POSTERIOR NORMAL APPROXIMATION OF REAL-TIME DEGRADATION MODELING USING LAPLACE APPROXIMATION

by

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Inspirational quote:

Life is good for only two things:

To study mathematics and to teach it.

− Simon-Denis Poisson

Dedication

To my creator

To my late father Ali Mahmoud Jawad, my mother Jamilé Kassem to whom I am grateful for their sacrifices. To my wife Khaleelah and my daughter Shadiah, who brought so much joy in my life.

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Abstract

Posterior Normal Approximation of Real-Time

Degradation Modeling Using Laplace Approximation

Supervising Professor: Dr. Shan Sun-Mitchell

Preventing failure that can cause delays or catastrophe, has been the focus and motivation for engineers, and other establishments that deals with heavy and light machinery, equipment, and devices. One of the biggest challenges, is accuracy and heavy computations of remaining useful life distribution. In this thesis we will use Laplace Approximations (LA) to avoid relying on complicated numerical computations, in calculating the remaining useful life distribution (RLD). LA is useful method to approximate the posterior distribution of Bayesian formula that incorporates linear degradation model and prior distribution.

This proposed approach is applicable to various degradation models composed of univariate and bivariate stochastic parameters that form the models, symmetric and non-symmetric prior believes, and different symmetrical error.

Under LA technique, we are able to normally approximate the posterior distribution with its proper parameters, and then implement Bernstein distribution using those parameters to calculate the residual life distribution. In addition, the mean squared error (MSE) of the parameters estimator is considered.

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Abbreviations

N-N: Normal error and Normal Prior.

N-BN: Normal error and Bivariate Normal Prior

N-JN: Normal error and Joint Normal Prior with $\rho_o = 0$

N-G: Normal error and Gamma Prior.

DE-G: Double Exponential error and Gamma Prior

DE-JN: Double Exponential error and joint Normal Prior with $\rho_o = 0$

DE-BN: Double Exponential error and Bivariate Normal Prior.

CM: Condition Monitoring.

RLD: Residual Life distribution.

Chapter 1

Motivation

1.1 Introduction

Performance degradation of components and devices is and always will be a challenge to technicians and engineers. One way of monitoring the health of components, is to obtain live data of device health through sensors in terms of amplitude signal from a working device. Change in signal will reflect a change in the health of the device, such as cracks, vibrations, and current draw etc., all this happens while the device is functioning. The following list includes some condition monitoring (CM) techniques applied in the industries: [16]

- Vibration analysis and diagnostics
- Lubricant analysis
- Acoustic emission (airborne ultrasound)
- Infrared thermography
- Ultrasound testing (material thickness/flaw testing)
- Motor condition monitoring and motor current signature analysis (MCSA)
- Model-based voltage and current systems (MBVI systems)

When data is collected, it will be used in mathematical degradation model to estimate residual life of intended device. This model will contain stochastic parameters with known distributions, hence we will use Bayesian updating technique that uses the collected data to derive posterior parameters. This technique requires a prior distribution assumption and the linear degradation model. This approach works well when the prior has symmetrical distribution. However, when the prior is skewed, this technique becomes harder to implement, and requires longer evaluation and calculation time. In this research, we apply Laplace approximation, to approximate the posterior distribution with normal distribution, using symmetric and non-symmetric priors (i.e. gamma prior) with different symmetrically distributed errors. Once the posterior distribution is derived, the residual life of the device is calculated using truncated Bernstein distribution.

1.2 Concept of Bernstein Distribution

Starting with the duality relationship [1]:

$$P(T > t) = P(W(t) < w_{cr}) \quad (1.1)$$

Where W(t) is the total wear function at instant t, calculated by:

$$W(t) = W_i + \theta t, \ t \epsilon T \qquad (1.2)$$

W_i is the initial wear random variable at time t = 0.

θ is the degradation rate of W(t). It is a random variable as well.

From equation (1), the component will fail when it reaches a critical threshold w_{cr}. w_i and θ are assumed to be statistically independent and normally distributed, due to the large number of data collected. Hence W(t) has a normal distribution with parameters μ(W(t)) and $\sigma^2(W(t))$ such that

$W(t) \sim N[\mu(W(t), \sigma^2(W(t))]$. Therefore,

$$P(T > t) = P(W(t) < w_{cr}) = 1- P(T < t)$$

$$\Rightarrow P(T < t) = 1 - P(W(t) < w_{cr})$$

Standardizing $W(t)$ *and* w_{cr}, yields

$$1 - P\left[\frac{W(t) - \mu[W(t)]}{\sigma^2[W(t)]} < \frac{w_{cr} - \mu[W(t)]}{\sigma^2[W(t)]}\right]$$

Representing in the normal distribution form, we get

$$= \Phi\left[\frac{w_{cr} - \mu[W(t)]}{\sqrt{\sigma^2[W(t)]}}\right]$$

This is called Bernstein distribution [1], where μ[W(t)] and $\sigma^2[W(t)]$ are the expected value and the variance of the distribution, respectively. These are represented by the following

$$\mu[W(t)] = E[w_i] + E[\theta t] = \mu[w_i] + t\mu[\theta]$$

$$\sigma^2[W(t)] = V[w_i] + V[\theta t] = \sigma^2[w_i] + t^2\sigma^2[\theta]$$

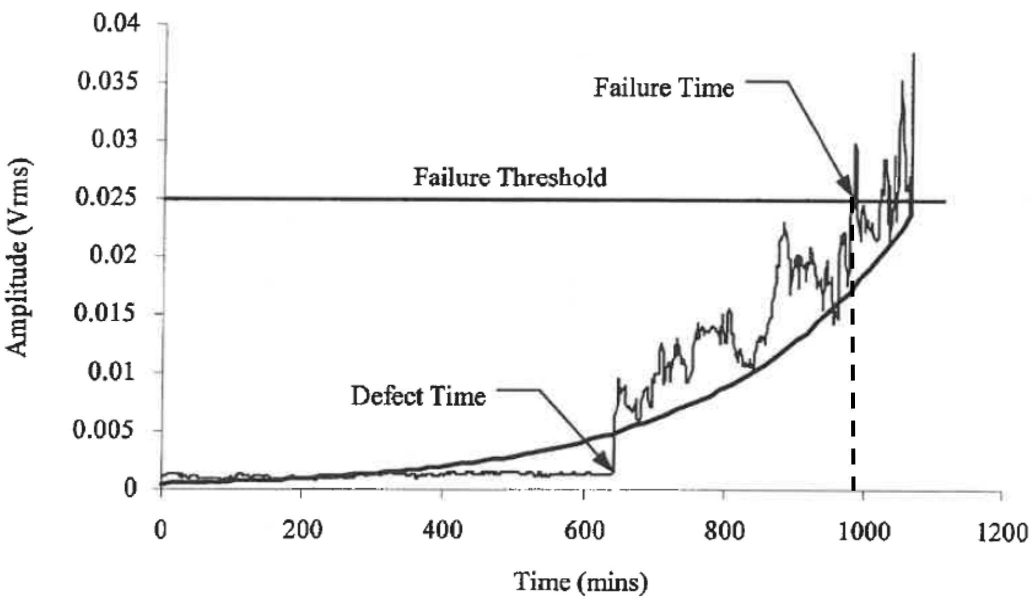
1.3 Problem Definitions

- Degradation Signal is live collected patterns of physical changes associated with the degradation process.
- Residual Life is the remaining life length till device fails
- Residual life Distribution: For a nonnegative random variable, X, the RLD denoted as $R_X(t)$, is defined by:

$$R_X(t) = P(X > x + t | X > x) \qquad (1.3)$$

In this research, we will assume the exponential degradation model for i = 1,2, … …

$$S(t_i) = \theta exp\big(\beta t_i + \varepsilon(t_i)\big) \ (1.4)$$



From (1.4) we deduce two degradation models:

Univariate case

$$S(t_i) = \mathrm{B}t_i + \varepsilon(t_i) \qquad (1.5)$$

Where the slope B is stochastic parameter, and $\varepsilon(t_i)$ is a random error.

And jointly distributed bivariate case

$$S(t_i) = \Theta + \mathrm{B}t_i + \varepsilon(t_i) \ (1.6)$$

Where the intercept Θ and slope B are stochastic parameter, and $\varepsilon(t_i)$ is a random error.

Let $\boldsymbol{S} = (S(t_1) = s_1, S(t_2) = s_2, \dots \dots, S(t_k) = s_k)$ be the observed degradations signals up to time t_k, where $t_i \geq 0$, and let D denote the failure threshold.

Using (1.5) the RLD denoted by

$$R_S(t) = P(S(t + t_n) \geq D|\boldsymbol{S})$$

$$= P(\Theta(t + t_n) + \varepsilon(t + t_n) \geq D|\boldsymbol{S})$$

The above probability requires the distribution (pdf) of $\Theta|\boldsymbol{S}$. To do this we will use the following Bayesian formula:

$$p(\theta|\boldsymbol{S}) = \frac{f(S|\theta)\pi(\theta)}{\int f(S|\theta)\pi(\theta)d(\theta)}$$

Since

$$\int f(S|\theta)\pi(\theta)d(\theta)$$

Is the normalizing constant, we can use the proportionality relationship;

$$p(\theta|\boldsymbol{S}) \propto f(S|\theta)\pi(\theta) \quad (1.7)$$

Where $p(\theta|\boldsymbol{S})$ is called the posterior distribution, $f(S|\theta)$ is the likelihood function, and $\pi(\theta)$ is the prior distribution.

In Chakraborty, *et al* paper [2], The wear function presented in this paper is referred to as the degradation linear model and represented by the following equation:

$$S_i = \emptyset + \theta t_i + \varepsilon_i$$

Where $\theta \sim N(\mu_0, \sigma_0^2)$ and $\varepsilon_i \sim N(0, \sigma^2)$. ε_i is the error caused by noise picked up by the monitoring sensors. Similar model was presented by Gebraeel, *et. Al.* [3] (2005). In [2], [3], and [4], Bayesian updating technique was used to obtain the degradation signal information updating the distributions of stochastic parameters for linear degradation model. The authors of [3] used matching coefficients method to show the posterior of $\theta \sim N(\mu_p, \sigma_p^2)$ given by the following equations:

$$\mu_p = \left(\frac{\sum_{i=1}^{k} S_i t_i}{\sum_{i=1}^{k} t_i^2}\right)\left(\frac{\sigma_0^2}{\sigma_0^2 + \left(\sigma^2/\sum_{i=1}^{k} t_i^2\right)}\right) + \mu_0\left(\frac{\left(\sigma^2/\sum_{i=1}^{k} t_i^2\right)}{\sigma_0^2 + \left(\sigma^2/\sum_{i=1}^{k} t_i^2\right)}\right) \quad (1.8)$$

$$\sigma_p^2 = \frac{\sigma_0^2\left(\sigma^2/\sum_{i=1}^{k} t_i^2\right)}{\sigma_0^2 + \left(\sigma^2/\sum_{i=1}^{k} t_i^2\right)} \quad (1.9)$$

Bernstein distribution is used to estimate the residual-life distribution RLD and is given by the following equation:

$$P(L_T \leq t|S_1, S_2, \dots \dots, S_k) = \frac{\Phi[g(t)]}{1 - \Phi[g(0)]} \quad (1.10)$$

$$g(t) = \frac{\mu_p(t + t_k) + \emptyset - T}{\sqrt{(t + t_k)^2\sigma_p^2 + \sigma^2}} \quad (1.11)$$

Equation (3) is truncated at zero to get a probability area of one after time t = 0. This is because the normal distribution gives the variation of random variable between $-\infty$ to $+\infty$, starting components tests at time t = 0 and disregarding the negative values of θ.

We will show that (5) and (6) can be obtained using the Modal approximation theorem. For the modal approximation to work we must have a unimodal distribution. Matching technique works fine when the right side of (1.7) is Gaussian distribution.

In [2] also, the authors decided to consider Gamma distribution as their prior, since it provides more flexibility in capturing the characteristics of real world sensory data. Various degrees of skewness were explored.

Again, the degradation model

$$S_i = \theta t_i + \varepsilon(t_i) \quad (1.12)$$

$$\theta \sim \Gamma(\alpha, \beta) \ and \ iid \ \varepsilon_i \sim N(0, \sigma^2) \ for \ i = 1,2, \dots \dots, k$$

They obtain the following posterior distribution

$$f(\theta|S_1, S_2, \dots \dots, S_k) = \frac{\theta^{\alpha-1}}{c} exp\left[-\frac{1}{2\sigma_1^2}(\theta - \mu_1)^2\right], \qquad \theta \in \mathbb{R}^+ \quad (1.13)$$

where

$$c = \int_{t=0}^{\infty} \theta^{\alpha-1} exp\left[-\frac{1}{2\sigma_1^2}(\theta - \mu_1)^2\right] d\theta$$

and

$$\mu_1 = \frac{b}{2a}, \qquad \sigma_1^2 = \frac{1}{2a}, \qquad a = \frac{1}{2\sigma^2}\sum_{i=1}^{k} t_i^2\,, \qquad b = \frac{1}{\sigma^2}\sum_{i=1}^{k} S_i t_i - \frac{1}{\beta}$$

Then the authors calculated the distribution of the residual life L_τ of the signal given by:

$$P(L_T \leq t|S_1, S_2, \dots \dots, S_k) = 1 - P(L_T > t|S_1, S_2, \dots \dots, S_k)$$

$$= 1 - \int_{y=0}^{+\infty} \frac{y^{\alpha-1}}{c(t+t_k)^\alpha} exp\left[\left(-\frac{1}{2\sigma_1^2}(\theta-\mu_1)^2\right)^2\right] \times \Phi\left(\frac{T-y}{\sigma}\right) dy \qquad (1.14)$$

Since c is the normalizing constant for the posterior distribution of θ, the authors did no find a closed form. Therefore this RLD requires numerical integration to calculate at any time t. Moreover, Φ is involved, the integrand becomes more complicated function, this requires the use of fine intervals to evaluate it numerically. Hence, this leads to slower evaluation times.

1.4 Normal approximation to the posterior distribution

To overcome slower evaluation time, and the closed form issue, we propose to apply Laplace approximation. If the posterior distribution $p(\theta|X)$ is unimodal with symmetric or non-symmetric, it can be approximated with Gaussian distribution. By taking the logarithm of the posterior density $log[p(\theta|S)]$, and using a Taylor expansion centered at the posterior mode $\hat{\theta}$, this expansion can be approximated by a quadratic function of θ. A full proof of this concept will be verified in chapter 2.

Chapter 2

Laplace Approximation

Laplace approximation method is a way to find a Gaussian approximation of the posterior. As it is known, in Gaussian distribution, mode=mean=median. Hence this approximation will be centered on the mode of the distribution of the posterior. To calculate this mode, it involves calculating the derivative of logarithm of the posterior, and then setting the derivative equal to zero. The following steps

2.1 Laplace Approximation

By Bayes rule:

$$p(\boldsymbol{\theta}|\boldsymbol{X}) = \frac{\tilde{p}(\boldsymbol{\theta}|\boldsymbol{X})}{\mathcal{Z}} = \frac{p(\boldsymbol{X}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{\mathcal{Z}} \qquad (2.1)$$

$$\mathcal{Z} = \int p(\boldsymbol{X}|\boldsymbol{\theta})p(\boldsymbol{\theta})d(\boldsymbol{\theta}) \qquad (2.2)$$

Laplace approximation aims at finding a Gaussian approximation $q(\boldsymbol{\theta}|\boldsymbol{X})$ centered on the mode of the distribution $p(\boldsymbol{\theta}|\boldsymbol{X})$. The approximate posterior distribution is:

$$for\ \boldsymbol{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}\ and\ \boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_k \end{bmatrix}$$

$$\tilde{p}(\boldsymbol{\theta}|\boldsymbol{X}) = p(\boldsymbol{X}|\boldsymbol{\theta})p(\boldsymbol{\theta})\ non-normalized\ posterior, and\ f(\theta) = \ log[\tilde{p}(\boldsymbol{\theta}|\boldsymbol{X})]$$

This approximation works as follows:

Using Taylor expansion on $log\ \tilde{p}(\boldsymbol{\theta}|\boldsymbol{X})\ at\ \boldsymbol{\theta} = \boldsymbol{\theta_o}$, we have:

$$log[\tilde{p}(\boldsymbol{\theta}|\boldsymbol{X})] \approx log\left[\tilde{p}(\boldsymbol{\theta}|\boldsymbol{X})_{\boldsymbol{\theta}=\boldsymbol{\theta_o}}\right] + J(\boldsymbol{\theta}|\boldsymbol{X})_{\boldsymbol{\theta}=\boldsymbol{\theta_o}}(\boldsymbol{\theta} - \boldsymbol{\theta_o})^T + \frac{1}{2}(\boldsymbol{\theta} - \boldsymbol{\theta_o})^T H(\boldsymbol{\theta}|\boldsymbol{X})_{\boldsymbol{\theta}=\boldsymbol{\theta_o}}(\boldsymbol{\theta} - \boldsymbol{\theta_o}) + O(\boldsymbol{\theta_o}|\boldsymbol{X}) \ (3)$$

Where $J(\boldsymbol{\theta}|\boldsymbol{X})_{\boldsymbol{\theta}=\boldsymbol{\theta_o}} = \nabla log[\tilde{p}(\boldsymbol{\theta}|\boldsymbol{X})]_{\boldsymbol{\theta}=\boldsymbol{\theta_o}} and\ H(\boldsymbol{\theta}|\boldsymbol{X})_{\boldsymbol{\theta}=\boldsymbol{\theta_o}} = -\left[\nabla^2 log\left(\tilde{p}(\boldsymbol{\theta}|\boldsymbol{X})\right)_{\boldsymbol{\theta}=\boldsymbol{\theta_o}}\right]^{-1} = \left[\left[\frac{\partial^2 f}{\partial \boldsymbol{\theta}^2}\right]_{\boldsymbol{\theta}=\boldsymbol{\theta_o}}\right]^{-1}$

Note that the second derivative of $H(\boldsymbol{\theta}|\boldsymbol{X})$ at its maxima converges to a negative number. Then for

$$\frac{\partial^2 f}{\partial \boldsymbol{\theta}^2} = \begin{bmatrix} \frac{\partial^2 f}{\partial \theta_1^2} & \frac{\partial^2 f}{\partial \theta_1 \partial \theta_2} & \cdots \\ \frac{\partial^2 f}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 f}{\partial \theta_2^2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

J(.) and H(.) are the Jacobian and Hessian matrix of $log\left(\tilde{p}(\boldsymbol{\theta}|\boldsymbol{X})\right)$ evaluated at $\boldsymbol{\theta_o}$ respectively. *O*(.) stands for the higher order terms of the Taylor series expansion. Since the logarithm of Gaussian distribution is a quadratic function we can ignore the higher order terms and (3) will become:

$$log\left(\tilde{p}(\boldsymbol{\theta}|\boldsymbol{X})\right) \approx log\left[\tilde{p}(\boldsymbol{\theta}|\boldsymbol{X})_{\boldsymbol{\theta}=\boldsymbol{\theta_o}}\right] + J(\boldsymbol{\theta}|\boldsymbol{X})_{\boldsymbol{\theta}=\boldsymbol{\theta_o}}(\boldsymbol{\theta}-\boldsymbol{\theta_o}) + \frac{1}{2!}(\boldsymbol{\theta}-\boldsymbol{\theta_o})^T H(\boldsymbol{\theta}|\boldsymbol{X})_{\boldsymbol{\theta}=\boldsymbol{\theta_o}}(\boldsymbol{\theta}-\boldsymbol{\theta_o}) \qquad (2.4)$$

If we let the expansion point $\boldsymbol{\theta_o}$ to be the local maximum of $log\left(\tilde{p}(\boldsymbol{\theta}|\boldsymbol{X})\right)$, the Jacobian matrix J(.) will vanish at the local maximum of the distribution. Thus (4) will be reduced to the following:

$$log\left(\tilde{p}(\boldsymbol{\theta}|\boldsymbol{X})\right) \approx log\left[\tilde{p}(\boldsymbol{\theta}|\boldsymbol{X})_{\boldsymbol{\theta}=\boldsymbol{\theta_o}}\right] + \frac{1}{2!}(\boldsymbol{\theta}-\boldsymbol{\theta_o})^T H(\boldsymbol{\theta}|\boldsymbol{X})_{\boldsymbol{\theta}=\boldsymbol{\theta_o}}(\boldsymbol{\theta}-\boldsymbol{\theta_o}) \qquad (2.5)$$

Exponentiation of (5) will lead to:

$$\tilde{p}(\boldsymbol{\theta}|\boldsymbol{X}) = EXP\left[log\left(\tilde{p}(\boldsymbol{\theta}|\boldsymbol{X})\right)\right] \approx \tilde{p}(\boldsymbol{\theta}|\boldsymbol{X})_{\boldsymbol{\theta}=\boldsymbol{\theta_o}} EXP\left[\frac{1}{2!}(\boldsymbol{\theta}-\boldsymbol{\theta_o})^T H(\boldsymbol{\theta}|\boldsymbol{X})_{\boldsymbol{\theta}=\boldsymbol{\theta_o}}(\boldsymbol{\theta}-\boldsymbol{\theta_o})\right] \qquad (2.6)$$

Define

$$\Sigma^{-1} = \left[-H(\underline{\boldsymbol{\theta}}|\boldsymbol{X})_{\boldsymbol{\theta}=\boldsymbol{\theta_o}}\right]$$

For (2.6) to be a multivariate normal distribution with a mean vector of θ_o and a covariance matrix

$$\Sigma = -\left[\nabla^2 log\left(\tilde{p}(\boldsymbol{\theta}|\boldsymbol{X})\right)_{\boldsymbol{\theta}=\boldsymbol{\theta_o}}\right]^{-1} = \left[\left(\frac{\partial^2 f}{\partial \boldsymbol{\theta}^2}\right)_{\boldsymbol{\theta}=\boldsymbol{\theta_o}}\right]^{-1}$$

The normalizing constant for the posterior distribution is

$$\mathcal{Z} = \int_\theta EXP\left[log[\tilde{P}(\boldsymbol{\theta}|\boldsymbol{X})]\right] d\boldsymbol{\theta}$$

$$\approx \tilde{p}(\boldsymbol{\theta_o}|\boldsymbol{X})\int_{\theta} EXP\left[-\frac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\theta_o})^T[\boldsymbol{\Sigma}^{-1}](\boldsymbol{\theta}-\boldsymbol{\theta_o})\right]d\boldsymbol{\theta}$$

$$= \tilde{p}(\boldsymbol{\theta_o}|\boldsymbol{X})\sqrt{(2\pi)^n|\boldsymbol{\Sigma}^{-1}|}\int_{\theta}\frac{1}{\sqrt{(2\pi)^n|\boldsymbol{\Sigma}^{-1}|}}EXP\left[-\frac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\theta_o})^T[\boldsymbol{\Sigma}^{-1}](\boldsymbol{\theta}-\boldsymbol{\theta_o})\right]d\boldsymbol{\theta}$$

$$= \tilde{p}(\boldsymbol{\theta_o}|\boldsymbol{X})\sqrt{(2\pi)^n|\boldsymbol{\Sigma}^{-1}|}$$

Where n is the dimension of the unknown variables $\boldsymbol{\theta}$ and $|\boldsymbol{\Sigma}^{-1}|$ is the determinant of $\boldsymbol{\Sigma}^{-1}$.

The normalized posterior distribution is now expressed approximately as

$$p(\boldsymbol{\theta}|\boldsymbol{X}) \approx \frac{\tilde{p}(\boldsymbol{\theta_o}|\boldsymbol{X})EXP\left[-\frac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\theta_o})^T[\boldsymbol{\Sigma}^{-1}](\boldsymbol{\theta}-\boldsymbol{\theta_o})\right]}{\tilde{p}(\boldsymbol{\theta_o}|\boldsymbol{X})\sqrt{(2\pi)^n|\boldsymbol{\Sigma}^{-1}|}}$$

With the normalizing constant $\mathcal{Z}$, the Gaussian approximation for $p(\boldsymbol{\theta}|\boldsymbol{X})$ is:

$$q(\boldsymbol{\theta}) = \frac{1}{\sqrt{(2\pi)^n|\boldsymbol{\Sigma}^{-1}|}}EXP\left[-\frac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\theta_o})^T[\boldsymbol{\Sigma}^{-1}](\boldsymbol{\theta}-\boldsymbol{\theta_o})\right]$$

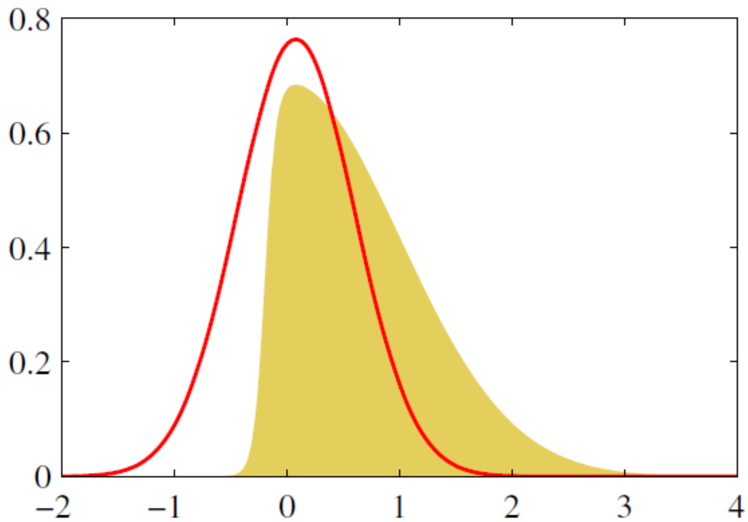


Figure (1): The above plot shows the normalized distribution $p(\theta)$ in yellow. Laplace approximation centered at mode θ_o of $p(\theta)$ in red. [2]

By Laplace approximation $(\boldsymbol{\theta}|\boldsymbol{X}) \sim N(\boldsymbol{\theta}, \boldsymbol{\Sigma}^{-\mathbf{1}})$ [4], we can apply the Modal approximation based on the multivariate normal distribution, to find the mode, $\boldsymbol{\theta_o}\ of\ (\boldsymbol{\theta}|\boldsymbol{X})$.

For univariate distribution

$$q(\theta) = \frac{1}{\sqrt{2\pi}\sigma} EXP\left[-\frac{1}{2\sigma^2}(\theta - \theta_o)^2\right]$$

Where

$$\sigma^2 = -\left(\left[\frac{\partial^2 log\, p(\theta|X)}{\partial\theta^2}\right]_{\theta=\theta_o}\right)^{-1}$$

Chapter 3

Laplace Approximation applied on univariate distribution

In this chapter, we apply Laplace approximation method to various cases of error and prior distributions on the univariate degradation model:

$$S_i = \theta t_i + \varepsilon_i$$

3.1 Univariate Normal Prior and Normal Error

Recall, the Bayesian equation:

$$f(\theta|S_i) = \frac{f(S_i|\theta) \cdot f(\theta)}{f(S_i)} \propto f(S_i|\theta) \cdot f(\theta)$$

Components of the equation are:

The likelihood function:

$$L(S_i|\theta) \propto \prod_{i=1}^{k} Exp\left\{-\frac{1}{2\sigma^2}[S_i^2 - 2S_i\theta t_i + \theta^2 t_i^2]\right\}$$

The prior function:

$$f(\theta) \propto Exp\left\{-\frac{1}{\sigma_0^2}[\theta^2 - 2\theta\mu_0 + \mu_0^2]\right\}$$

The product of

$$L(S_i|\theta) \cdot f(\theta) \propto Exp\left\{-\frac{1}{2}\left[\theta^2\left(\frac{1}{\sigma_0^2} + \frac{\sum_{i=1}^{k} t_i^2}{\sigma^2}\right) - \theta\left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=1}^{k} S_i t_i}{\sigma^2}\right) + \left(\frac{\mu_0^2}{\sigma_0^2} + \frac{S_i^2}{\sigma^2}\right)\right]\right\}$$

By Laplace approximation $(\theta|S_i) \sim N\left(\hat{\theta}, \Sigma^{-1}\right)$ [4], we can apply the Modal approximation based on the multivariate normal distribution, to find the mode, $\hat{\theta}\ of\ (\theta|S_i)$. $\hat{\theta}$ is the θ where $p(\theta|S_i)$ is max, this is solved by setting

$$\frac{\partial \log(f(S_i|\theta) \cdot f(\theta))}{\partial\theta} = 0$$

and solve for $\hat{\theta}$ [5].

$$\frac{\partial \log(f(S_i|\theta) \cdot f(\theta))}{\partial \theta} = -\frac{2}{2}\left[\theta\left(\frac{1}{\sigma_0^2} + \frac{\sum_{i=1}^{k} t_i^2}{\sigma^2}\right) - \left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=1}^{k} S_i t_i}{\sigma^2}\right)\right] = 0$$

$$\Rightarrow \theta\left(\frac{1}{\sigma_0^2} + \frac{\sum_{i=1}^{k} t_i^2}{\sigma^2}\right) = \left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=1}^{k} S_i t_i}{\sigma^2}\right)$$

$$\Rightarrow \theta\left(\frac{\sigma^2 + \sigma_0^2 \sum_{i=1}^{k} t_i^2}{\sigma_0^2 \sigma^2}\right) = \left(\frac{\sigma^2 \mu_0 + \sigma_0^2 \sum_{i=1}^{k} S_i t_i}{\sigma_0^2 \sigma^2}\right)$$

$$\Rightarrow \hat{\theta} = \frac{\sigma^2 \mu_0 + \sigma_0^2 \sum_{i=1}^{k} S_i t_i}{\sigma^2 + \sigma_0^2 \sum_{i=1}^{k} t_i^2}$$

$$\Rightarrow \hat{\theta} = \frac{\sigma^2 \mu_0}{\sigma^2 + \sigma_0^2 \sum_{i=1}^{k} t_i^2} + \frac{\sigma_0^2 \sum_{i=1}^{k} S_i t_i}{\sigma^2 + \sigma_0^2 \sum_{i=1}^{k} t_i^2}$$

$$\Rightarrow \hat{\theta} = \left(\frac{\sum_{i=1}^{k} S_i t_i}{\sum_{i=1}^{k} t_i^2}\right)\left(\frac{\sigma_0^2}{\sigma_0^2 + \left(\sigma^2/\sum_{i=1}^{k} t_i^2\right)}\right) + \mu_0\left(\frac{\left(\sigma^2/\sum_{i=1}^{k} t_i^2\right)}{\sigma_0^2 + \left(\sigma^2/\sum_{i=1}^{k} t_i^2\right)}\right) \qquad (11)$$

Next, we calculate the posterior variance:

$$\Sigma^{-1} = \left[-\frac{\partial^2 \log(L(S_i|\theta) \cdot f(\theta))}{\partial \theta^2}\Big|_{\theta=\hat{\theta}}\right]^{-1} \text{ [5].}$$

$$= -\left[-\left(\frac{1}{\sigma_0^2} + \frac{\sum_{i=1}^{k} t_i^2}{\sigma^2}\right)\right]^{-1}$$

$$= \left[\frac{\sigma^2 + \sigma_0^2 \sum_{i=1}^{k} t_i^2}{\sigma_0^2 \sigma^2}\right]^{-1}$$

$$= \frac{\sigma_0^2 \sigma^2}{\sigma^2 + \sigma_0^2 \sum_{i=1}^{k} t_i^2} \qquad (12)$$

Thus, equations (11) and (12) asymptotically matches (5) and (6).

3.2 Gamma Prior and Laplace Error

We will extend the Modal approximation for different priors, i.e. double exponential error and Gamma prior. As we stated above, the next phase of this research is to consider $\varepsilon(t_i)\ iid\ Laplace\ (0, b), \Theta \sim \Gamma(\alpha, \beta)$.

$$L[f(S_i|\theta)] \propto \prod_{i=1}^{k} Exp\left\{-\frac{1}{b}|S_i - \theta t_i|\right\} = Exp\left\{-\frac{1}{b}\sum_{i=1}^{k}|S_i - \theta t_i|\right\}$$

Need to rearrange the observations such that for some k we have

$$\varepsilon_k = S_k - \theta t_k \geq 0 \ or\ \varepsilon_k = S_k - \theta t_k < 0$$

Thus, we may write

$$\sum_{k=1}^{n}|S_k - \theta t_k| = \sum_{k=1}^{n}(S_k - \theta t_k) \cdot \mathbb{1}_{(\varepsilon_k \leq 0)} - \sum_{k=1}^{n}(S_k - \theta t_k) \cdot \mathbb{1}_{(\varepsilon_k > 0)}$$

The product of

$$L[f(S_i|\theta)] \cdot f(\theta) \propto \theta^{\alpha-1} Exp\left\{-\frac{1}{b}\left[\sum_{k=1}^{n}(S_k - \theta t_k) \cdot \mathbb{1}_{(\varepsilon_k \leq 0)} - \sum_{k=1}^{n}(S_k - \theta t_k) \cdot \mathbb{1}_{(\varepsilon_k > 0)} + \frac{\theta b}{\beta}\right]\right\}$$

$$= \theta^{\alpha-1} Exp\left\{-\frac{1}{b}\left[\sum_{k=1}^{n} S_k \cdot 1_{(\varepsilon_k \leq 0)} - \sum_{k=1}^{n} \theta t_k \cdot \mathbb{1}_{(\varepsilon_k \leq 0)} - \sum_{k=1}^{n} S_k \cdot \mathbb{1}_{(\varepsilon_k > 0)} + \sum_{k=1}^{n} \theta t_k \cdot \mathbb{1}_{(\varepsilon_k > 0)} + \frac{\theta b}{\beta}\right]\right\}$$

$$= \theta^{\alpha-1} Exp\left\{-\frac{1}{b}\left[\sum_{k=1}^{n} S_k \cdot \mathbb{1}_{(\varepsilon_k \leq 0)} - S_k \cdot \mathbb{1}_{(\varepsilon_k > 0)} + \theta\left(\sum_{k=1}^{n} t_k \cdot \mathbb{1}_{(\varepsilon_k > 0)} - t_k \cdot \mathbb{1}_{(\varepsilon_k \leq 0)} + \frac{b}{\beta}\right)\right]\right\}$$

Take the log of the above expression then we have the following:

$$(\alpha - 1)\log(\theta) - \frac{1}{b}\left[\sum_{k=1}^{n} S_k \cdot \mathbb{1}_{(\varepsilon_k \leq 0)} - S_k \cdot \mathbb{1}_{(\varepsilon_k > 0)} + \theta\left(\sum_{k=1}^{n} t_k \cdot \mathbb{1}_{(\varepsilon_k > 0)} - \sum_{k=1}^{n} t_k \cdot \mathbb{1}_{(\varepsilon_k \leq 0)} + \frac{b}{\beta}\right)\right]$$

$$\frac{\partial \log(L(S_i|\theta) \cdot f(\theta))}{\partial \theta} = \frac{(\alpha - 1)}{\theta} - \frac{1}{b}\left(\sum_{k=1}^{n} t_k \cdot \mathbb{1}_{(\varepsilon_k > 0)} - \sum_{k=1}^{n} t_k \cdot \mathbb{1}_{(\varepsilon_k \leq 0)} + \frac{b}{\beta}\right) = 0$$

$$\Rightarrow \frac{(\alpha - 1)}{\theta} = \frac{1}{b}\left(\sum_{k=1}^{n} t_k \cdot \mathbb{1}_{(\varepsilon_k > 0)} - \sum_{k=1}^{n} t_k \cdot \mathbb{1}_{(\varepsilon_k \leq 0)} + \frac{b}{\beta}\right)$$

$$\Longrightarrow \hat{\theta} = \frac{\alpha - 1}{\frac{1}{b}\left(\sum_{k=1}^{n} t_k \cdot \mathbb{1}_{(\varepsilon_k > 0)} - \sum_{k=1}^{n} t_k \cdot \mathbb{1}_{(\varepsilon_k \le 0)} + \frac{b}{\beta}\right)}$$

$$\hat{\sigma}^2 = \left[-\frac{\partial^2 \log(L(S_i|\theta) \cdot f(\theta))}{\partial \theta^2} |_{\theta=\hat{\theta}}\right]^{-1}$$

$$= \frac{\left(\frac{1}{b}\right)^2 \left(\sum_{k=1}^{n} t_k \cdot \mathbb{1}_{(\varepsilon_k > 0)} - t_k \cdot \mathbb{1}_{(\varepsilon_k \le 0)} + \frac{b}{\beta}\right)^2}{\alpha - 1} \; for\ \alpha > 1$$

Thus, the approximated posterior distribution is $\Theta|S \dot{\sim} N\left(\hat{\theta}, \hat{\sigma}^2\right)$.

With the mean and variance both approximated, the Bernstein distribution will be used to calculate the remaining life distribution.

3.3 Gamma Prior and Normal Error

The same procedures are used to approximate posterior distribution with $\varepsilon \sim N(0, \sigma^2)$ and $\theta \sim \Gamma(\alpha, \beta)$

$$p(\theta|\boldsymbol{S}) \propto \theta^{\alpha-1} exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{k} S_i^2 + \theta b - \theta^2 a\right\} \text{ where } \theta \in \Theta = \mathbb{R}^+ \cup \{0\}$$

Solve for

$$\frac{\partial \log p\ (\theta|\boldsymbol{S})}{\partial \theta} = 0 \Rightarrow \hat{\theta} = \frac{-b \pm \sqrt{b^2 + 8ac}}{-4a}$$

$$\hat{\sigma}^2 = -\left(\left[\frac{\partial^2 \log(p(\theta|\boldsymbol{S}\ )}{\partial \theta^2}\right]_{\theta=\hat{\theta}}\right)^{-1} = \frac{1}{\left(c/\hat{\theta}^2\right) + 2a}$$

$$\Theta|S \dot{\sim} N\left(\frac{-b - \sqrt{b^2 + 8ac}}{-4a}, \frac{1}{\left(c/\hat{\theta}^2\right) + 2a}\right)$$

Where

$$b = \frac{1}{\sigma^2}\sum_{i=1}^{k} S_i t_i - \frac{1}{\beta}, \qquad a = \frac{1}{2\sigma^2}\sum_{i=1}^{k} t_i^2,\ \ and\ \ c = \alpha - 1$$

With the mean and variance both approximated, the Bernstein distribution will be used to calculate the remaining life distribution.

Chapter 4

Laplace Approximation applied on bivariate distribution

In this chapter, we apply Laplace approximation method to:

4.1 Bivariate Normal prior $\pi(\Theta, \mathrm{B})$

The model of our continuous degradation signal w.r.t time is the following:

$$L_i = \Theta + \mathrm{B}t_i + \varepsilon_i \Longrightarrow \varepsilon_i = L_i - \Theta - \mathrm{B}t_i \ ,$$

is normally distributed random error.

Where $\binom{\Theta}{\mathrm{B}} \sim N\left(\binom{\mu_o}{\mu_1}, \begin{pmatrix} \sigma_o^2 & \rho_o\sigma_o\sigma_1 \\ \rho_o\sigma_o\sigma_1 & \sigma_1^2 \end{pmatrix}\right)$ $and\ \varepsilon\ iid \sim N(0, \sigma^2)$

By Bayes definition we have the following: $P(\theta', \beta|L_i) \propto f(L_i|\theta', \beta)\pi(\theta', \beta)$

$$f(L_i|\theta', \beta) = \frac{1}{\sqrt{2\pi\sigma^2}} EXP\left[-\sum_{i=1}^{n} \frac{1}{2}\left(\frac{L_i - \theta' - \beta t_i}{\sigma}\right)^2\right]$$

Let $A = 2\pi\sigma_o\sigma_1\sqrt{(1-\rho_o^2)}$

$$\pi(\theta', \beta) = \frac{1}{A} EXP\left[-\frac{1}{2(1-\rho_0^2)}\left[\frac{(\theta' - \mu_o)^2}{\sigma_o^2} - \frac{2\rho_o(\theta' - \mu_o)(\beta - \mu_1)}{\sigma_o\sigma_1} + \frac{(\beta - \mu_1)^2}{\sigma_1^2}\right]\right]$$

$$[L_i - (\theta' + \beta t_i)]^2 = [L_i^2 - 2L_i(\theta' + \beta t_i) + (\theta' + \beta t_i)^2]$$

$$f(L_i|\theta', \beta) = \frac{1}{\sqrt{2\pi\sigma^2}}\left[-\sum_{i=1}^{n} \frac{1}{2\sigma^2}\left(L_i^2 - 2L_i\theta' - 2\beta L_i t_i + \theta'^2 + 2\theta'\beta t_i + \beta^2 t_i^2\right)\right]$$

$$\pi(\theta', \beta) = \frac{1}{A} EXP\left\{-\frac{1}{2(1-\rho_0^2)}\left[\frac{\theta'^2 - 2\theta'\mu_o + \mu_o^2}{\sigma_o^2} - \frac{2\rho_o(\theta'\beta - \theta'\mu_1 - \mu_o\beta + \mu_o\mu_1)}{\sigma_o\sigma_1} + \frac{\beta^2 - 2\beta\mu_1 + \mu_1^2}{\sigma_1^2}\right]\right\}$$

$$f(L_i|\theta',\beta)\pi(\theta',\beta) \propto EXP\left\{\begin{array}{c} \sum_{i=1}^{n}\left(-\frac{L_i^2}{2\sigma^2}+\frac{L_i\theta'}{\sigma^2}+\frac{\beta L_i t_i}{\sigma^2}-\frac{\theta'^2}{2\sigma^2}-\frac{\theta'\beta t_i}{\sigma^2}-\frac{\beta^2 t_i^2}{2\sigma^2}\right)-\frac{\theta'^2}{2(1-\rho_o^2)\sigma_o^2} \\ +\frac{\theta'\mu_o}{(1-\rho_o^2)\sigma_o^2}-\frac{\mu_o^2}{2(1-\rho_o^2)\sigma_o^2}+\frac{\rho_o\theta'\beta}{(1-\rho_o^2)\sigma_o\sigma_1}-\frac{\rho_o\theta'\mu_1}{(1-\rho_o^2)\sigma_o\sigma_1} \\ -\frac{\rho_o\mu_o\beta}{(1-\rho_o^2)\sigma_o\sigma_1}+\frac{\rho_o\mu_o\mu_1}{(1-\rho_o^2)\sigma_o\sigma_1}-\frac{\beta^2}{2(1-\rho_o^2)\sigma_1^2}+\frac{\beta\mu_1}{(1-\rho_o^2)\sigma_1^2}-\frac{\mu_1^2}{2(1-\rho_o^2)\sigma_1^2} \end{array}\right.$$

Collecting like terms

$$(*1)\ -\frac{n\theta'^2}{2\sigma^2}-\frac{\theta'^2}{2(1-\rho_o^2)\sigma_o^2}$$

$$= -\frac{1}{2}\left[\frac{n\theta'^2(1-\rho_o^2)\sigma_o^2+\theta'^2\sigma^2}{(1-\rho_o^2)\sigma^2\sigma_o^2}\right]$$

$$= -\frac{\theta'^2}{2}\left[\frac{n(1-\rho_o^2)\sigma_o^2+\sigma^2}{(1-\rho_o^2)\sigma^2\sigma_o^2}\right]$$

$$(*2)\ -\frac{\sum_{i=1}^{n}\beta^2 t_i^2}{2\sigma^2}-\frac{\beta^2}{2(1-\rho_o^2)\sigma_1^2}$$

$$= -\frac{1}{2}\left[\frac{\sum_{i=1}^{n}\beta^2(1-\rho_o^2)t_i^2\sigma_1^2+\beta^2\sigma^2}{(1-\rho_o^2)\sigma^2\sigma_1^2}\right]$$

$$= -\frac{1}{2}\left[\frac{\sum_{i=1}^{n}\beta^2(1-\rho_o^2)t_i^2\sigma_1^2+\beta^2\sigma^2}{(1-\rho_o^2)\sigma^2\sigma_1^2}\right]$$

$$= -\frac{\beta^2}{2}\left[\frac{\sum_{i=1}^{n}(1-\rho_o^2)t_i^2\sigma_1^2+\sigma^2}{(1-\rho_o^2)\sigma^2\sigma_1^2}\right]$$

$$(*3)\ \frac{\sum_{i=1}^{n}L_i\theta'}{\sigma^2}+\frac{\theta'\mu_o}{(1-\rho_o^2)\sigma_o^2}-\frac{\rho_o\theta'\mu_1}{(1-\rho_o^2)\sigma_o\sigma_1}$$

$$= \left[\frac{\sum_{i=1}^{n}\theta'(1-\rho_o^2)L_i\sigma_o^2\sigma_1+\theta'\mu_o\sigma^2\sigma_1-\theta'\rho_o\mu_1\sigma^2\sigma_o}{(1-\rho_o^2)\sigma^2\sigma_o^2\sigma_1}\right]\frac{\sigma_1}{\sigma_1}$$

$$= \theta'\left[\frac{\sum_{i=1}^{n}(1-\rho_o^2)L_i\sigma_o^2\sigma_1^2+\mu_o\sigma^2\sigma_1^2-\rho_o\mu_1\sigma^2\sigma_o\sigma_1}{(1-\rho_o^2)\sigma^2\sigma_o^2\sigma_1^2}\right]$$

$$(*4)\ \frac{\sum_{i=1}^{n}\beta L_i t_i}{\sigma^2}-\frac{\rho_o\mu_o\beta}{(1-\rho_o^2)\sigma_o\sigma_1}+\frac{\beta\mu_1}{(1-\rho_o^2)\sigma_1^2}$$

$$= \beta \left[\frac{\sum_{i=1}^{n} (1-\rho_o^2) L_i t_i \sigma_o \sigma_1^2 - \rho_o \mu_o \sigma^2 \sigma_1 + \mu_1 \sigma^2 \sigma_o}{(1-\rho_o^2)\sigma^2 \sigma_o \sigma_1^2} \right] \frac{\sigma_o}{\sigma_o}$$

$$= \beta \left[\frac{\sum_{i=1}^{n} (1-\rho_o^2) L_i t_i \sigma_o^2 \sigma_1^2 + \mu_1 \sigma^2 \sigma_o^2 - \rho_o \mu_o \sigma^2 \sigma_o \sigma_1}{(1-\rho_o^2)\sigma^2 \sigma_o^2 \sigma_1^2} \right]$$

$$(*5) - \frac{\sum_{i=1}^{n} \theta' \beta t_i}{\sigma^2} + \frac{\rho_o \theta' \beta}{(1-\rho_o^2)\sigma_o \sigma_1}$$

$$= \theta' \beta \left[\frac{\rho_o \sigma^2 - \sum_{i=1}^{n} (1-\rho_o^2) t_i \sigma_o \sigma_1}{(1-\rho_o^2)\sigma^2 \sigma_o \sigma_1} \right] \left[\frac{\sigma_o \sigma_1}{\sigma_o \sigma_1} \right]$$

$$= \theta' \beta \left[\frac{\rho_o \sigma^2 \sigma_o \sigma_1 - \sum_{i=1}^{n} (1-\rho_o^2) t_i \sigma_o^2 \sigma_1^2}{(1-\rho_o^2)\sigma^2 \sigma_o^2 \sigma_1^2} \right]$$

Combine 1,2,3,4, and 5 we have the following

$$(L_i|\theta', \beta)\pi(\theta', \beta)$$

$$\propto EXP\left(-\frac{1}{2} \left\{ \begin{array}{c} \theta'^2 \left[\frac{n(1-\rho_o^2)\sigma_o^2 + \sigma^2}{(1-\rho_o^2)\sigma^2\sigma_o^2} \right] - 2\theta' \left[\frac{\sum_{i=1}^{n}(1-\rho_o^2) L_i \sigma_o^2 \sigma_1^2 + \mu_o \sigma^2 \sigma_1^2 - \rho_o \mu_1 \sigma^2 \sigma_o \sigma_1}{(1-\rho_o^2)\sigma^2 \sigma_o^2 \sigma_1^2} \right] \\ -2\theta'\beta \left[\frac{\rho_o \sigma^2 \sigma_o \sigma_1 - \sum_{i=1}^{n}(1-\rho_o^2) t_i \sigma_o^2 \sigma_1^2}{(1-\rho_o^2)\sigma^2 \sigma_o^2 \sigma_1^2} \right] \\ \beta^2 \left[\frac{\sum_{i=1}^{n}(1-\rho_o^2) t_i^2 \sigma_1^2 + \sigma^2}{(1-\rho_o^2)\sigma^2 \sigma_1^2} \right] - 2\beta \left[\frac{\sum_{i=1}^{n}(1-\rho_o^2) L_i t_i \sigma_o^2 \sigma_1^2 + \mu_1 \sigma^2 \sigma_o^2 - \rho_o \mu_o \sigma^2 \sigma_o \sigma_1}{(1-\rho_o^2)\sigma^2 \sigma_o^2 \sigma_1^2} \right] \end{array} \right. \right)$$

Next, we will take the derivative of $log\ p(\Theta, \mathrm{B}|L_i)$ and set it equal to zero to find $\underset{\theta\in\Theta,\beta\in\mathrm{B}}{\operatorname{argmax}}\ log\ p(\Theta, \mathrm{B}|L_i)$, where

$\widehat{\Theta}\ and\ \widehat{\mathrm{B}}$ are estimated mean vectors.

$$\frac{\partial log f(L_i|\theta', \beta)\pi(\theta', \beta)}{\partial \theta'} = 0$$

$$-\theta' \left[\frac{n(1-\rho_o^2)\sigma_o^2 + \sigma^2}{(1-\rho_o^2)\sigma^2\sigma_o^2} \right] + \frac{\sum_{i=1}^{n}(1-\rho_o^2) L_i \sigma_o^2 \sigma_1^2 + \mu_o \sigma^2 \sigma_1^2 - \rho_o \mu_1 \sigma^2 \sigma_o \sigma_1}{(1-\rho_o^2)\sigma^2 \sigma_o^2 \sigma_1^2}$$
$$+ \beta \left[\frac{\sum_{i=1}^{n}(1-\rho_o^2) t_i \sigma_o^2 \sigma_1^2 - \rho_o \sigma^2 \sigma_o \sigma_1}{(1-\rho_o^2)\sigma^2 \sigma_o^2 \sigma_1^2} \right] = 0$$

$$\Rightarrow \theta' \left[\frac{n(1-\rho_o^2)\sigma_o^2 + \sigma^2}{(1-\rho_o^2)\sigma^2\sigma_o^2} \right]$$

$$= \frac{\sum_{i=1}^{n}(1-\rho_o^2)L_i\sigma_o^2\sigma_1^2 + \mu_o\sigma^2\sigma_1^2 - \rho_o\mu_1\sigma^2\sigma_o\sigma_1}{(1-\rho_o^2)\sigma^2\sigma_o^2\sigma_1^2} + \beta\left[\frac{\sum_{i=1}^{n}(1-\rho_o^2)t_i\sigma_o^2\sigma_1^2 - \rho_o\sigma^2\sigma_o\sigma_1}{(1-\rho_o^2)\sigma^2\sigma_o^2\sigma_1^2}\right]$$

$$\Longrightarrow \hat{\theta}' = \frac{\sum_{i=1}^{n}(1-\rho_o^2)L_i\sigma_o^2\sigma_1^2 + \mu_o\sigma^2\sigma_1^2 - \rho_o\mu_1\sigma^2\sigma_o\sigma_1 + \hat{\beta}[\sum_{i=1}^{n}(1-\rho_o^2)t_i\sigma_o\sigma_1 - \rho_o\sigma^2]\sigma_o\sigma_1}{[n(1-\rho_o^2)\sigma_o^2 + \sigma^2]\sigma_1^2}$$

$$\frac{\partial logf(L_i|\theta',\beta)\pi(\theta',\beta)}{\partial\beta} = 0$$

$$\beta\left[\frac{\sum_{i=1}^{n}(1-\rho_o^2)t_i^2\sigma_1^2 + \sigma^2}{(1-\rho_o^2)\sigma^2\sigma_1^2}\right] + \frac{\sum_{i=1}^{n}(1-\rho_o^2)L_it_i\sigma_o^2\sigma_1^2 + \mu_1\sigma^2\sigma_o^2 - \rho_o\mu_o\sigma^2\sigma_o\sigma_1}{(1-\rho_o^2)\sigma^2\sigma_o^2\sigma_1^2}$$
$$+ \theta\left[\frac{\sum_{i=1}^{n}(1-\rho_o^2)t_i\sigma_o^2\sigma_1^2 - \rho_o\sigma^2\sigma_o\sigma_1}{(1-\rho_o^2)\sigma^2\sigma_o^2\sigma_1^2}\right] = 0$$

$$\Longrightarrow \beta\left[\frac{\sum_{i=1}^{n}(1-\rho_o^2)t_i^2\sigma_1^2 + \sigma^2}{(1-\rho_o^2)\sigma^2\sigma_1^2}\right]$$
$$= \frac{\sum_{i=1}^{n}(1-\rho_o^2)L_it_i\sigma_o^2\sigma_1^2 + \mu_1\sigma^2\sigma_o^2 - \rho_o\mu_o\sigma^2\sigma_o\sigma_1}{(1-\rho_o^2)\sigma^2\sigma_o^2\sigma_1^2} + \theta\left[\frac{\sum_{i=1}^{n}(1-\rho_o^2)t_i\sigma_o^2\sigma_1^2 - \rho_o\sigma^2\sigma_o\sigma_1}{(1-\rho_o^2)\sigma^2\sigma_o^2\sigma_1^2}\right]$$

$$\Longrightarrow \hat{\beta} = \frac{\sum_{i=1}^{n}(1-\rho_o^2)L_it_i\sigma_o^2\sigma_1^2 + \mu_1\sigma^2\sigma_o^2 - \rho_o\mu_o\sigma^2\sigma_o\sigma_1 + \hat{\theta}[\sum_{i=1}^{n}(1-\rho_o^2)t_i\sigma_o\sigma_1 - \rho_o\sigma^2]\sigma_o\sigma_1}{[\sum_{i=1}^{n}(1-\rho_o^2)t_i^2\sigma_1^2 + \sigma^2]\sigma_o^2}$$

Next, we will find covariance matrix $\hat{\Sigma}^{-1}$ by taking the second derivative at the point $\hat{\theta}\ and\ \hat{\beta}$. The second derivative is negative.

$$\frac{\partial^2 f(\theta',\beta)}{\partial(\theta',\beta)^2} = \begin{bmatrix} \frac{\partial^2 f(\theta',\beta)}{\partial\theta'^2} & \frac{\partial^2 f(\theta',\beta)}{\partial\theta'\partial\beta} \\ \frac{\partial^2 f(\theta',\beta)}{\partial\beta\partial\theta'} & \frac{\partial^2 f(\theta',\beta)}{\partial\beta^2} \end{bmatrix}$$

$$\frac{\partial^2 f(\theta',\beta)}{\partial\theta'^2} = -\left[\frac{n(1-\rho_o^2)\sigma_o^2 + \sigma^2}{(1-\rho_o^2)\sigma^2\sigma_o^2}\right]$$

$$\frac{\partial^2 f(\theta',\beta)}{\partial\theta'\partial\beta} = \frac{\partial^2 f(\theta',\beta)}{\partial\beta\partial\theta'} = \left[\frac{\sum_{i=1}^{n}(1-\rho_o^2)t_i\sigma_o^2\sigma_1^2 - \rho_o\sigma^2\sigma_o\sigma_1}{(1-\rho_o^2)\sigma^2\sigma_o^2\sigma_1^2}\right]$$

$$\frac{\partial^2 f(\theta',\beta)}{\partial\beta^2} = -\left[\frac{\sum_{i=1}^{n}(1-\rho_o^2)t_i^2\sigma_1^2 + \sigma^2}{(1-\rho_o^2)\sigma^2\sigma_1^2}\right]$$

$$-\frac{\partial^2 f(\theta',\beta)}{\partial(\theta',\beta)^2} = \begin{bmatrix} \frac{n(1-\rho_o^2)\sigma_o^2+\sigma^2}{(1-\rho_o^2)\sigma^2\sigma_o^2} & -\frac{(1-\rho_o^2)t_i\sigma_o^2\sigma_1^2-\rho_o\sigma^2\sigma_o\sigma_1}{(1-\rho_o^2)\sigma^2\sigma_o^2\sigma_1^2} \\ -\frac{(1-\rho_o^2)t_i\sigma_o^2\sigma_1^2-\rho_o\sigma^2\sigma_o\sigma_1}{(1-\rho_o^2)\sigma^2\sigma_o^2\sigma_1^2} & \frac{(1-\rho_o^2)t_i^2\sigma_1^2+\sigma^2}{(1-\rho_o^2)\sigma^2\sigma_1^2} \end{bmatrix}$$

$$\hat{\Sigma}^{-1} = \begin{bmatrix} \frac{n(1-\rho_o^2)\sigma_o^2+\sigma^2}{(1-\rho_o^2)\sigma^2\sigma_o^2} & -\frac{\sum_{i=1}^n(1-\rho_o^2)t_i\sigma_o^2\sigma_1^2-\rho_o\sigma^2\sigma_o\sigma_1}{(1-\rho_o^2)\sigma^2\sigma_o^2\sigma_1^2} \\ -\frac{\sum_{i=1}^n(1-\rho_o^2)t_i\sigma_o^2\sigma_1^2-\rho_o\sigma^2\sigma_o\sigma_1}{(1-\rho_o^2)\sigma^2\sigma_o^2\sigma_1^2} & \frac{\sum_{i=1}^n(1-\rho_o^2)t_i^2\sigma_1^2+\sigma^2}{(1-\rho_o^2)\sigma^2\sigma_1^2} \end{bmatrix}^{-1}$$

The inverse for a non-zero determinant of the above positive definite matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

Define

$$a = \frac{n(1-\rho_o^2)\sigma_o^2+\sigma^2}{(1-\rho_o^2)\sigma^2\sigma_o^2}, b = c = -\frac{\sum_{i=1}^n(1-\rho_o^2)t_i\sigma_o^2\sigma_1^2-\rho_o\sigma^2\sigma_o\sigma_1}{(1-\rho_o^2)\sigma^2\sigma_o^2\sigma_1^2}, and\ \ d = \frac{\sum_{i=1}^n(1-\rho_o^2)t_i^2\sigma_1^2+\sigma^2}{(1-\rho_o^2)\sigma^2\sigma_1^2}$$

$$\hat{\sigma}_{\theta'}^2 = \frac{d}{ad-bc}$$

$$\Longrightarrow \hat{\sigma}_{\theta'}^2 = \frac{[\sum_{i=1}^n(1-\rho_o^2)t_i^2\sigma_1^2+\sigma^2](1-\rho_o^2)\sigma^2\sigma_o^2}{[n(1-\rho_o^2)\sigma_o^2+\sigma^2][\sum_{i=1}^n(1-\rho_o^2)t_i^2\sigma_1^2+\sigma^2]-[\sum_{i=1}^n(1-\rho_o^2)t_i\sigma_o\sigma_1-\rho_o\sigma^2]^2}$$

$$\hat{\sigma}_{\beta}^2 = \frac{a}{ad-bc}$$

$$\Longrightarrow \hat{\sigma}_{\beta}^2 = \frac{[n(1-\rho_o^2)\sigma_o^2+\sigma^2](1-\rho_o^2)\sigma^2\sigma_1^2}{[n(1-\rho_o^2)\sigma_o^2+\sigma^2][\sum_{i=1}^n(1-\rho_o^2)t_i^2\sigma_1^2+\sigma^2]-[\sum_{i=1}^n(1-\rho_o^2)t_i\sigma_o\sigma_1-\rho_o\sigma^2]^2}$$

$$\hat{\sigma}_{\theta'\beta} = -\frac{b}{ad-bc}, \qquad \sigma_{\beta\theta'} = -\frac{c}{ad-bc}, and\ \sigma_{\beta\theta'} = \sigma_{\theta'\beta}$$

$$\hat{\sigma}_{\theta'\beta} = -\frac{[(1-\rho_o^2)t_i\sigma_o\sigma_1-\rho_o\sigma^2](1-\rho_o^2)\sigma^2\sigma_o\sigma_1}{[n(1-\rho_o^2)\sigma_o^2+\sigma^2][\sum_{i=1}^n(1-\rho_o^2)t_i^2\sigma_1^2+\sigma^2]-[\sum_{i=1}^n(1-\rho_o^2)t_i\sigma_o\sigma_1-\rho_o\sigma^2]^2}$$

$$\hat{\rho}_{\theta'\beta} = \frac{\sigma_{\theta'\beta}}{\sigma_\theta\sigma_\beta}$$

$$\Longrightarrow \hat{\rho}_{\theta'\beta} = \frac{\sum_{i=1}^{n}(\rho_o^2 - 1)t_i\sigma_o\sigma_1 + \rho_o\sigma^2}{\sqrt{n(1-\rho_o^2)\sigma_o^2 + \sigma^2}\sqrt{\sum_{i=1}^{n}(1-\rho_o^2)t_i^2\sigma_1^2 + \sigma^2}}$$

Thus $(\mathbf{\Theta}, \mathbf{B})|\boldsymbol{S} \dot{\sim} N_2\left(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}}^{-1}\right)$

where $\hat{\boldsymbol{\mu}} = \begin{pmatrix}\hat{\theta} \\ \hat{\beta}\end{pmatrix}$ and $\hat{\boldsymbol{\Sigma}}^{-1} = \begin{pmatrix}\hat{\sigma}_{\theta'}^2 & \hat{\rho}_{\theta'\beta}\hat{\sigma}_{\theta'}\hat{\sigma}_{\beta} \\ \hat{\rho}_{\theta'\beta}\hat{\sigma}_{\theta'}\hat{\sigma}_{\beta} & \hat{\sigma}_{\beta}^2\end{pmatrix}$

4.2 Joint Normal prior $\boldsymbol{\pi}(\boldsymbol{\Theta}, \mathbf{B})$

Where $\begin{pmatrix}\mathbf{\Theta} \\ \mathbf{B}\end{pmatrix} \sim N\left(\begin{pmatrix}\mu_o \\ \mu_1\end{pmatrix}, \begin{pmatrix}\sigma_o^2 & 0 \\ 0 & \sigma_1^2\end{pmatrix}\right)$ $and\ \varepsilon\ iid \sim N(0, \sigma^2)$

By Bayes definition we have the following: $f(\theta', \beta|L_i) \propto f(L_i|\theta', \beta)\pi(\theta')\pi(\beta)$

$$\pi(\theta')\pi(\beta) \propto EXP\left[-\frac{1}{2}\left(\frac{(\theta' - \mu_o)^2}{\sigma_o^2} + \frac{(\beta - \mu_1)^2}{\sigma_1^2}\right)\right]$$

$$f(L_i|\theta', \beta) = \frac{1}{\sqrt{2\pi\sigma^2}}\left[-\sum_{i=1}^{n}\frac{1}{2\sigma^2}\left(L_i^2 - 2L_i\theta' - 2\beta L_i t_i + {\theta'}^2 + 2\theta'\beta t_i + \beta^2 t_i^2\right)\right]$$

$$f(L_i|\theta', \beta)\pi(\theta')\pi(\beta) \propto EXP\left\{\begin{matrix} -\frac{1}{2}\left[\frac{{\theta'}^2 - 2\theta'\mu_o + \mu_o^2}{\sigma_o^2} + \frac{\beta^2 - 2\beta\mu_1 + \mu_1^2}{\sigma_1^2}\right] \\ -\sum_{i=1}^{n}\frac{1}{2\sigma^2}\left(L_i^2 - 2L_i\theta' - 2\beta L_i t_i + {\theta'}^2 + 2\theta'\beta t_i + \beta^2 t_i^2\right)\end{matrix}\right\}$$

Next, we will take the derivative of $log\ p(\Theta, \mathrm{B}|L_i)$ and set it equal to zero to find $\underset{\theta\in\Theta, \beta\in\mathrm{B}}{\operatorname{argmax}} log\ p(\Theta, \mathrm{B}|L_i)$, where

$\hat{\Theta}\ and\ \hat{\mathrm{B}}$ are estimated mean vectors.

$$\frac{\partial log f(L_i|\theta', \beta)\pi(\theta')\pi(\beta)}{\partial\theta'} = 0$$

$$\hat{\theta}' = \frac{(\sigma^2\mu_o + \sigma_o^2\sum_{i=1}^{n}L_i)(\sigma^2 + \sigma_1^2\sum_{i=1}^{n}t_i^2) - (\sigma_o^2\sum_{i=1}^{n}t_i)(\sigma_1^2\sum_{i=1}^{n}L_i t_i + \sigma^2\mu_1)}{(\sigma^2 + n\sigma_o^2)(\sigma^2 + \sigma_1^2\sum_{i=1}^{n}t_i^2) - (\sigma_o^2\sum_{i=1}^{n}t_i)(\sigma_1^2\sum_{i=1}^{n}t_i)}$$

$$\frac{\partial logf(L_i|\theta',\beta)\pi(\theta')\pi(\beta)}{\partial\beta} = 0$$

$$\hat{\beta} = \frac{(\sigma^2\mu_1 + \sigma_1^2\sum_{i=1}^n L_i t_i)(\sigma^2 + n\sigma_o^2) - (\sigma_1^2\sum_{i=1}^n t_i)(\sigma_o^2\sum_{i=1}^n L_i + \sigma^2\mu_o)}{(\sigma^2 + n\sigma_o^2)(\sigma^2 + \sigma_1^2\sum_{i=1}^n t_i^2) - (\sigma_o^2\sum_{i=1}^n t_i)(\sigma_1^2\sum_{i=1}^n t_i)}$$

$$\hat{\sigma}_{\theta'}^2 = \frac{\sigma^2\sigma_o^2(\sigma_1^2\sum_{i=1}^n t_i^2 + \sigma^2)}{(\sigma^2 + n\sigma_o^2)(\sigma^2 + \sigma_1^2\sum_{i=1}^n t_i^2) - (\sigma_o^2\sum_{i=1}^n t_i)(\sigma_1^2\sum_{i=1}^n t_i)}$$

$$\hat{\sigma}_{\beta}^2 = \frac{\sigma^2\sigma_1^2(n\sigma_o^2 + \sigma^2)}{(\sigma^2 + n\sigma_o^2)(\sigma^2 + \sigma_1^2\sum_{i=1}^n t_i^2) - (\sigma_o^2\sum_{i=1}^n t_i)(\sigma_1^2\sum_{i=1}^n t_i)}$$

$$\hat{\sigma}_{\theta'\beta} = -\frac{\sigma^2\sigma_o^2\sigma_1^2\sum_{i=1}^n t_i}{(\sigma^2 + n\sigma_o^2)(\sigma^2 + \sigma_1^2\sum_{i=1}^n t_i^2) - (\sigma_o^2\sum_{i=1}^n t_i)(\sigma_1^2\sum_{i=1}^n t_i)}$$

$$\hat{\rho}_{\theta'\beta} = \frac{\hat{\sigma}_{\theta'\beta}}{\hat{\sigma}_{\theta}\hat{\sigma}_{\beta}} = -\frac{\sigma_o\sigma_1\sum_{i=1}^n t_i}{\sqrt{(\sigma_1^2\sum_{i=1}^n t_i^2 + \sigma^2)}\sqrt{(\sigma^2 + n\sigma_o^2)}}$$

Notice that $\hat{\rho}_{\theta'\beta}$ is the posterior correlation coefficient, even thought, Θ, B are independent random vectors. The posterior model parameters correlation is due to updating the signal, therefore $-1 \le \hat{\rho}_{\theta'\beta} \le 0$.

Thus $(\boldsymbol{\Theta}, \mathbf{B})|\boldsymbol{S} \dot{\sim} N_2\left(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}}^{-1}\right)$

Where $\hat{\boldsymbol{\mu}} = \begin{pmatrix}\hat{\theta}\\ \hat{\beta}\end{pmatrix}$ and $\hat{\boldsymbol{\Sigma}}^{-1} = \begin{pmatrix}\hat{\sigma}_{\theta'}^2 & \hat{\rho}_{\theta'\beta}\hat{\sigma}_{\theta'}\hat{\sigma}_{\beta}\\ \hat{\rho}_{\theta'\beta}\hat{\sigma}_{\theta'}\hat{\sigma}_{\beta} & \hat{\sigma}_{\beta}^2\end{pmatrix}$

4.3 Joint normal prior and Double exponential error

Proposed Method for D-JN. $\varepsilon(t_i)\ iid\ Laplace\ (0, b)$, $\begin{pmatrix}\boldsymbol{\Theta}\\ \mathbf{B}\end{pmatrix} \sim N\left[\begin{pmatrix}\mu_o\\ \mu_1\end{pmatrix}, \begin{pmatrix}\sigma_o^2 & 0\\ 0 & \sigma_1^2\end{pmatrix}\right]$ $\boldsymbol{\Theta} \perp \mathbf{B}$

$$[f(S_i|\theta)] \propto \prod_{k=1}^n Exp\left\{-\frac{1}{b}|S_i - \theta - \beta t_k|\right\}$$

$$= Exp\left\{-\frac{1}{b}\sum_{i=1}^k |S_i - \theta - \beta t_k|\right\}$$

Need to rearrange the observations such that for some k we have

$$\varepsilon_k = S_k - \theta - \beta t_k < 0 \; or \; \varepsilon_k = S_k - \theta - \beta t_k \geq 0$$

Thus, we may write

$$\sum_{k=1}^{n} |S_k - \theta - \beta t_k| = \sum_{k=1}^{n} (S_k - \theta - \beta t_k) \cdot \mathbb{1}_{(\varepsilon_k \geq 0)} - \sum_{k=1}^{n} (S_k - \theta - \beta t_k) \cdot \mathbb{1}_{(\varepsilon_k < 0)}$$

The product of

$$L[f(S_k|\theta, \beta)] \cdot \pi(\theta) \cdot \pi(\beta)$$
$$\propto Exp\left\{-\frac{1}{b}\left[\sum_{k=1}^{n} (S_k - \theta - \beta t_k) \cdot \mathbb{1}_{(\varepsilon_k \geq 0)} - \sum_{k=1}^{n} (S_k - \theta - \beta t_k) \cdot \mathbb{1}_{(\varepsilon_k < 0)}\right]\right\}$$
$$\cdot Exp\left\{-\frac{1}{2\sigma_o^2}(\theta - \mu_o)^2 - \frac{1}{2\sigma_1^2}(\beta - \mu_1)^2\right\}$$

Take the log of the above expression then we have the following:

$$-\frac{1}{b}\left[\sum_{k=1}^{n} S_k \cdot 1_{(\varepsilon_k \geq 0)} - \sum_{k=1}^{n} \theta \cdot \mathbb{1}_{(\varepsilon_k \geq 0)} - \sum_{k=1}^{n} \beta \cdot t_k \cdot \mathbb{1}_{(\varepsilon_k \geq 0)}\right.$$
$$\left. - \sum_{k=1}^{n} S_k \cdot \mathbb{1}_{(\varepsilon_k < 0)} + \sum_{k=1}^{n} \theta \cdot \mathbb{1}_{(\varepsilon_k < 0)} + \sum_{k=1}^{n} \beta \cdot t_k \cdot \mathbb{1}_{(\varepsilon_k < 0)}\right] - \frac{\theta^2}{2 \cdot \sigma_o^2} + \frac{\theta \cdot \mu_o}{\sigma_o^2} - \frac{\mu_o^2}{2 \cdot \sigma_o^2} - \frac{\beta^2}{2 \cdot \sigma_1^2}$$
$$+ \frac{\beta \cdot \mu_1}{\sigma_1^2} - \frac{\mu_1^2}{2 \cdot \sigma_1^2}$$

$$= -\frac{\theta^2}{2 \cdot \sigma_o^2} + \frac{\theta \cdot \mu_o}{\sigma_o^2} - \frac{\mu_o^2}{2 \cdot \sigma_o^2} - \frac{\beta^2}{2 \cdot \sigma_1^2} + \frac{\beta \cdot \mu_1}{\sigma_1^2} - \frac{\mu_1^2}{2 \cdot \sigma_1^2} + \frac{1}{b}\sum_{k=1}^{n} S_k \cdot \left[\mathbb{1}_{(\varepsilon_k \geq 0)} - \mathbb{1}_{(\varepsilon_k < 0)}\right]$$
$$+ \frac{\theta}{b}\sum_{k=1}^{n}\left[\mathbb{1}_{(\varepsilon_k \geq 0)} - \mathbb{1}_{(\varepsilon_k < 0)}\right] + \frac{\beta}{b}\sum_{k=1}^{n} t_k \cdot \left[\mathbb{1}_{(\varepsilon_k \geq 0)} - \mathbb{1}_{(\varepsilon_k < 0)}\right]$$

$$\frac{\partial \log\{[f(S_k|\theta, \beta)] \cdot \pi(\theta) \cdot \pi(\beta)\}}{\partial \theta} = \frac{1}{b}\sum_{k=1}^{n}\left[\mathbb{1}_{(\varepsilon_k \geq 0)} - \mathbb{1}_{(\varepsilon_k < 0)}\right] - \frac{\theta}{\sigma_o^2} + \frac{\mu_o}{\sigma_o^2} = 0$$

$$\Longrightarrow \hat{\theta} = \mu_o + \frac{\sigma_o^2}{b}\sum_{k=1}^{n}\left[\mathbb{1}_{(\varepsilon_k \geq 0)} - \mathbb{1}_{(\varepsilon_k < 0)}\right]$$

$$\frac{\partial \log\{[f(S_k|\theta,\beta)]\cdot\pi(\theta)\cdot\pi(\beta)\}}{\partial\beta} = \frac{1}{b}\sum_{k=1}^{n}\left[\mathbb{1}_{(\varepsilon_k\geq 0)} - \mathbb{1}_{(\varepsilon_k<0)}\right] - \frac{\beta}{\sigma_1^2} + \frac{\mu_1}{\sigma_1^2} = 0$$

$$\Longrightarrow \hat{\beta} = \mu_1 + \frac{\sigma_1^2}{b}\sum_{k=1}^{n} t_k \cdot \left[\mathbb{1}_{(\varepsilon_k\geq 0)} - \mathbb{1}_{(\varepsilon_k<0)}\right]$$

$$-\frac{\partial^2 \log\{[f(S_k|\theta,\beta)]\cdot\pi(\theta)\cdot\pi(\beta)\}}{\partial(\theta,\beta)^2} = \begin{bmatrix} \frac{\partial^2 \log\{L[f(S_k|\theta,\beta)]\cdot\pi(\theta)\cdot\pi(\beta)\}}{\partial\theta^2} & \frac{\partial^2 \log\{L[f(S_k|\theta,\beta)]\cdot\pi(\theta)\cdot\pi(\beta)\}}{\partial\theta\partial\beta} \\ \frac{\partial^2 \log\{L[f(S_k|\theta,\beta)]\cdot\pi(\theta)\cdot\pi(\beta)\}}{\partial\beta\partial\theta} & \frac{\partial^2 \log\{L[f(S_k|\theta,\beta)]\cdot\pi(\theta)\cdot\pi(\beta)\}}{\partial\beta^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sigma_o^2} & 0 \\ 0 & \frac{1}{\sigma_1^2} \end{bmatrix}$$

$$\hat{\Sigma}^{-1} = \left[-\frac{\partial^2 \log\{[f(S_k|\theta,\beta)]\cdot\pi(\theta)\cdot\pi(\beta)\}}{\partial(\theta,\beta)^2}\right]^{-1} = \begin{pmatrix} \sigma_o^2 & 0 \\ 0 & \sigma_1^2 \end{pmatrix}$$

Where

$\hat{\sigma}_\theta^2 = \sigma_o^2, \qquad \hat{\sigma}_\beta^2 = \sigma_1^2, and\ \hat{\sigma}_{\theta\beta} = 0 \Longrightarrow \hat{\rho}_{\theta\beta} = 0$

Hence $(\boldsymbol{\theta},\boldsymbol{\beta}|\boldsymbol{S_i}) \dot{\sim} N_2\left(\begin{pmatrix}\hat{\theta}\\ \hat{\beta}\end{pmatrix}, \hat{\boldsymbol{\Sigma}}^{-1}\right)$

4.4 Bivariate Normal prior and Double exponential error distributions

Proposed Method for D-BN. $\varepsilon(t_i)\ iid\ Laplace\ (0,b), \begin{pmatrix}\boldsymbol{\Theta}\\ \mathbf{B}\end{pmatrix} \sim N\left[\begin{pmatrix}\mu_o\\ \mu_1\end{pmatrix}, \begin{pmatrix}\sigma_o^2 & \rho_o\sigma_o\sigma_1\\ \rho_o\sigma_o\sigma_1 & \sigma_1^2\end{pmatrix}\right]$

$$\hat{\theta} = \mu_o\sigma_1 + \sigma_o\rho_o\left(\hat{\beta} - \mu_1\right) + \frac{(1-\rho_o^2)\sigma_o^2\sigma_1}{b}\sum_{k=1}^{n}\left[\mathbb{1}_{(\varepsilon_k\geq 0)} - \mathbb{1}_{(\varepsilon_k<0)}\right]$$

$$\hat{\beta} = \mu_1 + \frac{\rho_o \mu_1 \sigma_1}{\sigma_o} + \frac{(1-\rho_o^2)\sigma_1^2}{b} \sum_{k=1}^{n} t_k \cdot \left[\mathbb{1}_{(\varepsilon_k \geq 0)} - \mathbb{1}_{(\varepsilon_k < 0)}\right]$$

$$\hat{\Sigma}^{-1} = \left[-\frac{\partial^2 \log\{[f(S_k|\theta,\beta)] \cdot \pi(\theta,\beta)\}}{\partial(\theta,\beta)^2}\right]^{-1} = \begin{pmatrix} \sigma_o^2 & \rho_o\sigma_o\sigma_1 \\ \rho_o\sigma_o\sigma_1 & \sigma_1^2 \end{pmatrix}$$

And

$$(\boldsymbol{\theta}, \boldsymbol{\beta} | \boldsymbol{S_i}) \dot{\sim} N_2\left(\begin{pmatrix} \hat{\theta} \\ \hat{\beta} \end{pmatrix}, \widehat{\boldsymbol{\Sigma}}^{-1}\right).$$

Chapter 5

Mean Asymptotic Mean Square Error

In this chapter, we will calculate the asymptotic mean square error. This will help in finding how good are our estimates of the Laplace approximation with the different priors and error distributions. Asymptotic mean square error is the convergence in quadratic mean q.m. of the parameter estimation in comparison to the true parameter. In mathematical terms it is $\lim_{n\to\infty} E\left[\left(\hat{X}_n - X\right)^2\right] = 0$, this measures the expected square distance between $\hat{X}_n$ and X, it is called the mean square error given by $MSE\left(\hat{X}_n\right) = E\left[\left(\hat{X}_n - X\right)^2\right]$. In this thesis, we need to show, $\lim_{n\to\infty}\left\{Var\left(\hat{\theta}_n\right) + \left[E\left(\hat{\theta}_n\right) - \theta\right]^2\right\} = 0$. In this thesis, we will show that $\theta = E(\theta|S) \to \hat{\theta}$. But when it is substituted in MSE equation, it is now a random variable, hence we will take the E(MSE).

5.1

As stated in the introduction of this chapter, we will investigate the asymptotic true parameter that is to show that $E(\theta|S) \to \hat{\theta}$, in other word we are asymptotically calculating the expected value of the posterior and then taking the limit as n tends to infinity. This work will be proved in the next theorem.

Theorem 5.1.1

Let $\Theta \subset \mathbb{R}$, $f(\boldsymbol{S}|\theta)$ is the likelihood function, and $\pi(\theta)$ is the prior density.

Let $nh(\theta) = n\left[\frac{1}{n}\log f(S|\theta) + \frac{1}{n}\log \pi(\theta)\right]$ continuous on a closed and bounded set Θ.

If there exists a unique maximum, $\hat{\theta}$ then:

$$\lim_{n\to\infty} E(\theta|\boldsymbol{S}) = \lim_{n\to\infty} \frac{\int_{-\infty}^{\infty} \theta \cdot EXP[nh(\theta)]d\theta}{\int_{-\infty}^{\infty} EXP[nh(\theta)]d\theta} = \hat{\theta}$$

In particular

$$E\left[MSE\left(\hat{\theta}_n\right)\right] = E\left[var\left(\hat{\theta}_n\right)\right] + E\left[E\left(\hat{\theta}_n\right) - \hat{\boldsymbol{\Theta}}\right]^2$$

$$= 2 \cdot var\left(\hat{\theta}_n\right)$$

Proof:

By Bayes' law:

$$f(\theta|\boldsymbol{S}) = \frac{f(\boldsymbol{S}|\theta)\pi(\theta)}{\int_{-\infty}^{\infty} f(\boldsymbol{S}|\theta)\pi(\theta)d\theta}$$

Expectation of the posterior

$$E(\theta|\boldsymbol{S}) = \frac{\int_{-\infty}^{\infty} \theta f(\boldsymbol{S}|\theta)\pi(\theta)d\theta}{\int_{-\infty}^{\infty} f(\boldsymbol{S}|\theta)\pi(\theta)d\theta}$$

Let

$$nh(\theta) = n\left(\frac{1}{n}\log f(S|\theta) + \frac{1}{n}\log \pi(\theta)\right)$$

$$\Longrightarrow E(\theta|\boldsymbol{S}) = \frac{\int_{-\infty}^{\infty} \theta \cdot EXP[nh(\theta)]d\theta}{\int_{-\infty}^{\infty} EXP[nh(\theta)]d\theta}$$

Solve

$$\int_{-\infty}^{\infty} \theta \cdot \{EXP[nh(\theta)]\}d\theta$$

Next, we take Taylor expansion of $h(\theta)$

$$h(\theta) = h(\hat{\theta}) + h'(\hat{\theta})(\theta - \hat{\theta}) + \frac{h''(\hat{\theta})}{2!}(\theta - \hat{\theta})^2 + \frac{h'''(\hat{\theta})}{3!}(\theta - \hat{\theta})^3 + \frac{h^4(\hat{\theta})}{4!}(\theta - \hat{\theta})^4 + \cdots$$

let $x = \theta - \hat{\theta} \Longrightarrow \theta = x + \hat{\theta}$ where $\hat{\theta} = \arg\max_{\theta \in \Theta} h(\theta)$

$$\Longrightarrow \int_{-\infty}^{\infty} (x + \hat{\theta}) \cdot EXP[nh(\hat{\theta})] \cdot EXP[nh''(\hat{\theta})x^2] \cdot EXP\left[\frac{nh'''(\hat{\theta})}{3!}x^3\right] \cdot EXP\left[\frac{nh^4(\hat{\theta})}{4!}x^4\right]dx$$

$$\Longrightarrow \int_{-\infty}^{\infty} (x + \hat{\theta}) \cdot EXP[nh(\hat{\theta})] \cdot EXP[nh''(\hat{\theta})x^2] \cdot EXP\left[\frac{nh'''(\hat{\theta})}{3!}x^3 + \frac{nh^4(\hat{\theta})}{4!}x^4\right]dx$$

Use

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$\Rightarrow EXP\left[nh\left(\hat{\theta}\right)\right]\cdot\int_{-\infty}^{\infty}\left(x+\hat{\theta}\right)\cdot EXP\left[nh''\left(\hat{\theta}\right)x^2\right]$$

$$\cdot\left[1+\frac{nh'''\left(\hat{\theta}\right)}{3!}x^3+\frac{nh^4\left(\hat{\theta}\right)}{4!}x^4+\frac{1}{2!}\cdot\left(\frac{nh'''\left(\hat{\theta}\right)}{3!}x^3+\frac{nh^4\left(\hat{\theta}\right)}{4!}x^4\right)^2\right]dx$$

$$=EXP\left[nh\left(\hat{\theta}\right)\right]\cdot\int_{-\infty}^{\infty}\left(x+\hat{\theta}\right)\cdot EXP\left[nh''\left(\hat{\theta}\right)x^2\right]$$

$$\cdot\left[1+\frac{nh'''\left(\hat{\theta}\right)}{6}x^3+\frac{nh^4\left(\hat{\theta}\right)}{24}x^4+\frac{n^2}{72}h'''\left(\hat{\theta}\right)^2x^6+\frac{n^2}{144}h'''\left(\hat{\theta}\right)h^4\left(\hat{\theta}\right)x^7+\frac{n^2}{1152}h^4\left(\hat{\theta}\right)^2x^8\right]dx$$

$$=\hat{\theta}\cdot EXP\left[nh\left(\hat{\theta}\right)\right]\left[\int_{-\infty}^{\infty}e^{nh''(\hat{\theta})x^2}dx+\int_{-\infty}^{\infty}e^{nh''(\hat{\theta})x^2}\frac{nh'''\left(\hat{\theta}\right)}{6}x^3dx+\int_{-\infty}^{\infty}e^{nh''(\hat{\theta})x^2}\frac{nh^4\left(\hat{\theta}\right)}{24}x^4dx\right.$$

$$+\int_{-\infty}^{\infty}e^{nh''(\hat{\theta})x^2}\frac{n^2}{72}h'''\left(\hat{\theta}\right)^2x^6dx+\int_{-\infty}^{\infty}e^{nh''(\hat{\theta})x^2}\frac{n^2}{144}h'''\left(\hat{\theta}\right)h^4\left(\hat{\theta}\right)x^7dx$$

$$\left.+\int_{-\infty}^{\infty}e^{nh''(\hat{\theta})x^2}\frac{n^2}{1152}h^4\left(\hat{\theta}\right)^2x^8dx\right]$$

$$+EXP\left[nh\left(\hat{\theta}\right)\right]\left[\int_{-\infty}^{\infty}EXP\left[nh''\left(\hat{\theta}\right)x^2\right]xdx\right.$$

$$+\int_{-\infty}^{\infty}EXP\left[nh''\left(\hat{\theta}\right)x^2\right]\frac{nh'''\left(\hat{\theta}\right)}{6}x^4dx+\int_{-\infty}^{\infty}EXP\left[nh''\left(\hat{\theta}\right)x^2\right]\frac{nh^4\left(\hat{\theta}\right)}{24}x^5dx$$

$$+\int_{-\infty}^{\infty}EXP\left[nh''\left(\hat{\theta}\right)x^2\right]\frac{n^2}{72}h'''\left(\hat{\theta}\right)^2x^7dx+\int_{-\infty}^{\infty}EXP\left[nh''\left(\hat{\theta}\right)x^2\right]\frac{n^2}{144}h'''\left(\hat{\theta}\right)h^4\left(\hat{\theta}\right)x^8dx$$

$$\left.+\int_{-\infty}^{\infty}EXP\left[nh''\left(\hat{\theta}\right)x^2\right]\frac{n^2}{1152}h^4\left(\hat{\theta}\right)^2x^9dx\right]$$

Use the definition of the integral of Gamma function to solve the above expression.

$$\int_{-\infty}^{\infty}\left[EXP(tx^2)\right]x^{2\cdot n}dx=t^{-n-\frac{1}{2}}*\frac{(2\cdot n)!}{n!\cdot 2^{2\cdot n}}\cdot\pi^{\frac{1}{2}}$$

Where the integrals of odd powers vanish given that the Gaussian is an even function.

$$\Rightarrow \hat{\theta}\cdot EXP\left[nh(\hat{\theta})\right]\left[\int_{-\infty}^{\infty} EXP\left[nh''(\hat{\theta})x^2\right]dx + \int_{-\infty}^{\infty} EXP\left[nh''(\hat{\theta})x^2\right]\frac{nh^4(\hat{\theta})}{24}x^4dx\right.$$

$$\left. + \int_{-\infty}^{\infty} EXP\left[nh''(\hat{\theta})x^2\right]\frac{n^2}{72}h'''(\hat{\theta})^2x^6dx + \int_{-\infty}^{\infty} EXP\left[nh''(\hat{\theta})x^2\right]\frac{n^2}{1152}h^4(\hat{\theta})^2x^8dx\right]$$

$$+EXP\left[nh(\hat{\theta})\right]\left[\int_{-\infty}^{\infty} EXP\left[nh''(\hat{\theta})x^2\right]\frac{nh'''(\hat{\theta})}{6}x^4dx + \int_{-\infty}^{\infty} EXP\left[nh''(\hat{\theta})x^2\right]\frac{n^2}{144}h'''(\hat{\theta})h^4(\hat{\theta})x^8dx\right]$$

$$\Rightarrow \int_{-\infty}^{\infty}(x+\hat{\theta})\cdot e^{nh(\hat{\theta})}\cdot e^{nh''(\hat{\theta})x^2}\cdot\left[e^{\frac{nh'''(\hat{\theta})}{3!}(x)^3} + e^{\frac{nh^4(\hat{\theta})}{4!}(x)^4}\right]dx$$

$$= \hat{\theta}\cdot EXP\left[nh(\hat{\theta})\right]\left[\frac{\sqrt{-2\pi}}{\sqrt{nh''(\hat{\theta})}} + \frac{\sqrt{-2\pi}}{\sqrt{nh''(\hat{\theta})}}\frac{n}{24}\frac{4}{n^2}\frac{1}{h''(\hat{\theta})^2}\frac{4!}{2!\cdot 16}h^4(\hat{\theta})\right.$$

$$\left. + \frac{\sqrt{-2\pi}}{\sqrt{nh''(\hat{\theta})}}\frac{n^2}{72}\frac{8}{n^3}\frac{1}{h''(\hat{\theta})^3}\frac{6!}{3!\cdot 2^6}h'''(\hat{\theta})^2 + \frac{\sqrt{-2\pi}}{\sqrt{nh''(\hat{\theta})}}\frac{n^2}{1152}\frac{16}{n^4}\frac{1}{h''(\hat{\theta})^4}\frac{8!}{4!\cdot 2^8}h^4(\hat{\theta})^2\right]$$

$$+ EXP\left[nh(\hat{\theta})\right]\left[\frac{\sqrt{-2\pi}}{\sqrt{nh''(\hat{\theta})}}\frac{n}{6}\frac{4}{n^2}\frac{1}{h''(\hat{\theta})^2}\frac{4!}{2!\cdot 16}h'''(\hat{\theta}) + \frac{\sqrt{-2\pi}}{\sqrt{nh''(\hat{\theta})}}\frac{n^2}{144}\frac{16}{n^4}\frac{1}{h''(\hat{\theta})^4}\frac{8!}{4!\cdot 2^8}h^4(\hat{\theta})\right]$$

$$= \hat{\theta}\cdot EXP\left[nh(\hat{\theta})\right]\frac{\sqrt{-2\pi}}{\sqrt{nh''(\hat{\theta})}}\left[1 + \frac{1}{8n}\frac{h^4(\hat{\theta})}{h''(\hat{\theta})^2} + \frac{5}{24n}\frac{h'''(\hat{\theta})^2}{h''(\hat{\theta})^3} + \frac{35}{384n^2}\frac{h^4(\hat{\theta})^2}{h''(\hat{\theta})^4}\right]$$

$$+ EXP\left[nh(\hat{\theta})\right]\frac{\sqrt{-2\pi}}{\sqrt{nh''(\hat{\theta})}}\left[\frac{1}{2n}\frac{h'''(\hat{\theta})}{h''(\hat{\theta})^2} + \frac{35}{48n^2}\frac{h^4(\hat{\theta})}{h''(\hat{\theta})^4}\right]$$

$$= \hat{\theta} \cdot EXP\left[nh(\hat{\theta})\right] \frac{\sqrt{-2\pi}}{\sqrt{nh''(\hat{\theta})}} \left[1 + O\left(\frac{1}{n}\right)\right] + e^{nh(\hat{\theta})} \frac{\sqrt{-2\pi}}{\sqrt{nh''(\hat{\theta})}} O\left(\frac{1}{n}\right)$$

Solve

$$\int_{-\infty}^{\infty} EXP[nh(\theta)]d\theta = EXP\left[nh(\hat{\theta})\right] \frac{\sqrt{-2\pi}}{\sqrt{nh''(\hat{\theta})}} \left[1 + \frac{1}{8n}\frac{h^4(\hat{\theta})}{h''(\hat{\theta})^2} + \frac{5}{24n}\frac{h'''(\hat{\theta})^2}{h''(\hat{\theta})^3} + \frac{35}{384n^2}\frac{h^4(\hat{\theta})^2}{h''(\hat{\theta})^4}\right]$$

$$= EXP\left[nh(\hat{\theta})\right] \frac{\sqrt{-2\pi}}{\sqrt{nh''(\hat{\theta})}} \left[1 + O\left(\frac{1}{n}\right)\right]$$

$$\Longrightarrow E(\theta|\boldsymbol{S}) = \frac{\hat{\theta} \cdot EXP\left[nh(\hat{\theta})\right] \frac{\sqrt{-2\pi}}{\sqrt{nh''(\hat{\theta})}} \left[1 + O\left(\frac{1}{n}\right)\right]}{EXP\left[nh(\hat{\theta})\right] \frac{\sqrt{-2\pi}}{\sqrt{nh''(\hat{\theta})}} \left[1 + O\left(\frac{1}{n}\right)\right]} + \frac{EXP\left[nh(\hat{\theta})\right] \frac{\sqrt{-2\pi}}{\sqrt{nh''(\hat{\theta})}} O\left(\frac{1}{n}\right)}{EXP\left[nh(\hat{\theta})\right] \frac{\sqrt{-2\pi}}{\sqrt{nh''(\hat{\theta})}} \left[1 + O\left(\frac{1}{n}\right)\right]}$$

$$= \frac{\hat{\theta}\left[1 + O\left(\frac{1}{n}\right)\right]}{\left[1 + O\left(\frac{1}{n}\right)\right]} + \frac{O\left(\frac{1}{n}\right)}{\left[1 + O\left(\frac{1}{n}\right)\right]} \leq \frac{\hat{\theta}\left[1 + O\left(\frac{1}{n}\right)\right]}{\left[1 + O\left(\frac{1}{n}\right)\right]} + \frac{O\left(\frac{1}{n}\right)}{1}$$

$$= \hat{\theta}\left[1 + O\left(\frac{1}{n}\right)\right] + O\left(\frac{1}{n}\right) = \hat{\theta}\left[1 + O\left(\frac{1}{n}\right)\right]$$

$$\lim_{n\to\infty} E(\theta|\boldsymbol{S}) = \lim_{n\to\infty} \hat{\theta}\left[1 + O\left(\frac{1}{n}\right)\right] = \hat{\theta} \quad \blacksquare$$

$$\lim_{n\to\infty} E\left[MSE(\hat{\theta}_n)\right] = \lim_{n\to\infty} \left\{E\left[var(\hat{\theta}_n)\right] + E\left[E(\hat{\theta}_n) - \hat{\boldsymbol{\Theta}}\right]^2\right\}$$

$$= \lim_{n\to\infty} 2 \cdot var(\hat{\theta}_n) \quad \blacksquare$$

5.2

In this section we will investigate the expectation of the mean integrated squared error for the case of univariate normal distributed prior and error. We call this case $E\left[MSE(\hat{\theta})\right]$ For N-N case.

Theorem 5.2.1

Given $S_i = \theta t_i + \varepsilon_i$ is a linear degradation model with degradation signal time $t_i > 0,$ and iid $N(0, \sigma^2)$ error terms $\varepsilon_i, i = 1, 2, \dots, k$ if $\theta \sim N(\mu_o, \sigma_o^2)$ then

$$\hat{\theta} = \left(\frac{\sum_{i=1}^{k} S_i t_i}{\sum_{i=1}^{k} t_i^2}\right)\left(\frac{\sigma_0^2}{\sigma_0^2 + \left(\sigma^2/\sum_{i=1}^{k} t_i^2\right)}\right) + \mu_0\left(\frac{\left(\sigma^2/\sum_{i=1}^{k} t_i^2\right)}{\sigma_0^2 + \left(\sigma^2/\sum_{i=1}^{k} t_i^2\right)}\right)$$

the consistent asymptotic posterior mean.

Proof:

For this proof we use convergence in quadratic mean, which is the strongest form of consistency.

Let

$$\hat{\theta} = \left(\frac{\sum_{i=1}^{k} S_i t_i}{\sum_{i=1}^{k} t_i^2}\right)\left(\frac{\sigma_0^2}{\sigma_0^2 + \left(\sigma^2/\sum_{i=1}^{k} t_i^2\right)}\right) + \mu_0\left(\frac{\left(\sigma^2/\sum_{i=1}^{k} t_i^2\right)}{\sigma_0^2 + \left(\sigma^2/\sum_{i=1}^{k} t_i^2\right)}\right)$$

$$var(\hat{\theta}) = \left(\frac{\sigma_0^2}{\sigma_0^2 \sum_{i=1}^{k} t_i^2 + \sigma^2}\right)^2 \sum_{i=1}^{k} var(S_i t_i)$$

$$= \left(\frac{\sigma_0^2}{\sigma_0^2 \sum_{i=1}^{k} t_i^2 + \sigma^2}\right)^2 \sum_{i=1}^{k} var\big((\theta t_i + \varepsilon_i) t_i\big)$$

$$= \left(\frac{\sigma_0^2}{\sigma_0^2 \sum_{i=1}^{k} t_i^2 + \sigma^2}\right)^2 \sum_{i=1}^{k} \left(t_i^4 var(\theta) + t_i^2 var(\varepsilon_i)\right)$$

$$= \left(\frac{\sigma_0^2}{\sigma_0^2 \sum_{i=1}^{k} t_i^2 + \sigma^2}\right)^2 \sum_{i=1}^{k} (t_i^4 \sigma_o^2 + t_i^2 \sigma^2)$$

$$= \frac{\sigma_0^4 \sum_{i=1}^{k} (t_i^4 \sigma_o^2 + t_i^2 \sigma^2)}{\left(\sigma_0^2 \sum_{i=1}^{k} t_i^2 + \sigma^2\right)^2} \le \frac{\sum_{i=1}^{k} t_i^4 \sigma_o^2}{\left(\sum_i^k t_i^2\right)^2} + \frac{\sigma^2 \sum_i^k t_i^2}{\sum_i^k t_i^2 \cdot \sum_i^k t_i^2} = \frac{\sum_{i=1}^{k} t_i^4 \sigma_o^2}{\left(\sum_i^k t_i^2\right)^2} + \frac{\sigma^2}{\sum_i^k t_i^2}$$

N.T.S

$$\sum_{i=1}^{k} t_i^4 \le \left(\sum_{i}^{k} t_i^2\right)^2$$

Proof by induction

$k = 2 \qquad\qquad t_1^4 + t_2^4 \leq t_1^4 + t_2^4 + 2t_1^2 t_2^2$

Assume it is true for $n = k$ that is $(t_1^4 + t_2^4 + \cdots + t_k^4) \leq (t_1^2 + t_2^2 + \cdots + t_k^2)^2$

N.T.S

It is true for $n = k + 1$

$$\underbrace{t_1^4 + t_2^4 + \cdots + t_k^4}_{A} + t_{k+1}^4 \leq \underbrace{(t_1^2 + t_2^2 + \cdots + t_k^2)^2}_{B} + t_{k+1}^4$$

$$= B^2 + (t_{k+1}^2)^2 \leq (B + t_{k+1}^2)^2$$

$$= (t_1^2 + \cdots + t_k^2 + t_{k+1}^2)^2 \quad ▪$$

Thus

$$Var(\hat{\theta}) \leq \frac{\sum_{i=1}^k t_i^4 \sigma_o^2}{\left(\sum_i^k t_i^2\right)^2} + \frac{\sigma^2}{\sum_i^k t_i^2}$$

$$= \frac{\sum_{i=1}^k t_i^4 \sigma_o^2}{\sum_i^k t_i^4 + \sum_{i \neq j} \sum_{j=1}^k t_i^2 t_j^2} + \frac{\sigma^2}{\sum_i^k t_i^2} \quad for \begin{array}{l} i = 1,2,\cdots,k \\ j = 1,2,\cdots,k \end{array}$$

$$\Rightarrow \lim_{k \to \infty} E\left[MSE(\hat{\theta})\right] = \lim_{k \to \infty} 2 \cdot var(\hat{\theta}) \to 0 \quad ▪$$

5.3

In this section we will investigate the expectation of the mean integrated squared error for the case of univariate normal distributed error and Gamma prior. We call this case $E\left[MSE(\hat{\theta})\right]$ For N-G case.

Theorem 5.3.1

Given $S_i = \theta t_i + \varepsilon_i$ is a linear degradation model with degradation signal time $t_i > 0,$ and iid $N(0, \sigma^2)$ error terms $\varepsilon_i, i = 1, 2, \ldots, k$ if $\theta \sim Gamma(\alpha, \beta)$ then

$$\hat{\theta} = \frac{-\left(\frac{1}{\sigma^2}\sum_{i=1}^{k} S_i t_i - \frac{1}{\beta}\right) - \sqrt{\left(\frac{1}{\sigma^2}\sum_{i=1}^{k} S_i t_i - \frac{1}{\beta}\right)^2 + 8(\alpha - 1)\frac{1}{\sigma^2}\sum_{i=1}^{k} t_i^2}}{-4\frac{1}{2\sigma^2}\sum_{i=1}^{k} t_i^2}$$

the consistent asymptotic posterior mean.

Proof:

For this proof we use convergence in quadratic mean, which is the strongest form of consistency.

Let

$$\hat{\theta} = \frac{-\left(\frac{1}{\sigma^2}\sum_{i=1}^{k} S_i t_i - \frac{1}{\beta}\right) - \sqrt{\left(\frac{1}{\sigma^2}\sum_{i=1}^{k} S_i t_i - \frac{1}{\beta}\right)^2 + 8(\alpha - 1)\frac{1}{\sigma^2}\sum_{i=1}^{k} t_i^2}}{-4\frac{1}{2\sigma^2}\sum_{i=1}^{k} t_i^2}$$

$$\leq \frac{-\left(\frac{1}{\sigma^2}\sum_{i=1}^{k} S_i t_i - \frac{1}{\beta}\right) - \sqrt{\left(\frac{1}{\sigma^2}\sum_{i=1}^{k} S_i t_i - \frac{1}{\beta}\right)^2}}{-2\frac{1}{\sigma^2}\sum_{i=1}^{k} t_i^2}$$

$$= \frac{-\left(\frac{1}{\sigma^2}\sum_{i=1}^{k} S_i t_i - \frac{1}{\beta}\right) - \left(\frac{1}{\sigma^2}\sum_{i=1}^{k} S_i t_i - \frac{1}{\beta}\right)}{-2\frac{1}{\sigma^2}\sum_{i=1}^{k} t_i^2} = \frac{-\frac{1}{\sigma^2}\sum_{i=1}^{k} S_i t_i + \frac{1}{\beta}}{-\frac{1}{\sigma^2}\sum_{i=1}^{k} t_i^2}$$

$$var(\hat{\theta}) = var\left(\frac{-\frac{1}{\sigma^2}\sum_{i=1}^{k} S_i t_i}{-\frac{1}{\sigma^2}\sum_{i=1}^{k} t_i^2}\right) + var\left(\frac{\frac{1}{\beta}}{-\frac{1}{\sigma^2}\sum_{i=1}^{k} t_i^2}\right)$$

$$= \frac{1}{\left(\sum_{i=1}^{k} t_i^2\right)^2}\sum_{i=1}^{k} var(S_i t_i) = \frac{1}{\left(\sum_{i=1}^{k} t_i^2\right)^2}\sum_{i=1}^{k} var[(\theta t_i + \varepsilon)t_i]$$

$$= \frac{1}{\left(\sum_{i=1}^{k} t_i^2\right)^2}\left[\sum_{i=1}^{k} t_i^4 \frac{\alpha}{\beta^2} + \sum_{i=1}^{k} t_i^2 \sigma_o^2\right]$$

$$= \frac{\sum_{i=1}^{k} t_i^4 \frac{\alpha}{\beta^2}}{\left(\sum_i^k t_i^2\right)^2} + \frac{\sigma_o^2}{\sum_i^k t_i^2} = \frac{\sum_{i=1}^{k} t_i^4 \frac{\alpha}{\beta^2}}{\sum_i^k t_i^4 + \sum_{i \neq j} \sum_{j=1}^{k} t_i^2 t_j^2} + \frac{\sigma^2}{\sum_i^k t_i^2} \quad for \begin{matrix} i = 1,2,\cdots,k \\ j = 1,2,\cdots,k \end{matrix}$$

$$\Longrightarrow \lim_{k\to\infty} E\left[MSE\left(\hat{\theta}\right)\right] = \lim_{k\to\infty} 2 \cdot var\left(\hat{\theta}\right) \to 0 \quad \blacksquare$$

5.4

In this section we will investigate the expectation of the mean integrated squared error for the case of Gamma distributed prior and Laplace error. We call this case $E\left[MSE\left(\hat{\theta}\right)\right]$ For D-G case.

5.4.1

For the Laplace-Gamma case the posterior mean estimator,

$$\hat{\theta} = \frac{\alpha - 1}{\frac{1}{b}\left(\sum_{k=1}^{n} t_k \cdot \mathbb{1}_{(\varepsilon_k > 0)} - \sum_{k=1}^{n} t_k \cdot \mathbb{1}_{(\varepsilon_k \leq 0)} + \frac{b}{\beta}\right)}$$

we will investigate mean squared convergence. First, we need to find $V\left(\hat{\theta}\right)$ then take the limit as n tends to infinity.

Let

$$X = \frac{1}{b} \cdot \left(\sum_{k=1}^{n} t_k \cdot \mathbb{1}_{(\varepsilon_k > 0)} - \sum_{k=1}^{n} t_k \cdot \mathbb{1}_{(\varepsilon_k \leq 0)} + \frac{b}{\beta}\right)$$

Define $g(X) = \frac{1}{X}$

Then $Var[g(x)] = E[g(X)^2] - E[g(X)]^2$

Use Taylor series expansion of $g(X) = \frac{1}{X}$ about the mean value μ_x then take expected value of both sides of the equation

$$\Longrightarrow E[g(X)] = E\left[g(X)|_{X=\mu_x}\right] + E\left[(X - \mu_x) g(x)'|_{X=\mu_x}\right] + E\left[\frac{1}{2!}(X - \mu_x)^2 g(X)'' \Big|_{X=\mu_x}\right] + \cdots$$

Where

$$E(X - \mu_x) = E[X] - \mu_X = \mu_X - \mu_X = 0$$

$$\Longrightarrow E[g(X)] = E\left[g(X)|_{X=\mu_x}\right] + E\left[\frac{1}{2!}(X-\mu_x)^2 g(X)''\Big|_{X=\mu_x}\right] + \cdots$$

$$\approx g(\mu_x) + \sigma_X^2 g''(\mu_X)$$

$$\Longrightarrow E\left(\frac{1}{X}\right) \approx \frac{1}{\mu_X} + \frac{\sigma_X^2}{\mu_X^3}$$

$$\Longrightarrow \left[E\left(\frac{1}{X}\right)\right]^2 \approx \left(\frac{1}{\mu_X} + \frac{\sigma_X^2}{\mu_X^3}\right)^2$$

$$= \left(\frac{1}{\mu_X}\right)^2 + 2\frac{\sigma_X^2}{\mu_X^4} + \frac{\sigma_X^4}{\mu_X^6}$$

$$E[g^2(X)] = E\left[g^2(X)|_{X=\mu_x}\right] + E\left[(X-\mu_x)g^2(x)'|_{X=\mu_x}\right] + E\left[\frac{1}{2!}(X-\mu_x)^2 g^2(X)''\Big|_{X=\mu_x}\right] + \cdots$$

$$\approx g^2(\mu_x) + 0 + \sigma_X^2[g'(\mu_X)]^2 + g(\mu_X)g''(\mu_X)$$

$$\Longrightarrow E\left[\left(\frac{1}{X}\right)^2\right] \approx \left(\frac{1}{\mu_X}\right)^2 + \sigma_X^2\frac{1}{\mu_X^4} + \sigma_X^2\frac{1}{\mu_X}\frac{2}{\mu_X^3}$$

$$\Longrightarrow Var\left(\frac{1}{X}\right) \approx \sigma_X^2\frac{1}{\mu_X^4} - \frac{\sigma_X^4}{\mu_X^6} \approx \sigma_X^2\frac{1}{\mu_X^4} \qquad for\ \sigma_X^4 \ll \sigma_X^2$$

$$\Longrightarrow Var(\hat{\theta}) = (\alpha-1)^2 \cdot Var\left(\frac{1}{X}\right) \qquad Let\ c = \alpha - 1 > 0$$

$$\approx \frac{Var\left[\frac{1}{b}\left(\sum_{k=1}^n t_k \cdot \mathbb{1}_{(\varepsilon_k>0)} - \sum_{k=1}^n t_k \cdot \mathbb{1}_{(\varepsilon_k\le 0)} + \frac{b}{\beta}\right)\right]}{\left\{E\left[\frac{1}{b}\left(\sum_{k=1}^n t_k \cdot \mathbb{1}_{(\varepsilon_k>0)} - \sum_{k=1}^n t_k \cdot \mathbb{1}_{(\varepsilon_k\le 0)} + \frac{b}{\beta}\right)\right]\right\}^4} \cdot c^2$$

$$= \frac{\frac{4}{b^2} \cdot Var\left(\sum_{k=1}^n t_k \cdot \mathbb{1}_{(\varepsilon_k>0)}\right)}{\frac{1}{b^4} \cdot \left[2 \cdot E\left(\sum_{k=1}^n t_k \cdot \mathbb{1}_{(\varepsilon_k>0)}\right) - \sum_{k=1}^n t_k + \frac{b}{\beta}\right]^4}$$

$$= \frac{4 \cdot b^2 \cdot \sum_{k=1}^n t_k^2 \cdot Var\left(\mathbb{1}_{(\varepsilon_k>0)}\right)}{\left[2 \cdot E\left(\sum_{k=1}^n t_k \cdot \mathbb{1}_{(\varepsilon_k>0)}\right) - \sum_{k=1}^n t_k + \frac{b}{\beta}\right]^4}$$

$$= \frac{4 \cdot b^2 \cdot \sum_{k=1}^{n} t_k^2 \cdot Var\left(\mathbb{1}_{(\varepsilon_k > 0)}\right)}{\left[2 \cdot \sum_{k=1}^{n} t_k \cdot E\left(\mathbb{1}_{(\varepsilon_k > 0)}\right) - \sum_{k=1}^{n} t_k + \frac{b}{\beta}\right]^4}$$

$$= \frac{4 \cdot b^2 \cdot \sum_{k=1}^{n} t_k^2 \cdot \left\{E\left(\mathbb{1}_{(\varepsilon_k > 0)}\right) - \left[E\left(\mathbb{1}_{(\varepsilon_k > 0)}\right)\right]^2\right\}}{\left[2 \cdot \sum_{k=1}^{n} t_k \cdot \frac{1}{2} - \sum_{k=1}^{n} t_k + \frac{b}{\beta}\right]^4}$$

$$= \frac{4 \cdot b^2 \cdot \sum_{k=1}^{n} t_k^2 \cdot \left\{\frac{1}{2} - \left[\frac{1}{2}\right]^2\right\}}{\left[\sum_{k=1}^{n} t_k - \sum_{k=1}^{n} t_k + \frac{b}{\beta}\right]^4}$$

$$= \frac{b^2 \cdot \sum_{k=1}^{n} t_k^2}{\left[\frac{b}{\beta}\right]^4} = \frac{\beta^4 \cdot \sum_{k=1}^{n} t_k^2}{b^2} \to \infty \ as \ k \to \infty$$

$\Longrightarrow Var\left(\hat{\theta}\right) \longrightarrow \infty \ as \ k \longrightarrow \infty \ Var\left(\hat{\theta}\right)$ does not converge in quadratic mean. Next, we investigate the asymptotic consistency of the estimator.

Definition 5.4.1

Let $\{X_x\}_{n=1}^{\infty}$ be a sequence of random variables. The sequence converges in probability to a random variable X if $X \ if \ \forall \varepsilon > 0, \lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$. This relationship is represented by $X_n \overset{p}{\to} X \ as \ n \to \infty$. []

In this research we used the following:

Theorem 5.4.1

Let $\mu_n = ES_n, \ \sigma_n^2 = var(S_n)$. If $\sigma_n^2 / b_n^2 \to 0$ then

$$\frac{S_n - \mu_n}{b_n} \to 0 \qquad \text{in probability. [13]}$$

Theorem 5.4.2

Let $V\left(\hat{\theta}\right) = \frac{\beta^4 \cdot \sum_{k=1}^{n} t_k^2}{b^2}$ take $a_n = O(n^3)$, by theorem 5.4.1 we have:

$$\frac{\hat{\theta}}{n^3} \overset{P}{\to} \frac{2\beta^3}{b^2}$$

Furthermore, let

$$X_n = \frac{\hat{\theta}}{n^3} \; and \; c = \frac{2\beta^3}{b^2}$$

$$then \;\; X_n \xrightarrow{d} c \;\; \Longrightarrow \lim_{n\to\infty} F_n(x) = \begin{cases} 0 & if \quad x < c \\ 1 & if \quad x \geq c \end{cases}$$

Proof:

$$Var\left(\frac{\hat{\theta}}{n^3}\right) = \frac{1}{n^6} \cdot Var(\hat{\theta}) = \frac{1}{n^6} \cdot \frac{\beta^4 \cdot \sum_{k=1}^{n} t_k^2}{b^2} \to 0 \; as \; n \to \infty$$

Recall

$$E\left(\frac{1}{X}\right) \approx \frac{1}{\mu_X} + \frac{\sigma_X^2}{\mu_X^3}$$

$$\mu_X = E\left[\frac{1}{b}\left(\sum_{k=1}^{n} t_k \cdot \mathbb{1}_{(\varepsilon_k > 0)} - \sum_{k=1}^{n} t_k \cdot \mathbb{1}_{(\varepsilon_k \leq 0)} + \frac{b}{\beta}\right)\right] = \frac{1}{\beta}$$

$$\sigma_X^2 = \frac{1}{b^2} \cdot \sum_{k=1}^{n} t_k^2$$

$$\Longrightarrow E\left(\frac{1}{X}\right) \approx \beta + \frac{\frac{1}{b^2} \cdot \sum_{k=1}^{n} t_k^2}{\frac{1}{\beta^3}} = \beta + \frac{\beta^3 \cdot \sum_{k=1}^{n} t_k^2}{b^2}$$

Then, by

$$\frac{\hat{\theta} - \mu_n}{a_n} \xrightarrow{p} 0$$

$$\Longrightarrow \frac{\hat{\theta}}{n^3} - \left(\frac{\beta}{n^3} + \frac{\beta^3 \cdot \sum_{k=1}^{n} t_k^2}{n^3 b^2}\right) = \frac{\hat{\theta}}{n^3} - \left(\frac{\beta}{n^3} + \frac{\beta^3(nt_1^2 + 2n^3 - 3n^2 + n)}{n^3 b^2} + \frac{n^2 t_1 - n t_1}{n^3}\right)$$

$$\Longrightarrow \frac{\hat{\theta}}{n^3} - \frac{2\beta^3}{b^2} \xrightarrow{p} 0$$

$$\Longrightarrow \frac{\hat{\theta}}{n^3} \xrightarrow{p} \frac{2\beta^3}{b^2} \qquad \blacksquare$$

Furthermore, since c is a constant, then by convergence of X_n to c in distribution is equivalent to convergence in probability. [20]

$$\text{Let } X_n = \frac{\hat{\theta}}{n^3} \ and \ k = \frac{2\beta^3}{b^2}$$

$$\text{then } X_n \overset{d}{\to} k \implies \lim_{n\to\infty} F_n(x) = \begin{cases} 0 & if \quad x < k \\ 1 & if \quad x \geq k \end{cases} \qquad \blacksquare$$

5.5.1

For the Laplace-Joint Normal distribution case with **Θ ⊥ B**, the posterior mean estimator,

$$\hat{\theta} = \mu_o + \frac{\sigma_o^2}{b} \cdot \sum_{k=1}^{n} \left[\mathbb{1}_{(\varepsilon_k \leq 0)} - \mathbb{1}_{(\varepsilon_k > 0)} \right]$$

$$\hat{\beta} = \mu_1 + \frac{\sigma_1^2}{b} \cdot \sum_{k=1}^{n} \left[t_k \cdot \mathbb{1}_{(\varepsilon_k \leq 0)} - t_k \cdot \mathbb{1}_{(\varepsilon_k > 0)} \right]$$

we will check consistency using mean squared convergence. First, we need to find $V(\hat{\theta})$ and $V(\hat{\beta})$ then take the limit as n tends to infinity. We call these $\boldsymbol{E}[MSE(\hat{\theta})]$ and $\boldsymbol{E}[MSE(\hat{\beta})]$ for D-JN Θ ⊥ B case.

For $\boldsymbol{E[MSE(\hat{\theta})]}$ for D-JN Θ ⊥ B case.

Let

$$Var(\hat{\theta}) = 4 \cdot \frac{\sigma_o^4}{b^2} \cdot Var\left(\sum_{k=1}^{n} \mathbb{1}_{(\varepsilon_k \leq 0)} \right)$$

$$but \ \varepsilon_k \ are \ iid \ Laplace \ (0, b)$$

$$\implies Var(\hat{\theta}) = 4 \cdot \frac{\sigma_o^4}{b^2} \cdot \sum_{k=1}^{n} Var\left(\mathbb{1}_{(\varepsilon_k \leq 0)}\right)$$

$$= 4 \cdot \frac{\sigma_o^4}{b^2} \cdot \sum_{k=1}^{n} \left\{ E\left[\mathbb{1}_{(\varepsilon_k \leq 0)}\right] - E\left[\mathbb{1}_{(\varepsilon_k \leq 0)}\right]^2 \right\}$$

$$= 4 \cdot \frac{\sigma_o^4}{b^2} \cdot \left\{ \sum_{k=1}^{n} [P(\varepsilon_k \leq 0) - P(\varepsilon_k \leq 0)^2] \right\}$$

$$= 4 \cdot \frac{\sigma_o^4}{b^2} \cdot n \cdot \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{\sigma_o^4}{b^2} \cdot n$$

$\Longrightarrow Var(\hat{\theta}) \longrightarrow \infty \; as \; n \longrightarrow \infty \; Var(\hat{\theta})$ does not converge in quadratic mean.

Theorem 5.5.1.1

Let $V(\hat{\theta}) = 4 \cdot \frac{\sigma_o^4}{b^2} \cdot Var\left(\sum_{k=1}^{n} \mathbb{1}_{(\varepsilon_k \leq 0)}\right)$ take $a_n = O(n)$, by theorem 5.4.1 we have:

$$\frac{\hat{\theta}}{n} \xrightarrow{P} 0$$

Furthermore, let

$$X_n = \frac{\hat{\theta}}{n}$$

$$then \;\; X_n \xrightarrow{d} 0 \;\; \Longrightarrow \lim_{n \to \infty} F_n(x) = \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad x \geq 0 \end{cases}$$

Proof:

Taking $a_n = O(n)$

$$Var\left(\frac{\hat{\theta}}{n}\right) = \frac{1}{n^2} \cdot Var(\hat{\theta}) = \frac{1}{n^2} \cdot \frac{\sigma_o^4}{b^2} \cdot n \to 0 \; as \; n \to \infty$$

$$\mu_n = \mu_o + \frac{\sigma_o^2}{b} \cdot \sum_{k=1}^{n} E\left[\mathbb{1}_{(\varepsilon_k \leq 0)} - \mathbb{1}_{(\varepsilon_k > 0)}\right]$$

$$= \mu_o + \frac{\sigma_o^2}{b} \cdot \sum_{k=1}^{n} 2 \cdot E\left[\mathbb{1}_{(\varepsilon_k \leq 0)}\right] - \frac{\sigma_o^2}{b} . n$$

$$= \mu_o + \frac{\sigma_o^2}{b} \cdot \sum_{k=1}^{n} 2 \cdot P(\varepsilon_k \leq 0) - \frac{\sigma_o^2}{b} . n$$

$$= \mu_o + \frac{\sigma_o^2}{b} \cdot \sum_{k=1}^{n} 2 \cdot \frac{1}{2} - \frac{\sigma_o^2}{b} . n$$

$$= \mu_o$$

Then, by

$$\frac{\hat{\theta} - \mu_n}{a_n} \xrightarrow{p} 0$$

$$\Longrightarrow \frac{\hat{\theta}}{n} - \frac{\mu_o}{n} = \xrightarrow{p} 0$$

$$\frac{\mu_o}{n} \to 0 \;\Longrightarrow \frac{\mu_o}{n} \xrightarrow{p} 0 \; and \; \frac{\hat{\theta}}{n} - \frac{\mu_o}{n} + \frac{\mu_o}{n} \xrightarrow{p} 0 + 0$$

$$\Longrightarrow \frac{\hat{\theta}}{n} \xrightarrow{p} 0 \qquad \blacksquare$$

Furthermore, if we let $X_n = \frac{\hat{\theta}}{n}$ then $X_n \xrightarrow{d} 0 \;\; \Longrightarrow \lim_{n\to\infty} F_n(x) = \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad x \geq 0 \end{cases}$ $\blacksquare$

5.5.2

$E\big[MSE\big(\hat{\boldsymbol{\beta}}\big)\big]$ for D-JN Θ ⊥ B case.

In this part we will check the consistency of $\hat{\beta}$.

Recall

$$\hat{\beta} = \mu_1 + \frac{\sigma_1^2}{b} \cdot \sum_{k=1}^{n} \left[t_k \cdot \mathbb{1}_{(\varepsilon_k \leq 0)} - t_k \cdot \mathbb{1}_{(\varepsilon_k > 0)}\right]$$

$$Var\big(\hat{\beta}\big) = 4 \cdot \frac{\sigma_1^4}{b^2} \cdot Var\left(\sum_{k=1}^{n} t_k \cdot \mathbb{1}_{(\varepsilon_k \leq 0)}\right)$$

$but\ \varepsilon_k\ are\ iid\ Laplace\ (0, b)$

$$\Longrightarrow Var\big(\hat{\beta}\big) = 4 \cdot \frac{\sigma_1^4}{b^2} \cdot \sum_{k=1}^{n} t_k^2 \cdot Var\big(\mathbb{1}_{(\varepsilon_k \leq 0)}\big)$$

$$= 4 \cdot \frac{\sigma_1^4}{b^2} \cdot \sum_{k=1}^{n} t_k^2 \cdot \left\{E\left[\mathbb{1}_{(\varepsilon_k \leq 0)}\right] - E\left[\mathbb{1}_{(\varepsilon_k \leq 0)}\right]^2\right\}$$

$$= 4 \cdot \frac{\sigma_1^4}{b^2} \cdot \left\{\sum_{k=1}^{n} t_k^2 \cdot [P(\varepsilon_k \leq 0) - P(\varepsilon_k \leq 0)^2]\right\}$$

$$= 4 \cdot \frac{\sigma_1^4}{b^2} \cdot \sum_{k=1}^{n} t_k^2 \cdot \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\sigma_1^4}{b^2} \cdot \sum_{k=1}^{n} t_k^2$$

$\Longrightarrow Var(\hat{\beta}) \longrightarrow \infty \; as \; n \longrightarrow \infty \; Var(\hat{\beta})$ does not converge in quadratic mean

Theorem 5.5.1.2

Let $V(\hat{\beta}) = 4 \cdot \frac{\sigma_1^4}{b^2} \cdot Var\left(\sum_{k=1}^{n} t_k \cdot \mathbb{1}_{(\varepsilon_k \le 0)}\right)$ take $a_n = O(n^2)$, by theorem 5.4.1 we have:

$$\frac{\hat{\beta}}{n^2} \xrightarrow{P} 0$$

Furthermore, let

$$X_n = \frac{\hat{\beta}}{n^2}$$

$$then \;\; X_n \xrightarrow{d} 0 \;\; \Longrightarrow \lim_{n \to \infty} F_n(x) = \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad x \ge 0 \end{cases}$$

Proof:

Take $a_n = O(n^2)$

$$Var\left(\frac{\hat{\beta}}{n^2}\right) = \frac{1}{n^4} \cdot Var(\hat{\beta}) = \frac{1}{n^4} \cdot \frac{\sigma_1^4}{b^2} \cdot \sum_{k=1}^{n} t_k^2 \to 0 \; as \; n \to \infty$$

$$\mu_n = \mu_1 + \frac{\sigma_1^2}{b} \cdot \sum_{k=1}^{n} t_k \cdot E\left[\mathbb{1}_{(\varepsilon_k \le 0)} - \mathbb{1}_{(\varepsilon_k > 0)}\right]$$

$$= \mu_1 + \frac{\sigma_1^2}{b} \cdot \sum_{k=1}^{n} 2 \cdot E\left[\mathbb{1}_{(\varepsilon_k \le 0)}\right] - \frac{\sigma_1^2}{b} . \sum_{k=1}^{n} t_k$$

$$= \mu_1 + \frac{\sigma_o^2}{b} \cdot \sum_{k=1}^{n} 2 \cdot t_k \cdot P(\varepsilon_k \le 0) - \frac{\sigma_1^2}{b} . \sum_{k=1}^{n} t_k$$

$$= \mu_1 + \frac{\sigma_1^2}{b} \cdot \sum_{k=1}^{n} 2 \cdot \frac{1}{2} \cdot t_k - \frac{\sigma_1^2}{b} . \sum_{k=1}^{n} t_k = \mu_1$$

Then, by

$$\frac{\hat{\beta} - \mu_n}{a_n} \xrightarrow{p} 0$$

$$\Longrightarrow \frac{\hat{\beta}}{n^2} - \frac{\mu_1}{n^2} = \xrightarrow{p} 0$$

$$\frac{\mu_1}{n^2} \to 0 \ \Longrightarrow \frac{\mu_1}{n^2} \xrightarrow{p} 0 \ and \ \frac{\hat{\beta}}{n^2} - \frac{\mu_1}{n^2} + \frac{\mu_1}{n^2} \xrightarrow{p} 0 + 0$$

$$\Longrightarrow \frac{\hat{\beta}}{n^2} \xrightarrow{p} 0 \qquad \blacksquare$$

Furthermore, if we let $X_n = \frac{\hat{\beta}}{n^2}$ then $X_n \xrightarrow{d} 0 \ \Longrightarrow \lim_{n\to\infty} F_n(x) = \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad x \geq 0 \end{cases}$ $\blacksquare$

5.6

For the Laplace-Bivariate Normal distribution case with Θ, B dependent, the posterior mean estimator

$$\hat{\theta} = \frac{\sigma_o^2 \sigma_1}{b}\left[2\sum_{k=1}^{n} \mathbb{1}_{(\varepsilon_k \geq 0)} - n\right] + \mu_o + \frac{\sigma_o \sigma_1 \rho_o}{b}\left[2\sum_{k=1}^{n} t_k \mathbb{1}_{(\varepsilon_k \geq 0)} - \sum_{k=1}^{n} t_k\right]$$

$$\hat{\beta} = \frac{{\sigma_1}^2}{b}\left[2\sum_{k=1}^{n} t_k \mathbb{1}_{(\varepsilon_k \geq 0)} - \sum_{k=1}^{n} t_k\right] + \mu_1 + \frac{\sigma_o \sigma_1^2 \rho_o}{b}\left[2\sum_{k=1}^{n} \mathbb{1}_{(\varepsilon_k \geq 0)} - n\right]$$

we will check consistency using mean squared convergence. First, we need to find $V(\hat{\theta})$ and $V(\hat{\beta})$ then take the limit as n tends to infinity. We denote these $E[MSE(\hat{\theta})]$ and $E[MSE(\hat{\beta})]$ for D-BN Θ, B dependent case.

5.6.1

$E[MSE(\hat{\theta})]$ for D-BN Θ, B dependent case.

In this part we will show for divergence in quadratic mean

$$Var(\hat{\theta}) = \left[\frac{\sigma_o^4 \sigma_1^2}{b^2} \cdot n + \frac{\sigma_o^2 \sigma_1^2 \rho_o^2}{b^2} \cdot \sum_{k=1}^{n} t_k^2\right] \to \infty \ as \ n \to \infty$$

Theorem 5.6.1.1

Let $Var(\hat{\theta}) = \left[\frac{\sigma_o^4\sigma_1^2}{b^2} \cdot n + \frac{\sigma_o^2\sigma_1^2\rho_o^2}{b^2} \cdot \sum_{k=1}^{n} t_k^2\right]$ and take $a_n = O(n)$, by theorem 5.4.1 we have:

$$\frac{\hat{\theta}}{n} \xrightarrow{P} 0$$

Furthermore, let

$$X_n = \frac{\hat{\theta}}{n}$$

$$then\ \ X_n \xrightarrow{d} 0 \ \ \Longrightarrow \lim_{n\to\infty} F_n(x) = \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad x \geq 0 \end{cases}$$

Proof:

Take $a_n = O(n^2)$

$$Var\left(\frac{\hat{\theta}}{n^2}\right) = \frac{1}{n^4} \cdot Var(\hat{\theta}) = \frac{1}{n^4} \cdot \frac{\sigma_o^4\sigma_1^2}{b^2} \cdot n + \frac{1}{n^4} \cdot \frac{\sigma_o^2\sigma_1^2\rho_o^2}{b^2} \cdot \sum_{k=1}^{n} t_k^2 \to 0\ as\ n \to \infty$$

$\mu_n = \mu_o$

Then, by

$$\frac{\hat{\theta} - \mu_n}{a_n} \xrightarrow{p} 0$$

$$\Longrightarrow \frac{\hat{\theta}}{n^2} - \frac{\mu_o}{n^2} = \xrightarrow{p} 0$$

$$\frac{\mu_o}{n^2} \to 0 \ \Longrightarrow \frac{\mu_o}{n^2} \xrightarrow{p} 0\ and\ \frac{\hat{\theta}}{n^2} - \frac{\mu_o}{n^2} + \frac{\mu_o}{n^2} \xrightarrow{p} 0 + 0$$

$$\Longrightarrow \frac{\hat{\theta}}{n^2} \xrightarrow{p} 0 \qquad \blacksquare$$

Furthermore, if we let $X_n = \frac{\hat{\theta}}{n^2}$ then $X_n \xrightarrow{d} 0 \ \Longrightarrow \lim_{n\to\infty} F_n(x) = \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad x \geq 0 \end{cases}$ ∎

5.6.2

$E[MSE(\hat{\beta})]$ for D-BN Θ, B dependent case.

In this part we will show for divergence in quadratic mean of the estimator $\hat{\beta}$

$$Var(\hat{\beta}) = \left[\frac{\sigma_o^2\sigma_1^4\rho_o^2}{b^2}\cdot n + \frac{\sigma_1^4}{b^2}\cdot\sum_{k=1}^{n} t_k^2\right] \to \infty \; as \; n \to \infty$$

Theorem 5.6.2.1

Let $Var(\hat{\beta}) = \left[\frac{\sigma_o^2\sigma_1^4\rho_o^2}{b^2}\cdot n + \frac{\sigma_1^4}{b^2}\cdot\sum_{k=1}^{n} t_k^2\right]$ take $a_n = O(n^2)$, by theorem 5.4.1 we have:

$$\frac{\hat{\beta}}{n^2} \xrightarrow{P} 0$$

Furthermore, let

$$X_n = \frac{\hat{\beta}}{n^2}$$

$$then \;\; X_n \xrightarrow{d} 0 \;\; \Longrightarrow \lim_{n\to\infty} F_n(x) = \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad x \geq 0 \end{cases}$$

Proof:

Take $a_n = O(n^2)$

$$Var\left(\frac{\hat{\beta}}{n^2}\right) = \frac{1}{n^4}\cdot Var(\hat{\beta}) = \frac{1}{n^4}\left[\frac{\sigma_o^2\sigma_1^4\rho_o^2}{b^2}\cdot n + \frac{\sigma_1^4}{b^2}\cdot\sum_{k=1}^{n} t_k^2\right] \to 0 \; as \; n \to \infty$$

$\mu_n = \mu_1$

Then, by

$$\frac{\hat{\theta} - \mu_n}{a_n} \xrightarrow{p} 0$$

$$\Longrightarrow \frac{\hat{\theta}}{n^2} - \frac{\mu_o}{n^2} = \xrightarrow{p} 0$$

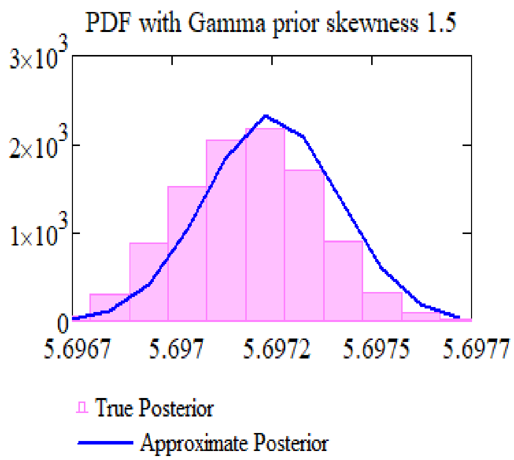
$$\frac{\mu_o}{n^2} \to 0 \implies \frac{\mu_o}{n^2} \xrightarrow{p} 0 \; and \; \frac{\hat{\beta}}{n^2} - \frac{\mu_1}{n^2} + \frac{\mu_1}{n^2} \xrightarrow{p} 0 + 0$$

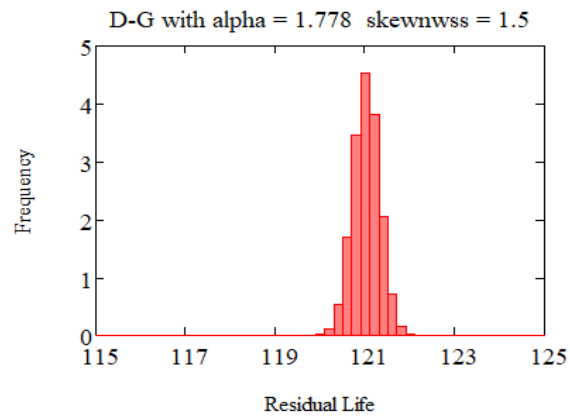
$$\implies \frac{\hat{\beta}}{n^2} \xrightarrow{p} 0 \qquad \blacksquare$$

Furthermore, if we let $X_n = \frac{\hat{\beta}}{n^2}$ then $X_n \xrightarrow{d} 0 \implies \lim_{n\to\infty} F_n(x) = \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad x \geq 0 \end{cases}$ ■

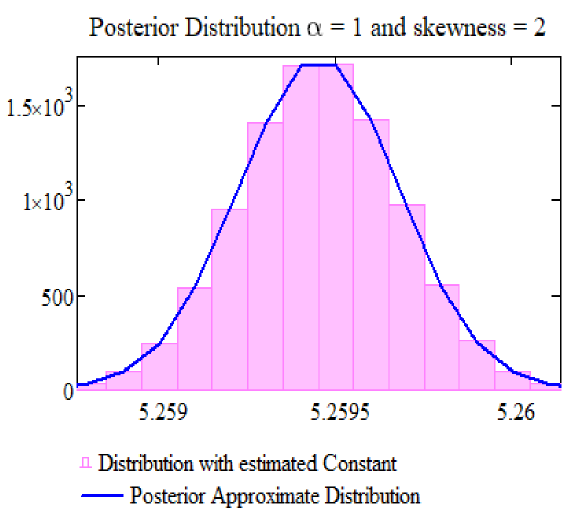
Chapter 6

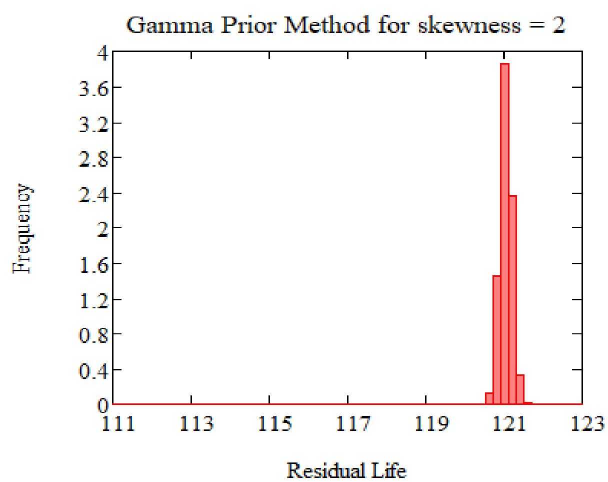
Results



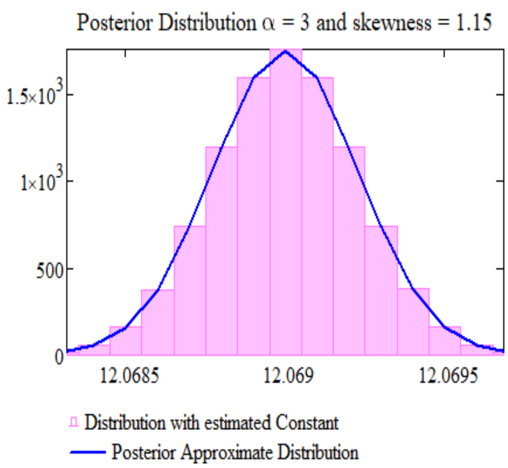


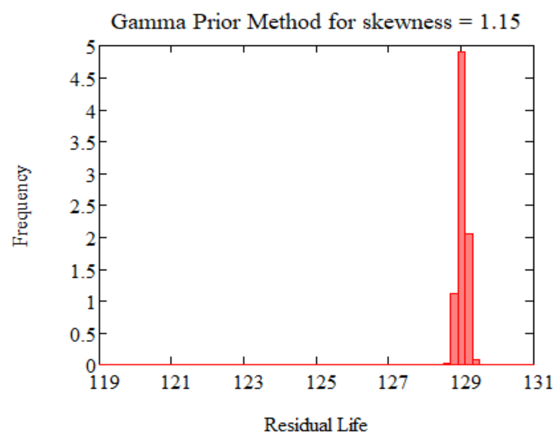
$$\gamma := 1.5 \quad \beta := 1.12 \quad b := 0.5 \quad \alpha := \left(\frac{2}{\gamma}\right)^2 = 1.778$$





$$\gamma := 2 \qquad \beta := 1.12 \quad \alpha := \left(\frac{2}{\gamma}\right)^2 = 1$$





$$\gamma := 1.15 \qquad \beta := 1.12 \qquad \alpha := \left(\frac{2}{\gamma}\right)^2 = 3.025$$

Comparison of variance of error and variance of the posterior N-G

Skewness	$\boldsymbol{\sigma_\varepsilon^2}$	$\boldsymbol{\sigma_p^2}$ **(Gebraeel)**	$\boldsymbol{\sigma_\varepsilon^2/\sigma_p^2}$ **(Gebraeel)**	$\boldsymbol{\sigma_p^2}$ **(Laplace)**	$\boldsymbol{\sigma_\varepsilon^2/\sigma_p^2}$ **(Laplace)**
5.16	0.16	1×10^{-3}	145.45	5.955×10^{-8}	2.687×10^{6}
3.65	0.18	8×10^{-4}	202.23	6.7×10^{-8}	2.687×10^{6}
2.39	0.20	9×10^{-4}	210.52	7.44×10^{-8}	2.687×10^{6}
2	0.15	8×10^{-4}	180.72	5.583×10^{-8}	2.687×10^{6}
1.63	0.17	8×10^{-4}	200.00	6.327×10^{-8}	2.687×10^{6}
1.26	0.19	1×10^{-3}	190.00	7.072×10^{-8}	2.687×10^{6}
1.15	0.15	8×10^{-4}	168.54	5.211×10^{-8}	2.687×10^{6}

Note that the ratio $\boldsymbol{\sigma_\varepsilon^2/\sigma_p^2}$ is always very high. Thus, we conclude that in these cases the effect of the posterior on the RL is very small in comparison to the error Gebraeel had on average 200 observations and we did the same.

Chapter 7

Conclusions and Future Work

7.1 Conclusion

This focus of this dissertation is to show that we can approximate the posterior distribution with normal distribution. This approach is applied to the distribution generated by Bayes formula, by multiplying the likelihood which contain degradation model with prior distribution. The intended degradation model houses the stochastic parameters which in turn describes the characteristics of a degrading component or a device. The main contribution of this method is to reduce dependence on heavy numerical computation.

Throughout this work we applied Laplace approximation to obtain the posterior distribution. Several types of prior distributions and two types of error distributions. This method is used on N-N, N-G, DE-G with a univariate degradation model, and N-JN, N-BN, DE-JN, DE-BN. After obtaining the asymptotic normal distribution of posterior, we tested consistency of the point estimate of each case. We used asymptotic mean square error (AMSE). In some cases, the AMSE converges to zero when n tends to infinity, hence our estimator is consistent, in other cases AMSE has the trivial upper bound, in this case we used the weaker type of consistency. Convergence in probability is used, we were able to show convergence to a constant in some cases to zero as n tends to infinity. Hence a degenerate distribution is deduced.

7.2 Future Work

Laplace approximation works well with unimodal distribution. For future research, it would be interesting to consider bimodal and multimodal posterior distribution, severely skewed prior distribution, and multivariate degradation model.

Chapter 8

Mathcad Codes

8.1 Simulation Codes for Double exponential error and Gamma prior

$\gamma := 1.1 \qquad \beta := 1.12 \qquad b := 0.5 \qquad \alpha := \left(\frac{2}{\gamma}\right)^2 = 3.306$

$\text{rgam}(m, \alpha, \beta) := \dfrac{\text{rgamma}(1, \alpha)}{\frac{1}{\beta}}$

$\text{tstart} := 0 \qquad \text{tend} := 300$

$K := \text{ceil}(\text{tend} - \text{tstart})$

$\theta := \text{rgam}(1, \alpha, \beta)_0$

$K = 300$

$i := 1..K$

$t_i := \text{tstart} + i$

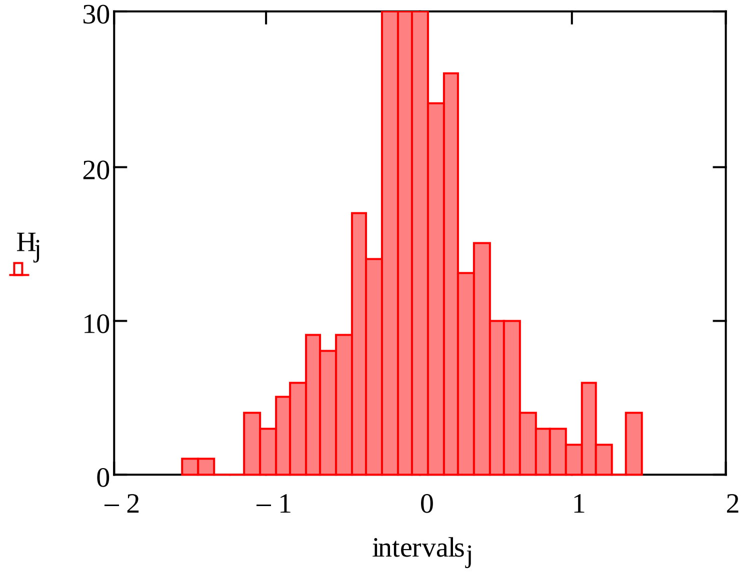
$r_i := \text{runif}(1, 0, 1)_0$

$$\varepsilon_i := \begin{cases} \log(2 \cdot r_i) & \text{if } r_i \le \frac{1}{2} \\ -\log(2 - 2 \cdot r_i) & \text{if } r_i > \frac{1}{2} \end{cases}$$

$j := 0..30$

$\text{intervals}_j := -1.5 + 0.1 \cdot j$

$H := \text{hist}(\text{intervals}, \varepsilon)$



Double Exponential distribution plot

$$S_i := \theta \cdot t_i + \varepsilon_i$$

$$a_i := \frac{S_i}{t_i}$$

$$M(x) := \sum_i \text{if}\left[x \le a_i, \frac{1}{b} \cdot \left(S_i - x \cdot t_i\right), 0\right]$$

$$N(x) := \sum_i \text{if}\left[x > a_i, \frac{1}{b} \cdot \left(S_i - x \cdot t_i\right), 0\right]$$

$$g(x, \alpha, \beta) := x^{\alpha - 1} \cdot e^{-\left(M(x) - N(x) + \frac{x}{\beta}\right)}$$

$$p(x,\alpha,\beta) := x^{\alpha-1} \cdot e^{\frac{-1}{b} \cdot \sum_i \left|S_i - x \cdot t_i\right| - \frac{x}{\beta}}$$

$$H(x) := \sum_i \text{if}\left(x \le a_i, \frac{-t_i}{b}, 0\right) \qquad T(x) := \sum_i \text{if}\left(x > a_i, \frac{-t_i}{b}, 0\right)$$

$$W(x) := \frac{\alpha - 1}{x} - H(x) + T(x) + \frac{1}{\beta}$$

$$W(x) := \frac{\alpha - 1}{x} - \left(\sum_i \text{if}\left(x \le a_i, \frac{-t_i}{b}, 0\right) - \sum_i \text{if}\left(x > a_i, \frac{-t_i}{b}, 0\right) + \frac{1}{\beta}\right)$$

$x := \alpha$

Given

$W(x) = 0$

$\mu_p := \text{Minerr}(x) = 4.474188361737699$

$r := \mu_p = 4.474188361737699$

$$v_i := \frac{a_{i-1} - a_i}{5K}$$

$h := \max(v) \qquad h = 2.23 \times 10^{-4}$

$$fcd(r,h) := \frac{W(r+h) - W(r-h)}{2 \cdot h}$$

$fcd(r,h) = -4 \times 10^{7}$

$$\sigma_p := \sqrt{\frac{-1}{fcd(r,h)}} = 0.00015811169235373$$

$\theta = 4.47405113653772$

$$m := \begin{cases} \mu_p - 3\sigma_p & \text{if } \mu_p - 3\sigma_p > 0 \\ 0 & \text{otherwise} \end{cases}$$

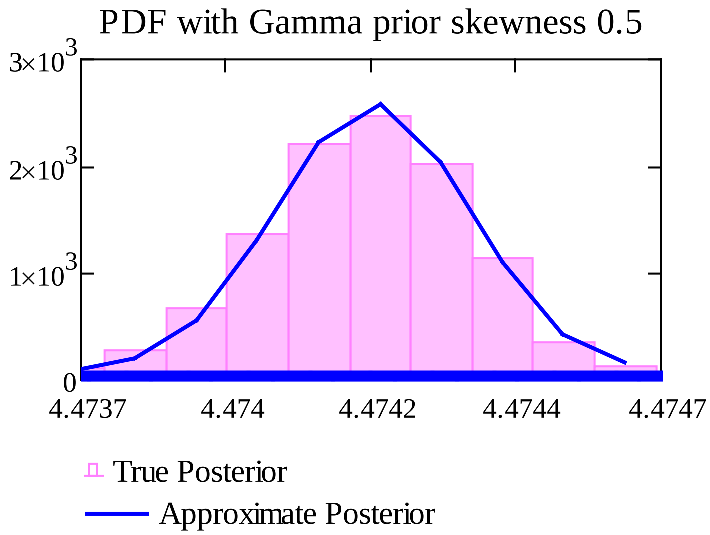
$$NC := \int_0^m x^{\alpha-1} \cdot e^{\frac{-1}{b} \cdot \sum_i |S_i - x \cdot t_i| - \frac{x}{\beta}} \, dx \ldots = 0$$
$$+ \int_m^{\mu_p + 3\cdot\sigma_p} x^{\alpha-1} \cdot e^{\frac{-1}{b} \cdot \sum_i |S_i - x \cdot t_i| - \frac{x}{\beta}} \, dx \ldots$$
$$+ \int_{\mu_p + 3\cdot\sigma_p}^{\infty} x^{\alpha-1} \cdot e^{\frac{-1}{b} \cdot \sum_i |S_i - x \cdot t_i| - \frac{x}{\beta}} \, dx$$

$\frac{p(\mu_p, \alpha, \beta)}{NC} = 2.406 \times 10^3$ $\quad$ $\text{dnorm}(\mu_p, \mu_p, \sigma_p) = 2.523 \times 10^3$

$$R := \frac{p(\mu_p, \alpha, \beta)}{\text{dnorm}(\mu_p, \mu_p, \sigma_p)} = 0$$

$NC = 0$

$x := 0, 0.0001 .. \mu_p + 3\cdot\sigma_p$



$t := 0 .. tend$

$t_k := 92$

$\mu_p = 4.474$

$\sigma_p^{\,2} = 2.5 \times 10^{-8}$

$D := 0.036$

$$g(t) := \frac{\mu_p \cdot (t - t_k) - D}{\sqrt{(t - t_k)^2 \cdot \sigma_p^{\,2} + 2 \cdot b^2}}$$

$g(0) = -582.05388881$

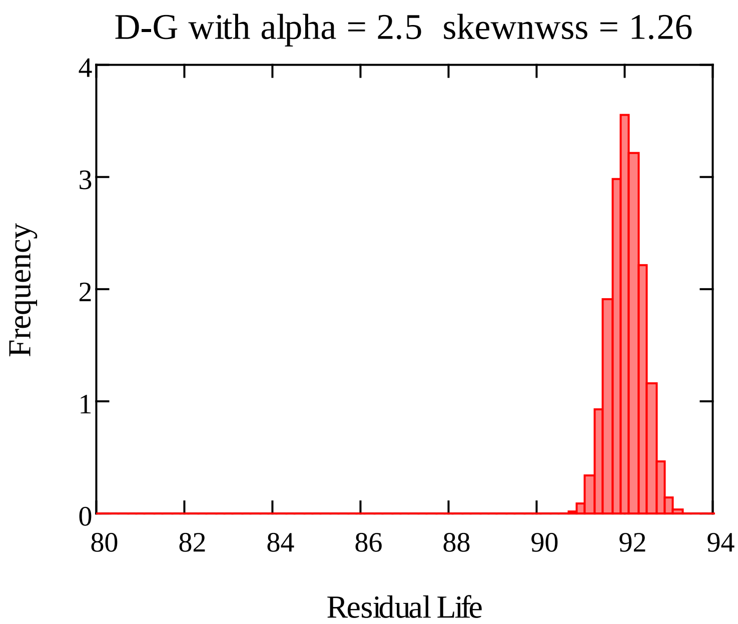
$t := 0 .. tend$

$$\frac{cnorm(g(t)) - cnorm(g(0))}{1 - cnorm(g(0))}$$

$t := 0, 0.1 .. tend$

$$f(t) := \frac{\mathrm{dnorm}\left[g(t), \mu_p\cdot(t-t_k), \sqrt{(t-t_k)^2\cdot\sigma_p^2 + 2\cdot b^2}\right]\cdot\left[\frac{\mu_p}{\sqrt{2\cdot b^2+\sigma_p^2\cdot(t-t_k)^2}} + \frac{\sigma_p^2\cdot(t-t_k)\cdot\left[D-\mu_p\cdot(t-t_k)\right]}{\left[2\cdot b^2+\sigma_p^2\cdot(t-t_k)^2\right]^{\frac{3}{2}}}\right]}{1-\mathrm{cnorm}(g(0))}$$

$t := 0, 0.2 .. \mathrm{tend}$



8.2 Simulation Codes for Gaussian error and Gamma prior

$\gamma := 1.26$ $\beta := 1.12$ $\mu := 0$ $\sigma := \sqrt{0.19}$ $\alpha := \left(\frac{2}{\gamma}\right)^2 = 2.52$

$\mathrm{tstart} := 0$ $\mathrm{tend} := 200$

$K := \mathrm{ceil}(\mathrm{tend} - \mathrm{tstart})$ $K = 200$

$\mathrm{rgam}(m, \alpha, \beta) := \frac{\mathrm{rgamma}(1, \alpha)}{\frac{1}{\beta}}$

$i := 1 .. K$

$t_i := \mathrm{tstart} + i$

$\varepsilon_i := \mathrm{rnorm}(1, \mu, \sigma)_0$

$\theta := \mathrm{rgam}(1, \alpha, \beta)_0$ $\mu_\theta := \alpha\cdot\beta = 2.822$

$\theta = 0.962$

$$S_i := \theta \cdot t_i + \varepsilon_i$$

$$g(x,\alpha,\beta) := x^{\alpha-1} \cdot e^{-\left[\frac{1}{2\cdot\sigma^2}\cdot\sum_{i=1}^{K}(S_i)^2\right]+\left[\left[\frac{1}{\sigma^2}\cdot\sum_{i=1}^{K}(S_i\cdot t_i)\right]-\frac{1}{\beta}\right]\cdot x-\left[\frac{1}{2\cdot\sigma^2}\cdot\sum_{i=1}^{K}(t_i)^2\cdot x^2\right]}$$

$$a := \left[\frac{1}{2\sigma^2}\cdot\sum_{i=1}^{K}(t_i)^2\right] \qquad b := \left[\frac{1}{\sigma^2}\cdot\sum_{i=1}^{K}(S_i\cdot t_i)-\frac{1}{\beta}\right] \qquad c := \alpha - 1$$

$$\mu_1 := \frac{-b-\sqrt{b^2+8\cdot a\cdot c}}{-4\cdot a} = 0.961$$

$$\mu_1 := \frac{-\left[\frac{1}{\sigma^2}\cdot\sum_{i=1}^{K}(S_i\cdot t_i)-\frac{1}{\beta}\right]-\sqrt{\left[\frac{1}{\sigma^2}\cdot\sum_{i=1}^{K}(S_i\cdot t_i)-\frac{1}{\beta}\right]^2+8\left[\frac{1}{2\sigma^2}\cdot\sum_{i=1}^{K}(t_i)^2\right]\cdot(\alpha-1)}}{-4\cdot\left[\frac{1}{2\sigma^2}\cdot\sum_{i=1}^{K}(t_i)^2\right]} = 0.961$$

$$\mu_2 := \frac{-b+\sqrt{b^2+8\cdot a\cdot c}}{-4\cdot a} = -1.118\times 10^{-7}$$

$$\mu_2 := \frac{-\left[\frac{1}{\sigma^2}\cdot\sum_{i=1}^{K}(S_i\cdot t_i)-\frac{1}{\beta}\right]+\sqrt{\left[\frac{1}{\sigma^2}\cdot\sum_{i=1}^{K}(S_i\cdot t_i)-\frac{1}{\beta}\right]^2+8\left[\frac{1}{2\sigma^2}\cdot\sum_{i=1}^{K}(t_i)^2\right]\cdot(\alpha-1)}}{-4\cdot\left[\frac{1}{2\sigma^2}\cdot\sum_{i=1}^{K}(t_i)^2\right]} = -1.118\times 10^{-7}$$

$$\sigma_1 := \sqrt{-\frac{1}{-2a - \frac{c}{\mu_1^{\,2}}}} = 2.659 \times 10^{-4} \qquad \sigma_2 := \sqrt{-\frac{1}{-2a - \frac{c}{\mu_2^{\,2}}}} = 9.071 \times 10^{-8}$$

$$f_1(x) := \text{dnorm}\left(x, \mu_1, \sigma_1\right)$$

$$EC := \frac{g\left(\mu_1, \alpha, \beta\right)}{f_1\left(\mu_1\right)} = 0$$

$$a := \begin{cases} \mu_1 - 3\sigma_1 & \text{if } \mu_1 - 3\sigma_1 > 0 \\ 0 & \text{otherwise} \end{cases}$$

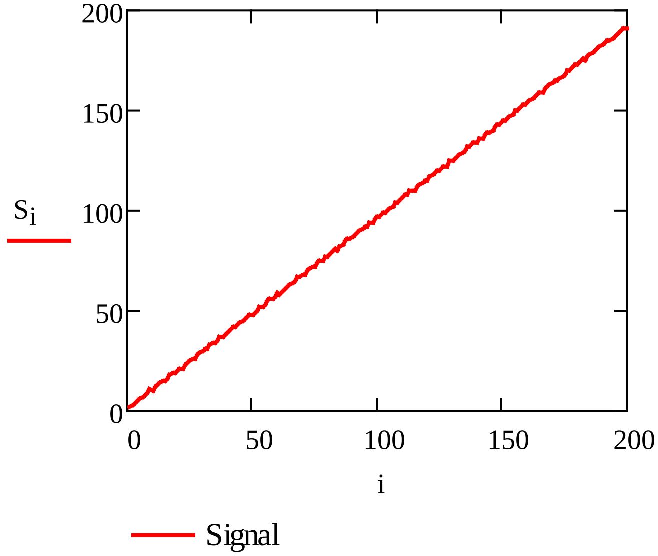
$$\mu_1 - 3\sigma_1 = 0.96$$

$$a = 0.96$$

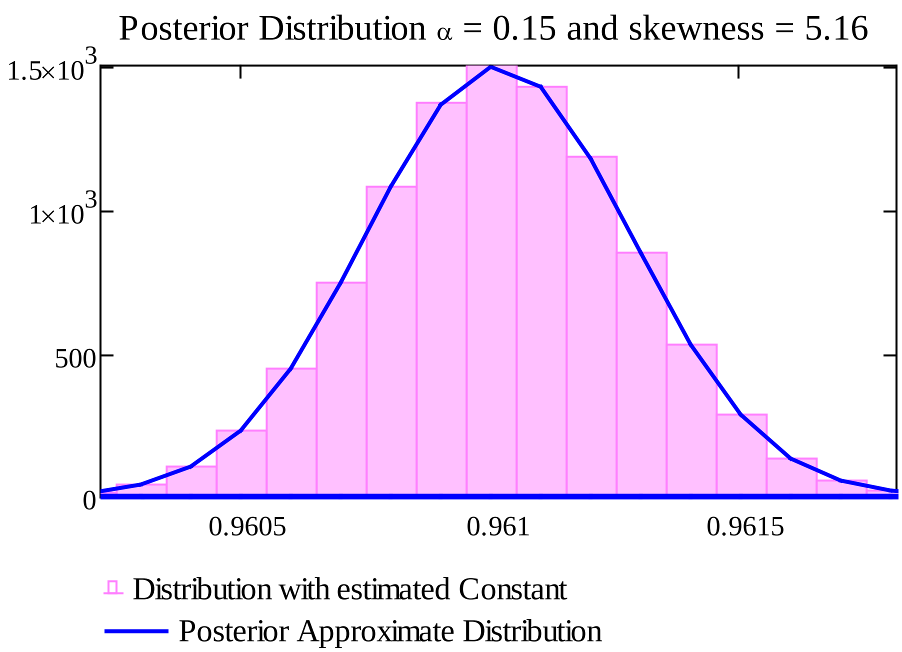
$$NC := \int_0^a x^{\alpha-1} \cdot e^{-\left[\frac{1}{2\cdot\sigma^2}\cdot\sum_{i=1}^{K}\left(S_i\right)^2\right] + \left[\left[\frac{1}{\sigma^2}\cdot\sum_{i=1}^{K}\left(S_i\cdot t_i\right)\right] - \frac{1}{\beta}\right]\cdot x - \left[\frac{1}{2\cdot\sigma^2}\cdot\sum_{i=1}^{K}\left(t_i\right)^2\cdot x^2\right]}\, dx \ldots$$

$$+ \int_a^{\mu_1+3\sigma_1} x^{\alpha-1} \cdot e^{-\left[\frac{1}{2\cdot\sigma^2}\cdot\sum_{i=1}^{K}\left(S_i\right)^2\right] + \left[\left[\frac{1}{\sigma^2}\cdot\sum_{i=1}^{K}\left(S_i\cdot t_i\right)\right] - \frac{1}{\beta}\right]\cdot x - \left[\frac{1}{2\cdot\sigma^2}\cdot\sum_{i=1}^{K}\left(t_i\right)^2\cdot x^2\right]}\, dx \ldots$$

$$+ \int_{\mu_1+3\sigma_1}^{\infty} x^{\alpha-1} \cdot e^{-\left[\frac{1}{2\cdot\sigma^2}\cdot\sum_{i=1}^{K}\left(S_i\right)^2\right] + \left[\left[\frac{1}{\sigma^2}\cdot\sum_{i=1}^{K}\left(S_i\cdot t_i\right)\right] - \frac{1}{\beta}\right]\cdot x - \left[\frac{1}{2\cdot\sigma^2}\cdot\sum_{i=1}^{K}\left(t_i\right)^2\cdot x^2\right]}\, dx$$



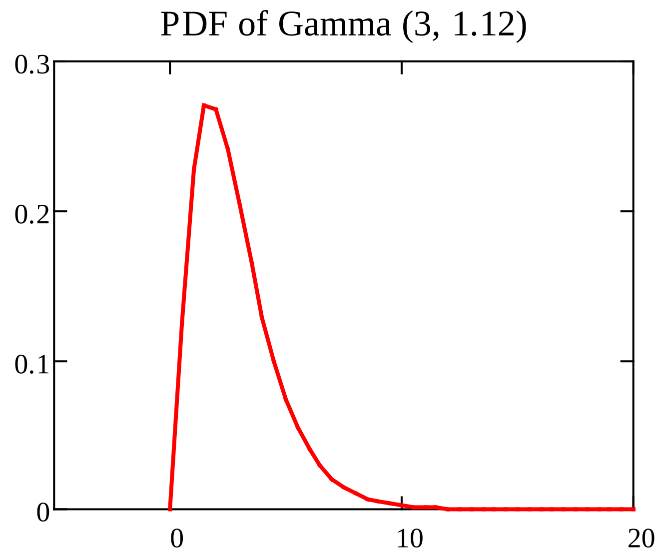
$x := 0, 0.0001 .. \theta + 1$



$$gam(s) := \frac{s^{\alpha-1}}{\Gamma(\alpha)\cdot\beta^{\alpha}}\cdot e^{-\frac{s}{\beta}}$$

$s := 0, 0.5 .. 50$

$(\alpha - 1)\cdot\beta = 1.702$



$t_k := 129$

$\mu_1 = 0.961$

$\sigma_1^{\,2} = 7.072 \times 10^{-8}$

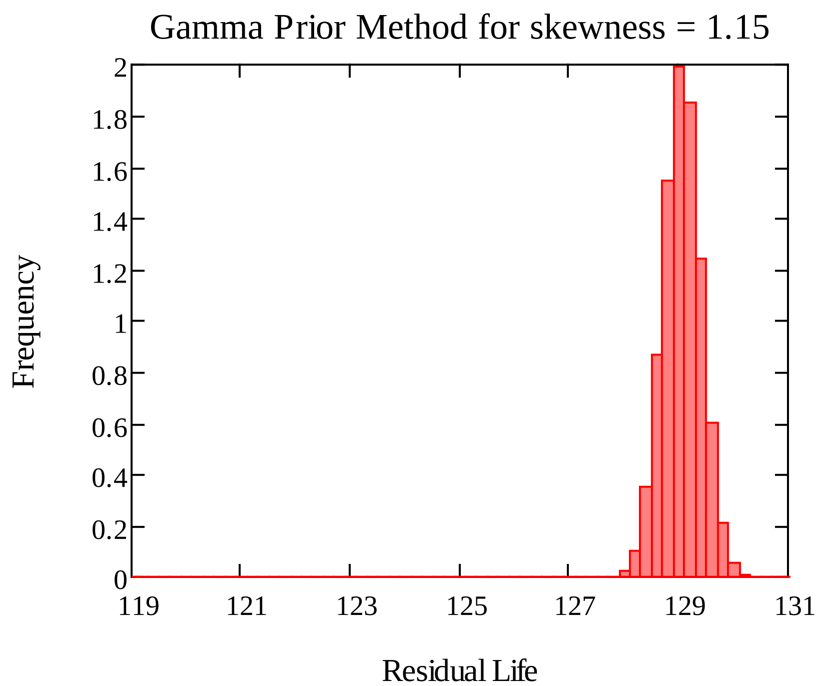
$\sigma^2 = 0.19$

$D := 0.03$

$$g(t) := \frac{\mu_1\cdot\left(t - t_k\right) - D}{\sqrt{\left(t - t_k\right)^2\cdot\sigma_1^{\,2} + \sigma^2}}$$

$t := 0, 0.2 .. tend$

$$f(t) := \frac{\mathrm{dnorm}\left[g(t), \mu_1\cdot\left(t - t_k\right), \sqrt{\left(t - t_k\right)^2\cdot\sigma_1^{\,2} + \sigma^2}\right]\cdot\left[\frac{\mu_1}{\sqrt{\sigma^2 + \sigma_1^{\,2}\cdot\left(t - t_k\right)^2}} + \frac{\sigma_1^{\,2}\cdot\left(t - t_k\right)\cdot\left[D - \mu_1\cdot\left(t - t_k\right)\right]}{\left[\sigma^2 + \sigma_1^{\,2}\cdot\left(t - t_k\right)^2\right]^{\frac{3}{2}}}\right]}{1 - \mathrm{cnorm}(g(0))}$$


Gamma Prior Method for skewness = 1.15
Frequency
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
119
121
123
125
127
129
131
Residual Life

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