_{13}M_{24}-M_{14}M_{23})\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.5.38. $-M_{13}M_{14}^2M_{23}q^8+M_{13}^2M_{14}M_{24}q^8+M_{13}^2M_{34}^2q^4-M_{14}^2M_{34}^2q^4+2M_{13}M_{14}^2M_{23}q^4-2M_{13}^2M_{14}M_{24}q^4-M_{13}M_{14}^2M_{23}+M_{13}^2M_{14}M_{24}$

5.4.5.39. $M_{13}M_{14}\left(q^4+1\right)\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.5.40. $M_{13}M_{14}\left(q^4+1\right)\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.5.41. $M_{13}M_{14}\left(q^4+1\right)\left(M_{13}M_{23}q^8+M_{34}^2q^4-2M_{13}M_{23}q^4+M_{13}M_{23}\right)$

5.4.5.42. $M_{14}M_{23}-M_{13}M_{24}+M_{12}M_{34}$

5.4.5.43. $M_{24}\left(M_{12}M_{23}M_{24}q^8-2M_{12}M_{23}M_{24}q^4-M_{23}^2M_{34}q^4+M_{24}^2M_{34}q^4+M_{12}M_{23}M_{24}\right)$

5.4.5.44. $M_{23}\left(M_{12}M_{23}M_{24}q^8-2M_{12}M_{23}M_{24}q^4-M_{23}^2M_{34}q^4+M_{24}^2M_{34}q^4+M_{12}M_{23}M_{24}\right)$

5.4.5.45. $M_{12}M_{14}M_{23}M_{24}q^{16}-4M_{12}M_{14}M_{23}M_{24}q^{12}-M_{14}M_{23}^2M_{34}q^{12}-M_{24}M_{34}^3q^8$
$+6M_{12}M_{14}M_{23}M_{24}q^8+2M_{14}M_{23}^2M_{34}q^8-4M_{12}M_{14}M_{23}M_{24}q^4-M_{14}M_{23}^2M_{34}q^4$
$+M_{12}M_{14}M_{23}M_{24}$

5.4.5.46. $M_{12}M_{13}M_{23}M_{24}q^{16}-4M_{12}M_{13}M_{23}M_{24}q^{12}+M_{13}M_{24}^2M_{34}q^{12}+M_{23}M_{34}^3q^8$
$+6M_{12}M_{13}M_{23}M_{24}q^8-2M_{13}M_{24}^2M_{34}q^8-4M_{12}M_{13}M_{23}M_{24}q^4$
$+M_{13}M_{24}^2M_{34}q^4+M_{12}M_{13}M_{23}M_{24}$

5.4.5.47. $M_{12}M_{13}M_{14}M_{24}q^{16} - 4M_{12}M_{13}M_{14}M_{24}q^{12} + M_{13}^2M_{24}M_{34}q^{12} + M_{14}M_{34}^3q^8$
$+ 6M_{12}M_{13}M_{14}M_{24}q^8 - 2M_{13}^2M_{24}M_{34}q^8 - 4M_{12}M_{13}M_{14}M_{24}q^4$
$+ M_{13}^2M_{24}M_{34}q^4 + M_{12}M_{13}M_{14}M_{24}$

5.4.5.48. $M_{12}M_{13}M_{14}M_{23}q^{16} - 4M_{12}M_{13}M_{14}M_{23}q^{12} - M_{14}^2M_{23}M_{34}q^{12} - M_{13}M_{34}^3q^8$
$+ 6M_{12}M_{13}M_{14}M_{23}q^8 + 2M_{14}^2M_{23}M_{34}q^8 - 4M_{12}M_{13}M_{14}M_{23}q^4 - M_{14}^2M_{23}M_{34}q^4$
$+ M_{12}M_{13}M_{14}M_{23}$

5.4.5.49. $M_{14}\left(M_{12}M_{13}M_{14}q^8 - 2M_{12}M_{13}M_{14}q^4 + M_{13}^2M_{34}q^4 - M_{14}^2M_{34}q^4 + M_{12}M_{13}M_{14}\right)$

5.4.5.50. $M_{13}\left(M_{12}M_{13}M_{14}q^8 - 2M_{12}M_{13}M_{14}q^4 + M_{13}^2M_{34}q^4 - M_{14}^2M_{34}q^4 + M_{12}M_{13}M_{14}\right)$

5.4.5.51. $M_{14}M_{23}M_{24}^2q^{16} + M_{13}M_{23}^2M_{24}q^{16} + 4M_{12}^2M_{23}M_{24}q^{16} + 4M_{14}M_{23}^3q^{12}$
$+ 4M_{13}M_{24}^3q^{12} - 4M_{14}M_{23}M_{24}^2q^{12} + 2M_{23}M_{24}M_{34}^2q^{12} - 4M_{13}M_{23}^2M_{24}q^{12}$
$- 16M_{12}^2M_{23}M_{24}q^{12} - 8M_{14}M_{23}^3q^8 - 8M_{13}M_{24}^3q^8 + 6M_{14}M_{23}M_{24}^2q^8$
$+ 4M_{23}M_{24}M_{34}^2q^8 + 6M_{13}M_{23}^2M_{24}q^8 + 24M_{12}^2M_{23}M_{24}q^8 + 4M_{14}M_{23}^3q^4$
$+ 4M_{13}M_{24}^3q^4 - 4M_{14}M_{23}M_{24}^2q^4 + 2M_{23}M_{24}M_{34}^2q^4 - 4M_{13}M_{23}^2M_{24}q^4$
$- 16M_{12}^2M_{23}M_{24}q^4 + M_{14}M_{23}M_{24}^2 + M_{13}M_{23}^2M_{24} + 4M_{12}^2M_{23}M_{24}$

5.4.5.52. $4M_{12}^2M_{13}M_{14}q^{16} + M_{13}^2M_{14}M_{23}q^{16} + M_{13}M_{14}^2M_{24}q^{16} + 2M_{13}M_{14}M_{34}^2q^{12}$
$- 16M_{12}^2M_{13}M_{14}q^{12} + 4M_{14}^3M_{23}q^{12} - 4M_{13}^2M_{14}M_{23}q^{12} + 4M_{13}^3M_{24}q^{12}$
$- 4M_{13}M_{14}^2M_{24}q^{12} + 4M_{13}M_{14}M_{34}^2q^8 + 24M_{12}^2M_{13}M_{14}q^8 - 8M_{14}^3M_{23}q^8$
$+ 6M_{13}^2M_{14}M_{23}q^8 - 8M_{13}^3M_{24}q^8 + 6M_{13}M_{14}^2M_{24}q^8 + 2M_{13}M_{14}M_{34}^2q^4$
$- 16M_{12}^2M_{13}M_{14}q^4 + 4M_{14}^3M_{23}q^4 - 4M_{13}^2M_{14}M_{23}q^4 + 4M_{13}^3M_{24}q^4$
$- 4M_{13}M_{14}^2M_{24}q^4 + 4M_{12}^2M_{13}M_{14} + M_{13}^2M_{14}M_{23} + M_{13}M_{14}^2M_{24}$

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Biographical Statement

Richard Gene Chandler Jr is the only child of Richard and Cynthia Chandler. He has lived his life, to date, in the Dallas-Fort Worth metroplex and received his high-school diploma from Haltom High School in 2006; he graduated fifteenth in a class of five-hundred.

Richard then attended Tarrant County Community College where he received his Associate in Arts in 2008. He then attended the University of Texas at Arlington and received his Bachelors of Arts in Mathematics and Secondary Teaching Certification in 2011.

After graduation, he returned to the University of Texas at Arlington to begin his doctoral studies. During this time, he researched under the supervision of Dr. Michaela Vancliff and became the coordinator for the Emerging Scholars Program under Dr. James Epperson. He earned his Ph.D. in Mathematics in May 2016.

Richard's research interests lie in noncommuative algebra and algebraic geometry, with particular interest in generic quadratic quantum $\mathbb{P}^3$s and algebras arising in quantum mechanics from Lie-type algebras.