A STUDY ON TRAVELING WAVE SOLUTIONS IN THE SHALLOW-WATER-TYPE SYSTEMS

by

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ABSTRACT

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The study of water waves reveals the physical principles of many phenomena of scientific and engineering interest. In this dissertation I consider three models: two-component Camassa-Holm system(2CH), generalized two-component Camassa-Holm equation(g2CH) and rotation-Camassa-Holm equation(R-CH). In the first part, we consider the stability of the Camassa-Holm peakons and antipeakons in the dynamics of the two-component Camassa-Holm system. The second part shows that the train of N-smooth traveling waves of this system is dynamically stable to perturbations in energy space with a range of parameters. In the third part, we formally derive the simplified phenomenological models with the Coriolis effect due to the Earth's rotation and justify rigorously that the solutions of these models are well approximated by the solutions of the rotation-Camassa-Holm equation. Furthermore, we demonstrate nonexistence of the Camassa-Holm-type peaked solution and classify various localized traveling-wave solutions to the rotation-Camassa-Holm equation.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

This dissertation is composed of some results on traveling-wave solution of three different models: two-component Camassa-Holm equation, generalized two-component Camassa-Holm equation, rotation-Camassa-Holm equation. I will introduce these models and their problem setting in this chapter.

1.1 Two-component Camassa-Holm system.

The two-component Camassa-Holm (2CH) system arising from the shallow-water waves with a background of constant vorticity [20, 54, 57] has the following form

$$\begin{cases} u_t - u_{xxt} + \kappa u_x + 3uu_x - (2u_x u_{xx} + uu_{xxx}) + \rho\rho_x = 0, \quad t > 0, x \in \mathbb{R}, \kappa \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, \quad t > 0, x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $u(t,x)$ represents the horizontal velocity component and $\rho(t,x)$ is related to the free surface elevation. The interaction between the free surface and the horizontal velocity component causes the wave-breaking phenomena, see for example [12, 30, 34, 35, 36, 37, 38, 59]. It was also shown in [53, 59] that the system captures smooth traveling wave with a single crest profile and presenting exponential decay at spatial infinity.

It is known that the 2CH system is formally integrable [42, 57]. Through the integrability, system (1.1) can be written as a compatibility condition of two linear systems (Lax pair) with a spectral parameter ζ, that is

$$\Psi_{xx} = \left[-\zeta^2\rho^2 + \zeta\left(u - u_{xx} + \frac{\kappa}{2}\right) + \frac{1}{4}\right]\Psi, \qquad \Psi_t = \left(\frac{1}{2\zeta} - u\right)\Psi_x + \frac{1}{2}u_x\Psi,$$

and has a bi-Hamiltonian structure corresponding to the following Hamiltonian functionals

$$H_1(u,\rho) = \int \left(u^2 + u_x^2 + \rho^2\right)\, dx, \qquad H_2(u,\rho) = \int \left(u^3 + uu_x^2 + u\rho^2 + \kappa u^2\right)\, dx.$$

By using the functional H_2, system (1.1) has the following abstract Hamiltonian form:

$$\partial_t \begin{pmatrix} u \\ \rho \end{pmatrix} = JH_2'(u,\rho), \tag{1.2}$$

where $H_2'(u,\rho) = (\delta H_2/\delta u,\, \delta H_2/\delta\rho)^T$ represents the variational derivative of the functional H_2 and J is a closed skew symmetric operator performing as

$$J = \frac{1}{2}\begin{pmatrix} -\partial_x(1-\partial_x^2)^{-1} & 0 \\ 0 & -\partial_x \end{pmatrix}.$$

Also system (1.1) admits two Casimirs: $\int \rho\, dx$ and $\int (u - u_{xx})\, dx$.

While letting $\rho = 0$, the 2CH system (1.1) reduces to the classical Camassa-Holm (CH) equation [8, 31]

$$u_t - u_{xxt} + \kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1.3}$$

which is a model proposed for the unidirectional propagation of long water waves in the shallow-water approximation to the Euler equations of inviscid incompressible fluid flow [8, 21, 44]. Equation (1.3) is completely integrable using the inverse scattering transformation [15, 19, 22] and it has bi-Hamiltonian structure [8, 54]. This model has attracted so much attention in more than two decades since it employs two remarkable features. One feature is that the CH equation (1.3) has wave breaking phenomena [8, 14, 17, 18, 49], i.e. the solution remains bounded while its slope becomes unbounded in finite time [58]. In [5] and [6], the authors showed that the solutions can be uniquely continued after breaking as either global conservative or global dissipative weak solution. Another important feature is that it admits peaked traveling waves [8, 46, 47], when $\kappa = 0$. The wave profile of so-called “peakon” is shaped like $\varphi_c(t,x) = c\, e^{-|x-ct|}$, $c > 0$. In particular, when $c < 0$, it is recognized as “antipeakon”. The first derivative of such peaked wave is smooth except at the peak, where it has a jump discontinuity.

In view of the CH equation (1.3), when $\kappa = 0$ and $p_1^0, \cdots, p_N^0 > 0$ and $q_1^0 < \cdots < q_N^0$, the 2CH system (1.1) possesses the multi-peakons profile $(\psi(t,x), 0)$ on $\mathbb{R}$ with

$$\psi(t,x) = \sum_{i=1}^{N} p_i(t) e^{-|x-q_i(t)|},$$

where $p_i(t)$ and $q_i(t)$ satisfy the Hamiltonian system

$$\begin{cases} \dot{p}_i = \sum\limits_{j=1}^{N} p_i p_j \mathrm{sign}(q_i - q_j) e^{-|q_i-q_j|}, \\ \dot{q}_i = \sum\limits_{j=1}^{N} p_j e^{-|q_i-q_j|}, \end{cases} \tag{1.4}$$

with the corresponding initial data $p_i(0) = p_i^0$ and $q_i(0) = q_i^0$, $i = 1, \cdots, N$. In [3] (see also [4, 41]), the asymptotic behavior of the multipeakons is investigated, including the limits as t tends to $+\infty$ and $-\infty$ of $p_i(t)$ and $\dot{q}_i(t)$ as well as the qualitative properties. If $p_1^0, \cdots, p_k^0$ are negative real numbers and $p_{k+1}^0, \cdots, p_N^0$ are positive real numbers, it admits the multi-antipeakon-peakons profile $(\vartheta(t,x), 0)$, where $\vartheta(t,x) = \sum_{i=1}^{N} p_i(t) e^{-|x-q_i(t)|}$, $p_i(t)$ and $q_i(t)$ satisfy the Hamiltonian system (1.4), $p_1(t), \cdots,\ p_k(t) < 0$, $p_{k+1}(t), \cdots, p_N(t) > 0$ and $q_1(t) < \cdots < q_N(t)$. It is worthwhile to mention that $(\psi(t,x), 0)$ and $(\vartheta(t,x), 0)$ are not classical solutions of the 2CH system (1.1) due to non-smoothness. They should be regarded as weak solutions, since system (1.1) with $\kappa = 0$ can be written in the following form

$$
\begin{cases}
u_t + uu_x + \partial_x p * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right) = 0, \\
\\
\rho_t + (\rho u)_x = 0,
\end{cases}
\tag{1.5}
$$

where $p(x) = e^{-|x|}/2$ is the corresponding kernel of the convolution operator $(1 - \partial_x^2)^{-1}$. The associated Hamiltonian functionals for (1.5) are

$$
E(u, \rho) = \int \left(u^2 + u_x^2 + \rho^2\right) \, dx \quad \text{and} \quad F(u, \rho) = \int \left(u^3 + uu_x^2 + u\rho^2\right) \, dx. \tag{1.6}
$$

Inspired by the similarity between the 2CH system (1.5) and the CH equation (1.3) with $\kappa = 0$, we are wondering how much the dynamical properties of a variety of the CH traveling waves under the 2CH evolution processing has in common with those in the scalar equation, especially on the stability issue. The orbital stability of the CH peakons $\varphi_c(t, x)$ [23, 24, 25], multi-peakons $\psi(t, x)$ [27] and multi-antipeakon-peakons $\vartheta(t, x)$ [28] seems to suggest the analogous result to the wave profiles $(\varphi_c, 0)$, $(\psi, 0)$ and $(\vartheta, 0)$ for the 2CH system (1.5). However, the interaction between two components u and ρ in (1.5) and (1.6) makes it non-trivial to verify. To extend the theory from a scalar equation to a system we have the following difficulties. Firstly, the parameter ρ may affect the translation of the N-peakons while using the modulation argument. Secondly, with two components, the energy functional has different formula, which provides the challenge to show the almost monotonicity. Thirdly, it is complicate to extend local and global estimates from a scalar to a vector. By using the two conservation laws and a fine analysis carefully, we overcome these difficulties with suitable initial condition. It is worth noticing that in [12], the variational characterization and the orbital stability of the wave patterns $(\varphi_c, 0)$ in the dynamics of the 2CH system are established. In this article, we are concerned with the orbital stability of multi-wave patterns $(\psi, 0)$ and $(\vartheta, 0)$ under the 2CH dynamics.

According to the collision theory between two peakons [4, 8, 9, 41], it follows that the multipeakons finally will be well ordered. Therefore, with well-posedness results, we focus on the stability of ordered trains of the N-CH-peakons $(\sum_{i=1}^{N}\varphi_{c_i}, 0)$ first. The case for $(\vartheta, 0)$ can be treated similarly. It is worth recalling that the general framework for the proof of the stability of the N-smooth traveling waves has two principal ingredients [26, 29, 50, 51]: one is the almost monotonicity of the functionals which describe the energy at the right of ith bump, for $i = 1, 2, \cdots, N$, the other one is the local coercivity of the Hessian operator of $c_iE_i - F_i$ around $(\varphi_{c_i}, 0)$, where E_i and F_i are localized conservation laws (defined in (2.31)). In [50], the orbital stability of the trains of N smooth traveling waves of the 2CH system was proved. However, the proof of the stability of the trains of the N-CH-peakons does not fit into the general framework, due to the non-differentiability of $(\varphi_{c_i}, 0)$, which fails the spectral analysis.

The method that we adopt here to prove the orbital stability of the trains of ordered N-CH-peakons has the flavor in the Lyapunov sense. This direct approach is initially introduced in [24] for the stability of single peakon solution to CH equation (1.3). In view of the conservation law E in (1.6), we expect the orbital stability of these kind wave pattern in the energy space $X = H^1(\mathbb{R}) \times L^2(\mathbb{R})$ with small perturbation. To fill the gap mentioned above, we construct a so-called localized Lyapunov function $P(M_i; u, \rho)$, where $M_i = \max_x u$ is the local maximum near i-th bump. Furthermore, $P(M_i; u, \rho)$ gives the following estimate

$$(M_i + 2c_i)(M_i - c_i)^2 \lesssim \frac{3}{2}|\left(E_i(u,\rho) - E_i(\varphi_{c_i}, 0)\right) M_i| + \frac{3}{2}|F_i(u,\rho) - F_i(\varphi_{c_i}, 0)|.$$

After applying the almost monotonicity of the localized energy functions and the conservation laws E and F, we achieve that $\sum_{i=1}^{N} c_i|M_i - c_i|$ is small enough, which implies the stability result. As to the CH-multi-antipeakon-peakons, by taking advantage of the invariance of the system with respect to the change of $u(x,t) \mapsto$

$-u(-x,t), \rho(x,t) \mapsto -\rho(-x,t)$, we apply the same approach to the N-CH-peakons and N-CH-antipeakons separately, then combine them together to prove the orbital stability of the ordered trains of the N-CH-antipeakon-peakons.

1.2 Generalized two-component Camassa-Holm system.

The generalized two-component Camassa-Holm (g2CH) system arising from the shallow-water theory with nonzero constant vorticity [11, 42] can be written in the form

$$\begin{cases} u_t - u_{xxt} - Au_x + 3uu_x - \sigma\left(2u_x u_{xx} + uu_{xxx}\right) + \rho\rho_x = 0, \quad t > 0, x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, \quad t > 0, x \in \mathbb{R}, \end{cases} \tag{1.7}$$

where $u(x,t)$ is connected with the horizontal velocity and $\rho(x,t)$ is related to the free surface elevation from equilibrium (or scalar density) with the boundary conditions $u \to 0$ and $\rho \to 1$ as $|x| \to \infty$. The scalar constant $A > 0$ features a linear underlying shear flow, which implies the system (1.7) describes the interaction between surface gravity wave and a mean flow. The parameter σ, a real dimensionless constant, provides the competition in the fluid convection between nonlinear steepening and the amplification due to stretching.

Our motivation is to investigate the stability issue for multi-solitary waves to system (1.7) with an effect of the parameter σ. It was shown in [13] that there exist smooth traveling waves $\boldsymbol{\varphi}_c$ (defined in Definition 3.1.1, Chapter 3) of system (1.7) if the parameter $\sigma \leq 1$, $c > \frac{-A+\sqrt{A^2+4}}{2}$ and these single traveling waves are orbitally stable in the energy space by spectral analysis. Inspired by an approach in [29] to deal with stability of N-smooth traveling waves of the CH equation, our goal in this chapter is to study the orbital stability of a train of N-smooth traveling waves in the dynamics of the g2CH system.

For integrability, we will adopt another form of the g2CH system (1.7), where $\eta \overset{\text{def}}{=} \rho - 1$, that is,

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x - \sigma\left(2u_x u_{xx} + uu_{xxx}\right) + \left(1+\eta\right)\eta_x = 0, \\ \eta_t + \left((1+\eta)u\right)_x = 0. \end{cases} \tag{1.8}$$

In view of the conservation laws for system (1.8),

$$E(u,\eta) = \frac{1}{2}\int_{\mathbb{R}} \left(u^2 + u_x^2 + \eta^2\right)\, dx, \tag{1.9}$$

$$F(u,\eta) = \frac{1}{2}\int_{\mathbb{R}} \left(u^3 + \sigma u u_x^2 + 2u\eta + u\eta^2 - Au^2\right)\, dx, \tag{1.10}$$

we expect the orbital stability of N-smooth traveling waves of the g2CH system in the sense of the energy space $X = H^1(\mathbb{R}) \times L^2(\mathbb{R})$. Note that existence for the smooth traveling waves and their stability naturally rely on the particular σ and A. By choosing $\sigma = 1$, the orbital stability of N-smooth traveling waves for system (1.1) is included while discussing the same topic on the g2CH system.

To establish the stability result for N-smooth traveling waves of the g2CH system, we develop the approaches employed in [29] for the CH equation and combine with the general strategy initiated in [51] for generalized KdV case. It is worth recalling that the general framework requires three principal ingredients: modulation argument, almost monotonicity and local coercivity. Note that the stability issue of the multi-solitons of system (1.8) is more subtle than the CH case, because of the interaction between the two components u and η of solution $\boldsymbol{u}$ in system (1.8). To deal with these difficulties, one needs to provide a new and effective approach to control the superposition of the error terms with respect to the solution $\boldsymbol{u}$ and each smooth traveling wave $\boldsymbol{\varphi}_{c_i}$ in the local energy space. Intrigued by the classical method, we expect to find the orbit of N-smooth traveling waves by the modulation argument, i.e. for each instant t there exist a series of translation $\{\tilde{x}_i\}_{i=1}^N$ such that $\sum_{i=1}^N \boldsymbol{\varphi}_{c_i}(x - \tilde{x}_i)$

lies close to $u(x)$ in X-norm under certain assumptions. Fortunately, the implicit function theorem, which is crucial to the modulation argument, works regardless of the dimension of the space and it also provides a new orthogonal condition connected with both elements in the remaining term of the solution. For the property of almost monotonicity, a more delicate work is needed to compare with the CH equation, not only because the energy function is in terms of both u and η, but also because it depends on the two parameters σ and $A > 0$. As to the local coercivity, we focus on the second differential operator of $cE - F$ around a traveling wave $\boldsymbol{\varphi}_c$ which is a 2×2 matrix. In order to apply the spectral analysis to this operator, restriction on σ is required. Hence, as long as one can extend the proof of these ingredients from a scalar to a vector, and choose the parameters properly, it is possible to establish Theorem 3.1.6.

1.3 Rotation-Camassa-Holm equation.

It is observed that in certain ranges of scales in the geophysical water waves fluid dynamics is primarily influenced by the interaction of gravity and the Earth's rotation. Consider now that water flows are incompressible and inviscid with a constant density ρ and no surface tension, and the interface between the air and the water is a free surface. Then such a motion of water flow occupying a domain Ω_t in $\mathbb{R}^3$ under the influence of the gravity g and the Coriolis force due to the Earth's rotation can be described by the Euler's equations [32], namely,

$$\begin{cases} \vec{u}_t + (\vec{u} \cdot \nabla)\,\vec{u} + 2\vec{\Omega} \times \vec{u} = -\frac{1}{\rho}\nabla P + \vec{g}, \quad x \in \Omega_t, \\ \nabla \cdot \vec{u} = 0, \quad x \in \Omega_t, \\ \vec{u}|_{t=0} = \vec{u_0}, \quad x \in \Omega_0, \end{cases} \tag{1.11}$$

where $\vec{u} = (u, v, w)^T$ is the fluid velocity, $P(t, x, y, z)$ is the pressure in the fluid, $\vec{g} = (0, 0, -g)^T$ with $g \approx 9.8m/s^2$ the constant gravitational acceleration at the Earth's surface, and $\vec{\Omega} = (0,\ \Omega_0 \cos\phi, \Omega_0 \sin\phi)^T$, with the rotational frequency $\Omega_0 \approx 73 \cdot 10^{-6}$rad/s and the local latitude ϕ, is the angular velocity vector which is directed along the axis of rotation of the rotating reference frame. We adopt a rotating framework with the origin located at a point on the Earth's surface, with the x-axis chosen horizontally due east, the y-axis horizontally due north and the z-axis upward. We now focus on two-dimensional flows, moving in the zonal direction along the equator independent of the y-coordinate, in other words, $v \equiv 0$ throughout the flow. In 2D case, consider here waves at the surface of water with a flat bed, and assume that $\Omega_t = \{(x, z) : 0 < z < h_0 + \eta(t, x)\}$, where h_0 is the typical depth of the water and $\eta(t, x)$ measures the deviation from the average level. Under the f-plane approximation ($\sin\phi \approx 0,\ \phi \ll 1$), the motion of inviscid irrotational fluid near the Equator in the region $0 < z < h_0 + \eta(t, x)$ with a constant density ρ is described by the Euler's equations in two dimensions [16, 32],

$$\begin{cases} u_t + uu_x + wu_z + 2\Omega_0 w = -\frac{1}{\rho} P_x, \\ w_t + uw_x + ww_z - 2\Omega_0 u = -\frac{1}{\rho} P_z - g, \end{cases} \tag{1.12}$$

the incompressibility of the fluid,

$$u_x + w_z = 0, \tag{1.13}$$

and the irrotational condition,

$$u_z - w_x = 0. \tag{1.14}$$

The pressure is written as

$$P(t, x, z) = P_a + \rho g(h_0 - z) + p(t, x, z),$$

where P_a is the constant atmosphere pressure, and p is a pressure variable measuring the hydrostatic pressure distribution.

The dynamic condition posed on the surface $z = h_0 + \eta$ yields $P = P_a$. Then there appears that

$$p = \rho g \eta. \tag{1.15}$$

Meanwhile, the kinematic condition on the surface is given by

$$w = \eta_t + u\eta_x, \quad \text{when} \quad z = h_0 + \eta(t, x). \tag{1.16}$$

Finally, we pose "no-flow" condition at the flat bottom $z = 0$, that is,

$$w|_{z=0} = 0. \tag{1.17}$$

There are many shallow water models as appropriate approximations to the full Euler dynamics when the water depth is small compared to the horizontal wavelength scale [2, 21]. We denote the amplitude parameter ε and the shallowness parameter μ respectively by

$$\varepsilon = a/h_0, \qquad \mu = h_0^2/\lambda^2, \tag{1.18}$$

where a is the typical amplitude of the wave and λ is the typical wavelength. It is known that the KdV and BBM models provide good asymptotic approximations of unidirectional solutions of the irrotational two-dimensional water waves problem in (1.11) without the Coriolis effect on the Boussinesq regime $\mu \ll 1$, $\varepsilon = O(\mu)$ [1]. To describe more accurately the motion of these unidirectional waves, it is shown in [21] that the Camassa-Holm (CH) equation in the Camassa-Holm scaling, $\mu \ll 1$, $\varepsilon = O(\sqrt{\mu})$, could be valid higher order approximations to the governing equation for full water waves in the long time scaling $O(\frac{1}{\varepsilon})$. The CH equation [8, 31] (see also [21, 43]) has brought up much attention recently, since it is completely integrable with infinity conservation laws [8, 31] and can present the phenomenon of wave breaking

[14, 18] (i.e. the solution remains bounded, but its slope becomes unbounded in finite time).

Analogous to the CH equation, there is a model equation with the Coriolis effect called the rotation-Camassa-Holm (R-CH) equation which is recently derived from the irrotational two-dimensional shallow water [40].

According to the magnitude of the physical quantities, we introduce dimensionless quantities as follows

$$x \to \lambda x, \quad z \to h_0 z, \quad \eta \to a\eta, \quad t \to \frac{\lambda}{\sqrt{gh_0}} t,$$

and

$$u \to \sqrt{gh_0} u, \quad w \to \sqrt{\mu g h_0} w, \quad p \to \rho g h_0 p.$$

And under the influence of the Earth rotation, we introduce

$$\Omega = \sqrt{\frac{h_0}{g}} \Omega_0. \tag{1.19}$$

Furthermore, considering whenever $\varepsilon \to 0$,

$$u \to 0, \quad w \to 0, \quad p \to 0,$$

that is, u, w and p are proportional to the wave amplitude. In this case, we choose a scaling

$$u \to \varepsilon u, \quad w \to \varepsilon w, \quad p \to \varepsilon p. \tag{1.20}$$

Therefore the governing equations become

$$u_t + \varepsilon(u u_x + w u_z) + 2\Omega w = -p_x \qquad \text{in} \quad 0 < z < 1 + \varepsilon\eta(t, x), \tag{1.21a}$$

$$\mu\left(w_t + \varepsilon(u w_x + w w_z)\right) - 2\Omega u = -p_z \qquad \text{in} \quad 0 < z < 1 + \varepsilon\eta(t, x), \tag{1.21b}$$

$$u_x + w_z = 0 \qquad \text{in} \quad 0 < z < 1 + \varepsilon\eta(t, x), \tag{1.21c}$$

$$u_z - \mu w_x = 0 \qquad \text{in} \quad 0 < z < 1 + \varepsilon\eta(t, x), \tag{1.21d}$$

$$p = \eta \qquad \text{on} \quad z = 1 + \varepsilon\eta(t,x), \tag{1.21e}$$

$$w = \eta_t + \varepsilon u \eta_x \qquad \text{on} \quad z = 1 + \varepsilon\eta(t,x), \tag{1.21f}$$

$$w = 0 \qquad \text{on} \quad z = 0. \tag{1.21g}$$

Choosing the suitable independent variables $\xi \overset{\text{def}}{=} \varepsilon^{\frac{1}{2}}(x - ct)$, $\tau \overset{\text{def}}{=} \varepsilon^{\frac{3}{2}} t$, where $c \overset{\text{def}}{=} \sqrt{1+\Omega^2} - \Omega$, it was derived in [40], after a double asymptotic expansion with respect to ε and μ, that the free surface $\eta = \eta(\tau, \xi)$ is governed by the equation

$$\begin{aligned} 2(\Omega + c)\eta_\tau + 3c^2\eta\eta_\xi + \frac{c^2}{3}\mu\eta_{\xi\xi\xi} + A_1\varepsilon\eta^2\eta_\xi + A_2\varepsilon^2\eta^3\eta_\xi + A_5\varepsilon^3\eta^4\eta_\xi \\ = \varepsilon\mu\Big[A_3\eta_\xi\eta_{\xi\xi} + A_4\eta\eta_{\xi\xi\xi}\Big] + O(\varepsilon^4, \mu^2) \end{aligned} \tag{1.22}$$

with $A_1 \overset{\text{def}}{=} \frac{3c^2(c^2-2)}{(c^2+1)^2}$, $A_2 \overset{\text{def}}{=} -\frac{c^2(2-c^2)(c^6-7c^4+5c^2-5)}{(c^2+1)^4}$, $A_3 \overset{\text{def}}{=} \frac{-c^2(9c^4+16c^2-2)}{3(c^2+1)^2}$, $A_4 \overset{\text{def}}{=} \frac{-c^2(3c^4+8c^2-1)}{3(c^2+1)^2}$, $A_5 \overset{\text{def}}{=} \frac{c^2(c^2-2)(3c^{10}+228c^8-540c^6-180c^4-13c^2+42)}{12(c^2+1)^6}$, and the function $\eta = \eta(\tau, \xi)$ with respect to the horizontal component of the velocity u under the Camassa-Holm regime $\varepsilon = O(\sqrt{\mu})$ is given by

$$\eta = \frac{1}{c}u + \gamma_1\varepsilon u^2 + \gamma_2\varepsilon^2 u^3 + \gamma_3\varepsilon^3 u^4 + \gamma_4\varepsilon\mu u_{\xi\xi} + O(\varepsilon^4, \mu^2), \tag{1.23}$$

where $\gamma_1 = \frac{2-c^2}{2c^2(c^2+1)}$, $\gamma_2 = \frac{(c^2-1)(c^2-2)(2c^2+1)}{2c^3(c^2+1)^3}$, $\gamma_3 = -\frac{(c^2-1)^2(c^2-2)(21c^4+16c^2+4)}{8c^4(c^2+1)^5}$, and $\gamma_4 = \frac{-(3c^4+6c^2-5)}{12c(c^2+1)^2}$. By this crucial relation between the velocity component u and the free surface component η, it then enables us to formally derive the R-CH equation [40] in the form

$$\begin{aligned} \partial_t u - \beta\mu\partial_t u_{xx} + cu_x + 3\alpha\varepsilon u u_x - \beta'\mu u_{xxx} + \omega_1\varepsilon^2 u^2 u_x + \omega_2\varepsilon^3 u^3 u_x \\ = \alpha\beta\varepsilon\mu(2u_x u_{xx} + u u_{xxx}), \end{aligned} \tag{1.24}$$

where $\alpha \overset{\text{def}}{=} \frac{c^2}{1+c^2}$, $\beta' \overset{\text{def}}{=} \frac{c(c^4+6c^2-1)}{6(c^2+1)^2}$, $\beta \overset{\text{def}}{=} \frac{3c^4+8c^2-1}{6(c^2+1)^2}$, $\omega_1 \overset{\text{def}}{=} \frac{-3c(c^2-1)(c^2-2)}{2(1+c^2)^3}$, and $\omega_2 \overset{\text{def}}{=} \frac{(c^2-2)(c^2-1)^2(8c^2-1)}{2(1+c^2)^5}$ satisfying $c \to 1$, $\beta \to \frac{5}{12}$, $\beta' \to \frac{1}{4}$, $\omega_1, \omega_2 \to 0$ and $\alpha \to \frac{1}{2}$ when $\Omega \to 0$.

Let $m \stackrel{\text{def}}{=} (1-\beta\mu\partial_x^2)u$. We now rewrite the above equation in terms of the evolution of m, namely,

$$\partial_t m + \alpha\varepsilon(um_x + 2mu_x) + cu_x - \beta'\mu u_{xxx} + \omega_1\varepsilon^2 u^2 u_x + \omega_2\varepsilon^3 u^3 u_x = 0, \tag{1.25}$$

which has the following two conserved quantities

$$E(u) = \frac{1}{2}\int_{\mathbb{R}} u^2 + \beta\mu u_x^2\, dx,$$

and

$$F(u) = \frac{1}{2}\int_{\mathbb{R}} cu^2 + \alpha\varepsilon u^3 + \beta'\mu u_x^2 + \frac{\omega_1\varepsilon^2}{6}u^4 + \frac{\omega_2\varepsilon^3}{10}u^5 + \alpha\beta\varepsilon\mu u u_x^2\, dx.$$

Denote that

$$B_1 \stackrel{\text{def}}{=} \partial_x(1-\beta\mu\partial_x^2), \qquad \text{and}$$

$$B_2 \stackrel{\text{def}}{=} \partial_x((\alpha\varepsilon m + \frac{c}{2})\cdot) + (\alpha\varepsilon m + \frac{c}{2})\partial_x - \beta'\mu\partial_x^3 + \frac{2}{3}\omega_1\varepsilon^2\partial_x(u\partial_x^{-1}(u\partial_x\cdot)) + \frac{5}{8}\omega_2\varepsilon^3\partial_x(u^{\frac{3}{2}}\partial_x^{-1}(u^{\frac{3}{2}}\partial_x\cdot)).$$

A simple calculation then reveals that the R-CH equation (1.24) can be written as

$$m_t = -B_1\frac{\delta F}{\delta m} = -B_2\frac{\delta E}{\delta m},$$

where B_1 and B_2 are two skew-symmetric operators. It is shown in [40] that the solutions of the R-CH equation (1.24) are uniformly bounded in suitable Sobolev norms for all small values of ε and μ defined in (1.18) with a valid time scale $O(\frac{1}{\varepsilon})$.

It is the first subject of present study to investigate whether or not the R-CH equation is a valid approximating model to the governing equations for water waves with the Coriolis effect. Our aim is to prove the relevance of the R-CH equation as a good model for the propagation of shallow-water waves with effect of the Coriolis forcing. To this end, we first use formal asymptotic procedures to derive the Green-Naghdi equations with the Coriolis effect in the shallow-water scaling ($\mu \ll 1$, without any assumptions on ε) from the governing equations of water waves (1.12) for one-dimensional surfaces and flat bottoms. This enables us to derive the KdV and BBM equations with effect of the Earth's rotation in the long-wave regime ($\varepsilon = O(\mu)$). Our investigation will first focus on the study that under what conditions the unidirectional solutions of the rotational KdV-type equation is well approximated by the solutions of the R-CH equation. To justify the R-CH approximation, one can use the solution of the R-CH equation to rewrite the corresponding rotation-KdV equation with the residual term, then show rigorously that the approximation error to the difference of the two solutions remains small up to $O(\mu^2)$ in the solution space over a long time scale. The argument can be approached by establishing the uniform boundedness of the solution up to the long time scaling $O(\frac{1}{\varepsilon})$. On the other hand, we will give a rigorous justification of solution of the classical CH as an approximation to that of the R-CH equations in a long time scale when the effect of the Earth's rotation vanishes. This can be done by showing that, for any given time interval, solutions of the R-CH equation are the Cauchy sequences in terms of the small rotation parameters using the conservation laws and the uniform bounded estimates of the solution.

Anther interesting issue to investigate here is concerned with the traveling-wave solutions of (4.55) in the form $u(t, x) = \varphi_\sigma(x - \sigma t)$, $\sigma \in \mathbb{R}$ for the function $\varphi_\sigma : \mathbb{R} \to \mathbb{R}$ such that $\varphi_\sigma \to 0$ as $|x| \to \infty$. It is known that the traveling-wave solution of the classical CH equation appears to be a weakly peaked soliton [8], which is one of interesting features for the CH-type equation. It was also found that the mCH equation, as the dual equation of the mKdV equation, admits peaked solitons [39]. As one can see that the R-CH equation (4.55) can be regarded as a perturbation of the CH equation by the weak Coriolis forcing. A natural question remains that how the Coriolis forcing affects the propagation of the traveling waves, in particular, the peaked solitons. To this end, it is of interest to study and classify the traveling wave solutions of (1.24), leading to some new types of nonanalytic traveling wave solutions, but nonexistence of the classical CH-type peaked solitons.

Later Lenells [46, 47] used a suitable framework for weak solutions to classify all weak traveling waves of the CH equation. However it is unclear whether the R-CH equation (1.24) with the weak Coriolis effect supports traveling waves with singularities. Using a natural weak formulation of (1.24), we can define exactly in what sense the peaked and cusped traveling waves are solutions. In fact, it turns out that the equation for φ takes the form $\varphi_x^2 = R(\varphi)$, where R is a rational function. A standard phase-plane analysis determines the behavior of solution near the zeros and poles of R. In fact, peaked traveling waves exist when R has a removable pole and cusped traveling waves correspond to when R has a non-removable pole. Due to the added rotational term, the numerator of R contains cubic polynomial $f(\varphi)$ whose root distribution is quite complicated. By analyzing each possible case carefully, we show here peaked and cusped traveling waves do exist for (1.24), but the CH-type peakon does not exist.

CHAPTER 2

STABILITY OF THE CAMASSA-HOLM MULTI-PEAKONS IN TWO-COMPONENT CAMASSA-HOLM SYSTEM

2.1 Stability of multipeakons

2.1.1 Basic definitions and results

In this chapter, we focus on the following Cauchy problem for system (1.5) written as

$$\begin{cases} u_t + uu_x + \partial_x p * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right) = 0, \\ \rho_t + (\rho u)_x = 0, \qquad t \in \mathbb{R},\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \rho(0,x) = \rho_0(x), \qquad x \in \mathbb{R}, \end{cases} \tag{2.1}$$

Prior to a discussion of the stability issue should be the well-posedness results. Actually, the local and global well-posedness results in varies cases were established in [30, 35, 37]. Hence, we recall the following result without proof, which suffices to develop the stability theory here.

Proposition 2.1.1. *[37] Let* $(u_0, \rho_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}), s > 3/2$. *Then there exists a time* $T > 0$ *such that the initial-value problem of system* (2.1) *has a unique solution* $(u, \rho) \in C([0,T); H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R}))$ *with* $(u(0), \rho(0)) = (u_0, \rho_0)$. *The solution* (u, ρ) *depends continuously on the initial value* (u_0, ρ_0) *and the maximal time of existence* $T > 0$ *is independent of* s. *If* $\rho_0(x) > 0$ *for all* $x \in \mathbb{R}$, *then* $T = +\infty$ *and the solution* (u, ρ) *is global. In additional, the functionals* $E(u, \rho)$ *and* $F(u, \rho)$ *defined in* (1.6) *are independent of the existence time* t.

The following definition of the orbital stability of a single traveling wave solution to the 2CH system was introduced in [12].

Definition 2.1.1. Let (φ_c, ψ_c) be a traveling wave of (1.5) with speed $c > 0$. Then (φ_c, ψ_c) is orbitally stable if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $(u_0, \rho_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > 3/2$, with $\|(u_0, \rho_0) - (\varphi_c, \psi_c)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} \leq \delta$, the corresponding solution $(u(t), \rho(t))$ of (1.5) with initial data (u_0, ρ_0) satisfies

$$\sup_{0<t<T} \inf_{r\in\mathbb{R}} \|(u(t,\cdot), \rho(t,\cdot)) - (\varphi_c(\cdot - r), \psi_c(\cdot - r))\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} \leq \varepsilon,$$

where T is the maximal existence time.

One of the main purpose in this chapter is to prove the orbital stability of the N-peaked traveling waves $(\sum_{i=1}^N \varphi_{c_i}, 0) = (\sum_{i=1}^N c_i e^{-|x|}, 0)$ of system (1.5) in energy space $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ with small perturbation.

Theorem 2.1.2. *Let $c_1, \cdots, c_N$ be N velocities such that $0 < c_1 < \cdots < c_N$. There exist $A > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$ such that if $(u_0, \rho_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ for $s > 3/2$ and*

$$\|u_0 - \sum_{i=1}^N \varphi_{c_i}(\cdot - z_i^0)\|_{H^1(\mathbb{R})} + \|\rho_0\|_{L^2(\mathbb{R})} \leq \varepsilon^2, \tag{2.2}$$

for some $0 < \varepsilon < \varepsilon_0$ and $z_i^0 - z_{i-1}^0 \geq L$, where $L > L_0$, then for the corresponding solution $(u, \rho) \in C([0,T); H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R}))$ of the Cauchy problem for the 2CH system (1.5) with initial data $(u, \rho)|_{t=0} = (u_0, \rho_0)$, there exist $\xi_1(t), \cdots, \xi_N(t) \in \mathbb{R}$, such that

$$\sup_{0<t<T} \left(\|u(t,\cdot) - \sum_{i=1}^N \varphi_{c_i}(\cdot - \xi_i(t))\|_{H^1(\mathbb{R})} + \|\rho(t,\cdot)\|_{L^2(\mathbb{R})} \right) \leq A\sqrt{\varepsilon}, \tag{2.3}$$

where $\xi_i(t) - \xi_{i-1}(t) > L/2$ and T depends only on initial data (u_0, ρ_0).

Remark 1. Notice that it is not a regular proof of Lyapunov stability [55]. A direct computation infers that the difference between $\|u_0 - \sum_{i=1}^N \varphi_{c_i}(\cdot - z_i^0)\|^2_{H^1(\mathbb{R})} + \|\rho_0\|^2_{L^2(\mathbb{R})}$ and $\|u(t,\cdot) - \sum_{i=1}^N \varphi_{c_i}(\cdot - \xi_i(t))\|^2_{H^1(\mathbb{R})} + \|\rho(t,\cdot)\|^2_{L^2(\mathbb{R})}$ is $\sum_{i=1}^N c_i|M_i - c_i|$. The estimate on Lyapunov function $P(M_i; u, \rho)$ shows that $\sum_{i=1}^{N} c_i|M_i - c_i| = O(\varepsilon)$ if the initial distance is bounded by $O(\varepsilon^2)$. Hence, the later distance is bounded by $O(\sqrt{\varepsilon})$. No matter what the order is for the initial data, the later distance should always be controlled by the $\frac{1}{4}$ of the initial order.".

Due to the collision theory between two peakons, stability of the trains of peakons provides the stability result on multipeakons.

Corollary 2.1.3. *Let $p_1^0, \cdots, p_N^0$ be N positive real numbers, and $q_1^0 < \cdots < q_N^0$. For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $(u_0, \rho_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > \frac{3}{2}$ satisfies*

$$\|u_0 - \sum_{j=1}^{N} p_j^0 e^{|\cdot - q_j^0|}\|_{H^1(\mathbb{R})} + \|\rho_0\|_{L^2(\mathbb{R})} \leq \delta, \quad \text{and} \quad \rho_0(x) > 0, \tag{2.4}$$

then

$$\forall t \in \mathbb{R}, \quad \inf_{0<p_1<\cdots<p_N, q_1<\cdots<q_N} \|u(t,\cdot) - \sum_{j=1}^{N} p_j e^{|\cdot - q_j|}\|_{H^1(\mathbb{R})} + \|\rho\|_{L^2(\mathbb{R})} \leq \varepsilon, \tag{2.5}$$

where $p_i(t)$ and $q_i(t)$ satisfy the Hamiltonian system (1.4).

Moreover, there exists $T > 0$, such that

$$\forall t \geq T, \quad \inf_{q_1<\cdots<q_N} \|u(t,\cdot) - \sum_{j=1}^{N} \lambda_j e^{|\cdot - q_j|}\|_{H^1(\mathbb{R})} + \|\rho\|_{L^2(\mathbb{R})} \leq \varepsilon, \tag{2.6}$$

and

$$\forall t \leq -T, \quad \inf_{q_1<\cdots<q_N} \|u(t,\cdot) - \sum_{j=1}^{N} \lambda_{N+1-j} e^{|\cdot - q_j|}\|_{H^1(\mathbb{R})} + \|\rho\|_{L^2(\mathbb{R})} \leq \varepsilon, \tag{2.7}$$

where $0 < \lambda_1 < \cdots < \lambda_N$ are the eigenvalues of the matrix $\left(p_j^0 e^{-|q_i^0 - q_j^0|/2}\right)_{1 \leq i,j \leq N}$.

2.1.2 Preliminary lemmas

Inspired by the idea adopted in [24, 27], we present the general strategy of the proof to establish Theorem 2.1.2 as follows. For $\alpha > 0$ and $L > 0$, we define the following neighborhood of $(\sum_{i=1}^N \varphi_{c_i}, 0)$ with spatial shifts x_i that satisfies $x_i - x_{i-1} \geq L$ for $i = 1, \cdots, N$,

$$U(\alpha, L) = \left\{ (u,\rho) \in H^1 \times L^2, \inf_{x_i - x_{i-1} > L} \left(\|u - \sum_{i=1}^N \varphi_{c_i}(\cdot - x_i)\|_{H^1(\mathbb{R})} + \|\rho\|_{L^2(\mathbb{R})} \right) < \alpha \right\}.$$

The purpose is to prove that there exist $A > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$ such that for any $L > L_0$, $0 < \varepsilon < \varepsilon_0$ and any initial data $(u_0, \rho_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > 3/2$ satisfying (2.2), then the corresponding solution $(u(t), \rho(t))$ belongs to $U\left(A(\sqrt{\varepsilon} + L^{-1/8}), L/2\right)$ for all $t \in [0, T)$, where T is the maximal existence time and A is independent of time.

By the continuity of the map $t \mapsto (u(t), \rho(t))$ from $[0, T)$ into $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ where $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \subset H^1(\mathbb{R}) \times L^2(\mathbb{R})$, to prove Theorem 2.1.2 is sufficient to prove that if (u_0, ρ_0) satisfies (2.2) and if for some $0 < t^* < T$,

$$(u(t), \rho(t)) \in U\left(A(\sqrt{\varepsilon} + L^{-1/8}),\ L/2\right), \quad \forall t \in [0, t^*], \tag{2.8}$$

then

$$(u(t^*), \rho(t^*)) \in U\left(\frac{A}{2}(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \frac{2L}{3}\right). \tag{2.9}$$

Henceforth, we assume (2.8) holds for some $0 < \varepsilon < \varepsilon_0$ and $L > L_0$, where A, ε_0 and L_0 are specified later, and demonstrate (2.9).

The proof of Theorem 2.1.2 relies on a series of lemmas. Firstly, we will show that the solution (u, ρ) of system (1.5) satisfying (2.8) has the following properties: $u(t)$ remains close to the sum of N modulated peakons in $H^1(\mathbb{R})$ where each peakon is away from others for $t \in [0, t^*]$ by using a modulation argument, and $\rho(t)$ is around 0 in $L^2(\mathbb{R})$ according to the conservation law (1.6).

Lemma 2.1.4. *Let the initial data* (u_0, ρ_0) *satisfy the assumption* (2.2) *in Theorem 2.1.2. There exist* $\alpha_0 > 0$ *and* $L_0 > 0$ *depending only on* $\{c_i\}_{i=1}^N$ *such that if* $0 < \alpha < \alpha_0$ *and* $L > L_0$*, the corresponding solution* $(u(t), \rho(t))$ *satisfies for some* $0 < t^* < T$

$$(u(t), \rho(t)) \in U\left(\alpha, \frac{L}{2}\right), \forall t \in [0, t^*], \tag{2.10}$$

then there exist unique C^1 *functions* $\tilde{x}_i(t) : [0, t^*] \to \mathbb{R}, i = 1, \cdots, N$*, such that denote* $v(t, x)$ *by*

$$v(t) = u(t) - \sum_{i=1}^{N} R_i(t), \quad \textit{where } R_i(t, \cdot) = \varphi_{c_i}(\cdot - \tilde{x}_i(t)),$$

we have the orthogonal condition as

$$\int_{\mathbb{R}} v(t)\partial_x R_i(t)\, dx = 0, \tag{2.11}$$

and the following properties for all $i \in \{1, 2, \cdots, N\}$ *and* $t \in [0, t^*]$:

$$\|v(t)\|_{H^1(\mathbb{R})} \leq O(\sqrt{\alpha}), \tag{2.12}$$

$$\left|\dot{\tilde{x}}_i(t) - c_i\right| \leq O(\sqrt{\alpha}) + O(L^{-1}), \tag{2.13}$$

$$\tilde{x}_i(t) - \tilde{x}_{i-1}(t) \geq \frac{3L}{4} + \frac{(c_i - c_{i-1})}{2} \cdot t, \quad i \geq 2. \tag{2.14}$$

Furthermore, defining $\mathcal{J}_i(t) = [y_i(t), y_{i+1}(t)]$*, with*

$$y_1 = -\infty, \quad y_{N+1} = +\infty \quad \textit{and} \quad y_i(t) = \frac{\tilde{x}_{i-1}(t) + \tilde{x}_i(t)}{2},\ i = 2, \cdots, N, \tag{2.15}$$

it holds

$$|\xi_i(t) - \tilde{x}_i(t)| = O(1), \tag{2.16}$$

where $\xi_i(t)$ *are any points such that*

$$u(t, \xi_i(t)) = \max_{x \in \mathcal{J}_i(t)} u(t, x), \quad t \in [0, t^*],\ i = 1, \cdots, N. \tag{2.17}$$

Proof. We use the standard modulation argument to discover the translations of N-peakons. Let $Z = (z_1, \cdots, z_N) \in \mathbb{R}^N$ be fixed such that $z_i - z_{i-1} > L/2$ for some $L > 0$. Set $R_Z(\cdot) = \sum_{i=1}^N \varphi_{c_i}(\cdot - z_i)$. For $0 < \delta_0 < 1$, we define the function

$$Y : \prod_{i=1}^N (-\delta_0, \delta_0) \times B_{H^1}(R_Z, \delta_0) \to \mathbb{R}^N,$$

$$(y_1, \cdots, y_N, u) \longmapsto \left(Y^1(y_1, \cdots, y_N, u), \cdots, Y^N(y_1, \cdots, y_N, u)\right),$$

with

$$Y^j(y_1, \cdots, y_N, u) = \int_{\mathbb{R}} \left(u - \sum_{i=1}^N \varphi_{c_i}(\cdot - z_i - y_i)\right) \partial_x \varphi_{c_j}(\cdot - z_j - y_j)\, dx,$$

where $B_{H^1}(R_Z, \delta_0)$ is the ball in $H^1(R)$ with center R_Z and radius δ_0. Notice that, due to ρ is around ground state, its effect on the translations of N-peakons can be neglected. To apply the implicit function theorem, two facts are needed. One fact is that function Y should be C^1 mapping which can be proved by dominated convergence theorem. The other fact is that the matrix of the first partial derivatives at $(0, \cdots, 0, R_Z)$ should be invertible. For $j = 1, \cdots, N$

$$\frac{\partial Y^j}{\partial y_j}(y_1, \cdots, y_N, u) = \int_{\mathbb{R}} \left(u_x - \sum_{i=1, i\neq j}^N \partial_x \varphi_{c_i}(\cdot - z_i - y_i)\right) \partial_x \varphi_{c_j}(\cdot - z_j - y_j)\, dx,$$

$$\frac{\partial Y^j}{\partial y_i}(y_1, \cdots, y_N, u) = \int_{\mathbb{R}} \partial_x \varphi_{c_i}(\cdot - z_i - y_i) \partial_x \varphi_{c_j}(\cdot - z_j - y_j)\, dx, \quad \text{where} \quad i \neq j.$$

Hence,

$$\frac{\partial Y^j}{\partial y_j}(0, \cdots, 0, R_Z) = \|\partial_x \varphi_{c_j}\|_{L^2(\mathbb{R})}^2 \geq c_1^2,$$

and

$$\frac{\partial Y^j}{\partial y_i}(0, \cdots, 0, R_Z) = \int_{\mathbb{R}} \partial_x \varphi_{c_i}(\cdot - z_i) \cdot \partial_x \varphi_{c_j}(\cdot - z_j)\, dx \leq O(e^{-\frac{L}{4}}).$$

Thus, there exists $L_0 > 0$ such that if $L > L_0$, we have

$$D_{(y_1, \cdots, y_N)} Y(0, \cdots, 0, R_z) = D + P,$$

where D is an invertible diagonal matrix with $\|D^{-1}\| \leq (c_1)^{-2}$ and $\|P\| \leq O(e^{-L/4})$, which implies $D_{(y_1,\cdots,y_N)}Y(0,\cdots,0,R_Z)$ is invertible with an inverse matrix of norm smaller than $2(c_1)^{-2}$. Therefore, by the implicit function theorem, there exist $0 < \beta_0 < \delta_0$ and C^1 functions $(y_1(u),\cdots,y_N(u))$ from $B_{H^1}(R_Z,\beta_0)$ to a neighborhood of $(0,\cdots,0)$ which are uniquely determined, such that

$$Y(y_1(u),\cdots,y_N(u),u) = 0, \quad \forall u \in B_{H^1}(R_Z,\beta_0).$$

Moreover, there exists $K_0 > 0$ such that if $u \in B_{H^1}(R_Z,\beta)$ with $0 < \beta < \beta_0$, then

$$\sum_{i=1}^{N} |y_i(u)| \leq K_0\beta, \tag{2.18}$$

where K_0 and β_0 depends on c_1 and L_0. For $u \in B_{H^1}(R_Z,\beta_0)$, setting $\tilde{x}_i(t) = z_i + y_i$ and $\beta_0 \leq \min\{L_0/(8K_0),\delta_0\}$ infers

$$\tilde{x}_i(u) - \tilde{x}_{i-1}(u) = z_i - z_{i-1} + y_i(u) - y_{i-1}(u) \geq \frac{L}{2} - 2K_0\beta_0 \geq \frac{L}{4}. \tag{2.19}$$

Then, we define the modulation of $(u,\rho) \in U(\alpha,L/2)$ for $L > L_0$ and $0 < \alpha < \alpha_0$ at a fix time t. Indeed, for $0 < \alpha < \alpha_0$, $U(\alpha,L/2)$ can be covered as follows

$$U\left(\alpha,\frac{L}{2}\right) \subset \bigcup_{Z\in\mathbb{R}^N, z_i - z_{i-1} > \frac{L}{2}} B_{H^1}(R_Z,2\alpha) \times B_{L^2}(0,2\alpha).$$

Additionally, the modulation of u is uniquely defined due to the uniqueness in the implicit function theorem.

Thus, we can define the modulation of the solution $(u(t),\rho(t))$ of the 2CH system satisfying $(u(t),\rho(t)) \in U(\alpha,L/2)$ for all $t \in [0,t^*]$ by setting $i = 1,\cdots,N$ and

$$\tilde{x}_i(t) = \tilde{x}_i(u(t)), \quad v(t) = u(t) - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - \tilde{x}_i(t)),$$

where v satisfies the orthogonal condition $\langle v, \partial_x R_i\rangle_{H^{-1},H^1} = 0$, that is

$$\int_{\mathbb{R}} \Big(u(t) - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - \tilde{x}_i(t))\Big)\partial_x \varphi_{c_i}(\cdot - \tilde{x}_i(t))\,dx = 0, \quad i = 1,\cdots,N.$$

According to the translation $\tilde{x}_i(t)$ defined above, using (2.18), triangle inequality, the following estimate holds

$$\begin{aligned}
\|v(t)\|_{H^1(\mathbb{R})} &\le \|u(t) - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - z_i)\|_{H^1(\mathbb{R})} + \sum_{i=1}^{N} \|\varphi_{c_i}(\cdot - z_i) - \varphi_{c_i}(\cdot - z_i - y_i)\|_{H^1(\mathbb{R})} \\
&\le \alpha + \sum_{i=1}^{N} \Bigg(\|\varphi_{c_i}(\cdot - z_i)\|^2_{H^1(\mathbb{R})} + \|\varphi_{c_i}(\cdot - z_i - y_i)\|^2_{H^1(\mathbb{R})} \\
&\qquad - 2 \int_{\mathbb{R}} \varphi_{c_i}(\cdot - z_i)\varphi_{c_i}(\cdot - z_i - y_i)\, dx \\
&\qquad - 2 \int_{\mathbb{R}} \partial_x\varphi_{c_i}(\cdot - z_i)\partial_x\varphi_{c_i}(\cdot - z_i - y_i)\, dx \Bigg)^{\frac{1}{2}} \\
&\le \alpha + 2 \sum_{i=1}^{N} |c_i| \cdot |y_i(u)|^{\frac{1}{2}} \\
&\le O(\sqrt{\alpha}).
\end{aligned}$$

Attention is now turn to the speed of $\tilde{x}_i$. In order to show it stays close to c_i, we adopt the following property of a single peakon

$$\partial_x^2 R_j(t) = R_j(t) - 2\, c_j\, \delta(\tilde{x}_j(t)). \tag{2.20}$$

Differentiating orthogonality condition with respect to t, we derive that

$$\begin{aligned}
\int_{\mathbb{R}} v_t \partial_x R_i = \dot{\tilde{x}}_i \langle \partial_x^2 R_i, v\rangle_{H^{-1},H^1} &= \dot{\tilde{x}}_i \langle R_i - 2c_i\delta(\tilde{x}_i(t)), v\rangle_{H^{-1},H^1} \\
&\le \dot{\tilde{x}}_i O(\|v\|_{H^1(\mathbb{R})}) \\
&\le (\dot{\tilde{x}}_i - c_i) O(\|v\|_{H^1(\mathbb{R})}) + O(\|v\|_{H^1(\mathbb{R})}).
\end{aligned} \tag{2.21}$$

Substituting u by $v + \sum_{i=1}^{N} R_i$ in the first equation of system (1.5) leads to

$$\begin{aligned}
&\left(1 - \partial_x^2\right) v_t + \sum_{i=1}^{N} (1 - \partial_x^2)\partial_t R_i \\
&= -\frac{1}{2}(1 - \partial_x^2)\partial_x \left((v + \sum_{i=1}^{N} R_i)^2 \right) - \partial_x \left((v + \sum_{i=1}^{N} R_i)^2 + \frac{1}{2}(v_x + \sum_{i=1}^{N} \partial_x R_i)^2 + \frac{1}{2}\rho^2 \right).
\end{aligned}$$

Since $(R_i, 0)$ is a solution to system (1.5), there holds

$$\partial_t R_i + (\dot{\tilde{x}}_i - c_i)\partial_x R_i + R_i \partial_x R_i + (1-\partial_x^2)^{-1}\partial_x \left(R_i^2 + \frac{1}{2}(\partial_x R_i)^2 \right) = 0.$$

Combining the two identities mentioned above, we infer that v satisfies the following condition on $[0, t^*]$,

$$\begin{aligned} v_t - \sum_{i=1}^{N}(\dot{\tilde{x}}_i - c_i)\partial_x R_i = -\frac{1}{2}\partial_x\Big(v + \sum_{i=1}^{N} R_i)^2 - \sum_{i=1}^{N} R_i^2\Big) - (1-\partial_x^2)^{-1}\partial_x\Big((v + \sum_{i=1}^{N} R_i)^2 \\ - \sum_{i=1}^{N} R_i^2 + \frac{1}{2}(v_x + \sum_{i=1}^{N}\partial_x R_i)^2 - \frac{1}{2}\sum_{i=1}^{N}(\partial_x R_i)^2\Big) - (1-\partial_x^2)^{-1}\partial_x(\frac{1}{2}\rho^2)\,. \end{aligned}$$

Taking the L^2-inner product with $\partial_x R_i$, then using integration by parts, the exponential decay of R_i and its first order derivative, we deduce

$$\begin{aligned} &\int_{\mathbb{R}} \left(\dot{\tilde{x}}_i - c_i\right) \partial_x R_i \cdot \partial_x R_i \, dx \\ &= \int_{\mathbb{R}} v_t \cdot \partial_x R_i \, dx - \int_{\mathbb{R}} \sum_{j\neq i} \left(\dot{\tilde{x}}_j - c_j\right) \partial_x R_j \cdot \partial_x R_i \, dx \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} \partial_x \left((v + \sum_{i=1}^{N} R_i)^2 - \sum_{i=1}^{N} R_i^2 \right) \cdot \partial_x R_i \, dx \\ &\quad + \int_{\mathbb{R}} (1-\partial_x^2)^{-1}\partial_x\Big((v + \sum_{i=1}^{N} R_i)^2 - \sum_{i=1}^{N} R_i^2 + \frac{1}{2}(v_x + \sum_{i=1}^{N}\partial_x R_i)^2 \\ &\qquad\qquad - \frac{1}{2}\sum_{i=1}^{N}(\partial_x R_i)^2\Big) \cdot \partial_x R_i \, dx \\ &\quad + \int_{\mathbb{R}} (1-\partial_x^2)^{-1}\partial_x(\frac{1}{2}\rho^2) \cdot \partial_x R_i \, dx \\ &:= \int_{\mathbb{R}} v_t \cdot \partial_x R_i \, dx - \int_{\mathbb{R}} \sum_{j\neq i} \left(\dot{\tilde{x}}_i - c_i\right) \partial_x R_j \cdot \partial_x R_i \, dx + V_1 + V_2 + V_3. \end{aligned} \tag{2.22}$$

We intend now to estimate V_1, V_2 and V_3 in the following steps. Firstly, considering $V_1 = \frac{1}{2}\int_{\mathbb{R}} \partial_x \left((v + \sum_{i=1}^{N} R_i)^2 - \sum_{i=1}^{N} R_i^2 \right) \cdot \partial_x R_i \, dx$, we denote

$$Q_1 = v^2 + 2v\sum_{j=1}^{N} R_j + \left(\sum_{j=1}^{N} R_j\right)^2 - \sum_{j=1}^{N} R_j^2. \tag{2.23}$$

Using integration by parts and (2.20), there holds

$$2V_1 = 2c_i Q_1(t, \tilde{x}_i(t)) - \int_{\mathbb{R}} Q_1 R_i(t)\, dx. \tag{2.24}$$

According to the following embedding formula

$$\|v\|_{L^\infty(\mathbb{R})} \leq \frac{\sqrt{2}}{2} \|v\|_{H^1(\mathbb{R})} \leq O(\sqrt{\alpha}),$$

we know that Q_1 and $\int_{\mathbb{R}} Q_1 R_i(t)\, dx$ can be estimated as

$$|Q_1| \leq \big(O(\sqrt{\alpha}) + O(1)\big)\, O(\sqrt{\alpha}) + O(e^{-\frac{L}{8}}),$$

$$\int_{\mathbb{R}} Q_1 R_i(t)\, dx \leq \big(O(\sqrt{\alpha}) + O(1)\big)\, O(\sqrt{\alpha}) + O(e^{-\frac{L}{8}}).$$

Hence,

$$V_1 \leq \big(O(\sqrt{\alpha}) + O(1)\big)\, O(\sqrt{\alpha}) + O(e^{-\frac{L}{8}}).$$

Secondly, for V_2, using integration by parts and the decay of R_i and its first order derivative, it performs as

$$V_2 = -\int_{\mathbb{R}} (1-\partial_x^2)^{-1}\partial_x^2 \Big((v + \sum_{i=1}^N R_i)^2 - \sum_{i=1}^N R_i^2 + \frac{1}{2}(v_x + \sum_{i=1}^N \partial_x R_i)^2 - \frac{1}{2}\sum_{i=1}^N (\partial_x R_i)^2 \Big) \cdot R_i\, dx.$$

In this case, let

$$P = (v + \sum_{i=1}^N R_i)^2 - \sum_{i=1}^N R_i^2 + \frac{1}{2}(v_x + \sum_{i=1}^N \partial_x R_i)^2 - \frac{1}{2}\sum_{i=1}^N (\partial_x R_i)^2.$$

From (2.12) and $(1-\partial_x^2)^{-1}P = \frac{1}{2}e^{-|x|} * P$, which along with Höder's inequality give rise to

$$\begin{aligned} |V_2| &= \left| \int_{\mathbb{R}} R_i \left[\left(1-\partial_x^2\right)^{-1} P - P \right] dx \right| \\ &\leq C \int_{\mathbb{R}} \left(v^2 + vR_X + \frac{1}{2}v_x^2 + \frac{1}{2} v R_{X,x} \right) dx + O(L^{-1}) \\ &\leq O(\|v\|_{H^1(\mathbb{R})}) + O(L^{-1}) \\ &\leq O(\sqrt{\alpha}) + O(L^{-1}). \end{aligned} \tag{2.25}$$

Finally, the bound for V_3 is determined as follows

$$V_3 = \int_{\mathbb{R}} (1-\partial_x^2)^{-1}\partial_x(\frac{1}{2}\rho^2)\partial_x R_i\, dx \leq C\int_{\mathbb{R}} \rho^2\, dx \leq O(\alpha^2). \tag{2.26}$$

Combining (2.21) and (2.24)-(2.26), we obtain

$$|\dot{\tilde{x}}_i - c_i| \left(\|\partial_x R_i\|^2_{L^2(\mathbb{R})} + O(\sqrt{\alpha}) \right) \leq O(\sqrt{\alpha}) + O(e^{-\frac{L}{8}}).$$

Taking α small enough and L large enough, it infers that $|\dot{\tilde{x}}_i - c_i| \leq O(\sqrt{\alpha}) + O(e^{-L})$.

Furthermore, in light of $z_i^0 - z_{i-1}^0 \geq L$, there obtains

$$\begin{aligned}
\tilde{x}_i(t) - \tilde{x}_{i-1}(t) &= \tilde{x}_i(0) - \tilde{x}_{i-1}(0) + \left(\dot{\tilde{x}}_i(s) - \dot{\tilde{x}}_{i-1}(s)\right) t \\
&= \tilde{x}_i(0) - \tilde{x}_{i-1}(0) + \left(\dot{\tilde{x}}_i(s) - c_i + c_{i-1} - \dot{\tilde{x}}_{i-1}(s)\right) t + (c_i - c_{i-1})t \\
&\geq \frac{3L}{4} + \frac{1}{2}(c_i - c_{i-1})t,
\end{aligned}$$

which yields the estimate (2.14).

Let $x = \tilde{x}_i(t)$. From (2.12) and (2.14), we deduce

$$|u(t, \tilde{x}_i(t))| = |c_i| + O(\sqrt{\alpha}) + O(e^{-\frac{L}{4}}) \geq \frac{3|c_i|}{4}.$$

On the other hand, for $x \in [\tilde{x}_i(t) - \frac{L}{4}, \tilde{x}_i(t) + \frac{L}{4}] \setminus (\tilde{x}_i(t) - 2, \tilde{x}_i(t) + 2)$, the following estimate holds

$$|u(t,x)| \leq |c_i|e^{-2} + O(\sqrt{\alpha}) + O(e^{-\frac{L}{4}}) \leq \frac{|c_i|}{2}.$$

In conclusion, ξ_i belongs to $[\tilde{x}_i(t) - 2, \tilde{x}_i(t) + 2]$. This completes the proof of Lemma 2.1.4. □

Then effort is devoted to show functionals describing the energy at the right side of i-th bump are almost monotonic, for $i = 1, \cdots, N$. To prove this property, we start with the introduction of weight functions. Let Ψ be a C^∞ function, such that

$$\begin{cases} 0 < \Psi(x) < 1,\ \Psi'(x) > 0, & x \in \mathbb{R}, \\ |\Psi'''(x)| \leq 10\Psi'(x), & x \in [-1, 1], \end{cases} \tag{2.27}$$

and

$$\Psi(x) = \begin{cases} e^{-|x|}, & x < -1, \\ 1 - e^{-|x|}, & x > 1. \end{cases} \tag{2.28}$$

Set $\Psi_K = \Psi(\cdot/K)$, $K > 0$. Define the weight functions $\Phi_i = \Phi_i(t,x)$, by

$$\Phi_1 = 1 - \Psi_{2,K}, \qquad \Phi_N = \Psi_{N,K}, \qquad \Phi_i = \Psi_{i,K} - \Psi_{i+1,K}, \quad i = 2, \cdots, N-1,$$

where $\Psi_{i,K} = \Psi_K(x - y_i(t))$ with $y_i(t)$ defined in (2.15). Obviously, $\sum_{i=1}^{N} \Phi_i(t,x) = 1$, for $t \in [0, t^*]$. Taking $L > 0$ and $L/K > 0$ large enough, we have

$$|1 - \Phi_i| \leq 4e^{-\frac{L}{4K}}, \quad \text{on } \left[\tilde{x}_i - \frac{L}{4}, \tilde{x}_i + \frac{L}{4}\right], \tag{2.29}$$

and

$$|\Phi_i| \leq 4e^{-\frac{L}{4K}}, \quad \text{on } \left[\tilde{x}_j - \frac{L}{4}, \tilde{x}_j + \frac{L}{4}\right], \quad \text{for } j \neq i. \tag{2.30}$$

Then, we introduce the localized conservation laws of E and F in terms of weight functions, for $i = 1, \cdots, N$,

$$E_i(u,\rho) = \int_{\mathbb{R}} \left(u^2 + u_x^2 + \rho^2\right) \Phi_i \, dx, \quad \text{and} \quad F_i(u,\rho) = \int_{\mathbb{R}} \left(u^3 + uu_x^2 + u\rho^2\right) \Phi_i \, dx. \tag{2.31}$$

Moreover, for simplicity, we set

$$\sigma_0 = \frac{1}{4} \min(c_1, c_2 - c_1, \cdots, c_N - c_{N-1}). \tag{2.32}$$

Lemma 2.1.5. *Let (u,ρ) be a solution of system (1.5) such that $(u,\rho) \in U(\alpha, \frac{L}{2})$ on $[0,t^*]$ where $\{\tilde{x}_i(t)\}_{i=1}^N$ are defined in Lemma 2.1.4. Then there exist $\alpha_0 > 0$ and $L_0 > 0$ only depending on $\{c_i\}_{i=1}^N$ such that if $0 < \alpha < \alpha_0$ and $L > L_0$, for $4 \leq K = O(L^{1/2})$, it follows that*

$$I_{j,K}(t) - I_{j,K}(0) \leq C \left(\|u_0\|_{H^1(\mathbb{R})}^2 + \|\rho_0\|_{L^2(\mathbb{R})}^2\right)^{\frac{3}{2}} e^{-\frac{L}{8K}}, \tag{2.33}$$

for $j = \{2, \cdots, N\}$ and $t \in [0,t^*]$, where $I_{j,K}(t) = \int_{\mathbb{R}} (u^2 + u_x^2 + \rho^2) \Psi_{j,K} \, dx$.

Proof. We know system (1.5) can be written as the following abstract Hamiltonian form

$$\begin{pmatrix} u \\ \rho \end{pmatrix}_t = \frac{1}{2} \begin{pmatrix} -\partial_x(1-\partial_x^2)^{-1} & 0 \\ 0 & -\partial_x \end{pmatrix} \begin{pmatrix} F'_u \\ F'_\rho \end{pmatrix}, \tag{2.34}$$

where

$$F'_u = 3u^2 - u_x^2 - 2uu_{xx} + \rho^2, \quad \text{and} \quad F'_\rho = 2u\rho. \tag{2.35}$$

Fixing j, differentiating $I_{j,K}$ with respect to t and using (2.34), we get

$$\begin{aligned} \frac{d}{dt} I_{j,K}(t) = &- \dot{y}_j \int_{\mathbb{R}} (u^2 + u_x^2 + \rho^2)\partial_x \Psi_{j,K}\, dx + 2 \int_{\mathbb{R}} u(u_t - u_{txx})\Psi_{j,K}\, dx \\ &- 2 \int_{\mathbb{R}} u u_{tx} \partial_x \Psi_{j,K}\, dx + 2 \int_{\mathbb{R}} \rho\rho_t \Psi_{j,K}\, dx \\ = &- \dot{y}_j \int_{\mathbb{R}} (u^2 + u_x^2 + \rho^2)\partial_x \Psi_{j,K}\, dx - \int_{\mathbb{R}} u \Psi_{j,K} \partial_x F'_u\, dx \\ &+ \int_{\mathbb{R}} u \partial_x \Psi_{j,K} (1-\partial_x^2)^{-1} \partial_x^2 F'_u\, dx - \int_{\mathbb{R}} \rho \Psi_{j,K} \partial_x F'_\rho\, dx \\ = &- \dot{y}_j \int_{\mathbb{R}} (u^2 + u_x^2 + \rho^2)\partial_x \Psi_{j,K}\, dx + \int_{\mathbb{R}} u_x \Psi_{j,K} F'_u\, dx \\ &+ \int_{\mathbb{R}} u \partial_x \Psi_{j,K} (1-\partial_x^2)^{-1} F'_u\, dx + \int_{\mathbb{R}} \rho_x \Psi_{j,K} F'_\rho\, dx + \int_{\mathbb{R}} \rho \partial_x \Psi_{j,K} F'_\rho\, dx \\ := &- \dot{y}_j \int_{\mathbb{R}} (u^2 + u_x^2 + \rho^2)\partial_x \Psi_{j,K}\, dx + J_1(t) + J_2(t) + J_3(t) + J_4(t). \end{aligned} \tag{2.36}$$

Substituting (2.35) into (2.36) and using integration by parts, $J_1(t)$, $J_2(t)$, $J_3(t)$ and $J_4(t)$ become

$$\begin{aligned} J_1(t) = &\int_{\mathbb{R}} u_x \Psi_{j,K} \left(3u^2 - u_x^2 - 2uu_{xx} + \rho^2\right)\, dx \\ = &- \int_{\mathbb{R}} \partial_x \Psi_{j,K} u^3\, dx + \int_{\mathbb{R}} \partial_x \Psi_{j,K} u u_x^2\, dx + \int_{\mathbb{R}} \Psi_{j,K} u_x \rho^2\, dx, \\ J_2(t) = &\int_{\mathbb{R}} u \partial_x \Psi_{j,K} (1-\partial_x^2)^{-1} \left(3u^2 - u_x^2 - 2uu_{xx} + \rho^2\right)\, dx \\ = &\int_{\mathbb{R}} \partial_x \Psi_{j,K} u^3\, dx + \int_{\mathbb{R}} u \partial_x \Psi_{j,K} (1-\partial_x^2)^{-1} \left(2u^2 + u_x^2 + \rho^2\right)\, dx, \\ J_3(t) = &\, 2 \int_{\mathbb{R}} \rho_x \Psi_{j,K} u\rho\, dx = - \int_{\mathbb{R}} \Psi_{j,K} u_x \rho^2\, dx - \int_{\mathbb{R}} \partial_x \Psi_{j,K} u \rho^2\, dx, \end{aligned}$$

$$J_4(t) = 2\int_{\mathbb{R}} \rho\partial_x\Psi_{j,K} u\rho\, dx = 2\int_{\mathbb{R}} \partial_x\Psi_{j,K} u\rho^2\, dx.$$

Combining $J_1(t)$, $J_2(t)$, $J_3(t)$ and $J_4(t)$, we can rewrite (2.36) as follows

$$\begin{aligned}
\frac{d}{dt}I_{j,K}(t) = & \frac{d}{dt}\int_{\mathbb{R}} \left(u^2 + u_x^2 + \rho^2\right)\Psi_{j,K}\, dx \\
&= -\dot{y}_j(t)\int_{\mathbb{R}} \left(u^2 + u_x^2 + \rho^2\right)\partial_x\Psi_{j,K}\, dx + \int_{\mathbb{R}} \left(uu_x^2\right)\partial_x\Psi_{j,K}\, dx \qquad (2.37)\\
&\quad + \int_{\mathbb{R}} \left(u\rho^2\right)\partial_x\Psi_{j,K}\, dx + \int_{\mathbb{R}} u\cdot\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}\left(2u^2 + u_x^2 + \rho^2\right)\, dx.
\end{aligned}$$

In view of (2.9), $\dot{y}_j(t)$ employs the following estimate

$$\begin{aligned}
-\dot{y}_j(t) = & -\frac{\dot{\tilde{x}}_j(t) - c_j}{2} - \frac{\dot{\tilde{x}}_{j-1}(t) - c_{j-1}}{2} - \frac{c_j + c_{j-1}}{2} \\
\le & -\frac{c_{j-1} + c_j}{2} + O(\sqrt{\alpha}) + O(L^{-1}) < -\frac{c_1}{2},
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{d}{dt}I_{j,K}(t) \le & -\frac{c_1}{2}\int_{\mathbb{R}} \left(u^2 + u_x^2 + \rho^2\right)\partial_x\Psi_{j,K}\, dx + \int_{\mathbb{R}} uu_x^2\cdot\partial_x\Psi_{j,K}\, dx \\
&+ \int_{\mathbb{R}} u\rho^2\cdot\partial_x\Psi_{j,K}\, dx + \int_{\mathbb{R}} u\cdot\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}\left(2u^2 + u_x^2 + \rho^2\right)\, dx \\
:= & -\frac{c_1}{2}\int_{\mathbb{R}} \left(u^2 + u_x^2 + \rho^2\right)\partial_x\Psi_{j,K}\, dx + Q_1(t) + Q_2(t), \qquad (2.38)
\end{aligned}$$

where $Q_1(t) = \int_{\mathbb{R}} u(u_x^2+\rho^2)\partial_x\Psi_{j,K}\, dx$, $Q_2(t) = \int_{\mathbb{R}} u\cdot\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}\left(2u^2 + u_x^2 + \rho^2\right)\, dx$. For further estimates, we define the interval D_j by

$$D_j = \left[\tilde{x}_{j-1}(t) + \frac{L}{4}, \tilde{x}_j(t) - \frac{L}{4}\right],$$

and divide $\mathbb{R}$ by $\mathbb{R} = D_j \cup D_j^c$. The two crucial estimates related to D_j and D_j^c are listed in the following. For $x \in D_j$,

$$\begin{aligned}
\|u(t)\|_{L^\infty(D_j)} \le & \sum_{i=1}^N \|\varphi_{c_i}(\cdot - \tilde{x}_i(t))\|_{L^\infty(D_j)} + \|u - \sum_{i=1}^N \varphi_{c_i}(\cdot - \tilde{x}_i(t))\|_{L^\infty(D_j)} \\
\le & O(e^{-\frac{L}{8}}) + O(\sqrt{\alpha}). \qquad (2.39)
\end{aligned}$$

For $x \in D_j^c$, by using (2.13), (2.15) and the definition of $\Psi_{j,K}$, we know

$$|x - y_j(t)| \geq \frac{\tilde{x}_j(t) - \tilde{x}_{j-1}(t)}{2} - \frac{L}{4} \geq \frac{(c_j - c_{j-1})t}{4} + \frac{L}{8} \geq \sigma_0 t + \frac{L}{8},$$

which implies, for $K = O(\sqrt{L})$ and sufficiently large L_0,

$$\left| \frac{x - y_j(t)}{K} \right| \geq \frac{\sigma_0 t + \frac{L}{8}}{K} > 1.$$

Hence, there holds

$$\partial_x \Psi_{j,K}(t,x) = \frac{1}{K} \Psi'_{j,K}\left(\frac{x - y_j(t)}{K}\right) \leq \frac{1}{K} e^{-\frac{1}{K}(\sigma_0 t + \frac{L}{8})}, \quad x \in D_j^c. \tag{2.40}$$

Firstly, let us consider $Q_1(t)$ by splitting the domain into two parts. For $x \in D_j^c$, from the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and (2.40), we deduce that

$$\begin{aligned} \int_{D_j^c} u(u_x^2 + \rho^2) \partial_x \Psi_{j,K} \, dx &\leq \frac{1}{K} e^{-\frac{1}{K}(\sigma_0 t + \frac{L}{8})} \cdot \|u\|_{L^\infty(\mathbb{R})} \cdot \int_{D_j^c} (u_x^2 + \rho^2) \, dx \\ &\leq \frac{C}{K} (\|u_0\|_{H^1(\mathbb{R})}^2 + \|\rho_0\|_{L^2(\mathbb{R})}^2)^{\frac{3}{2}} \cdot e^{-\frac{1}{K}(\sigma_0 t + \frac{L}{8})}. \end{aligned}$$

For $x \in D_j$, by (2.39) we know

$$\begin{aligned} \int_{D_j} u(u_x^2 + \rho^2) \partial_x \Psi_{j,K} \, dx &\leq \|u\|_{L^\infty(D_j)} \cdot \int_{D_j} (u_x^2 + \rho^2) \partial_x \Psi_{j,K} \, dx \\ &\leq (O(e^{-\frac{L}{8}}) + O(\sqrt{\alpha}))(u_x^2 + \rho^2) \partial_x \Psi_{j,K}. \end{aligned}$$

Thus, for $\alpha \ll 1$ and $L \gg 1$, one obtains

$$Q_1(t) \leq \frac{c_1}{4} \int_{\mathbb{R}} \left(u^2 + u_x^2 + \rho^2\right) \partial_x \Psi_{j,K} \, dx + \frac{C}{K} (\|u_0\|_{H^1(\mathbb{R})}^2 + \|\rho_0\|_{L^2(\mathbb{R})}^2)^{\frac{3}{2}} \cdot e^{-\frac{1}{K}(\sigma_0 t + \frac{L}{8})}. \tag{2.41}$$

Secondly, in order to analyze $Q_2(t)$, a method similar to the previous case is applied here. For $x \in D_j^c$, by the property of convolution and (2.40), we know

$$\begin{aligned} &\int_{D_j^c} u \cdot \partial_x \Psi_{j,K} (1 - \partial_x^2)^{-1} \left(2u^2 + u_x^2 + \rho^2\right) \, dx \\ &\leq \|u\|_{L^\infty(D_j^c)} \cdot e^{-\frac{1}{K}(\sigma_0 t + \frac{L}{8})} \cdot \int_{D_j^c} (1 - \partial_x^2)^{-1} \left(2u^2 + u_x^2 + \rho^2\right) \, dx \\ &\leq \|u\|_{L^\infty(\mathbb{R})} \cdot e^{-\frac{1}{K}(\sigma_0 t + \frac{L}{8})} \cdot \int_{D_j^c} e^{-|x|} * \left(2u^2 + u_x^2 + \rho^2\right) \, dx \\ &\leq \frac{C}{K} (\|u_0\|_{H^1(\mathbb{R})}^2 + \|\rho_0\|_{L^2(\mathbb{R})}^2)^{\frac{3}{2}} \cdot e^{-\frac{1}{K}(\sigma_0 t + \frac{L}{8})}. \end{aligned}$$

For $x \in D_j$, by the assumption (2.27), it is deduced that

$$(1-\partial_x^2)\partial_x\Psi_{j,K}(t,x) = \partial_x\Psi_{j,K}(t,x) - \frac{1}{K^3}\Psi'''\left(\frac{x-y_j(t)}{K}\right) \geq \left(1-\frac{10}{K^2}\right)\partial_x\Psi_{j,K}(t,x),$$

which implies, for $K \geq 4$,

$$(1-\partial_x^2)^{-1}\partial_x\Psi_{j,K}(t,x) \leq \left(1-\frac{10}{K^2}\right)^{-1}\partial_x\Psi_{j,K}(t,x).$$

Hence, for $K \geq 4$ and $\Psi'(x) > 0$, we have

$$\begin{aligned}
&\int_{D_j} u\left((1-\partial_x^2)^{-1}(2u^2+u_x^2+\rho^2)\right)\partial_x\Psi_{j,K}\,dx \\
&\leq 2\|u\|_{L^\infty(D_j)}\int_{\mathbb{R}}\left(u^2+u_x^2+\rho^2\right)(1-\partial_x^2)^{-1}\partial_x\Psi_{j,K}\,dx \\
&\leq (O(\sqrt{\alpha})+O(e^{-\frac{L}{8}}))\int_{\mathbb{R}}\left(u^2+u_x^2+\rho^2\right)\partial_x\Psi_{j,K}\,dx \\
&\leq \frac{c_1}{4}\int_{\mathbb{R}}\left(u^2+u_x^2+\rho^2\right)\partial_x\Psi_{j,K}\,dx.
\end{aligned}$$

Consequently, for $\alpha \ll 1$ and $L \gg 1$, there holds

$$Q_2(t) \leq \frac{c_1}{4}\int_{\mathbb{R}}\left(u^2+u_x^2+\rho^2\right)\partial_x\Psi_{j,K}\,dx + \frac{C}{K}(\|u_0\|^2_{H^1(\mathbb{R})}+\|\rho_0\|^2_{L^2(\mathbb{R})})^{\frac{3}{2}}\cdot e^{-\frac{1}{K}(\sigma_0 t+\frac{L}{8})}. \tag{2.42}$$

Moreover, through (2.38), (2.41) and (2.42), one obtain

$$\frac{d}{dt}I_{j,K}(t) \leq \frac{C}{K}(\|u_0\|^2_{H^1(\mathbb{R})}+\|\rho_0\|^2_{L^2(\mathbb{R})})^{\frac{3}{2}}\cdot e^{-\frac{1}{K}(\sigma_0 t+\frac{L}{8})}.$$

By Gronwall's inequality on $[0,t^*]$, we know the following estimate holds

$$\begin{aligned}
I_{j,K}(t)-I_{j,K}(0) \leq& \frac{C}{K}(\|u_0\|^2_{H^1(\mathbb{R})}+\|\rho_0\|^2_{L^2(\mathbb{R})})^{\frac{3}{2}}\int_0^t e^{-\frac{1}{K}\left(\sigma_0 s+\frac{L}{8}\right)}\,ds \\
\leq& \frac{C}{\sigma_0}(\|u_0\|^2_{H^1(\mathbb{R})}+\|\rho_0\|^2_{L^2(\mathbb{R})})^{\frac{3}{2}}e^{-\frac{L}{8K}}.
\end{aligned}$$

This completes the proof of Lemma 2.1.5. □

Three lemmas regarding to the local and global estimates of the traveling waves to system (1.5) are introduced in the following. The first lemma is to show the relation between the localized conservation laws E_i, F_i defined in (2.31) and the localized maximum of u at a fixed time.

Lemma 2.1.6. *Given N real numbers $\tilde{x}_1 < \cdots < \tilde{x}_N$ with $\tilde{x}_i - \tilde{x}_{i-1} \geq 3L/4$. Define interval $\mathcal{J}_i$ as in (2.15). Assume for any fixed function $(u, \rho) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > 3/2$, there exists $\xi_i \in \mathcal{J}_i$, where $i = 1, \cdots, N$, such that*

$$u(\xi_i) = \max_{x \in \mathcal{J}_i} u(x) := M_i \quad \text{and} \quad |\xi_i - \tilde{x}_i| = O(1).$$

Then, for each $i = 1, \cdots, N$, it holds

$$F_i(u, \rho) \leq M_i E_i(u, \rho) - \frac{2}{3} M_i^3 + (\|u_0\|_{H^1(\mathbb{R})}^2 + \|\rho_0\|_{L^2(\mathbb{R})}^2)^{\frac{3}{2}} O(L^{-\frac{1}{2}}). \tag{2.43}$$

Proof. Let $i \in \{1, \cdots, N\}$ be fixed. Following with the strategy in [24], we define

$$g(x) = \begin{cases} u(x) - u_x(x), & x < \xi_i, \\ u(x) + u_x(x), & x > \xi_i. \end{cases}$$

Using the integration by parts, we derive that

$$\begin{aligned} \int_{\mathbb{R}} g^2(x) \Phi_i(x)\, dx &= \int_{-\infty}^{\xi_i} (u - u_x)^2 \Phi_i\, dx + \int_{\xi_i}^{\infty} (u + u_x)^2 \Phi_i\, dx \\ &= \int_{\mathbb{R}} \left(u^2 + u_x^2\right) \Phi_i\, dx - 2M_i^2 \Phi_i(\xi_i) + \int_{-\infty}^{\xi_i} u^2 \partial_x \Phi_i\, dx \\ &\quad - \int_{\xi_i}^{\infty} u^2 \partial_x \Phi_i\, dx, \\ \int_{\mathbb{R}} u(x) g^2(x) \Phi_i(x)\, dx &= \int_{-\infty}^{\xi_i} u(u - u_x)^2 \Phi_i\, dx + \int_{\xi_i}^{\infty} u(u + u_x)^2 \Phi_i\, dx \\ &= \int_{\mathbb{R}} \left(u^3 + u u_x^2\right) \Phi_i\, dx - \frac{4}{3} M_i^3 \Phi_i(\xi_i) + \frac{2}{3} \int_{-\infty}^{\xi_i} u^3 \partial_x \Phi_i\, dx \\ &\quad - \frac{2}{3} \int_{\xi_i}^{\infty} u^3 \partial_x \Phi_i\, dx. \end{aligned}$$

Then the localized conservation laws E_i and F_i have the following forms

$$E_i(u, \rho) = \int_{\mathbb{R}} \left(g^2 + \rho^2\right) \Phi_i\, dx + 2M_i^2 \Phi_i(\xi_i) - \int_{-\infty}^{\xi_i} u^2 \partial_x \Phi_i\, dx + \int_{\xi_i}^{\infty} u^2 \partial_x \Phi_i\, dx,$$

and

$$F_i(u, \rho) = \int_{\mathbb{R}} u \left(g^2 + \rho^2\right) \Phi_i\, dx + \frac{4}{3} M_i^3 \Phi_i(\xi_i) - \frac{2}{3} \int_{-\infty}^{\xi_i} u^3 \partial_x \Phi_i\, dx + \frac{2}{3} \int_{\xi_i}^{\infty} u^3 \partial_x \Phi_i\, dx.$$

According to the definition of M_i and integration by parts, a direct computation shows that

$$\begin{aligned}
F_i(u,\rho) &= \int_{\mathcal{J}_i} u\left(g^2+\rho^2\right)\Phi_i\,dx + \int_{\mathcal{J}_i^c} u\left(g^2+\rho^2\right)\Phi_i\,dx \\
&\quad + \frac{4}{3}M_i^3\Phi_i(\xi_i) - \frac{2}{3}\int_{-\infty}^{\xi_i} u^3\partial_x\Phi_i\,dx + \frac{2}{3}\int_{\xi_i}^{\infty} u^3\partial_x\Phi_i\,dx \\
&\le M_i\int_{\mathbb{R}}\left(g^2+\rho^2\right)\Phi_i\,dx + \int_{\mathcal{J}_i^c} u\left(g^2+\rho^2\right)\Phi_i\,dx \\
&\quad + \frac{4}{3}M_i^3\Phi_i(\xi_i) - \frac{2}{3}\int_{-\infty}^{\xi_i} u^3\partial_x\Phi_i\,dx + \frac{2}{3}\int_{\xi_i}^{\infty} u^3\partial_x\Phi_i\,dx \\
&\le M_i\left(E_i(u,\rho) - 2M_i^2\Phi_i(\xi_i) + \int_{-\infty}^{\xi_i} u^2\partial_x\Phi_i\,dx - \int_{\xi_i}^{\infty} u^2\partial_x\Phi_i\,dx\right) \\
&\quad + \int_{\mathcal{J}_i^c} u\left(g^2+\rho^2\right)\Phi_i\,dx + \frac{4}{3}M_i^3\Phi_i(\xi_i) - \frac{2}{3}\int_{-\infty}^{\xi_i} u^3\partial_x\Phi_i\,dx + \frac{2}{3}\int_{\xi_i}^{\infty} u^3\partial_x\Phi_i\,dx \\
&\le M_iE_i(u,\rho) - \frac{2}{3}M_i^3\Phi_i(\xi_i) + \int_{\mathcal{J}_i^c} u\left(g^2+\rho^2\right)\Phi_i\,dx + M_i\int_{-\infty}^{\xi_i} u^2\partial_x\Phi_i\,dx \\
&\quad - M_i\int_{\xi_i}^{\infty} u^2\partial_x\Phi_i\,dx - \frac{2}{3}\int_{-\infty}^{\xi_i} u^3\partial_x\Phi_i\,dx + \frac{2}{3}\int_{\xi_i}^{\infty} u^3\partial_x\Phi_i\,dx. \qquad (2.44)
\end{aligned}$$

Due to the construction of Φ_i and the exponential decay of Ψ, taking $K=\sqrt{L}/8$, there exists a constant $C>0$, such that $|\partial_x\Phi_i| \le C/K = O(L^{-\frac{1}{2}})$. On the other hand, $|\xi_i - \tilde{x}_i| = O(1)$ implies

$$|1-\Phi_i(\xi_i)| \le 4e^{-\frac{L}{4K}} \le O(L^{-\frac{1}{2}}).$$

Hence, in view of the conserved quantities E and F defined in (1.6), we have

$$F_i(u,\rho) \le M_iE_i(u,\rho) - \frac{2}{3}M_i^3 + (\|u_0\|_{H^1(\mathbb{R})}^2 + \|\rho_0\|_{L^2(\mathbb{R})}^2)^{\frac{3}{2}}O(L^{-\frac{1}{2}}),$$

which completes the proof of Lemma 2.1.6. □

Next, we present a global identity to show the difference between the smooth traveling wave (u,ρ) and N peaked traveling waves $(\sum_{i=1}^N \varphi_{c_i}, 0)$ to system (1.5) in energy space $H^1(\mathbb{R})\times L^2(\mathbb{R})$.

Lemma 2.1.7. *Let $(z_1, \cdots, z_N) \in \mathbb{R}^N$ such that $z_i - z_{i-1} > L/2$ with $L > 0, i = 1, \cdots, N$. Then for any solution $(u, \rho) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}), s > \frac{3}{2}$ to system* (1.5) *, there holds*

$$E(u,\rho)-\sum_{i=1}^{N} E(\varphi_{c_i},0) = \|u(x)-\sum_{i=1}^{N} R_{z_i}(x)\|^2_{H^1(\mathbb{R})}+\|\rho\|^2_{L^2(\mathbb{R})}+4\sum_{i=1}^{N} c_i(u(z_i)-c_i)+O(e^{-\frac{L}{4}}). \tag{2.45}$$

Proof. With the integration by parts, it is deduced that

$$\begin{aligned}
&\|u(x) - \sum_{i=1}^{N} R_{z_i}(x)\|^2_{H^1(\mathbb{R})} + \|\rho\|^2_{L^2(\mathbb{R})} \\
&= \int_{\mathbb{R}} \left(u(x) - \sum_{i=1}^{N} R_{z_i}(x)\right)^2 + \left(u(x) - \sum_{i=1}^{N} R_{z_i}(x)\right)_x^2 + (\rho(x))^2 \, dx \\
&= \int_{\mathbb{R}} \left(u^2 + u_x^2 + \rho^2\right) dx + \int_{\mathbb{R}} \left(\left(\sum_{i=1}^{N} R_{z_i}(x)\right)^2 + \left(\sum_{i=1}^{N} R_{z_i}(x)\right)_x^2 + 0^2\right) dx \\
&\quad - 2\sum_{i=1}^{N} \int_{\mathbb{R}} \left(u\varphi_{c_i}(\cdot - z_i) + u_x \partial_x \varphi_{c_i}(\cdot - z_i)\right) dx \\
&= E(u,\rho) + E(\sum_{i=1}^{N} R_{z_i}, 0) - 2\sum_{i=1}^{N} \int_{\mathbb{R}} u\varphi_{c_i}(\cdot - z_i)\, dx \\
&\quad + 2\sum_{i=1}^{N} \int_{z_i}^{\infty} u_x\varphi_{c_i}(\cdot - z_i)\, dx - 2\sum_{i=1}^{N} \int_{-\infty}^{z_i} u_x\varphi_{c_i}(\cdot - z_i)\, dx \\
&= E(u,\rho) + E(\sum_{i=1}^{N} R_{z_i}, 0) - 4\sum_{i=1}^{N} c_i u(z_i).
\end{aligned} \tag{2.46}$$

Notice that we have the facts $\int_{\mathbb{R}} R_{z_i}^2 + (R_{z_i})_x^2 \, dx = 2c_i^2$ and $z_i - z_{i-1} > \frac{L}{2}$, which infer that

$$E(\sum_{i=1}^{N} R_{z_i}, 0) = \sum_{i=1}^{N} E(R_{z_i}, 0) + O\left(e^{-\frac{L}{4}}\right) = 2\sum_{i=1}^{N} c_i^2 + O\left(e^{-\frac{L}{4}}\right). \tag{2.47}$$

In conclusion, combining (2.46) and (2.47), there obtains the identity

$$E(u,\rho)-\sum_{i=1}^{N} E(\varphi_{c_i},0) = \|u(x)-\sum_{i=1}^{N} R_{z_i}(x)\|^2_{H^1(\mathbb{R})}+\|\rho\|^2_{L^2(\mathbb{R})}+4\sum_{i=1}^{N} c_i(u(z_i)-c_i)+O\left(e^{-\frac{L}{4}}\right).$$

□

In view of the identity (2.45), one key to establish the stability result is to control of the difference between the localized solution $u(z_i)$ and the maximum of each single peakon, where the translation z_i will be determined as ξ_i later.

Lemma 2.1.8. *Let* (u,ρ) *be the solution of system* (1.5) *such that* $(u,\rho)\in U(\alpha,\frac{L}{2})$ *on* $[0,t^*]$ *with initial data* (u_0,ρ_0) *satisfying* $\|u_0-\sum_{i=1}^{N}\varphi_{c_i}(\cdot-z_i^0)\|_{H^1(\mathbb{R})}+\|\rho_0\|_{L^2(\mathbb{R})}<\varepsilon^2$. *For* $i\in\{1,\cdots,N\}$, *set*

$$M_i(t)=\max_{x\in\mathcal{J}_i(t)}u(t,x)=u(t,\xi_i(t)),\quad \forall t\in[0,t^*].$$

Then, we have the estimate

$$\sum_{i=1}^{N}c_i\,|M_i-c_i|\le O(\varepsilon)+O(L^{-\frac{1}{4}}),\quad \forall t\in[0,t^*],\tag{2.48}$$

where the constant $O(\cdot)$ *depend on* $(c_i)_{i=1}^N$ *and* $(\|u_0\|^2_{H^1(\mathbb{R})}+\|\rho_0\|^2_{L^2(\mathbb{R})})^{\frac{3}{2}}$.

Proof. Since $R_{Z^0}=\sum_{i=1}^{N}\varphi_{c_i}(\cdot-z_i^0)$ and $\|u_0-R_{Z^0}\|_{H^1(\mathbb{R})}+\|\rho_0\|_{L^2(\mathbb{R})}<\varepsilon^2$, it is deduced from Minkowski inequality that

$$\begin{aligned}|E(u_0,\rho_0)-E(R_{Z^0},0)|&=|\|u_0\|^2_{H^1(\mathbb{R})}+\|\rho_0\|^2_{L^2(\mathbb{R})}-\|R_{Z^0}\|^2_{H^1(\mathbb{R})}|\\&\le\|u_0-R_{Z^0}\|^2_{H^1(\mathbb{R})}+\|\rho_0\|^2_{L^2(\mathbb{R})}\le O\left(\varepsilon^2\right),\end{aligned}$$

which implies

$$\begin{aligned}E(u,\rho)&=E(u_0,\rho_0)\le|E(u_0,\rho_0)-E(R_{Z^0},0)|+E(R_{Z^0},0)\\&\le\sum_{i=1}^{N}E(\varphi_{c_i},0)+O\left(\varepsilon^2\right)+O\left(e^{-\frac{L}{4}}\right).\end{aligned}\tag{2.49}$$

According to Lemma 2.1.6, the following inequality holds, for $i=1,\cdots,N$

$$M_i^3-\frac{3}{2}M_iE_i(u,\rho)+\frac{3}{2}F_i(u,\rho)\le\left(\|u_0\|^2_{H^1(\mathbb{R})}+\|\rho_0\|^2_{L^2(\mathbb{R})}\right)^{\frac{3}{2}}O\left(L^{-\frac{1}{2}}\right).\tag{2.50}$$

Define a cubic polynomial P with respect to $y\in\mathbb{R}$, for $(u,\rho)\in H^s\times H^{s-1}$, $s>\frac{3}{2}$,

$$P(y_i;u,\rho)=y_i^3-\frac{3}{2}y_iE_i(u,\rho)+\frac{3}{2}F_i(u,\rho).\tag{2.51}$$

In particular, for $E_i(\varphi_{c_i},0)=2c_i^2$ and $F_i(\varphi_{c_i},0)=\frac{4}{3}c_i^3$, we obtain

$$P(y_i;\varphi_{c_i},0)=y_i^3-\frac{3}{2}y_iE_i(\varphi_{c_i},0)+\frac{3}{2}F_i(\varphi_{c_i},0)=y_i^3-3y_ic_i^2+2c_i^3. \tag{2.52}$$

Taking $y_i(t)=M_i=u(t,\xi_i(t))$ in (2.51) and (2.52), a direct calculation gives rise to

$$(M_i-c_i)^2(M_i+2c_i)=P(M_i;u,\rho)+\frac{3}{2}\left(E_i(u,\rho)-E_i(\varphi_{c_i},0)\right)M_i-\frac{3}{2}\left(F_i(u,\rho)-F_i(\varphi_{c_i},0)\right).$$

Taking summation from $i=1$ to N and using the estimate (2.50), there appears the inequality

$$\begin{aligned}
\sum_{i=1}^{N}(M_i-c_i)^2(M_i+2c_i)&\le\sum_{i=1}^{N}\frac{3}{2}\left(E_i(u,\rho)-E_i(\varphi_{c_i},0)\right)M_i-\sum_{i=1}^{N}\frac{3}{2}\left(F_i(u,\rho)-F_i(\varphi_{c_i},0)\right)\\
&\quad+(\|u_0\|_{H^1(\mathbb{R})}^2+\|\rho_0\|_{L^2(\mathbb{R})}^2)^{\frac{3}{2}}O\left(L^{-\frac{1}{2}}\right)\\
&\le\frac{3}{2}\sum_{i=1}^{N}\left(E_i(u,\rho)-E_i(u_0,\rho_0)\right)M_i+\frac{3}{2}\sum_{i=1}^{N}\left(E_i(u_0,\rho_0)-E_i(\varphi_{c_i},0)\right)M_i\\
&\quad-\sum_{i=1}^{N}\frac{3}{2}\left(F_i(u,\rho)-F_i(\varphi_{c_i},0)\right)+(\|u_0\|_{H^1(\mathbb{R})}^2+\|\rho_0\|_{L^2(\mathbb{R})}^2)^{\frac{3}{2}}O\left(L^{-\frac{1}{2}}\right)\\
&:=K_1+K_2+K_3+(\|u_0\|_{H^1(\mathbb{R})}^2+\|\rho_0\|_{L^2(\mathbb{R})}^2)^{\frac{3}{2}}O\left(L^{-\frac{1}{2}}\right).
\end{aligned} \tag{2.53}$$

In view of (2.53), to proceed the proof of (2.48), attention is now given to the estimates on K_1, K_2 and K_3. Let us start with K_1. Using the Abel transformation and conservation law E, we deduce that

$$\begin{aligned}
\frac{3}{2}\sum_{i=1}^{N}\left(E_i(u,\rho)-E_i(u_0,\rho_0)\right)M_i&=\frac{3}{2}\Big(M_N\cdot\sum_{i=1}^{N}\left(E_i(u,\rho)-E_i(u_0,\rho_0)\right)\Big)\\
&\quad-\frac{3}{2}\sum_{j=1}^{N-1}(M_{j+1}-M_j)\cdot\sum_{i=1}^{j}\left(E_i(u,\rho)-E_i(u_0,\rho_0)\right)\\
&=-\frac{3}{2}\sum_{j=1}^{N-1}(M_{j+1}-M_j)\cdot\left(I_{j+1,K}(t)-I_{j+1,K}(0)\right).
\end{aligned}$$

Since N peakons are ordered, by the definition of M_j, we know $\sum_{j=1}^{N-1}(M_{j+1}-M_j)$ is positive and bounded above. Let $K=L^{1/2}/8$. From Lemma 2.1.5, it is derived that

$$|K_1|\le O(e^{-\frac{L}{8K}})=O(L^{-\frac{1}{2}}). \tag{2.54}$$

For K_2, from the assumption in Lemma 2.1.8, the exponential decay of φ_{c_i} and Φ_i and the definition of E_i, we know

$$\begin{aligned}
&\sum_{i=1}^{N}|E_i(u_0,\rho_0)-E_i(\varphi_{c_i},0)| \\
&\le \sum_{i=1}^{N}\left|\int_{\mathcal{J}_i}(u_0^2+u_{0x}^2+\rho_0^2)\Phi_i\,dx-\int_{\mathcal{J}_i}(\varphi_{c_i}^2+\varphi_{c_i,x}^2)\,dx\right|+O(L^{-\frac{1}{2}}) \\
&\le \sum_{i=1}^{N}\left|\|u_0\|_{H^1(\mathcal{J}_i)}^2-\|\varphi_{c_i}\|_{H^1(\mathcal{J}_i)}^2\right|+O(\varepsilon^2)+O(L^{-\frac{1}{2}}) \\
&\le \sum_{i=1}^{N}\left(\|u_0-\sum_{i=1}^{N}\varphi_{c_i}(\cdot-z_i^0)\|_{H^1(\mathbb{R})}+2\sum_{j=1}^{N}\sqrt{2c_j^2}\right) \\
&\quad\cdot(\|u_0-\sum_{i=1}^{N}\varphi_{c_i}(\cdot-z_i^0)\|_{H^1(\mathcal{J}_i(0))}+\sum_{j=1,j\neq i}^{N}\|\varphi_{c_j}\|_{H^1(\mathcal{J}_i(0))})+O(\varepsilon^2)+O(L^{-\frac{1}{2}}) \\
&\le O(\varepsilon^2)+O(L^{-\frac{1}{2}}).
\end{aligned} \tag{2.55}$$

On the other hand, $M_i(t)$ is bounded for any $i=1\cdots,N$, since

$$M_i^2(t)\le\|u(t,x)\|_{L^\infty(\mathbb{R})}^2\le\frac{1}{2}E(u_0,\rho_0)\le\frac{1}{2}\sum_{i=1}^{N}E(\varphi_{c_i},0)+O(\varepsilon^2)+O(e^{-\frac{L}{4}})\le\sum_{i=1}^{N}c_i^2. \tag{2.56}$$

Associating (2.54) with (2.56), there holds

$$|K_2|\le\frac{3}{2}\cdot\max_i M_i\cdot\sum_{i=1}^{N}|E_i(u_0,\rho_0)-E_i(\varphi_{c_i},0)|\le O(\varepsilon^2)+O(L^{-\frac{1}{2}}). \tag{2.57}$$

As to K_3, from the Sobolev embedding theorem and conservation law F, it is deduced that

$$\begin{aligned}
&\left|F(u,\rho)-F(\sum_{i=1}^{N}\varphi_{c_i}(\cdot-z_i^0),0)\right| \\
&=\left|\int_{\mathbb{R}}u_0^3+u_0u_{0,x}^2+u_0\rho_0^2-(\sum_{i=1}^{N}\varphi_{c_i}(\cdot-z_i^0))^3-(\sum_{i=1}^{N}\varphi_{c_i}(\cdot-z_i^0))(\sum_{i=1}^{N}\varphi_{c_i}(\cdot-z_i^0))_x^2\,dx\right| \\
&\le\int_{\mathbb{R}}\left|u_0^3-(\sum_{i=1}^{N}\varphi_{c_i}(\cdot-z_i^0))^3\right|dx+\int_{\mathbb{R}}\left|u_0u_{0,x}^2-(\sum_{i=1}^{N}\varphi_{c_i}(\cdot-z_i^0))(\sum_{i=1}^{N}\varphi_{c_i}(\cdot-z_i^0))_x^2\right|dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}} |u_0\rho_0^2|\,dx\\
&\le \int_{\mathbb{R}} \Big|(u_0-\sum_{i=1}^N \varphi_{c_i}(\cdot-z_i^0))(u_0^2+u_0(\sum_{i=1}^N \varphi_{c_i}(\cdot-z_i^0))+(\sum_{i=1}^N \varphi_{c_i}(\cdot-z_i^0))^2)\Big|\,dx\\
&+\int_{\mathbb{R}} \Big|(u_0-\sum_{i=1}^N \varphi_{c_i}(\cdot-z_i^0))u_{0,x}^2+(\sum_{i=1}^N \varphi_{c_i}(\cdot-z_i^0))(u_{0,x}^2-(\sum_{i=1}^N \varphi_{c_i}(\cdot-z_i^0))_x^2)\Big|\,dx\\
&+\int_{\mathbb{R}} |u_0\rho_0^2|\,dx\\
&\le \|u_0-\sum_{i=1}^N \varphi_{c_i}(\cdot-z_i^0)\|_{L^\infty(\mathbb{R})}\cdot\frac{3}{2}\int_{\mathbb{R}} u_0^2+(\sum_{i=1}^N \varphi_{c_i}(\cdot-z_i^0))^2\,dx\\
&+\int_{\mathbb{R}} \Big|(u_0-\sum_{i=1}^N \varphi_{c_i}(\cdot-z_i^0))u_{0,x}^2\Big|\,dx\\
&+\int_{\mathbb{R}} \Big|(\sum_{i=1}^N \varphi_{c_i}(\cdot-z_i^0))(u_{0,x}-\sum_{i=1}^N \varphi_{c_i,x}(\cdot-z_i^0))(u_{0,x}+\sum_{i=1}^N \varphi_{c_i,x}(\cdot-z_i^0))\Big|\,dx\\
&+\|u_0\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}} \rho_0^2\,dx\\
&\le \frac{3}{2}\|u_0-\sum_{i=1}^N \varphi_{c_i}(\cdot-z_i^0)\|_{L^\infty(\mathbb{R})}\Big(\|u_0\|_{L^2}^2+\|\sum_{i=1}^N \varphi_{c_i}(\cdot-z_i^0)\|_{L^2(\mathbb{R})}^2\Big)\\
&+\|u_0-\sum_{i=1}^N \varphi_{c_i}(\cdot-z_i^0)\|_{L^\infty(\mathbb{R})}\|u_0\|_{H^1(\mathbb{R})}\\
&+\|\sum_{i=1}^N \varphi_{c_i}(\cdot-z_i^0)\|_{L^\infty(\mathbb{R})}\|u_{0,x}-\sum_{i=1}^N \varphi_{c_i,x}(\cdot-z_i^0)\|_{L^2(\mathbb{R})}^2\|u_{0,x}+\sum_{i=1}^N \varphi_{c_i,x}(\cdot-z_i^0)\|_{L^2(\mathbb{R})}^2\\
&+\|u_0\|_{L^\infty(\mathbb{R})}\|\rho_0\|_{L^2(\mathbb{R})}^2.
\end{aligned}
$$

Due to the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and u_0, $\sum_{i=1}^N \varphi_{c_i}$ are in $H^1(\mathbb{R})$, by (2.2) and (2.12), we derive that

$$
|F(u_0,\rho_0)-F(\sum_{i=1}^N \varphi_{c_i}(\cdot-z_i^0),0)| \le O(\varepsilon^2),
$$

which along with $z_i^0 - z_{i-1}^0 \ge L/2$ gives rise to

$$
|K_3| \ \le \frac{3}{2}\Big|\sum_{i=1}^N \Big(F_i(u,\rho)-F_i(\varphi_{c_i},0)\Big)\Big|
$$

$$\leq \frac{3}{2}\Big|F(u_0,\rho_0) - F\Big(\sum_{i=1}^{N}\varphi_{c_i}(\cdot - z_i^0), 0\Big)\Big| + \frac{3}{2}\Big|F\Big(\sum_{i=1}^{N}\varphi_{c_i}(\cdot - z_i^0), 0\Big) - F\Big(\sum_{i=1}^{N}\varphi_{c_i}, 0\Big)\Big|$$

$$\leq O(\varepsilon^2) + O(e^{-\frac{L}{4}}) \tag{2.58}$$

Consequently, substituting (2.54),(2.57) and (2.58) into (2.53), we have

$$\sum_{i=1}^{N}(M_i - c_i)^2(M_i + 2c_i) \leq O(\varepsilon^2) + O(L^{-\frac{1}{2}}).$$

Then the desired result in Lemma 2.1.8 follows. □

2.1.3 Proof of stability theorem

With the preparation in the previous subsections, the proof of stability of the trains of N-peakons in system (1.5) is provided as well as the conclusion on multipeakons through the asymptotic analysis.

Proof of Theorem 2.1.2. In light of (2.9), it suffices to show that there exist $z_1 < \cdots < z_N$ satisfying $z_i - z_{i-1} > L/2$ and a constant $C > 0$ independent of A at $t = t^*$ such that

$$\|u(t^*, x) - \sum_{i=1}^{N}\varphi_{c_i}(x - z_i(t^*))\|_{H^1(\mathbb{R})} + \|\rho(t^*, x)\|_{L^2(\mathbb{R})} \leq C(\sqrt{\varepsilon} + L^{-\frac{1}{8}}).$$

Let $z_i = \xi_i$. By (2.14) and (2.16) in lemma 2.1.4, we have

$$\begin{aligned}\xi_i(t^*) - \xi_{i-1}(t^*) &\geq \tilde{x}_i(t^*) - \tilde{x}_{i-1}(t^*) + |\xi_i(t^*) - \tilde{x}_i(t^*)| - |\xi_{i-1}(t^*) - \tilde{x}_{i-1}(t^*)| \\ &\geq \frac{3L}{4} - \frac{L}{6} > \frac{2L}{3}.\end{aligned}$$

By (2.45) in Lemma 2.1.7 and (2.48) in Lemma 2.1.8, there holds

$$\begin{aligned}&\|u(t^*, x) - \sum_{i=1}^{N}\varphi_{c_i}(x - \xi_i(t^*))\|_{H^1(\mathbb{R})}^2 + \|\rho(t^*, x)\|_{L^2(\mathbb{R})}^2 \\ &= E(u(t^*), \rho(t^*)) - \sum_{i=1}^{N}E(\varphi_{c_i}, 0) - 4\sum_{i=1}^{N}c_i(M_i - c_i) + O(e^{-\frac{L}{4}})\end{aligned}$$

$$\leq E(u_0,\rho_0) - \sum_{i=1}^{N} E(\varphi_{c_i},0) + 4\sum_{i=1}^{N} c_i|M_i - c_i| + O(e^{-\frac{L}{4}})$$
$$\leq O(\varepsilon^2) + O(\varepsilon) + O(L^{-\frac{1}{4}})$$
$$\leq O(\varepsilon) + O(L^{-\frac{1}{4}}).$$

Hence, for $0 < \varepsilon < \varepsilon_0, L > L_0$ where $0 < \varepsilon_0 \ll 1, L_0 \gg 1$ are depending only on $(c_i)_{i=1}^N$, we conclude that

$$\|u(t^*,x) - \sum_{i=1}^{N} \varphi_{c_i}(x - \xi_i(t^*))\|_{H^1(\mathbb{R})} + \|\rho(t^*,x)\|_{L^2(\mathbb{R})} \leq C(\sqrt{\varepsilon} + L^{-\frac{1}{8}}).$$

where C is independent of A. Then taking $A = 2C$ concludes the proof of Theorem 2.1.2. □

Proof of Corollary 2.1.4. Since $p_1^0, \cdots, p_N^0$ are positive real numbers and $q_1^0 < \cdots < q_N^0$, then we know that these relations hold for any time t and different peakons never overlap [41]. From the asymptotics of p_i and q_i [4], we deduce that there exists $T > 0$ such that $q_i(T) - q_{i-1}(T) > L$ and $q_i(-T) - q_{i-1}(-T) > L$ with $L > L_0$ large enough. Hence, according to the invariant transformation $(t,x) \mapsto (-t,-x)$ of system (1.5) and the continuity argument while proving Theorem 2.1.2, there exists $\delta > 0$ such that if (u_0, ρ_0) satisfies initial condition (2.4), then for any time t, it holds

$$\|u(t,x) - \sum_{j=1}^{N} p_j e^{|\cdot - q_j|}\|_{H^1(\mathbb{R})} + \|\rho(t,x)\|_{L^2(\mathbb{R})} \leq \frac{\varepsilon}{2A},$$

which implies the orbital stability result (2.5). Furthermore, it is shown in [4] that

$$\lim_{t\to+\infty} p_i(t) = \lim_{t\to+\infty} \dot{q}_i(t) = \lambda_i \tag{2.59}$$

and

$$\lim_{t\to-\infty} p_i(t) = \lim_{t\to-\infty} \dot{q}_i(t) = \lambda_{N+1-i}, \tag{2.60}$$

where $0 < \lambda_1 < \cdots < \lambda_N$ are the eigenvalues of the matrix $\left(p_j(0)e^{-|q_i(0)-q_j(0)|/2}\right)_{i,j}$. Hence, it is straight forward to obtain (2.6) and (2.7). □

2.2 Stability of multi-antipeakons-peakons

2.2.1 Basic definitions and results

As outlined in the introduction, equation (1.3) admits peakon as well as antipeakon when $\kappa = 0$. Inspired by the stability of N-antipeakon-peakons concerned in [28], we now show the stability results of antipeakons and peakons in a dynamic system (1.5).

Theorem 2.2.1. *Let $c_1, \cdots, c_N$ be N non-vanishing velocities such that $c_1 < \cdots < c_k < 0 < c_{k+1} < \cdots < c_N$. There exist $A > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$ such that if $(u, \rho) \in C([0,T); H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R}))$, for $s > \frac{3}{2}$ is a solution of the 2CH system* (1.5) *with initial data $(u, \rho)|_{t=0} = (u_0, \rho_0)$ satisfying*

$$\|u_0 - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - z_i^0)\|_{H^1(\mathbb{R})} + \|\rho_0\|_{L^2(\mathbb{R})} \leq \varepsilon^2, \tag{2.61}$$

for some $0 < \varepsilon < \varepsilon_0$ and $z_i^0 - z_{i-1}^0 \geq L$, with $L > L_0$, then there exist $\xi_1(t), \cdots, \xi_N(t) \in \mathbb{R}$, such that

$$\sup_{0<t<T} \left(\|u(t,\cdot) - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - \xi_i(t))\|_{H^1(\mathbb{R})} + \|\rho(t,\cdot)\|_{L^2(\mathbb{R})} \right) \leq A(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \tag{2.62}$$

where $\xi_i(t) - \xi_{i-1}(t) > \frac{L}{2}$ and T depends only on initial data (u_0, ρ_0).

Moreover, the analogue stability result of CH-multi-antipeakon-peakons for the system (1.5) is discovered as well.

Corollary 2.2.2. *Let $p_1^0, \cdots, p_k^0$ be k negative real numbers, $p_{k+1}^0, \cdots, p_N^0$ be $N-k$ positive real numbers and $q_1^0 < \cdots < q_N^0$ be N real numbers. For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $(u_0, \rho_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > \frac{3}{2}$ satisfies*

$$\|u_0 - \sum_{j=1}^{N} p_j^0 \exp(\cdot - q_j^0)\|_{H^1(\mathbb{R})} + \|\rho_0\|_{L^2(\mathbb{R})} \leq \delta, \quad \text{and} \quad \rho_0(x) > 0,$$

then

$$\forall t > 0, \quad \inf_{p_1<\cdots<p_k<0<p_{k+1}<\cdots<p_N, q_1<\cdots<q_N} \|u(t,\cdot) - \sum_{j=1}^{N} p_j \exp(\cdot - q_j)\|_{H^1(\mathbb{R})} + \|\rho(t,\cdot)\|_{L^2(\mathbb{R})} \leq \varepsilon,$$

where $p_i(t)$ and $q_i(t)$ satisfy the Hamiltonian system (1.4).

Moreover, there exists $T > 0$, such that

$$\forall t \geq T, \quad \inf_{q_1 < \cdots < q_N} \|u(t,\cdot) - \sum_{j=1}^{N} \lambda_j \exp(\cdot - q_j)\|_{H^1(\mathbb{R})} + \|\rho(t,\cdot)\|_{L^2(\mathbb{R})} \leq \varepsilon,$$

where $\lambda_1 < \cdots < \lambda_N$ are the eigenvalues of the matrix $\left(p_j^0 e^{-|q_i^0 - q_j^0|/2}\right)_{1 \leq i,j \leq N}$.

2.2.2 Preliminary lemmas

The proof of Theorem 2.2.1 will be based on a series of lemmas. Firstly, we present the translation of each peakon and antipeakon analogous to Lemma 2.1.4. The detailed proof refers to Lemma 2.1.4 as well.

Lemma 2.2.3. *Let (u_0, ρ_0) be the initial value to system* (1.5) *satisfying* (2.61). *There exist $\alpha_0 > 0$ and $L_0 > 0$ such that for all $0 < \alpha < \alpha_0$ and $0 < L_0 < L$, if $(u, \rho) \in U(\alpha, \frac{L}{2})$ on $[0, t^*]$ for some $0 < t^* < T$, then there exist C^1 functions $\tilde{x}_1(t), \cdots, \tilde{x}_N(t) \in \mathbb{R}$ defined on $[0, t^*]$ such that for all $t \in [0, t^*]$,*

$$\|u(t,\cdot) - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - \tilde{x}_i(t))\|_{H^1(\mathbb{R})} + \|\rho(t,\cdot)\|_{L^2(\mathbb{R})} = O(\sqrt{\alpha}), \tag{2.63}$$

$$|\dot{\tilde{x}}_i(t) - c_i| \leq O(\sqrt{\alpha}) + O(L^{-1}), \; i = 1, \cdots, N, \tag{2.64}$$

and

$$\tilde{x}_i(t) - \tilde{x}_{i-1}(t) \geq \frac{3L}{4} + \frac{(c_i - c_{i-1})t}{2}, \; i \geq 2. \tag{2.65}$$

Moreover, for $i = 1, \cdots, N$, it holds

$$|\xi_i(t) - \tilde{x}_i(t)| = O(1). \tag{2.66}$$

where $\xi_i(t) \in [\tilde{x}_i(t) - \frac{L}{4}, \tilde{x}_i(t) + \frac{L}{4}]$ is any point such that

$$|u(t, \xi_i(t))| = \max_{[\tilde{x}_i(t) - \frac{L}{4}, \tilde{x}_i(t) + \frac{L}{4}]} |u(t)|. \tag{2.67}$$

Secondly, we state the almost monotonicity of energy functionals which are different with those in Lemma 2.1.5. Due to (2.43) is no longer hold, where $M_i = 0$ for antipeakon part, we will consider all positive bumps and negative bumps separately. Define the energy functionals $I_{j,\lambda}(t)$ as follows

$$I_{j,\lambda}(t) = I_{j,\lambda,K}(u,\rho) = \int_{\mathbb{R}} \left[\left(u^2 + u_x^2 + \rho^2\right) - \lambda\left(u^3 + uu_x^2 + u\rho^2\right)\right] \Psi_{j,K} dx, \tag{2.68}$$

where $\Psi_{j,K}(t,x) = \Psi_K(x - y_j(t))$, $j = k+1, \cdots, N$, with $y_j(t)$ defined by

$$y_{k+1}(t) = \tilde{x}_{k+1}(0) + c_{k+1}t/2 - L/4, \qquad y_{N+1}(t) = +\infty,$$

and

$$y_i(t) = \frac{\tilde{x}_{i-1}(t) + \tilde{x}_i(t)}{2}, i = k+2, \cdots, N.$$

To prove the almost monotonicity property, we start with the following lemma.

Lemma 2.2.4. *Assume g is a smooth space function. Then there holds*

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} (u^3 + uu_x^2 + u\rho^2) g dx \\ &\quad = \int_{\mathbb{R}} (\frac{1}{4}u^4 + u^2u_x^2) g' dx + \int_{\mathbb{R}} u^2 h g' dx + \int_{\mathbb{R}} (h^2 - h_x^2) g' dx + \int_{\mathbb{R}} u^2\rho^2 g' dx, \end{aligned} \tag{2.69}$$

where $h = (1-\partial_x^2)^{-1}(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2)$.

Proof. Since g is a smooth space function, using integration by parts and (2.34), we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} (u^3 + uu_x^2 + u\rho^2) g dx \\ &= \int_{\mathbb{R}} 3u^2 u_t g dx + \int_{\mathbb{R}} u_t u_x^2 g dx + \int_{\mathbb{R}} 2uu_x u_{xt} g dx + \int_{\mathbb{R}} u_t \rho^2 g dx + \int_{\mathbb{R}} 2u\rho\rho_t g dx \\ &= 2\int_{\mathbb{R}} u_t(u^2 + \frac{u_x^2}{2}) g dx + \int_{\mathbb{R}} (u_t - u_{txx}) u^2 g dx - \int_{\mathbb{R}} u_{tx} u^2 g' dx + \int_{\mathbb{R}} u_t \rho^2 g dx \\ &\qquad + \int_{\mathbb{R}} 2u\rho\rho_t g dx \\ &= 2\int_{\mathbb{R}} u_t(u^2 + \frac{u_x^2}{2} + \frac{\rho^2}{2}) g dx + \int_{\mathbb{R}} (u_t - u_{txx}) u^2 g dx - \int_{\mathbb{R}} u_{tx} u^2 g' dx + \int_{\mathbb{R}} 2u\rho\rho_t g dx \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Notice that (2.35) implies $F_u' = (1-\partial_x^2)u^2 + 2u^2 + u_x^2 + \rho^2$. It follows that I_1 can be written as:

$$\begin{aligned}
I_1 &= 2\int_{\mathbb{R}} u_t(u^2 + \frac{u_x^2}{2} + \frac{\rho^2}{2})g dx \\
&= 2\int_{\mathbb{R}} \left[-uu_x - (1-\partial_x^2)^{-1}\partial_x\left(u^2 + \frac{u_x^2}{2} + \frac{\rho^2}{2}\right)\right](u^2 + \frac{u_x^2}{2} + \frac{\rho^2}{2})g dx.
\end{aligned}$$

From $h = (1-\partial_x^2)^{-1}\left(u^2 + \frac{u_x^2}{2} + \frac{\rho^2}{2}\right)$, it is deduced that

$$\begin{aligned}
I_1 = &-2\int_{\mathbb{R}} uu_x(u^2 + \frac{u_x^2}{2} + \frac{\rho^2}{2})g dx - 2\int_{\mathbb{R}} gh_x(1-\partial_x^2)h dx \\
= &-2\int_{\mathbb{R}} u^3u_x g dx - \int_{\mathbb{R}} uu_x^3 g dx - \int_{\mathbb{R}} uu_x\rho^2 g dx - 2\int_{\mathbb{R}} hh_x g dx + 2\int_{\mathbb{R}} h_x h_x x g dx \\
= &\frac{1}{2}\int_{\mathbb{R}} u^4 g' dx - \int_{\mathbb{R}} uu_x^3 g dx - \int_{\mathbb{R}} uu_x\rho^2 g dx + \int_{\mathbb{R}} (h^2 - h_x^2)g' dx.
\end{aligned}$$

According to integration by parts and the abstract Hamiltonian structure (2.34) of system (1.5), I_2 performs as follows:

$$\begin{aligned}
I_2 = &\int_{\mathbb{R}} \left[-\frac{3}{2}(u^2)_x + \frac{1}{2}(u^2)_{xxx} - \frac{1}{2}(u_x^2)_x - \frac{1}{2}(\rho^2)_x\right]u^2 g dx \\
= &-3\int_{\mathbb{R}} u^3u_x g dx - \frac{1}{2}\int_{\mathbb{R}} \partial_x(u_x^2)u^2 g dx + \frac{1}{2}\int_{\mathbb{R}} \partial_x^3(u^2)u^2 g dx - \frac{1}{2}\int_{\mathbb{R}} (\rho^2)xu^2 g dx \\
= &\frac{3}{4}\int_{\mathbb{R}} u^4 g' dx + \int_{\mathbb{R}} uu_x^3 g dx + \frac{1}{2}\int_{\mathbb{R}} u^2u_x^2 g' dx - \frac{1}{2}\int_{\mathbb{R}} \partial_x^2(u^2)\partial_x(u^2) g dx \\
&-\frac{1}{2}\int_{\mathbb{R}} \partial_x^2(u^2)u^2 g' dx - \frac{1}{2}\int_{\mathbb{R}} (\rho^2)_x u^2 g dx \\
= &\frac{3}{4}\int_{\mathbb{R}} u^4 g' dx + \int_{\mathbb{R}} uu_x^3 g dx + \frac{1}{2}\int_{\mathbb{R}} u^2u_x^2 g' dx - \frac{1}{4}\int_{\mathbb{R}} (\partial_x(u^2))^2 g' dx \\
&+\int_{\mathbb{R}} \partial_x(u^2)uu_x g' dx + \frac{1}{2}\int_{\mathbb{R}} \partial_x(u^2)u^2 g'' dx - \frac{1}{2}\int_{\mathbb{R}} (\rho^2)_x u^2 g dx \\
= &\frac{3}{4}\int_{\mathbb{R}} u^4 g' dx + \int_{\mathbb{R}} uu_x^3 g dx + \frac{1}{2}\int_{\mathbb{R}} u^2u_x^2 g' dx + \int_{\mathbb{R}} u^2u_x^2 g' dx \\
&+2\int_{\mathbb{R}} u^2u_x^2 g' dx + \int_{\mathbb{R}} u^3u_x g'' dx - \frac{1}{2}\int_{\mathbb{R}} (\rho^2)_x u^2 g dx \\
= &\frac{3}{4}\int_{\mathbb{R}} u^4 g' dx - \frac{1}{4}\int_{\mathbb{R}} u^4 g''' dx + \frac{7}{2}\int_{\mathbb{R}} u^2u_x^2 g' dx + \int_{\mathbb{R}} uu_x^3 g dx - \frac{1}{2}\int_{\mathbb{R}} (\rho^2)_x u^2 g dx.
\end{aligned}$$

Proceeding similarly, we derive that

$$
\begin{aligned}
I_3 =& \int_{\mathbb{R}} \partial_x \left[uu_x + (1-\partial_x^2)^{-1}\partial_x \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 \right) \right] u^2 g' dx \\
=& \int_{\mathbb{R}} \partial_x (uu_x) u^2 g' dx + \int_{\mathbb{R}} (1-\partial_x^2)^{-1}\partial_x \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 \right) u^2 g' dx \\
=& -2\int_{\mathbb{R}} u^2 u_x^2 g' dx - \int_{\mathbb{R}} u^3 u_x g'' dx - \int_{\mathbb{R}} u^2 \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 \right) g' dx + \int_{\mathbb{R}} u^2 h g' dx \\
=& -2\int_{\mathbb{R}} u^2 u_x^2 g' dx + \frac{1}{4}\int_{\mathbb{R}} u^4 g''' dx - \int_{\mathbb{R}} u^4 g' dx - \frac{1}{2}\int_{\mathbb{R}} u^2 u_x^2 g' dx - \frac{1}{2}\int_{\mathbb{R}} u^2 \rho^2 g' dx \\
& + \int_{\mathbb{R}} u^2 h g' dx \\
=& -\frac{5}{2}\int_{\mathbb{R}} u^2 u_x^2 g' dx + \frac{1}{4}\int_{\mathbb{R}} u^4 g''' dx - \int_{\mathbb{R}} u^4 g' dx - \frac{1}{2}\int_{\mathbb{R}} u^2 \rho^2 g' dx + \int_{\mathbb{R}} u^2 h g' dx.
\end{aligned}
$$

From (2.35), I_4 employs the following structure

$$
\begin{aligned}
I_4 =& -\int_{\mathbb{R}} 2u\rho(\rho u)_x g dx - 2\int_{\mathbb{R}} u\rho(\rho u_x + \rho_x u) g dx = -2\int_{\mathbb{R}} uu_x \rho^2 g dx - 2\int_{\mathbb{R}} u^2 \rho\rho_x g dx \\
=& -\int_{\mathbb{R}} u^2 (\rho^2)_x g dx + \int_{\mathbb{R}} u^2 (\rho^2) g' dx - 2\int_{\mathbb{R}} u^2 \rho\rho_x g dx = \int_{\mathbb{R}} u^2 \rho^2 g' dx.
\end{aligned}
$$

In conclusion, combining I_1, I_2, I_3 and I_4, one obtains

$$
\begin{aligned}
I_1 + & I_2 + I_3 + I_4 \\
&= \frac{1}{2}\int_{\mathbb{R}} u^4 g' dx - \int_{\mathbb{R}} uu_x^3 g dx - \int_{\mathbb{R}} uu_x \rho^2 g dx + \int_{\mathbb{R}} (h^2 - h_x^2) g' dx \\
&= \frac{3}{4}\int_{\mathbb{R}} u^4 g' dx - \frac{1}{4}\int_{\mathbb{R}} u^4 g''' dx + \frac{7}{2}\int_{\mathbb{R}} u^2 u_x^2 g' dx + \int_{\mathbb{R}} uu_x^3 g dx - \frac{1}{2}\int_{\mathbb{R}} (\rho^2)_x u^2 g dx \\
&\qquad - \frac{5}{2}\int_{\mathbb{R}} u^2 u_x^2 g' dx + \frac{1}{4}\int_{\mathbb{R}} u^4 g''' dx - \int_{\mathbb{R}} u^4 g' dx - \frac{1}{2}\int_{\mathbb{R}} u^2 \rho^2 g' dx + \int_{\mathbb{R}} u^2 h g' dx \\
&\qquad + \int_{\mathbb{R}} u^2 \rho^2 g' dx \\
&= \frac{1}{4}\int_{\mathbb{R}} u^4 g' dx + \int_{\mathbb{R}} u^2 u_x^2 g' dx + \int_{\mathbb{R}} u^2 \rho^2 g' dx + \int_{\mathbb{R}} u^2 h g' dx + \int_{\mathbb{R}} (h^2 - h_x^2) g' dx.
\end{aligned}
$$

Hence, the proof of Lemma 2.2.4 is completed. □

Lemma 2.2.5. *Let* $(u, \rho) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ *with* $s > \frac{3}{2}$ *be a solution of system (1.5) satisfying (2.63) on* $[0, t^*]$*. There exist* $\alpha_0 > 0$ *and* $L_0 > 0$ *only depending on*

$c_{k+1}, c_{k+2}, \cdots, c_N$ *such that, if* $0 < \alpha < \alpha_0$ *and* $L > L_0$*, then for any* $4 \le K = O(L^{1/2})$ *and* $0 \le \lambda \le \frac{2}{c_{k+1}}$*, there holds*

$$I_{j,\lambda,K}(t) - I_{j,\lambda,K}(0) \le O(e^{-\frac{\sigma_0 L}{8K}}), \text{ for } j \in \{k+1, \cdots, N\} \text{ and } t \in [0, t^*]. \quad (2.70)$$

Proof. Similar to the almost monotonicity studied in previous section, the estimate on $\frac{d}{dt} I_{j,\lambda,K}(t)$ is crucial. Denote $h = (1 - \partial_x^2)^{-1}(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2)$. Applying $g = \Psi_{j,K}$, $j \ge k+1$ to Lemma 2.2.4, which along with (2.37) gives rise to

$$\begin{aligned}
\frac{d}{dt} I_{j,\lambda,K} =& \frac{d}{dt}\int_{\mathbb{R}} \Psi_{j,K}\left[(u^2 + u_x^2 + \rho^2) - \lambda(u^3 + uu_x^2 + u\rho^2)\right] dx \\
=& -\dot{y}_j \int_{\mathbb{R}} \partial_x \Psi_{j,K}\left[(u^2 + u_x^2 + \rho^2) - \lambda(u^3 + uu_x^2 + u\rho^2)\right] dx \\
&+ \int_{\mathbb{R}} uu_x^2 \partial_x \Psi_{j,K} dx + \int_{\mathbb{R}} u\rho^2 \partial_x \Psi_{j,K} dx + 2\int_{\mathbb{R}} uh \partial_x \Psi_{j,K} dx \\
&- \lambda\Big[\int_{\mathbb{R}} (\frac{1}{4}u^4 + u^2u_x^2)\partial_x \Psi_{j,K} dx + \int_{\mathbb{R}} u^2 h \partial_x \Psi_{j,K} dx \\
&+ \int_{\mathbb{R}} (h^2 - h_x^2)\partial_x \Psi_{j,K} dx + \int_{\mathbb{R}} u^2\rho^2 \partial_x \Psi_{j,K} dx\Big] \\
=& -\dot{y}_j \int_{\mathbb{R}} \partial_x \Psi_{j,K}(u^2 + u_x^2 + \rho^2) dx - \lambda \int_{\mathbb{R}} \partial_x \Psi_{j,K}(h^2 - h_x^2) dx \\
&+ \int_{\mathbb{R}} \partial_x \Psi_{j,K}(2u - \lambda u^2) h dx + \int_{\mathbb{R}} \partial_x \Psi_{j,K}\Big[u(u_x^2 + \rho^2) \\
&+ \lambda\Big(\dot{y}_j(u^3 + uu_x^2 + u\rho^2) - (\frac{1}{4}u^4 + u^2u_x^2 + u^2\rho^2)\Big)\Big] dx \\
:=& -\dot{y}_j \int_{\mathbb{R}} \partial_x \Psi_{j,K}(u^2 + u_x^2 + \rho^2) dx + J_1 + J_2 + J_3 \\
\le& -\frac{c_{k+1}}{2} \int_{\mathbb{R}} \partial_x \Psi_{j,K}(u^2 + u_x^2 + \rho^2) dx + J_1 + J_2 + J_3,
\end{aligned}$$

where

$$\begin{aligned}
J_1 =& -\lambda \int_{\mathbb{R}} \partial_x \Psi_{j,K}(h^2 - h_x^2) dx, \quad J_2 = \int_{\mathbb{R}} \partial_x \Psi_{j,K}(2u - \lambda u^2) h dx, \\
J_3 =& \int_{\mathbb{R}} \partial_x \Psi_{j,K}\left[u(u_x^2 + \rho^2) + \lambda\left(\dot{y}_j(u^3 + uu_x^2 + u\rho^2) - (\frac{1}{4}u^4 + u^2u_x^2 + u^2\rho^2)\right)\right] dx.
\end{aligned}$$

By repeating the same procedure in Lemma 2.1.5, one obtain $J_3 \leq 0$ and $J_i \leq \frac{c_{k+1}}{4}\int_{\mathbb{R}} \Psi'_{j,K}(u^2+u_x^2+\rho^2)dx + \frac{C}{K}e^{-\frac{1}{K}(\sigma_0 t+L/8)}$, where $i=1,2$, which implies

$$\frac{d}{dt}I_{j,\lambda,K} \leq \frac{C}{K}\left(\|u_0\|^2_{H^1(\mathbb{R})}+\|\rho_0\|^2_{L^2(\mathbb{R})}\right)e^{-\frac{1}{K}(\sigma_0 t+\frac{L}{8})},$$

Hence

$$I_{j,\lambda,K}(t) - I_{j,\lambda,K}(0) \leq O(e^{-\frac{\sigma_0 L}{8K}}). \tag{2.71}$$

This then completes the proof of Lemma 2.2.4. □

Thirdly, we establish the local and global estimates analogous to Lemma 2.1.6, Lemma 2.1.7 and Lemma 2.1.8.

Lemma 2.2.6. *Given $N-k$ real numbers $y_{k+1} < \cdots < y_N$ with $y_i - y_{i-1} \geq \frac{2L}{3}$, for $i=k+1,\cdots,N$. Define interval $\mathcal{J}_i = (y_i - \frac{L}{4}, y_{i+1}+\frac{L}{4})$. Assume for any fixed function $(u,\rho) \in H^s(\mathbb{R})\times H^{s-1}(\mathbb{R})$ with $s>3/2$, there exists $\xi_i \in \mathcal{J}_i$, where $i=k+1,\cdots,N$, such that*

$$u(\xi_i) = \max_{x\in\mathcal{J}_i} u(x) := M_i \quad \text{and} \quad |\xi_i - \tilde{x}_i| = O(1).$$

Then, for each $i=k+1,\cdots,N$, there holds

$$F_i(u,\rho) \leq M_i E_i(u,\rho) - \frac{2}{3}M_i^3 + (\|u_0\|^2_{H^1(\mathbb{R})}+\|\rho_0\|^2_{L^2(\mathbb{R})})^{\frac{3}{2}}O(L^{-\frac{1}{2}}). \tag{2.72}$$

Lemma 2.2.7. *For any $(\xi_1,\cdots,\xi_N)\in\mathbb{R}^N$, with $\xi_k < y_{k+1}-\frac{L}{4}$ and $\xi_i - \xi_{i-1} > \frac{L}{2}$, $i=k+1,\cdots,N$, and any $(u,\rho)\in H^s(\mathbb{R})\times H^{s-1}(\mathbb{R}), s>\frac{3}{2}$, we have*

$$E_i(u,\rho) - E(\varphi_{c_i},0) = E_i(u-R_X,\rho) + 4c_i(M_i - c_i) + \left(\|u\|^2_{H^1(\mathbb{R})}+\|\rho\|^2_{L^2(\mathbb{R})}\right)O(L^{-1/2}). \tag{2.73}$$

Lemma 2.2.8. *Let (u,ρ) be the solution of system (1.5) such that $(u,\rho)\in U(\alpha,\frac{L}{2})$ on $[0,t^*]$, with initial data (u_0,ρ_0) satisfying (2.61). Denote $M_i(t) = \max_{x\in\mathcal{J}_i(t)} u(t,x) = u(t,\xi_i(t))$, for all $t\in[0,t^*]$ and $i\in k+1,\cdots,N$. Then we have*

$$\sum_{i=k+1}^{N} c_i|M_i - c_i| \leq O(\varepsilon)+O(L^{-\frac{1}{4}}), \text{ for all } t\in[0,t^*], \tag{2.74}$$

where $O(\cdot)$ depends on $(c_i)_{i=k+1}^N$ and $\left(\|u\|^2_{H^1(\mathbb{R})}+\|\rho\|^2_{L^2(\mathbb{R})}\right)^{\frac{3}{2}}$.

2.2.3 Proof of stability theorem

The following presents the detail proof of Theorem 2.2.1.

Proof of Theorem 2.2.1. In view of the continuity argument, our attention will focus on the estimate at $t=t^*$ in the following procedure. Let $K=\sqrt{L}/8$. Taking summation for i from $k+1$ to N of (2.73), from Lemma 2.2.8, it is deduced that

$$I_{k+1,0}(t^*)-\sum_{i=k+1}^N E(\varphi_{c_i},0)=\sum_{i=k+1}^N E_i\left(u(t^*)-R_{X(t^*)},\rho(t^*)-0\right)+O(\varepsilon)+O(e^{-\frac{L}{4}}). \tag{2.75}$$

The monotonicity of $I_{k+1,0}(t^*)$ respect to t ensures that

$$\sum_{i=k+1}^N E_i(u(t^*)-R_{X(t^*)},\rho(t^*)-0)\le I_{k+1,0}(0)-\sum_{i=k+1}^N E(\varphi_{c_i},0)+O(\varepsilon)+O(e^{-\frac{L}{4}}).$$

In view of (2.63), we reveal

$$\sum_{i=k+1}^N E_i(u(t^*)-R_{X(t^*)},\rho(t^*)-0)\le O(\varepsilon)+O(e^{-\frac{L}{4}}). \tag{2.76}$$

Finally, following (2.75) and (2.76), it is deduced that

$$I_{k+1,0}(t^*)=\sum_{i=k+1}^N E(\varphi_{c_i},0)+O(\varepsilon)+O(e^{-\frac{L}{4}}). \tag{2.77}$$

It is worth noticing that the system (1.5) is invariant by the change of $u(x,t)\mapsto -u(-x,t)$ and $\rho(x,t)\mapsto-\rho(-x,t)$. Then for any $(u,\rho)\in H^s(\mathbb{R})\times H^{s-1}(\mathbb{R})$, there holds

$$I^-_{k,0}(t^*):=\int_{\mathbb{R}}\Psi(y_k(t^*)-x)[u^2(x)+u_x^2(x)+\rho^2(x)]\,dx,\quad y_k(t^*)=\tilde{x}_k(0)+c_kt^*/2+L/4,$$

where u describes only negative bumps. With the same process used above, we establish the following two estimates

$$\sum_{i=1}^k E_i(u(t^*)-R_{X(t^*)},\rho(t^*)-0)\le O(\varepsilon)+O(e^{-\frac{L}{4}}), \tag{2.78}$$

and

$$I_{k,0}^{-}(t^*) = \sum_{i=1}^{k} E(\varphi_{c_i}, 0) + O(\varepsilon) + O(e^{-\frac{L}{4}}). \tag{2.79}$$

Hence, from (2.77) and (2.79), it is derived that

$$\begin{aligned} I_{k,0}^{-}(t^*) + I_{k+1,0}(t^*) &= \sum_{i=1}^{N} E(\varphi_{c_i}, 0) + O(\varepsilon) + O(e^{-\frac{L}{4}}) = E(u_0, \rho_0) + O(\varepsilon) + O(e^{-\frac{L}{4}}) \\ &= E(u(t^*), \rho(t^*)) + O(\varepsilon) + O(e^{-\frac{L}{4}}), \end{aligned}$$

which implies

$$\int_{\mathbb{R}} [1 - \Psi(y_k(t^*) - x) - \Psi(x - y_{k+1}(t^*))]\,(u^2 + u_x^2 + \rho^2)\,dx = O(\varepsilon) + O(e^{-\frac{L}{4}}).$$

It is worth to remark that the way how we construct Ψ infers $|1 - \Psi(y_k(t^*) - x) - \Psi(x - y_{k+1}(t^*))| \le O(e^{-\frac{L}{2}})$ for $x \in (-\infty, y_k - \frac{L}{4}] \cup [y_{k+1} + \frac{L}{4}, +\infty)$. In addition, the exponential decay of φ_{c_i} and Lemma 2.2.3 lead to

$$\int_{y_k - L/4}^{y_{k+1}+L/4} |R_X|^2 + |\partial_x R_X|^2\,dx \le O(e^{-\frac{L}{4}}).$$

Hence, we obatin

$$\begin{aligned} &\int_{\mathbb{R}} \Big(1 - \Psi(y_k(t^*) - x) - \Psi(x - y_{k+1}(t^*))\Big)\,\big((u - R_X)^2 + (u_x - \partial_x R_X)^2 + (\rho^2 - 0)\big)\,dx \\ &\quad = O(\varepsilon) + O(e^{-\frac{L}{4}}). \end{aligned} \tag{2.80}$$

By using (2.76), (2.78) and (2.80), we conclude

$$E\Big(u(t^*) - R_{X(t^*)}, \rho(t^*) - 0\Big) = C(\varepsilon + e^{-\frac{L}{4}}),$$

where the positive C depends only on $\{c_i\}_{i=1}^N$ and $E(u_0, \rho_0)$, not on A. Consequently, the theorem is proved by choosing $A = 2C$. □

With Theorem 2.2.1 in hand, by repeating the asymptotic analysis in the proof of Corollary 2.1.3, we may readily prove Corollary 2.2.2. We omit the details of the proof of this result.

CHAPTER 3

STABILITY OF THE TRAINS OF N-SMOOTH TRAVELING WAVE TO THE GENERALIZED TWO-COMPONENT CAMASSA-HOLM SYSTEM

3.1 Basic definition and results

In this chapter, we mainly consider the Cauchy problem of the two-component CH system on the real line, that is

$$\begin{cases} u_t - u_{xxt} - Au_x + 3uu_x - \sigma\left(2u_xu_{xx} + uu_{xxx}\right) + (1+\eta)\,\eta_x = 0, \\ \eta_t + \left((1+\eta)\,u\right)_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\ (u(x,0), \eta(x,0)) = (u_0(x), \eta_0(x)). \end{cases} \tag{3.1}$$

Denote

$$p(x) := \frac{1}{2}e^{-|x|}, \quad x \in \mathbb{R}.$$

Then for all $f \in L^2(\mathbb{R})$,

$$(1-\partial_x^2)^{-1} f = p * f, \tag{3.2}$$

where '$*$' denotes the spatial convolution. Using this notation, system (1.8) can be now written in the following form

$$\begin{cases} u_t + \sigma uu_x + \partial_x p * \left(-Au + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1}{2}\left(1+\eta\right)^2\right) = 0, \\ \eta_t + \left((1+\eta)u\right)_x = 0. \end{cases} \tag{3.3}$$

Logically, prior to a discussion of stability as formulated above in terms of perturbations of the initial data should be a theory for the initial-valued problem itself. This is a subject that has attracted a lot of attention and it is not our purpose to provide a survey of results. The following local existence result suffices for the stability theory developed here (see [11]). More subtle results are available in some cases but these do not concern us here.

Proposition 3.1.1. *[11] Let* $(u_0, \eta_0) \in H^s \times H^{s-1}$, $s > \frac{3}{2}$. *Then there exist a maximal time* $T = T(u_0, \eta_0) > 0$ *and a unique solution* (u, η) *of* (3.1) *in* $C([0,T); H^s \times H^{s-1}) \cap C^1([0,T); H^{s-1} \times H^{s-2})$. *Moreover, the solution depends continuously on the initial data and* T *is independent of* s. *In addition, the functionals* $E(u,\eta)$ *and* $F(u,\eta)$ *defined in* (1.9) *and* (1.10) *are independent of the existence time* t.

In the case $\sigma = 0$, we have the following global existence of the solutions.

Proposition 3.1.2. *[13] Let* $\sigma = 0$. *If* $(u_0, \eta_0) \in H^s \times H^{s-1}$ *with* $s > \frac{3}{2}$, *then there exist a unique global solution* (u, η) *of* (3.1) *in* $C([0,\infty); H^s \times H^{s-1}) \cap C^1([0,\infty); H^{s-1} \times H^{s-2})$. *Moreover, the solution depends continuously on the initial data. In addition, the functionals* $E(u,\eta)$ *and* $F(u,\eta)$ *defined in* (1.9) *and* (1.10) *are independent of the existence time* t.

Next, we provide the definition of traveling wave of system (3.3) and the condition for existence of traveling waves.

Definition 3.1.1. A vector function is a traveling wave of (3.3) if it has the form

$$\boldsymbol{\varphi}_c(x,t) = (\varphi_c(x-ct), \psi_c(x-ct)) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}), \quad c \in \mathbb{R}$$

with φ_c and ψ_c vanishing at infinity along with their first and second derivatives.

One can check that a traveling wave of (3.3) satisfies

$$\begin{cases} \Big(- c\varphi + \frac{\sigma}{2}\varphi^2 + p * \big(-A\varphi + \frac{3-\sigma}{2}\varphi^2 + \frac{\sigma}{2}\varphi_x^2 + \frac{1}{2}(1+\psi)^2\big) \Big)_x = 0, \\ \Big(- c\psi + (1+\psi)\varphi\Big)_x = 0. \end{cases} \tag{3.4}$$

Integrating the above system and applying $(1 - \partial_x^2)$ to the first equation, we have

$$\begin{cases} -(c+A)\varphi + c\varphi_{xx} + \frac{3}{2}\varphi^2 = \sigma\varphi\varphi_{xx} + \frac{\sigma}{2}\varphi_x^2 - \frac{1}{2}(1+\psi)^2 + \frac{1}{2}, \\ -c\psi + (1+\psi)\varphi = 0. \end{cases} \tag{3.5}$$

The following result on existence of traveling waves of (3.3) was given in [13].

Proposition 3.1.3. *[13] Let $\sigma \leq 1$ and assume $c > \frac{-A+\sqrt{A^2+4}}{2}$. Then there exists a smooth traveling wave $\boldsymbol{\varphi}_c = (\varphi_c, \psi_c)$ of (3.3), which decays exponentially to zero at infinity.*

Remark 2. The smooth traveling wave $\boldsymbol{\varphi}_c$ exists either $c > A_1$ or $c < A_2$, where A_1 and A_2 are two roots of the equation $y^2 + Ay - 1 = 0$. Without loss of generality, we only consider the case $c > A_1 = \frac{-A+\sqrt{A^2+4}}{2}$ in this article.

Moreover, we introduce those results associated with the stability of single smooth traveling wave of (3.3). Begin with the dual space of X: $X^* = H^{-1}(\mathbb{R}) \times L^2(\mathbb{R})$, we can define a natural isomorphism I from X to X^*

$$I = \begin{pmatrix} 1 - \partial_x^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then combining the map I and the definition of $H^1(\mathbb{R})$ norm and $L^2(\mathbb{R})$ norm, the pair $\langle \cdot, \cdot \rangle$ between X and X^* can be presented as

$$\langle I\boldsymbol{u}, \boldsymbol{v} \rangle = (u, v)_{H^1(\mathbb{R})} + (\eta, w)_{L^2(\mathbb{R})},$$

where $\boldsymbol{u} = (u, \eta) \in X, \boldsymbol{v} = (v, w) \in X^*$.

According to the definition of norm in X, the quantity $E(\boldsymbol{u})$, which is an invariant functional of (3.1) with respect to time, can be written as

$$E(\boldsymbol{u}) = \frac{1}{2} (\boldsymbol{u}, \boldsymbol{u}) = \frac{1}{2} \|\boldsymbol{u}\|_X^2, \quad u \in X.$$

Furthermore, taking advantage of the conservation law $F(\boldsymbol{u})$, system (3.3) has following abstract Hamiltonian form:

$$\partial_t \boldsymbol{u} = JF'(\boldsymbol{u}), \tag{3.6}$$

where J is a closed skew symmetric operator given by

$$J = \begin{pmatrix} -\partial_x(1-\partial_x^2)^{-1} & 0 \\ 0 & -\partial_x \end{pmatrix},$$

and $F'(\boldsymbol{u}) : X \to X^*$ is the variational derivative of F in X at $\boldsymbol{u}$.

Attention is now given to the definition of orbital stability of a single traveling wave.

Definition 3.1.2. The traveling wave $\boldsymbol{\varphi}_c$ of (3.3) is stable in X if for any $\varepsilon > 0$, there exists $\delta > 0$ and $s_0 \in \mathbb{R}$ such that for any $\boldsymbol{u}_0 \in X$ satisfying

$$\|\boldsymbol{u}_0 - \boldsymbol{\varphi}_c(\cdot - s_0)\|_X < \delta,$$

and if $\boldsymbol{u} \in C([0,T);X)$ for some $0 < T \leq \infty$ is a solution of (3.3) with $\boldsymbol{u}(0) = \boldsymbol{u}_0$, then

$$\inf_{s\in\mathbb{R}} \|\boldsymbol{u}(t) - \boldsymbol{\varphi}_c(\cdot - s)\|_X < \varepsilon, \qquad \forall\, t \in [0,T).$$

Otherwise, the traveling wave $\boldsymbol{\varphi}_c$ is said to be unstable in X.

One key to the stability of a single traveling wave is the coercivity of H_c which is the second differential operator of $cE - F$ around single traveling wave $\boldsymbol{\varphi}_c$. We here rephrase their results and refer the readers to [13] for details.

Lemma 3.1.4. *[13]. Assume $\sigma \leq 1$ and $c > A_1$. Let $\boldsymbol{\varphi}_c(x,t)$ be a smooth traveling wave of* (3.3)*. Then there exists a constant $k = k(c) > 0$ such that*

$$\langle H_c(\boldsymbol{\psi}), \boldsymbol{\psi}\rangle \geq k\,\|\boldsymbol{\psi}\|_X^2, \tag{3.7}$$

for all $\boldsymbol{\psi} \in X$ satisfying $(\boldsymbol{\varphi}_c, \boldsymbol{\psi}) = (\boldsymbol{\varphi}_c', \boldsymbol{\psi}) = 0$.

Lemma 3.1.5. *[13]. Let $\sigma \leq 1$ and $c > A_1$. All those smooth traveling waves of* (3.3) *are orbitally stable in the energy space X.*

Remark 3. The exactly form of H_c in (3.7) is

$$H_c = cE''(\boldsymbol{\varphi}_c) - F''(\boldsymbol{\varphi}_c) = \begin{pmatrix} L_c & -(1+\eta) \\ -(1+\eta) & c-\varphi \end{pmatrix},$$

where

$$L_c = -\partial_x\left((c-\sigma\varphi)\partial_x\right) - 3\varphi + \sigma\varphi_{xx} + c + A.$$

The proofs of Lemma 3.1.4 and Lemma 3.1.5 are based on the spectrum analysis of H_c and the convexity of $d(c) = cE(\boldsymbol{\varphi}_c) - F(\boldsymbol{\varphi}_c)$ both requiring $\sigma \leq 1$. In the following section, we will show the property of local coercivity via Lemma 3.1.4. Therefore, $\sigma \leq 1$ is a prerequisite to establish the stability of the train of N-smooth traveling waves.

The goal of this chapter is to show the orbital stability of the trains of N-smooth traveling waves with the assumption on initial profile which guarantees the existence of solution.

The N-smooth traveling waves $\sum_{i=1}^{N} \boldsymbol{\varphi}_{c_i}$ of system (1.8) can be shown to be orbitally stable in the energy space X, which is the principal result of the present paper.

Theorem 3.1.6. *(Main Result) Suppose $\sigma \leq 1$. Let $c_1, c_2, \cdots, c_N$ be N speeds, such that $0 < \max\{\frac{-A+\sqrt{A^2+4}}{2}, \frac{A+\sqrt{A^2+2}}{2}\} < c_1 < \cdots < c_N$. Let $s > \frac{3}{2}$. If $M > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$, such that for any $\boldsymbol{u_0} = (u_0, \eta_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ satisfies*

$$\left\| \boldsymbol{u}_0 - \sum_{i=1}^{N} \boldsymbol{\varphi}(\cdot - z_j^0) \right\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} \leq \varepsilon, \tag{3.8}$$

for some $0 < \varepsilon < \varepsilon_0$ and $z_j^0 - z_{j-1}^0 \geq L$ with $L \geq L_0$, then for the corresponding solution $\boldsymbol{u} = (u, \eta) \in C([0,T); H^s \times H^{s-1}) \cap C^1([0,T); H^{s-1} \times H^{s-2})$ of the

Cauchy problem for the g2CH system (1.8) *with initial data* $\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0$*, there exist* $\tilde{x}_1(t), \tilde{x}_2(t), \cdots, \tilde{x}_N(t) \in \mathbb{R}$*, such that*

$$\sup_{0<t<T} \left\| \boldsymbol{u}(t,\cdot) - \sum_{i=1}^{N} \boldsymbol{\varphi}(\cdot - \tilde{x}_i(t)) \right\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} \leq M(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \tag{3.9}$$

where T *depends only on initial data* $\boldsymbol{u_0}$*.*

Remark 4. It is noted that the assumptions in Theorem 3.1.6 ensures that the train of N-smooth traveling waves of different speeds is arranged in increasing order and sufficiently decoupled and as time goes by the phase between two traveling waves will be enlarged.

Remark 5. In Theorem 3.1.6, the life-time span T may be infinite if $\sigma = 0$, since we have the global solution in that situation according to Proposition 3.1.2.

3.2 Preliminary lemmas

For $\alpha > 0$ and $L > 0$, we define the following neighborhood of N-smooth traveling waves $\sum_{i=1}^{N} \boldsymbol{\varphi}_{c_i}$ with spatial shifts x_i that satisfies $x_i - x_{i-1} \geq L$ for every $i = 1, \cdots, N$,

$$U(\alpha, L) = \left\{ \boldsymbol{u} \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), \inf_{x_i - x_{i-1} > L} \left\| \boldsymbol{u} - \sum_{i=1}^{N} \boldsymbol{\varphi}_{c_i}(\cdot - x_i) \right\|_X < \alpha \right\}.$$

We want to prove that there exist $M > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$ such that for any $L > L_0$, $0 < \varepsilon < \varepsilon_0$ and any initial data $\boldsymbol{u}_0 \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > \frac{3}{2}$ satisfies initial condition (3.8), then the corresponding solution $\boldsymbol{u}(t)$ belongs to $U\left(M(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \frac{L}{2}\right)$ for all $t \in [0, T)$, where T is the maximal existence time and M is independent of time t. By the continuity of the map $t \mapsto \boldsymbol{u}(t)$ from $[0, T)$ into $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ where $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \subset H^1(\mathbb{R}) \times L^2(\mathbb{R})$, one can demonstrate Theorem 3.1.6 as long as the following proposition is proved.

Proposition 3.2.1. *Let $\sigma, c_1, c_2, \cdots, c_N$ meet the assumption in Theorem 3.1.6. There exist $M > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$ such that for any $\boldsymbol{u}_0 \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > \frac{3}{2}$, if*

$$\left\| \boldsymbol{u}_0 - \sum_{i=1}^{N} \boldsymbol{\varphi}(\cdot - z_j^0) \right\|_X \leq \varepsilon,$$

for some $0 < \varepsilon < \varepsilon_0$ and $z_j^0 - z_{j-1}^0 \geq L$ with $L \geq L_0$ and if for some $0 < t^* < T$, the solution $\boldsymbol{u} \in C([0,T); H^s \times H^{s-1}) \cap C^1([0,T); H^{s-1} \times H^{s-2})$ of Cauchy problem (3.1) with initial value $\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0$ satisfies

$$\boldsymbol{u}(t) \in U\left(M(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \frac{L}{2}\right), \quad \forall t \in [0, t^*], \tag{3.10}$$

then

$$\boldsymbol{u}(t^*) \in U\left(\frac{M}{2}(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \frac{2L}{3}\right). \tag{3.11}$$

where M, L_0 and ε_0 are independent of t^*.

We briefly prove that Proposition 3.2.1 implies the stability result in Theorem 3.1.6. Let M, L_0, ε_0 be chosen as in Proposition 3.2.1 and let $\boldsymbol{u}_0$ satisfy the initial condition (3.8). Then, by continuity of $\boldsymbol{u}(t) \in H^s \times H^{s-1}$, we have $\boldsymbol{u}(t) \in U\left(M(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \frac{L}{2}\right)$ for $t \in [0, \kappa]$ for some $\kappa > 0$. Let

$$t^* = \sup\left\{t \geq 0, \boldsymbol{u}(t') \in U\left(M(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \frac{L}{2}\right), \forall t' \in [0, t]\right\}.$$

Assume $t^* < T$. According to Proposition 3.2.1, we have $\forall t \in [0, t^*], \boldsymbol{u}(t) \in U\left(\frac{M}{2}(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \frac{2L}{3}\right)$. By continuity, there exists $\tau > 0$ such that $\forall t \in [0, t^* + \tau], \boldsymbol{u}(t) \in U\left(\frac{M}{2}(\sqrt{\varepsilon} + L^{-\frac{1}{8}}), \frac{2L}{3}\right)$, which contradicts the definition of t^*. Hence, we conclude $t^* \geq T$ which implies the stability consequence of the trains of N-smooth traveling waves.

In this part, three properties of system (3.3) are established. We start with the modulation argument that as long as $\boldsymbol{u}$ stays in the neighborhood $U(\alpha, \frac{L}{2})$ of the sum of N modulated traveling waves, where $\alpha = O(\sqrt{\varepsilon} + L^{-\frac{1}{8}})$, it can be decomposed as the sum of N modulated traveling waves plus a vector function $\boldsymbol{v} = (v, w)$ which is an infinitesimal in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$, that is,

$$\begin{cases} u(t,x) = \sum_{i=1}^N \varphi_{c_i}(\cdot - \tilde{x}_i(t)) + v(t,x), \\ \eta(t,x) = \sum_{i=1}^N \psi_{c_i}(\cdot - \tilde{x}_i(t)) + w(t,x). \end{cases}$$

Lemma 3.2.2. *Let $\alpha_0 > 0$ and $L_0 > 0$. For any $0 < \alpha < \alpha_0$ and $L > L_0$, if the initial data $\boldsymbol{u}_0 = (u_0, \eta_0)$ satisfies the assumption (3.8) given in Theorem 3.1.6 and the solution $\boldsymbol{u} \in U(\alpha, \frac{L}{2})$ on $[0, t^*]$, then there exist C^1 functions $\tilde{x}_i(t) : [0, t^*] \to \mathbb{R}, i = 1, \cdots, N$ and $t \in [0, t^*]$, such that*

$$\int_{\mathbb{R}} v(t)(1 - \partial_x^2)\partial_x R_i(t)\, dx + \int_{\mathbb{R}} w(t)\partial_x P_i(t)\, dx = 0, \tag{3.12}$$

where $\boldsymbol{R}_i(t,x) = (R_i(t,x), P_i(t,x)) = (\varphi_{c_i}(x - \tilde{x}_i(t)), \psi_{c_i}(x - \tilde{x}_i(t))) = \boldsymbol{\varphi}_i(t,x)$,

$$\|\boldsymbol{v}(t)\|_X \leq O(\sqrt{\alpha}), \tag{3.13}$$

$$\left|\dot{\tilde{x}}_i(t) - c_i\right| \leq O(\sqrt{\alpha}) + O(L^{-1}), \tag{3.14}$$

$$\tilde{x}_i(t) - \tilde{x}_{i-1}(t) \geq \frac{3L}{4} + \frac{(c_i - c_{i-1})}{2} \cdot t, \quad i \geq 2, \tag{3.15}$$

where $\dot{\tilde{x}}_i(t)$ means the derivative of $\tilde{x}_i(t)$ respect to time t.

Proof. We use the standard modulation argument to discover the translations of N-smooth traveling waves. Let $Z = (z_1, \cdots, z_N) \in \mathbb{R}^N$ be fixed such that $z_i - z_{i-1} > \frac{L}{2}$ and set $\boldsymbol{R}_Z(\cdot) = \sum_{i=1}^N \boldsymbol{\varphi}_{c_i}(\cdot - z_i)$. For $0 < \delta_0 < 1$, we define the function

$$Y : \prod_{i=1}^N (-\delta_0, \delta_0) \times B_{H^1}(R_Z, \delta_0) \times B_{L^2}(P_Z, \delta_0) \to \mathbb{R}^N,$$

$$(y_1, \cdots, y_N, u, \eta) \longmapsto \left(Y^1(y_1, \cdots, y_N, u, \eta), \cdots, Y^N(y_1, \cdots, y_N, u, \eta)\right),$$

with

$$Y^j(y_1,\cdots,y_N,u,\eta)=\int_{\mathbb{R}}\left(u-\sum_{i=1}^N\varphi_{c_i}(\cdot-z_i-y_i)\right)(1-\partial_x^2)\partial_x\varphi_{c_j}(\cdot-z_j-y_j)\,dx$$
$$+\int_{\mathbb{R}}\left(\eta-\sum_{i=1}^N\psi_{c_i}(\cdot-z_i-y_i)\right)\partial_x\psi_{c_j}(\cdot-z_j-y_j)\,dx.$$

where $B_{H^1}(R_Z,\delta_0)$ is the ball in $H^1(\mathbb{R})$ with center R_Z and radius δ_0, $B_{L^2}(P_Z,\delta_0)$ is the ball in $L^2(\mathbb{R})$ with center P_Z and radius δ_0. To apply the implicit function theorem, two facts are needed. One fact is that function Y should be C^1 mapping which can be proved by dominated convergence theorem. The other fact is that the matrix of all first-order partial derivatives of function Y at $(0,\cdots,0,R_Z,P_Z)$ should be invertible. For $j=1,\cdots,N$,

$$\frac{\partial Y^j}{\partial y_j}(y_1,\cdots,y_N,u,\eta)=\int_{\mathbb{R}}\left(u_x-\sum_{i=1,i\neq j}^N\partial_x\varphi_{c_i}(\cdot-z_i-y_i)\right)(1-\partial_x^2)\partial_x\varphi_{c_j}(\cdot-z_j-y_j)\,dx$$
$$+\int_{\mathbb{R}}\left(\eta_x-\sum_{i=1,i\neq j}^N\partial_x\psi_{c_i}(\cdot-z_i-y_i)\right)\partial_x\psi_{c_j}(\cdot-z_j-y_j)\,dx,$$

$$\frac{\partial Y^j}{\partial y_i}(y_1,\cdots,y_N,u,\eta)=\int_{\mathbb{R}}\partial_x\varphi_{c_i}(\cdot-z_i-y_i)(1-\partial_x^2)\partial_x\varphi_{c_j}(\cdot-z_j-y_j)\,dx$$
$$+\int_{\mathbb{R}}\partial_x\psi_{c_i}(\cdot-z_i-y_i)\partial_x\psi_{c_j}(\cdot-z_j-y_j)\,dx,\quad\text{where}\quad i\neq j,$$

$$\frac{\partial Y^j}{\partial u}(y_1,\cdots,y_N,u,\eta)=\int_{\mathbb{R}}(1-\partial_x^2)\partial_x\varphi_{c_j}(\cdot-z_j-y_j)\,dx,$$
$$\frac{\partial Y^j}{\partial \eta}(y_1,\cdots,y_N,u,\eta)=\int_{\mathbb{R}}\partial_x\psi_{c_j}(\cdot-z_j-y_j)\,dx.$$

Hence,

$$\frac{\partial Y^j}{\partial y_j}(0,\cdots,0,R_Z,P_Z)=\|\partial_x\varphi_{c_j}\|^2_{H^1(\mathbb{R})}+\|\partial_x\psi_{c_j}\|^2_{L^2(\mathbb{R})},$$
$$\frac{\partial Y^j}{\partial y_i}(0,\cdots,0,R_Z,P_Z)=\left(\partial_x\varphi_{c_i}(\cdot-z_i),\partial_x\varphi_{c_j}(\cdot-z_j)\right)_{H^1(\mathbb{R})}$$
$$+\left(\partial_x\psi_{c_i}(\cdot-z_i),\partial_x\psi_{c_j}(\cdot-z_j)\right)_{L^2(\mathbb{R})}.$$

Furthermore, there exists $L_0 > 0$ such that if $L > L_0$ large enough, we have

$$D_{(y_1,\cdots,y_N)}Y(0,\cdots,0,R_z,P_Z) = D + P \neq 0,$$

where D is an invertible diagonal matrix with

$$\|D^{-1}\| \leq \min_{j=1,\cdots,N}\left(\|\partial_x\varphi_{c_j}\|^2_{H^1(\mathbb{R})} + \|\partial_x\psi_{c_j}\|^2_{L^2(\mathbb{R})}\right)^{-1},$$

and P is a matrix with

$$\|P\| \leq O(e^{-\frac{L}{4}}),$$

which implies $D_{(y_1,\cdots,y_N)}Y(0,\cdots,0,R_Z,P_Z)$ is invertible. Therefore, by the implicit function theorem, there exist $0 < \beta_0 < \delta_0$ and C^1 functions $(y_1(u,\eta),\cdots,y_N(u,\eta))$ from $B(\boldsymbol{R}_Z,\delta_0)$ to a neighborhood of $(0,\cdots,0)$ which are uniquely determined, such that

$$Y(y_1(u,\eta),\cdots,y_N(u,\eta),u,\eta) = 0, \quad \forall (u,\eta) \in B_{H^1}(R_Z,\beta_0) \times B_{L^2}(P_Z,\beta_0).$$

Moreover, there exists $K_0 > 0$ such that if $(u,\eta) \in B_{H^1}(R_Z,\beta_0) \times B_{L^2}(P_Z,\beta_0)$ with $0 < \beta < \beta_0$ the following holds,

$$\sum_{i=1}^{N}|y_i(\boldsymbol{u})| = \sum_{i=1}^{N}|y_i(u,\eta)| \leq K_0\beta, \tag{3.16}$$

where K_0 and β_0 depends on c_1 and L_0. For $(u,\eta) \in B_{H^1}(R_Z,\beta_0) \times B_{L^2}(P_Z,\beta_0)$, setting $\tilde{x}_i(\boldsymbol{u}) = z_i + y_i(\boldsymbol{u})$ and $\beta_0 \leq \min\{\frac{L_0}{8K_0},\delta_0\}$ infers

$$\tilde{x}_i(\boldsymbol{u}) - \tilde{x}_{i-1}(\boldsymbol{u}) = z_i - z_{i-1} + y_i(\boldsymbol{u}) - y_{i-1}(\boldsymbol{u}) \geq \frac{L}{2} - 2K_0\beta_0 \geq \frac{L}{4}. \tag{3.17}$$

Then, we define the modulation of $\boldsymbol{u} = (u,\eta) \in U(\alpha,\frac{L}{2})$ for $L > L_0$ and $0 < \alpha < \alpha_0$ at a fix time t. Indeed, for $0 < \alpha < \alpha_0$, $U(\alpha,\frac{L}{2})$ can be covered as follows

$$U\left(\alpha,\frac{L}{2}\right) \subset \bigcup_{Z\in\mathbb{R}^N, z_i - z_{i-1} > \frac{L}{2}} B_{H^1}(R_Z,2\alpha) \times B_{L^2}(P_Z,2\alpha).$$

Additionally, the modulation of $\boldsymbol{u}$ is uniquely defined due to the uniqueness in the implicit function theorem.

Thus, we define the modulation of the solution $\boldsymbol{u}(t) = (u(t), \eta(t))$ of system (3.3) satisfying $\boldsymbol{u}(t) \in U(\alpha, \frac{L}{2})$ for all $t \in [0, t^*]$ by setting $i = 1, \cdots, N$ and

$$\tilde{x}_i(t) = \tilde{x}_i(\boldsymbol{u}(t)), \quad \boldsymbol{v}(t) = \boldsymbol{u}(t) - \sum_{i=1}^{N} \boldsymbol{\varphi}_{c_i}(\cdot - \tilde{x}_i(t)),$$

where $\boldsymbol{v}$ satisfies the orthogonal condition

$$\int_{\mathbb{R}} v(t)(1-\partial_x^2)\partial_x R_i(t)\, dx + \int_{\mathbb{R}} w(t)\partial_x P_i(t)\, dx = 0,$$

According to the translation $\tilde{x}_i(t)$ defined above, using (3.16), triangle inequality and the smoothness of $\boldsymbol{\varphi}_{c_i}$, the following estimate holds

$$\begin{aligned}
\|\boldsymbol{v}(t)\|_X \leq & \|\boldsymbol{u}(t) - \sum_{i=1}^{N} \boldsymbol{\varphi}_{c_i}(\cdot - z_i)\|_X + \sum_{i=1}^{N} \|\boldsymbol{\varphi}_{c_i}(\cdot - z_i) - \boldsymbol{\varphi}_{c_i}(\cdot - z_i - y_i(\boldsymbol{u}(t)))\|_X \\
& \leq \alpha + \sum_{i=1}^{N} \Bigg(4E(\boldsymbol{u}) - 2\int_{\mathbb{R}} \varphi_{c_i}(\cdot - z_i)\varphi_{c_i}(\cdot - z_i - y_i(\boldsymbol{u}(t)))\, dx \\
& \qquad - 2\int_{\mathbb{R}} \partial_x\varphi_{c_i}(\cdot - z_i)\partial_x\varphi_{c_i}(\cdot - z_i - y_i(\boldsymbol{u}(t)))\, dx \\
& \qquad - 2\int_{\mathbb{R}} \psi_{c_i}(\cdot - z_i)\psi_{c_i}(\cdot - z_i - y_i(\boldsymbol{u}(t)))\, dx \Bigg)^{\frac{1}{2}} \\
& \leq \alpha + O(\sum_{i=1}^{N} |y_i(\boldsymbol{u})|^{\frac{1}{2}}) \leq O(\sqrt{\alpha}).
\end{aligned}$$

Attention is now turn to the speed of $\tilde{x}_i(t)$. In order to show it stays close to c_i, we differentiate the orthogonality condition with respect to t,

$$\begin{aligned}
\left| \int_{\mathbb{R}} v_t(1-\partial_x^2)\partial_x R_i\, dx + \int_{\mathbb{R}} w_t\partial_x P_i\, dx \right| &= \left| \dot{\tilde{x}}_i \left(\int_{\mathbb{R}} v(1-\partial_x^2)\partial_x^2 R_i\, dx + \int_{\mathbb{R}} w\partial_x^2 P_i\, dx \right) \right| \\
&\leq \left| \dot{\tilde{x}}_i \right| O(\|\boldsymbol{v}\|_X) \\
&\leq \left| \dot{\tilde{x}}_i - c_i \right| O(\|\boldsymbol{v}\|_X) + O(\|\boldsymbol{v}\|_X). \qquad (3.18)
\end{aligned}$$

Substituting $\boldsymbol{u}$ by $\boldsymbol{v} + \sum_{i=1}^{N} \boldsymbol{R}_i$ in the system (3.3) leads to

$$\begin{aligned}(1-\partial_x^2)v_t + \sum_{i=1}^{N}(1-\partial_x^2)\partial_t R_i = -\frac{\sigma}{2}(1-\partial_x^2)\partial_x\left((v+\sum_{i=1}^{N}R_i)^2\right)\\ -\partial_x\Bigg(-A(v+\sum_{i=1}^{N}R_i) + \frac{3-\sigma}{2}(v+\sum_{i=1}^{N}R_i)^2 + \frac{\sigma}{2}(v_x+\sum_{i=1}^{N}\partial_x R_i)^2\\ +(w+\sum_{i=1}^{N}P_i) + \frac{1}{2}(w+\sum_{i=1}^{N}P_i)^2\Bigg),\end{aligned}\tag{3.19}$$

and

$$w_t + \sum_{i=1}^{N}\partial_t P_i = -\partial_x\left((v+\sum_{i=1}^{N}R_i) + (v+\sum_{i=1}^{N}R_i)(w+\sum_{i=1}^{N}P_i)\right).\tag{3.20}$$

Since $\boldsymbol{R}_i$ satisfies (3.4), i.e.

$$\begin{aligned}(1-\partial_x^2)\partial_t R_i + (\dot{\tilde{x}}_i - c_i)(1-\partial_x^2)\partial_x R_i = -\frac{\sigma}{2}(1-\partial_x^2)\partial_x(R_i^2) + A\partial_x R_i - \frac{3-\sigma}{2}\partial_x(R_i^2)\\ -\frac{\sigma}{2}(\partial_x R_i)^2 - \partial_x P_i - P_i\partial_x P_i,\end{aligned}\tag{3.21}$$

and

$$\partial_t P_i + (\dot{\tilde{x}}_i - c_i)\partial_x P_i + \partial_x R_i + \partial_x(R_i P_i) = 0,\tag{3.22}$$

combining (3.19)-(3.22), we infer that $\boldsymbol{v}(t) = (v(t), w(t))$ satisfies the following conditions on $[0, t^*]$,

$$\begin{aligned}(1-\partial_x^2)v_t - \sum_{i=1}^{N}(\dot{\tilde{x}}_i - c_i)(1-\partial_x^2)\partial_x R_i = -\frac{\sigma}{2}(1-\partial_x^2)\partial_x\left[(v+\sum_{i=1}^{N}R_i)^2 - \sum_{i=1}^{N}R_i^2\right]\\ -\partial_x\Bigg[\frac{3-\sigma}{2}\Big((v+\sum_{i=1}^{N}R_i)^2 - \sum_{i=1}^{N}R_i^2\Big) - Av + \frac{\sigma}{2}\Big((v_x+\sum_{i=1}^{N}\partial_x R_i)^2\\ -\sum_{i=1}^{N}(\partial_x R_i)^2\Big) + w + \frac{1}{2}(w+\sum_{i=1}^{N}P_i)^2 - \frac{1}{2}\sum_{i=1}^{N}P_i^2\Bigg],\end{aligned}\tag{3.23}$$

and

$$w_t - \sum_{i=1}^{N}(\dot{\tilde{x}}_i - c_i)\partial_x P_i = -\partial_x\left(v + (v+\sum_{i=1}^{N}R_i)(w+\sum_{i=1}^{N}P_i) - \sum_{i=1}^{N}R_i P_i\right).\tag{3.24}$$

Taking L^2-inner product for (3.23) with $\partial_x R_i$ and (3.24) with $\partial_x P_i$, respectively, then using integration by parts, the exponential decay of $\boldsymbol{R_i}$ and its derivatives to simplify these two equations. By (3.13),(3.17) and (3.18), letting α_0 small enough and $L_0 > \sigma^2$ large enough, there holds

$$|\dot{\tilde{x}}_i - c_i|(\|\partial_x \boldsymbol{R}_i\|_X + O(\alpha)) \leq O(\sqrt{\alpha}) + O(e^{-L}).$$

Consequently, we demonstrate (3.14).

Furthermore, by the assumption $z_j^0 - z_{j-1}^0 \geq L$ given in Theorem 3.1.6, it is deduced that

$$\begin{aligned}
\tilde{x}_i(t) - \tilde{x}_{i-1}(t) &= \tilde{x}_i(0) - \tilde{x}_{i-1}(0) + \left(\dot{\tilde{x}}_i(s) - \dot{\tilde{x}}_{i-1}(s)\right) t \\
&= \tilde{x}_i(0) - \tilde{x}_{i-1}(0) + \left(\dot{\tilde{x}}_i(s) - c_i + c_{i-1} - \dot{\tilde{x}}_{i-1}(s)\right) t + (c_i - c_{i-1})t \\
&\geq \frac{3L}{4} + \frac{1}{2}(c_i - c_{i-1})t.
\end{aligned}$$

Hence, we complete the proof of Lemma 3.2.2. □

Next, we prove the almost monotonicity of the functionals which describe the energy at the right of ith bump, for $i = 1, 2, \cdots, N$. To construct these functionals, we begin with the introduction of weight functions.

Let Ψ be a C^∞ function, such that

$$\begin{cases} 0 < \Psi(x) < 1, \ \Psi'(x) > 0, \quad x \in \mathbb{R}, \\ |\Psi'''(x)| \leq 10\Psi'(x), \quad x \in [-1, 1], \end{cases} \tag{3.25}$$

and

$$\Psi(x) = \begin{cases} e^{-|x|}, & x < -1, \\ 1 - e^{-|x|}, & x > 1. \end{cases} \tag{3.26}$$

Set $\Psi_K = \Psi(\cdot/K)$, $K > 0$. Define the weight functions $\Phi_i = \Phi_i(t, x)$, by

$$\Phi_1 = 1 - \Psi_{2,K}, \qquad \Phi_N = \Psi_{N,K}, \qquad \Phi_i = \Psi_{i,K} - \Psi_{i+1,K}, \quad i = 2, \cdots, N-1. \tag{3.27}$$

where $\Psi_{i,K} = \Psi_K(x - y_i(t))$ with $y_i(t)$ defined

$$y_1 = -\infty, \quad y_{N+1} = +\infty \quad \text{and} \quad y_i(t) = \frac{\tilde{x}_{i-1}(t) + \tilde{x}_i(t)}{2}, \; i = 2, \cdots, N. \tag{3.28}$$

Obviously, $\sum_{i=1}^{N} \Phi_i(t,x) = 1$, for $t \in [0, t^*]$. Taking $L > 0$ and $L/K > 0$ large enough, we have

$$|1 - \Phi_i| \leq 4e^{-\frac{L}{4K}}, \quad \text{on } \left[\tilde{x}_i - \frac{L}{4}, \tilde{x}_i + \frac{L}{4}\right], \tag{3.29}$$

and

$$|\Phi_i| \leq 4e^{-\frac{L}{4K}} \quad \text{on } \left[\tilde{x}_j - \frac{L}{4}, \tilde{x}_j + \frac{L}{4}\right], \quad \text{for } j \neq i. \tag{3.30}$$

Then, we introduce the localized conservation laws of E and F in terms of weight functions, for $i = 1, \cdots, N$,

$$E_i(\boldsymbol{u}) = E_i(u, \eta) = \frac{1}{2} \int_{\mathbb{R}} \left(u^2 + u_x^2 + \eta^2\right) \Phi_i \, dx, \tag{3.31}$$

$$F_i(\boldsymbol{u}) = F_i(u, \eta) = \frac{1}{2} \int_{\mathbb{R}} \left(u^3 + \sigma u u_x^2 + 2u\eta + u\eta^2 - Au^2\right) \Phi_i \, dx. \tag{3.32}$$

Moreover, for simplicity, we set

$$\sigma_0 = \frac{1}{4} \min(c_1, c_2 - c_1, \cdots, c_N - c_{N-1}). \tag{3.33}$$

Lemma 3.2.3. *Let $\boldsymbol{u}$ be the solution of the system (3.3) such that $\boldsymbol{u} \in U(\alpha, \frac{L}{2})$ on $[0, t^*]$ where $\{\tilde{x}_i(t)\}_{i=1}^N$ are defined in Lemma 3.2.2. There exist $\alpha_0 > 0$ and $L_0 > 0$ depending on $\{c_i\}_{i=1}^N$ and σ such that if $0 < \alpha < \alpha_0$ and $L > L_0$, for $\max\{4, \sigma^2\} \leq K = O(L^{1/2})$, it follows that*

$$I_{j,K}(t) - I_{j,K}(0) \leq O(e^{-\frac{L}{4K}}), \tag{3.34}$$

for all $j = 2, \cdots, N$ and $t \in [0, t^*]$, where $I_{j,K} = \frac{1}{2} \int_{\mathbb{R}} (u^2 + u_x^2 + \eta^2) \Psi_{j,K} \, dx$.

Proof. We know system (3.3) can be written in abstract Hamiltonian form as (3.6), that is

$$\begin{pmatrix} u \\ \eta \end{pmatrix}_t = \begin{pmatrix} -\partial_x (1 - \partial_x^2)^{-1} & 0 \\ 0 & -\partial_x \end{pmatrix} \begin{pmatrix} F'_u \\ F'_\eta \end{pmatrix}, \tag{3.35}$$

where

$$F'_u = \frac{3}{2}u^2 - \frac{\sigma}{2}u_x^2 - \sigma u u_{xx} - Au + \eta + \frac{1}{2}\eta^2, \qquad F'_\eta = u + u\eta. \tag{3.36}$$

Fixing j, differentiating $I_{j,K}$ with respect to t and using (3.35), we get

$$\begin{aligned}
\frac{d}{dt}I_{j,K}(t) = & -\frac{\dot{y}_j}{2}\int_{\mathbb{R}}(u^2+u_x^2+\eta^2)\partial_x\Psi_{j,K}\,dx + \int_{\mathbb{R}} u(u_t - u_{txx})\Psi_{j,K}\,dx \\
& - \int_{\mathbb{R}} u u_{tx}\partial_x\Psi_{j,K}\,dx + \int_{\mathbb{R}}\eta\eta_t\Psi_{j,K}\,dx \\
= & -\frac{\dot{y}_j}{2}\int_{\mathbb{R}}(u^2+u_x^2+\eta^2)\partial_x\Psi_{j,K}\,dx - \int_{\mathbb{R}} u\Psi_{j,K}\partial_x F'_u\,dx \\
& + \int_{\mathbb{R}} u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}\partial_x^2 F'_u\,dx - \int_{\mathbb{R}}\eta\Psi_{j,K}\partial_x F'_\eta\,dx \\
= & -\frac{\dot{y}_j}{2}\int_{\mathbb{R}}(u^2+u_x^2+\eta^2)\partial_x\Psi_{j,K}\,dx + \int_{\mathbb{R}} u_x\Psi_{j,K}F'_u\,dx \\
& + \int_{\mathbb{R}} u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}F'_u\,dx + \int_{\mathbb{R}}\eta_x\Psi_{j,K}F'_\eta\,dx + \int_{\mathbb{R}}\eta\partial_x\Psi_{j,K}F'_\eta\,dx \\
:= & -\frac{\dot{y}_j}{2}\int_{\mathbb{R}}(u^2+u_x^2+\eta^2)\partial_x\Psi_{j,K}\,dx + J_1(t)+J_2(t)+J_3(t)+J_4(t).
\end{aligned} \tag{3.37}$$

Substituting (3.36) into (3.37) and using integration by parts, $J_1(t)$, $J_2(t)$, $J_3(t)$ and $J_4(t)$ become

$$\begin{aligned}
J_1(t) = & \int_{\mathbb{R}} u_x\Psi_{j,K}\left(\frac{3}{2}u^2 - \frac{\sigma}{2}u_x^2 - \sigma u u_{xx} - Au + \eta + \frac{1}{2}\eta^2\right)\,dx \\
= & -\frac{1}{2}\int_{\mathbb{R}}\partial_x\Psi_{j,K}u^3\,dx + \frac{\sigma}{2}\int_{\mathbb{R}}\partial_x\Psi_{j,K}u u_x^2\,dx + \frac{A}{2}\int_{\mathbb{R}}\partial_x\Psi_{j,K}u^2 + \int_{\mathbb{R}}\Psi_{j,K}u_x\eta\,dx \\
& + \frac{1}{2}\int_{\mathbb{R}}\Psi_{j,K}u_x\eta^2\,dx, \\
J_2(t) = & \int_{\mathbb{R}} u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}\left(\frac{3}{2}u^2 - \frac{\sigma}{2}u_x^2 - \sigma u u_{xx} - Au + \eta + \frac{1}{2}\eta^2\right)\,dx \\
= & \int_{\mathbb{R}} u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}\left(\frac{3}{2}u^2 + \frac{\sigma}{2}u_x^2 - \frac{\sigma}{2}(u^2)_{xx} - Au + \eta + \frac{1}{2}\eta^2\right)\,dx \\
= & \int_{\mathbb{R}} u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}\left(\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 - Au + \eta + \frac{1}{2}\eta^2\right)\,dx \\
& + \frac{\sigma}{2}\int_{\mathbb{R}}\partial_x\Psi_{j,K}u^3\,dx, \\
J_3(t) = & \int_{\mathbb{R}}\eta_x\Psi_{j,K}\left(u + u\eta\right)\,dx
\end{aligned}$$

$$= -\int_{\mathbb{R}} \Psi_{j,K} u_x \eta \, dx - \int_{\mathbb{R}} \partial_x \Psi_{j,K} u\eta \, dx - \frac{1}{2}\int_{\mathbb{R}} \Psi_{j,K} u_x \eta^2 \, dx - \frac{1}{2}\int_{\mathbb{R}} \partial_x \Psi_{j,K} u\eta^2 \, dx,$$

$$J_4(t) = \int_{\mathbb{R}} \eta \partial_x \Psi_{j,K} \left(u + u\eta\right) \, dx = \int_{\mathbb{R}} \partial_x \Psi_{j,K} u\eta \, dx + \int_{\mathbb{R}} \partial_x \Psi_{j,K} u\eta^2 \, dx.$$

Combining $J_1(t)$, $J_2(t)$, $J_3(t)$ and $J_4(t)$, we can simplify $J(t) = J_1(t) + J_2(t) + J_3(t) + J_4(t)$ as follows

$$\begin{aligned} J(t) =& \frac{A}{2}\int_{\mathbb{R}} u^2 \partial_x \Psi_{j,K} \, dx - A\int_{\mathbb{R}} u\partial_x \Psi_{j,K}(1-\partial_x^2)^{-1} u \, dx \\ &+ \frac{1}{2}\int_{\mathbb{R}} u\partial_x \Psi_{j,K}\Big((\sigma-1)u^2 + \sigma u_x^2 + \eta^2\Big) \, dx + \int_{\mathbb{R}} u\partial_x \Psi_{j,K}(1-\partial_x^2)^{-1}\eta \, dx \\ &+ \int_{\mathbb{R}} u\partial_x \Psi_{j,K}(1-\partial_x^2)^{-1}\left(\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1}{2}\eta^2\right) \, dx \\ =& \frac{A}{2}\int_{\mathbb{R}} u^2 \partial_x \Psi_{j,K} \, dx - A\int_{\mathbb{R}} u\partial_x \Psi_{j,K}(1-\partial_x^2)^{-1} u \, dx + Q_1(t) + Q_2(t) + Q_3(t). \end{aligned} \tag{3.38}$$

To deal with the first two terms in (3.38), we adopt the same trick in [29]. Setting $h = (1-\partial_x^2)^{-1}u$ infers

$$\begin{aligned} -A\int_{\mathbb{R}} u\partial_x \Psi_{j,K}(1-\partial_x^2)^{-1} u \, dx =& -A\int_{\mathbb{R}} h\partial_x \Psi_{j,K}(1-\partial_x^2) h \, dx \\ =& -A\int_{\mathbb{R}} (h^2 + h_x^2)\partial_x \Psi_{j,K} \, dx + \frac{A}{2}\int_{\mathbb{R}} h^2 \partial_x^3 \Psi_{j,K} \, dx. \end{aligned}$$

According to the definition of h, the following identity holds

$$\int_{\mathbb{R}} u^2 \Psi_{j,K} \, dx = \int_{\mathbb{R}} (h^2 + h_{xx}^2 + 2h_x^2)\partial_x \Psi_{j,K} \, dx - \int_{\mathbb{R}} h^2 \partial_x^3 \Psi_{j,K} \, dx.$$

Then for $m > \frac{A}{2}$, we have

$$\begin{aligned} -A\int_{\mathbb{R}} \left(h^2 + h_x^2\right)\partial_x \Psi_{j,K} \, dx =& \left(-\frac{A}{2} + m\right)\int_{\mathbb{R}} \left(h^2 + h_x^2\right)\partial_x \Psi_{j,K} \, dx \\ &- \left(m + \frac{A}{2}\right)\int_{\mathbb{R}} \left(h^2 + h_x^2\right)\partial_x \Psi_{j,K} \, dx \\ \leq& \left(-\frac{A}{2} + m\right)\int_{\mathbb{R}} u^2 \partial_x \Psi_{j,K} \, dx + \left(-\frac{A}{2} + m\right) \cdot \\ &\int_{\mathbb{R}} h^2 \partial_x^3 \Psi_{j,K} \, dx - \left(m + \frac{A}{2}\right)\int_{\mathbb{R}} \left(h^2 + h_x^2\right)\partial_x \Psi_{j,K} \, dx, \end{aligned}$$

which implies

$$-A\int_{\mathbb{R}} u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}u\ dx \leq (-\frac{A}{2}+m)\int_{\mathbb{R}} u^2\partial_x\Psi_{j,K}\ dx + m\int_{\mathbb{R}} h^2\partial_x^3\Psi_{j,K}\ dx - (m+\frac{A}{2})\int_{\mathbb{R}} h^2\partial_x\Psi_{j,K}\ dx.$$

It follows from (3.25) that for $K>0$, if

$$\frac{10m}{K^2} \leq (m+\frac{A}{2}) \Leftrightarrow K \geq \sqrt{\frac{10m}{m+\frac{A}{2}}},$$

then

$$-A\int_{\mathbb{R}} u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}u\ dx \leq (-\frac{A}{2}+m)\int_{\mathbb{R}} u^2\partial_x\Psi_{j,K}\ dx. \tag{3.39}$$

To consider the upper bound for Q_1, Q_2 and Q_3, we divide $\mathbb{R}$ into two parts: $D_j \cup D_j^c$, where $D_j = [\tilde{x}_{j-1}+\frac{L}{4}, \tilde{x}_j - \frac{L}{4}]$. The two crucial estimates related to D_j and D_j^c are listed in the following. For $x \in D_j$,

$$\begin{aligned}\|u(t)\|_{L^\infty(D_j)} &\leq \sum_{i=1}^N \|\varphi_{c_i}(\cdot - \tilde{x}_i(t))\|_{L^\infty(D_j)} + \|u - \sum_{i=1}^N \varphi_{c_i}(\cdot - \tilde{x}_i(t))\|_{L^\infty(D_j)} \\ &\leq O(e^{-\frac{L}{8}}) + O(\sqrt{\alpha}),\end{aligned} \tag{3.40}$$

For $x \in D_j^c$, taking advantage of (3.14), (3.28) and the definition of $\Psi_{j,K}$, we know

$$|x-y_j(t)| \geq \frac{\tilde{x}_j(t)-\tilde{x}_{j-1}(t)}{2} - \frac{L}{4} \geq \frac{(c_j-c_{j-1})t}{4} + \frac{L}{8} \geq \sigma_0 t + \frac{L}{8},$$

which implies, for $K = O(\sqrt{L})$ and sufficiently large L_0,

$$\left|\frac{x-y_j(t)}{K}\right| \geq \frac{\sigma_0 t + \frac{L}{8}}{K} > 1.$$

Hence, there holds

$$\partial_x\Psi_{j,K}(t,x) = \frac{1}{K}\Psi'_{j,K}(\frac{x-y_j(t)}{K}) \leq \frac{1}{K}e^{-\frac{1}{K}(\sigma_0 t+\frac{L}{8})}, \quad x \in D_j^c. \tag{3.41}$$

For $\alpha_0 > 0$ small enough and $L_0 > \sigma^2$ large enough, since $\partial_x\Psi_{j,K}, u^2, u_x^2, \eta^2$ are all positive, then

$$Q_1(t) = \frac{1}{2}\int_{D_j} u\partial_x\Psi_{j,K}\Big((\sigma-1)u^2 + \sigma u_x^2 + \eta^2\Big)\ dx$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{D_j^c} u\partial_x\Psi_{j,K}\Big((\sigma-1)u^2+\sigma u_x^2+\eta^2\Big)\,dx\\
\le\ &\frac{1}{2}\max\{|\sigma-1|,|\sigma|,1\}\,\|u(t)\|_{L^\infty(D_j)}\int_{D_j}\partial_x\Psi_{j,K}\Big(u^2+u_x^2+\eta^2\Big)\,dx\\
&+\frac{1}{2}\max\{|\sigma-1|,|\sigma|,1\}\,\|u(t)\|_{L^\infty(D_j^c)}\sup_{x\in D_j^c}|\partial_x\Psi_{j,K}(x-y_j(t))|\cdot\\
&\int_{D_j^c}\Big(u^2+u_x^2+\eta^2\Big)\,dx\\
\le\ &\frac{C}{K}\,\|\boldsymbol{u}_0\|_X^3\,e^{-\frac{\sigma_0 t+\frac{L}{8}}{K}}+\frac{\sigma_0}{8}\int_{\mathbb{R}}\partial_x\Psi_{j,K}\Big(u^2+u_x^2+\eta^2\Big)\,dx.
\end{aligned}
$$

For $Q_2(t)$, by Cauchy inequality and Hölder's inequality we know

$$
\begin{aligned}
\int_{\mathbb{R}} u\partial_x\Psi_{j,K}(1-\partial_x^2)^{-1}\eta\,dx &= \int_{\mathbb{R}} u\partial_x\Psi_{j,K}p*\eta\,dx\\
&\le \frac{1}{2a}\int_{\mathbb{R}} u^2\partial_x\Psi_{j,K}\,dx+\frac{a}{2}\int_{\mathbb{R}}\partial_x\Psi_{j,K}(p*\eta)^2\,dx\\
&\le \frac{1}{2a}\int_{\mathbb{R}} u^2\partial_x\Psi_{j,K}\,dx+\frac{a}{4}\int_{\mathbb{R}}\partial_x\Psi_{j,K}(p*\eta^2)\,dx,
\end{aligned}
\tag{3.42}
$$

where

$$
\begin{aligned}
(p*\eta)^2(x) &= \left[\frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\eta(y)\,dy\right]^2\\
&\le \frac{1}{4}\left(\int_{\mathbb{R}} e^{-|x-y|}\,dy\right)\left(\int_{\mathbb{R}} e^{-|x-y|}\eta^2(y)\,dy\right)\\
&= \frac{1}{2}\left(p*\eta^2\right)(x).
\end{aligned}
$$

For $x\in D_j^c$, by the property of convolution and (3.41), the following inequality holds

$$
\begin{aligned}
\int_{D_j^c}\partial_x\Psi_{j,K}p*\eta^2\,dx &\le \sup_{x\in D_j^c}|\partial_x\Psi_{j,K}|\int_{D_j^c} p*\left(u^2+u_x^2+\eta^2\right)\,dx\\
&\le \frac{1}{2}\sup_{x\in D_j^c}|\partial_x\Psi_{j,K}|\int_{\mathbb{R}} e^{-|x|}dx\int_{\mathbb{R}}\left(u^2+u_x^2+\eta^2\right)\,dx\\
&\le \frac{C}{K}\,\|\boldsymbol{u}_0\|_X^2\,e^{-\frac{\sigma_0 t+\frac{L}{8}}{K}}.
\end{aligned}
\tag{3.43}
$$

By the given condition (3.25), we have

$$
\left(1-\partial_x^2\right)\partial_x\Psi_{j,K}\ge\left(1-\frac{10}{K^2}\right)\partial_x\Psi_{j,K}.
$$

which is equivalent to

$$\left(1-\partial_x^2\right)^{-1}\partial_x\Psi_{j,K} \leq \left(1-\frac{10}{K^2}\right)^{-1}\partial_x\Psi_{j,K}, \quad \text{for} \quad K\geq 4. \tag{3.44}$$

Hence, for $x\in D_j$,

$$\begin{aligned}\int_{D_j}\partial_x\Psi_{j,K}(p*\eta^2)\,dx &\leq \int_{D_j}\eta^2\left(1-\partial_x^2\right)^{-1}\partial_x\Psi_{j,K}\,dy\\ &\leq \left(1-\frac{10}{K^2}\right)^{-1}\int_{\mathbb{R}}\eta^2\partial_x\Psi_{j,K}\,dx.\end{aligned} \tag{3.45}$$

Combining (3.42), (3.43) and (3.45), we obtain

$$Q_2(t)\leq \frac{1}{2a}\int_{\mathbb{R}}u^2\partial_x\Psi_{j,K}\,dx+\frac{a}{4}\left(1-\frac{10}{K^2}\right)^{-1}\int_{\mathbb{R}}\eta^2\partial_x\Psi_{j,K}\,dx+\frac{C}{K}\left\|\boldsymbol{u}_0\right\|_X^2e^{-\frac{\sigma_0t+\frac{L}{8}}{K}}.$$

$Q_3(t)$ can be treated with the same process. For $x\in D_j^c$, we deduce from (3.41) that

$$\begin{aligned}&\int_{D_j^c}u\partial_x\Psi_{j,K}\left(1-\partial_x^2\right)^{-1}\left(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2+\frac{1}{2}\eta^2\right)\,dx\\ &\leq \frac{1}{2}\max\{|3-\sigma|,|\sigma|,1\}\left\|u\right\|_{L^\infty}\sup_{x\in D_j^c}\left|\partial_x\Psi_{j,K}(x-y_j(t))\right|\int_{\mathbb{R}}p*\left(u^2+u_x^2+\eta^2\right)\,dx\\ &=\frac{1}{4}\max\{|3-\sigma|,|\sigma|,1\}\left\|u\right\|_{L^\infty}\sup_{x\in D_j^c}\left|\partial_x\Psi_{j,K}(x-y_j(t))\right|\int_{\mathbb{R}}e^{-|x|}*\left(u^2+u_x^2+\eta^2\right)\,dx\\ &\leq \frac{C}{K}\left\|\boldsymbol{u}_0(t)\right\|_X^3e^{-\frac{\sigma_0t+\frac{L}{8}}{K}}.\end{aligned}$$

For $x\in D_j$, using equation (3.40) and taking $L_0>\sigma^2$ large enough, we infer

$$\begin{aligned}&\int_{D_j}u\partial_x\Psi_{j,K}\left(1-\partial_x^2\right)^{-1}\left(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2+\frac{1}{2}\eta^2\right)\,dx\\ &\leq \frac{1}{2}\max\{|3-\sigma|,|\sigma|,1\}\left\|u\right\|_{L^\infty}\int_{D_j}\partial_x\Psi_{j,K}\left(1-\partial_x^2\right)^{-1}\left(u^2+u_x^2+\eta^2\right)\,dx\\ &\leq \frac{1}{4}\max\{|3-\sigma|,|\sigma|,1\}\left\|u\right\|_{L^\infty}\int_{\mathbb{R}}\partial_x\Psi_{j,K}e^{-|x|}*\left(u^2+u_x^2+\eta^2\right)\,dx\\ &\leq \frac{\sigma_0}{8}\int_{\mathbb{R}}\partial_x\Psi_{j,K}\Big(u^2+u_x^2+\eta^2\Big)\,dx.\end{aligned}$$

Thus, the estimates on D_j^c and D_j guarantee that

$$Q_3(t)\leq \frac{C}{K}\left\|\boldsymbol{u}_0\right\|_X^3e^{-\frac{\sigma_0t+\frac{L}{8}}{K}}+\frac{\sigma_0}{8}\int_{\mathbb{R}}\partial_x\Psi_{j,K}\Big(u^2+u_x^2+\eta^2\Big)\,dx.$$

Substituting the estimates of $Q_1(t)$, $Q_2(t)$, and $Q_3(t)$ and (3.39) into (3.38), we derive that

$$J(t) \leq \frac{C}{K}\left(\|\boldsymbol{u}_0(t)\|_X^3 + \|\boldsymbol{u}_0(t)\|_X^2\right) e^{-\frac{\sigma_0 t + \frac{L}{8}}{K}} + \frac{\sigma_0}{4}\int_{\mathbb{R}} \partial_x \Psi_{j,K}\left(u^2 + u_x^2 + \eta^2\right) dx$$
$$+ \left(m + \frac{1}{2a}\right)\int_{\mathbb{R}} u^2 \partial_x \Psi_{j,K}\, dx + \frac{a}{4}\left(1 - \frac{10}{K^2}\right)^{-1}\int_{\mathbb{R}} \eta^2 \partial_x \Psi_{j,K}\, dx.$$

For $\alpha_0 > 0$ small enough and $L_0 > \sigma^2$ large enough, by the definition of σ_0 in (3.33) and equation (3.14), we have

$$-\frac{\dot{y}_j(t)}{2} = -\frac{\dot{\tilde{x}}_{j-1}(t) - c_{j-1}}{4} - \frac{\dot{\tilde{x}}_j(t) - c_j}{4} - \frac{c_{j-1} + c_j}{4} \leq -\frac{c_1 + \sigma_0}{2}.$$

Associating the above estimate with $J(t)$ and (3.37), there obtains the inequality

$$\frac{d}{dt} I_{j,K}(t) \leq \frac{C}{K}\left(\|\boldsymbol{u}_0(t)\|_X^3 + \|\boldsymbol{u}_0(t)\|_X^2\right) e^{-\frac{\sigma_0 t + \frac{L}{8}}{K}} - \frac{\sigma_0}{4}\int_{\mathbb{R}} \partial_x \Psi_{j,K}\left(u^2 + u_x^2 + \eta^2\right) dx$$
$$+ \left(-\frac{c_1}{2} + \frac{a}{4}\left(1 - \frac{10}{K^2}\right)^{-1}\right)\int_{\mathbb{R}} \eta^2 \partial_x \Psi_{j,K}\, dx$$
$$+ \left(m - \frac{c_1}{2} + \frac{1}{2a}\right)\int_{\mathbb{R}} u^2 \partial_x \Psi_{j,K}\, dx.$$

Let m and a satisfy the following conditions

$$m - \frac{c_1}{2} + \frac{1}{2a} \leq 0, \quad -\frac{c_1}{2} + \frac{a}{4}\left(1 - \frac{10}{K^2}\right)^{-1} \leq 0, \quad m > \frac{A}{2}. \tag{3.46}$$

Then

$$\frac{d}{dt} I_{j,K}(t) \leq \frac{C}{K} e^{-\frac{\sigma_0 t + \frac{L}{8}}{K}} - \frac{\sigma_0}{4}\int_{\mathbb{R}} \partial_x \Psi_{j,K}\left(u^2 + u_x^2 + \eta^2\right) dx.$$

The desired results (3.34) now follows by applying Gronwall's inequality. □

Remark 6. We can find such m and a to meet the conditions in (3.46) as follows. Let

$$f(x) = \frac{A + x + \sqrt{(A+x)^2 + 2}}{2}.$$

Since $c_1 > \frac{A + \sqrt{A^2+2}}{2} = f(0)$ and $f(x)$ is increasing for $x > 0$, then there exists $\epsilon_0 > 0$ such that

$$f(0) < f(\epsilon_0) < c_1, \quad 0 < \epsilon_0 < \frac{c_1 - A}{2}, \quad \epsilon_0 \to 0.$$

Choose $m = \frac{A}{2} + \epsilon_0$, then from the first two condition in (3.46), to find such a, we need to show the following inequality is true

$$\frac{1}{c_1 - A - 2\epsilon_0} \leq 2c_1(1 - \frac{10}{K^2}).$$

Actually, since $c_1 > \frac{A+\sqrt{A^2+2}}{2}$, taking $K \geq \sqrt{\frac{20c_1(c_1-A-2\epsilon_0)}{2c_1(c_1-A-2\epsilon_0)-1}}$, we have the inequality mentioned above holds, which implies such m and a exist.

In the next lemma, we present another version of Lemma 3.7 in [13], where the assumption is not strictly orthogonal. This is helpful while proving the local coercivity of the linear operator $H_c : X \to X^*$ at $\boldsymbol{\varphi}_c$, $\forall c \in \{c_i\}_{i=1}^N$.

Lemma 3.2.4. *Assume $\delta > 0, \sigma \leq 1$ and $\boldsymbol{\varphi}_c = (\varphi_c, \psi_c)$ is a smooth traveling wave of* (3.3)*. There exists a constant $C_\delta > 0$ such that*

$$\langle H_c \boldsymbol{\zeta}, \boldsymbol{\zeta} \rangle \geq C_\delta \left\| \boldsymbol{\zeta} \right\|_X^2 ,$$

for all $\boldsymbol{\zeta} \in X$ satisfying $|\langle \boldsymbol{\zeta}, \boldsymbol{\nu} \rangle| + |\langle \boldsymbol{\zeta}, \partial_x \boldsymbol{\nu} \rangle| \leq \delta \left\| \boldsymbol{\zeta} \right\|_X$, where *$\boldsymbol{\nu} = ((1 - \partial_x^2)\, \varphi_c, \psi_c)$.*

Proof. According to the spectrum analysis of H_c, let $\partial_c \boldsymbol{\varphi}$ be decomposed into $a_0 \boldsymbol{\chi} + b_0 \boldsymbol{\varphi}' + a_0 \boldsymbol{p}_0$ with $a_0 > 0$, where $\boldsymbol{\chi}$ is the eigenvector of negative eigenvalue, i.e $H_c \boldsymbol{\chi} = -|\lambda_1| \boldsymbol{\chi}$, $\boldsymbol{\varphi}'$ is the eigenvector of eigenvalue 0, i.e. $H_c \boldsymbol{\varphi}' = 0$, $\boldsymbol{p}_0$ is the positive subspace of H_c. Without loss of generality, we assume $\|\boldsymbol{\chi}\| = 1$. Therefore,

$$\begin{aligned}
\langle H_c \partial_c \boldsymbol{\varphi}, \partial_c \boldsymbol{\varphi} \rangle =& \langle H_c \left(a_0 \boldsymbol{\chi} + b_0 \boldsymbol{\varphi}' + a_0 \boldsymbol{p}_0 \right), a_0 \boldsymbol{\chi} + b_0 \boldsymbol{\varphi}' + a_0 \boldsymbol{p}_0 \rangle \\
=& \, a_0^2 \langle H_c \boldsymbol{\chi}, \boldsymbol{\chi} \rangle + a_0^2 \langle H_c \boldsymbol{p}_0, \boldsymbol{p}_0 \rangle \\
=& \, - a_0^2 |\lambda_1| + a_0^2 \langle H_c \boldsymbol{p}_0, \boldsymbol{p}_0 \rangle.
\end{aligned}$$

Since $\langle H_c \partial_c \boldsymbol{\varphi}, \partial_c \boldsymbol{\varphi} \rangle = -d''(c) < 0$, then $\langle H_c \boldsymbol{p}_0, \boldsymbol{p}_0 \rangle < |\lambda_1|$. Let $\varepsilon > 0$, such that

$$\langle H_c \boldsymbol{p}_0, \boldsymbol{p}_0 \rangle \leq \frac{1}{1+\varepsilon} |\lambda_1| .$$

By the assumption we know $|(\boldsymbol{\zeta}, \boldsymbol{\varphi})_{H^1 \times L^2}| + |(\boldsymbol{\zeta}, \boldsymbol{\varphi}')_{H^1 \times L^2}| = |\langle \boldsymbol{\zeta}, \boldsymbol{\nu} \rangle| + |\langle \boldsymbol{\zeta}, \partial_x \boldsymbol{\nu} \rangle| \leq \delta \left\| \boldsymbol{\zeta} \right\|_X$. Then there exists $\varepsilon = \varepsilon(\delta)$ such that $\boldsymbol{\zeta}$ has the decomposition $a\boldsymbol{\chi} + \varepsilon \boldsymbol{\varphi}' + \boldsymbol{p_1}$

with $\boldsymbol{p}_1$ in the positive subspace of H_c. If $a = 0$, then $\boldsymbol{\zeta} = \varepsilon\boldsymbol{\varphi}' + \boldsymbol{p}_1$. It is easily to check that $\langle H_c\boldsymbol{\zeta}, \boldsymbol{\zeta}\rangle \geq C_\delta \|\boldsymbol{\zeta}\|_X^2$. If $a \neq 0$, then $\boldsymbol{\zeta} = a\boldsymbol{\chi} + \varepsilon\boldsymbol{\varphi}' + a\boldsymbol{p}$ with $\boldsymbol{p}$ in the positive subspace of H_c. Differentiating $\langle cE'(\boldsymbol{\varphi}_c) - f'(\boldsymbol{\varphi}_c), \boldsymbol{\zeta}\rangle = 0$ with respect to c, we have $\langle H_c\partial_c\boldsymbol{\varphi}, \boldsymbol{\zeta}\rangle + \langle E'(\boldsymbol{\varphi}_c), \boldsymbol{\zeta}\rangle = 0$, i.e.

$$\langle H_c\partial_c\boldsymbol{\varphi}, \boldsymbol{\zeta}\rangle = \langle -a_0 |\lambda_1| \boldsymbol{\chi} + a_0 H_c\boldsymbol{p}_0, a\boldsymbol{\chi} + \varepsilon\boldsymbol{\varphi}' + a\boldsymbol{p}\rangle,$$

which implies

$$|\langle H_c\boldsymbol{p}_0, \boldsymbol{p}\rangle| \leq |\lambda_1| + \frac{\delta \|\boldsymbol{\zeta}\|_X}{|aa_0|}.$$

Consequently,

$$\begin{aligned}
\langle H_c\boldsymbol{\zeta}, \boldsymbol{\zeta}\rangle &= \langle H_c\left(a\boldsymbol{\chi} + \varepsilon\boldsymbol{\varphi}' + a\boldsymbol{p}\right), a\boldsymbol{\chi} + \varepsilon\boldsymbol{\varphi}' + a\boldsymbol{p}\rangle \\
&= -a^2 |\lambda_1| + a^2\langle H_c\boldsymbol{p}, \boldsymbol{p}\rangle \\
&\geq -a^2 |\lambda_1| + a^2\frac{\langle H_c\boldsymbol{p}_0, \boldsymbol{p}\rangle^2}{\langle H_c\boldsymbol{p}_0, \boldsymbol{p}_0\rangle} \\
&\geq \varepsilon a^2 |\lambda_1| = \varepsilon |\lambda_1| \|a\boldsymbol{\chi}\|_X^2.
\end{aligned}$$

In fact, the function $\Gamma(f, g) = \langle H_c f, g\rangle$ is a nonnegative sesquilinear form on positive subspace of H_c. Thus, we have the Cauchy-Schwarz inequality

$$|\Gamma(f, g)|^2 \leq \langle H_c f, f\rangle\langle H_c g, g\rangle.$$

On the other hand, in view of $\boldsymbol{p}$ in the positive subspace of H_c, there exists $\lambda_3 > 0$, such that $\langle H_c\boldsymbol{p}, \boldsymbol{p}\rangle \geq \lambda_3 \|\boldsymbol{p}\|_X^2$ for all $\boldsymbol{p}$ orthogonal to $\boldsymbol{\chi}$ and $\boldsymbol{\varphi}'$, which implies

$$\langle H_c\boldsymbol{\zeta}, \boldsymbol{\zeta}\rangle \geq -|\lambda_1| \|a\boldsymbol{\chi}\|_X^2 + \lambda_3 \|a\boldsymbol{p}\|_X^2.$$

Then, it follows that for $k = \frac{\varepsilon}{\varepsilon+2}\min\{|\lambda_1|, \lambda_3\}$,

$$\langle H_c\boldsymbol{\zeta}, \boldsymbol{\zeta}\rangle \geq k\left(\|a\boldsymbol{\chi}\|_X^2 + \|a\boldsymbol{p}\|_X^2\right).$$

Since ε and δ are small enough comparing with k, we obtain

$$\langle H_c\boldsymbol{\zeta}, \boldsymbol{\zeta}\rangle \geq C_\delta \|\boldsymbol{\zeta}\|_X^2,$$

which completes the proof of this lemma. $\square$

Now, it remains to show the last property in this subsection. Taking advantage of Lemma 3.2.4, we will prove that H_{c_i} is a local coercive operator as follows.

Lemma 3.2.5. *There exist $\sigma \leq 1$, $\delta > 0$, $C_\delta > 0$ and $C > 0$ depending only on c_1, such that for all $c \geq c_1$, $\Theta \in C^2(\mathbb{R}) > 0$ and $\boldsymbol{\zeta} = (\zeta, \xi) \in X$ satisfying*

$$\left|\langle\sqrt{\Theta}\boldsymbol{\zeta}, \boldsymbol{\nu}\rangle\right| + \left|\langle\sqrt{\Theta}\boldsymbol{\zeta}, \partial_x\boldsymbol{\nu}\rangle\right| \leq \delta \left\|\boldsymbol{\zeta}\right\|_X,$$

and

$$\left|\frac{(\Theta')^2}{4\Theta}\right| + c\left|\sigma\Theta'\right| + \left|\frac{\Theta''}{2}\right| \leq \min\left\{\frac{1}{4}, \frac{C_\delta}{4(c + c|\sigma|)}\right\}\Theta,$$

where $\boldsymbol{\nu} = ((1 - \partial_x^2)\,\varphi_c, \psi_c)$, it holds

$$\begin{aligned}\int_{\mathbb{R}} \Big[\Theta\left((c - \sigma\varphi_c)(\partial_x\zeta)^2 + (-3\varphi_c + \sigma\varphi_{c,xx} + c + A)\,\zeta^2\right) - 2\Theta\frac{c}{c - \varphi}\zeta\xi \\ + \Theta\,(c - \varphi)\,\xi^2 + \sigma\Theta'\zeta^2\varphi_{c,x}\Big]\,dx \geq C\int_{\mathbb{R}}\Theta\left(\zeta^2 + \zeta_x^2 + \xi^2\right)\,dx.\end{aligned} \tag{3.47}$$

Proof. Following with the strategy in [29], we have

$$\begin{aligned}
&\langle H_c\sqrt{\Theta}\boldsymbol{\zeta}, \sqrt{\Theta}\boldsymbol{\zeta}\rangle_{L^2\times L^2} = \langle H_c\sqrt{\Theta}\begin{pmatrix}\zeta\\ \xi\end{pmatrix}, \sqrt{\Theta}\begin{pmatrix}\zeta\\ \xi\end{pmatrix}\rangle_{L^2\times L^2}\\
&= \langle\begin{pmatrix}(-\partial_x\left((c - \sigma\varphi_c)\partial_x\right) - 3\varphi_c + \sigma\varphi_{c,xx} + c + A)\sqrt{\Theta}\zeta - \frac{c}{c-\varphi}\sqrt{\Theta}\xi\\ -\frac{c}{c-\varphi}\sqrt{\Theta}\zeta + (c - \varphi)\sqrt{\Theta}\xi\end{pmatrix}, \begin{pmatrix}\sqrt{\Theta}\zeta\\ \sqrt{\Theta}\xi\end{pmatrix}\rangle_{L^2\times L^2}\\
&= \int_{\mathbb{R}}\Theta\left[(c - \sigma\varphi_c)(\partial_x\zeta)^2 + (-3\varphi_c + \sigma\varphi_{c,xx} + c + A)\,\zeta^2\right]\,dx + \int_{\mathbb{R}}\frac{c - \sigma\varphi_c}{4}\frac{(\Theta')^2}{\Theta}\zeta^2\,dx\\
&\quad + \int_{\mathbb{R}}(c - \sigma\varphi_c)\,\Theta'\zeta\partial_x\zeta\,dx - 2\int_{\mathbb{R}}\Theta\frac{c}{c - \varphi_c}\zeta\xi\,dx + \int_{\mathbb{R}}\Theta\,(c - \varphi_c)\,\xi^2\,dx\\
&= \int_{\mathbb{R}}\Bigg[\Theta\left((c - \sigma\varphi_c)(\partial_x\zeta)^2 + (-3\varphi_c + \sigma\varphi_{c,xx} + c + A)\,\zeta^2\right) - 2\Theta\frac{c}{c - \varphi_c}\zeta\xi + \Theta\,(c - \varphi_c)\,\xi^2\\
&\quad + \sigma\Theta'\zeta^2\varphi_{c,x}\Bigg]\,dx + \int_{\mathbb{R}}\left[(c - \sigma\varphi_c)\left(\frac{(\Theta')^2}{4\Theta} - \frac{\Theta''}{2}\right) - \frac{\sigma}{2}\varphi_{c,x}\Theta'\right]\zeta^2\,dx
\end{aligned}$$

$$:= \Lambda + \int_{\mathbb{R}} \left[(c - \sigma\varphi_c) \left(\frac{(\Theta')^2}{4\Theta} - \frac{\Theta''}{2} \right) - \frac{\sigma}{2} \varphi_{c,x} \Theta' \right] \zeta^2 \, dx,$$

and

$$\left\| \sqrt{\Theta} \begin{pmatrix} \zeta \\ \xi \end{pmatrix} \right\|_X^2 = \int_{\mathbb{R}} \left(\sqrt{\Theta}\zeta \right)^2 + \left(\partial_x \left(\sqrt{\Theta}\zeta \right) \right)^2 + \left(\sqrt{\Theta}\xi \right)^2 \, dx$$
$$= \int_{\mathbb{R}} \Theta \left(\zeta^2 + (\partial_x \zeta)^2 + \xi^2 \right) \, dx + \int_{\mathbb{R}} \left(\frac{(\Theta')^2}{4\Theta} - \frac{\Theta''}{2} \right) \zeta^2 \, dx.$$

Due to Lemma 3.2.4, there exist $\delta > 0$ and $C_\delta > 0$, such that if for $c \geq c_1$

$$|\langle \boldsymbol{\zeta}, \boldsymbol{\nu} \rangle_{L^2 \times L^2}| + |\langle \boldsymbol{\zeta}, \partial_x \boldsymbol{\nu} \rangle_{L^2 \times L^2}| \leq \delta \|\boldsymbol{\zeta}\|_X, \text{ where } \boldsymbol{\nu} = \left(\left(1 - \partial_x^2 \right) \varphi_c, \psi_c \right), \tag{3.48}$$

then

$$\langle H_c \boldsymbol{\zeta}, \boldsymbol{\zeta} \rangle_{L^2 \times L^2} \geq C_\delta \|\boldsymbol{\zeta}\|_X^2 . \tag{3.49}$$

Hence, taking $\boldsymbol{\zeta} = \sqrt{\Theta}\boldsymbol{\zeta}$, integrating with the conditions that φ_c is bounded by 0 and $c - A_1$, $|\varphi_c'|$ is bounded by $c - A_1$ and

$$\left| \frac{(\Theta')^2}{4\Theta} \right| + c \left| \sigma\Theta' \right| + \left| \frac{\Theta''}{2} \right| \leq \min \left\{ \frac{1}{4}, \frac{C_\delta}{4(c + c \|\sigma\|)} \right\} \Theta,$$

we deduce the following inequality

$$\Lambda + \frac{C_\delta}{4} \int_{\mathbb{R}} \Theta\zeta^2 \, dx \geq C_\delta \int_{\mathbb{R}} \Theta \left(\zeta^2 + (\partial_x \zeta)^2 + \xi^2 \right) \, dx - \min \left\{ \frac{1}{4}, \frac{C_\delta}{4(c + c|\sigma|)} \right\} \int_{\mathbb{R}} \Theta\zeta^2 \, dx,$$

which implies there exists C such that

$$\Lambda \geq C \int_{\mathbb{R}} \Theta \left(\zeta^2 + (\partial_x \zeta)^2 + \xi^2 \right) \, dx.$$

This completes the proof of Lemma 3.2.5. □

According to Proposition 3.2.1, to prove the main theorem is to prove inequality (3.11), where $x_i(t)$ are taken as $\tilde{x}_i(t)$ in Lemma 3.2.2 and M will be determined later. Indeed, the following three lemmas contribute to the estimate of $\boldsymbol{v}$ at time $t = t^*$.

When there is no ambiguity, we write $\boldsymbol{u} = \boldsymbol{u}(t^*)$, $X = (\tilde{x}_1, \cdots, \tilde{x}_N) = (\tilde{x}_1(t^*), \cdots, \tilde{x}_N(t^*))$, $E_i(\boldsymbol{u}) = E_i(\boldsymbol{u}(t^*))$, $F_i(\boldsymbol{u}) = E_i(\boldsymbol{u}(t^*))$ and so forth. Define $\boldsymbol{\mu}_i = (\mu_i, \tau_i) \in X$, for $i = 1, 2, \cdots, N$, by

$$\boldsymbol{u} = (1 + a_i)\boldsymbol{R}_X + \boldsymbol{\mu}_i, \text{ satisfying } \langle E_i'(\boldsymbol{R}_X), \boldsymbol{\mu}_i\rangle_{L^2\times L^2} = 0, \tag{3.50}$$

where

$$\begin{aligned}\boldsymbol{R}_X =& (R_X, P_X) = \sum_{i=1}^N \boldsymbol{R}_i = (\sum_{i=1}^N R_i(\cdot), \sum_{i=1}^N P_i(\cdot)) \\ =& \left(\sum_{i=1}^N \varphi_{c_i}(\cdot - \tilde{x}_i(t^*)), \sum_{i=1}^N \psi_{c_i}(\cdot - \tilde{x}_i(t^*))\right).\end{aligned}$$

By (3.29) and (3.30), and the exponential decay of $\boldsymbol{\varphi}_{c_i}$, we have

$$\begin{aligned}\langle E_i'(\boldsymbol{R}_X), \boldsymbol{R}_X\rangle =& \langle E_i'(\boldsymbol{\varphi}_{c_i}), \boldsymbol{\varphi}_{c_i}\rangle + O(e^{-\frac{L}{4}}) \\ =& \ \|\boldsymbol{\varphi}_{c_i}\|_X^2 + O(e^{-\frac{L}{4}}) \\ >& \frac{1}{2}\|\boldsymbol{\varphi}_{c_i}\|_X^2,\end{aligned} \tag{3.51}$$

which implies $\boldsymbol{\mu}_i$ $(i = 1, \ldots, N)$ are well defined.

We also set $\boldsymbol{v} = (v, w) = \boldsymbol{u} - \boldsymbol{R}_X$. The goal of this subsection is to show $\|\boldsymbol{v}\|_X \leq O(\sqrt{\varepsilon} + e^{-\frac{L}{8}})$. If $\|\boldsymbol{v}\|_X \leq \sqrt{\varepsilon} + e^{-\frac{L}{8}}$, then we can set $M = 2$ to reach the conclusion (3.11). Hence, we subsequently assume $\|\boldsymbol{v}\|_X \geq \sqrt{\varepsilon} + e^{-\frac{L}{8}}$.

By computing the variational derivatives of localized conservation laws E_i and F_i, we know

$$\begin{cases}(E_i)''_{uu} = \Phi_i(1 - \partial_{xx}) - \Phi_i'\partial_x, \\ (E_i)''_{u\eta} = (E_i)''_{\eta u} = 0, \\ (E_i)''_{\eta\eta} = \Phi_i,\end{cases}$$

and

$$\begin{cases} (F_i)''_{uu} = \left(3u - \sigma u_x \partial_x - \sigma u_{xx} - \sigma u \partial_x^2 - A\right)\Phi_i - \sigma\left(u_x \Phi_i' + u\Phi_i' + \partial_x\right), \\ (F_i)''_{u\eta} = (F_i)''_{\eta u} = (\eta + 1)\,\Phi_i, \\ (F_i)''_{\eta\eta} = u\Phi_i. \end{cases}$$

Then the localized Hessian operator H_{c_i} has following form:

$$H_{c_i} = c_i E''(\boldsymbol{\varphi}_{c_i}) - F''\left(\boldsymbol{\varphi}_{c_i}\right) = \begin{pmatrix} L_{c_i} & -\Phi_i(1+\psi_{c_i}) \\ -\Phi_i(1+\psi_{c_i}) & \Phi_i\left(c_i - \varphi_{c_i}\right) \end{pmatrix}, \tag{3.52}$$

where

$$L_{c_i} = -\partial_x\left(\Phi_i(c_i - \sigma\varphi_{c_i})\partial_x\right) + \Phi_i\left(-3\varphi_{c_i} + \sigma\varphi_{c_i,xx} + c_i + A\right) + \sigma\partial_x\varphi_{c_i}.$$

For abbreviation, we use H_i instead of H_{c_i}, L_i instead of L_{c_i}.

The following lemma gives the estimate of a_i and $\boldsymbol{\mu}_i$ through the almost monotonicity property.

Lemma 3.2.6. *Let $\boldsymbol{u} = \boldsymbol{R}_X + \boldsymbol{v} = (1+a_i)\boldsymbol{R}_X + \boldsymbol{\mu}_i$ satisfy $\langle E_i'(\boldsymbol{R}_X), \boldsymbol{\mu}_i\rangle = 0$. Assume $\|\boldsymbol{v}\|_X \geq \sqrt{\varepsilon} + e^{-\frac{L}{8}}$. Then for $i = 1, \cdots, N$,*

$$|a_i| \leq O(\|\boldsymbol{v}\|_X^2), \tag{3.53}$$

and

$$\|\boldsymbol{\mu}_i\|_X \sim \|\boldsymbol{v}\|_X. \tag{3.54}$$

Proof. From the definition of localized conservation laws (3.31) and (3.32) and the property of Φ_i (3.29) and (3.30), we know

$$I_{j,K}\left(\boldsymbol{u}(t^*)\right) = \sum_{j=i}^{N} E_j\left(\boldsymbol{u}(t^*)\right), \text{ for } i = 2, \cdots, N, \tag{3.55}$$

$$E\left(\boldsymbol{u}(t^*)\right) = \sum_{j=1}^{N} E_j\left(\boldsymbol{u}(t^*)\right). \tag{3.56}$$

In light of Taylor formula and equation (3.50), we infer that for $t = t^*$

$$\begin{aligned}\sum_{j=1}^{N} E_j(\boldsymbol{u}) &= \sum_{j=1}^{N} E_j(\boldsymbol{R}_X) + \sum_{j=1}^{N} \langle E_j'(\boldsymbol{R}_X), \boldsymbol{v}\rangle + O(\|\boldsymbol{v}\|_X^2) \\ &= \sum_{j=1}^{N} E_j(\boldsymbol{\varphi}_{c_j}) + \sum_{j=1}^{N} a_j \langle E_j'(\boldsymbol{R}_X), \boldsymbol{R}_X\rangle + O(\|\boldsymbol{v}\|_X^2).\end{aligned} \tag{3.57}$$

Moreover, the way how we construct $\Psi_{j,K}$ implies

$$E_j(\boldsymbol{R}_k) \le O(e^{-\frac{L}{4}}), \quad \text{for} \quad j \ne k, \quad \text{and} \quad E_j(\boldsymbol{R}_j) = E(\boldsymbol{\varphi}_{c_j}) + O(e^{-\frac{L}{4}}). \tag{3.58}$$

Note that the conservation law $E(\boldsymbol{u})$ and almost monotonicity of $I_{j,K}$ provide

$$\begin{aligned}I_{j,K}(\boldsymbol{u}(t^*)) &\le I_{j,K}(\boldsymbol{u}_0) + O(e^{-\frac{L}{4}}) \le \sum_{j=i}^{N} E(\boldsymbol{\varphi}_{c_j}) + O(e^{-\frac{L}{4}}) + O(\varepsilon), \\ E(\boldsymbol{u}(t^*)) &= E(\boldsymbol{u}_0) = \sum_{j=1}^{N} E(\boldsymbol{\varphi}_{c_j}) + O(e^{-\frac{L}{4}}) + O(\varepsilon).\end{aligned} \tag{3.59}$$

Hence, from (3.55)-(3.59), we deduce

$$\sum_{j=i}^{N} a_j \langle E_j'(\boldsymbol{R}_X), \boldsymbol{R}_X\rangle \le O(\|\boldsymbol{v}\|_X^2) \quad i = 1, \cdots, N. \tag{3.60}$$

Similarly, in view of conservation law $F(\boldsymbol{u})$ and Taylor's formula, it follows that for $t = t^*$

$$\begin{aligned}F(\boldsymbol{u}(t^*)) &= \sum_{i=1}^{N} F_i(\boldsymbol{u}(t^*)) = \sum_{i=1}^{N} F_i(\boldsymbol{R}_X) + \sum_{i=1}^{N} \langle F_i'(\boldsymbol{R}_X), \boldsymbol{v}\rangle + O(\|\boldsymbol{v}\|_X^2) \\ &= \sum_{i=1}^{N} F_i(\boldsymbol{\varphi}_{c_i}) + \sum_{i=1}^{N} \langle F_i'(\boldsymbol{R}_X), \boldsymbol{v}\rangle + O(\|\boldsymbol{v}\|_X^2) + O(e^{-\frac{L}{4}}),\end{aligned} \tag{3.61}$$

and

$$F(\boldsymbol{u}(t^*)) = F(\boldsymbol{u}_0) = \sum_{j=1}^{N} F(\boldsymbol{\varphi}_{c_j}) + O(e^{-\frac{L}{4}}) + O(\varepsilon). \tag{3.62}$$

Thus, from equation (3.61) and (3.62), there holds

$$\sum_{i=1}^{N} \langle F_i'(\boldsymbol{R}_X), \boldsymbol{v}\rangle = O(\|\boldsymbol{v}\|_X^2). \tag{3.63}$$

On the other hand, by the identity $F'(\boldsymbol{\varphi}_c) = cE'(\boldsymbol{\varphi}_c)$, (3.29) and (3.30), we know

$$\|F_i'(\boldsymbol{R}_X) - c_iE_i'(\boldsymbol{R}_X)\|_{X^*} \leq \|F'(\boldsymbol{\varphi}_{c_i}) - c_iE'(\boldsymbol{\varphi}_{c_i})\|_{X^*} + O(e^{-\frac{L}{4}}) \leq O(e^{-\frac{L}{4}}),$$

which implies

$$\begin{aligned}\sum_{i=1}^{N}\langle F_i'(\boldsymbol{R}_X), \boldsymbol{v}\rangle &= \sum_{i=1}^{N}\langle F_i'(\boldsymbol{R}_X) - c_iE_i'(\boldsymbol{R}_X), \boldsymbol{v}\rangle + \sum_{i=1}^{N}\langle c_iE_i'(\boldsymbol{R}_X), \boldsymbol{v}\rangle \\ &= \sum_{i=1}^{N}\langle F_i'(\boldsymbol{R}_X) - c_iE_i'(\boldsymbol{R}_X), \boldsymbol{v}\rangle + \sum_{i=1}^{N} c_ia_i\langle E_i'(\boldsymbol{R}_X), \boldsymbol{R}_X\rangle \\ &= \sum_{i=1}^{N} c_ia_i\langle E_i'(\boldsymbol{R}_X), \boldsymbol{R}_X\rangle + O(\|\boldsymbol{v}\|_X^2).\end{aligned} \tag{3.64}$$

Hence, combining (3.63) and (3.64), we obtain

$$\sum_{i=1}^{N} c_ia_i\langle E_i'(\boldsymbol{R}_X), \boldsymbol{R}_X\rangle = O(\|\boldsymbol{v}\|_X^2).$$

Using the Abel Transformation, the above estimate yields

$$\sum_{i=1}^{N}(c_i - c_{i-1})\sum_{j=i}^{N} a_j\langle E_j'(\boldsymbol{R}_X), \boldsymbol{R}_X\rangle + c_1\sum_{i=1}^{N} a_i\langle E_i'(\boldsymbol{R}_X), \boldsymbol{R}_X\rangle = O(\|\boldsymbol{v}\|_X^2). \tag{3.65}$$

Furthermore, associating (3.60) with (3.65), we infer

$$|a_i\langle E_i'(\boldsymbol{R}_X), \boldsymbol{R}_X\rangle| \leq O(\|\boldsymbol{v}\|_X^2), \forall i = 1, \cdots, N,$$

which implies

$$|a_i| \leq O(\|\boldsymbol{v}\|_X^2).$$

Consequently, by (3.50), we have

$$\|\boldsymbol{\mu}_i\|_X \sim \|\boldsymbol{v}\|_X.$$

This competes the proof of Lemma 3.2.6. □

The lemma in the following shows that $E_i(\boldsymbol{\mu}_i)$ is bounded above by the property of local coercivity of H_i.

Lemma 3.2.7. *Let a_i and $\boldsymbol{\mu}_i$ satisfy Lemma 3.2.6. We have*

$$\langle H_i(\boldsymbol{R}_X)\boldsymbol{\mu}_i, \boldsymbol{\mu}_i\rangle \geq C_\delta E_i(\boldsymbol{\mu}_i), \quad i = 1, \cdots, N, \tag{3.66}$$

where E_i and H_i are defined in (3.31) *and* (3.52), *respectively.*

Proof. Notice that (3.66) is equivalent to (3.47) while letting $\Theta = \Phi_i$, $\boldsymbol{\zeta} = \boldsymbol{\mu}_i$, $\boldsymbol{\nu} = ((1-\partial_x^2)R_i, P_i)$ and $H_c = H_i$ in Lemma 3.2.5. Hence, in order to apply this lemma, we claim that for $\boldsymbol{\nu} = ((1-\partial_x^2)R_i, P_i)$,

$$\left|\langle\sqrt{\Phi_i}\boldsymbol{\mu}_i, \partial_x\boldsymbol{\nu}\rangle\right| + \left|\langle\sqrt{\Phi_i}\boldsymbol{\mu}_i, \boldsymbol{\nu}\rangle\right| \leq \left(O(\sqrt{\varepsilon}) + O(e^{-\frac{L}{8}})\right)\|\boldsymbol{\mu}_i\|_X.$$

In fact,

$$\begin{aligned}
\left\langle\sqrt{\Phi_i}\boldsymbol{\mu}_i, \partial_x\boldsymbol{\nu}\right\rangle =& \left\langle\boldsymbol{\mu}_i, \partial_x\boldsymbol{\nu}\right\rangle + \left\langle\left(\sqrt{\Phi_i}-1\right)\boldsymbol{\mu}_i, \partial_x\boldsymbol{\nu}\right\rangle \\
=& \left\langle\mu_i, (1-\partial_x^2)\partial_x R_i\right\rangle + \left\langle\tau_i, \partial_x P_i\right\rangle + \left\langle\left(\sqrt{\Phi_i}-1\right)\tau_i, \partial_x P_i\right\rangle \\
&+ \left\langle\left(\sqrt{\Phi_i}-1\right)\mu_i, (1-\partial_x^2)\partial_x R_i\right\rangle \\
=& -a_i\left\langle R_i, (1-\partial_x^2)\partial_x R_i\right\rangle - a_i\left\langle P_i, \partial_x P_i\right\rangle + \left\langle\left(\sqrt{\Phi_i}-1\right)\tau_i, \partial_x P_i\right\rangle \\
&+ \left\langle\left(\sqrt{\Phi_i}-1\right)\mu_i, (1-\partial_x^2)\partial_x R_i\right\rangle,
\end{aligned}$$

Then, by (3.29) and (3.30), we deduce

$$\left|\left\langle\sqrt{\Phi_i}\boldsymbol{\mu}_i, \partial_x\boldsymbol{\nu}\right\rangle\right| \leq \left(O(\sqrt{\varepsilon}) + O(e^{-\frac{L}{8}})\right)\|\boldsymbol{\mu}_i\|_X.$$

On the other hand, adopting the same method, we have

$$\begin{aligned}
\left\langle\sqrt{\Phi_i}\boldsymbol{\mu}_i, \boldsymbol{\nu}\right\rangle &= \left\langle\boldsymbol{\mu}_i, \boldsymbol{\nu}\right\rangle + \left\langle\left(\sqrt{\Phi_i}-1\right)\boldsymbol{\mu}_i, \boldsymbol{\nu}\right\rangle \\
&= \left\langle\mu_i, (1-\partial_x^2)R_i\right\rangle + \left\langle\tau_i, P_i\right\rangle + \left\langle\left(\sqrt{\Phi_i}-1\right)\mu_i, (1-\partial_x^2)R_i\right\rangle \\
&\quad + \left\langle\left(\sqrt{\Phi_i}-1\right)\tau_i, P_i\right\rangle \\
&= \langle E_i'(R_X), \boldsymbol{\mu}_i\rangle - \int_{\mathbb{R}}(1-\Phi_i)\left(\mu_i R_i + \partial_x\mu_i\partial_x R_i + \tau_i P_i\right)\, dx \\
&\quad - \sum_{j=1, j\neq i}^{N}\int_{\mathbb{R}}(1-\Phi_i)\left(\mu_i R_j + \partial_x\mu_i\partial_x R_j + \tau_i P_j\right)\, dx \\
&\quad + \left\langle\left(\sqrt{\Phi_i}-1\right)\mu_i, (1-\partial_x^2)R_i\right\rangle + \left\langle\left(\sqrt{\Phi_i}-1\right)\tau_i, P_i\right\rangle,
\end{aligned}$$

which infers

$$\left|\left\langle \sqrt{\Phi_i}\boldsymbol{\mu}_i, \boldsymbol{\nu} \right\rangle\right| \leq \left(O(\sqrt{\varepsilon}) + O(e^{-\frac{L}{8}})\right) \|\boldsymbol{\mu}_i\|_X .$$

Hence, according to Lemma 3.2.5, we know (3.66) holds, which implies the proof of Lemma 3.2.7 is compete. □

In the last lemma, we prove that $\sum_{i=1}^{N}\langle H_i(\boldsymbol{R}_X)\boldsymbol{\mu}_i, \boldsymbol{\mu}_i\rangle$, which is the upper bound of $E_i(\boldsymbol{\mu}_i)$, can be controlled by an infinitesimal with the application of conservation laws and Taylor formula.

Lemma 3.2.8. *Let a_i and $\boldsymbol{\mu}_i$ satisfy Lemma 3.2.6. If E_i, F_i are the two localized conservation laws of system* (3.3) *and H_i is defined in* (3.52), *then*

$$\sum_{i=1}^{N}\langle H_i(\boldsymbol{R}_X)\boldsymbol{\mu}_i, \boldsymbol{\mu}_i\rangle = O(\|\boldsymbol{v}\|_X^3) + O(e^{-\frac{L}{4}}) + O(\varepsilon). \tag{3.67}$$

Proof. By Taylor formula, we know

$$\begin{aligned}
\sum_{i=1}^{N}(c_iE_i(\boldsymbol{u}) - F_i(\boldsymbol{u})) =& \sum_{i=1}^{N}(c_iE_i(\boldsymbol{R}_X) - F_i(\boldsymbol{R}_X)) + \sum_{i=1}^{N}\langle c_iE_i'(\boldsymbol{R}_X) - F_i'(\boldsymbol{R}_X), \boldsymbol{v}_i\rangle \\
&+ \frac{1}{2}\sum_{i=1}^{N}\langle c_iE_i''(\boldsymbol{R}_X) - F_i''(\boldsymbol{R}_X), \boldsymbol{v}_i^2\rangle \\
&+ O(\|\boldsymbol{v}\|_X^3) + O(e^{-\frac{L}{4}}) \\
=& \sum_{i=1}^{N}(c_iE_i(\boldsymbol{R}_X) - F_i(\boldsymbol{R}_X)) + \frac{1}{2}\sum_{i=1}^{N}\langle c_iE_i''(\boldsymbol{R}_X) - F_i''(\boldsymbol{R}_X), \boldsymbol{v}_i^2\rangle \\
&+ O(\|\boldsymbol{v}\|_X^3) + O(e^{-\frac{L}{4}}).
\end{aligned}$$

More precisely, the decomposition of $\boldsymbol{v}$ and (3.50) ensure

$$\begin{aligned}
\sum_{i=1}^{N}(c_iE_i(\boldsymbol{u}) - F_i(\boldsymbol{u})) =& \sum_{i=1}^{N}(c_iE_i(\boldsymbol{R}_X) - F_i(\boldsymbol{R}_X)) + \frac{1}{2}\sum_{i=1}^{N}\langle H_i(\boldsymbol{R}_X)\boldsymbol{\mu}_i, \boldsymbol{\mu}_i\rangle \\
&+ \sum_{i=1}^{N}a_i\langle H_i(\boldsymbol{R}_X)\boldsymbol{R}_X, \boldsymbol{\mu}_i\rangle + \sum_{i=1}^{N}\frac{a_i^2}{2}\langle H_i(\boldsymbol{R}_X)\boldsymbol{R}_X, \boldsymbol{R}_X\rangle \\
&+ O(\|\boldsymbol{v}\|_X^3) + O(e^{-\frac{L}{4}}).
\end{aligned}$$

which implies

$$\begin{aligned}\sum_{i=1}^N \langle H_i(\boldsymbol{R}_X)\boldsymbol{\mu}_i, \boldsymbol{\mu}_i\rangle =& \sum_{i=1}^N 2\left[c_iE_i(\boldsymbol{u}) - c_iE_i(\boldsymbol{R}_X)\right] - \sum_{i=1}^N 2\left[F_i(\boldsymbol{u}) - F_i(\boldsymbol{R}_X)\right] \\ & - \sum_{i=1}^N 2a_i\langle H_i(\boldsymbol{R}_X)\boldsymbol{R}_X, \boldsymbol{\mu}_i\rangle - \sum_{i=1}^N a_i^2\langle H_i(\boldsymbol{R}_X)\boldsymbol{R}_X, \boldsymbol{R}_X\rangle \\ & + O(\|\boldsymbol{v}\|_X^3) + O(e^{-\frac{L}{4}}).\end{aligned} \tag{3.68}$$

For the first term on the left side of (3.68), Abel's transformation infers

$$\begin{aligned}\sum_{i=1}^N c_iE_i(\boldsymbol{u}) &= \sum_{i=2}^N (c_i - c_{i-1})\sum_{j=i}^N E_j(\boldsymbol{u}) + c_1\sum_{j=1}^N E_j(\boldsymbol{u}) \\ &= \sum_{i=2}^N (c_i - c_{i-1})\, I_i(t^*, \boldsymbol{u}(t^*)) + c_1E(\boldsymbol{u}(t^*)) \\ &\le \sum_{i=2}^N (c_i - c_{i-1})\, I_i(0, \boldsymbol{u}_0) + c_1E(\boldsymbol{u}_0) + O(e^{-\frac{L}{4}}) \\ &\le \sum_{i=2}^N (c_i - c_{i-1})\sum_{j=i}^N E(\boldsymbol{\varphi}_{c_j}) + c_1\sum_{j=1}^N E(\boldsymbol{\varphi}_{c_j}) + O(e^{-\frac{L}{4}}) + O(\varepsilon) \\ &\le \sum_{i=1}^N c_iE(\boldsymbol{\varphi}_{c_i}) + O(e^{-\frac{L}{4}}) + O(\varepsilon) \\ &\le \sum_{i=1}^N c_iE_i(\boldsymbol{R}_X) + O(e^{-\frac{L}{4}}) + O(\varepsilon).\end{aligned} \tag{3.69}$$

For the second term on the left side of (3.68), by conservation law we obtain

$$\sum_{i=1}^N F_i(\boldsymbol{u}) = \sum_{i=1}^N F_i(\boldsymbol{u}_0) = \sum_{i=1}^N F_i(\boldsymbol{R}_X) + O(e^{-\frac{L}{4}}) + O(\varepsilon). \tag{3.70}$$

The estimates for the third term and fourth term on the left side of (3.68) follow with Lemma 3.2.6,

$$\sum_{i=1}^N a_i\langle H_i(\boldsymbol{R}_X)\boldsymbol{R}_X, \boldsymbol{\mu}_i\rangle = O(\|\boldsymbol{v}\|_X^3) + O(e^{-\frac{L}{4}}) + O(\varepsilon), \tag{3.71}$$

and

$$\sum_{i=1}^N a_i^2\langle H_i(\boldsymbol{R}_X)\boldsymbol{R}_X, \boldsymbol{R}_X\rangle = O(\|\boldsymbol{v}\|_X^3) + O(e^{-\frac{L}{4}}) + O(\varepsilon). \tag{3.72}$$

In consequence, substituting (3.69)-(3.72) into (3.68), we complete the proof of Lemma 3.2.8. □

3.3 Proof of stability theorem

Proof of Theorem 3.1.6. According to equation (3.66) and (3.67), we have

$$\sum_{i=1}^{N} E_i(\boldsymbol{\mu}_i) \leq \sum_{i=1}^{N} \langle H_i(\boldsymbol{R}_X)\boldsymbol{\mu}_i, \boldsymbol{\mu}_i \rangle = O(\|\boldsymbol{v}\|_X^3) + O(e^{-\frac{L}{4}}) + O(\varepsilon). \tag{3.73}$$

Meanwhile, by the definition of $\boldsymbol{\mu}_i$ (3.50) and the property of (3.53) the following estimate holds

$$\sum_{i=1}^{N} E_i(\boldsymbol{\mu}_i) = \sum_{i=1}^{N} E_i(\boldsymbol{v}) + O(\|\boldsymbol{v}\|_X^3). \tag{3.74}$$

Hence, in light of

$$\sum_{i=1}^{N} E_i(\boldsymbol{v}) = E(\boldsymbol{v}) = \|\boldsymbol{v}\|_X^2 , \tag{3.75}$$

there exists a constant C, such that

$$\|\boldsymbol{v}\|_X \leq O(e^{-\frac{L}{8}}) + O(\sqrt{\varepsilon}) \leq C(e^{-\frac{L}{8}} + \sqrt{\varepsilon}). \tag{3.76}$$

Let M be $2C$ in (3.10). This then concludes the result of Proposition 3.2.1. Consequently, the proof of Theorem 3.1.6 is complete. □

CHAPTER 4

A SHALLOW-WATER MODELING WITH THE CORIOLIS EFFECT AND TRAVELING WAVES

4.1 Model equations in the rotational shallow water

In this chapter, we consider the incompressible geophysical fluid dynamics with the Coriolis effect. We will first establish the Green-Naghdi equations with effect of the Coriolis forcing in shallow water.

4.1.1 Derivation of the rotation-Green-Naghdi equations

It is known that the Green-Naghdi equations (or the Serre-Green-Naghdi equations) [33, 56] are the first order approximation of the 2D governing water wave equations in the shallow-water scaling ($\mu \ll 1$, $\varepsilon = O(1)$) [2, 33]. It is the one-dimensional surface wave system coupled with free surface elevation η to the vertically averaged horizontal component of the velocity $\bar{u}$ in the form

$$\begin{cases} \eta_t + ((1+\varepsilon\eta)\bar{u})_x = 0, \\ \bar{u}_t + \eta_x + \varepsilon\bar{u}\bar{u}_x = \frac{\mu}{3(1+\varepsilon\eta)}\left((1+\varepsilon\eta)^3(\bar{u}_{xt} + \varepsilon\bar{u}\bar{u}_{xx} - \varepsilon\bar{u}_x^2)\right)_x + O(\mu^2). \end{cases}$$

It is our purpose here to establish equations with the effect of the Earth rotation analogous to the classical Green-Naghdi equations. This is the starting point our derivation of other lower order approximation models with the Coriolis effect. These equations, so called the rotation-Green-Naghdi (R-GN) equations, are now proposed in the following.

$$\begin{cases} \eta_t + ((1+\varepsilon\eta)\bar{u})_x = 0, \\ \bar{u}_t + \eta_x + \varepsilon\bar{u}\bar{u}_x + 2\Omega\eta_t = \frac{\mu}{3(1+\varepsilon\eta)}\left((1+\varepsilon\eta)^3(\bar{u}_{xt} + \varepsilon\bar{u}\bar{u}_{xx} - \varepsilon\bar{u}_x^2)\right)_x + O(\mu^2). \end{cases} \tag{4.1}$$

It is noted that the R-GN model in (4.1) (for the solution $(\eta, \bar{u})$) is locally well-posed in the Sobolev space $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ with $s > \frac{3}{2}$ [10], while the case of without the Coliolis effect was studied in [48]. In what follows, we are going to formally demonstrate derivation of the R-GN equations (4.1) in the above approximated from the governing equations in the f-plane (1.12).

4.1.1.1 The first equation of the rotation-Green-Naghdi equations

This is the same as in the classical Green-Naghdi equations [33]. Let $\bar{u}$ be the average horizontal velocity,

$$\bar{u}(t,x) \stackrel{\text{def}}{=} \frac{1}{h}\int_0^h u(t,x,z)\, dz, \tag{4.2}$$

where $h = h(t,x) = 1 + \varepsilon\eta(t,x)$. We multiply (4.2) by h and differentiate it with respect to x to find

$$(h\bar{u})_x = \int_0^h u_x\, dz + \varepsilon\eta_x u_h,$$

where $u_h = u(t,x,z)|_{z=h}$. According to (1.21c), u_x can be substituted by $-w_z$. Then with the information provided by (1.21f) and (1.21g), the above equation can be written as

$$\eta_t + (h\bar{u})_x = 0, \tag{4.3}$$

or

$$h_t + \varepsilon(h\bar{u})_x = 0. \tag{4.4}$$

4.1.1.2 The second equation of the rotation-Green-Naghdi equations

In this subsection, we will derive the second equation of the R-GN model equations, where the assumptions on the pressure field play the crucial role. To this end, we require the shallowness parameter $\mu \ll 1$, but without any assumption on ε.

The process will be divided into two parts. Firstly, let

$$u(t,x,z) = u_0(t,x,z) + \mu u_1(t,x,z) + O(\mu^2). \tag{4.5}$$

We are going to present u, w, p in terms of u_0, h, z and find the equation related to u_0 and h only. For the linear problem ($\mu \to 0$), the expression in (1.21d) implies $u_{0,z} = 0$. Hence, u_0 is a function independent of z, i.e. $u_0 = u_0(x,t)$. From (1.21c) and (1.21d), we have

$$\mu u_{xx} + u_{zz} = 0,$$

which implies

$$\begin{aligned} \mu^0 : \ & u_{0,zz} = 0, \\ \mu^1 : \ & u_{0,xx} = -u_{1,zz}. \end{aligned} \tag{4.6}$$

Considering $u_0 = u_0(t,x)$, the equation of order μ^1 implies

$$u_1 = -\frac{z^2}{2} u_{0,xx} + z\Psi(t,x),$$

where $\Psi(t,x)$ is an arbitrary function. Therefore,

$$u = u_0 - \mu \frac{z^2}{2} u_{0,xx} + \mu z \Psi(t,x) + O(\mu^2). \tag{4.7}$$

Applying (4.7), (1.21c) and (1.21g) to the identity $w = w|_{z=0} + \int_0^z w_{z'}\, dz'$, we obtain

$$w = -z u_{0,x} + \mu \frac{z^3}{6} u_{0,xxx} - \mu \frac{z^2}{2} \Psi_x(t,x) + O(\mu^2). \tag{4.8}$$

Now substituting u and w into (1.21b) gives

$$-p_z = -\mu z (u_{0,xt} + \varepsilon u_0 u_{0,xx} - \varepsilon u_{0,x}^2) - 2\Omega u_0 + \mu \Omega z^2 u_{0,xx} - 2\Omega \mu z \Psi + O(\mu^2).$$

In light of this equation, (1.21e) and the identity $p = p|_{z=h} - \int_z^h p_{z'}\, dz'$, we get

$$p = \eta - \frac{\mu}{2}(h^2 - z^2)(u_{0,xt} + \varepsilon u_0 u_{0,xx} - \varepsilon u_{0,x}^2) - 2\Omega(h - z)u_0 + \frac{\mu}{3}\Omega(h^3 - z^3)u_{0,xx} - \mu\Omega(h^2 - z^2)\Psi + O(\mu^2).$$

So far the expressions of u, w, p are discovered. To find the equation related to u_0 and h relies on these expressions and (1.21a). Actually, on the one hand, we differentiate the equation above by x, which yields

$$\begin{aligned} p_x = \eta_x &- \frac{\mu}{2}(h^2 - z^2)(u_{0,xt} + \varepsilon u_0 u_{0,xx} - \varepsilon u_{0,x}^2)_x - \mu h h_x(u_{0,xt} + \varepsilon u_0 u_{0,xx} - \varepsilon u_{0,x}^2) \\ &- 2\Omega h_x u_0 - 2\Omega(h - z)u_{0,x} + \mu\Omega h^2 h_x u_{0,xx} + \frac{\mu}{3}\Omega(h^3 - z^3)u_{0,xxx} \\ &- 2\mu\Omega h h_x \Psi - \mu\Omega(h^2 - z^2)\Psi_x + O(\mu^2). \end{aligned}$$

On the other hand, we use (4.7) and (4.8) in (1.21a) to deduce

$$\begin{aligned} u_{0,t} - \mu\frac{z^2}{2}u_{0,xxt} + \mu z\Psi_t + \varepsilon u_0 u_{0,x} + \varepsilon\mu\frac{z^2}{2}(u_{0,x}u_{0,xx} - u_0 u_{0,xxx}) + \varepsilon\mu z u_0 \Psi_x \\ -2\Omega z u_{0,x} + \mu\frac{\Omega z^3}{3}u_{0,xxx} - \mu\Omega z^2\Psi_x = -p_x + O(\mu^2). \end{aligned}$$

Combining the last two equations, we have

$$\begin{aligned} u_{0,t} + \mu z\Psi_t + \varepsilon u_0 u_{0,x} + \varepsilon\mu z u_0 \Psi_x + \eta_x &= \frac{\mu}{2}h^2(u_{0,xt} + \varepsilon u_0 u_{0,xx} - \varepsilon u_{0,x}^2)_x \\ &+ \mu h h_x(u_{0,xt} + \varepsilon u_0 u_{0,xx} - \varepsilon u_{0,x}^2) + 2\Omega(h_x u_0 + h u_{0,x}) - \mu\Omega h^2 h_x u_{0,xx} \\ &- \frac{\mu}{3}\Omega h^3 u_{0,xxx} + 2\mu\Omega h h_x\Psi + \mu\Omega h^2\Psi_x + O(\mu^2). \end{aligned}$$

Then integrating the above equation with respect to z from 0 to h, we find the equation related to u_0, η, h, i.e.

$$\begin{aligned} u_{0,t} + \mu\frac{h}{2}\Psi_t + \varepsilon u_0 u_{0,x} + \varepsilon\mu\frac{h}{2}u_0\Psi_x + \eta_x &= \frac{\mu}{2}h^2(u_{0,xt} + \varepsilon u_0 u_{0,xx} - \varepsilon u_{0,x}^2)_x \\ &+ \mu h h_x(u_{0,xt} + \varepsilon u_0 u_{0,xx} - \varepsilon u_{0,x}^2) + 2\Omega(h_x u_0 + h u_{0,x}) - \mu\Omega h^2 h_x u_{0,xx} \\ &- \frac{\mu}{3}\Omega h^3 u_{0,xxx} + 2\mu\Omega h h_x\Psi + \mu\Omega h^2\Psi_x + O(\mu^2). \end{aligned} \tag{4.9}$$

Secondly, to reveal this is the second equation of the R-GN equations, we will introduce the average horizontal velocity $\bar{u}$ to replace u. The relation between $\bar{u}$ and u_0 is

$$\bar{u} = u_0 - \mu\frac{h^2}{6}u_{0,xx} + \mu\frac{h}{2}\Psi + O(\mu^2), \tag{4.10}$$

which can be obtained by using the function u's expression (4.7) in the definition of $\bar{u}$ (4.2). Now we show more facts about $\bar{u}$ and u_0.

1. Invoking (4.10), we obtain

$$\mu\bar{u} = \mu u_0 + O(\mu^2), \tag{4.11}$$

$$\bar{u}_t = u_{0,t} - \mu\frac{h^2}{6}u_{0,xxt} - \mu\frac{hh_t}{3}u_{0,xx} + \mu\frac{h_t}{2}\Psi + \mu\frac{h}{2}\Psi_t + O(\mu^2). \tag{4.12}$$

Similarly,

$$\frac{\mu}{3h}\left(h^3(\bar{u}_{xt} + \varepsilon\bar{u}\bar{u}_{xx} - \varepsilon\bar{u}_x^2)\right)_x = \frac{\mu}{3h}\left(h^3(u_{0,xt} + \varepsilon u_0 u_{0,xx} - \varepsilon u_{0,x}^2)\right)_x + O(\mu^2). \tag{4.13}$$

2. Next, we show the fact that

$$\begin{aligned}\varepsilon\bar{u}\bar{u}_x = \varepsilon u_0 u_{0,x} + \mu\frac{hh_t}{3}u_{0,xx} - \mu\frac{h_t}{2}\Psi + \varepsilon\mu\frac{h}{2}u_0\Psi_x \\ - \mu\frac{h^2}{6}(\varepsilon u_0 u_{0,xx} - \varepsilon u_{0,x}^2)_x + O(\mu^2).\end{aligned} \tag{4.14}$$

To confirm this, we start with (4.10), which implies

$$\begin{aligned}\varepsilon\bar{u}\bar{u}_x = \varepsilon u_0 u_{0,x} - \varepsilon\mu\frac{h^2}{6}u_0 u_{0,xxx} - \varepsilon\mu\frac{hh_x}{3}u_0 u_{0,xx} + \varepsilon\mu\frac{h_x}{2}u_0\Psi \\ + \varepsilon\mu\frac{h}{2}u_0\Psi_x - \varepsilon\mu\frac{h^2}{6}u_{0,x}u_{0,xx} + \varepsilon\mu\frac{h}{2}u_{0,x}\Psi + O(\mu^2).\end{aligned}$$

Now according to Eq.(4.4), we have

$$-\mu\frac{h_t}{2}\Psi = \frac{\mu}{2}\Psi(\varepsilon h_x\bar{u} + \varepsilon h\bar{u}_x) = \frac{\varepsilon\mu}{2}h_x u_0\Psi + \frac{\varepsilon\mu}{2}hu_{0,x}\Psi + O(\mu^2). \tag{4.15}$$

The form of $\varepsilon\bar{u}\bar{u}_x$ then becomes

$$\varepsilon\bar{u}\bar{u}_x = \varepsilon u_0 u_{0,x} - \varepsilon\mu\frac{h^2}{6}u_0 u_{0,xxx} - \varepsilon\mu\frac{hh_x}{3}u_0 u_{0,xx} - \mu\frac{h_t}{2}\Psi + \varepsilon\mu\frac{h}{2}u_0\Psi_x - \varepsilon\mu\frac{h^2}{6}u_{0,x}u_{0,xx} + O(\mu^2),$$

which in turn implies that

$$\begin{aligned}\varepsilon\bar{u}\bar{u}_x = \varepsilon u_0 u_{0,x} - \varepsilon\mu\frac{h^2}{6}u_0 u_{0,xxx} - \varepsilon\mu\frac{h^2}{3}u_{0,x}u_{0,xx} - \varepsilon\mu\frac{hh_x}{3}u_0 u_{0,xx} \\ +\varepsilon\mu\frac{h^2}{6}u_{0,x}u_{0,xx} - \mu\frac{h_t}{2}\Psi + \varepsilon\mu\frac{h}{2}u_0\Psi_x + O(\mu^2).\end{aligned}$$

In addition, it is inferred from (4.4) that

$$\mu\frac{hh_t}{3}u_{0,xx} = -\varepsilon\mu\frac{h^2}{3}u_{0,x}u_{0,xx} - \varepsilon\mu\frac{hh_x}{3}u_0 u_{0,xx} + O(\mu^2). \tag{4.16}$$

Consequently, we get (4.14) by plugging (4.16) into the expression of $\varepsilon\bar{u}\bar{u}_x$.

3. By (4.3) and (4.10), it is found that

$$\begin{aligned}-2\Omega\eta_t = 2\Omega(h\bar{u})_x =& 2\Omega(hu_0)_x - \mu\Omega h^2 h_x u_{0,xx} - \frac{\mu}{3}\Omega h^3 u_{0,xxx} \\ &+ 2\mu\Omega hh_x\Psi + \mu\Omega h^2\Psi_x + O(\mu^2).\end{aligned} \tag{4.17}$$

With these facts in hand, we will establish the second equation in the R-GN equations as follows. To this end, subtracting $\frac{\mu}{6}h^2(u_{0,xt} + \varepsilon u_0 u_{0,xx} - \varepsilon u_{0,x}^2)_x$ on both sides of (4.9), we get

$$\begin{aligned}&u_{0,t} - \frac{\mu}{6}h^2(u_{0,xt} + \varepsilon u_0 u_{0,xx} - \varepsilon u_{0,x}^2)_x + \mu\frac{h}{2}\Psi_t + \varepsilon u_0 u_{0,x} + \varepsilon\mu\frac{h}{2}u_0\Psi_x + \eta_x \\ &= \frac{\mu}{3}h^2(u_{0,xt} + \varepsilon u_0 u_{0,xx} - \varepsilon u_{0,x}^2)_x + \mu hh_x(u_{0,xt} + \varepsilon u_0 u_{0,xx} - \varepsilon u_{0,x}^2) \\ &\quad + 2\Omega(hu_0)_x - \mu\Omega h^2 h_x u_{0,xx} - \frac{\mu}{3}\Omega h^3 u_{0,xxx} + 2\mu\Omega hh_x\Psi + \mu\Omega h^2\Psi_x + O(\mu^2),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&u_{0,t} - \mu\frac{h^2}{6}u_{0,xxt} - \mu\frac{hh_t}{3}u_{0,xx} + \mu\frac{h_t}{2}\Psi + \mu\frac{h}{2}\Psi_t \\ &\quad + \varepsilon u_0 u_{0,x} + \mu\frac{hh_t}{3}u_{0,xx} - \mu\frac{h_t}{2}\Psi + \varepsilon\mu\frac{h}{2}u_0\Psi_x - \mu\frac{h^2}{6}(\varepsilon u_0 u_{0,xx} - \varepsilon u_{0,x}^2)_x + \eta_x \\ &= \frac{\mu}{3}h^2(u_{0,xt} + \varepsilon u_0 u_{0,xx} - \varepsilon u_{0,x}^2)_x + \mu hh_x(u_{0,xt} + \varepsilon u_0 u_{0,xx} - \varepsilon u_{0,x}^2) \\ &\quad + 2\Omega(hu_0)_x - \mu\Omega h^2 h_x u_{0,xx} - \frac{\mu}{3}\Omega h^3 u_{0,xxx} + 2\mu\Omega hh_x\Psi + \mu\Omega h^2\Psi_x + O(\mu^2).\end{aligned}$$

In light of (4.12), (4.13), (4.14) and (4.17), the second equation of R-GN equations has the form as

$$\bar{u}_t + \eta_x + \varepsilon\bar{u}\bar{u}_x + 2\Omega\eta_t = \frac{\mu}{3(1+\varepsilon\eta)}\big((1+\varepsilon\eta)^3(\bar{u}_{xt} + \varepsilon\bar{u}\bar{u}_{xx} - \varepsilon\bar{u}_x^2)\big)_x + O(\mu^2).$$

With the transport equation in (4.3), there appears the R-GN equations (4.1).

Remark 7. It is observed that presumably from the irrotational condition (1.21d), (4.7) and (4.8), the function Ψ appeared in the expression of (4.10) could be zero.

For simplicity, we will drop the superscript 'bar' in $\bar{u}$ in the following subsections.

4.1.2 The rotation-Korteweg-de Vries and the rotation-Benjamin-Bona-Mahony equations

In view of the derivation of the R-GN equations in the previous subsection, our attention is now to turned to the cases for the Korteweg-de Vries (KdV) and the Benjamin-Bona-Mahony (BBM) equations. We derive the rotation-Korteweg-de Vries (R-KdV) and the rotation-Benjamin-Bona-Mahony (R-BBM) equations in this subsection. With the expression of εv and the second equation in (4.30), we obtain

$$u_t + cu_x + \varepsilon\frac{3c^2}{2(c^2+1)}(u^2)_x - \mu\frac{c^2}{3\,(c^2+1)}u_{xxt} = O(\varepsilon^2, \varepsilon\mu, \mu^2), \tag{4.18}$$

which is actually the R-BBM equation. Replacing u_{xxt} by $-cu_{xxx} + O(\varepsilon, \mu)$, it is the R-KdV equation, namely

$$u_t + cu_x + \varepsilon\frac{3}{2}\cdot\frac{c^2}{c^2+1}(u^2)_x + \mu\frac{c^3}{3(c^2+1)}u_{xxx} = O(\varepsilon^2, \varepsilon\mu, \mu^2). \tag{4.19}$$

4.1.2.1 Assume $\eta = \frac{1}{c}u + \varepsilon v$ and determine the expression of v

In view of the R-GN equations in (4.1), we readily check that the leading order expansion with respect to two small independent parameters ε and μ gives the following Boussinesq system with the Earth rotation

$$\begin{cases} \eta_t + ((1+\varepsilon\eta)\,u)_x = 0, \\ u_t + \eta_x + \varepsilon u u_x + 2\Omega\eta_t = \frac{\mu}{3} u_{xxt} + O(\varepsilon\mu, \mu^2). \end{cases} \tag{4.20}$$

Consider now the linear terms in (4.20) in terms of ε and μ given by

$$\begin{cases} \eta_t + u_x = O(\varepsilon, \mu), \\ u_t + \eta_x + 2\Omega\eta_t = O(\varepsilon, \mu). \end{cases} \tag{4.21}$$

This formula in turn implies that

$$\begin{cases} \eta_{tt} - \eta_{xx} - 2\Omega\eta_{xt} = O(\varepsilon, \mu), \\ u_{tt} - u_{xx} - 2\Omega u_{xt} = O(\varepsilon, \mu). \end{cases} \tag{4.22}$$

Solving the second order linear partial differential equation, we have the following relations

$$\begin{cases} \eta = \eta_1(x - ct) + \eta_2(x + (c + 2\Omega)t) + O(\varepsilon, \mu), \\ u = u_1(x - ct) + u_2(x + (c + 2\Omega)t) + O(\varepsilon, \mu), \end{cases} \tag{4.23}$$

where $c = \sqrt{1 + \Omega^2} - \Omega$. For simplicity, we only consider the waves move towards to the right side, i.e.

$$\begin{cases} \eta = \eta\,(x - ct) + O(\varepsilon, \mu), \\ u = u\,(x - ct) + O(\varepsilon, \mu), \end{cases} \tag{4.24}$$

which implies

$$\begin{cases} \eta_t = -c\eta_x + O(\varepsilon, \mu), \\ u_t = -c u_x + O(\varepsilon, \mu). \end{cases} \tag{4.25}$$

According to (4.21) and (4.24), we let

$$\eta = \frac{1}{c} u + \varepsilon v. \tag{4.26}$$

Since $v = v(u, u_x, \cdots)$, we conclude the relation between v_t and v_x from (4.25) by

$$v_t = -c v_x + O(\varepsilon, \mu). \tag{4.27}$$

Therefore, (4.20) can be converted to

$$\begin{cases} \left(\frac{1}{c}u+\varepsilon v\right)_t + \left(\left(1+\varepsilon\left(\frac{1}{c}u+\varepsilon v\right)\right)u\right)_x = O(\mu^2), \\ u_t + \left(\frac{1}{c}u+\varepsilon v\right)_x + \varepsilon u u_x - 2\Omega\left(\left(1+\varepsilon(\frac{1}{c}u+\varepsilon v)\right)u\right)_x - \frac{\mu}{3}u_{xxt} = O(\varepsilon\mu,\mu^2). \end{cases} \tag{4.28}$$

Multiplying the first equation in (4.28) by c and reorganizing the system with order of parameters ε and μ, we have

$$\begin{cases} u_t + cu_x + \varepsilon\left(cv_t + (u^2)_x\right) + \varepsilon^2 c\,(uv)_x = O(\mu^2), \\ u_t + cu_x + \varepsilon v_x + \varepsilon u u_x - \varepsilon\frac{2\Omega}{c}\left(u^2\right)_x - \varepsilon^2\, 2\Omega\ (uv)_x - \frac{\mu}{3}u_{xxt} = O(\varepsilon\mu,\mu^2). \end{cases} \tag{4.29}$$

Using the relation between c and Ω, and truncating the system up to order of $O(\varepsilon,\mu)$, the system (4.29) is then simplified as

$$\begin{cases} u_t + cu_x + \varepsilon\left(cv_t\right) + \varepsilon\left(u^2\right)_x = O(\varepsilon^2,\mu^2), \\ u_t + cu_x + \varepsilon v_x + \varepsilon\frac{3c^2-2}{2c^2}\left(u^2\right)_x - \frac{\mu}{3}u_{xxt} = O(\varepsilon^2,\varepsilon\mu,\mu^2). \end{cases} \tag{4.30}$$

We now derive the expression of εv. To this end, using the first equation in (4.30) to subtract the second equation in (4.30) gives

$$\varepsilon\left(cv_t - v_x\right) + \varepsilon\frac{2-c^2}{2c^2}\left(u^2\right)_x + \frac{\mu}{3}u_{xxt} = O(\varepsilon^2,\varepsilon\mu,\mu^2). \tag{4.31}$$

Plugging (4.27) into (4.31) yields

$$-\varepsilon\left(c^2+1\right)v_x + \varepsilon\frac{2-c^2}{2c^2}\left(u^2\right)_x + \frac{\mu}{3}u_{xxt} = O(\varepsilon^2,\varepsilon\mu,\mu^2). \tag{4.32}$$

Consequently, integrating (4.32) with respect to x, we have

$$\varepsilon v = \varepsilon\frac{2-c^2}{2c^2(c^2+1)}u^2 + \frac{\mu}{3\left(c^2+1\right)}u_{xt} + O(\varepsilon^2,\varepsilon\mu,\mu^2). \tag{4.33}$$

With the expression of εv and the second equation in (4.30), we obtain

$$\begin{aligned} u_t + cu_x + \varepsilon\left(\frac{1}{2\left(c^2+1\right)} + \frac{2\Omega}{c\left(c^2+1\right)}\right)\left(u^2\right)_x + \frac{\mu}{3\left(c^2+1\right)}u_{xxt} & \\ +\varepsilon\left(\frac{1}{2} - \frac{2\Omega}{c}\right)\left(u^2\right)_x - \frac{\mu}{3}u_{xxt} &= O(\varepsilon^2,\varepsilon\mu,\mu^2). \end{aligned} \tag{4.34}$$

which implies

$$\begin{aligned} u_t + cu_x + \varepsilon\left(\frac{1}{2} + \frac{1}{2\,(c^2+1)} - \frac{2\Omega c}{(c^2+1)}\right)\left(u^2\right)_x \\ -\mu\left(\frac{1}{3} - \frac{1}{3\,(c^2+1)}\right)u_{xxt} &= O(\varepsilon^2, \varepsilon\mu, \mu^2). \end{aligned} \tag{4.35}$$

This is actually the R-BBM equation

$$u_t + cu_x + \varepsilon\left(\frac{3}{2}\,\frac{c^2}{c^2+1}\right)\left(u^2\right)_x - \mu\left(\frac{c^2}{3\,(c^2+1)}\right)u_{xxt} = O(\varepsilon^2, \varepsilon\mu, \mu^2). \tag{4.36}$$

If replace u_{xxt} by $-cu_{xxx} + O(\varepsilon, \mu)$, it is the R-KdV equation in the following form

$$u_t + cu_x + \varepsilon\left(\frac{3}{2}\,\frac{c^2}{c^2+1}\right)\left(u^2\right)_x + \mu\left(\frac{c^3}{3\,(c^2+1)}\right)u_{xxx} = O(\varepsilon^2, \varepsilon\mu, \mu^2), \tag{4.37}$$

Adopting the system (4.30), taking derivative with respect to x of the first equation, and taking derivative with respect to t of the second equation, we have

$$\begin{cases} u_{xt} + cu_{xx} + \varepsilon\,(cv_{tx}) + \varepsilon\,(u^2)_{xx} = O(\varepsilon^2, \varepsilon\mu, \mu^2), \\ u_{tt} + cu_{xt} + \varepsilon v_{xt} + \varepsilon\frac{3c^2-2}{2c^2}\,(u^2)_{xt} - \frac{\mu}{3}u_{xxtt} = O(\varepsilon^2, \varepsilon\mu, \mu^2). \end{cases} \tag{4.38}$$

After cancelling u_{xt}, we deduce

$$u_{tt} - c^2u_{xx} + \varepsilon\left(1-c^2\right)v_{xt} + \varepsilon\frac{3c^2-2}{2c^2}\left(u^2\right)_{xt} - \varepsilon c\left(u^2\right)_{xx} - \frac{\mu}{3}u_{xxtt} + O(\varepsilon^2, \varepsilon\mu, \mu^2). \tag{4.39}$$

Applying (4.25) and (4.33) to the above equation, one can also derive the following rotation-improved-Boussinesq equation in the unidirectional case.

$$\begin{aligned} u_{tt} - c^2u_{xx} + \varepsilon\frac{(1-c^2)(2-c^2)}{2c^2(c^2+1)}\left(u^2\right)_{xt} + \frac{\mu\,(1-c^2)}{3\,(c^2+1)}u_{xxtt} \\ +\varepsilon\frac{3c^2-2}{2c^2}\left(u^2\right)_{xt} - \varepsilon c\left(u^2\right)_{xx} - \frac{\mu}{3}u_{xxtt} &= O(\varepsilon^2, \varepsilon\mu, \mu^2). \end{aligned} \tag{4.40}$$

4.2 Justification of the rotation-Camassa-Holm equation

4.2.1 Uniform estimates for the solutions of the Rotation-Camassa-Holm equation

Our attention in this section is now turned to the uniform boundedness of the solution to the R-CH equation. Denote

$$\|f\|^2_{X^{s+1}_\mu} \stackrel{\text{def}}{=} \|f\|^2_{H^s} + \mu\,\beta\|f_x\|^2_{H^s}, \text{ where } \mu > 0, \beta > 0,$$

and the KdV regime

$$\mathcal{K}_{C_0,C_0',C_0''} = \{(\varepsilon,\mu)|\ 0 < C_0\mu \le \varepsilon \le C_0'\mu \le C_0''\},$$

for some given constants $C_0, C_0', C_0'' > 0$, and the Camassa-Holm regime

$$\mathcal{P}_{\mu_0,M} = \{(\varepsilon,\mu)\,|\ 0 < \mu \le \mu_0, 0 < \varepsilon \le M\sqrt{\mu}\}\,,$$

for given constants $\mu_0, M > 0$. The uniform estimate for the solution of the R-CH equation (1.24) was established already in [21] for a more general CH equation.

Proposition 4.2.1. *([21]) Assume that $\mu_0 > 0$, $M > 0$, $s > \frac{3}{2}$, $\Omega > 0$, $c = c(\Omega) = \sqrt{1+\Omega^2} - \Omega$, $\alpha = \frac{c^2}{c^2+1}$, $\beta > 0$, $\beta - \frac{1}{c}\beta' = \frac{1}{3}\frac{c^2}{c^2+1}$, , $\omega_1, \omega_2 \in \mathbb{R}$, and $u_0 \in H^{s+1}(\mathbb{R})$. Then there exists a positive time $T > 0$ and a unique family of solutions $\{u^{\varepsilon,\mu}\}_{(\varepsilon,\,\mu)\in\mathcal{P}_{\mu_0,M}}$ to the Cauchy problem of the R-CH equation* (1.24) *with the initial value $u0 = u_0$ bounded in $C\left([0,\frac{T}{\varepsilon}]; X_\mu^{s+1}(\mathbb{R})\right) \cap C^1\left([0,\frac{T}{\varepsilon}]; X_\mu^{s}(\mathbb{R})\right)$. Moreover, there holds for all $t \in [0,\frac{T}{\varepsilon}]$*

$$\|u^{\varepsilon,\mu}(t)\|_{H^s} + \|u_t^{\varepsilon,\mu}(t)\|_{H^{s-1}} \le C,$$

with the constant C independent of ε and μ.

In view of the proof of Proposition 4.2.1, one can establish the similar uniform boundedness result for the R-KdV case.

Proposition 4.2.2. *Assume that $C_0 > 0$, $C_0' > 0$, $C_0'' > 0$, $s > \frac{3}{2}$, $c = c(\Omega) = \sqrt{1+\Omega^2} - \Omega$, $\Omega > 0$, and $u_0 \in H^{s+1}(\mathbb{R})$. Then there exists a unique family of solutions $\{u^{\varepsilon,\mu}\}_{(\varepsilon,\,\mu)\in\mathcal{K}_{C_0,C_0',C_0''}} \in C(\mathbb{R}; H^{s+1}(\mathbb{R})) \cap C^1(\mathbb{R}; H^{s-2}(\mathbb{R}))$ to the Cauchy problem of the R-KdV equation* (4.37) *with the initial value $u(0) = u_0$. Moreover, there exists a positive time $T > 0$ such that for all $t \in [0,\frac{T}{\varepsilon}]$*

$$\|u^{\varepsilon,\mu}(t)\|_{H^{s+1}} + \|u_t^{\varepsilon,\mu}(t)\|_{H^{s-2}} \le C,$$

with the constant C independent of ε and μ.

We now turn next to the uniform boundedness of the solution to the R-CH equation (1.24), which needs the following lemma.

Lemma 4.2.3 (Commutator estimates [45])**.** *Let* $\Lambda^s := (1-\partial_x^2)^{-\frac{s}{2}}$ *with* $s > 0$. *Then the following two estimates are true:*

(i) $\|[\Lambda^s, f]g\|_{L^2(\mathbb{R})} \le C(\|f\|_{H^s}\|g\|_{L^\infty(\mathbb{R})} + \|f_x\|_{L^\infty(\mathbb{R})}\|g\|_{H^{s-1}(\mathbb{R})})$;

(ii) $\|[\Lambda^s, f]g\|_{L^2(\mathbb{R})} \le C\|f_x\|_{H^{q_0}(\mathbb{R})}\|g\|_{H^{s-1}(\mathbb{R})}, \quad \forall\, 0 \le s \le q_0+1,\ q_0 > \frac{1}{2}$,

where all the constants Cs *are independent of* f *and* g.

Lemma 4.2.4. *Given* $\varepsilon > 0$, $\mu > 0$. *Let* $u_0 \in X_\mu^{s+1}(\mathbb{R})$ *with* $s > \frac{3}{2}$. *Assume that* $u \in C([0,T^*); X_\mu^{s+1}(\mathbb{R})) \cap C^1([0,T^*); X_\mu^s(\mathbb{R}))$ *is the solution of the R-CH equation* (1.24) *with initial data* u_0. *Then there is a positive constant* $\Omega_1 > 0$ *and* $0 < T_0 < T^*$ *such that* $\forall\ 0 \le \Omega \le \Omega_1$ *and* $\forall\ 0 \le t \le T_0/\varepsilon$, *there holds*

$$\|u(t)\|_{X_\mu^{s+1}} \le \|u_0\|_{X_\mu^{s+1}} e^{C\|u_0\|_{X_\mu^{s+1}}(1+\|u_0\|^2_{X_\mu^{s+1}})\varepsilon t}, \tag{4.41}$$

where the positive constant C *is independent of* Ω.

Proof. Notice that $\beta\mu(2u_xu_{xx} + uu_{xxx}) = \beta\mu\partial_x^2(uu_x) - \frac{1}{2}\beta\mu\partial_x(u_x^2)$. We rewrite the R-CH equation (1.24) in the form,

$$\begin{aligned} \partial_t u - \beta\mu u_{xxt} + c\partial_x u + \frac{3\varepsilon\, c^2}{c^2+1} uu_x - \beta'\mu u_{xxx} + \omega_1\varepsilon^2 u^2 u_x + \omega_2\varepsilon^3 u^3 u_x \\ - \alpha\beta\varepsilon\mu\partial_x^2(uu_x) + \frac{\alpha\beta\varepsilon\mu}{2}(u_x^2)_x = 0. \end{aligned} \tag{4.42}$$

Hence, applying the operator Λ^s to equation (4.42) and then taking the L^2-inner product with $\Lambda^s u$ yield

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\Lambda^s u\|_{L^2}^2 + \beta\mu\|\partial_x\Lambda^s u\|_{L^2}^2) \\ &= -\frac{3\varepsilon\, c^2}{c^2+1}\int_{\mathbb{R}} \Lambda^s(uu_x)\cdot\Lambda^s u\, dx - \int_{\mathbb{R}} (\omega_1\varepsilon^2\Lambda^s(u^2u_x) + \omega_2\varepsilon^3\Lambda^s(u^3u_x))\,\Lambda^s u\, dx \\ &\quad + \alpha\varepsilon\beta\mu\int_{\mathbb{R}} \Lambda^s\partial_x^2(uu_x)\cdot\Lambda^s u\, dx + \frac{\alpha\beta\varepsilon\mu}{2}\int_{\mathbb{R}} \Lambda^s(u_x^2)\Lambda^s u_x\, dx \equiv \sum_{i=1}^4 I_i. \end{aligned} \tag{4.43}$$

Thanks to the commutator process, we can estimate to obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}} \Lambda^s(uu_x)\cdot\Lambda^s u\,dx\right| = \left|\int_{\mathbb{R}} u\Lambda^s\partial_x u\cdot\Lambda^s u\,dx + \int_{\mathbb{R}}[\Lambda^s,u]u_x\cdot\Lambda^s u\,dx\right| \\ &\le \frac{1}{2}\|u_x\|_{L^\infty}\|\Lambda^s u\|_{L^2}^2 + \|[\Lambda^s,u]u_x\|_{L^2}\|\Lambda^s u\|_{L^2} \lesssim \|u_x\|_{L^\infty}\|\Lambda^s u\|_{L^2}^2,\end{aligned} \tag{4.44}$$

where use has been made of the commutator estimates in Lemma 4.2.3 to deal with the commutator term $\|[\Lambda^s,u]u_x\|_{L^2}$. It thus transpires that

$$|I_1| \le C\varepsilon\|u_x\|_{L^\infty}\|u\|_{H^s}^2.$$

Similarly, we may get that for $s>3/2$, $n=2,3$,

$$\begin{aligned}&\left|\int_{\mathbb{R}} \Lambda^s(u^n u_x)\cdot\Lambda^s u\,dx\right| \le \frac{1}{2}\left|\int_{\mathbb{R}}(u^n)_x(\Lambda^s u)^2\,dx\right| + \|\Lambda^s u\|_{L^2}\|[\Lambda^s,u^n]u_x\|_{L^2} \\ &\le C\|(u^n)_x\|_{L^\infty}\|u\|_{HS}^2 + C\|u^n\|_{H^s}\|u_x\|_{L^\infty}\|u\|_{H^s} \le C\|u\|_{H^s}^{n+2}.\end{aligned}$$

It thus follows that

$$|I_2| \le C\varepsilon(\|u\|_{H^s}^2 + \varepsilon\|u\|_{H^s}^3)\|u\|_{H^s}^2.$$

For I_3, we deduce from integration by parts and Lemma 4.2.3 that

$$\begin{aligned}&\left|\int_{\mathbb{R}} \Lambda^s\partial_x^2(uu_x)\cdot\Lambda^s u\,dx\right| \le \left|\int_{\mathbb{R}} \Lambda^s(u_x^2)\cdot\Lambda^s u_x\,dx\right| + \left|\int_{\mathbb{R}} \Lambda^s(uu_{xx})\cdot\Lambda^s u_x\,dx\right| \\ &\le \|\Lambda^s u_x\|_{L^2}\|\Lambda^s(u_x^2)\|_{L^2} + \frac{1}{2}\|u_x\|_{L^\infty}\|\Lambda^s u_x\|_{L^2}^2 + \|[\Lambda^s,u](u_x)_x\|_{L^2}\|\Lambda^s u_x\|_{L^2} \\ &\lesssim \|u_x\|_{L^\infty}\|\Lambda^s u_x\|_{L^2}^2,\end{aligned}$$

which gives rise to

$$|I_3| \le C\alpha\beta\varepsilon\mu\|u_x\|_{L^\infty}\|\Lambda^s u_x\|_{L^2}^2 \le C\alpha\beta\varepsilon\mu\|u_x\|_{H^s}^2\|u\|_{H^s}.$$

While for I_4, applying Hölder's inequality gives

$$|I_4| \le C\alpha\beta\varepsilon\mu\|u_x^2\|_{H^s}\|u_x\|_{H^s} \le C\alpha\beta\varepsilon\mu\|u_x\|_{H^s}^2\|u\|_{H^s}.$$

In consequence, it is found from all the estimates above that for $s>3/2$,

$$\frac{d}{dt}(\|u\|_{H^s}^2 + \beta\mu\|u_x\|_{H^s}^2) \le C\varepsilon\|u\|_{H^s}(1+\|u\|_{H^s}^2)(\|u\|_{H^s}^2 + \beta\mu\|u_x\|_{H^s}^2). \tag{4.45}$$

Taking $T_0 > 0$, such that $\sqrt{2}C\|u_0\|_{X_\mu^{s+1}}(1+2\|u_0\|^2_{X_\mu^{s+1}})\varepsilon T_0 < \ln\frac{3}{2}$, we claim that

$$\|u(t)\|^2_{H^s} + \beta\mu\|u_x\|^2_{H^s} \leq 2\|u_0\|^2_{X_\mu^{s+1}}, \forall t \in [0, T_0]. \tag{4.46}$$

By using the bootstrap argument, it thus follows from (4.45) that $\forall t \in [0, T_0]$,

$$\begin{aligned}
&\|u(t)\|^2_{H^s} + \beta\mu\|u_x(t)\|^2_{H^s} \leq \|u_0\|^2_{X_\mu^{s+1}} e^{C\varepsilon\int_0^t(\|u(t)\|_{H^s}+\|u\|^3_{H^s})}d\tau \\
&\leq \|u_0\|^2_{X_\mu^{s+1}} e^{\sqrt{2}C\|u_0\|_{X_\mu^{s+1}}(1+2\|u_0\|^2_{X_\mu^{s+1}})\varepsilon t} \leq \frac{3}{2}\|u_0\|^2_{X_\mu^{s+1}},
\end{aligned}$$

which implies (4.2.4). This completes the proof of Lemma 4.2.4. □

4.2.2 Justification of the approximation between the R-KdV equation and the R-CH equation

This subsection contains two parts. The first part is to justify rigorously the approximation between the solution of R-KdV equation and the solution of R-CH equation (1.24). The second part concerns the limit issue as $\Omega \to 0$ for the R-CH equation. The justification result could be stated in the following.

Theorem 4.2.5. *Assume that $\mu_0 > 0$, $M > 0$, $C_0 > 0$, $C_0' > 0$, $C_0'' > 0$, $\beta > 0$, $\beta - \frac{1}{c}\beta' = \frac{1}{3}\frac{c^2}{c^2+1}$, $c = c(\Omega) := \sqrt{1+\Omega^2} - \Omega$, $w > 0$, $\alpha = \frac{1}{2}\frac{c}{\Omega+c}$, and $\Omega > 0$. Let $u_0 \in H^{s+6}(\mathbb{R})$ with $s > \frac{1}{2}$, $(\varepsilon, \mu) \in \mathcal{K}_{C_0,C_0',C_0''}$, and $u^{\varepsilon,\mu}$ and $v^{\varepsilon,\mu}$ be the strong solutions of the R-CH equation (1.24) and the R-KdV equation (4.37) with the same initial value u_0, respectively. Then there exists the time $T > 0$ such that*

$$\|u^{\varepsilon,\mu}(t) - v^{\varepsilon,\mu}(t)\|_{H^s} \leq C\mu^2 t, \tag{4.47}$$

for all $t \in [0, \frac{T}{\varepsilon}]$.

Proof. For fixed $(\varepsilon, \mu) \in \mathcal{K}_{C_0,C_0',C_0''}$, let $r^{\varepsilon,\mu}(t) \overset{\text{def}}{=} v^{\varepsilon,\mu}(t) - u^{\varepsilon,\mu}(t)$, $r_0^{\varepsilon,\mu} \equiv 0$. Owing to Propositions 4.2.1 and 4.2.2, it is noted that the guaranteed existence time for $u^{\varepsilon,\mu}$ is $\frac{T}{\varepsilon}$ for some positive time T independent of ε and μ. For simplicity, we drop the indices ε and μ in u, v and r.

By the definition of r, it is observed that $r(t)$ solves that

$$\begin{cases} r_t + cr_x + 3\varepsilon\frac{c^2}{c^2+1}\left(ur_x + rv_x\right) + \mu\frac{c^3}{3(c^2+1)}r_{xxx} + \mu(\beta' + \frac{c^3}{3(c^2+1)})u_{xxx} \\ \qquad +\beta\mu u_{xxt} = \omega_1\varepsilon^2 u^2 u_x + \omega_2\varepsilon^3 u^3 u_x - \alpha\beta\varepsilon\mu(2u_x u_{xx} + uu_{xxx}), \\ r|_{t=0} = 0. \end{cases} \tag{4.48}$$

Notice that u satisfies

$$\begin{aligned} u_t + cu_x = -3\varepsilon\frac{c^2}{c^2+1}uu_x + \mu\beta' u_{xxx} + \mu\beta u_{xxt} - (\omega_1\varepsilon^2 u^2 u_x + \omega_2\varepsilon^3 u^3 u_x) \\ + \alpha\beta\varepsilon\mu(2u_x u_{xx} + uu_{xxx}) \equiv F. \end{aligned}$$

We then deduce from the identity $\beta - \frac{1}{c}\beta' = \frac{1}{3}\frac{c^2}{c^2+1}$ that

$$\mu\left(\beta' + \frac{c^3}{3(c^2+1)}\right)u_{xxx} + \beta\mu u_{xxt} = c\mu\beta u_{xxx} + \mu\beta u_{xxt} = \mu\beta(u_t + cu_x)_{xx} = \mu\beta\, F_{xx}.$$

It then follows that

$$\begin{aligned} r_t + cr_x + 3\varepsilon\frac{c^2}{c^2+1}\left(ur_x + rv_x\right) + \mu\frac{c^3}{3(c^2+1)}r_{xxx} \\ = -\mu\beta\, F_{xx} + \omega_1\varepsilon^2 u^2 u_x + \omega_2\varepsilon^3 u^3 u_x - \alpha\beta\varepsilon\mu(2u_x u_{xx} + uu_{xxx}). \end{aligned} \tag{4.49}$$

Energy estimate implies that $\forall\ s > \frac{1}{2}$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Lambda^s r\|_{L^2}^2 = &-3\varepsilon\frac{c^2}{c^2+1}\left(\int_{\mathbb{R}}\Lambda^s\left(u\, r_x\right)\cdot\Lambda^s r\, dx + \int_{\mathbb{R}}\Lambda^s\left(rv_x\right)\cdot\Lambda^s r\, dx\right) \\ &- \mu\beta\langle\Lambda^s F_{xx}, \Lambda^s r\rangle_{L^2} + \langle\omega_1\varepsilon^2\Lambda^s(u^2u_x) + \omega_2\varepsilon^3\Lambda^s(u^3u_x), \Lambda^s r\rangle_{L^2} \\ &- \alpha\beta\varepsilon\mu\langle\Lambda^s\left(2u_x u_{xx} + uu_{xxx}\right), \Lambda^s r\rangle_{L^2}. \end{aligned} \tag{4.50}$$

Thanks to

$$\begin{aligned} \int_{\mathbb{R}}\Lambda^s\left(u\, r_x\right)\cdot\Lambda^s r\, dx &= \int_{\mathbb{R}} u\,\Lambda^s r_x\cdot\Lambda^s r\, dx + \int_{\mathbb{R}}[\Lambda^s, u]r_x\cdot\Lambda^s r\, dx \\ &= -\frac{1}{2}\int_{\mathbb{R}} u_x\left(\Lambda^s r\right)^2\, dx + \int_{\mathbb{R}}[\Lambda^s, u]r_x\cdot\Lambda^s r\, dx, \end{aligned}$$

and Lemma 4.2.3, we can estimate as above that

$$\left|\int_{\mathbb{R}}\Lambda^s(ur_x)\cdot\Lambda^s r\, dx\right| \lesssim \|u_x\|_{L^\infty}\|\Lambda^s r\mathbb{R}|_{L^2}^2 + \|\Lambda^s r\|_{L^2}\cdot\|u_x\|_{H^s}\|r_x\|_{H^{s-1}} \lesssim \|u_x\|_{H^s}\|r\|_{H^s}^2.$$

On the other hand, the Hölder inequality and Moser-type estimate [52] show that

$$\left|\int_{\mathbb{R}} \Lambda^s (rv_x) \cdot \Lambda^s r \, dx\right| \lesssim \|rv_x\|_{H^s} \|r\|_{H^s} \lesssim \|r\|_{H^s}^2 \|v_x\|_{H^s},$$

$$|\langle \Lambda^s (u^n u_x), \Lambda^s r\rangle_{L^2}| \lesssim \|u^n u_x\|_{H^s} \|r\|_{H^s} \lesssim \|u\|_{H^s}^n \|u_x\|_{H^s} \|r\|_{H^s} \quad (\text{for } n = 2, 3),$$

$$|\langle \Lambda^s (2u_x u_{xx} + uu_{xxx}), \Lambda^s r\rangle_{L^2}| \lesssim (\|u_x\|_{H^s} \|u_{xx}\|_{H^s} + \|u\|_{H^s} \|u_{xxx}\|_{H^s}) \|r\|_{H^s},$$

and

$$|\langle \Lambda^s F_{xx}, \Lambda^s r\rangle_{L^2}| \lesssim \|r\|_{H^s} \Big(\varepsilon \|u\|_{H^{s+2}} \|u_x\|_{H^{s+2}} + \mu \|u_{xxx}\|_{H^{s+2}} + \mu \|u_{xxt}\|_{H^{s+2}}$$
$$+ \varepsilon^2 \|u\|_{H^{s+2}}^2 \|u_x\|_{H^{s+2}} + \varepsilon\mu (\|u_x\|_{H^{s+2}} \|u_{xx}\|_{H^{s+2}} + \|u\|_{H^{s+2}} \|u_{xxx}\|_{H^{s+2}})\Big).$$

It thus follows from Propositions 4.2.1 and 4.2.2 that

$$\begin{aligned}\frac{d}{dt} \|r\|_{H^s}^2 &\leq C\varepsilon(\|u\|_{H^{s+1}} + \|v\|_{H^{s+1}}) \|r\|_{H^s}^2 \\ &\quad + C(\varepsilon^2 + \mu^2)\|r\|_{H^s}\Big(\|u\|_{H^{s+5}}^4 + \|u\|_{H^{s+5}} + \|u_t\|_{H^{s+4}}\Big) \\ &\leq C\left(\varepsilon \|r\|_{H^s}^2 + \left(\varepsilon^2 + \mu^2\right) \|r\|_{H^s}\right).\end{aligned}$$

As $\|r_0\|_{H^s} = 0$, Gronwall's inequality in turn implies that $\forall\, t \in [0, \frac{T}{\varepsilon}]$

$$\|r(t)\|_{H^s} \leq \frac{\varepsilon^2 + \mu^2}{\varepsilon}(e^{\frac{\varepsilon tC}{2}} - 1) \leq Ce^{\frac{\varepsilon tC}{2}} t(\varepsilon^2 + \mu^2) \leq Ct(\varepsilon^2 + \mu^2),$$

where the constants C and C' are independent of ε and μ. This then allows the conclusion from the definition of $T_0^{\varepsilon,\mu}$ that

$$\|r(t)\|_{H^s} \leq C\left(\varepsilon^2 + \mu^2\right) t \leq C\mu^2 t, \ \forall\ 0 < t \leq \frac{T}{\varepsilon},$$

which gives rise to (4.47), and ends the proof of Theorem 4.2.5. □

A similar conclusion is also valid for the lower order approximations to the R-BBM equation. This result is given in the following.

Corollary 4.2.6. *Assume that* $\mu_0 > 0$*,* $M > 0$*,* $C_0 > 0$*,* $C_0' > 0$*,* $C_0'' > 0$*,* $\beta > 0$*,* $\beta - \frac{1}{c}\beta' = \frac{1}{3}\frac{c^2}{c^2+1}$*,* $c = c(\Omega) := \sqrt{1+\Omega^2} - \Omega$*,* $\omega_1, \omega_2 \in \mathbb{R}$*,* $\alpha = \frac{c^2}{c^2+1}$*, and* $\Omega > 0$*. Let* $u_0 \in H^{s+6}(\mathbb{R})$ *with* $s > \frac{1}{2}$*,* $(\varepsilon, \mu) \in \mathcal{K}_{C_0, C_0', C_0''}$*, and* $u^{\varepsilon,\mu}$ *and* $v^{\varepsilon,\mu}$ *be the strong solutions of the R-CH equation* (1.24) *and the R-BBM equation with the same initial data* u_0*, respectively. Then there exists* $T > 0$ *such that*

$$\|u^{\varepsilon,\mu}(t) - v^{\varepsilon,\mu}(t)\|_{H^s} \leq C\mu^2 t, \tag{4.51}$$

for all $t \in [0, \frac{T}{\varepsilon}]$*.*

Next, our attention will be turned to the limit issue as $\Omega \to 0$ for the R-CH equation. In the case $\Omega = 0$, the R-CH equation is reduced to the CH equation in the following form,

$$(1 - \frac{5}{12}\mu\partial_x^2)u_t + \partial_x u + \frac{3}{2}\varepsilon u u_x - \frac{1}{4}\mu u_{xxx} = \frac{5}{24}\varepsilon\mu(2u_x u_{xx} + u u_{xxx}) \tag{4.52}$$

We may get the convergence theorem of the R-CH equation (1.24) as follows.

Theorem 4.2.7. *Let* $u_0 \in H^s(\mathbb{R})$ *with* $s \geq 3$*. Assume that* u^c *and* u *are solutions of the R-CH equation* (1.24)*and the CH equation* (4.52) *with the same initial value* u_0*. Then, for any fixed common existence time* $T_0 > 0$*, there hold*

$$u^c \rightharpoonup u \ \textit{weak} * \textit{in}\ L^\infty([0, T_0]; H^s), \text{ as } \Omega \to 0,$$

and

$$u^c \to u \text{ in } C([0, T_0]; H^{s'})\, (\forall\, 0 \leq s' < s) \text{ as } \Omega \to 0.$$

The following lemma is crucial to achieve the result in Theorem 4.2.7.

Lemma 4.2.8. *Let* $u_0 \in H^s(\mathbb{R})$ *with* $s \geq 3$*. Assume that two functions* $u^{(1)}, u^{(2)} \in C([0, T_0]; H^s) \cap C^1([0, T_0]; H^{s-1})$ *for a positive time* $T_0 > 0$ *are solutions of the R-CH equation* (1.24) *with the same initial data* u_0 *and the different rotation parameters* Ω_1 *and* Ω_2 *respectively. Then there holds that*

$$\|u^{(1)} - u^{(2)}(t)\|_{C([0,T_0];H^1)} = O(|\Omega_1 - \Omega_2|).$$

Proof. Notice that $u^{(i)}$, $\forall\, i = 1, 2$, solves the equation

$$\partial_t u^{(i)} - \mu\beta_i u^{(i)}_{xxt} + c_i u^{(i)}_x + 3\varepsilon \frac{c_i^2}{c_i^2+1} u^{(i)} u^{(i)}_x - \mu\beta_i' u^{(i)}_{xxx} + \omega_1^i \varepsilon^2 \Big(u^{(i)}\Big)^2 u^{(i)}_x + \omega_2^i \varepsilon^3 \Big(u^{(i)}\Big)^3 u^{(i)}_x = \alpha_i\beta_i\varepsilon\mu\Big(2u^{(i)}_x u^{(i)}_{xx} + u^{(i)} u^{(i)}_{xxx}\Big)$$

where $c_i = c(\Omega_i), \beta_i = \beta(\Omega_i), \beta_i' = \beta'(\Omega_i), \omega_1^i = \omega_1(\Omega_i), \omega_2^i = \omega_2(\Omega_i)$ and $\alpha_i = \alpha(\Omega_i)$. Denote that $u^{(1,2)} \stackrel{\text{def}}{=} u^{(1)} - u^{(2)}$. It is then found that

$$\begin{aligned}
&\partial_t u^{(1,2)} - \mu\beta_1 u^{(1,2)}_{xxt} + c_1 u^{(1,2)}_x - \mu\beta_1' u^{(1,2)}_{xxx} + h^{(1,2)} u^{(1,2)}_x + h^{(1,2)}_x u^{(1,2)} \\
&\quad + \left(3\varepsilon \frac{(c_1-c_2)(c_1+c_2)}{(c_1^2+1)(c_2^2+1)} u^{(2)} + \varepsilon^2(\omega_1^1 - \omega_1^2)(u^{(2)})^2 + \varepsilon^3(\omega_2^1 - \omega_2^2)(u^{(2)})^3\right) u^{(2)}_x \\
&\quad + (c_1 - c_2) u^{(2)}_x - \mu(\beta_1' - \beta_2') u^{(2)}_{xxx} - \mu(\beta_1 - \beta_2) u^{(2)}_{xxt} \\
&= \alpha_1\beta_1\varepsilon\mu\Big(2u^{(1,2)}_x u^{(1)}_{xx} + 2u^{(2)}_x u^{(1,2)}_{xx} + u^{(1,2)} u^{(1)}_{xxx} + u^{(2)} u^{(1,2)}_{xxx}\Big) \\
&\quad + \varepsilon\mu\Big((\alpha_1 - \alpha_2)\beta_1 + \alpha_2(\beta_1 - \beta_2)\Big)\Big(2u^{(2)}_x u^{(2)}_{xx} + u^{(2)} u^{(2)}_{xxx}\Big),
\end{aligned}$$

with $u^{(1,2)}|_{t=0} = 0$, where $h^{(1,2)} = \frac{3\varepsilon}{2}\frac{c_1^2}{c_1^2+1} v^{(1,2)} + \frac{\omega_1\varepsilon^2}{3} q^{(1,2)} + \frac{\omega_2\varepsilon^3}{4} w^{(1,2)}$ with $v^{(1,2)} \stackrel{\text{def}}{=} u^{(1)} + u^{(2)}$, $q^{(1,2)} \stackrel{\text{def}}{=} (u^{(1)})^2 + u^{(1)}u^{(2)} + (u^{(2)})^2$, and $w^{(1,2)} \stackrel{\text{def}}{=} (u^{(1)})^3 + (u^{(1)})^2 u^{(2)} + u^{(1)}(u^{(2)})^2 + (u^{(2)})^3$.

Taking the L^2-inner product between (4.53) and $u^{(1,2)}$, and then using integration by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u^{(1,2)}\|^2_{L^2} + \mu\beta_1\|u^{(1,2)}_x\|^2_{L^2}) \\
&= \int_{\mathbb{R}} \Big(((\alpha_1 - \alpha_2)\beta_1 + \alpha_2(\beta_1 - \beta_2))\varepsilon\mu(2u^{(2)}_x u^{(2)}_{xx} + u^{(2)} u^{(2)}_{xxx}) - (c_1 - c_2) u^{(2)}_x\Big) u^{(1,2)}\, dx\, dx \\
&\quad + \int_{\mathbb{R}} \left(\mu(\beta_1' - \beta_2') u^{(2)}_{xxx} + \mu(\beta_1 - \beta_2) u^{(2)}_{xxt} - 3\varepsilon \frac{(c_1+c_2)(c_1-c_2)}{(c_1^2+1)(c_2^2+1)} u^{(2)} u^{(2)}_x\right) u^{(1,2)}\, dx \\
&\quad - \int_{\mathbb{R}} \Big(\varepsilon^2(\omega_1^1 - \omega_1^2)(u^{(2)})^2 u^{(2)}_x - \varepsilon^3(\omega_2^1 - \omega_2^2)(u^{(2)})^3 u^{(2)}_x\Big) u^{(1,2)}\, dx \\
&\quad - \frac{1}{2}\int_{\mathbb{R}} (u^{(1,2)})^2 h^{(1,2)}\, dx + \frac{1}{2}\alpha_1\beta_1\varepsilon\mu \int_{\mathbb{R}} u^{(2)}_{xxx}(u^{(1,2)})^2 - u^{(2)}_x (u^{(1,2)}_x)^2\, dx.
\end{aligned}$$

Using the uniform boundedness estimate of the solution in (4.2.4) with the parameters Ω_1 and Ω_2 , it is then deduced that

$$
\begin{aligned}
&\frac{d}{dt}(\|u^{(1,2)}\|_{L^2}^2+\mu\beta_1\|u_x^{(1,2)}\|_{L^2}^2)\\
\le& C|\Omega_1-\Omega_2|\|u^{(1,2)}\|_{L^2}\Big((1+\|u^{(2)}\|_{L^\infty}^3)\|u_x^{(2)}\|_{H^2}+\beta_2\mu\|u_{xxt}^{(2)}\|_{L^2}\Big)\\
&+C(\|u^{(1,2)}\|_{L^2}^2+\mu\beta_1\|u_x^{(1,2)}\|_{L^2}^2)\Big((1+\|u^{(1)}\|_{W^{1,\infty}}^3+\|u^{(2)}\|_{W^{1,\infty}}^3+\beta_2\mu\|u_{xxx}^{(2)}\|_{L^\infty}\Big)\\
\le& C_0\Big(\|u^{(1,2)}\|_{L^2}^2+\mu\beta_1\|u_x^{(1,2)}\|_{L^2}^2\Big)+C_2|\Omega_1-\Omega_2|\|u^{(1,2)}\|_{L^2}.
\end{aligned}
$$

Therefore, using Gronwall's inequality for fixed $T_0>0$ yields

$$\|u^{(1,2)}(t)\|_{L^2}^2+\mu\beta_1\|u_x^{(1,2)}(t)\|_{L^2}^2\le C|\Omega_1-\Omega_2|^2\quad\forall\, t\in[0,T_0],$$

which follows that

$$\|u^{(1,2)}\|_{C([0,T_0];H^1)}\le C|\Omega_1-\Omega_2|.$$

We thus finish the proof of the lemma. □

Proof of Theorem 4.2.7. Thanks to Lemma 4.2.8, we deduce from the interpolation inequality that $\forall\, 0<s'<s$

$$
\begin{aligned}
\|u^{(c_1)}-u^{(c_2)}\|_{C([0,T];H^{s'})}&\le C\|u^{(c_1)}-u^{(c_2)}\|_{C([0,T];L^2)}^{\frac{s-s'}{s}}\|u^{(c_1)}-u^{(c_2)}\|_{C([0,T];H^s)}^{\frac{s'}{s}}\\
&\le C|\Omega_1-\Omega_2|^{1-\frac{s'}{s}},
\end{aligned}
\tag{4.53}
$$

which in turn implies that

$$\{u^{c(\Omega)}\}_{\Omega>0}\text{ is a Cauchy net in }C([0,T];H^{s'}),\text{ as }\Omega\to 0,\text{ for any }0<s'<s.$$

Therefore, there exists a function $u\in C([0,T];H^{s'})$ (with $0\le s'<s$), such that

$$u^{(c)}\to u\text{ in }C([0,T];H^{s'}),\quad\text{as }\Omega\to 0.$$

Furthermore, by the Banach algebra estimate and uniform boundedness for u and u^c in $H^{s'}(\mathbb{R})$, we get for any $\frac{5}{2}<s'<s$

$$\left\|\frac{c^2}{c^2+1}u^cu_x^c-\frac{1}{2}uu_x\right\|_{H^{s'-1}}$$

$$\le \frac{c^2}{c^2+1}\|u^c(u^c_x-u_x)\|_{H^{s'-1}} + \left\|\left(\frac{c^2}{c^2+1}u^c-\frac{1}{2}u\right)u_x\right\|_{H^{s'-1}} + \left\|\left(\frac{c^2}{c^2+1}u^c-\frac{1}{2}u\right)u_x\right\|_{H^{s'-1}}$$
$$\le C(\|u^c\|_{H^{s'-1}}\|u^c-u\|_{H^{s'}} + \|u\|_{H^{s'}}\|u^c-u\|_{H^{s'-1}})$$
$$\le C\|u^c-u\|_{H^{s'}},$$

which yields

$$3\varepsilon\frac{c^2}{c^2+1}u^c\partial_x u^c \to \frac{3}{2}\varepsilon u\partial_x u \quad \text{in} \quad C([0,T];H^{s'-1}), \quad \text{as} \quad \Omega\to 0.$$

Similarly,

$$cu^c_x + \omega_1\varepsilon^2(u^c)^2u^c_x + \omega_2\varepsilon^3(u^c)^3u^c_x \to u_x \quad \text{in} \quad C([0,T];H^{s'-1}), \quad \text{as} \quad \Omega\to 0,$$
$$\mu\beta' u^c_{xxx} + \alpha\beta\varepsilon\mu(2u^c_x u^c_{xx} + u^c u^c_{xxx}) \to \frac{1}{4}\mu u_{xxx} + \frac{5}{24}\varepsilon\mu(2u_x u_{xx} + uu_{xxx})$$
$$\text{in} \quad C([0,T];H^{s'-3}), \quad \text{as} \quad \Omega\to 0,$$

which along with the R-CH equation (1.24) gives rise to

$$\partial_t u^c \to -(1-\frac{5}{12}\mu\partial_x^2)^{-1}\left(\partial_x u + \frac{3}{2}\varepsilon uu_x - \frac{1}{4}\mu u_{xxx} - \frac{5}{24}\varepsilon\mu(2u_xu_{xx}+uu_{xxx})\right)$$
$$\text{in} \quad C([0,T];H^{s'-1}), \quad \text{as} \quad \Omega\to 0. \tag{4.54}$$

On the other hand, from the R-CH equation, it is deduced that

$$\{\partial_t u^c\}_{\Omega>0} \text{ is uniformly bounded in } C([0,T];H^{s-1}), \quad \text{and}$$
$$\{u^c\}_{\Omega>0} \text{ is uniformly bounded in } C([0,T];H^{s}),$$

which along with the Banach-Alaoglu Theorem yields that there is a subsequence $\{\partial_t u^{c_i}\}_{i=0}^{\infty}$ (with $c_i = c(\Omega_i)$ and $\Omega_i \to 0$) of $\{\partial_t u^c\}_{\Omega>0}$ and a function $v^* \in L^\infty([0,T];H^s)$ such that

$$\partial_t u^{c_i} \rightharpoonup \partial_t v^* \text{ weakly } * \text{ in } L^\infty([0,T];H^{s-1}), \quad \text{and}$$
$$u^{c_i} \rightharpoonup v^* \text{ weakly } * \text{ in } L^\infty([0,T];H^{s}).$$

By the uniqueness of the distribution limit, we have $v^* = u$, which along with (4.54) implies that $u \in C([0,T];H^{s'}) \cap L^\infty([0,T];H^s)$ solves the CH equation (4.52) with initial value $u(0) = u_0$. By using a standard approximation argument and the uniqueness of the solution to (4.52) (see also Proposition 4.2.1), we may get the limit function $u \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$ solves the CH equation (4.52). This completes the proof of Theorem 4.2.7. □

4.3 Traveling-wave solutions

It is known there are two prominent features on the CH equation, wave breaking and wave peaking [8]. It is intriguing to know how these two effects manifest in the R-CH model. Our attention in this section is now turned to analyzing smooth and non-smooth localized traveling-wave solutions to the R-CH equation (1.24) with certain Coriolis effect.

4.3.1 Nonexistence of single peaked solution

Applying the transformation

$u_{\varepsilon,\mu}(t,x) = \alpha\varepsilon u(\sqrt{\beta\mu}\, t, \sqrt{\beta\mu}\, x)$ to (1.24), then $u_{\varepsilon,\mu}(t,x)$ solves

$$u_t - u_{xxt} + cu_x + 3uu_x - \frac{\beta_0}{\beta}u_{xxx} + \frac{\omega_1}{\alpha^2}u^2u_x + \frac{\omega_2}{\alpha^3}u^3u_x = 2u_xu_{xx} + uu_{xxx}. \qquad (4.55)$$

Our purpose here is to demonstrate the nonexistence of single CH-type peaked solution to the R-CH equation (4.55), which particularly has the form

$$u(t,x) = a(\sigma,t)e^{-|x-\sigma t|}, \quad \sigma \in \mathbb{R} \text{ and } a(\sigma,t) \in C(\mathbb{R}\times[0,T)). \qquad (4.56)$$

Note it is a weak function in $H^1(\mathbb{R})$. The weak solution of the R-CH equation (4.42) is defined in distribution sense.

Definition 4.3.1. Given initial data $u_0 \in H^1(\mathbb{R})$, the function $u \in C([0,T); H^1(\mathbb{R}))$ is said to be a weak solution to the initial-value problem

$$\begin{cases} u_t - u_{xxt} + cu_x + 3uu_x - \frac{\beta_0}{\beta}u_{xxx} + \frac{\omega_1}{\alpha^2}u^2u_x + \frac{\omega_2}{\alpha^3}u^3u_x = 2u_xu_{xx} + uu_{xxx}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \\ u \to 0, \text{ as } |x| \to \infty, \end{cases} \tag{4.57}$$

if it satisfies the following identity:

$$\int_0^T \int_{\mathbb{R}} \Big[u\varphi_t + \frac{1}{2}u^2\varphi_x + \frac{\beta_0}{\beta}u\varphi_x + p * \Big((c - \frac{\beta_0}{\beta})u + u^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\Big) \cdot \varphi_x\Big]dxdt + \int_{\mathbb{R}} u_0(x)\varphi(0,x)dx = 0,$$

for any smooth test function $\varphi(t,x) \in C_c^\infty([0,T) \times \mathbb{R})$. If u is a weak solution on $[0,T)$ for every $T > 0$, then it is called a global weak solution.

Theorem 4.3.1. *There is no any nonzero weak solution of* (4.57) *in the form* (4.56).

Proof. The proof of this theorem contains two cases. If the rotation parameter $\Omega = 0$ (i.e. $c = 1$), the equation only has smooth traveling-wave solutions only [9]. We now consider the case of $\Omega \neq 0$. We will give a proof in this case by a contradictory argument. Suppose that the R-CH equation admits the peaked solution in the form (4.56). Then, for all $t \in \mathbb{R}^+$, in the sense of distribution and $\partial_x u_a(t,x) = -\operatorname{sign}(x - \sigma t)u_a(t,x)$ belongs to $L^\infty(\mathbb{R})$. For any test function $\varphi(\cdot) \in C_c^\infty(\mathbb{R})$, by using integration by parts, we have

$$\int_{\mathbb{R}} \operatorname{sign}(y)e^{-|y|}\varphi(y)dy = \int_{-\infty}^0 -e^y\varphi(y)dy + \int_0^{+\infty} e^{-y}\varphi(y)dy = \int_{\mathbb{R}} e^{-|y|}\varphi'(y)dy.$$

Note that

$$\partial_t u_a(t,x) = \partial_t a(\sigma,t)e^{-|x-\sigma t|} + \sigma \operatorname{sign}(x - \sigma t)u_a(t,x) \in L^\infty(\mathbb{R}) \text{ for all } t \geq 0. \tag{4.58}$$

Hence, by integration by parts, we deduce that

$$
\begin{aligned}
&\int_0^{\infty}\int_{\mathbb{R}}\Big(u_a\varphi_t+\frac{1}{2}u_a^2\varphi_x+\frac{\beta_0}{\beta}u_a\varphi_x\Big)dxdt+\int_{\mathbb{R}}u_a(0,x)\varphi(0,x)dx\\
&\quad=-\int_0^{\infty}\int_{\mathbb{R}}\varphi\Big[\partial_t u_a+u_a\cdot\partial_x u_a+\frac{\beta_0}{\beta}\partial_x u_a\Big]dxdt\\
&\quad=-\int_0^{\infty}\int_{\mathbb{R}}\varphi\Big[\operatorname{sign}(x-\sigma t)u_a\cdot\sigma+\partial_t a(\sigma,t)e^{-|x-\sigma t|}-\operatorname{sign}(x-\sigma t)u_a^2\\
&\qquad\qquad-\frac{\beta_0}{\beta}\cdot\operatorname{sign}(x-\sigma t)u_a\Big]dxdt \qquad (4.59)\\
&\quad=-\int_0^{\infty}\int_{\mathbb{R}}\varphi\cdot\operatorname{sign}(x-\sigma t)u_a\cdot\Big[\sigma-\frac{\beta_0}{\beta}-u_a\Big]+\varphi\cdot\partial_t a(\sigma,t)e^{-|x-\sigma t|}dxdt.
\end{aligned}
$$

On the other hand, we know

$$
u=(1-\partial_x^2)^{-1}m=p*m,\ \text{ where } p(x)=\frac{1}{2}e^{-|x|},
$$

and the notation " $*$ " denotes the convolution product on $\mathbb{R}$, defined by

$$
\Big(f*g\Big)(x)=\int_{\mathbb{R}}f(y)g(x-y)dy.
$$

Hence,

$$
\begin{aligned}
&\int_0^{\infty}\int_{\mathbb{R}}\Big[(1-\partial_x^2)^{-1}\Big((c-\frac{\beta_0}{\beta})u_a+u_a^2+\frac{1}{2}u_{a,x}^2+\frac{\omega_1}{3\alpha^2}u_a^3+\frac{\omega_2}{4\alpha^3}u_a^4\Big)\cdot\partial_x\varphi\Big]dxdt\\
&\quad=-\int_0^{\infty}\int_{\mathbb{R}}\Big[\varphi\cdot\partial_x p*\Big((c-\frac{\beta_0}{\beta})u_a+u_a^2+\frac{1}{2}u_{a,x}^2+\frac{\omega_1}{3\alpha^2}u_a^3+\frac{\omega_2}{4\alpha^3}u_a^4\Big)\Big]dxdt
\end{aligned} \tag{4.60}
$$

It is noted that $\partial_x p(x)=-\frac{1}{2}\operatorname{sign}(x)e^{-|x|}$ for $x\in\mathbb{R}$. A simple computation reveals that

$$
\begin{aligned}
&\partial_x p*\Big((c-\frac{\beta_0}{\beta})u_a+u_a^2+\frac{1}{2}u_{a,x}^2+\frac{\omega_1}{3\alpha^2}u_a^3+\frac{\omega_2}{4\alpha^3}u_a^4\Big)(t,x)\\
&\quad=-\frac{1}{2}\int_{-\infty}^{+\infty}\operatorname{sign}(x-y)e^{-|x-y|}\cdot\Big[(c-\frac{\beta_0}{\beta})a(\sigma,t)e^{-|y-\sigma t|}+a^2(\sigma,t)e^{-2|y-\sigma t|}\\
&\qquad+\frac{1}{2}\operatorname{sign}^2(y-\sigma t)a^2(\sigma,t)e^{-2|y-\sigma t|}+\frac{\omega_1}{3\alpha^2}a^3(\sigma,t)e^{-3|y-\sigma t|}\\
&\qquad+\frac{\omega_2}{4\alpha^3}a^4(\sigma,t)e^{-4|y-\sigma t|}\Big]dy.
\end{aligned} \tag{4.61}
$$

When $x > \sigma t$, we split the right hand side of (4.61) into the following three parts.

$$
\begin{aligned}
&\partial_x p * \Big((c - \frac{\beta_0}{\beta})u_a + u_a^2 + \frac{1}{2}u_{a,x}^2 + \frac{\omega_1}{3\alpha^2}u_a^3 + \frac{\omega_2}{4\alpha^3}u_a^4\Big)(t,x) \\
&\quad = -\frac{1}{2}\Big(\int_{-\infty}^{\sigma t} + \int_{\sigma t}^{x} + \int_{x}^{+\infty}\Big) \operatorname{sign}(x-y)e^{-|x-y|} \cdot \Big[(c - \frac{\beta_0}{\beta})a(\sigma,t)e^{-|y-\sigma t|} \\
&\qquad + a^2(\sigma,t)e^{-2|y-\sigma t|} + \frac{1}{2}\operatorname{sign}^2(y-\sigma t)a^2(\sigma,t)e^{-2|y-\sigma t|} + \frac{\omega_1}{3\alpha^2}a^3(\sigma,t)e^{-3|y-\sigma t|} \\
&\qquad + \frac{\omega_2}{4\alpha^3}a^4(\sigma,t)e^{-4|y-\sigma t|}\Big]dy \\
&\quad =: I_1 + I_2 + I_3.
\end{aligned}
$$

In the case that $-\infty < y < \sigma t < x$, it follows that

$$
\begin{aligned}
I_1 =& -\frac{1}{2}\int_{-\infty}^{\sigma t} e^{-x+y}\Big[(c - \frac{\beta_0}{\beta})a(\sigma,t)e^{y-\sigma t} + \frac{3}{2}a^2(\sigma,t)e^{2(y-\sigma t)} \\
&+ \frac{\omega_1}{3\alpha^2}a^3(\sigma,t)e^{3(y-\sigma t)} + \frac{\omega_2}{4\alpha^3}a^4(\sigma,t)e^{4(y-\sigma t)}\Big]dy \\
=& -\frac{1}{2}(c - \frac{\beta_0}{\beta})a(\sigma,t)e^{-x-\sigma t}\int_{-\infty}^{\sigma t} e^{2y}dy - \frac{3}{4}a^2(\sigma,t)e^{-x-2\sigma t}\int_{-\infty}^{\sigma t} e^{3y}dy \\
&- \frac{\omega_1}{6\alpha^2}a^3(\sigma,t)e^{-x-3\sigma t}\int_{-\infty}^{\sigma t} e^{4y}dy - \frac{\omega_2}{8\alpha^3}a^4(\sigma,t)e^{-x-4\sigma t}\int_{-\infty}^{\sigma t} e^{5y}dy \\
=& -\frac{1}{4}\Big[(c - \frac{\beta_0}{\beta})a(\sigma,t) + a^2(\sigma,t) + \frac{\omega_1}{6\alpha^2}a^3(\sigma,t) + \frac{\omega_2}{10\alpha^3}a^4(\sigma,t)\Big]e^{-x+\sigma t}.
\end{aligned}
$$

For $\sigma t < y < x$, a direct computation gives that

$$
\begin{aligned}
I_2 =& -\frac{1}{2}\int_{\sigma t}^{x} e^{-x+y}\Big[(c - \frac{\beta_0}{\beta})a(\sigma,t)e^{-y+\sigma t} + \frac{3}{2}a^2(\sigma,t)e^{2(-y+\sigma t)} \\
&+ \frac{\omega_1}{3\alpha^2}a^3(\sigma,t)e^{3(-y+\sigma t)} + \frac{\omega_2}{4\alpha^3}a^4(\sigma,t)e^{4(-y+\sigma t)}\Big]dy \\
=&\Big[-\frac{1}{2}(c - \frac{\beta_0}{\beta})a(\sigma,t)\cdot(x-\sigma t) - \frac{3}{4}a^2(\sigma,t) - \frac{\omega_1}{12\alpha^2}a^3(\sigma,t) - \frac{\omega_2}{24\alpha^3}a^4(\sigma,t)\Big]e^{-x+\sigma t} \\
&+ \frac{3}{4}a^2(\sigma,t)e^{-2x+2\sigma t} + \frac{\omega_1}{12\alpha^2}a^3(\sigma,t)e^{-3x+3\sigma t} + \frac{\omega_2}{24\alpha^3}a^4(\sigma,t)e^{-4x+4\sigma t}.
\end{aligned}
$$

For $\sigma t < x < y < +\infty$, we have

$$\begin{aligned}I_3 =&\frac{1}{2}\int_x^{+\infty} e^{x-y}\Big[(c-\frac{\beta_0}{\beta})a(\sigma,t)e^{-y+\sigma t}+\frac{3}{2}a^2(\sigma,t)e^{2(-y+\sigma t)}\\&+\frac{\omega_1}{3\alpha^2}a^3(\sigma,t)e^{3(-y+\sigma t)}+\frac{\omega_2}{4\alpha^3}a^4(\sigma,t)e^{4(-y+\sigma t)}\Big]dy\\=&\frac{1}{4}(c-\frac{\beta_0}{\beta})a(\sigma,t)e^{-x+\sigma t}+\frac{1}{4}a^2(\sigma,t)e^{-2x+2\sigma t}+\frac{\omega_1}{24\alpha^2}a^3(\sigma,t)e^{-3x+3\sigma t}\\&+\frac{\omega_2}{40\alpha^3}a^4(\sigma,t)e^{-4x+4\sigma t}.\end{aligned}$$

Combining I_1, I_2 and I_3, for $x > \sigma t$, we have

$$\begin{aligned}&\partial_x p * \Big((c-\frac{\beta_0}{\beta})u_a+u_a^2+\frac{1}{2}u_{a,x}^2+\frac{\omega_1}{3\alpha^2}u_a^3+\frac{\omega_2}{4\alpha^3}u_a^4\Big)(t,x)\\&=\Big[-\frac{1}{2}(c-\frac{\beta_0}{\beta})a(\sigma,t)\cdot(x-\sigma t)-a^2(\sigma,t)-\frac{\omega_1}{8\alpha^2}a^3(\sigma,t)-\frac{\omega_2}{15\alpha^3}a^4(\sigma,t)\Big]e^{-x+\sigma t}\\&\quad+a^2(\sigma,t)e^{-2x+2\sigma t}+\frac{\omega_1}{8\alpha^2}a^3(\sigma,t)e^{-3x+3\sigma t}+\frac{\omega_2}{15\alpha^3}a^4(\sigma,t)e^{-4x+4\sigma t}.\end{aligned}$$

When $x \le \sigma t$, we split the right hand side of (4.61) into the following three parts.

$$\begin{aligned}&\partial_x p * \Big((c-\frac{\beta_0}{\beta})u_a+u_a^2+\frac{1}{2}u_{a,x}^2+\frac{\omega_1}{3\alpha^2}u_a^3+\frac{\omega_2}{4\alpha^3}u_a^4\Big)(t,x)\\&=-\frac{1}{2}\Big(\int_{-\infty}^{x}+\int_x^{\sigma t}+\int_{\sigma t}^{+\infty}\Big)\operatorname{sign}(x-y)e^{-|x-y|}\cdot\Big[(c-\frac{\beta_0}{\beta})a(\sigma,t)e^{-|y-\sigma t|}\\&\quad+a^2(\sigma,t)e^{-2|y-\sigma t|}+\frac{1}{2}\operatorname{sign}^2(y-\sigma t)a^2(\sigma,t)e^{-2|y-\sigma t|}+\frac{\omega_1}{3\alpha^2}a^3(\sigma,t)e^{-3|y-\sigma t|}\\&\quad+\frac{\omega_2}{4\alpha^3}a^4(\sigma,t)e^{-4|y-\sigma t|}\Big]dy\\&=: II_1+II_2+II_3.\end{aligned}$$

For $-\infty < y < x \le \sigma t$, a simple computation shows that

$$\begin{aligned}II_1 =&-\frac{1}{2}\int_{-\infty}^{x} e^{-x+y}\Big[(c-\frac{\beta_0}{\beta})a(\sigma,t)e^{y-\sigma t}+\frac{3}{2}a^2(\sigma,t)e^{2(y-\sigma t)}\\&+\frac{\omega_1}{3\alpha^2}a^3(\sigma,t)e^{3(y-\sigma t)}+\frac{\omega_2}{4\alpha^3}a^4(\sigma,t)e^{4(y-\sigma t)}\Big]dy\\=&-\frac{1}{4}(c-\frac{\beta_0}{\beta})a(\sigma,t)e^{x-\sigma t}-\frac{1}{4}a^2(\sigma,t)e^{2x-2\sigma t}-\frac{\omega_1}{24\alpha^2}a^3(\sigma,t)e^{3x-3\sigma t}\\&-\frac{\omega_2}{40\alpha^3}a^4(\sigma,t)e^{4x-4\sigma t}.\end{aligned}$$

For $x < y < \sigma t$, it is found that

$$\begin{aligned}II_2 =&\Big[\frac{1}{2}(c-\frac{\beta_0}{\beta})a(\sigma,t)\cdot(\sigma t-x)+\frac{3}{4}a^2(\sigma,t)+\frac{\omega_1}{12\alpha^2}a^3(\sigma,t)+\frac{\omega_2}{24\alpha^3}a^4(\sigma,t)\Big]e^{x-\sigma t}\\&-\frac{3}{4}a^2(\sigma,t)e^{2x-2\sigma t}-\frac{\omega_1}{12\alpha^2}a^3(\sigma,t)e^{3x-3\sigma t}-\frac{\omega_2}{24\alpha^3}a^4(\sigma,t)e^{4x-4\sigma t}.\end{aligned}$$

For $x \le \sigma t < y < +\infty$, it is easy to check that

$$II_3 =\Big[\frac{1}{4}(c-\frac{\beta_0}{\beta})a(\sigma,t)+\frac{1}{4}a^2(\sigma,t)+\frac{\omega_1}{24\alpha^2}a^3(\sigma,t)+\frac{\omega_2}{40\alpha^3}a^4(\sigma,t)\Big]e^{x-\sigma t}.$$

Combining II_1, II_2 and II_3, in the case $x \le \sigma t$ gives

$$\begin{aligned}&\partial_x p*\Big((c-\frac{\beta_0}{\beta})u_a+u_a^2+\frac{1}{2}u_{a,x}^2+\frac{\omega_1}{3\alpha^2}u_a^3+\frac{\omega_2}{4\alpha^3}u_a^4\Big)(t,x)\\&=\Big[\frac{1}{2}(c-\frac{\beta_0}{\beta})a(\sigma,t)\cdot(\sigma t-x)+a^2(\sigma,t)+\frac{\omega_1}{8\alpha^2}a^3(\sigma,t)+\frac{\omega_2}{15\alpha^3}a^4(\sigma,t)\Big]e^{x-\sigma t}\\&\quad-a^2(\sigma,t)e^{-2x+2\sigma t}-\frac{\omega_1}{8\alpha^2}a^3(\sigma,t)e^{-3x+3\sigma t}-\frac{\omega_2}{15\alpha^3}a^4(\sigma,t)e^{-4x+4\sigma t}.\end{aligned}$$

It is then inferred from these two cases that

$$\begin{aligned}&\partial_x p*\Big((c-\frac{\beta_0}{\beta})u_a+u_a^2+\frac{1}{2}u_{a,x}^2+\frac{\omega_1}{3\alpha^2}u_a^3+\frac{\omega_2}{4\alpha^3}u_a^4\Big)(t,x)\\&=\begin{cases}\Big[-\frac{1}{2}(c-\frac{\beta_0}{\beta})a(\sigma,t)\cdot(x-\sigma t)-a^2(\sigma,t)-\frac{\omega_1}{8\alpha^2}a^3(\sigma,t)-\frac{\omega_2}{15\alpha^3}a^4(\sigma,t)\Big]e^{-x+\sigma t}\\ \quad+a^2(\sigma,t)e^{-2x+2\sigma t}+\frac{\omega_1}{8\alpha^2}a^3(\sigma,t)e^{-3x+3\sigma t}+\frac{\omega_2}{15\alpha^3}a^4(\sigma,t)e^{-4x+4\sigma t}, \text{ if } x>\sigma t,\\ \Big[\frac{1}{2}(c-\frac{\beta_0}{\beta})a(\sigma,t)\cdot(\sigma t-x)+a^2(\sigma,t)+\frac{\omega_1}{8\alpha^2}a^3(\sigma,t)+\frac{\omega_2}{15\alpha^3}a^4(\sigma,t)\Big]e^{x-\sigma t}\\ \quad-a^2(\sigma,t)e^{-2x+2\sigma t}-\frac{\omega_1}{8\alpha^2}a^3(\sigma,t)e^{-3x+3\sigma t}-\frac{\omega_2}{15\alpha^3}a^4(\sigma,t)e^{-4x+4\sigma t}, \text{ if } x\le\sigma t.\end{cases}\end{aligned}$$

On the other hand, we have

$$\mathrm{sign}(x-\sigma t)u_a\Big[\sigma-\frac{\beta_0}{\beta}-u_a\Big](t,x)=\begin{cases}(\sigma-\frac{\beta_0}{\beta})a(\sigma,t)e^{-x+\sigma t}-a^2(\sigma^2)e^{-2x+2\sigma t}, & \text{if } x>\sigma t,\\ -(\sigma-\frac{\beta_0}{\beta})a(\sigma,t)e^{x-\sigma t}+a^2(\sigma^2)e^{2x-2\sigma t}, & \text{if } x\le\sigma t.\end{cases}$$

If the function in the form (4.56) is a weak solution of equation (4.55), then adding (4.59) to (4.60) together yields that

$$\begin{cases} \Big[\partial_t a(\sigma,t) - \frac{1}{2}(c-\frac{\beta_0}{\beta})a(\sigma,t)\cdot(x-\sigma t) + (\sigma-\frac{\beta_0}{\beta})a(\sigma,t) - a^2(\sigma,t) - \frac{\omega_1}{8\alpha^2}a^3(\sigma,t) \\ \quad - \frac{\omega_2}{15\alpha^3}a^4(\sigma,t)\Big]e^{-x+\sigma t} + \frac{\omega_1}{8\alpha^2}a^3(\sigma,t)e^{-3x+3\sigma t} + \frac{\omega_2}{15\alpha^3}a^4(\sigma,t)e^{-4x+4\sigma t} = 0, \text{ if } x > \sigma t, \\ \Big[\partial_t a(\sigma,t) + \frac{1}{2}(c-\frac{\beta_0}{\beta})a(\sigma,t)\cdot(\sigma t - x) - (\sigma-\frac{\beta_0}{\beta})a(\sigma,t) + a^2(\sigma,t) + \frac{\omega_1}{8\alpha^2}a^3(\sigma,t) \\ \quad + \frac{\omega_2}{15\alpha^3}a^4(\sigma,t)\Big]e^{x-\sigma t} - \frac{\omega_1}{8\alpha^2}a^3(\sigma,t)e^{-3x+3\sigma t} - \frac{\omega_2}{15\alpha^3}a^4(\sigma,t)e^{-4x+4\sigma t} = 0, \text{ if } x \le \sigma t. \end{cases}$$

By the linear independence of the functions $e^{-x+\sigma t}$, $e^{x-\sigma t}$, $e^{-3x+3\sigma t}$ and $e^{-4x+4\sigma t}$, the above condition holds if and only if

$$a(\sigma,t) = 0,$$

which provides a trivial solution of equation (4.55), $u_a(t,x) = 0$, thereby concluding the proof of Theorem 4.3.1. □

4.3.2 Classification of traveling-wave solutions

We now classify the traveling-wave solutions to the R-CH equation (4.55). For a traveling wave solution $\varphi(t,x) = \varphi(x-\sigma t)$ with speed σ, equation (4.55) takes the form

$$-\sigma\varphi_x + \sigma\varphi_{xxx} + c\varphi_x + 3\varphi\varphi_x - \frac{\beta_0}{\beta}\varphi_{xxx} + \frac{\omega_1}{\alpha^2}\varphi^2\varphi_x + \frac{\omega_2}{\alpha^3}\varphi^3\varphi_x = 2\varphi_x\varphi_{xx} + \varphi\varphi_{xxx}. \quad (4.62)$$

Integrating respect to spatial variable gives

$$(c-\sigma)\varphi + \frac{3}{2}\varphi^2 + \frac{\omega_1}{3\alpha^2}\varphi^3 + \frac{\omega_2}{4\alpha^3}\varphi^4 + (\sigma - \frac{\beta_0}{\beta})\varphi_{xx} - \frac{1}{2}(\varphi_x)^2 - \varphi\varphi_{xx} = 0, \quad (4.63)$$

for $|x| \to \infty$, $\varphi, \varphi_x, \varphi_{xx} \to 0$.

It is observed that

$$[(\varphi - \sigma + \frac{\beta_0}{\beta})^2]_{xx} = 2\varphi_x^2 + 2\varphi\varphi_{xx} - 2(\sigma - \frac{\beta_0}{\beta})\varphi_{xx}.$$

Equation (4.63) may take the following form

$$[(\varphi-\sigma+\frac{\beta_0}{\beta})^2]_{xx}=\varphi_x^2+3\varphi^2+\frac{2\omega_1}{3\alpha^2}\varphi^3+\frac{\omega_2}{2\alpha^3}\varphi^4+2(c-\sigma)\varphi. \tag{4.64}$$

Inspired by the approach of classification of the traveling-wave solutions to the classical Camassa-Holm equation [46], we can establish a similar result in the following lemma, which is related to the regularity of the traveling waves.

Lemma 4.3.2. *If φ is a traveling wave of equation* (4.55) *and $\varphi\in H^1(\mathbb{R})$, then*

$$(\varphi-\sigma+\frac{\beta_0}{\beta})^k\in C^j(\mathbb{R}\setminus\varphi^{-1}(\sigma-\frac{\beta_0}{\beta})),\quad \text{for}\quad \mathrm{k}\geq 2^{\mathrm{j}}.$$

Furthermore, we conclude

$$\varphi\in C^\infty(\mathbb{R}\setminus\varphi^{-1}(\sigma-\frac{\beta_0}{\beta})).$$

Proof. Let $v=\varphi-\sigma+\frac{\beta_0}{\beta}$. Then equation (4.64) infers

$$(v^2)_{xx}=v_x^2+p(v),$$

where $p(v)$ is a polynomial in v, more precisely

$$p(v)=2(c-\sigma)(v+\sigma-\frac{\beta_0}{\beta})+3(v+\sigma-\frac{\beta_0}{\beta})^2+\frac{2\omega_1}{3\alpha^2}(v+\sigma-\frac{\beta_0}{\beta})^3+\frac{\omega_2}{3\alpha^3}(v+\sigma-\frac{\beta_0}{\beta})^4.$$

From the assumption, we know $v\in H^1_{loc}(\mathbb{R})$, which gives rise to $v_x^2+p(v)\in L^1_{loc}(\mathbb{R})$ and $(v^2)_{xx}\in L^1_{loc}(\mathbb{R})$. Thus, $(v^2)_x\in W^{1,1}_{loc}(\mathbb{R})$. This implies that $(v^2)_x$ is absolutely continuous, and v^2, v^3 are belongs to $C^1(\mathbb{R}\setminus v^{-1}(0))$. Moreover,

$$\begin{aligned}(v^k)_{xx}=&[(v^k)_x]_x=[kv^{k-1}v_x]_x=[\frac{k}{2}v^{k-2}(2vv_x)]_x=\frac{k}{2}(v^{k-2})_x(v^2)_x+\frac{k}{2}v^{k-2}(v^2)_{xx}\\ =&k(k-2)v^{k-2}v_x^2+\frac{k}{2}v^{k-2}[v_x^2+p(v)]=k(k-\frac{3}{2})v^{k-2}v_x^2+\frac{k}{2}v^{k-2}p(v).\end{aligned}$$

For $k=3$, the right-hand side of the above equation is in $L^1_{loc}(\mathbb{R})$. Therefore, we have

$$v^3\in C^1(\mathbb{R})\setminus v^{-1}(0)).$$

Similarly, for $k \geq 4$, we obtain $(v^k)_{xx} = \frac{k}{4}(k-\frac{3}{2})v^{k-4}[(v^2)_x]^2 + \frac{k}{2}v^{k-2}p(v)$. Since $v^2 \in C^1(\mathbb{R}) \setminus v^{-1}(0))$, it follows that

$$v^k \in C^2(\mathbb{R}) \setminus v^{-1}(0)), \quad k \geq 4.$$

For $k \geq 8$, we have $v^{k-2}p(v) \in C^2(\mathbb{R}) \setminus v^{-1}(0))$, by previous conclusion as well as $v^4, v^{k-4} \in C^2(\mathbb{R}) \setminus v^{-1}(0))$. Since $v^{k-2}v_x^2 = \frac{1}{4}(v^4)_x \frac{1}{k-4}(v^{k-4})_x \in C^1(\mathbb{R}) \setminus v^{-1}(0))$, we have

$$v^k \in C^3(\mathbb{R}) \setminus v^{-1}(0)), \quad k \geq 8.$$

Extending these arguments to higher values of k, we prove that

$$v^k \in C^j(\mathbb{R} \setminus v^{-1}(0)), \quad \text{for} \quad \mathrm{k} \geq 2^{\mathrm{j}}.$$

This completes the proof of Lemma 4.3.2. □

We may rewrite equation (4.63) by multiplying by φ_x and integrating on $(-\infty, x]$

$$(c-\sigma)\varphi^2 + \varphi^3 + \frac{\omega_1}{6\alpha^2}\varphi^4 + \frac{\omega_2}{10\alpha^3}\varphi^5 + (\sigma - \frac{\beta_0}{\beta})\varphi_x^2 - \varphi\varphi_x^2 = 0,$$

which implies, if $\varphi \neq \sigma - \frac{\beta_0}{\beta}$ for all x,

$$\varphi_x^2 = \frac{\varphi^2[\frac{\omega_2}{10\alpha^3}\varphi^3 + \frac{\omega_1}{6\alpha^2}\varphi^2 + \varphi + (c-\sigma)]}{\varphi - \sigma + \frac{\beta_0}{\beta}} = \frac{\varphi^2 f(\varphi)}{\varphi - \sigma + \frac{\beta_0}{\beta}} := F(\varphi). \tag{4.65}$$

Applying the similar arguments as introduced in [46], we have the following conclusion.

1. When φ approaches a simple zero m of $F(\varphi)$ so that $F(m) = 0$ and $F'(m) \neq 0$, the solution φ of (4.65) satisfies

$$\varphi_x^2 = (\varphi - m)F'(m) + O((\varphi - m)^2), \text{ as } \varphi \to m, \tag{4.66}$$

where $f = O(g)$ as $x \to a$ means $|\frac{f(x)}{g(x)}|$ is bounded in some interval $[a-\varepsilon, a+\varepsilon]$ with $\varepsilon > 0$. Then, we have

$$\varphi(x) = m + \frac{1}{4}(x-x_0)^2 F'(m) + O((x-x_0)^4), \text{ as } x \to x_0, \tag{4.67}$$

where $\varphi(x_0) = m$.

2. If $F(\varphi)$ has a double zero at $\varphi = 0$ such that $F(0) = F'(0) = 0$ and $F''(0) > 0$, then

$$\varphi_x^2 = \varphi^2 F''(0) + O(\varphi^3), \text{ as } \varphi \to 0. \tag{4.68}$$

Hence,

$$\varphi = O\Big(\exp(-\sqrt{F''(0)}|x|) \Big), \text{ as } |x| \to \infty, \tag{4.69}$$

which implies $\varphi \to 0$ exponentially as $x \to \infty$.

3. If φ approaches a simple pole $\varphi(x_0) = \sigma - \frac{\beta_0}{\beta}$ of $F(\varphi)$, then

$$\varphi(x) - \sigma + \frac{\beta_0}{\beta} = \lambda|x - x_0|^{2/3} + O((x - x_0)^{4/3}), \quad \text{as} \quad x \to x_0, \tag{4.70}$$

$$\varphi_x = \begin{cases} \frac{2}{3}\lambda|x - x_0|^{-1/3} + O((x - x_0)^{1/3}), & \text{as} \quad x \downarrow x_0, \\ -\frac{2}{3}\lambda|x - x_0|^{-1/3} + O((x - x_0)^{1/3}), & \text{as} \quad x \uparrow x_0, \end{cases} \tag{4.71}$$

for some constant λ.

4. Peaked traveling waves occur when φ suddenly changes direction: $\varphi_x \mapsto -\varphi_x$ according to equation (4.65).

Based on discussion above on traveling wave solution of equation (4.55), we will classify the various travelling-wave solutions to (4.65).

In view of the expression of the function of f, one should consider the following three different cases.

$$\sqrt{\frac{\sqrt{19} - 4}{3}} < c < \frac{1}{\sqrt{8}}, \quad \frac{1}{\sqrt{8}} < c < 1 \quad \text{and} \quad c = \frac{1}{\sqrt{8}},$$

corresponding to $\omega_2 < 0$, $\omega_2 > 0$ and $\omega_2 = 0$, respectively.

Let us start with

$$f(\varphi) = a_3\varphi^3 + a_2\varphi^2 + a_1\varphi + a_0, \quad a_3 \neq 0,$$

where

$$a_3 = \frac{\omega_2}{10\alpha^3}, \quad a_2 = \frac{\omega_1}{6\alpha^2}, \quad a_1 = 1, \quad a_0 = c - \sigma.$$

Define that $\eta = \varphi + \frac{a_2}{3a_3}$, that is, $\varphi = \eta - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$. Then, by the property of cubic polynomial [7], we can rewrite

$$f(\varphi) = f(\eta) = \frac{\omega_2}{10\alpha^3}(\eta^3 + 3p\eta + 2q), \tag{4.72}$$

where

$$p = \frac{3a_3a_1 - a_2^2}{9a_3^2} = \frac{10}{3}\frac{\alpha^3}{\omega_2} - \frac{25}{81}\frac{\alpha^2\omega_1^2}{\omega_2^2}, \tag{4.73}$$

$$q = \frac{2a_2^3 - 9a_3a_2a_1 + 27a_3^2a_0}{54a_3^3} = \frac{125}{729}\frac{\alpha^3\omega_1^3}{\omega_2^3} - \frac{25}{9}\frac{\alpha^4\omega_1}{\omega_2^2} + 5(c-\sigma)\frac{\alpha^3}{\omega_2}. \tag{4.74}$$

The determinator of equation $f(\eta) = 0$ is defined by

$$\begin{aligned} D = q^2 + p^3 = &\frac{1000}{27}\frac{\alpha^9}{\omega_2^3} - \frac{625}{243}\frac{\alpha^8\omega_1^2}{\omega_2^4} - \frac{250}{9}(c-\sigma)\frac{\alpha^7\omega_1}{\omega_2^3} \\ &+ \frac{1250}{729}(c-\sigma)\frac{\alpha^6\omega_1^3}{\omega_2^4} + 25(c-\sigma)^2\frac{\alpha^6}{\omega_2^2}. \end{aligned} \tag{4.75}$$

Case I: $\sqrt{\frac{\sqrt{19}-4}{3}} < c < \frac{1}{\sqrt{8}}$. The restriction on c then yields that

$$\alpha > 0, \quad \beta_0 < 0, \quad \beta > 0, \quad \omega_1 < 0, \quad \omega_2 > 0.$$

Consider $D = 0$ as a quadratic equation of $c - \sigma$, that is

$$A(c-\sigma)^2 + B(c-\sigma) + C = 0, \tag{4.76}$$

where

$$A = \frac{25\alpha^6}{\omega_2^2}, \quad B = \frac{1250}{729}\frac{\alpha^6\omega_1^3}{\omega_2^4} - \frac{250}{9}\frac{\alpha^7\omega_1}{\omega_2^3}, \quad \text{and} \quad C = \frac{1000}{27}\frac{\alpha^9}{\omega_2^3} - \frac{625}{243}\frac{\alpha^8\omega_1^2}{\omega_2^4}.$$

It is obviously that $A > 0$. It is also observed from $c \in (\sqrt{\frac{\sqrt{19}-4}{3}}, \frac{1}{\sqrt{8}})$ that $8 - 139c^2 < 0$ and $8c^2 - 1 < 0$. This then implies that

$$B = \frac{250}{9}\frac{\alpha^6\omega_1}{\omega_2^3}\frac{c^2(8-139c^2)}{18(c^2+1)(8c^2-1)} < 0.$$

For C, it is easy to see that

$$C = \frac{125}{27}\frac{\alpha^8}{\omega_2^3}\frac{(128c^4 - 21c^2 + 10)}{2(c^2+1)(8c^2-1)} < 0.$$

Hence, the quadratic equation $D = 0$ has a negative solution y_1 and a positive solution y_2, where

$$y_1 = \frac{5}{9}\frac{\alpha\omega_1}{\omega_2} - \frac{25}{729}\frac{\omega_1^3}{\omega_2^2} - \frac{1}{2}\sqrt{\Delta} < 0,$$
$$y_2 = \frac{5}{9}\frac{\alpha\omega_1}{\omega_2} - \frac{25}{729}\frac{\omega_1^3}{\omega_2^2} + \frac{1}{2}\sqrt{\Delta} > 0, \text{ and}$$
$$\Delta = \Big(\frac{50}{729}\frac{\omega_1^3}{\omega_2^2} - \frac{10}{9}\frac{\alpha\omega_1}{\omega_2}\Big)^2 - 4\Big(\frac{40}{27}\frac{\alpha^3}{\omega_2} - \frac{25}{243}\frac{\alpha^2\omega_1^2}{\omega_2^2}\Big) > 0.$$

In addition, we know

1), If $y_1 < c - \sigma < y_2$, then $D < 0$;

2), If $c - \sigma = y_1$ or $c - \sigma = y_2$, then $D = 0$;

3), If $c - \sigma < y_1$ or $c - \sigma > y_2$, then $D > 0$.

The following theorem provides the classification of traveling wave solution to equation (4.55) when $D > 0$ and $\sqrt{\frac{\sqrt{19}-4}{3}} < c < \frac{1}{\sqrt{8}}$, i.e. $f(\varphi)$ has exactly one real root, which takes the form of $\eta_1 = \sqrt[3]{-q + \sqrt{q^2+p^3}} + \sqrt[3]{-q - \sqrt{q^2+p^3}}$.

Theorem 4.3.3. *Assume that* $\sqrt{\frac{\sqrt{19}-4}{3}} < c < \frac{1}{\sqrt{8}}$.

(1) Suppose $\sigma > c - y_1$.

- *If* $\sigma - \frac{\beta_0}{\beta} = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$, *then there is a peaked traveling wave solution* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$.
- *If* $\sigma - \frac{\beta_0}{\beta} > \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$, *then there is a smooth traveling wave solution* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$.
- *If* $\sigma - \frac{\beta_0}{\beta} < \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$, *then there is a cusped traveling wave solution* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$.

(2) Suppose $\sigma < \min\{\frac{\beta_0}{\beta}, c - y_2\}$ *and* $\gamma_2 = \frac{\omega_2}{10\alpha^3}(\frac{5}{9}\frac{\alpha\omega_1}{\omega_2} - \eta_1) > 0$.

- *If* $\sigma - \frac{\beta_0}{\beta} = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$, *then there is an antipeaked traveling wave solution*

$\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$.

- *If* $\sigma - \frac{\beta_0}{\beta} < \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$, *then there is a smooth traveling wave solution* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$.
- *If* $\sigma - \frac{\beta_0}{\beta} > \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$, *then there is an anticusped traveling wave solution* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$.

Proof. Since $\frac{\beta_0}{\beta} = \frac{c(c^4+6c^2-1)}{3c^4+8c^2-1}$, this implies $c - \frac{\beta_0}{\beta} = \frac{2c^3(c^2+1)}{3c^4+8c^2-1} > 0$, for $c \in (\sqrt{\frac{\sqrt{19}-4}{3}}, \frac{1}{\sqrt{8}})$. The decay of $\varphi(x)$ at infinity gives a necessary condition for the existence of traveling wave is

$$\lim_{|x|\to\infty} \frac{\frac{\omega_2}{10\alpha^3}\varphi^3 + \frac{\omega_1}{6\alpha^2}\varphi^2 + \varphi + (c-\sigma)}{\varphi - \sigma + \frac{\beta_0}{\beta}} \geq 0,$$

which implies

$$\begin{cases} c - \sigma \geq 0, \\ \frac{\beta_0}{\beta} - \sigma > 0, \end{cases} \quad \text{or} \quad \begin{cases} c - \sigma \leq 0, \\ \frac{\beta_0}{\beta} - \sigma < 0, \end{cases}$$

i.e. $\sigma < \frac{\beta_0}{\beta}$ or $\sigma \geq c$ (when $\sigma = c$, $D < 0$).

By the property of cubic equation, it is thereby inferred from $D > 0$ that $f = 0$ has one real root and two complex roots. Based on necessary condition for the existence of traveling wave, we will discuss the following two cases for $D > 0$: $\sigma > c$ and $\sigma < \frac{\beta_0}{\beta}$.

If $\sigma > c$, then

$$\begin{aligned} p &= \frac{10}{3}\frac{\alpha^3}{\omega_2} - \frac{25}{81}\frac{\alpha^2\omega_1^2}{\omega_2^2} = \frac{5}{3}\frac{\alpha^2 c^2}{\omega_2}\Big[\frac{(91c^2-2)}{6(c^2+1)(8c^2-1)}\Big] < 0, \\ q &= \frac{125}{729}\frac{\alpha^3\omega_1^3}{\omega_2^3} - \frac{25}{9}\frac{\alpha^4\omega_1}{\omega_2^2} + 5(c - \sigma\frac{\alpha^3}{\omega_2}) \\ &= \frac{25}{9}\frac{\alpha^3\omega_1}{\omega_2^2}\Big[\frac{c^2(8-139c^2)}{18(c^2+1)(8c^2-1)}\Big] + 5(c-\sigma)\frac{\alpha^3}{\omega_2} < 0, \end{aligned}$$

where $\omega_1 < 0$, $\omega_2 > 0$, $c - \sigma < 0$, $8c^2 - 1 < 0$, $91c^2 - 2 > 0$ and $8 - 139c^2 < 0$, since $c \in (\sqrt{\frac{\sqrt{19}-4}{3}}, \frac{1}{\sqrt{8}})$.

It then follows from the Cardano Formula that the real root of $f(\eta) = 0$ can be expressed as

$$\eta_1 = \sqrt[3]{-q + \sqrt{q^2 + p^3}} + \sqrt[3]{-q - \sqrt{q^2 + p^3}}. \tag{4.77}$$

Denote $u = \sqrt[3]{-q + \sqrt{q^2 + p^3}}$ and $v = \sqrt[3]{-q - \sqrt{q^2 + p^3}}$. Then the other two roots can be expressed as

$$\eta_2 = u(-\frac{1}{2} + \frac{\sqrt{3}}{2}i) + v(-\frac{1}{2} - \frac{\sqrt{3}}{2}i),$$
$$\eta_3 = u(-\frac{1}{2} - \frac{\sqrt{3}}{2}i) + v(-\frac{1}{2} + \frac{\sqrt{3}}{2}i).$$

Hence,

$$f(\eta) = \frac{\omega_2}{10\alpha^3}(\eta - \eta_1)(\eta - \eta_2)(\eta - \eta_3),$$

where $\eta_1 > 0$ and $(\eta - \eta_2)(\eta - \eta_3) > 0$. Substituting η by $\varphi + \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$ implies

$$f(\varphi) = \frac{\omega_2}{10\alpha^3}(\varphi + \frac{5}{9}\frac{\alpha\omega_1}{\omega_2} - \eta_1)(\varphi + \frac{5}{9}\frac{\alpha\omega_1}{\omega_2} - \eta_2)(\varphi + \frac{5}{9}\frac{\alpha\omega_1}{\omega_2} - \eta_3),$$

where $\eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2} > 0$ and $(\varphi + \frac{5}{9}\frac{\alpha\omega_1}{\omega_2} - \eta_2)(\varphi + \frac{5}{9}\frac{\alpha\omega_1}{\omega_2} - \eta_3) > 0$.
Let

$$\gamma_1 = \frac{\omega_2}{10\alpha^3}(\frac{5}{9}\frac{\alpha\omega_1}{\omega_2} - \eta_1) < 0, \tag{4.78}$$

and

$$Q(\varphi) = (\varphi + \frac{5}{9}\frac{\alpha\omega_1}{\omega_2} - \eta_2)(\varphi + \frac{5}{9}\frac{\alpha\omega_1}{\omega_2} - \eta_3). \tag{4.79}$$

Then equation (4.65) can be written as the following form

$$\varphi_x^2 = \frac{\varphi^2(\frac{\omega_2}{10\alpha^3}\varphi + \gamma_1)Q(\varphi)}{\varphi - \sigma + \frac{\beta_0}{\beta}} = \frac{\frac{\omega_2}{10\alpha^3}\varphi^2\left[\varphi - (\eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2})\right]Q(\varphi)}{\varphi - \sigma + \frac{\beta_0}{\beta}} := G_1(\varphi).$$

Hence, if $\sigma - \frac{\beta_0}{\beta} = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$, then φ suddenly changes direction from φ_x to $-\varphi_x$ at $\varphi = \sigma - \frac{\beta_0}{\beta}$ and $\varphi \to 0$ exponentially as $|x| \to \infty$, which give rise to the existence of a peaked traveling wave solution $\varphi > 0$ with $\max_{x\in\mathbb{R}} \varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$.

If $\sigma - \frac{\beta_0}{\beta} > \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$, then $G_1(\varphi)$ has a simple zero at $\varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$ and a double zero at $\varphi = 0$. In view of (4.66)-(4.69), there exists a smooth traveling wave solution $\varphi > 0$ with $\max\limits_{x\in\mathbb{R}} \varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$ and $\varphi \to 0$ exponentially as $|x| \to \infty$.

If $\sigma - \frac{\beta_0}{\beta} < \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$, then $G_1(\varphi)$ has a pole at $\varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$ and a double zero at $\varphi = 0$. It then implies from (4.68)-(4.71) that there exists a cusped traveling wave solution $\varphi > 0$ with $\max\limits_{x\in\mathbb{R}} \varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$ and $\varphi \to 0$ exponentially as $|x| \to \infty$.

In the case of $\sigma < \frac{\beta_0}{\beta}$, it is inferred from equation (4.72) that $p < 0$. From the property of φ decaying at infinity, we require $\frac{\omega_2}{10\alpha^3} \cdot q = c - \sigma > 0$, which implies $q > 0$. Equation (4.65) then can be written as

$$\varphi_x^2 = \frac{\varphi^2(\frac{\omega_2}{10\alpha^3}\varphi + \gamma_2)Q(\varphi)}{\varphi - \sigma + \frac{\beta_0}{\beta}} = \frac{\frac{\omega_2}{10\alpha^3}\varphi^2\Big[\varphi - (\eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2})\Big]Q(\varphi)}{\varphi - \sigma + \frac{\beta_0}{\beta}} := G_2(\varphi),$$

where $\gamma_2 = \frac{\omega_2}{10\alpha^3}(\frac{5}{9}\frac{\alpha\omega_1}{\omega_2} - \eta_1) > 0$, $\eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2} < 0$, and $\eta_1 < 0$ in this case.

Hence, if $\sigma - \frac{\beta_0}{\beta} = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$, then φ suddenly changes direction from φ_x to $-\varphi_x$ at $\varphi = \sigma - \frac{\beta_0}{\beta}$ and $\varphi \to 0$ exponentially as $|x| \to \infty$, which gives rise to the existence of an antipeaked traveling wave solution $\varphi < 0$ with $\min\limits_{x\in\mathbb{R}} \varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$.

If $\sigma - \frac{\beta_0}{\beta} < \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$, then $G_1(\varphi)$ has a simple zero at $\varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$ and a double zero at $\varphi = 0$. In view of (4.66)-(4.69), there exists a smooth traveling wave solution $\varphi < 0$ with $\min\limits_{x\in\mathbb{R}} \varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$ and $\varphi \to 0$ exponentially as $|x| \to \infty$.

If $\sigma - \frac{\beta_0}{\beta} > \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$, then $G_1(\varphi)$ has a pole at $\varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$ and a double zero at $\varphi = 0$. It is then deduced from (4.68)-(4.71) that these is $\varphi < 0$ with $\min\limits_{x\in\mathbb{R}} \varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$ and $\varphi \to 0$ exponentially as $|x| \to \infty$. This completes the proof of Theorem 4.3.3. □

Remark 8. Since the existence of peaked traveling wave solution requires $\sigma > c - y_1$ and $\sigma = \frac{\beta_0}{\beta} + \eta_1(\sigma) - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$ in Case (1). Due to $c > \frac{\beta_0}{\beta}$, which implies $\sigma < c + \eta_1(\sigma) - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$, then the necessary condition becomes $c + \eta_1(\sigma) - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2} > c - y_1$. As η_1 is positive, the above condition can be simplified as $y_1 > \dfrac{5}{9}\dfrac{\alpha\omega_1}{\omega_2}$, where $y_1 = \frac{5}{9}\frac{\alpha\omega_1}{\omega_2} - \frac{25}{729}\frac{\omega_1^3}{\omega_2^2} - \frac{1}{2}\sqrt{\Delta}$. This requires

$$0 < \Delta < \left(\frac{-50}{729}\frac{\omega_1^3}{\omega_2^2}\right)^2. \tag{4.80}$$

Actually,

$$\frac{50}{729}\frac{\omega_1^3}{\omega_2^2} = -\frac{50}{54}\frac{c^3(c^2-2)(c^2+1)}{(c^2-1)(8c^2-1)^2}, \tag{4.81}$$

$$\begin{aligned}\Delta = &\left(-\frac{50}{54}\frac{c^3(c^2-2)(c^2+1)}{(c^2-1)(8c^2-1)^2} + \frac{10}{3}\frac{c^3(c^2+1)}{(c^2-1)(8c^2-1)}\right)^2 \\ &- 4\left(\frac{80}{27}\frac{c^6(c^2+1)}{(c^2-2)(c^2-1)^2(8c^2-1)} - \frac{25}{27}\frac{c^6(c_1^2)^2}{(c^2-1)^2(8c^2-1)^2}\right).\end{aligned} \tag{4.82}$$

The leading order of $\Delta - \left(\frac{50}{729}\frac{\omega_1^3}{\omega_2^2}\right)^2$ is $(8c^2-1)^3$ and it has the following form

$$-\frac{500}{162}\frac{c^6(c^2-2)(c^2+1)^2}{(c^2-1)^2(8c^2-1)^3} < 0, \quad \text{when} \quad c \to \frac{1}{\sqrt{8}}^{-},$$

which implies (4.80) holds. This guarantees the existence of peaked traveling wave solution.

Case II: $\frac{1}{\sqrt{8}} < c \leq 1$. It is observed that the restriction on c gives that $\alpha > 0, \quad \beta_0 < 0, \quad \beta > 0, \quad \omega_1 < 0$, and $\omega_2 < 0$. In this case, equation (4.76) has a negative solution y_1 and a positive solution y_2. We could follow the similar proof in Theorem 4.3.3 to obtain the following results.

Theorem 4.3.4. *Assume* $\frac{1}{\sqrt{8}} < c \leq 1$.

(1) If $\sigma > c - y_1$*, then there is a smooth traveling wave solution* $\varphi < 0$ *with* $\min\limits_{x\in\mathbb{R}} \varphi = -\frac{10\alpha^3}{\omega_2}\gamma_3$ *where* $\gamma_3 = \frac{\omega_2}{10\alpha^3}\left(\frac{5}{9}\frac{\alpha\omega_1}{\omega_2} - \sqrt[3]{-q+\sqrt{q^2+p^3}} - \sqrt[3]{-q-\sqrt{q^2+p^3}}\right)$ *and a cusped traveling wave solution* $\varphi > 0$ *with* $\max\limits_{x\in\mathbb{R}} \varphi = \sigma - \frac{\beta_0}{\beta}$.

(2) If $\sigma < \min\{\frac{\beta_0}{\beta}, c - y_2\}$ *and* $\gamma_4 > 0$*, then there is a smooth traveling wave solution* $\varphi > 0$ *with* $\max\limits_{x\in\mathbb{R}} \varphi = -\frac{10\alpha^3}{\omega_2}\gamma_4$*, where* $\gamma_4 = \frac{\omega_2}{10\alpha^3}\left(\frac{5}{9}\frac{\alpha\omega_1}{\omega_2} - \sqrt[3]{-q+\sqrt{q^2+p^3}} + \sqrt[3]{-q-\sqrt{q^2+p^3}}\right)$ *and an anticusped traveling wave solution* $\varphi < 0$ *with* $\min\limits_{x\in\mathbb{R}} \varphi = \sigma - \frac{\beta_0}{\beta}$.

Proof. According to the decay of $\varphi(x)$ at infinity, a necessary condition for the existence of traveling wave is that $\sigma \geq c$ or $\sigma < \frac{\beta_0}{\beta}$, since $c > \frac{\beta_0}{\beta}$ for $\frac{1}{\sqrt{8}} < c \leq 1$. Note that $\sigma = c$ is not included since $D < 0$ when $\sigma = c$.

Case (1). If $\sigma > c$, we know $p < 0$ and $q > 0$ from (4.73) and (4.74). Then equation (4.65) has the following form

$$\varphi_x^2 = \frac{\varphi^2(\frac{\omega_2}{10\alpha^3}\varphi + \gamma_3)Q(\varphi)}{\varphi - \sigma + \frac{\beta_0}{\beta}} = \frac{\frac{\omega_2}{10\alpha^3}\varphi^2\Big[\varphi - (\eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2})\Big]Q(\varphi)}{\varphi - \sigma + \frac{\beta_0}{\beta}} := G_3(\varphi),$$

where $\gamma_3 = \frac{\omega_2}{10\alpha^3}(\frac{5}{9}\frac{\alpha\omega_1}{\omega_2} - \eta_1) < 0$, $\eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2} < 0$, and $\eta_1 < 0$ in this case.

Hence, for $\varphi < 0$, $G_3(\varphi)$ has a simple zero at $\varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$ and a double zero at $\varphi = 0$. According to (4.66), (4.67), (4.68), (4.69), there is a smooth traveling wave solution $\varphi < 0$ with $\min\limits_{x\in\mathbb{R}} \varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$ and $\varphi \to 0$ exponentially as $|x| \to \infty$.

For $\varphi > 0$, $G_3(\varphi)$ has a pole at $\varphi = \sigma - \frac{\beta_0}{\beta}$ and a double zero at $\varphi = 0$. From (4.68)-(4.71), there is a cusped traveling wave solution $\varphi > 0$ with $\max\limits_{x\in\mathbb{R}} \varphi = \sigma - \frac{\beta_0}{\beta}$ and $\varphi \to 0$ exponentially as $|x| \to \infty$.

Case (2). If $\sigma < \frac{\beta_0}{\beta}$, then we know $p < 0$ in (4.73). It then follows from the property of φ decaying at infinity that

$$q < 0 \text{ and } \frac{\omega_2}{10\alpha^3}\Big(\frac{5}{9}\frac{\alpha\omega_1}{\omega_2} - \sqrt[3]{-q + \sqrt{q^2 + p^3}} - \sqrt[3]{-q - \sqrt{q^2 + p^3}}\Big) > 0.$$

Similar to Case (1), we have

$$\varphi_x^2 = \frac{\varphi^2(\frac{\omega_2}{10\alpha^3}\varphi + \gamma_4)Q(\varphi)}{\varphi - \sigma + \frac{\beta_0}{\beta}} = \frac{\frac{\omega_2}{10\alpha^3}\varphi^2\Big[\varphi - (\eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2})\Big]Q(\varphi)}{\varphi - \sigma + \frac{\beta_0}{\beta}} := G_4(\varphi),$$

where $\gamma_4 = \frac{\omega_2}{10\alpha^3}(\frac{5}{9}\frac{\alpha\omega_1}{\omega_2} - \eta_1) < 0$, $\eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2} > 0$, and $\eta_1 > 0$ in this case.

Consequently, for $\varphi > 0$, $G_4(\varphi)$ has a simple zero at $\varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$ and a double zero at $\varphi = 0$. From (4.66)-(4.69), there is a smooth traveling wave solution $\varphi > 0$ with $\max\limits_{x\in\mathbb{R}} \varphi = \eta_1 - \frac{5}{9}\frac{\alpha\omega_1}{\omega_2}$ and $\varphi \to 0$ exponentially as $|x| \to \infty$.

For $\varphi < 0$, $G_4(\varphi)$ has a pole at $\varphi = \sigma - \frac{\beta_0}{\beta}$ and a double zero at $\varphi = 0$. It is then deduced from (4.68)-(4.71) there is a cusped traveling wave solution $\varphi < 0$ with $\min_{x\in\mathbb{R}} \varphi = \sigma - \frac{\beta_0}{\beta}$ and $\varphi \to 0$ exponentially as $|x| \to \infty$. This completes the proof of Theorem 4.3.4. □

Case III: $c = \frac{1}{\sqrt{8}}$. In this case, we have

$$\alpha = \frac{1}{9}, \quad \beta_0 = -\frac{5\sqrt{2}}{648}, \quad \beta = \frac{1}{162}, \quad \omega_1 = -\frac{35\sqrt{2}}{81}, \quad \omega_2 = 0,$$

which implies

$$\frac{\omega_1}{6\alpha^2} = -\frac{35\sqrt{2}}{6}, \quad c = \frac{\sqrt{2}}{4}, \quad \frac{\beta_0}{\beta} = -\frac{5\sqrt{2}}{4}.$$

Hence, equation (4.65) can be simplified as

$$\varphi_x^2 = \frac{\varphi^2[\varphi^2 - \frac{3\sqrt{2}}{35}\varphi - \frac{3\sqrt{2}}{35}(\frac{\sqrt{2}}{4} - \sigma)]}{\frac{3\sqrt{2}}{35}[(\sigma + \frac{5\sqrt{2}}{4}) - \varphi]}, \tag{4.83}$$

where $f(\varphi) = \varphi^2 - \frac{3\sqrt{2}}{35}\varphi - \frac{3\sqrt{2}}{35}(\frac{\sqrt{2}}{4} - \sigma)$. The following discussion enlists all possible distribution of the roots of f.

(a) $f(\varphi)$ has no zeros: If $\sigma > \frac{19\sqrt{2}}{70}$, then $(\frac{3\sqrt{2}}{35})^2 + 4 \cdot \frac{3\sqrt{2}}{35} \cdot (\frac{\sqrt{2}}{4} - \sigma) < 0$. Hence, $f(\varphi) > 0$.

(b) $f(\varphi)$ has a double zero: If $\sigma = \frac{19\sqrt{2}}{70}$, then $(\frac{3\sqrt{2}}{35})^2 + 4 \cdot \frac{3\sqrt{2}}{35} \cdot (\frac{\sqrt{2}}{4} - \sigma) = 0$. Hence, $f(\varphi) = (\varphi - \frac{3\sqrt{2}}{70})^2$.

(c) $f(\varphi)$ has two simple zeros: If $\sigma < \frac{19\sqrt{2}}{70}$, then $(\frac{3\sqrt{2}}{35})^2 + 4 \cdot \frac{3\sqrt{2}}{35} \cdot (\frac{\sqrt{2}}{4} - \sigma) > 0$. Hence, $f(\varphi) = (\varphi - M_1)(\varphi - M_2)$, where

$$M_1 = \frac{3\sqrt{2}}{70} - \frac{1}{2}\sqrt{\left(\frac{3\sqrt{2}}{35}\right)^2 + 4 \cdot \frac{3\sqrt{2}}{35} \cdot \left(\frac{\sqrt{2}}{4} - \sigma\right)},$$

$$M_2 = \frac{3\sqrt{2}}{70} + \frac{1}{2}\sqrt{\left(\frac{3\sqrt{2}}{35}\right)^2 + 4 \cdot \frac{3\sqrt{2}}{35} \cdot \left(\frac{\sqrt{2}}{4} - \sigma\right)},$$

and $M_1 < M_2$, $M_1 + M_2 > 0$. In addition, if $\sigma > \frac{\sqrt{2}}{4}$, then $M_1M_2 > 0$; If $\sigma = \frac{\sqrt{2}}{4}$, then $M_1 = 0$, $M_2 = \frac{3\sqrt{2}}{35}$; If $\sigma < \frac{\sqrt{2}}{4}$, then $M_1M_2 < 0$.

In view of equation (4.83), it is found from decay of φ at infinity that a necessary condition for the existence of traveling wave solution is

$$\lim_{|x|\to\infty} \frac{\varphi^2 - \frac{3\sqrt{2}}{35}\varphi - \frac{3\sqrt{2}}{35}(\frac{\sqrt{2}}{4} - \sigma)}{(\sigma + \frac{5\sqrt{2}}{4}) - \varphi} \geq 0,$$

which implies $\sigma < -\frac{5\sqrt{2}}{4}$ or $\sigma \geq \frac{\sqrt{2}}{4}$.

With the results established above, we are in the position to classify all traveling waves of equation (4.55) when $\omega_2 = 0$ for various σ.

Theorem 4.3.5. *Let* $c = \frac{1}{\sqrt{8}}$.

(1) If $\sigma < -\frac{5\sqrt{2}}{4}$*, i.e.* $f(\varphi)$ *has a negative root and a positive root, then:*

(1a) If $\sigma + \frac{5\sqrt{2}}{4} < M_1 < 0$*, then there is a smooth traveling wave solution* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = M_2$*;*

(1b) If $M_1 < \sigma + \frac{5\sqrt{2}}{4} < 0$*, then there is an anticusped traveling wave solution* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = \sigma + \frac{5\sqrt{2}}{4}$*;*

(1c) If $\sigma + \frac{5\sqrt{2}}{4} = M_1$*, then there is an antipeaked traveling wave solution* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = M_1 = \sigma + \frac{5\sqrt{2}}{4}$.

(2) If $\sigma = \frac{\sqrt{2}}{4}$*, i.e.* $f(\varphi)$ *has two simple roots:* $M_1 = 0$, $M_2 = \frac{3\sqrt{2}}{35}$*, there is no traveling wave solution.*

(3) If $\frac{\sqrt{2}}{4} < \sigma < \frac{19\sqrt{2}}{70}$*, i.e.* $f(\varphi)$ *has two simple roots:* $M_1 > M_2 > 0$, $M_2 = \frac{3\sqrt{2}}{35}$*, then there is a smooth traveling wave* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = M_1$.

(4) If $\sigma = \frac{19\sqrt{2}}{70}$*, i.e.* $f(\varphi)$ *has a double roots:* $M_1 = M_2 = \frac{3\sqrt{2}}{70}$*, then there is a smooth traveling wave* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = M_1 = M_2$ *and a cusped traveling wave solution* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = \frac{213\sqrt{2}}{140}$.

(5) If $\sigma > \frac{19\sqrt{2}}{70}$*, i.e.* $f(\varphi) > 0$ *for all* $\varphi(x) \in \mathbb{R}$*, then there is a cusped traveling wave solution* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = \sigma + \frac{5\sqrt{2}}{4}$.

Proof. Case (1), If $\sigma < -\frac{5\sqrt{2}}{4}$, then $f(\varphi)$ has two simple roots M_1 and M_2 satisfying $M_2 > 0 > M_1$, which implies

$$\varphi_x^2 = \frac{\varphi^2(\varphi - M_1)(\varphi - M_2)}{\frac{3\sqrt{2}}{35}[(\sigma + \frac{5\sqrt{2}}{4}) - \varphi]} := F_1(\varphi), \quad \text{where} \quad \sigma + \frac{5\sqrt{2}}{4} < 0. \tag{4.84}$$

From (4.84) we know that φ can not oscillate around zero near infinity. Let us consider the following two cases.

(1.1) $\varphi(x) > 0$ near $-\infty$. Then there is some x_0 sufficiently large negative so that $\varphi(x_0) = \varepsilon > 0$, with ε sufficiently small, and $\varphi_x(x_0) > 0$. Since $\sqrt{F_1(\varphi)}$ is locally Lipschitz in φ for $\varphi > 0$. Hence, there is a local solution to

$$\begin{cases} \varphi_x = \sqrt{F_1(\varphi)}, \\ \varphi(x_0) = \varepsilon. \end{cases}$$

on $[x_0 - L, x_0 + L]$ for some $L > 0$. Therefore from (4.66), (4.67), (4.68) and (4.69) we see that in this case we can obtain a smooth traveling wave solution with maximum height $\varphi = M_2$ and an exponential decay to zero at infinity

$$\varphi(x) = O\Big(\exp\Big(-\sqrt{\frac{\frac{3\sqrt{2}}{35}(\frac{\sqrt{2}}{4} - \sigma)}{\sigma + \frac{5\sqrt{2}}{4}}}\Big|x\Big|\Big)\Big) \quad \text{as} \quad |x| \to \infty. \tag{4.85}$$

(1.2) $\varphi(x) < 0$ near $-\infty$. Then there is some x_0 sufficiently large negative so that $\varphi(x_0) = -\varepsilon < 0$, with ε sufficiently small, and $\varphi_x(x_0) < 0$. Since $\sqrt{F_1(\varphi)}$ is locally Lipschitz in φ for $\sigma + \frac{5\sqrt{2}}{4} < \varphi < 0$, we an continue the local solution to $(-\infty, x_0 - L]$ with $\varphi(x) \to 0$ as $x \to -\infty$. As for $x \geq x_0 + L$, we can solve the initial value problem

$$\begin{cases} \varphi_x = -\sqrt{F_1(\varphi)}, \\ \varphi(x_0) = -\varepsilon. \end{cases}$$

If $M_1 < \sigma + \frac{5\sqrt{2}}{4} < 0$, the initial value problem can be solved all the way until $\varphi = \sigma + \frac{5\sqrt{2}}{4}$, which is a simple pole of $F_1(\varphi)$. From (4.68), (4.69), (4.70) and (4.71) we know that we can construct an anticusped solution with a cusp singularity at $\varphi = \sigma + \frac{5\sqrt{2}}{4}$ and $\varphi \to 0$ exponentially as $|x| \to \infty$;

If $M_1 = \sigma + \frac{5\sqrt{2}}{4} < 0$, then φ suddenly changes direction from φ_x to $-\varphi_x$ at $\varphi = M_1 = \sigma + \frac{5\sqrt{2}}{4}$, which implies the existence of an antipeaked solution $\varphi < 0$ with $\min_{x\in\mathbb{R}} \varphi(x) = M_1 = \sigma + \frac{5\sqrt{2}}{4}$ and $\varphi \to 0$ exponentially as $|x| \to \infty$.

Case (2), If $\sigma = \frac{\sqrt{2}}{4}$, then $f(\varphi)$ has two simple roots $M_1 = 0$ and $M_2 = \frac{3\sqrt{2}}{35}$, which implies

$$\varphi_x^2 = \frac{\varphi^3(\varphi - \frac{3\sqrt{2}}{35})}{\frac{3\sqrt{2}}{35}[\frac{3\sqrt{2}}{2} - \varphi]} := F_2(\varphi). \tag{4.86}$$

Notice that $\varphi(x) < 0$ near $-\infty$. Because $\varphi(x) \to 0$ as $x \to -\infty$, there is some x_0 sufficiently negative so that $\varphi(x_0) = -\varepsilon < 0$ with $\varepsilon > 0$ sufficiently small, and $\varphi_x(x_0) < 0$. From standard ODE theory, we can generate a unique local solution $\varphi(x)$ on $[x_0 - L, x_0 + L]$ for some $L > 0$. Due to

$$F_2'(\varphi) = \frac{\varphi^2\Big[-3\varphi^2 + \frac{216\sqrt{2}}{35}\varphi - \frac{27}{35}\Big]}{\frac{3\sqrt{2}}{35}(\frac{3\sqrt{2}}{2} - \varphi)^2} < 0, \quad \text{for} \quad \varphi < 0,$$

we know $F_2(\varphi)$ decreases for $\varphi < 0$. Since $\varphi_x(x_0) < 0$, φ decreases near x_0, which implies $F_2(\varphi)$ increases near x_0. Hence, by $F_2'(\varphi)$, φ_x decreases near x_0, and then φ and φ_x both decrease on $[x_0 - L, x_0 + L]$. Since $\sqrt{F_1(\varphi)}$ is locally Lipschitz in φ for $\varphi < 0$, we can continue the local solution to all $\mathbb{R}$ and obtain that $\varphi(x) \to -\infty$ as $x \to \infty$, which fails to be in H^1. Thus, there is no traveling wave solution in this case.

Case (3), If $\frac{\sqrt{2}}{4} < \sigma < \frac{19\sqrt{2}}{70}$, then $f(\varphi)$ has two simple roots M_2 and M_1, which implies

$$\varphi_x^2 = \frac{\varphi^2(\varphi - M_1)(\varphi - M_2)}{\frac{3\sqrt{2}}{35}[(\sigma + \frac{5\sqrt{2}}{4}) - \varphi]} := F_3(\varphi), \quad \text{where} \quad 0 < M_1 < M_2 < \sigma + \frac{5\sqrt{2}}{4}. \tag{4.87}$$

Similar to Case (1), we know that φ can not be oscillated around zero near infinity. Let us consider the following two cases.

(3.1) $\varphi(x) > 0$ near $-\infty$. Then the same analysis as used in the proof of Case (1.1) leads to the conclusion that there is a smooth traveling wave solution with maximum height $\varphi = M_1$ and an exponential decay to zero at infinity.

(3.2) $\varphi(x) < 0$ near $-\infty$. A directly computation shows that $F_3'(\varphi) < 0$, if $\varphi < 0$, where

$$F_3'(\varphi) = \frac{\Big[4\varphi^3 - 3(M_1+M_2)\varphi^2 + 2M_1M_2\varphi\Big](\sigma + \frac{5\sqrt{2}}{4}) - 3\varphi^4 + 2(M_1+M_2)\varphi^3 - 2M_1M_2\varphi^2}{(\frac{3\sqrt{2}}{35})^2\Big[(\sigma + \frac{5\sqrt{2}}{4} - \varphi)\Big]^2}.$$

Arguing as Case (2), there is no traveling wave solution in this case.

Case (4), If $\sigma = \frac{19\sqrt{2}}{70}$, then $f(\varphi)$ has a double root $M_1 = M_2 = \frac{3\sqrt{2}}{70}$, which implies

$$\varphi_x^2 = \frac{\varphi^2(\varphi - \frac{3\sqrt{2}}{70})^2}{\frac{3\sqrt{2}}{35}[\frac{213\sqrt{2}}{140} - \varphi]} := F_4(\varphi). \tag{4.88}$$

(4.1) $\varphi(x) > 0$ near $-\infty$. By the standard ODE theory in Case (1), if φ reaches its absolutely maximum at $\varphi = M_1 = M_2$, then there exists a smooth traveling wave solution with maximum height $\varphi = M_1 = M_2$ and an exponential decay to zero at infinity; if $\varphi = M_1 = M_2$ is not an absolutely maximum, then we obtain a cusped traveling wave solution with maximum height $\varphi = \frac{213\sqrt{2}}{140}$.

(4.2) $\varphi(x) < 0$ near $-\infty$. From

$$F_4'(\varphi) = \frac{\Big(-3\varphi^3 + \frac{876\sqrt{2}}{140}\varphi^2 - \frac{3582}{4900}\varphi + \frac{7668\sqrt{2}}{686000}\Big)\varphi}{\frac{3\sqrt{2}}{35}(\frac{213\sqrt{2}}{140} - \varphi)^2},$$

one can check that $F_5'(\varphi) < 0$ for $\varphi < 0$. Hence, similar to Case (2), there is no traveling wave solution in this case.

Case (5), If $\sigma > \frac{19\sqrt{2}}{70}$, then there is no real roots for $f(\varphi)$ and we know $f(\varphi) > 0$ for all $\varphi(x) \in \mathbb{R}$. Denote

$$\varphi_x^2 = \frac{\varphi^2 f(\varphi)}{\frac{3\sqrt{2}}{35}[(\sigma + \frac{5\sqrt{2}}{4}) - \varphi]} := F_5(\varphi). \tag{4.89}$$

The discussion in the previous section shows that there is no smooth traveling waves in this case and since $f(\varphi)$ has no zeros for $\varphi \in \mathbb{R}$, there can only exist cuspons or anticuspons. It is then inferred from (4.89) that φ can not oscillate around zero near infinity.

(5.1) $\varphi(x) > 0$ near $-\infty$. Then there is some x_0 sufficiently large negative so that $\varphi(x_0) = \varepsilon > 0$, with ε sufficiently small, and $\varphi_x(x_0) > 0$. $\sqrt{F_5(\varphi)}$ is locally Lipschitz in φ for $0 < \varphi < \sigma + \frac{5\sqrt{2}}{4}$, then $\sigma + \frac{5\sqrt{2}}{4}$ becomes a pole of $F_5(\varphi)$. Thus we obtain a traveling wave solution with a cusp at $\varphi = \sigma + \frac{5\sqrt{2}}{4}$ and decay exponentially.

(5.2) $\varphi(x) < 0$ near $-\infty$. A directly computation shows that $F_5'(\varphi) < 0$ for $\varphi < 0$. It is then adduced from the argument applied in Case (2) that there is no traveling-wave solution. Hence, the proof of Theorem 4.3.5 is completed. □

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BIOGRAPHICAL STATEMENT

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