REACHABLE SET COMPUTATION AND ANALYSIS FOR PERTURBED LINEAR SYSTEMS

by

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Dedicated to my Family.

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ABSTRACT

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Determination of the set of all possible states, which a system can attain, plays an important role in safety for critical application. Prior knowledge of this set for the complete run-time provides critical information about how a system may evolve, providing accurate information of all the states, which could violate constraints. The knowledge of these states, helps in estimating control input, which can control the system, such that, these states are eliminated from the reachable set.

Computation of reachable set of a dynamic system for a set of initial conditions can be easily performed, provided the analytical solution of the system for all initial conditions can be obtained. However, obtaining analytical solutions for nonlinear systems is a non-trivial task. Therefore, numerical methods are constructed, to obtain approximate solutions for these systems.

Owing to the recent advancements in computational technology, it is now possible to tackle nonlinear systems using numerical methods. The reduction in computational errors and the increase in the rate of computation have enhanced the quality of results obtained from discrete approximations of continuous systems. The iterative

property of these discrete approximations can be implemented in the form of algorithms. These algorithms, in turn, compute precise numerical solutions of systems for which analytical solutions are otherwise difficult to obtain.

The primary objective of this thesis is to formulate and construct algorithms to compute reachable sets for linear systems and extending these algorithms to compute reachable sets for linear systems with perturbations. The secondary objective is to apply and verify the algorithms on a real-world application, previously studied in the open literature, and to discuss the results obtained.

The computation of a reachable set is carried out in MATLAB® and the computed reachable sets for representative mathematical models of dynamic systems are presented and different ideas of reachable states are discussed.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

Figure | Page

List of Algorithms

CHAPTER 1

INTRODUCTION

1.1 Motivation and Background

Mathematical modelling has been a significant and an important tool for progress in various fields. “Seven Bridges of Königsberg”, is one of the most notable examples of this. The problem, after being solved by *Leonhard Euler*, gave rise to *Graph Theory* and laid foundation for *Topology*. The problem also gives a very abstract idea about reachability, “Given, 7 bridges connecting 4 lands around the city of Königsberg in Prussia (“System”). Is there a way to start from any of the lands (“Initial Condition”) and go through each of the bridges once and only once?(“Reachability?”)”. With the recent progress in mathematical modelling and computational technology, analysis of various properties of system has become easier. These properties are related to the states of the system, more specifically ‘dynamic systems’. These states are primarily defined as points in a normed vector space. Knowledge of all the possible values of these states for various initial conditions and applied inputs can be used in effective utilization of the systems.

The *Reachable Set* is the set of all possible values a state can take, for a given dynamic system, subject to the constraints over control input and the states. On the other hand, *Positive Invariant Set* is the set of all possible states, which can be propagated without violating state constraints. Various physical systems are represented by mathematical models to acquire information of the states these physical systems can attain. A survey paper written by F. Blanchini [1], describes the extensive research, which has been carried out to compute various invariant sets. While, other survey paper written by Oded Maler [2], compiles the research which has been carried out in the field of reachable sets.

The theory of reachable sets is extensively used in the context of unmanned vehicle systems. One of the primary intentions of the study is *Collision Avoidance* ([3],[4],[5]) and *Obstacle Avoidance* [6] - where computing the Reachable Set is used for confirming safety. *Motion Planning* [7] - utilises computation of optimal reachable set for effective planning. *Interaction between multiple unmanned vehicles* ([8],[9],[10]) - building upon a network of reachable sets and performing tasks related to cooperative control. *Error Tracking* for robust control ([11],[12]) - computation of forward reachable set of errors are carried out to estimate the actual state with respect to desired state. However, the extension of this theory is being carried into other research fields, as well. *"Computing the Projected Reachable Set of Stochastic Biochemical Reaction Networks Modeled by Switched Affine Systems"*, by F. Parise, M. E. Valcher and J. Lygeros [13], and *"Predicting Voltage Instability of Power System via Hybrid System Reachability Analysis"*, by Y. Susuki and T. Hikihara [14] are two of the examples of such extensions of the theory of reachable set into different application areas.

Recently, the study of computing reachable sets has received a significant impetus. New procedures and methods are being developed by extending the notion of a reachable state to fit different schema. These methods are based on level set methods [15], stochastic approach to reachable sets [10], flow-pipe based reachable sets [16], polyhedral set based reachable sets ([17],[18],[19]) and ellipsoidal set methods ([20],[21]). In this thesis **polyhedral** and **ellipsoidal** based approaches will be looked into.

The progress is not only being carried out by building various methods, but also by refining the computation methods and reducing approximation errors. These researches primarily centralize around linear systems while focusing upon other aspects of reachable set computation. These aspects include truncation errors from approximating continuous system equations to discrete system equations, and to increase efficiency in the computation of a reachable set for the system ([22],[23],[24]). A brief discussion over the truncation errors will be carried out in this thesis.

1.2 Outline

This thesis will encapsulate multiple methods of computation of reachable sets and invariant sets. These methods of computation will be built for two types of sets, namely, *Polyhedral Sets* and *Ellipsoidal Sets*. The mathematical formulation of these sets will be done for the linear systems and the algorithms will be built based upon the mathematical formulation. The algorithms will then be extended over to perturbed systems. Consequently, the reachable set will be computed for various systems and comparison of algorithms will be made. A concise preview of each chapter is given below -

- **Chapter-2 Mathematical Preliminaries** - This chapter will primarily focus upon various definitions and theorems. Discussion will highlight the stability of discrete approximation of continuous systems.
- **Chapter-3 Problem Statements** - This chapter will comprise of a brief discussion of the problem statements. It will provide an outlook over the structure of this thesis.

- **Chapter-4 Solution Methodology** - Mathematical formulation of reachable set and invariant set will be done for linear systems. Algorithms will be provided to compute these sets. Conditions of controllability will be derived for perturbed systems. The algorithms will be extended to compute reachable and invariant sets for perturbed systems. Utilising these theories, computation of forward reachable sets for a multi-rotor problem will be carried out.
- **Chapter-5 Application, Results and Discussion** - Results for all the problems stated in Chapter-3 will be stated and observed. A brief discussion will be carried out tying up the observations.
- **Chapter-6 Summary and Conclusion** - A concise review of the results and methods will be carried out. Concluding remarks will be made.
- **Chapter-7 Future Work** - Prospective notions for extending the methods and procedure will be debated upon.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

The definitions and theorems which shall be used for the entirety of the thesis, are discussed in this chapter. This way the discussion of a specific topic will not divert into explaining the preliminaries.

The chapter is divided into four sections for clarity. The sections are shown below:

1. Functional Analysis Preliminaries - Definition of norms and other matrix properties will be discussed.
2. Set Terminology - Basic literature of sets will be discussed
3. Mathematical model of Dynamic Systems - Various dynamic model and method of discretization will be discussed.
4. Controllable, Reachable and Invariant Sets - All the definitions needed for computation of Reachable Sets will be discussed.

2.1 Functional Analysis Preliminaries

These definitions are common preliminaries in functional analysis and can be found in [25],[26]. Thus, the discussion will mostly be done on the topic which needs specific attention.

Definition 2.1. (*Normed metric spaces or Banach spaces*) Let $\boldsymbol{X}$ denote a linear space over $\mathbb{R}$ or $\mathbb{C}$. A *norm* in $\boldsymbol{X}$ is a real valued function: $\boldsymbol{X} \to \mathbb{R}$, denoted as $\|\mathbf{x}\|$, for every $\mathbf{x} \in \boldsymbol{X}$, with the following properties,

1. *Positivity*,

$$\|\mathbf{x}\| > 0 \;\; \text{if,} \;\; \mathbf{x} \neq 0 \; ; \; \|\mathbf{x}\| = 0 \;\; \text{if,} \;\; \mathbf{x} = 0 \tag{2.1}$$

2. *Homogenity*, for all $\alpha \in \mathbb{R}$,

$$\|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\| \tag{2.2}$$

3. *Subadditivity*,

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|, \; \forall \mathbf{x}, \mathbf{y} \in \boldsymbol{X} \tag{2.3}$$

Example 2.1. (*Norms*) For any vector, $\mathbf{x} \in \boldsymbol{X}$, following are some example of norms,

$$\begin{aligned}
\|\mathbf{x}\|_1 &= \sum_{i=1}^{n} |x_i| && \text{(1-norm)} \\
\|\mathbf{x}\|_2 &= \sqrt{\sum_{i=1}^{n} |x_i|^2} && \text{(Euclidean norm)} \\
\|\mathbf{x}\|_p &= \sqrt[p]{\sum_{i=1}^{n} |x_i|^p} && \text{(p-norm for $p > 1$)} \\
\|\mathbf{x}\|_\infty &= \max_{\{i=1,2,\cdots,n\}} |x_i| && \text{(infinity norm)}
\end{aligned}$$

Remark 2.1. In all the upcoming sections of this thesis, $\| \cdot \|_2$ will be denoted as, $\| \cdot \|$.

Definition 2.2. (*Induced Matrix Norms*) Let, $\boldsymbol{A} : \boldsymbol{X} \to \boldsymbol{Y}$ be a bounded linear map of one Banach space $\boldsymbol{X}$ into another Banach space $\boldsymbol{Y}$, defined as $\boldsymbol{A}(\mathbf{x}) = \mathbf{A}\mathbf{x}$. The induced matrix norm, $\|\mathbf{A}\|_p$, is defined as,

$$\|\mathbf{A}\|_p = \sup_{\|\mathbf{x}\| \neq 0} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p} \tag{2.4}$$

For all $\mathbf{x} \in \boldsymbol{X}$.

Alternately,

$$\|\mathbf{A}\|_p = \sup_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\|_p \tag{2.5}$$

Definition 2.3. (*Condition Number*) For any invertible matrix, $\mathbf{A} \in \mathbb{R}^{n\times n}$, the condition number, $\kappa(\mathbf{A})$, is defined as,

$$\kappa(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{A}^{-1}\|$$

Definition 2.4. (*Resolvent set and Spectrum*)

1. The *Resolvent set* of $\mathbf{A} : \boldsymbol{X} \to \boldsymbol{Y}$ consists of, $\lambda \in \mathbb{C}$ for which, $[\lambda\mathbf{I} - \mathbf{A}]$ is invertible. The resolvent set is denoted by, $\rho(\mathbf{A})$.
2. The *Spectrum* of $\mathbf{A} : \boldsymbol{X} \to \boldsymbol{Y}$ consists of, $\lambda \in \mathbb{C}$ for which, $[\lambda\mathbf{I} - \mathbf{A}]$ is not invertible. The spectrum is denoted by, $\sigma(\mathbf{A})$.

Theorem 2.1. (*Gelfand's Theorem*) Let, $\mathbf{A} : \boldsymbol{X} \to \boldsymbol{Y}$ be a bounded linear map of one Banach space $\boldsymbol{X}$ into another Banach space $\boldsymbol{Y}$. Then,

1. The spectrum $\sigma(\mathbf{A})$ is a closed, bounded, non-empty set in $\mathbb{C}$.
2. The *Spectral Radius* of $\mathbf{A}$, denoted as $|\sigma(\mathbf{A})|$, is defined as,

$$|\sigma(\mathbf{A})| = \max_{\lambda\in\sigma(\mathbf{A})} |\lambda| \tag{2.6}$$

Thus,

$$|\sigma(\mathbf{A})| = \lim_{n\to\infty} \|\mathbf{A}^n\|^{1/n} \tag{2.7}$$

Proof of this theorem can be found in [25].

Theorem 2.2. (*Gronwall's Inequality*) Let $\{x_n\}_{n=0}^{\infty}$, $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be sequence of real numbers with $b_n \geq 0$, which satisfy,

$$x_n \leq a_n + \sum_{j=n_0}^{n-1} b_j x_j, \quad n = n_0, n_0 + 1, \cdots \tag{2.8}$$

For any integer $N > n_0$, let $S(n_0, N) = \{k$ where $x_k(\prod_{j=n_0}^{k-1}(1+b_j))^{-1}$ is maximized in $\{n_0, \cdots, N\}$.
Then, for any $\theta \in S(n_0, N)$,

$$x_n \leq a_\theta \prod_{j=n_0}^{k-1}(1 + b_j) \tag{2.9}$$

In particular,

$$x_n \leq \min\{a_\theta : \theta \in S(n_0, N)\} \prod_{j=n_0}^{k-1}(1 + b_j) \tag{2.10}$$

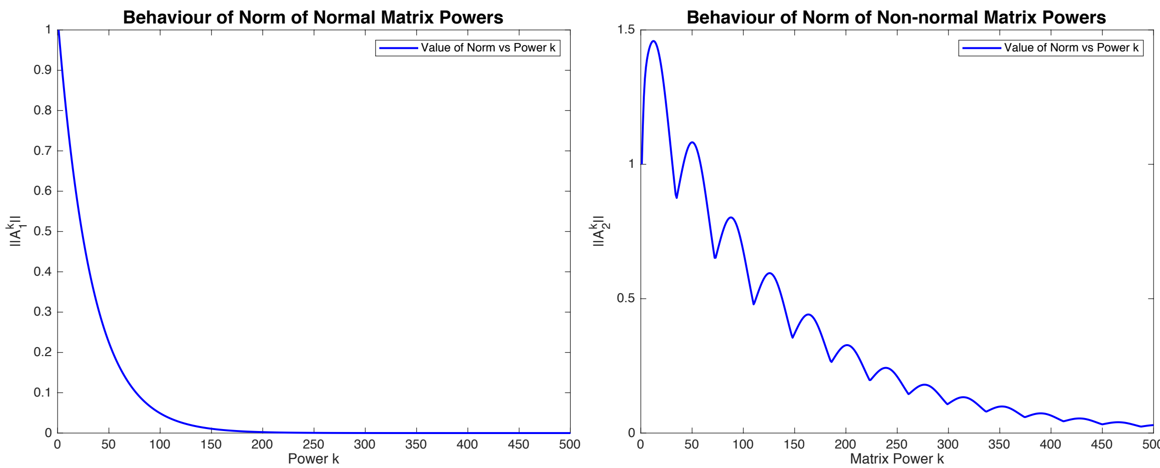
Proof of this theorem can be found in [27].

2.1.1 Boundedness of Norm of Matrix Powers

The discrete systems are iterative mappings of vector $\mathbf{x}_0$ ($\mathbf{x}_0 \in \boldsymbol{X}$). The mapping function $\mathbf{\Phi}$, also known as *State Transition Matrix* (discussed later), is used to map $\mathbf{x}_0$ to another vector $\mathbf{x}_1$. The vector $\mathbf{x}_1$ is again mapped to $\mathbf{x}_2$ using the same matrix $\mathbf{\Phi}$. Iterating this k times implies, that vector $\mathbf{x}_0$ was mapped to $\mathbf{x}_k$ using the matrix, $\mathbf{\Phi}^k$. Thus, to assure that vector $\mathbf{x}_k$ remains bounded, $\|\mathbf{\Phi}^k\|$ is required to be bounded.
(See [28] for more details.)
The topics which shall be discussed are - behaviour of norm of non-normal matrix powers, and the upper bound of norm of matrix powers.
Behaviour of Norm of Non-Normal Matrix Powers - To study this topic, two matrices will be considered and the behaviour of their norms will be compared. The matrices that are considered are shown below,

$$\mathbf{A}_1 = \begin{bmatrix} 0.97 & 0 & 0 \\ 0 & 0.85 & 0 \\ 0 & 0 & 0.6 \end{bmatrix} \text{ and } \mathbf{A}_2 = \begin{bmatrix} 1 & -0.1 & 0 \\ 0 & 1 & -0.1 \\ -0.4 & 0.2 & 0.4 \end{bmatrix} \tag{2.11}$$

The behaviour of norm for $\|\mathbf{A}_1^k\|$ and $\|\mathbf{A}_2^k\|$ show a huge difference. The difference being, $\|\mathbf{A}_2^k\|$ shows an oscillating nature, while, $\|\mathbf{A}_1^k\|$ has a strict decreasing nature.



(a) Norm of matrix $\mathbf{A}_1$ - $\|\mathbf{A}_1^k\|$ (b) Norm of matrix $\mathbf{A}_2$ - $\|\mathbf{A}_2^k\|$

Figure 2.1: Behaviour of Norm of Matrix Powers

This implies that the usage of $\|\mathbf{A}^k\| \leq \|\mathbf{A}\|^k$, becomes useless for non-normal matrices and **another method to upper bound these matrices** is necessary.

Remark 2.2. The matrix $\mathbf{A}_2$ is chosen in such a manner, because most of the matrices that will be dealt with are of this kind. For example, Euler 1-step discretization obtained from a continuous dynamical system,

$$\mathbf{A}_2 = \mathbf{I}_3 + \mathbf{A}_{\boldsymbol{\Delta}}\Delta t$$

where,

$$\mathbf{A}_{\boldsymbol{\Delta}}\Delta t = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -4 & 2 & -6 \end{bmatrix} 0.1$$

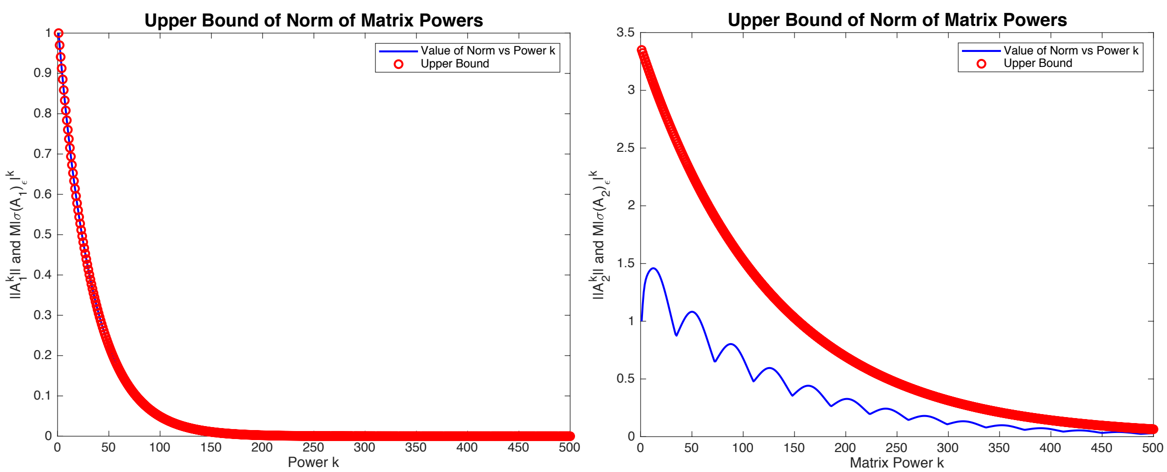
$\mathbf{A}_{\boldsymbol{\Delta}}$ are the type of matrices a normal system produces.

Upper Bound of Norm of Matrix Powers - Using Gelfand's Theorem, it can be asserted that, norm of matrix powers tend to 0 if spectral radius is less than 1. However, a substantial amount of research has been performed to study ϵ-pseudospectra ([29],[30]) of the matrices. The conclusion being, these matrices can be upper bounded in the following manner,

$$\|\mathbf{A}^k\| \leq M|\sigma(\mathbf{A})_\epsilon|^k \tag{2.12}$$

where, $M \geq 1$ and $|\sigma(\mathbf{A})_\epsilon| = |\sigma(\mathbf{A})| + \epsilon$ where $\epsilon > 0$, and $\sigma(\mathbf{A})_\epsilon$ belongs to ϵ-pseudospectra of $\mathbf{A}$.

The upper bounds of the same matrices mentioned in eqn.(2.11) are shown below,



(a) Upper Bound of - $\|\mathbf{A}_1^k\|$ (b) Upper Bound of - $\|\mathbf{A}_2^k\|$

Figure 2.2: Upper Bound of Norm of Matrix Powers

2.2 Set Terminology

All the definitions provided in this section are derived from ([31],[32]).

Definition 2.5. (*Convex Set*) A set $\mathcal{S} \subseteq \mathbf{R}^n$ is said to be convex if for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{S}$,

$$\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in \mathcal{S}, \quad 0 \leq \alpha \leq 1 \tag{2.13}$$

Definition 2.6. (*Convex Hull*) For a given set, $\mathcal{S}$, the convex hull is defined as the smallest convex set containing $\mathcal{S}$.

2.2.1 Set Operations for Convex Sets

Definition 2.7. (*Set Scaling*) Let, $\mathcal{A} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then, the scaled set of $\mathcal{A}$ denoted by set, $\alpha\mathcal{A}$, is defined as,

$$\alpha\mathcal{A} = \{\mathbf{x} = \alpha\mathbf{a}; \quad \forall\mathbf{a} \in \mathcal{A}\} \tag{2.14}$$

Definition 2.8. (*Set Addition/Minkowski Sum*) Let, $\mathcal{A}$, $\mathcal{B} \in \mathbb{R}^n$. Then, the Minkowski sum be denoted by set, $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$, is defined as,

$$\mathcal{C} = \{\mathbf{x} = \mathbf{a} + \mathbf{b} : \quad \forall\mathbf{a} \in \mathcal{A},\ \mathbf{b} \in \mathcal{B}\} \tag{2.15}$$

Definition 2.9. (*Set Subtraction/Erosion*) Let, $\mathcal{A}$, $\mathcal{B} \in \mathbb{R}^n$. Then, the erosion of $\mathcal{A}$ w.r.t. $\mathcal{B}$ be denoted by set, $\mathcal{D} = \mathcal{A} \ominus \mathcal{B}$, is defined as,

$$\mathcal{D} = \{\mathbf{x} :\ \mathbf{x} + \mathbf{b} \in \mathcal{A}, \quad \forall\ \mathbf{b} \in \mathcal{B}\} \tag{2.16}$$

2.2.2 Set Types

The type of sets which will be used in further discussion are - $\mathcal{H}$-polytope, $\mathcal{V}$-polytope, Zonotope, and Ellipsoidal Sets.

Definition 2.10. (*Halfspace*) Let, $\mathbf{H} \in \mathbb{R}^{1\times n}$, and $h \in \mathbb{R}$, then a halfspace in $\mathbb{R}^n$ is a set $\mathcal{X}$, defined as,

$$\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : \quad \mathbf{H}\mathbf{x} \leq h\} \tag{2.17}$$

Definition 2.11. (*Polyhedral Set*) A polyhedral set is the intersection of finitely many halfspaces. Let, $\mathbf{H} \in \mathbb{R}^{m\times n}$ and $\mathbf{h} \in \mathbb{R}^m$, then a polyhedral set in $\mathbb{R}^n$ is a set $\mathcal{X}$, defined as,

$$\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : \quad \mathbf{H}\mathbf{x} \leq \mathbf{h}\} \tag{2.18}$$

Definition 2.12. (*Polytope*)A bounded polyhedral set is called polytope.

Remark 2.3. Polytopes are divided into two types: $\mathcal{H}$-polytope, which are polytope defined using halfspaces and $\mathcal{V}$-polytope, which are defined using vertices of a polytope.

Remark 2.4. The initial set of states $\mathbf{x}$ and inputs $\mathbf{u}$ will be chosen as polytopes, unless specified otherwise. For $\mathbf{x}$ the polytope set will be represented as $\mathcal{X}$ and for $\mathbf{u}$ the polytope set will be represented as $\mathcal{U}$.

Definition 2.13. (*Zonotope*) A zonotope is a set such that,

$$\mathcal{Z} = \{\mathbf{x} \in \mathbb{R}^n : \quad \mathbf{x} = \mathbf{c} + \sum_{i=1}^{p} \alpha^i \mathbf{g}^i, \quad -1 \leq \alpha^i \leq 1\} \tag{2.19}$$

with $\mathbf{c}, \mathbf{g}^1, \cdots, \mathbf{g}^p \in \mathbb{R}^n$. The order of the zonotope is defined as $\frac{p}{n}$ and the short notation is, $\mathcal{Z} = \{\mathbf{c}, \mathbf{g}^1, \cdots, \mathbf{g}^p\}$.[18, 19]

Definition 2.14. (*Ellipsoidal Set*) Let, $\mathbf{P} \in \mathbb{R}^{n\times n}$ be a positive definite matrix and $\mathbf{c} \in \mathbb{R}^n$ be the center, then the Ellipsoidal Set, denoted by ξ, is defined as,

$$\xi = \{\mathbf{x} \in \mathbb{R}^n : \quad (\mathbf{x} - \mathbf{c})^\intercal \mathbf{P} (\mathbf{x} - \mathbf{c}) \leq 1\} \tag{2.20}$$

2.3 Mathematical Model of Dynamic System

2.3.1 Linear Dynamic System

The mathematical model of a SIMO linear time invariant (LTI) system, defined in continuous time domain is,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \end{aligned} \tag{2.21}$$

where: $\mathbf{x}(t) \in \mathcal{X} \subseteq \mathbb{R}^n$, $u(t) \in \mathcal{U}$, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times 1}$; $\mathbf{y}(t) \in \mathbb{R}^p$, $\mathbf{C} \in \mathbb{R}^{n\times n}$, $\mathbf{D} \in \mathbb{R}^{p\times 1}$. Assuming that the entire state is accessible, we note, $\mathbf{C} = \mathbf{I}_n$ (i.e, $\mathbf{I}_n = \mathbf{I}_{n\times n}$) and $\mathbf{D} = \mathbf{0}_{p\times 1}$.

The mathematical model of corresponding linear time invariant system, defined in discrete time domain is,

$$\mathbf{x}_{k+1} = \mathbf{\Phi}\mathbf{x}_k + \mathbf{\Gamma}u_k \tag{2.22}$$

where, $\mathbf{x}_k \in \mathcal{X}$, $u_k \in \mathcal{U}$, $\mathbf{\Phi}_k \in \mathbb{R}^{n\times n}$, $\mathbf{\Gamma}_k \in \mathbb{R}^n$

2.3.2 State Transition Matrix

For the LTI system described in eqn.(2.21), with input $u(t) = 0$, and initial condition $\mathbf{x}_0 = \mathbf{x}(t_0) \in \mathcal{X}$, the map, $\mathbf{\Phi} : \mathcal{X} \times \mathbb{R}_+ \times R_+ \to \mathcal{X}$, maps $\mathbf{x}_0$ to $\mathbf{x}(t) \in \mathcal{X}$. The matrix $\mathbf{\Phi}(t, t_0)$ is defined as state transition matrix for dynamic system in continuous time domain.

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0 \tag{2.23}$$

Remark 2.5. The state transition matrix satisfies the following differential equation,

$$\dot{\mathbf{\Phi}}(t, t_0) = \mathbf{A}\mathbf{\Phi}(t, t_0) \quad ; \qquad \mathbf{\Phi}(t_0, t_0) = \mathbf{I}$$

For the linear system described in eqn.(2.22), with input $u_k = 0,\ \forall k \in \mathbb{N}$, the map, $\mathbf{\Phi} : \mathcal{X} \to \mathcal{X}$, maps $\mathbf{x}_k \in \mathcal{X}$ to $\mathbf{x}_{k+1} \in \mathcal{X}$,

$$\mathbf{x}_{k+1} = \mathbf{\Phi}\mathbf{x}_k \tag{2.24}$$

Definition 2.15. (*Kalman Controllability Criterion*) - An event $(t, \mathbf{x})$ is controllable iff it can be transferred to 0 in finite time by an appropriate choice of the input function $u(t)$.[33]

Further discussion, over these topics, is carried out in depth in ([34],[33]).

2.3.3 Perturbed Dynamic System

Mathematical model of a Non-Linear Dynamic System - Let, $\mathbf{f} : \mathcal{X} \times \mathcal{U} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{X}$, then the mathematical model of a dynamic system in continuous time is shown below,

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, u, t, t_0) \tag{2.25}$$

Mathematical model of Non-Linear Dynamic System Affine in Input - Let, $\mathbf{f} : \mathcal{X} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{X}$, and $\mathbf{g} \circ u : \mathcal{U} \to \mathcal{X}$, then the mathematical model of the non-linear dynamic system affine in input in continuous time domain is shown below,

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, t, t_0) + \mathbf{g}(\mathbf{x}, t, t_0)u(t) \tag{2.26}$$

Perturbed Linear Dynamic Systems - Let, $\mathbf{A} : \mathcal{X} \to \mathcal{X}$, $\mathbf{B} : \mathcal{U} \to \mathcal{X}$, and $\mathbf{f} : \mathcal{X} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{X}$, then the mathematical model of perturbed linear dynamic system in continuous time domain is shown below,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{f}(\mathbf{x}, t, t_0) \tag{2.27}$$

The function $\mathbf{f}(\mathbf{x}, t, t_0)$, which shall be taken into consideration will be *Lipschitz Continuous.*

Definition 2.16. (*Lipschitz Continuity*) The function, $\mathbf{f} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, is globally Lipschitz, if and only if there exists a piecewise continuous function $\ell : \mathbb{R} \to \mathbb{R}_+$, if for all $\mathbf{x_1}, \mathbf{x_2} \in \mathbb{R}^n$, $t \in \mathbb{R}$,

$$||\mathbf{f}(\mathbf{x_1}, t) - \mathbf{f}(\mathbf{x_2}, t)|| \leq \ell(t)||\mathbf{x_1} - \mathbf{x_2}|| \tag{2.28}$$

The types of functions that will be dealt with, are time independent. Thus, the definition shall be used appropriately with those functions. The function, $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$, is locally Lipschitz, iff there exists $\ell \in \mathbb{R}_+$, for all $\mathbf{x_1}, \mathbf{x_2} \in \mathbb{R}^n$,

$$||\mathbf{f}(\mathbf{x_1}) - \mathbf{f}(\mathbf{x_2})|| \leq \ell||\mathbf{x_1} - \mathbf{x_2}|| \tag{2.29}$$

The dynamic system equation for perturbed system which shall be used is shown below,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{f}(\mathbf{x}) \tag{2.30}$$

2.3.4 Euler 1-step Discretization

The systems that shall be discussed in this thesis are continuous systems, which shall be discretized using Euler 1-step discretization. For example, the system given in eqn.(2.30), will be discretized using time step Δt as,

$$\frac{\mathbf{x}_{k+1} - \mathbf{x}_k}{\Delta t} = \mathbf{A}\mathbf{x}_k + \mathbf{B}u_k + \mathbf{f}(\mathbf{x}_k) \implies \mathbf{x}_{k+1} = (\mathbf{I} + \mathbf{A}\Delta t)\mathbf{x}_k + \mathbf{B}\Delta t u_k + \mathbf{f}(\mathbf{x}_k)\Delta t$$

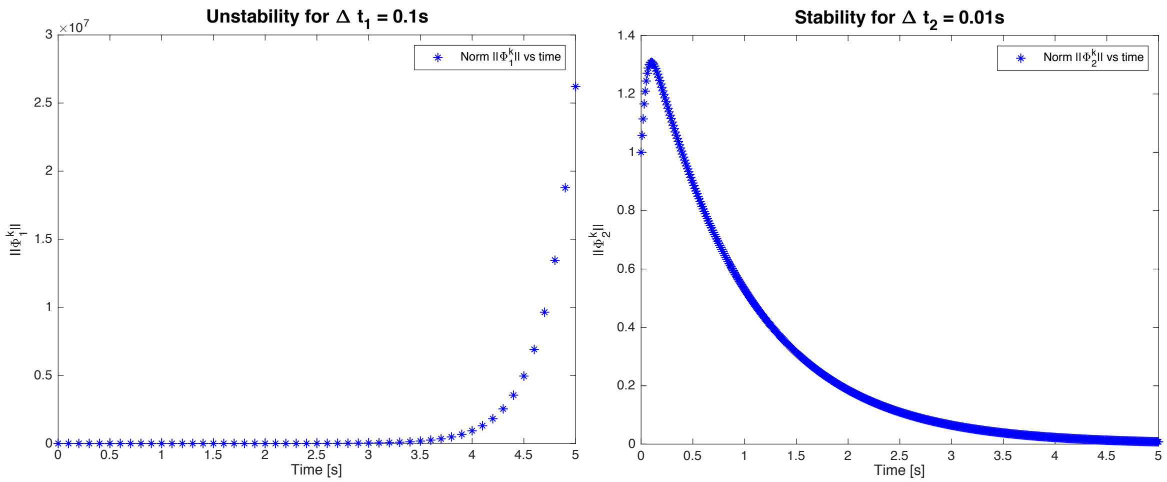
Additional discussion over this type of discretisation can be found in [35].

This discretization contains truncation errors, but, aside from truncation errors, the stability of the system is also affected. The choice of Δt affects the spectral radius of the discrete state transition matrix. This implies that, while discretizing, an appropriate time step needs to be chosen. For example, a stable system will be taken and discretized using $\Delta t_1 = 0.1s$ and $\Delta t_2 = 0.01s$ and simulated for total time $T = 5s$

and their norms will be compared over time. The norms give proof of existence of a unit vector which may go unbounded if mapped using the unstable discrete transition matrix. The continuous system matrix $\mathbf{A}$, shall be taken as,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -25 & -25 \end{bmatrix}$$

This is used to obtain, $\mathbf{\Phi}_1 = \mathbf{I}_2 + \mathbf{A}\Delta t_1$ and $\mathbf{\Phi}_2 = \mathbf{I}_2 + \mathbf{A}\Delta t_2$.



(a) Unstable for $\Delta t = 0.1s$

(b) Stable for $\Delta t = 0.01s$

Figure 2.3: Comparing Stability of Euler 1-step Discretized systems for different time step.

Thus, the choice over time step must be made so that, the stability of the system is not compromised Additional information on discretization affecting stability can be found in ([36],[37]).

2.4 Controllable, Reachable and Invariant Sets

The definitions mentioned in this section are adopted from [38],[32],[39].

Two types of systems, will be used for the following definitions:

1. Autonomous System -

$$\mathbf{x}_{k+1} = \mathbf{g}_a(\mathbf{x}_k) \tag{2.31}$$

2. Systems subject to external inputs -

$$\mathbf{x}_{k+1} = \mathbf{g}(\mathbf{x}_k, u_k) \tag{2.32}$$

Both the systems are subject to state and input constraints -

$$\mathbf{x}_k \in \mathcal{X} \ , \quad u_k \in \mathcal{U} \ , \quad \forall k \geq 0 \tag{2.33}$$

$\mathbf{x}_0 = \mathbf{x}(0)$ and $u_0 = u(0)$. The sets $\mathcal{X}$ and $\mathcal{U}$ are polyhedral sets. Let, $\mathcal{S} \subseteq \mathcal{X}$, be a set of initial states.

Definition 2.17. (*Precursor set for autonomous system*) For the autonomous system (2.31), the precursor set to the set $\mathcal{S}$ is denoted as,

$$\text{Pre}(\mathcal{S}) = \{\mathbf{x} \in \mathbf{R}^n \ : \ \mathbf{g}_a(\mathbf{x}) \in \mathcal{S}\}$$

Definition 2.18. (*Precursor Set*) For the system (2.32), the precursor set to the set $\mathcal{S}$ is denoted as,

$$\text{Pre}(\mathcal{S}) = \{\mathbf{x} \in \mathbf{R}^n \ : \ \exists u \in \mathcal{U} \ s.t. \ \mathbf{g}(\mathbf{x}, u) \in \mathcal{S}\}$$

Definition 2.19. (*Successor set for autonomous system*) For the autonomous system (2.31), the successor set from the set $\mathcal{S}$ is denoted as,

$$\text{Suc}(\mathcal{S}) = \{\mathbf{x} \in \mathbf{R}^n \ : \ \exists \mathbf{x}(0) \in \mathcal{S} \ s.t. \ \mathbf{x} = \mathbf{g}_a(\mathbf{x}(0))\}$$

Definition 2.20. (*Successor Set*) For the system (2.32), the successor set from the set $\mathcal{S}$ is denoted as,

$$\text{Suc}(\mathcal{S}) = \{\mathbf{x} \in \mathbf{R}^n \ : \ \exists \mathbf{x}(0) \in \mathcal{S}, \ \exists u(0) \in \mathcal{U} \ s.t. \ \mathbf{x} = \mathbf{g}(\mathbf{x}(0), u(0))\}$$

Remark 2.6. $\text{Pre}(\mathcal{S})$ and $\text{Suc}(\mathcal{S})$ are also called "One-step backward reachable set" and "One-step forward reachable set", respectively.

Definition 2.21. (*N-Step Controllable Set* $\mathcal{K}_N(\mathcal{S})$) For a given target set $\mathcal{S} \subseteq \mathcal{X}$, the N-step controllable set $\mathcal{K}_N(\mathcal{S})$ of the system (2.31), (2.32) subject to the constraints 2.33 is defined recursively as:

$$\mathcal{K}_j(\mathcal{S}) = \text{Pre}(\mathcal{K}_{j-1}(\mathcal{S})) \cap \mathcal{X}, \quad \mathcal{K}_0(\mathcal{S}) = \mathcal{S}, \qquad j \in \{1, \cdots, N\}$$

Definition 2.22. (*N-Step Reachable Set* $\mathcal{R}_N(\mathcal{X}_0)$) For a given initial set $\mathcal{X}_0 \subseteq \mathcal{X}$, the N-step reachable set $\mathcal{R}_N(\mathcal{X}_0)$ of the system (2.31), (2.32) subject to the constraints 2.33 is defined as:

$$\mathcal{R}_{i+1}(\mathcal{X}_0) = \text{Suc}(\mathcal{R}_i(\mathcal{X}_0)) \cap \mathcal{X}, \quad \mathcal{R}_0(\mathcal{X}_0) = \mathcal{X}_0, \qquad i = 0, 1, \cdots, N-1$$

Definition 2.23. (*Positive Invariant Set*) A set $\mathcal{O} \subseteq \mathcal{X}$ is said to be positive invariant set for the autonomous system (2.31) subject to constraints in (2.33), if

$$\mathbf{x}_k \in \mathcal{O} \;\Rightarrow\; \mathbf{g}_a(\mathbf{x}_k) \in \mathcal{O}, \quad \forall k \in \mathbb{N}_+.$$

Definition 2.24. (*Maximal Positive Invariant Set* $\mathcal{O}_\infty$) The set $\mathcal{O}_\infty \subseteq \mathcal{X}$ is the maximal positive invariant set of the autonomous system (2.31) subject to constraints in (2.33), if $\mathcal{O}_\infty$ is invariant and $\mathcal{O}_\infty$ contains all the invariant sets contained in $\mathcal{X}$.

Definition 2.25. (*Control Invariant Set*) A set $\mathcal{C} \subseteq \mathcal{X}$ is said to be control invariant set for the system (2.32) subject to constraints in (2.33), if

$$\mathbf{x}_k \in \mathcal{C} \;\Rightarrow\; \exists u_k \in \mathcal{U} \;\Rightarrow\; \mathbf{g}(\mathbf{x}_k, u_k) \in \mathcal{C}, \quad \forall k \in \mathbb{N}_+.$$

Definition 2.26. (*Maximal Control Invariant Set* $\mathcal{C}_\infty$) The set $\mathcal{C}_\infty \subseteq \mathcal{X}$ is the maximal control invariant set of the system (2.32) subject to constraints in (2.33), if $\mathcal{C}_\infty$ is control invariant and $\mathcal{C}_\infty$ contains all the control invariant sets contained in $\mathcal{X}$.

Two types of systems, will be used for the following definitions:

1. Autonomous System -

$$\mathbf{x}_{k+1} = \mathbf{g}_a(\mathbf{x}_k, \mathbf{w}_k) \tag{2.34}$$

2. Systems subject to external inputs -

$$\mathbf{x}_{k+1} = \mathbf{g}(\mathbf{x}_k, u_k, \mathbf{w}_k) \tag{2.35}$$

Both the systems are subject to disturbance $\mathbf{w}_k$ with state and input constraints -

$$\mathbf{x}_k \in \mathcal{X} \ , \quad u_k \in \mathcal{U} \ , \quad \mathbf{w}_k \in \mathcal{W} \ , \quad \forall k \geq 0 \tag{2.36}$$

$\mathbf{x}_0 = \mathbf{x}(0)$ and $u_0 = u(0)$. The sets $\mathcal{X}, \mathcal{U}$ and $\mathcal{W}$ are polyhedral sets.

Definition 2.27. (*Robust Positive Invariant Set*) A set $\mathcal{O} \subseteq \mathcal{X}$ is said to be robust positive invariant set for the autonomous system (2.34) subject to constraints in (2.36), if

$$\mathbf{x}_0 \in \mathcal{O} \ \Rightarrow \ \ \mathbf{x}_k \in \mathcal{O}, \ \ \forall \mathbf{w}_k \in \mathcal{W}, \ \ \forall k \in \mathbb{N}_+.$$

Definition 2.28. (*Maximal Robust Positive Invariant Set* $\mathcal{O}_\infty$) The set $\mathcal{O}_\infty \subseteq \mathcal{X}$ is the maximal robust positive invariant set of the autonomous system (2.34) subject to constraints in (2.36), if $\mathcal{O}_\infty$ is robust invariant and $\mathcal{O}_\infty$ contains all the robust invariant sets contained in $\mathcal{X}$.

Definition 2.29. (*Robust Control Invariant Set*) A set $\mathcal{C} \subseteq \mathcal{X}$ is said to be robust control invariant set for the system (2.35) subject to constraints in (2.36), if

$$\mathbf{x}_k \in \mathcal{C} \ \Rightarrow \ \exists u_k \in \mathcal{U} \ \Rightarrow \ \ \mathbf{g}(\mathbf{x}_k, u_k, \mathbf{w}_k) \in \mathcal{C}, \ \ \forall \mathbf{w}_k \in \mathcal{W}, \ \ \forall k \in \mathbb{N}_+.$$

Definition 2.30. (*Maximal Robust Control Invariant Set* $\mathcal{C}_\infty$) The set $\mathcal{C}_\infty \subseteq \mathcal{X}$ is the maximal robust control invariant set of the system (2.35) subject to constraints in (2.36), if $\mathcal{C}_\infty$ is robust control invariant and $\mathcal{C}_\infty$ contains all the robust control invariant sets contained in $\mathcal{X}$.

CHAPTER 3

PROBLEM STATEMENTS

Analysis of reachable set and control invariant set will be performed and a procedure to estimate these sets will be developed for various types of systems. The following topics will be discussed in the order provided below.

1. Maximal control invariant set for discrete system.
2. Reachable Set for LTI System.
3. Perturbed Linear System.
4. Perturbed Linear System with Compensation.

Each system will be used to build upon theory of reachable sets. Thus, the order will be maintained for all the subsequent chapters.

3.1 Maximal Control Invariant Set for Discrete system

For an unstable system, which is controllable, the maximal control invariant set will be determined. It will help in showing that, for an unstable system, given the state constraint and control input constraint, what the initial set should be such that, the states remain within the state constraints. This is done due to the constrained control input. Unconstrained control input has no problem over any controllable system. However once constrained, the states which can be controlled, require to be estimated.

For this section the problem has been derived from [38]. The reason for taking this problem is to begin the discussion over computation of invariant set and to extend this topic for perturbed system.

The system considered is in discrete state space form, and is shown below,

$$\mathbf{x}_{k+1} = \begin{bmatrix} 1.5 & 0 \\ 1 & -1.5 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k \tag{3.1}$$

This system is subject to state constraints ($\mathcal{X}$) and input constraints ($\mathcal{U}$), shown below,

$$\begin{aligned} \mathcal{X} &= \{\mathbf{x} \in \mathbb{R} : \quad \mathbf{H}_0 \mathbf{x} \leq \mathbf{h}_0\} \\ \mathcal{U} &= \{u \in \mathbb{R} : \quad \mathbf{H}_u u \leq \mathbf{h}_u\} \end{aligned} \tag{3.2}$$

where, for the state constraints,

$$\mathbf{H}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } \mathbf{h}_0 = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}$$

and, for the control input constraints,

$$\mathbf{H}_u = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{h}_u = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

3.2 Reachable Set for LTI System

The reachable set provides us the information of whether a given system under state and control input constraints, will ever be able to attain a desired state or a set of states. If the desired state or set of states is contained within the reachable set, those states are called reachable. Before proceeding with perturbed systems, the linear dynamic system shall be discussed. The procedure of building a reachable set will then be extended to perturbed systems. Furthermore, the maximal control invariant set will also be formulated so as to compare it with the maximal control invariant set of perturbed system.

The system discussed in this section is a Spring-Mass-Damper (SMD) system, with the system being stable. The state equation of this system is,

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -10x_1(t) - 5x_2(t) + u(t) \end{aligned} \tag{3.3}$$

Defining, $\mathbf{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^\intercal$, the eqn.(3.3) can be written in state space form as,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \tag{3.4}$$

where,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -10 & -5 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Now, for the given state and control input constraints, $\mathcal{X} \subset \mathbb{R}^2$ and $\mathcal{U} \subset \mathbb{R}$, the reachable set and control invariant set for system eqn.(3.4) is to be estimated.

The state constraint set $\mathcal{X}_0$ and control input constraint set $\mathcal{U}$ is as shown below,

$$\begin{aligned} \mathcal{X}_0 &= \{\mathbf{x} \in \mathbb{R} : \quad \mathbf{H}_0\mathbf{x} \leq \mathbf{h}_0\} \\ \mathcal{U} &= \{u \in \mathbb{R} : \quad \mathbf{H}_u u \leq \mathbf{h}_u\} \end{aligned} \tag{3.5}$$

where, for the state constraints,

$$\mathbf{H}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } \mathbf{h}_0 = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}$$

and, for the control input constraints,

$$\mathbf{H}_u = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{h}_u = \begin{bmatrix} u_{max} \\ u_{max} \end{bmatrix}$$

Various u_{max} will be chosen to obtain the result and will be mentioned in result section about the choice.

3.3 Perturbed Linear Dynamic System

The procedure to compute reachable set and control invariant set for perturbed system will be built upon. This will be done by extending the idea of reachable set and control invariant set from linear systems to perturbed systems.

The system under consideration, is the same Spring-Mass-Damper (SMD) system with additional perturbation $\ell\sin(x_2)$. The state equation of this system is shown below,

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -10x_1(t) - 5x_2(t) + u(t) + \ell\sin(x_2)\end{aligned} \tag{3.6}$$

Defining, $\mathbf{x}(t) = \begin{bmatrix} x_1(t) & x_2(t)\end{bmatrix}^\intercal$, $\mathbf{f}(\mathbf{x}) = \begin{bmatrix} 0 & \ell\sin(x_2)\end{bmatrix}^\intercal$, the eqn.(3.6) is written in state space form,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{f}(\mathbf{x}) \tag{3.7}$$

where,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -10 & -5 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The system (3.7) is discretized using Euler 1-step discretization, with time step Δt holding control input $u(k\Delta t) = u_k$ from time $k\Delta t \rightarrow (k+1)\Delta t$. This will be the only discretization scheme focused upon.

$$\frac{\mathbf{x}_{k+1} - \mathbf{x}_k}{\Delta t} = \mathbf{A}\mathbf{x}_k + \mathbf{B}u_k + \mathbf{f}(\mathbf{x}_k)$$

Thus, an approximate mathematical model of the perturbed system in discrete time is obtained.

$$\mathbf{x}_{k+1} = \mathbf{\Phi}\mathbf{x}_k + \mathbf{\Gamma}u_k + \mathbf{f}(\mathbf{x}_k)\Delta t \tag{3.8}$$

where, $\mathbf{\Phi} = (\mathbf{I} + \mathbf{A}\Delta t)$, and $\mathbf{\Gamma} = \mathbf{B}\Delta t$.

The function $\mathbf{f}(\mathbf{x}_k)$ is s.t. it is globally Lipschitz. Implying that,

$$\|\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2)\| \leq \ell\|\mathbf{x}_1 - \mathbf{x}_2\| \tag{3.9}$$

The coefficient of the perturbation $\sin(x_2)$, was chosen to be stated as ℓ for the same reason. The Lipschitz constant of $\sin(x)$ is 1, thus, $\ell\sin(x)$ will have Lipschitz constant ℓ.

3.4 Perturbed Linear Dynamic System with Compensation

The same theory shall now be extended to a practical problem - Multi-rotor System under First-order Aerodynamic Effects [12]. The aim is to estimate the outer approximated Forward Reachable Set (FRS) for the system. The purpose for estimating the outer approximated FRS rather than the exact reachable set, is to bypass handling the rotation matrix that appears in the perturbation. Furthermore, different sets will be used to compute and analyse the reachable sets.

For all the cases, prior to this, polytopes will be used, while in this case, ellipsoidal sets will be focused upon. Moreover, no constraints shall be considered for the states. The system formulation has already been done in few papers ([12],[40],[41],[42]). Thus, the formulation will not be discussed in depth, instead the system will be described directly. The system's dynamic equation is shown below -

$$\ddot{\mathbf{p}} = -ge_3 + \gamma\mathbf{z_b} - \hat{c}_d\mathbf{R\Pi R}^\intercal\dot{\mathbf{p}} - \tilde{c}_d\mathbf{R\Pi R}^\intercal\dot{\mathbf{p}} \tag{3.10}$$

Here, $\mathbf{p} \in \mathbb{R}^3$, is the position of the multi-rotor, $\gamma \in \mathbb{R}$, is the input, $e_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\intercal$, $\mathbf{z_b}$ is the multi-rotor body z-axis unit vector expressed in inertial frame, $\hat{c}_d \in \mathbb{R}$, is estimated coefficient of first-order aerodynamic effect, $\tilde{c}_d \in \mathbb{R}$, is the residual error while estimating $\hat{c}_d$, and other errors, $\mathbf{R} \in \mathbb{R}^{3\times3}$, is the rotation matrix of the multi-rotor and, $\mathbf{\Pi} = \mathbf{I}_3 - e_3 e_3^\intercal$.

Remark 3.1. The rotation dynamics is not mentioned, because the only property of rotation matrix, which will be utilised is - $\|\mathbf{R}\| = 1$. Furthermore, $\|\mathbf{\Pi}\| = 1$, proof of this is trivial. Thus, $\|\mathbf{R}\mathbf{\Pi}\mathbf{R}^\intercal\| \leq 1$.

Defining, $\mathbf{e_p}$ and $\mathbf{e_v}$ as the errors in position and velocity w.r.t. reference position, and velocity respectively. The reference trajectory position and velocity is denoted by, $\mathbf{p}_r$ and $\dot{\mathbf{p}}_r$, respectively. Therefore, $\mathbf{e_p} = \mathbf{p} - \mathbf{p}_r$ and $\mathbf{e_v} = \dot{\mathbf{p}} - \dot{\mathbf{p}}_r$.
The control input is designed by taking into account,

1. Feedback - $\mathbf{F}_{fb} = -\mathbf{K}_p \mathbf{e_p} - \mathbf{K}_v \mathbf{e_v}$.
2. Upward thrust - $g e_3$.
3. Reference acceleration - $\ddot{\mathbf{p}}_r$.
4. Compensation - $\hat{c}_d \dot{\mathbf{p}}$.

All these inputs are used to produce desired acceleration. The desired acceleration is used to compute the control input, γ, as normalised thrust and $\mathbf{z_b}^d$, as the desired thrust direction.

$$\begin{aligned} \ddot{\mathbf{p}}^d &= \mathbf{F}_{fb} + g e_3 + \ddot{\mathbf{p}}_r + \hat{c}_d \dot{\mathbf{p}} \\ \mathbf{z_b}^d &= \frac{\ddot{\mathbf{p}}^d}{\|\ddot{\mathbf{p}}^d\|} \\ \gamma &= (\ddot{\mathbf{p}}^d - \hat{c}_d \dot{\mathbf{p}})^\intercal \mathbf{z_b} \end{aligned} \tag{3.11}$$

Using aforementioned equations (3.10, 3.11), the equation for position dynamics is converted into set of equations of error dynamics. The position of multi-rotor is now defined with the help of error around the reference trajectory. A set of initial errors is considered and the forward reachable set of those initial set of errors will be determined. These sets determine where the multi-rotor will be, around the reference position. The dynamics in error form is shown below,

$$\begin{aligned} \dot{\mathbf{e}}_{\mathbf{p}} &= \mathbf{I}_3\mathbf{e}_{\mathbf{v}} \\ \dot{\mathbf{e}}_{\mathbf{v}} &= -\mathbf{K}_p\mathbf{e}_{\mathbf{p}} - \mathbf{K}_v\mathbf{e}_{\mathbf{v}} + \|\ddot{\mathbf{p}}^d\| s_\phi \mathbf{u} - \tilde{c}_d\mathbf{R}\mathbf{\Pi}\mathbf{R}^\intercal\dot{\mathbf{p}} \end{aligned} \tag{3.12}$$

The value of $|s_\phi| \leq 0.052$ ($\approx \sin(3°)$), this upper limit will be denoted by $\bar{s}_\phi = 0.052$.

Rewriting the system (3.12), using $\mathbf{e} = \begin{bmatrix} \mathbf{e}_{\mathbf{p}} & \mathbf{e}_{\mathbf{v}} \end{bmatrix}^\intercal$.

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{B}\|\ddot{\mathbf{p}}^d\| s_\phi \mathbf{u} - \begin{bmatrix} \mathbf{0}_{3,3} \\ \tilde{c}_d\mathbf{R}\mathbf{\Pi}\mathbf{R}^\intercal\dot{\mathbf{p}} \end{bmatrix} \tag{3.13}$$

Here,

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_{3,3} & \mathbf{I}_3 \\ -\mathbf{K}_p & -\mathbf{K}_v \end{bmatrix},\ (\mathbf{K}_p = 10\mathbf{I}_3,\ \mathbf{K}_v = 6\mathbf{I}_3) \text{ and } \mathbf{B} = \begin{bmatrix} \mathbf{0}_{3,3} \\ \mathbf{I}_3 \end{bmatrix}$$

Using Euler 1-step discretization, with time step $\Delta t = 0.05s$, eqn. (3.13) is discretized.

$$\mathbf{e}_{k+1} = \mathbf{\Phi}\mathbf{e}_k + \mathbf{\Gamma}\|\ddot{\mathbf{p}}_k^d\| s_\phi \mathbf{u}_k - \begin{bmatrix} \mathbf{0}_{3,3} \\ \tilde{c}_d\mathbf{R}_k\mathbf{\Pi}\mathbf{R}_k^\intercal\dot{\mathbf{p}}_k \end{bmatrix} \Delta t \tag{3.14}$$

$\mathbf{\Phi} = (\mathbf{I}_6 + \mathbf{A}\Delta t)$ and, $\mathbf{\Gamma} = \mathbf{B}\Delta t$.

The discrete equation (3.14), will be further divided into linear system equation, with $\bar{\mathbf{e}}$ being the linear counterpart, and $\delta\mathbf{e}$ being the correction, ie. $\mathbf{e} = \bar{\mathbf{e}} + \delta\mathbf{e}$. The set of errors form an ellipsoidal set, $\xi_0 = \{\mathbf{e}_0 \mid \mathbf{e}_0^\intercal\mathbf{P}_0\mathbf{e}_0 \leq 1\}$, where, $\mathbf{e}_0$ is the initial value of $\mathbf{e}$, and initial condition for δe is, $\delta e_0 = \mathbf{0}$. Thus, the initial condition for the linear set can be defined as, $\bar{\xi}_0 = \{\bar{\mathbf{e}}_0 \mid \bar{\mathbf{e}}_0^\intercal\mathbf{P}_0\bar{\mathbf{e}}_0 \leq 1\}$.

The reference trajectory for the system is taken as,

$$\begin{aligned}\mathbf{p}_r &= \begin{bmatrix} 5\sin(0.4\pi t) + 1 & 5\cos(0.4\pi t) & 1 \end{bmatrix}^\intercal \text{ m} \\ \dot{\mathbf{p}}_r &= \begin{bmatrix} 2\pi\cos(0.4\pi t) & -2\pi\sin(0.4\pi t) & 0 \end{bmatrix}^\intercal \text{ m s}^{-1}\end{aligned} \tag{3.15}$$

Where, t is time, with total run-time of $T = 5s$.

Two different sizes of initial sets for error are considered, so as to observe difference in the results, if it exists. The two set of initial errors are taken as,

$$\bar{\xi}_{0s} = \xi_{0s} = \{\bar{\mathbf{e}}_0 \mid \bar{\mathbf{e}}_0^\intercal \mathbf{P}_{0s} \bar{\mathbf{e}}_0 \leq 1\}, \; \mathbf{P}_{0s} = \begin{bmatrix} 100\mathbf{I}_3 & \mathbf{0}_{3,3} \\ \mathbf{0}_{3,3} & 25\mathbf{I}_3 \end{bmatrix} \tag{3.16}$$

$$\bar{\xi}_{0b} = \xi_{0b} = \{\bar{\mathbf{e}}_0 \mid \bar{\mathbf{e}}_0^\intercal \mathbf{P}_{0b} \bar{\mathbf{e}}_0 \leq 1\}, \; \mathbf{P}_{0b} = \begin{bmatrix} 1.56\mathbf{I}_3 & \mathbf{0}_{3,3} \\ \mathbf{0}_{3,3} & 0.39\mathbf{I}_3 \end{bmatrix} \tag{3.17}$$

The FRS of these initial sets need to be estimated for two cases:

1. Uncompensated Case - In this condition, $\hat{c}_d = 0$ and $\tilde{c}_d = 0.35$.
2. Compensated Case - In this condition, $\hat{c}_d = 0.30$ and $\tilde{c}_d = 0.05$.

CHAPTER 4

SOLUTION METHODOLOGY

The problems will be addressed in the same order as they were presented in Chapter 3. Each procedure obtained will either be used in further cases, or will be extended further to comply with the problem.

4.1 Maximal Control Invariant Set for Discrete system

The system can be represented as,

$$\mathbf{x}_{k+1} = \mathbf{\Phi}\mathbf{x}_k + \mathbf{\Gamma}u_k \tag{4.1}$$

where,

$$\mathbf{\Phi} = \begin{bmatrix} 1.5 & 0 \\ 1 & -1.5 \end{bmatrix} \text{ and } \mathbf{\Gamma} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.2}$$

The above mentioned system is unstable, yet controllable. If no condition is applied on u_k, every state can be kept within the state constraints. However, given the control input constraints, a limitation develops over the states that can be kept within the state constraints. Control invariant set is used in determining which states will never violate the state constraints, when constraint over control is applied. This is done to determine *Safe Sets*. To construct the maximal control invariant set, the algorithm was constructed and written in line with theory mentioned in these references ([38],[39]). The algorithm is given below,

Algorithm 4.1.1: Computation of Maximal Control Invariant Set

Data: $\mathcal{X}_0,\ \mathcal{U},\ \mathbf{\Phi},\ \mathbf{\Gamma}$

Result: Maximal Control Invariant Set $\mathcal{C}_\infty$

$k \leftarrow 0;$

$\mathcal{X}_{-k} \leftarrow \mathcal{X}_0;$

repeat

$\quad \mathcal{X}_{-(k+1)} \leftarrow (\mathcal{X}_{-k} \oplus ((-\mathbf{\Gamma}) \circ \mathcal{U})) \circ \mathbf{\Phi};$

$\quad \mathcal{C} \leftarrow \mathcal{X}_{-(k+1)} \cap \mathcal{X}_{-k};$

$\quad \mathcal{X}_{-(k+1)} \leftarrow \mathcal{C};$

$\quad k \leftarrow k+1;$

until $\mathcal{X}_{-k} = \mathcal{X}_{-k+1};$

$\mathcal{C}_\infty = \mathcal{C}$

4.2 Reachable Set for LTI System

The system is of the form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{4.3}$$

The discretization of the system (4.3) can be carried out using multiple methods. The method which will be focused upon is “Euler 1-step discretization”. The forward reachable set will be estimated, using different time steps. This is to show the difference that occurs in computation of forward reachable set due to large time step and small time step.

Using the discretization that was mentioned, the discrete system will have the form -

$$\mathbf{x}_{k+1} = \mathbf{\Phi}\mathbf{x}_k + \mathbf{\Gamma} u_k \tag{4.4}$$

where, $\mathbf{\Phi} = (\mathbf{I}_2 + \mathbf{A}\Delta t)$ and $\mathbf{\Gamma} = \mathbf{B}\Delta t$. The procedure to find reachable set of the set $\mathcal{X}_0$ will be discussed in parts, building up the whole procedure step-wise.

4.2.1 Reachable Set for Autonomous Linear System

The theory to build a reachable set for autonomous linear system is very common and can be referred to in [39]. However, the discussion over this method will still be made.

The reachable set of a Linear System with no control input (autonomous) shall be looked upon first. This system will be represented as,

$$\mathbf{x}_{k+1} = \mathbf{\Phi}\mathbf{x}_k \tag{4.5}$$

Now taking the initial set and state constraints, as $\mathcal{X}_0 = \{\mathbf{x} \in \mathbb{R}^n : \quad \mathbf{H}_0\mathbf{x} \leq \mathbf{h}_0\}$, where,

$$\mathbf{H}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } \mathbf{h}_0 = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \tag{4.6}$$

Considering that $\mathbf{\Phi}$ is invertible, the successor set of $\mathcal{X}_0$, can be obtained as follows,

$$\mathbf{x}_0 = \mathbf{\Phi}^{-1}\mathbf{x}_1$$

$$\mathbf{H}_0\mathbf{x}_0 = \mathbf{H}_0\mathbf{\Phi}^{-1}\mathbf{x}_1 \leq \mathbf{h}_0$$

Applying this condition,

$$\text{Suc}(\mathcal{X}_0) = \{\mathbf{x} \in \mathbb{R}^n : \quad \mathbf{H}_0\mathbf{\Phi}^{-1}\mathbf{x} \leq \mathbf{h}_0\} \tag{4.7}$$

The intersection of $\text{Suc}(\mathcal{X}_0)$ and $\mathcal{X}_0$ will give us the 1-step forward reachable set $\mathcal{R}_{a,1}$. The intersection implies that $\mathbf{x}$ must follow the constraints of $\mathcal{X}_0$ and $\text{Suc}(\mathcal{X}_0)$. These constraints can be concatenated to be written as a single set of constraint,

$$\mathbf{H}_1 = \begin{bmatrix} \mathbf{H}_0 \\ \mathbf{H}_0\mathbf{\Phi}^{-1} \end{bmatrix} \text{ and } \mathbf{h}_1 = \begin{bmatrix} \mathbf{h}_0 \\ \mathbf{h}_0 \end{bmatrix} \tag{4.8}$$

The 1-step forward reachable set can be represented using the new constraints given in eqn. (4.8), as follows,

$$\mathcal{R}_1 = \{\mathbf{x} \in \mathbb{R}^n : \quad \mathbf{H}_1\mathbf{x} \leq \mathbf{h}_1\} \tag{4.9}$$

Defining, $\mathcal{X}_1 = \mathcal{R}_1$, and iterating the same procedure for $\mathcal{X}_1$ as used for $\mathcal{X}_0$. The successor set $\text{Suc}(\mathcal{X}_1)$ will be given as follows,

$$\begin{aligned} \text{Suc}(\mathcal{X}_1) &= \{\mathbf{x} \in \mathbb{R}^n : \quad \mathbf{H}_1\mathbf{\Phi}^{-1}\mathbf{x} \leq \mathbf{h}_1\} \\ &= \left\{\mathbf{x} \in \mathbb{R}^n : \quad \begin{bmatrix} \mathbf{H}_0\mathbf{\Phi}^{-1} \\ \mathbf{H}_0\mathbf{\Phi}^{-2} \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} \mathbf{h}_0 \\ \mathbf{h}_0 \end{bmatrix} \right\} \end{aligned} \tag{4.10}$$

Taking the intersection, $\text{Suc}(\mathcal{X}_1) \cap \mathcal{X}_0$, the constraints are updated,

$$\mathbf{H}_2 = \begin{bmatrix} \mathbf{H}_0 \\ \mathbf{H}_0\mathbf{\Phi}^{-1} \\ \mathbf{H}_0\mathbf{\Phi}^{-2} \end{bmatrix} \text{ and } \mathbf{h}_2 = \begin{bmatrix} \mathbf{h}_0 \\ \mathbf{h}_0 \\ \mathbf{h}_0 \end{bmatrix} \tag{4.11}$$

The 2-step reachable set can be similarly represented as shown in eqn.(4.9),

$$\mathcal{R}_2 = \{\mathbf{x} \in \mathbb{R}^n : \quad \mathbf{H}_2\mathbf{x} \leq \mathbf{h}_2\} \tag{4.12}$$

Thus, iterating this over k-steps, produces the k-step reachable set $\mathcal{R}_k$. The set thus formed is represented as,

$$\mathcal{R}_k = \{\mathbf{x} \in \mathbb{R}^n : \quad \mathbf{H}_k\mathbf{x} \leq \mathbf{h}_k\} \tag{4.13}$$

Where,

$$\mathbf{H}_k = \begin{bmatrix} \mathbf{H}_0 \\ \mathbf{H}_0\mathbf{\Phi}^{-1} \\ \vdots \\ \mathbf{H}_0\mathbf{\Phi}^{-k} \end{bmatrix} \text{ and } \mathbf{h}_k = \begin{bmatrix} \mathbf{h}_0 \\ \mathbf{h}_0 \\ \vdots \\ \mathbf{h}_0 \end{bmatrix} \tag{4.14}$$

This is the basic procedure around which the n-step reachable set is formulated which leads to a discussion of feasibility of the method. The discussion is broken down as follows,

1. *The size of* $\mathbf{H}_k$ *and* $\mathbf{h}_k$ - Let's say, the full run-time of a system is $T = 10s$, and the time step is taken as $\Delta t = 0.01s$. Thus, the total number to steps will be, $k_f = 10^4$. If the size of our initial constraint matrix $\mathbf{H}_0$ is 4×2, the final constraint matrix $\mathbf{H}_k$ will be of size $40,000 \times 2$. Any higher run-time or lower discretization time step, will only increase this size.
2. *What if* $\mathbf{\Phi}$ *is not invertible?* - The formulation of the reachable set, was based on the idea that $\mathbf{\Phi}$ is invertible, but for non invertible matrix, the above formulation becomes useless.

To address these problems, the idea will be shifted from constraints to vertices of the polytope. The mapping done by the matrix $\mathbf{\Phi}$ is linear, which implies that mapping of a line using $\mathbf{\Phi}$, remains a line. Thus, taking two points from that given line and mapping them using $\mathbf{\Phi}$, the new points will be on the same mapped line. This implies, the polytope mapped using $\mathbf{\Phi}$, can be obtained using its vertices.

Remark 4.1. For all the discussion on vertices following notation will be used - $\mathcal{V}_k$ is vertices of $\mathcal{X}_k$.

To work with vertices, convex hull of all points will be needed. This will provide an appropriate number of vertices to form the polytope. The only problem being, there is no mathematical formulation of this method. Thus, an algorithm to compute the reachable set is developed, which will give the desired result. This algorithm was written in line with the theory mentioned in references [38],[39]. The algorithm is shown below,

---

Algorithm 4.2.1: Computation of Reachable Set of Linear Autonomous System

---

Data: $\mathcal{X}_0$, $\mathbf{\Phi}$, k_f.

Result: k_f-step Reachable Set $\mathcal{R}_{k_f}$.

$k \leftarrow 0$;

$\mathcal{X}_k \leftarrow \mathcal{X}_0$;

$\mathcal{V}_0 \leftarrow$ con2vert($\mathcal{X}_0$);

$\mathcal{V}_k \leftarrow \mathcal{V}_0$;

while $k \leq k_f$ **do**

  $\mathcal{V}_{k+1} \leftarrow \mathbf{\Phi} \circ \mathcal{V}_k$;

  $\mathcal{X}_{k+1} \leftarrow$ vert2con($\mathcal{V}_{k+1}$);

  $\mathcal{R} \leftarrow \mathcal{X}_{k+1} \cap \mathcal{X}_0$;

  $\mathcal{V}_{k+1} \leftarrow$ con2vert($\mathcal{R}$);

  $k \leftarrow k + 1$;

end

$\mathcal{R}_{k_f} = \mathcal{R}$

Remark 4.2. The functions described as **"con2vert"** and **"vert2con"**, are functions **con2vert.m**[43] and **vert2con.m**[44] written in MATLAB®, by *'Michael Kleder'*. Below are the use of each function:

1. **con2vert** - Function to obtain vertices of polytope from constraints of polytope.
2. **vert2con** - Function to obtain constraints of polytope from vertices of polytope.

These functions will be used to describe their roles in the algorithm.

Remark 4.3. For detailed information of Minkowski sum of polytopes, one can refer to [45].

Remark 4.4. For detailed information on polytopes and zonotopes and their approximate conversions, so as to implement changes in the algorithm s.t. computation time can be reduced further (see [46]).

4.2.2 Reachable Set for Linear System

The theory of building the procedure and algorithm of reachable sets for Linear Systems, was referred from [18],[19]. The procedure to construct reachable set of linear autonomous system shall be extended, for linear systems with control input. Firstly, the system (4.4) whose initial state is $\mathbf{x}_0 = 0$, needs to be considered. Secondly, the set of all the possible states reachable using the control, u_k, needs to be computed. Accordingly, from the eqn.(4.4),

$$\mathbf{x}_{k+1} = \mathbf{\Gamma} u_k + \mathbf{\Phi}\mathbf{\Gamma} u_{k-1} + \cdots + \mathbf{\Phi}^k \mathbf{\Gamma} u_0 \tag{4.15}$$

We know that, $|u_k| \leq u_{max}$. Using this, eqn.(4.15) is rewritten as,

$$\mathbf{x}_{k_f} = u_{max}[\mathbf{\Gamma}\alpha_{k_f-1} + \mathbf{\Phi}\mathbf{\Gamma}\alpha_{k_f-2} + \cdots + \mathbf{\Phi}^{k_f-1}\mathbf{\Gamma}\alpha_0], \quad \forall 0 \leq i \leq k_f - 1, \ \alpha_i \in [-1, 1] \tag{4.16}$$

The set of $\mathbf{x}_{k_f}$ which is formed is called a zonotope. Hence, the reachable set will be

$$\begin{aligned}\mathcal{R}_{k_f} = \{\mathbf{x}_{k_f} \in \mathbb{R}^n,\ u_k \in \mathcal{U},\ \alpha_k \in [-1,1],\ \forall 0 \leq k \leq k_f \in \mathbb{N}_0 : \\ \mathbf{x}_{k_f} = u_{max} \sum_{i=0}^{k_f-1} \mathbf{\Phi}^{(k_f-1)-i}\Gamma\alpha_i\}\end{aligned} \tag{4.17}$$

If the initial state is non-zero ($\mathbf{x}_0 \neq 0$), the zonotope will be centred around $\mathbf{\Phi}^{k_f}\mathbf{x}_0$. Thus, the reachable set will be,

$$\begin{aligned}\mathcal{R}_{k_f} = \{\mathbf{x}_{k_f} \in \mathbb{R}^n,\ u_k \in \mathcal{U},\ \alpha_k \in [-1,1],\ \forall 0 \leq k \leq k_f \in \mathbb{N}_0 : \\ \mathbf{x}_{k_f} = \mathbf{\Phi}^{k_f}\mathbf{x}_0 + u_{max} \sum_{i=0}^{k_f-1} \mathbf{\Phi}^{(k_f-1)-i}\Gamma\alpha_i\}\end{aligned} \tag{4.18}$$

Thereafter, if the initial state itself belongs to a set, and the reachable set for all those states is needed to be determined, then Minkowski sum of all the states $\mathbf{\Phi}^{k_f}\mathbf{x}_0$ and the zonotope needs to be performed. The algorithm was written in line with the theory mentioned in these references [38],[39]. The algorithm to estimate the forward reachable set of linear systems, is written below,

Remark 4.5. For boundedness of all states in reachable set, see [47].

Algorithm 4.2.2: Computation of Reachable Set of Linear System

Data: $\mathcal{X}_0,\ \mathcal{U},\ \mathbf{\Phi},\ \mathbf{\Gamma},\ k_f$

Result: k_f-step Reachable Set $\mathcal{R}_{k_f}$.

$k \leftarrow 0;$

$\mathcal{X}_k \leftarrow \mathcal{X}_0;$

$\mathcal{V}_0 \leftarrow$ con2vert$(\mathcal{X}_0);$

$\mathcal{V}_k \leftarrow \mathcal{V}_0;$

while $k \leq k_f$ **do**

  $\mathcal{V}_{k+1} \leftarrow (\mathbf{\Phi} \circ \mathcal{V}_k) \oplus (\mathbf{\Gamma} \circ \mathcal{U});$

  $\mathcal{X}_{k+1} \leftarrow$ vert2con$(\mathcal{V}_{k+1});$

  $\mathcal{R} \leftarrow \mathcal{X}_{k+1} \cap \mathcal{X}_0;$

  $\mathcal{V}_{k+1} \leftarrow$ con2vert$(\mathcal{R});$

  $k \leftarrow k + 1;$

end

$\mathcal{R}_{k_f} = \mathcal{R}$

4.3 Perturbed Linear Dynamic System

For the perturbed system (3.6), finding the appropriate condition over $\ell \sin(x_2)$ is first priority. Although, $(\mathbf{A}, \mathbf{B})$ are controllable, the value of ℓ determines if the state remains bounded. Thus, a general perturbation $\mathbf{f}(\mathbf{x})$ shall be taken and the condition over ℓ will be determined.

The choice over the time step for Euler 1-step discretization, will be carried out in linear dynamic system section. Thus, the discretized system (3.8) will be taken as it is.

4.3.1 Condition for Lipschitz Constant

The principle of superposition cannot be applied because of the nonlinear perturbation $\mathbf{f}(\mathbf{x}_k)$. Hence, the discrete equation is separated such that, $\mathbf{x}_k = \bar{\mathbf{x}}_k + \delta\mathbf{x}_k$. The separation is carried in such a way that, $\bar{\mathbf{x}}_k$ follows the linear dynamic system, and $\delta\mathbf{x}_k$ is the correction.

All the states in linear system will be denoted by $\bar{\mathbf{x}}_k$ and the states in non-linear system will be denoted by $\mathbf{x}_k$.

The discretization of both system is done using Euler 1-step discretization. Therefore, the system in discrete time becomes,

Linear system -

$$\bar{\mathbf{x}}_{k+1} = (\mathbf{I} + \mathbf{A}\Delta t)\bar{\mathbf{x}}_k + \mathbf{B}\Delta t u_k$$

Perturbed linear system -

$$\mathbf{x}_{k+1} = (\mathbf{I} + \mathbf{A}\Delta t)\mathbf{x}_k + \mathbf{B}\Delta t u_k + \mathbf{f}(\mathbf{x}_k)\Delta t$$

Comparing both systems,

$$\begin{aligned}\mathbf{x}_{k+1} - \bar{\mathbf{x}}_{k+1} &= \mathbf{\Phi}(\mathbf{x}_k - \bar{\mathbf{x}}_k) + \mathbf{f}(\mathbf{x}_k)\Delta t \\ &= \mathbf{\Phi}(\mathbf{x}_k - \bar{\mathbf{x}}_k) + (\mathbf{f}(\mathbf{x}_k) - \mathbf{f}(\bar{\mathbf{x}}_k))\Delta t + \mathbf{f}(\bar{\mathbf{x}}_k)\Delta t\end{aligned}$$

We know from Lipschitz condition that, $\|(\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2))\| \leq \ell\|(\mathbf{x}_1 - \mathbf{x}_2)\|$. This can be re-written as, $\|(\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2))\| = \alpha\ell\|(\mathbf{x}_1 - \mathbf{x}_2)\|$, where $0 \leq \alpha \leq 1$.

$$
\begin{aligned}
\mathbf{x}_{k+1} - \bar{\mathbf{x}}_{k+1} &= (\mathbf{\Phi}(\mathbf{x}_k - \bar{\mathbf{x}}_k) + \alpha_k\ell\|(\mathbf{x}_k - \bar{\mathbf{x}}_k)\|\hat{\mathbf{f}}_k\Delta t + \mathbf{f}(\bar{\mathbf{x}}_k)\Delta t \\
&= (\mathbf{\Phi}(\mathbf{x}_k - \bar{\mathbf{x}}_k) + \alpha_k\ell\hat{\mathbf{f}}_k\hat{\mathbf{e}}_k^{\intercal}\|(\mathbf{x}_k - \bar{\mathbf{x}}_k)\|\hat{\mathbf{e}}_k\Delta t + \mathbf{f}(\bar{\mathbf{x}}_k)\Delta t \\
&= (\mathbf{\Phi} + \alpha_k\ell\hat{\mathbf{f}}_k\hat{\mathbf{e}}_k^{\intercal}\Delta t)(\mathbf{x}_k - \bar{\mathbf{x}}_k) + \mathbf{f}(\bar{\mathbf{x}}_k)\Delta t \\
&= (\mathbf{\Phi} + \alpha_k\ell\mathbf{C}_k\Delta t)(\mathbf{x}_k - \bar{\mathbf{x}}_k) + \mathbf{f}(\bar{\mathbf{x}}_k)\Delta t
\end{aligned}
$$

where, $\mathbf{C}_k = \hat{\mathbf{f}}_k\hat{\mathbf{e}}_k^{\intercal}$.

Now, as stated earlier $\mathbf{x}_k = \bar{\mathbf{x}}_k + \delta\mathbf{x}_k$, thus, substituting, $\mathbf{x}_k - \bar{\mathbf{x}}_k = \delta\mathbf{x}_k$.

$$
\delta\mathbf{x}_{k+1} = (\mathbf{\Phi} + \alpha_k\ell\mathbf{C}_k\Delta t)\delta\mathbf{x}_k + \mathbf{f}(\bar{\mathbf{x}}_k)\Delta t
$$

The matrix, $(\mathbf{\Phi} + \alpha_k\ell\mathbf{C}_k\Delta t)$ changes at each step, thus, use of the Jordan canonical form is restricted. Therefore, $\alpha_k\ell\mathbf{C}_k\Delta t$ is separated completely and only $\mathbf{\Phi}\delta\mathbf{x}_k$ is eliminated.

$$
\begin{aligned}
\delta\mathbf{x}_{k+1} &= \mathbf{\Phi}\delta\mathbf{x}_k + \mathbf{f}(\bar{\mathbf{x}}_k)\Delta t + \alpha_k\ell\mathbf{C}_k\Delta t\delta\mathbf{x}_k \\
&= \mathbf{\Phi}\delta\mathbf{x}_k + \mathbf{f}(\bar{\mathbf{x}}_k)\Delta t + \alpha_k\ell\mathbf{C}_k\Delta t\delta\mathbf{x}_k \\
&= \mathbf{\Phi}^{k+1}\delta\mathbf{x}_0 + \sum_{i=0}^{k}(\mathbf{\Phi}^{(k-i)}\mathbf{f}(\bar{\mathbf{x}}_i) + \mathbf{\Phi}^{(k-i)}\alpha_i\ell\mathbf{C}_i\delta\mathbf{x}_i)\Delta t
\end{aligned}
$$

The initial condition for both systems is the same, $\mathbf{x}_0 = \bar{\mathbf{x}}_0$, which leads to $\delta\mathbf{x}_0 = 0$.

$$
\begin{aligned}
\delta\mathbf{x}_{k+1} &= \sum_{i=0}^{k}(\mathbf{\Phi}^{(k-i)}\mathbf{f}(\bar{\mathbf{x}}_i) + \mathbf{\Phi}^{(k-i)}\alpha_i\ell\mathbf{C}_i\delta\mathbf{x}_i)\Delta t \\
\|\delta\mathbf{x}_{k+1}\| &= \|\sum_{i=0}^{k}(\mathbf{\Phi}^{(k-i)}\mathbf{f}(\bar{\mathbf{x}}_i) + \mathbf{\Phi}^{(k-i)}\alpha_i\ell\mathbf{C}_i\delta\mathbf{x}_i)\Delta t\| \\
\|\delta\mathbf{x}_{k+1}\| &\leq \|\sum_{i=0}^{k}\mathbf{\Phi}^{(k-i)}\mathbf{f}(\bar{\mathbf{x}}_i)\Delta t\| + \sum_{i=0}^{k}\|\mathbf{\Phi}^{(k-i)}\|\|\alpha_i\ell\mathbf{C}_i\|\|\delta\mathbf{x}_i\|\Delta t
\end{aligned}
$$

It should be noted that, $0 \le \alpha_i \le 1$, $\|\mathbf{C}_i\| = 1$, $\forall i \ge 0$.

$$\|\delta\mathbf{x}_{k+1}\| \le \|\sum_{i=0}^{k} \mathbf{\Phi}^{(k-i)}\mathbf{f}(\bar{\mathbf{x}}_i)\Delta t\| + \ell(\sum_{i=0}^{k} \|\mathbf{\Phi}^{(k-i)}\|\|\delta\mathbf{x}_i\|)\Delta t \tag{4.19}$$

Denoting, $S_k = \|\sum_{i=0}^{k} \mathbf{\Phi}^{(k-i)}\mathbf{f}(\bar{\mathbf{x}}_i)\Delta t\|$, the equation is hereby, simplified to,

$$\|\delta\mathbf{x}_{k+1}\| \le S_k + \ell(\sum_{i=0}^{k} \|\mathbf{\Phi}^{(k-i)}\|\|\delta\mathbf{x}_i\|)\Delta t \tag{4.20}$$

Defining $\mathcal{D} \subseteq \mathbb{R}$ as a set of $\|\delta\mathbf{x}_k\|$, $\forall k \in \mathbb{N}_0$ (i.e., $\mathcal{D} = \{\|\delta\mathbf{x}_0\|, \|\delta\mathbf{x}_1\|, \cdots, \|\delta\mathbf{x}_{kf}\|\}$). For $\|\delta\mathbf{x}_k\|$ to be bounded, $\sup(\mathcal{D})$ must exist. Let, the supremum be denoted by $\delta_m \in \mathbb{R}_+$, i.e., $\|\delta\mathbf{x}_k\| \le \delta_m$, $\forall \|\delta\mathbf{x}_k\| \in \mathcal{D}$.

We also know that, $\|\mathbf{\Phi}^k\|$ will be bounded if the spectral radius is less than 1 ($|\sigma(\mathbf{\Phi})| < 1$). It is established that, $\|\mathbf{\Phi}^k\| \le M|\sigma(\mathbf{\Phi})_\epsilon|^k$, where $\sigma(\mathbf{\Phi})_\epsilon$ belongs to the ϵ-pseudospectra of $\mathbf{\Phi}$.

$$\begin{aligned}
\|\delta\mathbf{x}_{k+1}\| &\le S_k + \ell(\sum_{i=0}^{k} M|\sigma(\mathbf{\Phi})_\epsilon|^{(k-i)}\delta_m)\Delta t \\
&\le S_k + \left(\frac{1 - |\sigma(\mathbf{\Phi})|^{(k+1)}}{1 - |\sigma(\mathbf{\Phi})_\epsilon|}\right)\ell M \delta_m \Delta t \\
&\le S_k + \frac{\ell M \delta_m \Delta t}{1 - |\sigma(\mathbf{\Phi})_\epsilon|}
\end{aligned}$$

The above condition is for all $0 \le k \le k_f - 1$. Subtracting δ_m from both sides we get,

$$\|\delta\mathbf{x}_{k+1}\| - \delta_m \le S_k + \frac{\ell M \delta_m \Delta t}{1 - |\sigma(\mathbf{\Phi})_\epsilon|} - \delta_m$$

For some k, $\|\delta\mathbf{x}_{k+1}\| - \delta_m = 0$, leading to,

$$\begin{aligned}
0 &\le S_k + \frac{\ell M \delta_m \Delta t}{1 - |\sigma(\mathbf{\Phi})_\epsilon|} - \delta_m \\
\implies \left(1 - \frac{\ell M \Delta t}{1 - |\sigma(\mathbf{\Phi})_\epsilon|}\right)\delta_m &\le S_k
\end{aligned}$$

δ_m was defined as the supremum or the least upper bound of $\mathcal{D}$. Thus, having another number larger than δ_m can guarantee the existence of it. This leads to the condition shown below,

$$\boxed{\ell < \frac{1 - |\sigma(\mathbf{\Phi})_\epsilon|}{M\Delta t}} \tag{4.21}$$

It is not a necessary, but just a sufficient condition. The reason for this condition to be just sufficient, is that it guarantees existence but does not guarantee unboundedness when the condition is violated. If, say the range-space of $\mathbf{f}$ is the same as $\mathbf{B}u_k\Delta t$ and is bounded, it will be seen how this condition is not needed.

Remark 4.6. For verifying the condition over Lipschitz constant ℓ, the derivation of the same condition will be done using 'Gronwall's Inequality'. Using eqn.(4.20) and $\|\mathbf{\Phi}^k\| \leq M|\sigma(\Phi)_\epsilon|^k$, we get,

$$\|\delta\mathbf{x}_{k+1}\| \leq S_k + M\ell(\sum_{i=0}^{k} |\sigma(\Phi)_\epsilon|^{k-i}\|\delta\mathbf{x}_i\|)\Delta t$$

Dividing the equation by $|\sigma(\Phi)_\epsilon|^{k+1}$,

$$|\sigma(\Phi)_\epsilon|^{-(k+1)}\|\delta\mathbf{x}_{k+1}\| \leq |\sigma(\Phi)_\epsilon|^{-(k+1)}S_k + (\sum_{i=0}^{k} \frac{M\ell\Delta t}{|\sigma(\Phi)_\epsilon|}(|\sigma(\Phi)_\epsilon|^{-i}\|\delta\mathbf{x}_i\|))$$

Applying the 'Gronwall's Inequality',

$$|\sigma(\Phi)_\epsilon|^{-(k+1)}\|\delta\mathbf{x}_{k+1}\| \leq |\sigma(\Phi)_\epsilon|^{-(k+1)}S_m \prod_{i=0}^{k}(1 + \frac{M\ell\Delta t}{|\sigma(\Phi)_\epsilon|})$$

Multiplying $|\sigma(\Phi)_\epsilon|^{k+1}$ to the equation,

$$\|\delta\mathbf{x}_{k+1}\| \leq |\sigma(\Phi)_\epsilon|^{-1}S_m(|\sigma(\Phi)_\epsilon| + M\ell\Delta t)^k$$

For $\|\delta\mathbf{x}_{k+1}\|$, to be upper bounded, $(|\sigma(\Phi)_\epsilon| + M\ell\Delta t)$ needs to be less than 1, i.e., $(|\sigma(\Phi)_\epsilon| + M\ell\Delta t) \leq 1$. Thus,

$$\ell < \frac{1 - |\sigma(\mathbf{\Phi})_\epsilon|}{M\Delta t} \tag{4.22}$$

Thus, verifying the condition over Lipschitz constant ℓ.

4.3.2 Case 1 - $\mathbf{f}$ is in the Range-space of Control Input (i.e. $\mathbf{\Gamma}u$) and is Bounded

This is a very special case. The reason being, if $\mathbf{f}$ is bounded and in the range-space of control input, it is controllable no matter the value of Lipschitz constant. However, for the sake of completeness, the condition shall be derived. Considered below, are two systems for this case, linear system and perturbed system. The aim will be to determine, the additional control input needed for the reachable set of linear system to completely bound the reachable set of perturbed system. The discrete state transition matrix ($\mathbf{\Phi}$) and control matrix ($\mathbf{\Gamma}$) for both the systems shall be taken same. The condition over control input, for linear system shall be taken as, $\mathcal{U}_L = \{u_{k,L} \in \mathbb{R} : \ |u_{k,L}| \leq u_{max,L}\}$, and for perturbed system shall be taken as, $\mathcal{U}_N = \{u_{k,N} \in \mathbb{R} : \ |u_{k,N}| \leq u_{max,N}\}$.

1. Linear system.

$$\mathbf{x}_{k+1,L} = \mathbf{\Phi}\mathbf{x}_{k,L} + \mathbf{\Gamma}u_{k,L} \tag{4.23}$$

2. Perturbed System.

$$\mathbf{x}_{k+1,N} = \mathbf{\Phi}\mathbf{x}_{k,N} + \mathbf{\Gamma}u_{k,N} + \mathbf{\Gamma}f(\mathbf{x}_{k,N}) \tag{4.24}$$

Remark 4.7. $\mathbf{f}(\mathbf{x}_{k,N})$ is in range-space of $\mathbf{\Gamma}u$, thus, it is represented as $\mathbf{\Gamma}f(\mathbf{x}_{k,N})$.

Let, $\mathbf{x}_{k,N} - \mathbf{x}_{k,L} = \delta\mathbf{x}_k$, also let, $u_{k,N} - u_{k,L} = \Delta u_k$. Now, subtracting eqn.(4.23) from eqn.(4.24),

$$\delta\mathbf{x}_{k+1} = \mathbf{\Phi}\delta\mathbf{x}_k + \mathbf{\Gamma}\Delta u_k + \mathbf{\Gamma}f(\mathbf{x}_{k,N}) \tag{4.25}$$

Taking, $\mathbf{x}_{0,N} = \mathbf{x}_{0,L} \implies \delta\mathbf{x}_0 = 0$. Taking $k = 1$, it can be observed that,

$$\begin{aligned}\delta\mathbf{x}_1 &= \mathbf{\Gamma}\Delta u_1 + \mathbf{\Gamma}f(\mathbf{x}_{1,N}) \\ &= \mathbf{\Gamma}(\Delta u_1 + f(\mathbf{x}_{1,N}))\end{aligned}$$

Choosing Δu_1, s.t.,

$$\Delta u_1 + f(\mathbf{x}_{1,N}) = 0 \implies \delta\mathbf{x}_1 = 0 \tag{4.26}$$

It can be guaranteed that $\mathbf{x}_{1,L} = \mathbf{x}_{1,N}$, i.e., the state reachable by perturbed system after the 1st step, is also reachable by the Linear System with higher control.

Now, this procedure can be iterated for all k_f steps, giving us the condition over control law as such,

$$\Delta u_k + f(\mathbf{x}_{k,N}) = 0 \tag{4.27}$$

If the choice over Δu_k, is actually feasible, it will be possible to guarantee that, $\mathbf{x}_{k,L} = \mathbf{k}, \mathbf{N}$, i.e., the state reachable by perturbed system after k-steps, is also reachable by the Linear System with higher control.

Re-writing eqn. (4.27), by substituting Δu_k with $u_{k,N} - u_{k,L}$,

$$u_{k,L} = u_{k,N} + f(\mathbf{x}_{k,N})$$

As stated before, $\forall k$, $f(\mathbf{x}_{k,N})$ is bounded. Let, the bound on $f(\mathbf{x}_{k,N})$ be F, i.e. $f(\mathbf{x}_{k,N}) \leq F \ \forall 0 \leq k \leq k_f$. Now, the control input constraint of Linear system, $u_{max,L}$ is chosen s.t.,

$$|u_{k,L}| \leq u_{max,L} = u_{max,N} + F \tag{4.28}$$

The choice over $u_{max,L}$ mentioned in (4.28) is made such that, $\Delta u_k \geq f(\mathbf{x}_{k,N})$. This, implies that the choice of Δu_k, as mentioned in eqn. (4.27), is feasible.

Now, that the constraint over control input for linear system has been determined, it can be stated that -

'*All the States Reachable by Perturbed System (4.24), are also Reachable by the Linear System (4.23), given the constraint over control input $u_{max,L}$, is chosen as shown* *in eqn.(4.28). Thus, for bounded perturbations in range-space of* $\mathbf{\Gamma}u$*, the Perturbed System is controllable.'*

In other words, if $\mathcal{R}_{k_f,N}$ is reachable set of Perturbed System (4.24), and $\mathcal{R}_{k_f,L}$ be the reachable set of Linear System (4.23), $\mathcal{R}_{k_f,N} \subseteq \mathcal{R}_{k_f,L}$, if $u_{max,L} = u_{max,N} + F$.

Therefore, as the reachable set of the perturbed system is always upper bounded by the reachable set of the linear system, no condition over the Lipschitz constant ℓ, is hence, required for this case. The problem in focus of this section is exactly of this kind. Where, $\ell \sin(x_2)$ is a bounded function, thus, $-\infty < \ell < \infty$ works for the system.

Remark 4.8. Equation (4.28) will be used to evaluate u_{max} for computation of upper bounding reachable set.

4.3.3 Case 2 - $\mathbf{f}$ is in the Range-space of Control Input (i.e. $\mathbf{\Gamma}u$) and is Lipschitz Continuous

As in the previous case, it was demonstrated that perturbed system is bounded using, $\mathcal{R}_{k_f,N} \subseteq \mathcal{R}_{k_f,L}$ and $\mathcal{R}_{k_f,L}$ is bounded. The same procedure will be followed, and the condition over ℓ will be estimated for this case. Both identical systems shall be taken, but, the control law, for the linear system shall be taken little differently, i.e., $u_{k,L} = v_{k,L} + \mathbf{K}_k \mathbf{x}_{k,L}$, where, $|v_{k,L}| \leq v_{max,L}$. The perturbed system control is taken same, $|u_{k,N}| \leq u_{max,N}$. The systems shall be stated once again.

1. Linear system.

$$\mathbf{x}_{k+1,L} = \mathbf{\Phi}\mathbf{x}_{k,L} + \mathbf{\Gamma}(v_{k,L} + \mathbf{K}_k \mathbf{x}_{k,L}) \tag{4.29}$$

2. Perturbed System.

$$\mathbf{x}_{k+1,N} = \mathbf{\Phi}\mathbf{x}_{k,N} + \mathbf{\Gamma}u_{k,N} + \mathbf{\Gamma}f(\mathbf{x}_{k,N}) \tag{4.30}$$

Remark 4.9. $\mathbf{f}(\mathbf{x}_{k,N})$ is in range-space of $\mathbf{\Gamma}u$, thus, it is represented as $\mathbf{\Gamma}f(\mathbf{x}_{k,N})$. Let, $\mathbf{x}_{k,N} - \mathbf{x}_{k,L} = \delta\mathbf{x}_k$, also let, $u_{k,N} - v_{k,L} = \Delta u_k$. Now, subtracting eqn.(4.29) from eqn.(4.30),

$$\delta\mathbf{x}_{k+1} = \mathbf{\Phi}\delta\mathbf{x}_k + \mathbf{\Gamma}(\Delta u_k - \mathbf{K}_k\mathbf{x}_{k,L}) + \mathbf{\Gamma}f(\mathbf{x}_{k,N}) \tag{4.31}$$

Taking, $\mathbf{x}_{0,N} = \mathbf{x}_{0,L} \implies \delta\mathbf{x}_0 = 0$. Observing the system for $k = 1$,

$$\begin{aligned}\delta\mathbf{x}_1 &= \mathbf{\Gamma}(\Delta u_1 - \mathbf{K}_1\mathbf{x}_{1,L}) + \mathbf{\Gamma}f(\mathbf{x}_{1,N}) \\ &= \mathbf{\Gamma}(\Delta u_1 - \mathbf{K}_1\mathbf{x}_{1,L} + f(\mathbf{x}_{1,N}))\end{aligned}$$

Choosing Δu_1, s.t.,

$$\Delta u_1 - \mathbf{K}_1\mathbf{x}_{1,L} + f(\mathbf{x}_{1,N}) = 0 \implies \delta\mathbf{x}_1 = 0 \tag{4.32}$$

It can be guaranteed that $\mathbf{x}_{1,L} = \mathbf{x}_{1,N}$, i.e., the state reachable by the perturbed system after the 1st step, is also reachable by the linear system with higher control. This procedure is iterated for all k_f steps, giving us the condition over control law such as,

$$\Delta u_k - \mathbf{K}_k\mathbf{x}_{k,L} + f(\mathbf{x}_{k,N}) = 0 \tag{4.33}$$

If the choice over Δu_k, $\mathbf{K}_k$, is actually feasible, it will be possible to guarantee that, $\mathbf{x}_{k,L} = \mathbf{x}_{k,N}$, i.e., the state reachable by perturbed system after k-steps, is also reachable by the linear system with higher control.

Rewriting eqn. (4.33), by substituting Δu_k with $u_{k,N} - v_{k,L}$, and getting,

$$v_{k,L} + \mathbf{K}_k\mathbf{x}_{k,L} = u_{k,N} + f(\mathbf{x}_{k,N})$$

It is not known, if $f(\mathbf{0})$ will be 0.

$$v_{k,L} + \mathbf{K}_k\mathbf{x}_{k,L} = u_{k,N} + (f(\mathbf{x}_{k,N}) - f(\mathbf{0})) + f(\mathbf{0}) \tag{4.34}$$

Let, $f(\mathbf{0}) = F_0$, s.t., $|F_0| < \infty$.

The first part of control law $u_{k,L}$, $|v_{k,L}| \leq v_{max,L}$, is then chosen as shown below,

$$|v_{k,L}| \leq v_{max,L} = u_{max,N} + F_0 \tag{4.35}$$

This implies, Δu_k can be chosen s.t. $\Delta u_k = f(\mathbf{0})$. Focusing on the other half of the control law $\mathbf{K}_k\mathbf{x}_{k,L}$, it can be noted that $\mathbf{K}_k$ is chosen s.t., $\mathbf{x}_{k,L} = \mathbf{x}_{k,N} = \mathbf{x}_k$. Rewriting eqn. (4.34) and considering eqn. (4.35),

$$\mathbf{K}_k\mathbf{x}_k = (f(\mathbf{x}_k) - f(\mathbf{0}))$$

It is known that,

$$\|(f(\mathbf{x}_k) - f(\mathbf{0}))\| \leq \ell\|\mathbf{x}_k\| \tag{4.36}$$

Also, re-writing $\mathbf{x}_k = \|\mathbf{x}_k\|\hat{\mathbf{x}}_k$, where $\hat{\mathbf{x}}_k$ is the unit vector along $\mathbf{x}_k$,

$$(\mathbf{K}_k\hat{\mathbf{x}}_k)\|\mathbf{x}_k\| = (f(\mathbf{x}_k) - f(\mathbf{0})) \tag{4.37}$$

Using the definition of induced norms, it can be implied that $\|\mathbf{K}_k\| \geq (\mathbf{K}_k\hat{\mathbf{x}}_k)$. Hereby, the choice over $\mathbf{K}_k$ is feasible. This choice over v_k and $\mathbf{K}_k$, implies that the reachable set of the perturbed system will be a subset of reachable set of linear system.

However, this leads to a different problem with the linear system, where $\mathbf{\Phi} + \mathbf{\Gamma}\mathbf{K}_k$ is a *time-dependent* system. Thus, determining the condition over ℓ, which can keep $\mathbf{\Phi} + \mathbf{\Gamma}\mathbf{K}_k$ stable must be formulated. This way, it can be safely said that the perturbed system will be stable.

For stability a similar pattern is followed as given in *"Stability of time-varying linear system"*, [48].

Taking the autonomous system, as only stability is in question.

$$\mathbf{x}_{k+1} = \mathbf{\Phi}\mathbf{x}_k + \mathbf{\Gamma}\mathbf{K}_k\mathbf{x}_k \tag{4.38}$$

Substituting for $\mathbf{x}_k$, only in $\mathbf{\Phi x}_k$, and iterating this till $\mathbf{x}_0$,

$$\mathbf{x}_{k+1} = \mathbf{\Phi}^{k+1}\mathbf{x}_0 + \sum_{i=0}^{k} \mathbf{\Phi}^{k-i}\mathbf{\Gamma K}_i\mathbf{x}_i$$

Taking norm on both sides,

$$\|\mathbf{x}_{k+1}\| \leq M|\sigma(\Phi)_\epsilon|^{k+1}\|\mathbf{x}_0\| + \sum_{i=0}^{k} M|\sigma(\Phi)_\epsilon|^{k-i}\|\mathbf{\Gamma K}_i\|\|\mathbf{x}_i\|$$

Using $x_k = \|\mathbf{x}_{k+1}\|$,

$$x_{k+1} \leq M|\sigma(\Phi)_\epsilon|^{k+1}x_0 + \sum_{i=0}^{k} M|\sigma(\Phi)_\epsilon|^{k-i}\|\mathbf{\Gamma K}_i\|x_i$$

Dividing the whole equation by $|\sigma(\Phi)_\epsilon|^{k+1}$, and also substituting $\|\mathbf{\Gamma K}_i\| \leq \ell\|\Gamma\|$.

$$|\sigma(\Phi)_\epsilon|^{-(k+1)}x_{k+1} \leq Mx_0 + \sum_{i=0}^{k} \frac{M\ell\|\Gamma\|}{|\sigma(\Phi)_\epsilon|}(|\sigma(\Phi)_\epsilon|^{-i}x_i)$$

Using the Gronwall's inequality,

$$|\sigma(\Phi)_\epsilon|^{-(k+1)}x_{k+1} \leq Mx_0 \prod_{i=0}^{k}(1 + \frac{M\ell\|\Gamma\|}{|\sigma(\Phi)_\epsilon|}) \tag{4.39}$$

$$\implies x_{k+1} \leq M|\sigma(\Phi)_\epsilon|x_0(|\sigma(\Phi)_\epsilon| + M\ell\|\Gamma\|)^k \tag{4.40}$$

Thereupon, we get the required condition which is desired.

$$|\sigma(\Phi)_\epsilon| + M\ell\|\Gamma\| \leq 1 \tag{4.41}$$

Taking Γ as $\mathbf{B}\Delta t$ ($\|\mathbf{B}\|$ being 1), it again leads us to the same equation which had been derived, eqn.(4.21).

$$|\sigma(\Phi)_\epsilon| + M\ell\Delta t \leq 1 \tag{4.42}$$

Thus, confirming the result.

Now, that the condition over ℓ has been determined, it can be safely claimed that $\mathcal{R}_{k_f,N} \subseteq \mathcal{R}_{k_f,L}$, and that $\mathcal{R}_{k_f,L}$ will remain bounded, if the condition is followed.

Remark 4.10. For additional details, see [49],[50].

4.3.4 Computation of Reachable Set for Perturbed System

The computation of reachable set for perturbed system is done by directly extending algorithm (4.2.2), by incorporating $\mathbf{f}(\mathbf{x})\Delta t$ in the successor set of $\mathcal{X}_k$. The procedure that is followed is the direct result of definition (2.22) and theory mentioned in these references ([38],[39]). It should be noted that this method uses intersection to compute the reachable set. This process in turn can be represented as a process of elimination. Thus, the algorithm will only provide us with reachable states at specific time T. The discussion of results (5.3.1) obtained using this method, leads to question of all the reachable states within time T.

Algorithm 4.3.1: Computation of Reachable Set of Perturbed System

Data: $\mathcal{X}_0,\ \mathcal{U},\ \mathbf{\Phi},\ \mathbf{\Gamma},\ \mathbf{f}\Delta t,\ k_f$

Result: k_f-step Reachable Set $\mathcal{R}_{k_f}$.

$k \leftarrow 0$;

$\mathcal{X}_k \leftarrow \mathcal{X}_0$;

$\mathcal{V}_0 \leftarrow$ con2vert($\mathcal{X}_0$);

$\mathcal{V}_k \leftarrow \mathcal{V}_0$;

while $k \leq k_f$ **do**

$\quad \mathcal{V}_{k+1} \leftarrow \mathbf{\Phi} \circ (\mathcal{V}_k) \oplus \mathbf{\Gamma} \circ (\mathcal{U}) \oplus \mathbf{f}\Delta t \circ (\mathcal{V}_k)$;

$\quad \mathcal{X}_{k+1} \leftarrow$ vert2con($\mathcal{V}_{k+1}$);

$\quad \mathcal{R} \leftarrow \mathcal{X}_{k+1} \cap \mathcal{X}_0$;

$\quad \mathcal{V}_{k+1} \leftarrow$ con2vert($\mathcal{R}$);

$\quad k \leftarrow k + 1$;

end

$\mathcal{R}_{k_f} = \mathcal{R}$

4.3.5 Method-2 for - Computation of Total Reachable Set for Perturbed System

The Total Reachable Set shall be defined as a union of all the reachable sets of the perturbed system obtained after settling time. To compute a reachable set containing all the states, we shall begin with a reachable set obtained using algorithm (4.3.1) and define it as $\Delta\mathcal{X}_0$. The set $\Delta\mathcal{X}_0$ shall be called *Initiating Set*. Any set obtained by transitioning this set, over time will become part of the Reachable Set. This way all the states reachable at different times will be included within the total reachable set. The algorithm to compute reachable set was constructed, by adopting the theory of reachable set mentioned in [2],[17],[51],[52]. The algorithm to obtain the said set is given below.

Algorithm 4.3.2: Computation of Total Reachable Set of Perturbed System

Data: $\mathcal{X}_0,\ \Delta\mathcal{X}_0,\ \mathcal{U},\ \mathbf{\Phi},\ \mathbf{\Gamma},\ \mathbf{f}\Delta t,\ k_f$

Result: k_f-step Reachable Set $\mathcal{R}_{k_f}$.

$k \leftarrow 0$;

$\Delta\mathcal{X}_k \leftarrow \Delta\mathcal{X}_0$;

$\Delta\mathcal{V}_0 \leftarrow$ con2vert$(\Delta\mathcal{X}_0)$;

$\Delta\mathcal{V}_k \leftarrow \Delta\mathcal{V}_0$;

while $k \leq k_f$ **do**

$\Delta\mathcal{V}_{k+1} \leftarrow \mathbf{\Phi} \circ (\Delta\mathcal{V}_k) \oplus \mathbf{\Gamma} \circ (\mathcal{U}) \oplus \mathbf{f}\Delta t \circ (\Delta\mathcal{V}_k)$;

$\Delta\mathcal{X}_{k+1} \leftarrow$ vert2con$(\Delta\mathcal{V}_{k+1})$;

$\Delta\mathcal{X}_{k+1} \leftarrow \Delta\mathcal{X}_{k+1} \cup \Delta\mathcal{X}_k$;

$\mathcal{R} \leftarrow \Delta\mathcal{X}_{k+1}$;

$\Delta\mathcal{V}_{k+1} \leftarrow$ con2vert$(\mathcal{R})$;

$k \leftarrow k + 1$;

end

$\mathcal{R}_{k_f} = \mathcal{R}$

4.3.6 Computation of Maximal Robust Control Invariant Set for Perturbed System

Extending the algorithm of Maximal Control Invariant (MCI) set to build MCI set for perturbed systems is extremely complex. The function $\mathbf{f}(\mathbf{x})$ may not be a bijective function, thus, incorporating it into the H matrix for MCI set becomes highly complex. Thus, for the case of bounded function, the perturbation is treated as disturbance and this set of disturbance can be defined as $\mathcal{W}$. Accordingly, the idea of Maximal Robust Control Invariant Set (definition 2.30) is extended for perturbed systems. The algorithm is constructed, by adopting the theory of control invariant sets mentioned in [38],[39].

Algorithm 4.3.3: Computation of Maximal Robust Control Invariant Set

Data: $\mathcal{X}_0$, $\mathcal{U}$, $\mathcal{W}$, $\mathbf{\Phi}$, $\mathbf{\Gamma}$

Result: Maximal Control Invariant Set $\mathcal{C}_\infty$

$k \leftarrow 0$;

$\mathcal{X}_{-k} \leftarrow \mathcal{X}_0$;

repeat

$\mathcal{X}_{-(k+1)} \leftarrow ((\mathcal{X}_{-k} \ominus \mathcal{W}) \oplus ((-\mathbf{\Gamma}) \circ \mathcal{U})) \circ \mathbf{\Phi}$;

$\mathcal{C} \leftarrow \mathcal{X}_{-(k+1)} \cap \mathcal{X}_{-k}$;

$\mathcal{X}_{-(k+1)} \leftarrow \mathcal{C}$;

$k \leftarrow k + 1$;

until $\mathcal{X}_{-k} = \mathcal{X}_{-k+1}$;

$\mathcal{C}_\infty = \mathcal{C}$

4.4 Perturbed Linear Dynamic System with Compensation

The computation of Forward Reachable Set for this section will be executed in two segments. The first segment will be evaluating Forward Reachable Set (FRS) of errors in linear system dynamics and second segment will be evaluating the norm of correction to be added. The final FRS for the complete system will then be evaluated by doing minkowski sum of both the sets.

4.4.1 Linear Counterpart Forward Reachable Set

The discrete system mentioned in eqn. (3.14), has the non-linear term, $\tilde{c}_d \mathbf{R}_k \mathbf{\Pi} \mathbf{R}_k^\intercal \dot{\mathbf{p}} \Delta t$. The focus of this section, is solely on the linear part. As stated earlier, the states will be divided as such - $\mathbf{e}_k = \bar{\mathbf{e}}_k + \delta \mathbf{e}_k$, where, $\bar{\mathbf{e}}_k$ is the linear counterpart and $\delta \mathbf{e}_k$ being the correction. The linear system is represented as follows,

$$\bar{\mathbf{e}}_{k+1} = \mathbf{\Phi} \bar{\mathbf{e}}_k \tag{4.43}$$

The control is not included in linear system, as the control is the compensation for the non-linearity and will be looked upon with the error correction.

We now construct the FRS in the similar manner that we used to construct FRS in linear autonomous section (4.2.1). We know that the $\mathbf{\Phi}$ we have is invertible, therefore, it will be used to construct the ellipsoidal set for all the steps. This theory was referred from ([20],[21])

Taking a general ellipsoidal set for initial set,

$$\bar{\xi}_0 = \{\bar{\mathbf{e}}_0 \in \mathbb{R}^6 : \quad \bar{\mathbf{e}}_0^\intercal \mathbf{P}_0 \bar{\mathbf{e}}_0 \leq 1\} \tag{4.44}$$

Using the eqn.(4.43), we can say that,

$$\bar{\mathbf{e}}_0 = \mathbf{\Phi}^{-k} \bar{\mathbf{e}}_k \tag{4.45}$$

Constructing the ellipsoid condition using the eqn.(4.45),

$$\bar{\mathbf{e}}_0 \mathbf{P}_0 \bar{\mathbf{e}}_0 = \bar{\mathbf{e}}_k^\intercal (\mathbf{\Phi}^{-k})^\intercal \mathbf{P}_0 \mathbf{\Phi}^{-k} \bar{\mathbf{e}}_k \tag{4.46}$$

Defining $\mathbf{P}_k = (\mathbf{\Phi}^{-k})^\intercal \mathbf{P}_0 \mathbf{\Phi}^{-k}$, we get the k-step FRS.

$$\bar{\xi}_k = \{\bar{\mathbf{e}}_k \in \mathbb{R}^6 : \quad \bar{\mathbf{e}}_k^\intercal \mathbf{P}_k \bar{\mathbf{e}}_k \leq 1\} \tag{4.47}$$

4.4.2 Correction System Analysis

The error in perturbed system, as mentioned before, will be evaluated by adding the correction, $\delta\mathbf{e}$, to the linear errors, $\bar{\mathbf{e}}$ (note - $\mathbf{e} = \bar{\mathbf{e}} + \delta\mathbf{e}$). Using eqn. (3.14),(4.43), the system of correction is formulated.

$$\delta\mathbf{e}_{k+1} = \mathbf{\Phi}\delta\mathbf{e}_k + \mathbf{\Gamma}\|\ddot{\mathbf{p}}_k^d\| s_\phi \mathbf{u}_k - \begin{bmatrix} \mathbf{0}_{3,3} \\ \tilde{c}_d \mathbf{R}_k \mathbf{\Pi} \mathbf{R}_k^\intercal \dot{\mathbf{p}}_k \end{bmatrix} \Delta t \tag{4.48}$$

The non-linear function mentioned in eqn. (4.48), is Lipschitz continuous, ie. $\|\tilde{c}_d \mathbf{R}\mathbf{\Pi}\mathbf{R}^\intercal \dot{\mathbf{p}}\| \leq |\tilde{c}_d|\|\dot{\mathbf{p}}\|$. The Lipschitz constant, $\ell = |\tilde{c}_d|$.

The equation can be re-written as,

$$\delta\mathbf{e}_{k+1} = \mathbf{\Phi}\delta\mathbf{e}_k + \mathbf{\Gamma}(\|\ddot{\mathbf{p}}_k^d\| s_\phi \mathbf{u}_k - \tilde{c}_d \mathbf{R}_k \mathbf{\Pi} \mathbf{R}_k^\intercal \dot{\mathbf{p}}_k) \tag{4.49}$$

The term, $\mathbf{\Phi}\delta\mathbf{e}_k$ is completely eliminated by substituting previous steps (till, $\mathbf{\Phi}^{k+1}\delta\mathbf{e}_0$), and $\delta\mathbf{e}_0 = 0$.

$$\delta\mathbf{e}_{k+1} \leq \sum_{i=0}^{k} \mathbf{\Phi}^{(k-i)}\mathbf{\Gamma}\|(\|\ddot{\mathbf{p}}_i^d\| s_\phi \mathbf{u}_i - \tilde{c}_d \mathbf{R}_i \mathbf{\Pi} \mathbf{R}_i^\intercal \dot{\mathbf{p}}_i) \tag{4.50}$$

This will further be divided into equations for $\delta\mathbf{e}_{k,p}$, $\delta\mathbf{e}_{k,v}$, by indexing the matrices $(\mathbf{\Phi}^{(k-i)}\mathbf{\Gamma})$ into $(\mathbf{\Phi}^{(k-i)}\mathbf{\Gamma})_{1,3;1,3}$, $(\mathbf{\Phi}^{(k-i)}\mathbf{\Gamma})_{4,6;1,3}$,

$$\begin{aligned} \delta\mathbf{e}_{k+1,p} &= \sum_{i=0}^{k} (\mathbf{\Phi}^{(k-i)}\mathbf{\Gamma})_{1,3;1,3}(\|\ddot{\mathbf{p}}_i^d\| s_\phi \mathbf{u}_i - \tilde{c}_d \mathbf{R}_i \mathbf{\Pi} \mathbf{R}_i^\intercal \dot{\mathbf{p}}_i) \\ \delta\mathbf{e}_{k+1,v} &= \sum_{i=0}^{k} (\mathbf{\Phi}^{(k-i)}\mathbf{\Gamma})_{4,6;1,3}(\|\ddot{\mathbf{p}}_i^d\| s_\phi \mathbf{u}_i - \tilde{c}_d \mathbf{R}_i \mathbf{\Pi} \mathbf{R}_i^\intercal \dot{\mathbf{p}}_i) \end{aligned} \tag{4.51}$$

The norm of the eqn. (4.51), will be used to evaluate the FRS.

$$\begin{aligned} \|\delta\mathbf{e}_{k+1,p}\| &\leq \sum_{i=0}^{k} \|(\mathbf{\Phi}^{(k-i)}\mathbf{\Gamma})_{1,3;1,3}\|\|(\|\ddot{\mathbf{p}}_i^d\| s_\phi \mathbf{u}_i - \tilde{c}_d \mathbf{R}_i \mathbf{\Pi} \mathbf{R}_i^\intercal \dot{\mathbf{p}}_i)\| \\ \|\delta\mathbf{e}_{k+1,v}\| &\leq \sum_{i=0}^{k} \|(\mathbf{\Phi}^{(k-i)}\mathbf{\Gamma})_{4,6;1,3}\|\|(\|\ddot{\mathbf{p}}_i^d\| s_\phi \mathbf{u}_i - \tilde{c}_d \mathbf{R}_i \mathbf{\Pi} \mathbf{R}_i^\intercal \dot{\mathbf{p}}_i)\| \end{aligned} \tag{4.52}$$

It shall be noted, $\|\mathbf{R}\boldsymbol{\Pi}\mathbf{R}^\intercal\| = 1$. Furthermore, eqn. (4.51), depends upon $\dot{\mathbf{p}} = \mathbf{e}_v + \dot{\mathbf{p}}_r$. Let $\dot{\bar{\mathbf{p}}}$, be the linear component of $\dot{\mathbf{p}}$, s.t. $\dot{\mathbf{p}} - \dot{\bar{\mathbf{p}}} = \delta\mathbf{e_v}$. Therefore, $\dot{\bar{\mathbf{p}}} = \dot{\mathbf{p}}_r + \bar{\mathbf{e}}_\mathbf{v}$. The term $\|(\|\ddot{\mathbf{p}}_i^d\| s_\phi \mathbf{u}_i - \tilde{c}_d \mathbf{R}_i \boldsymbol{\Pi}\mathbf{R}_i^\intercal \dot{\mathbf{p}}_i)\|$ can be reduced to,

$$\|(\|\ddot{\mathbf{p}}_i^d\| s_\phi \mathbf{u}_i - \tilde{c}_d \mathbf{R}_i \boldsymbol{\Pi}\mathbf{R}_i^\intercal \dot{\mathbf{p}}_i)\| \leq (\Delta_i + \bar{s}_\phi \|\mathbf{F}_{fb}\| + (\bar{s}_\phi\|\hat{c}_d\| + \|\tilde{c}_d\|)(\|\bar{\mathbf{e}}_i\| + (\|\delta\mathbf{e}_{i,p}\|^2 + \|\delta\mathbf{e}_{i,v}\|^2)^{\frac{1}{2}})$$

Where, $\Delta_k = |\tilde{c}_d| \|\dot{\mathbf{p}}_k^r\| + \bar{s}_\phi \|g\mathbf{e}_3 + \hat{c}_d \dot{\mathbf{p}}_k^r + \ddot{\mathbf{p}}_k^r\|$.

Thus, the final set of equations which is obtained,

$$\begin{aligned} \|\delta\mathbf{e}_{k+1,p}\| \leq \sum_{i=0}^{k} \|(\boldsymbol{\Phi}^{(k-i)}\boldsymbol{\Gamma})_{1,3;1,3}\| (\Delta_i + \bar{s}_\phi \|\mathbf{F}_{fb}\| + (\bar{s}_\phi\|\hat{c}_d\| + \|\tilde{c}_d\|) \\ (\|\bar{\mathbf{e}}_i\| + (\|\delta\mathbf{e}_{i,p}\|^2 + \|\delta\mathbf{e}_{i,v}\|^2)^{\frac{1}{2}}) \\ \|\delta\mathbf{e}_{k+1,v}\| \leq \sum_{i=0}^{k} \|(\boldsymbol{\Phi}^{(k-i)}\boldsymbol{\Gamma})_{4,6;1,3}\| (\Delta_i + \bar{s}_\phi \|\mathbf{F}_{fb}\| + (\bar{s}_\phi\|\hat{c}_d\| + \|\tilde{c}_d\|) \\ (\|\bar{\mathbf{e}}_i\| + (\|\delta\mathbf{e}_{i,p}\|^2 + \|\delta\mathbf{e}_{i,v}\|^2)^{\frac{1}{2}}) \end{aligned} \tag{4.53}$$

The term Δ, is the only term which depends upon the input, as well as the non-linearity. The values of $\{k_p, k_v, \mathbf{p}_r, \dot{\mathbf{p}}_r, \xi_0\}$, are controller gain and system prerequisites. Using these values, and the values of estimated system parameters, Δ, can be pre-estimated. An error profile can be generated using various pre-determined values of Δ and the behaviour of the system can be observed.

These norms, themselves can be treated as n-circular sets. Let the set of $\delta\mathbf{e}_{k,p}$ be denoted as, $\delta\xi_{k,p}$. Accordingly, we can define this set as,

$$\begin{aligned} \delta\xi_{k,p} = \{\delta\mathbf{e}_{k,p} \in \mathbb{R}^6 : \quad (\delta\mathbf{e}_{k,p}^\intercal \delta\mathbf{e}_{k,p})^{\frac{1}{2}} \leq \sum_{i=0}^{k} \|(\boldsymbol{\Phi}^{(k-i)}\boldsymbol{\Gamma})_{1,3;1,3}\| (\Delta_i + \bar{s}_\phi \|\mathbf{F}_{fb}\| + \\ (\bar{s}_\phi\|\hat{c}_d\| + \|\tilde{c}_d\|)(\|\bar{\mathbf{e}}_i\| + (\|\delta\mathbf{e}_{i,p}\|^2 + \|\delta\mathbf{e}_{i,v}\|^2)^{\frac{1}{2}})\} \end{aligned} \tag{4.54}$$

The set $\delta\xi_{k,v}$ can also be formulated in the same procedure.

The reachable set of linear system can now be indexed into reachable set of position errors $\bar{\xi}_{k,p}$ and reachable set of velocity errors $\bar{\xi}_{k,v}$. Thus, the forward reachable set for the perturbed system can be denoted as $\xi_{k,p},\ \xi_{k,v}$. The set of position errors can thus, be represented as,

$$\xi_{k,p} = \bar{\xi}_{k,p} \oplus \delta\xi_{k,p} \tag{4.55}$$

Thus, after evaluating both linear system FRS and correction FRS, we can obtain perturbed system FRS.

4.4.3 Computation of Reachable Set for Perturbed System with Compensation

The ellipsoidal sets will be represented with the positive definite matrix and centre, used to define the set, for the algorithm. For example,

$$\xi(\mathbf{P}_0, \mathbf{c}) = \{\mathbf{e} \in \mathbb{R}^n : \quad (\mathbf{e} - \mathbf{c})^\intercal \mathbf{P}_0 (\mathbf{e} - \mathbf{c}) \leq 1\}$$

To be noted, $\|\xi(\mathbf{P}_0, \mathbf{c})\|$ implies, that all the states within the set $\xi(\mathbf{P}_0, \mathbf{c})$ are mapped to their respective norms.

Furthermore, $\max\{\|\xi(\mathbf{P}_0, \mathbf{c})\|\}$ implies, maximum of all the norms of states within the set $\xi(\mathbf{P}_0, \mathbf{c})$.

The requirement is to estimate the Forward Reachable Set of Position errors, thus, the states in the set will be split accordingly, using the 'index()' command. This command will split all the states within $\xi(\mathbf{P}, \mathbf{c}_p)_p$, into position states $\xi(\mathbf{P}, \mathbf{c}_v)_v$ and velocity states.

Remark 4.11. For computing ellipsoidal sets, see [53].

Remark 4.12. For Minkowski sum of different ellipsoids, see [54].

The algorithm to compute the Forward Reachable set is thus, given below,

Algorithm 4.4.1: Computation of Forward Reachable Set

Data: $\xi(\mathbf{P}_0, 0),\ \mathbf{\Phi},\ \mathbf{\Gamma},\ \Delta,\ \bar{s}_\phi,\ k_f,\ \mathbf{p}_r,\ \dot{\mathbf{p}}_r$

Result: Maximal Control Invariant Set $\mathcal{C}_\infty$

$k \leftarrow 0;$

$\|\delta\mathbf{e}_{k,p}\| \leftarrow 0;$

$\|\delta\mathbf{e}_{k,v}\| \leftarrow 0;$

$\xi(\mathbf{P}_k, 0) \leftarrow \xi(\mathbf{P}_0, 0);$

$\bar{\xi}(\mathbf{P}_k, 0) \leftarrow \xi(\mathbf{P}_0, 0);$

$\|\bar{\mathbf{e}}_k\| \leftarrow \max\{\|\bar{\xi}(\mathbf{P}_k, 0)\|);$

while $k \leq k_f$ **do**

$\quad \bar{\xi}(\mathbf{P}_{k+1}, 0) \leftarrow \bar{\xi}((\mathbf{\Phi}^{-1})^\intercal \mathbf{P}_k \mathbf{\Phi}^{-1}, 0);$

$\quad r_p \leftarrow \sum_{i=0}^{k} \|(\mathbf{\Phi}^{(k-i)}\mathbf{\Gamma})_{1,3;1,3}\|(\Delta_i + \bar{s}_\phi\|\mathbf{F}_{fb}\| + (\bar{s}_\phi\|\hat{c}_d\| + \|\tilde{c}_d\|)(\|\bar{\mathbf{e}}_i\| + (\|\delta\mathbf{e}_{i,p}\|^2 + \|\delta\mathbf{e}_{i,v}\|^2)^{\frac{1}{2}});$

$\quad r_v \leftarrow \sum_{i=0}^{k} \|(\mathbf{\Phi}^{(k-i)}\mathbf{\Gamma})_{4,6;1,3}\|(\Delta_i + \bar{s}_\phi\|\mathbf{F}_{fb}\| + (\bar{s}_\phi\|\hat{c}_d\| + \|\tilde{c}_d\|)(\|\bar{\mathbf{e}}_i\| + (\|\delta\mathbf{e}_{i,p}\|^2 + \|\delta\mathbf{e}_{i,v}\|^2)^{\frac{1}{2}});$

$\quad \delta\xi(\delta\mathbf{P}_{k+1}, 0)_p \leftarrow \delta\xi(\frac{\mathbf{I}_3}{r_p^2}, 0);$

$\quad \delta\xi(\delta\mathbf{P}_{k+1}, 0)_v \leftarrow \delta\xi(\frac{\mathbf{I}_3}{r_v^2}, 0);$

$\quad (\bar{\xi}(\mathbf{P}_{k+1}, 0)_p, \bar{\xi}(\mathbf{P}_{k+1}, 0)_v) \leftarrow \text{index}(\bar{\xi}(\mathbf{P}_{k+1}, 0));$

$\quad \xi(\mathbf{P}_{k+1}, 0)_p \leftarrow (\bar{\xi}(\mathbf{P}_{k+1}, 0)_p \oplus \delta\xi(\delta\mathbf{P}_{k+1}, 0)_p);$

$\quad \mathcal{R}_{k+1,p} \leftarrow \xi(\mathbf{P}_{k+1}, \mathbf{p}_{r,k+1})_p \cup \mathcal{R}_{k,p};$

$\quad \|\delta\mathbf{e}_{k+1,p}\| \leftarrow \max\{\|\delta\xi(\delta\mathbf{P}_{k+1}, 0)_p\|\};$

$\quad \|\delta\mathbf{e}_{k+1,v}\| \leftarrow \max\{\|\delta\xi(\delta\mathbf{P}_{k+1}, 0)_v\|\};$

$\quad \|\bar{\mathbf{e}}_{k+1}\| \leftarrow \max\{\|\bar{\xi}(\mathbf{P}_{k+1}, 0)\|\};$

$\quad k \leftarrow k + 1;$

end

$\mathcal{R}_{k_f} \leftarrow \mathcal{R}_k;$

CHAPTER 5

APPLICATION, RESULTS AND DISCUSSION

5.1 Maximal Control Invariant Set for Discrete system

The Maximal Control Invariant (MCI) Set for unstable discrete system (figure 5.1), was computed using Algorithm(4.1.1).

The input that was taken, is given below,

$$\mathbf{\Phi} = \begin{bmatrix} 1.5 & 0 \\ 1 & -1.5 \end{bmatrix}, \ \mathbf{\Gamma} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ k_f = 75 \tag{5.1}$$

The maximal control invariant set is computed within 75 steps. The polytope equation of the Maximal Control Invariant set was computed to be -

$$\mathcal{C}_\infty = \{\mathbf{x} \in \mathbb{R}^2 : \quad \mathbf{H}_f \mathbf{x} \leq \mathbf{h}_f\} \tag{5.2}$$

where,

$$\mathbf{H}_f = \begin{bmatrix} 2 & -3 \\ -2 & 3 \\ 0 & 2 \\ 0 & -2 \end{bmatrix} \text{ and } \mathbf{h}_f = \begin{bmatrix} 8 \\ 8 \\ 8 \\ 8 \end{bmatrix}$$

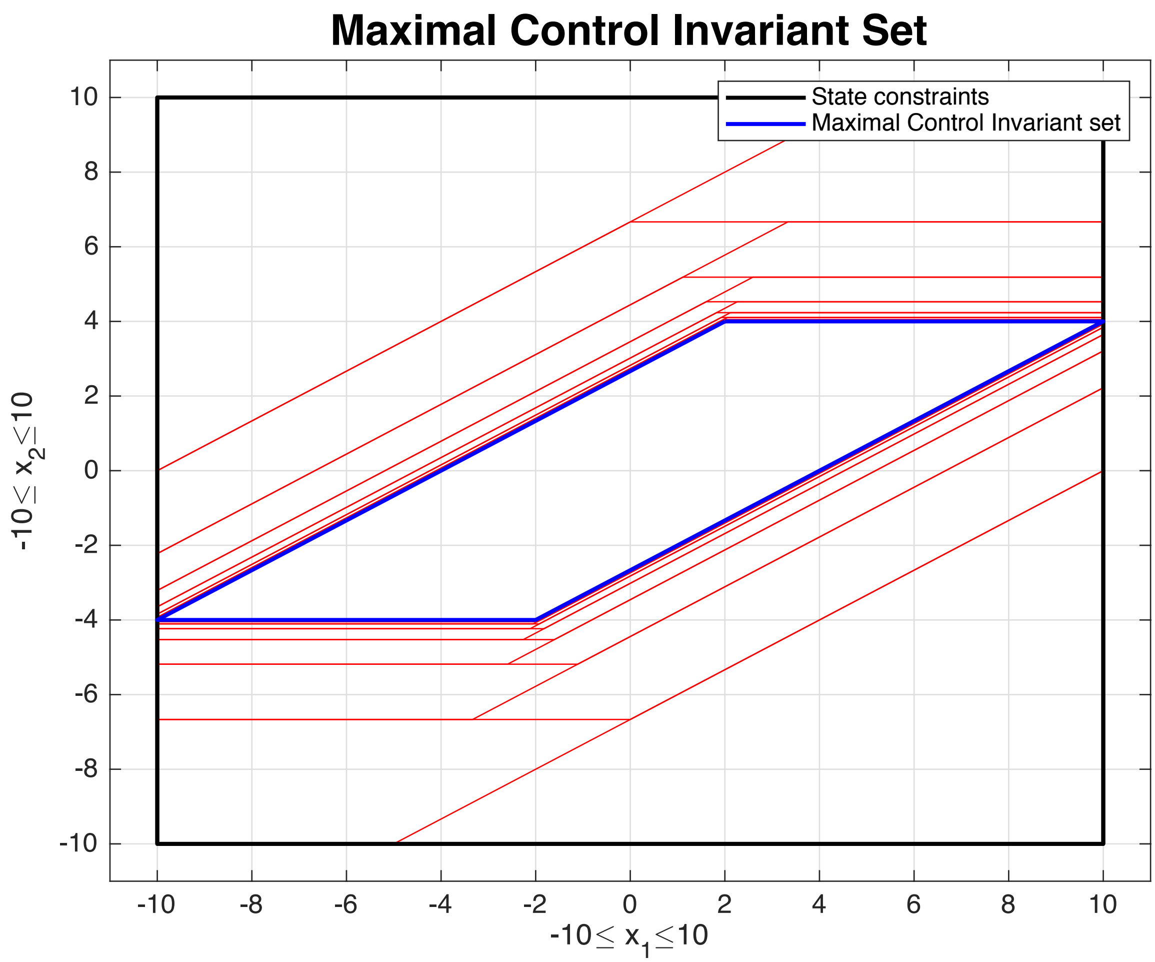


Figure 5.1: Maximal Controllable Set

Remark 5.1. As the number of constraints were less, it was feasible to write the obtained MCI set. However, for the other problems, these sets obtained have a large number of constraints, thus, no final set constraints are exhibited for other results.

5.2 Reachable Set for LTI System

5.2.1 Effect of Δt on Reachable Set

The choice of Δt affects the precision with which the reachable set is computed. The Euler 1-step discretization, produces truncation error in each step, thus, having a low Δt will generate more precise results, and it can be seen in the exhibited results.

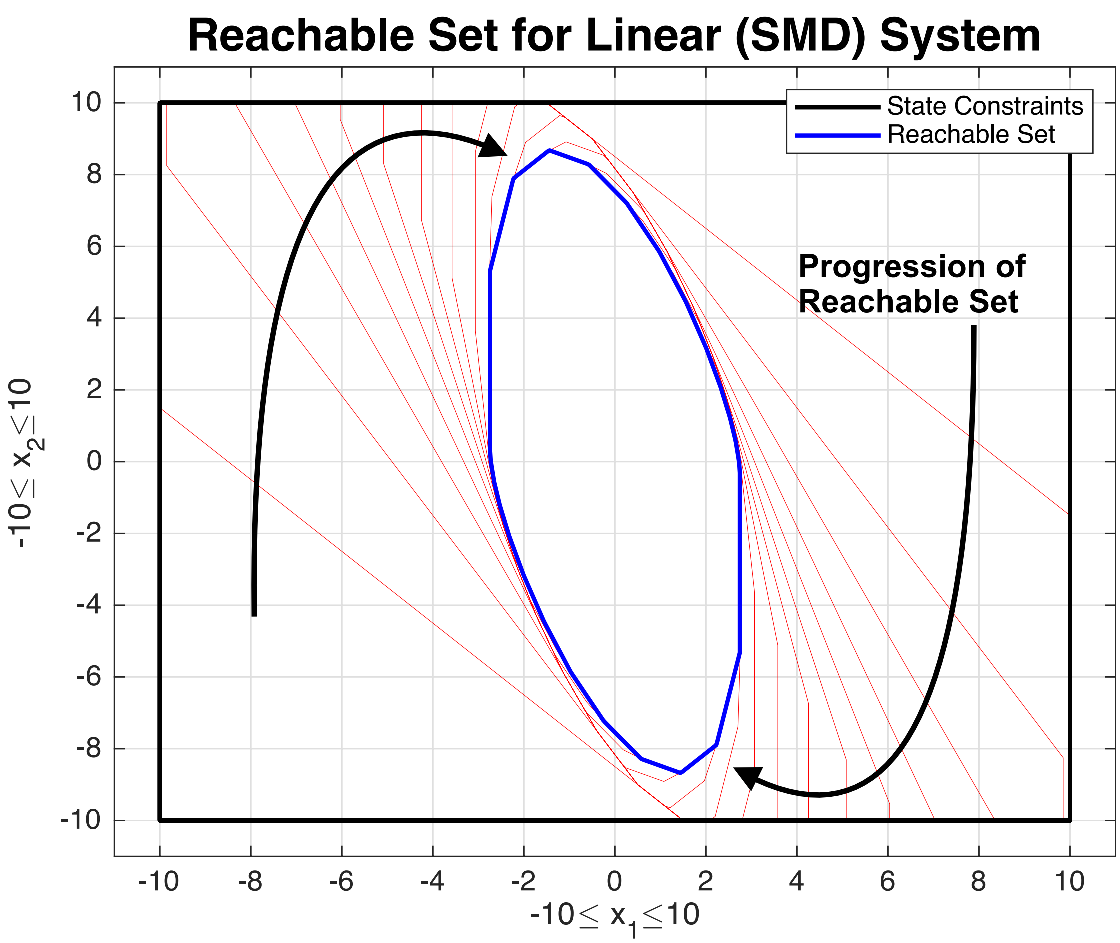


Figure 5.2: Reachable Set, when $|u_{max}| \leq 25,\ \ \Delta t = 0.1sec$

As it can be seen in fig.(5.2) and fig.(5.3), the size of the reachable set also gets affected and reduces for a lower Δt. However, this is the direct cause of truncation error, and is not a system property. The truncation error can be reduced by reducing Δt. Thus, $\Delta t = 0.01s$ shall be used in the considered perturbed system case.

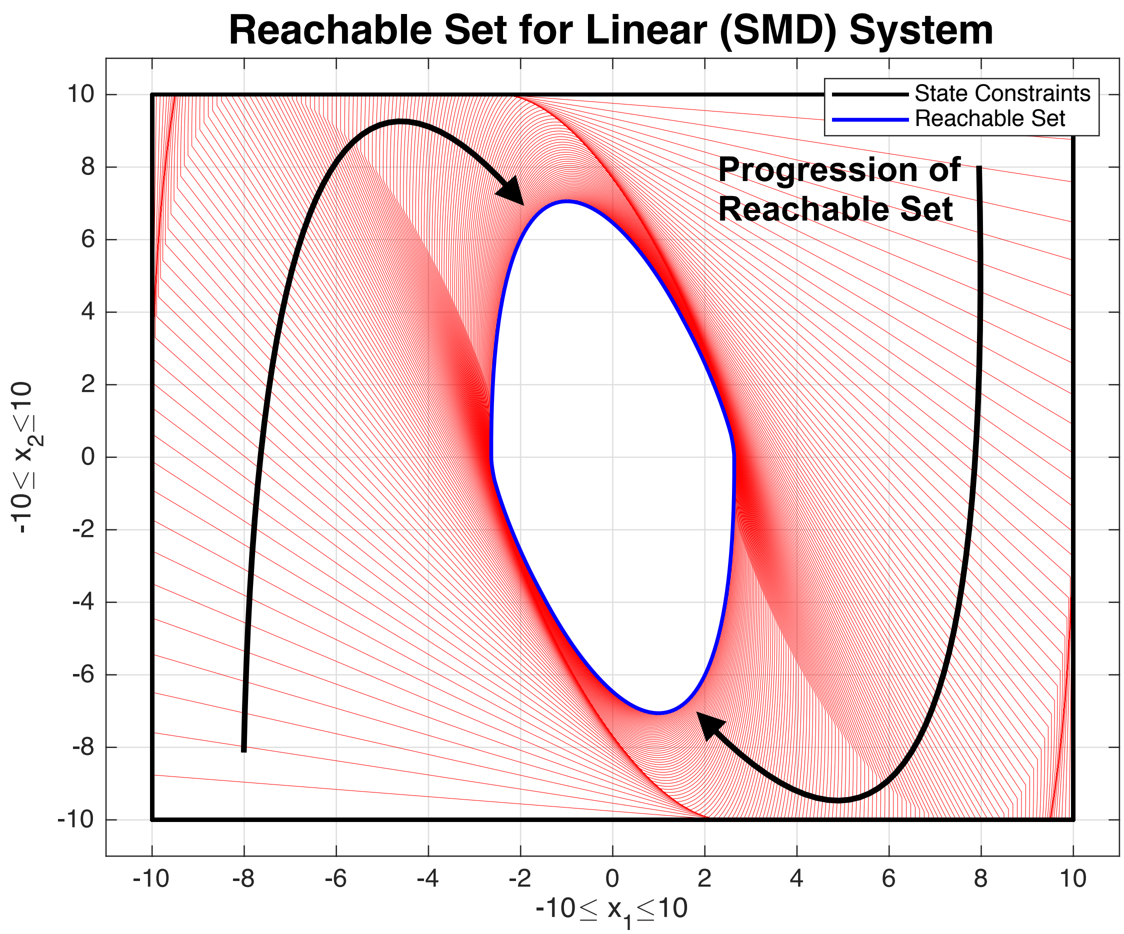


Figure 5.3: Reachable Set, when $|u_{max}| \leq 25$, $\Delta t = 0.01sec$

It should be noted that the coarseness of the reachable set, isn't because of truncation errors, but because zonotope is a discrete realization of the actual reachable set. Thus, lower time step allows for a finer realization of the actual reachable set.

5.2.2 Effect of u_{max} on Reachable Set

This is the direct result of what has been stated in section(4.2.2). The reachable set develops into a zonotope of control input and has less effect of initial states for higher run-time. In this case, $|u_k| \leq u_{max} = 5$ is the constraint over control which has been taken.

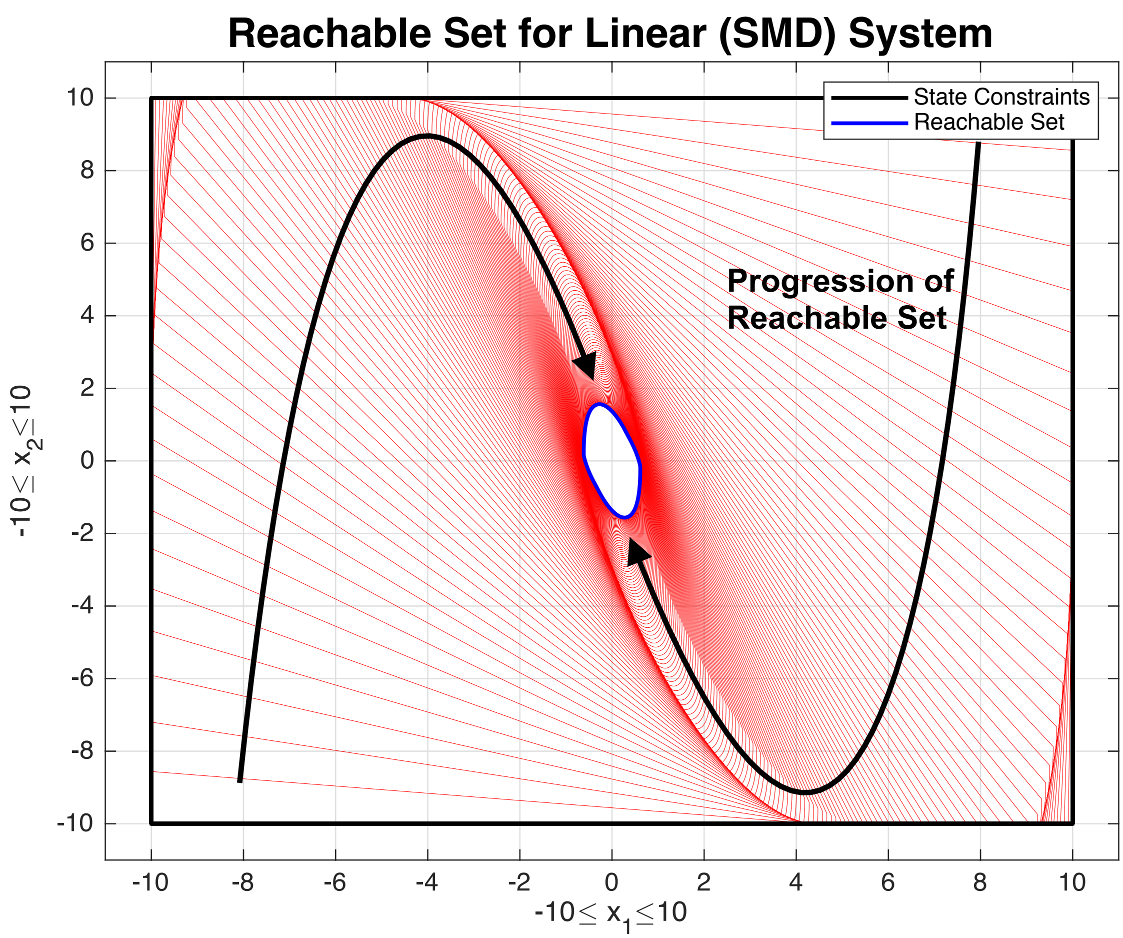


Figure 5.4: Reachable Set, when $|u_k| \leq u_{max} = 5, \ \Delta t = 0.01sec$

Increasing the magnitude of u_{max} from 5 to 25, the size of the reachable set is seen to increase. This increase in size is direct scaling due to control constraint u_{max}.

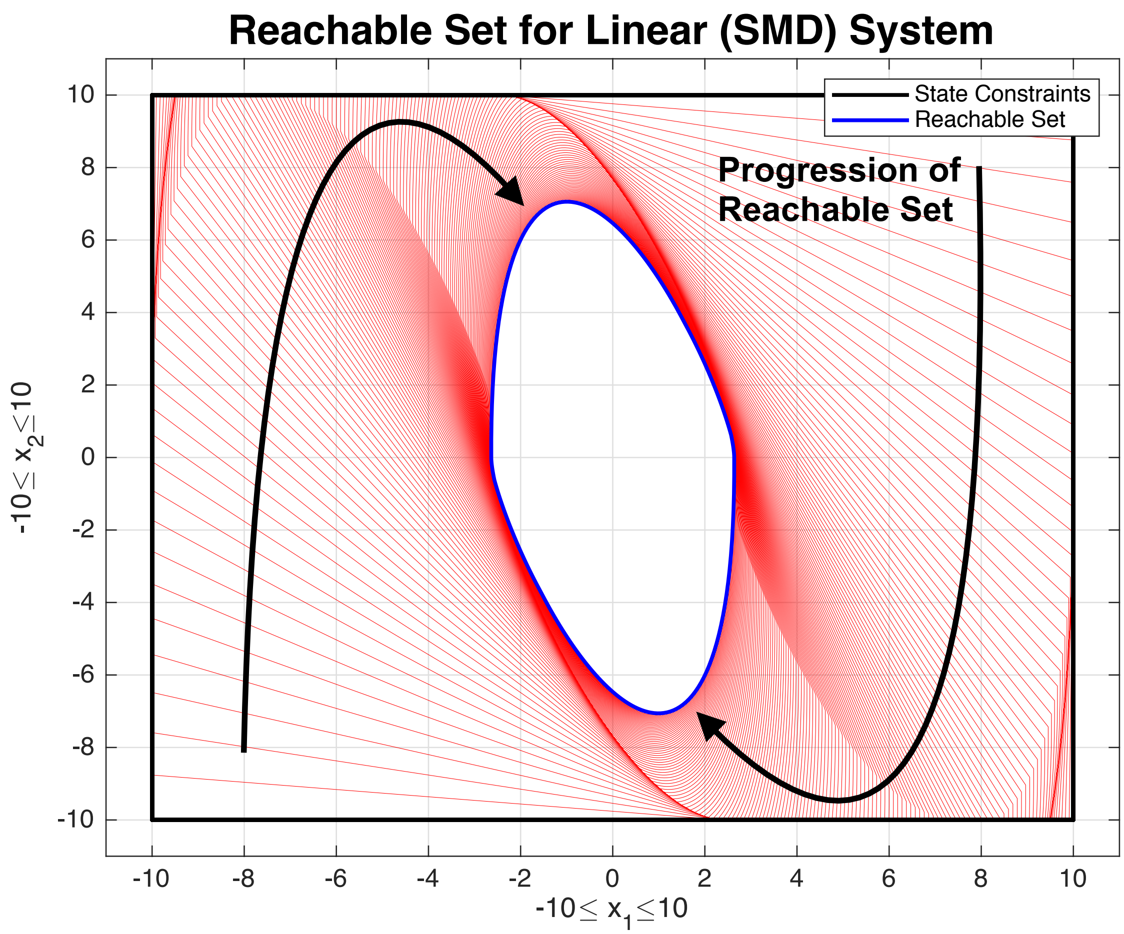


Figure 5.5: Reachable Set, when $|u_k| \leq u_{max} = 25,\ \Delta t = 0.01sec$

5.2.3 Maximal Positive Invariant and Maximal Control Invariant Set

As the system was stable, its maximal positive invariant set can be constructed. This is just a sub-case of MCI set for when $u_{max} \rightarrow 0$. Henceforth, the maximal positive invariant set will be referred to as, MCI set with 0 control input.

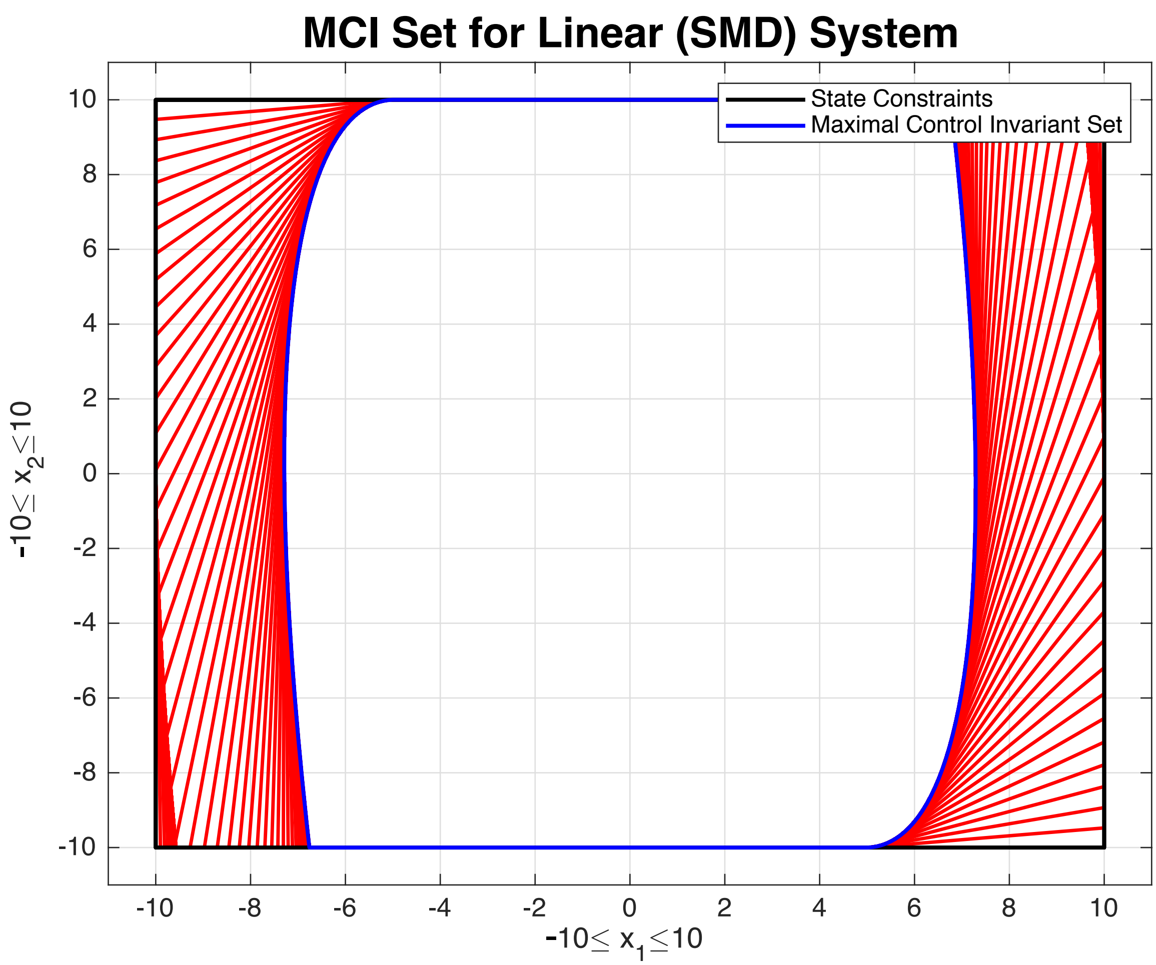


Figure 5.6: Maximal Positive Invariant Set

Increasing the constraint of control input, can help us contain more states within MCI Set. Furthermore, increasing it enough can make the total constraint set as MCI set.

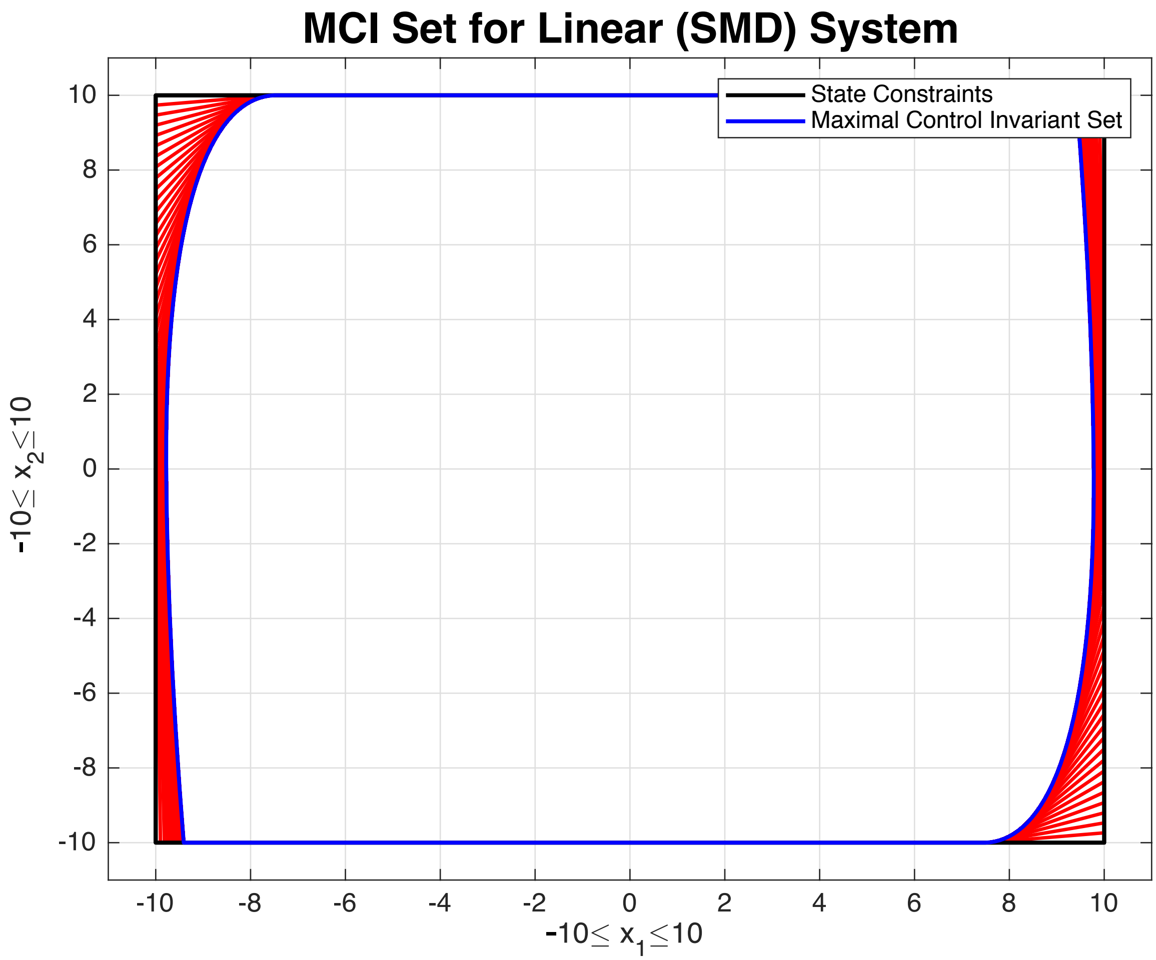


Figure 5.7: Maximal Control Invariant Set when $|u_k| \leq u_{max} = 25$

The figure. (5.8), will be used to describe the scheme of the different reachable sets, that shall be used to define the results in this section. The reason for this is, four different sets will be focused upon in this section (not including state constraint set). This color scheme will be used as reference to their respective set, and will be used in the same manner throughout the section (5.3). The color/line-style scheme is established for this section only.

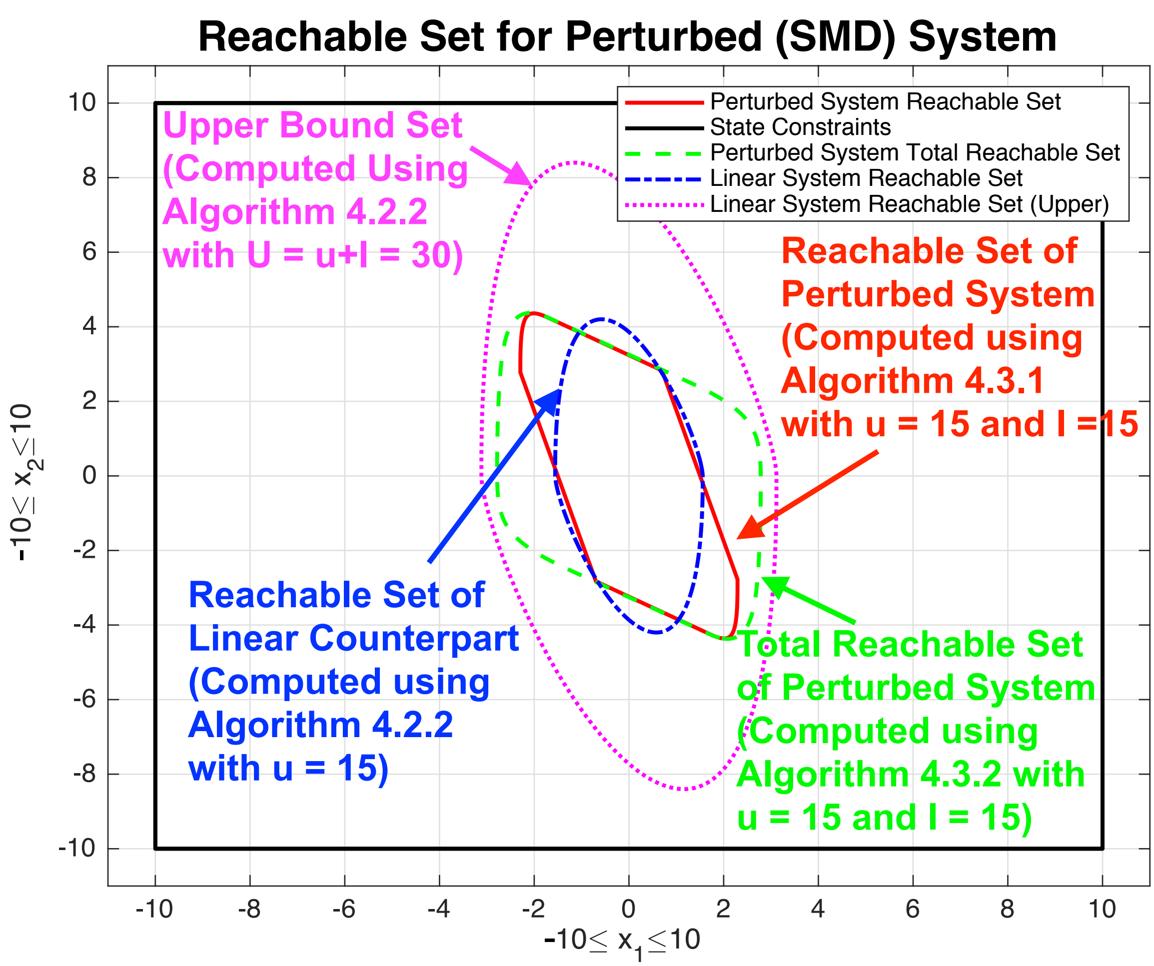
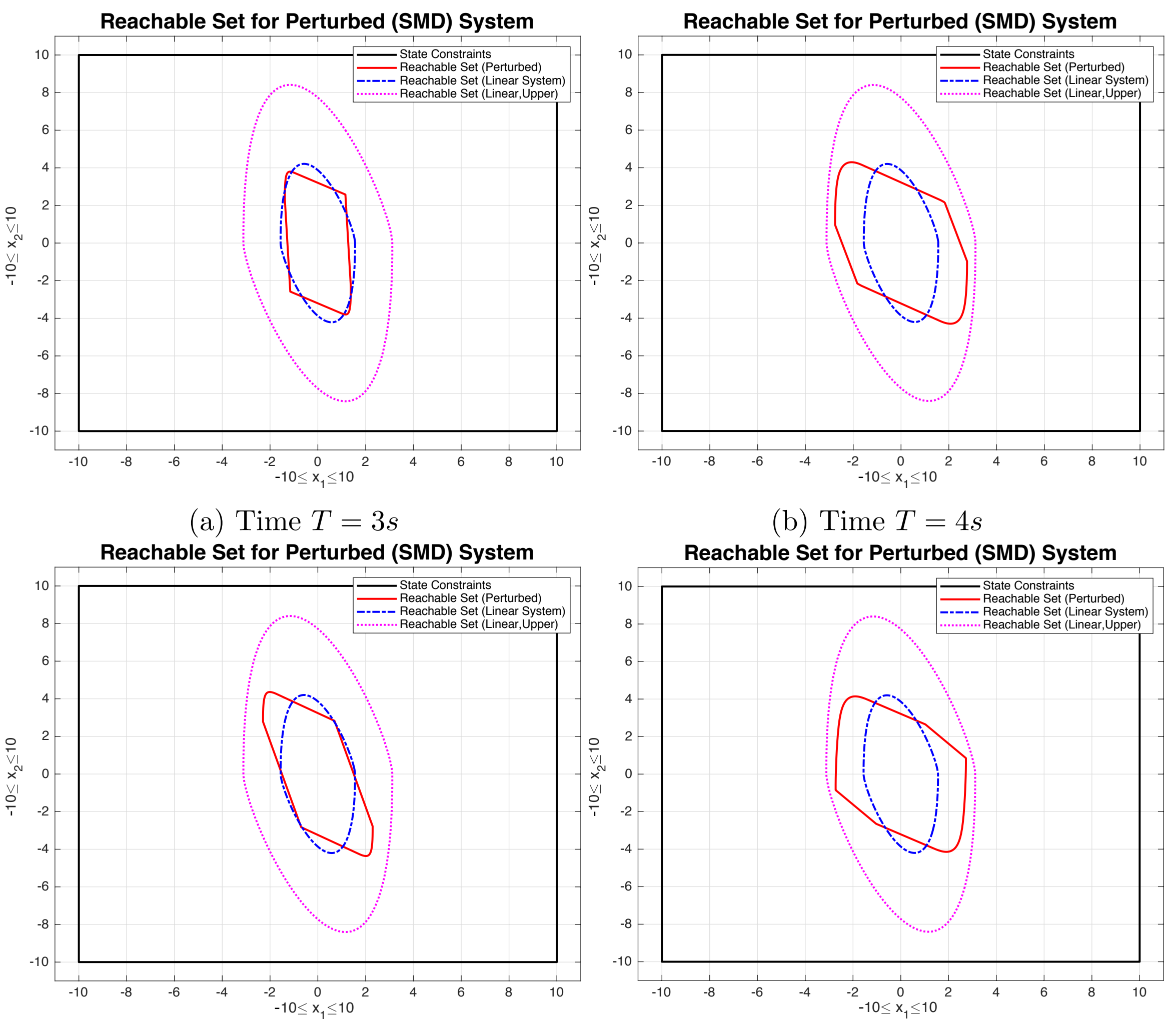


Figure 5.8: Reachable Set when $\ell = 15$, $|u_k| \leq 15$ at time $T = 5s$

5.3.1 Estimation of Forward Reachable Set of Perturbed System at Different times

The procedure to obtain Forward Reachable Set at time T mentioned in algorithm (4.3.1), computes an accurate reachable set, but has a drawback. For discussing the drawback, the results of Forward Reachable Sets shall be observed at different times. The figure (5.9), shows the FRS of perturbed system (in red) estimated at different times.



(a) Time $T = 3s$

(b) Time $T = 4s$

(c) Time $T = 5s$

(d) Time $T = 6s$

Figure 5.9: Comparison of Forward Reachable Set of Perturbed System at different times for $|u_k| \leq 15$ and $\ell = 15$

Listing the observations -

1. The reachable set of linear systems does not change over time but, the reachable set of perturbed system, keeps changing over time.
2. The outer reachable set always behaves as an upper bound to the reachable set of perturbed system.

Analysing these observation -

1. Reachable set of perturbed system obtained using this method, never realises the "Total Reachable Set". In other words, even if the reachable set at time T_1 is computed, the reachable set at time $T_2 > T_1$ cannot be confirmed to be contained within the the reachable set of T_1.
2. The reachable set of perturbed system will never go outside those bounds, thus, utilising the upper bound set can be helpful. However, the upper bound set also contains the states which are not reachable by the perturbed system. Thus, usage of the upper bound set, can *mislead* into thinking of a certain unreachable state as a reachable state.

This led to the problem of, *How to estimate the 'Total Reachable Set' of the perturbed system?* This set is discussed in the next subsection, where, a different approach over building the reachable set is used.

5.3.2 Method 2 - Estimation of Forward Reachable Set of Perturbed System

The Method-2 approach follows the idea of union of sets rather than intersection. Therefore, this method requires a small *initiating* subset of the reachable set, rather than a larger initial set, as the union method will never estimate a set boundary within the initial set itself. This reachable set of perturbed system obtained, will be called 'Total Reachable Set' (in green/dashed).

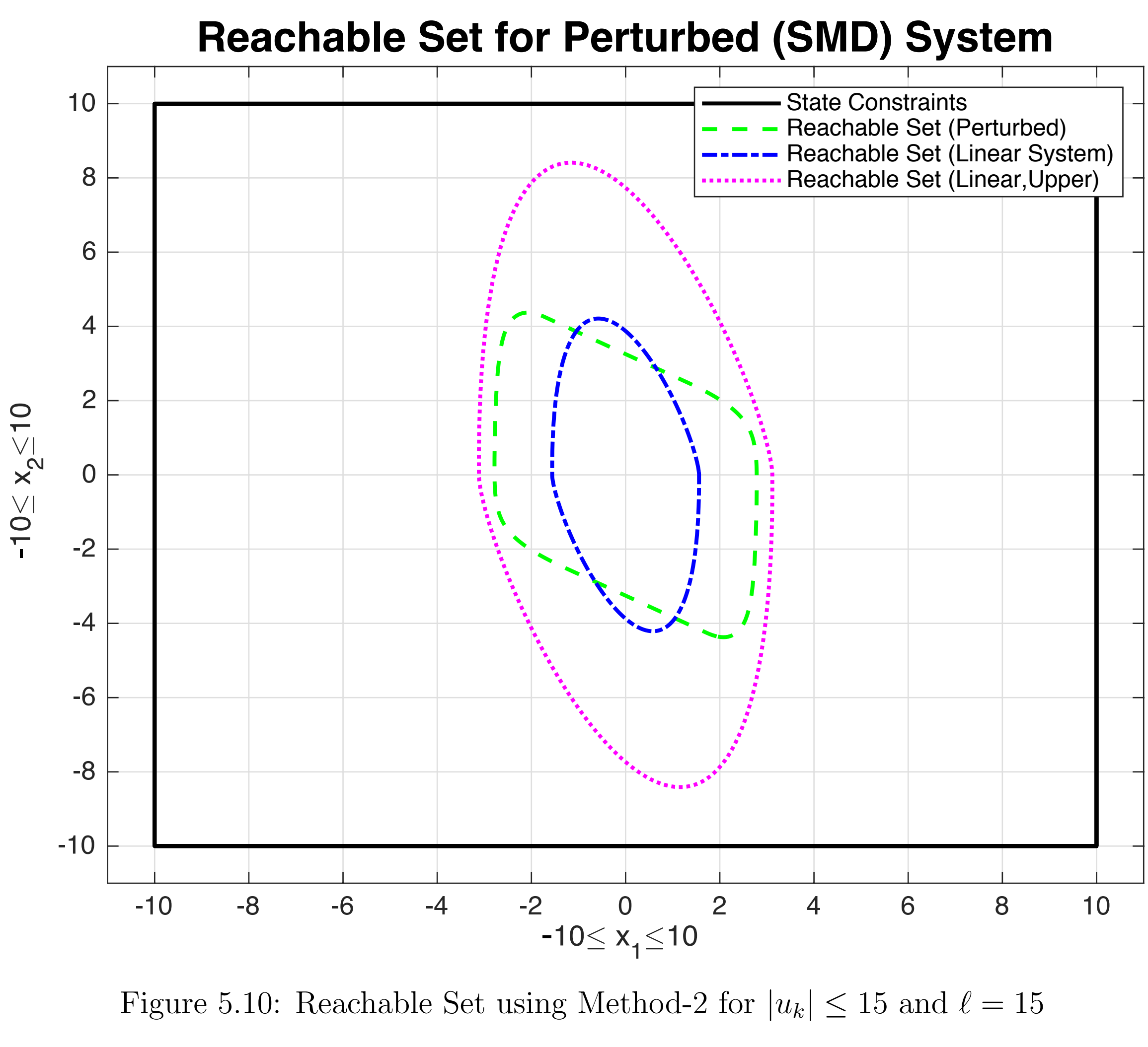


Figure 5.10: Reachable Set using Method-2 for $|u_k| \leq 15$ and $\ell = 15$

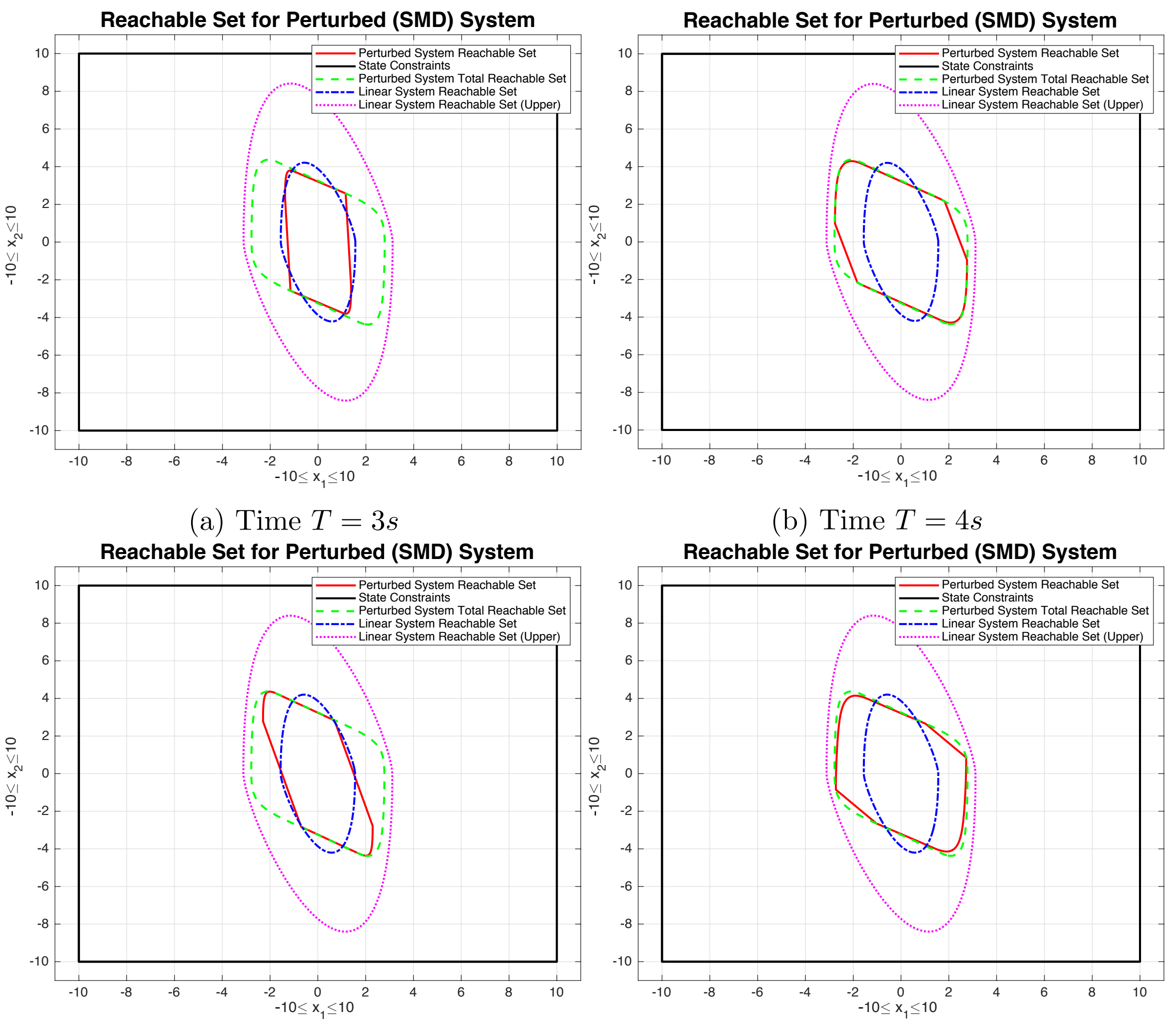
The reason for calling it 'Total Reachable Set' is, that the set boundary estimated using this method, takes into account all the states reachable at any given time $t \leq T$. This enables us to take into account, all the possible reachable sets of perturbed system within time T and build a union of all these sets.

The total reachable set provides the states which are reachable *within* time T. This eliminates all the unwanted unreachable states in the upper bound set.

Although, this method can determine all the reachable states, it has a drawback. The set of all reachable states does not imply all these states are reachable at time T. The comparison will be done and shown in the next subsection.

5.3.3 Comparison of both Reachable Sets

Having computed both these sets, the results can be combined and be subjected to comparison. This comparison is carried out to demonstrate that the FRS of perturbed system, beyond settling time, remains within the total reachable set.



(a) Time $T = 3s$

(b) Time $T = 4s$

(c) Time $T = 5s$

(d) Time $T = 6s$

Figure 5.11: Comparison of Forward Reachable Set of Perturbed System and Total Reachable Set of Perturbed System at different times for $|u_k| \leq 15$ and $\ell = 15$

Looking at these sets it can be confirmed that the algorithm produces an accurate Total Reachable set.

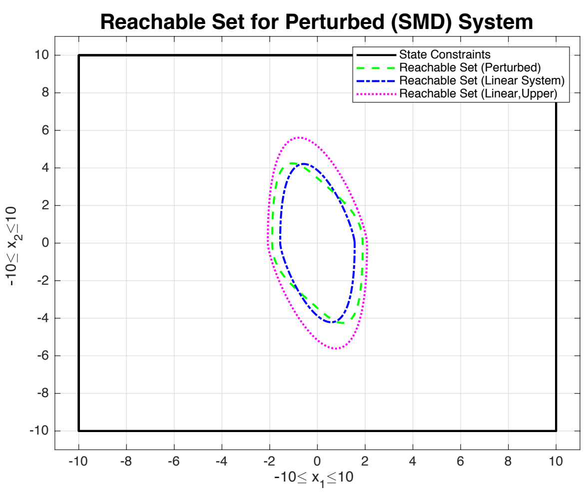
Comparing the result -

1. The primary reason, the upper bound set was discarded as an approximate set, was because, apart from containing the reachable states, it also contained non-reachable states. The total reachable set eliminates all the unreachable states.
2. The total reachable set, cannot guarantee that all the states contained in it can be reached *at* a given time T, the only way to know which state is reachable *at* time T, is to build the exact reachable set for that time.
3. The total reachable set, can guarantee that all states contained in it are reachable *within* time T, thus, even if that state is not reachable at time $T_1 \leq T$, it might be reachable at time $T_2 \leq T$, if it's contained in the total reachable set at time T.

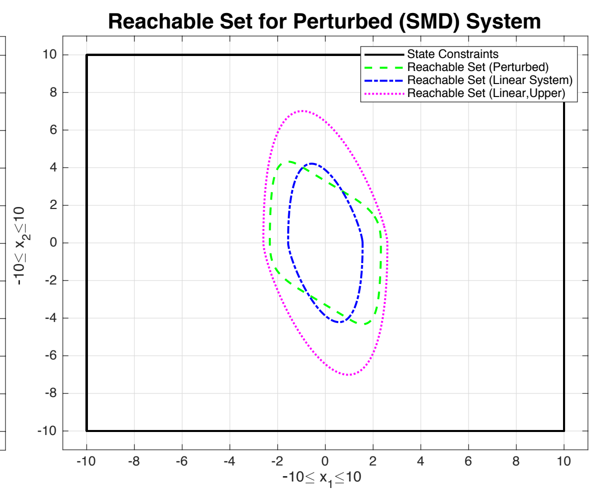
Thus, total reachable set and the forward reachable set both have their usage depending upon the need.

5.3.4 Size of Reachable Set with increasing ℓ for same Control Constraint

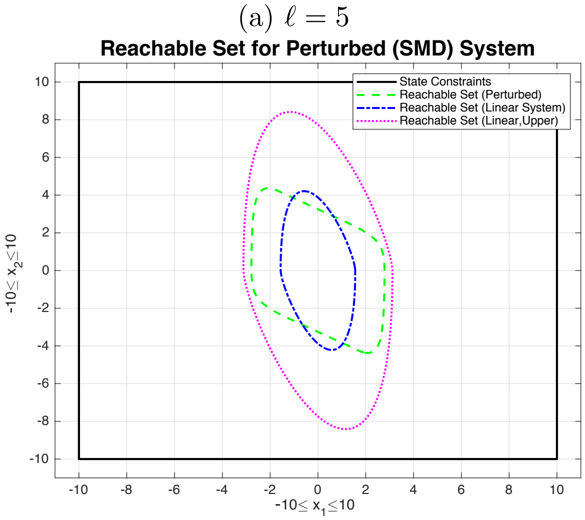
As discussed in the linear system, the size of the FRS of linear system depends upon the size of constraint on control input. Larger the control, larger is our forward reachable set. For the perturbed system, the size of the FRS also depends upon the value of perturbation bound (which in our case is ℓ). This was formulated in section (4.3.2).



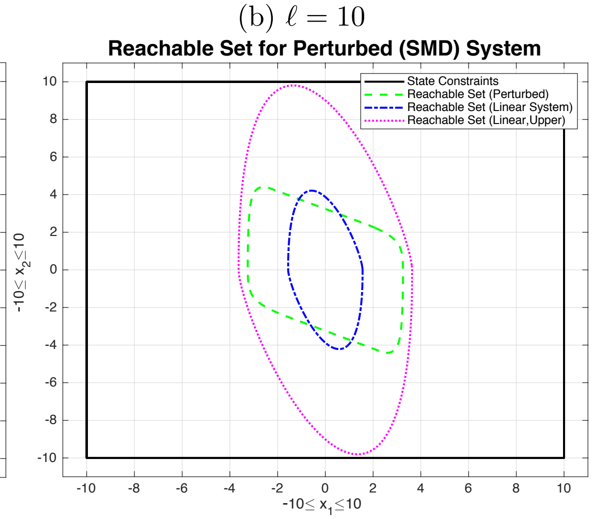
(a) $\ell = 5$



(b) $\ell = 10$



(c) $\ell = 15$

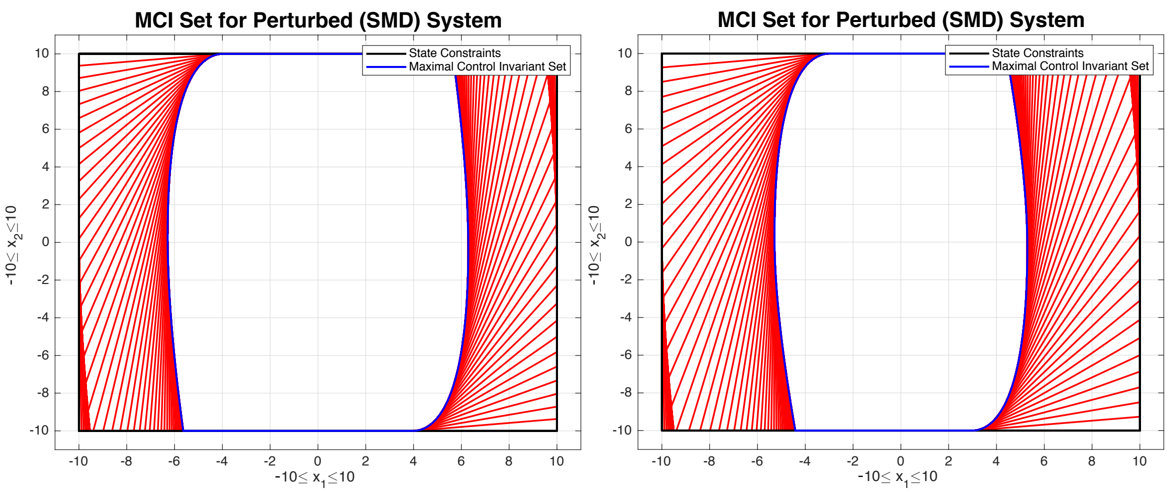


(d) $\ell = 20$

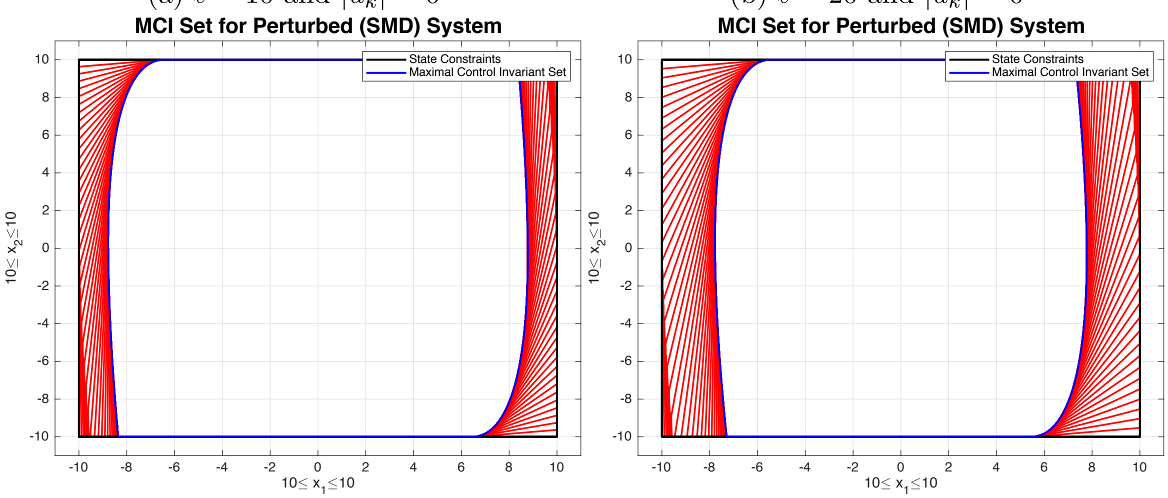
Figure 5.12: Comparison of Size of Total Reachable Set of Perturbed System for different ℓ where $|u_k| \leq 15$

5.3.5 Maximal Control Invariant (MCI) Set of the Perturbed System

The MCI set of the Perturbed System is built on the same lines as Maximal Robust Control Invariant Set. The reason for doing so was, that building a precursor set of a system with $\sin(x)$ is highly complex. $\sin(x)$ is a **many-to-one** function, thus, for any desired value of $\ell\sin(x_2)$, infinite possibility of $\mathbf{x}$ state exists, whose x_2, will satisfy the desired value.



(a) $\ell = 10$ and $|u_k| = 0$ (b) $\ell = 20$ and $|u_k| = 0$



(c) $\ell = 10$ and $|u_k| \leq 25$ (d) $\ell = 20$ and $|u_k| \leq 25$

Figure 5.13: MCI Set of Perturbed System for different ℓ and $|u_k|$

Although, these are Robust MCI sets, they are still subsets of the actual MCI sets. This implies that the states in these sets, will never violate the state constraint. Thus, it satisfies the conditions needed for the MCI set to follow.

A few things to observe -

1. The size of the MCI set obtained grows larger when control input constraints are increased. This implies that with high enough control input the whole state constraint set can become an MCI set. This follows in line with the linear MCI set.
2. The value of ℓ also affects the size of MCI set as the robustness was taken into account. Thus, the size decreases as the value of ℓ is increased.

5.4 Perturbed Linear Dynamic System (Compensation)

Following the procedure, FRS of the linear error counterpart will be estimated first. Then the norm of correction error at each step will be estimated. The norm itself can be interpreted as an equation of a circle, implying that the correction error set will be a circular set. The Minkowski sum of both these sets will yield us the total FRS of perturbed error system. These errors are centred around the reference position of that time step. Thus, each of these errors will be plotted around the respective reference position. The size of the set guarantees how close to the reference trajectory the multi-rotor will be.

5.4.1 FRS of the Linear Counterpart

All the states in FRS of a linear system tend to zero. The system which we have, has states which are errors of position and velocity. The error states will tend to 0 with increasing time. This implies that the reachable set of multi-rotor will tend to reference trajectory and converge on it.

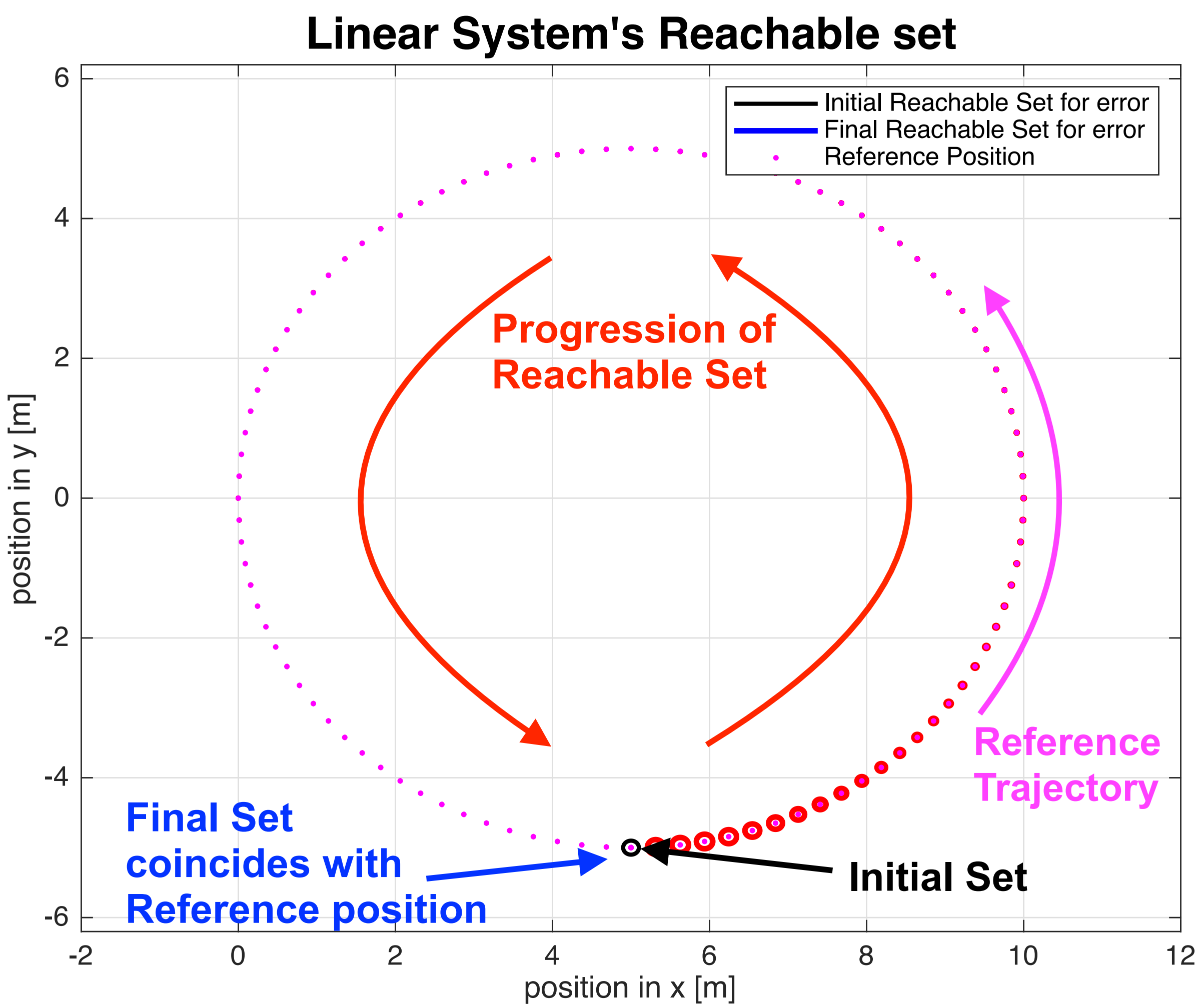


Figure 5.14: Reachable Set of ξ_{0s} for the Linear system

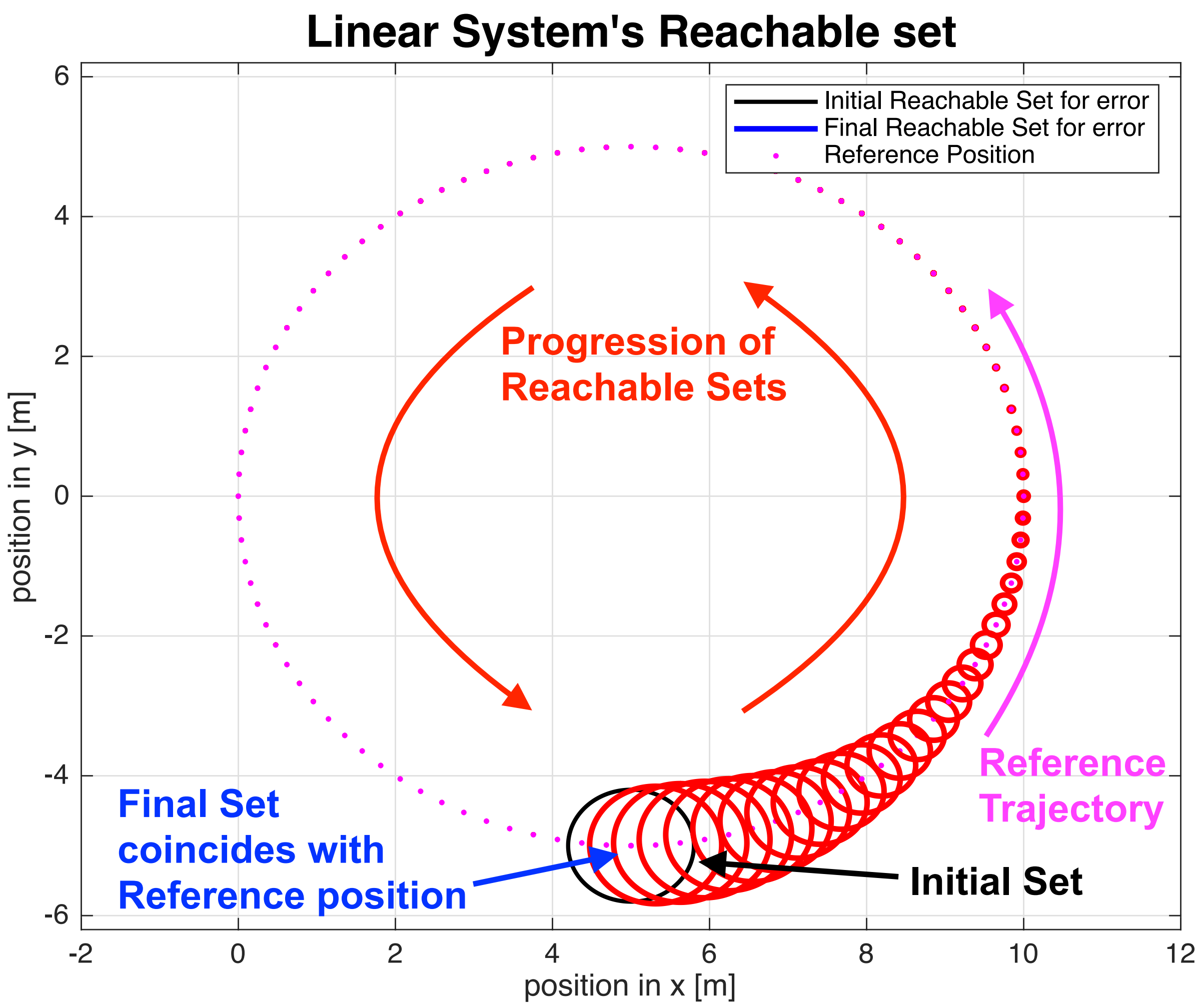


Figure 5.15: Reachable Set of ξ_{0b} for the Linear System

5.4.2 FRS of the Perturbed System (with and without compensation) for the Initial Set ξ_{0s}

The values of Δ taken are mentioned in the **legend**. The initial set ξ_{0s}, is used to plot the total error profile of the system for different values of Δ.

For the initial set of errors ξ_{0s}, the error is seen to grow, but is bounded despite the growth.

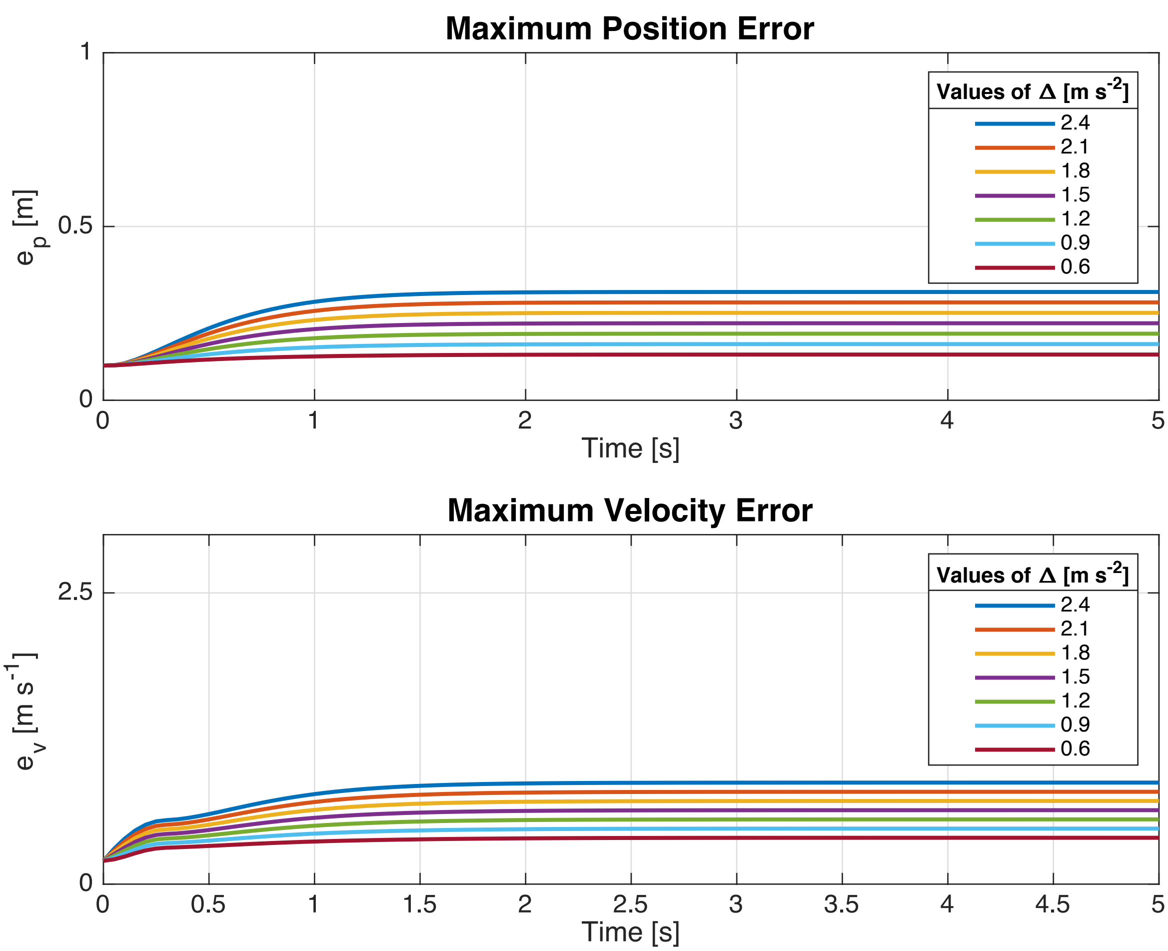


Figure 5.16: Error profile of position error and velocity error over time, for $|\tilde{c}_d| = 0.35$ and initial error sets : ξ_{0s} and different values of Δ[m s^{-2}]

The FRS of the initial set ξ_{0s} is estimated. Comparing the results of each initial set for two different cases - Uncompensated case and Compensated case.

The FRS of errors, for the initial set ξ_{0s} (uncompensated) - The size of the reachable set grows and stabilizes after a certain time. This, implies that the errors will grow but remain bounded.

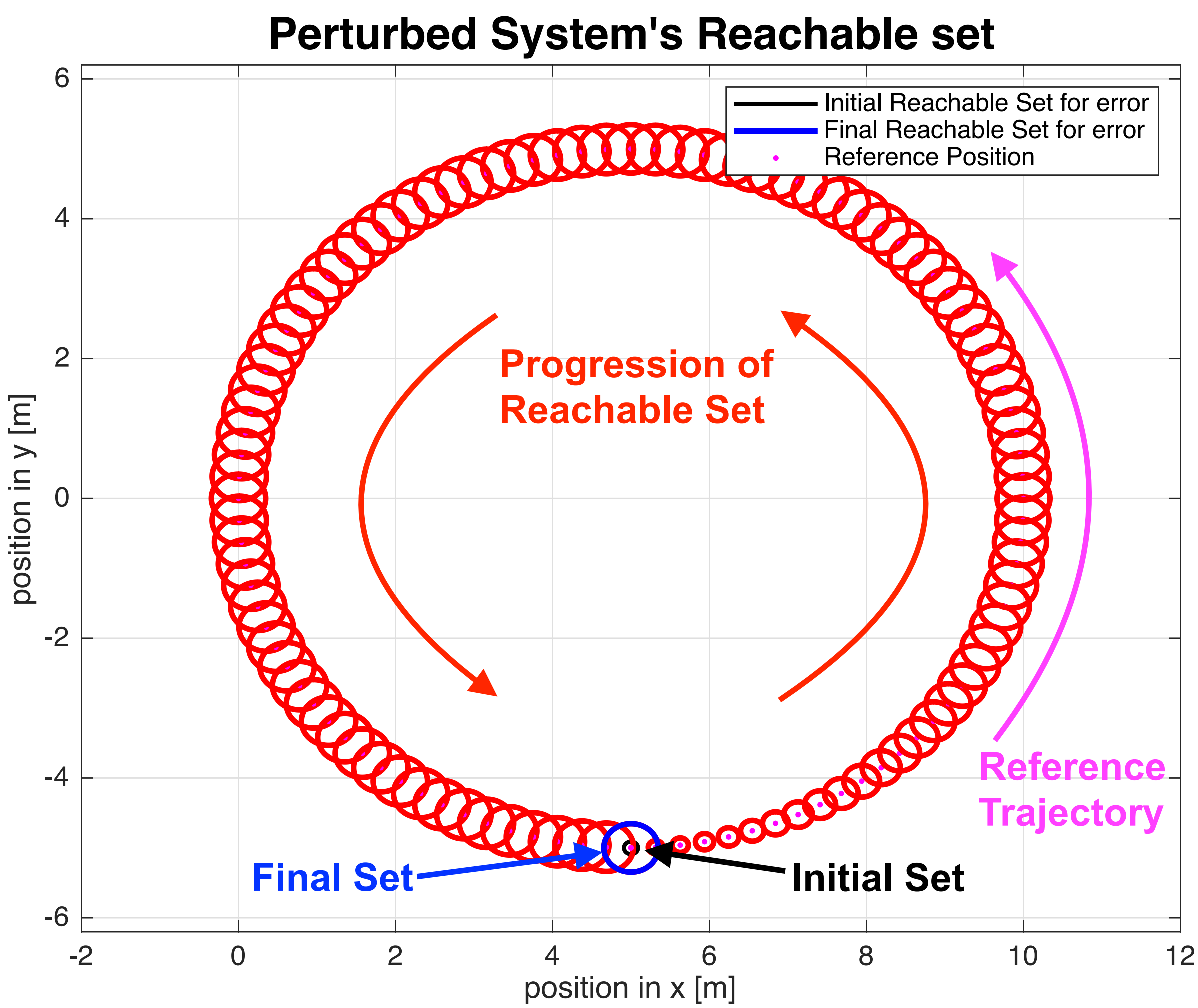


Figure 5.17: Reachable Set of ξ_{0s} uncompensated case

The FRS of errors, for the initial set ξ_{0s} (compensated) - The compensated case produces a much more smaller FRS. This implies that, the errors will be much less compared to the uncompensated case.

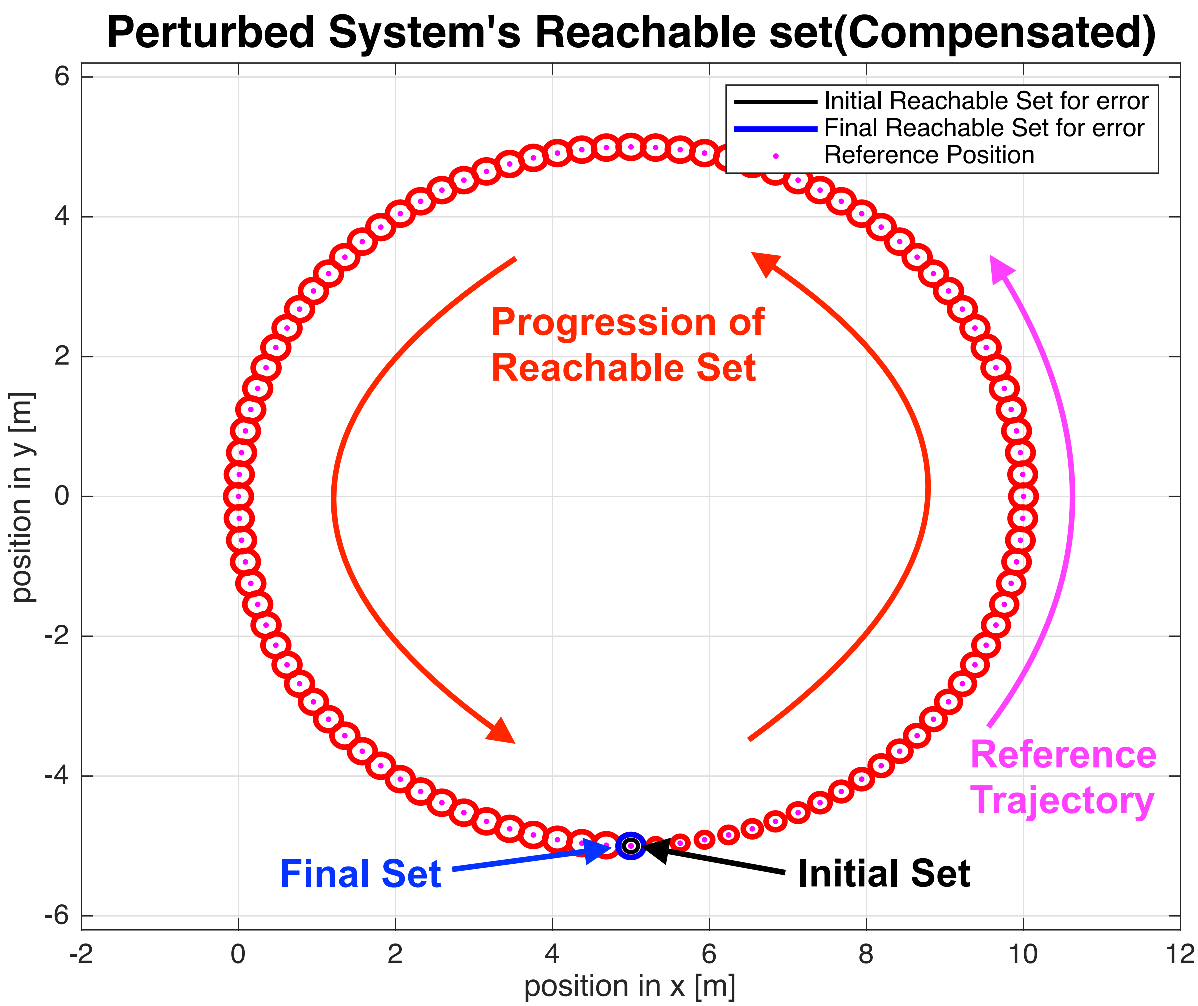


Figure 5.18: Reachable Set of ξ_{0s} compensated case

5.4.3 FRS of the Perturbed System (with and without compensation) for the Initial Set ξ_{0b}

The values of Δ taken are mentioned in the **legend**. The initial set ξ_{0b}, is used to plot the total error profile of the system for different values of Δ.

The error profile for the initial set ξ_{0b} - the error is seen to reduce over time, and converge, but doesn't converge to 0.

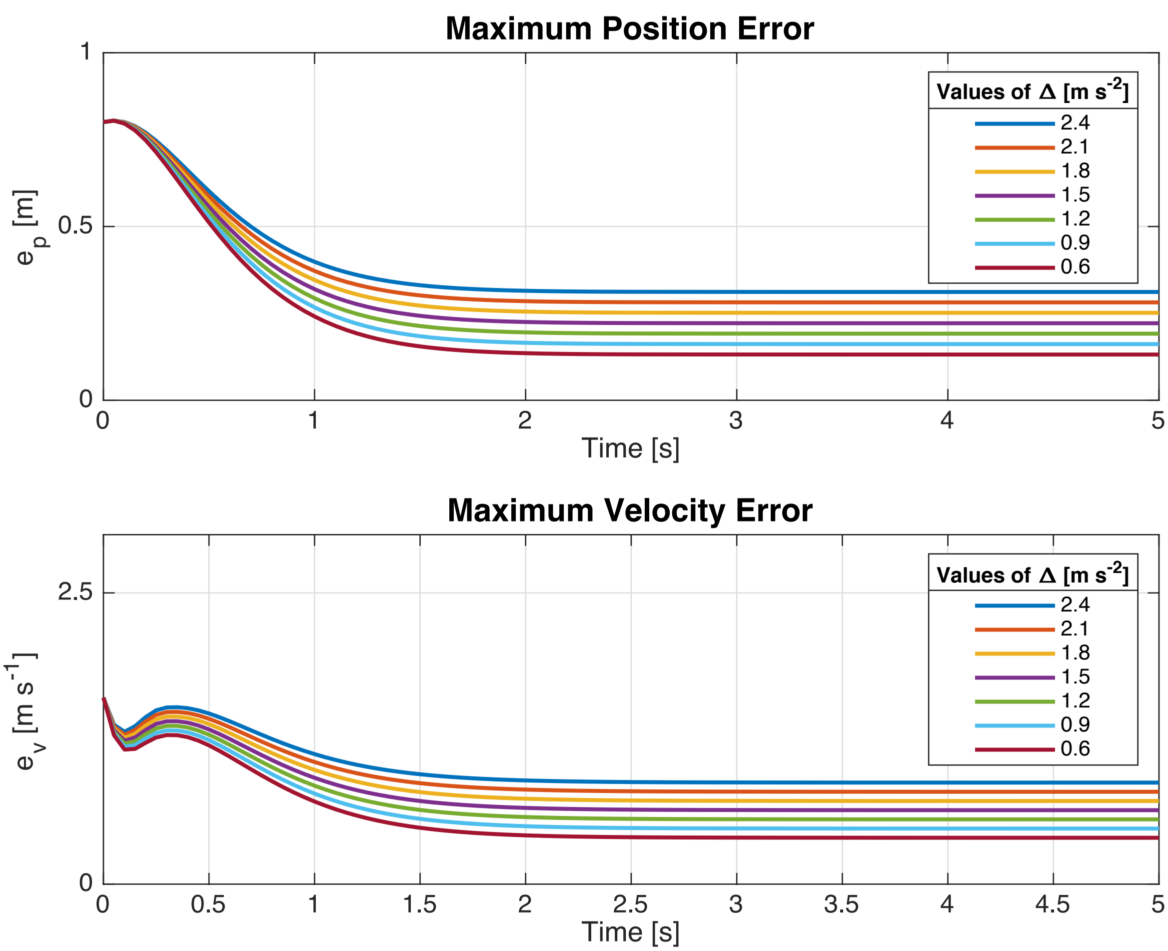


Figure 5.19: Error profile of position error and velocity error over time, for $|\tilde{c}_d| = 0.35$ and initial error sets : ξ_{0b} and different values of Δ[m s^{-2}]

The FRS of errors, for the initial set ξ_{0b} (uncompensated) - The set gradually decreases in size, but never converges with reference trajectory. This implies that, the size of set of final error does not depend upon initial set.

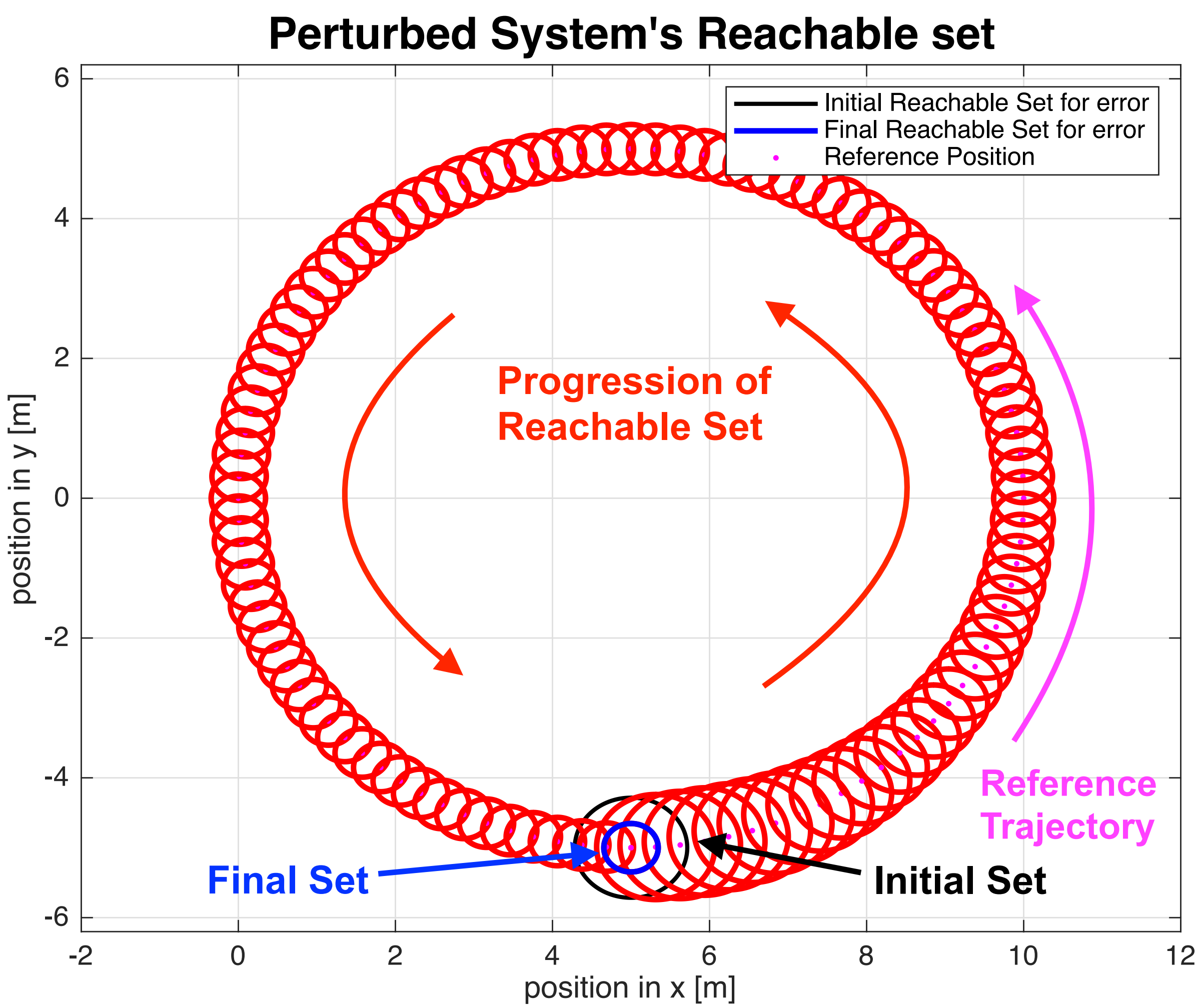


Figure 5.20: Reachable Set of ξ_{0b} uncompensated case

The FRS of errors, for the initial set ξ_{0b} (compensated) - For the compensated case, the size of the FRS reduces much more before stabilizing, compared to uncompensated case. This implies that, the compensation does affect the size of the set of final errors.

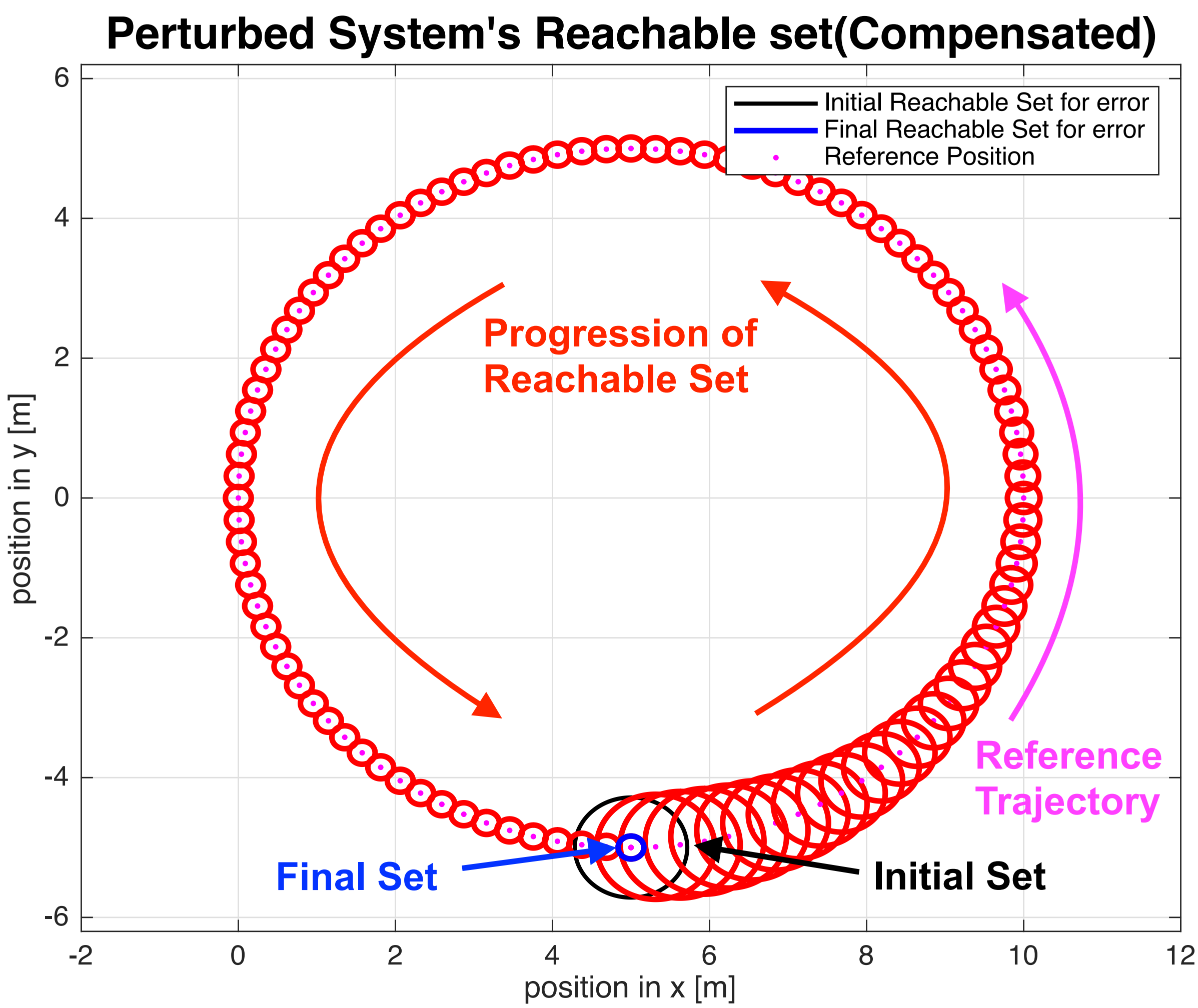


Figure 5.21: Reachable Set of ξ_{0b} compensated case

Final Remarks -

The error profile shows error converging in both the cases, for all the values of Δ. The values at which the respective errors of both initial sets $\{\xi_{0s},\ \xi_{0b}\}$ converge, seem to be independent of the initial set and show dependence over Δ.

All the combination of cases, proceed to show a similarity with the plot of error profile.

In both the cases (Compensated and Uncompensated), the compensated case, produces a smaller sized FRS. This shows that in the compensated case, errors of the multi-rotor will be lesser, thus, it will be closer to the reference trajectory, compared to the uncompensated case.

As defined earlier, $\hat{c}_d$ is estimation of the coefficient of first order aerodynamics effect, thus, better estimation and compensation of this effect implies lesser error. Therefore, depending upon desired error bound, the compensation should be chosen appropriately.

CHAPTER 6

SUMMARY AND CONCLUSION

The aim of the thesis, was solely to build *methods* to compute reachable and invariant sets for dynamic systems. These methods were constructed in the form of algorithms and two different aspects of computation of reachable sets was focused upon.

Initially, this thesis concentrated upon computation of reachable set of linear systems, and algorithms to compute these sets were constructed. The effect of time step Δt and control input $|u_k| \leq u_{max}$, was observed in the process of computing reachable sets. Choice over Δt was made by taking the results into consideration s.t., errors due to truncation would be reduced and improved computation of reachable sets could be performed. Furthermore, effects of control input was also observed to confirm the result with the literature.

The construction of algorithms for perturbed systems, were a result of extension of the reachable set theory of linear systems to perturbed systems. The extended method, produced the set which was exact computation of reachable set *at* time T but wasn't constant, i.e. it eliminated all the states which can be stated as reachable with other definitions of reachable state. The reason being the vagueness in definition of the term "Reachable State": "Reachable *at* time T", "Reachable *within* time T". The literature of reachable set tackles this problem by assuming both the ideas of reachable state whichever necessary. Therefore, Method-2 was constructed keeping in mind the second definition of reachable state.

The reachable sets were computed using both these methods and their results were compared. The comparison was made to show the difference in reachable set depending upon which definition of reachable state is assumed. The reachable sets at time T, could be stated as a subset of reachable set within time T. The usage of both these methods depend upon which definition of reachable state is accurate for a given circumstance.

The multi-rotor under first-order aerodynamic effect problem was adopted from [12] to utilise the theory of reachable set which was constructed. The challenge was to compute these forward reachable sets using ellipsoidal sets, rather than polytopes and zonotopes. The mathematical formulation was done so as to have a prior idea regarding the computation of these sets. The formulation was then implemented into the algorithm, which constructed forward reachable sets of error bounds of the multi-rotor. The obtained results showed decrease in error when compensation was implemented. Error profile was also generated to discuss implementation of compensation based on the desired error threshold.

Thus, the algorithms constructed in this thesis can be implemented to compute reachable set of the system, which can later be utilised to produce desired results.

CHAPTER 7

FUTURE WORK

All the work presented in this thesis can be extended to fulfill a number of specific reachability criteria. Some of the models, for which, these algorithms can be extended to are -

1. Trajectory Planning - Method-2 can compute all possible reachable states needed prior to trajectory planning. Thus, implementing other conditions over control can refine the method to provide accurate and robust trajectories.
2. Obstacle Avoidance - The primary algorithm was built upon the requirement of estimating reachable states at time T. Having the information over these exact states can improve implementation of obstacle avoidance maneuver.
3. Cooperative Control - Precomputing reachable set for all the unmanned vehicle can provide information of safety and feasibility. Implementing the idea of reachability would be extremely useful in this field, however, the complexity of building the algorithm increases drastically.

The condition which was derived for controllability of perturbed systems, given the perturbation is Lipschitz kind, can be extended to perturbation of Holder's kind. Thus, controllability of higher order perturbation can be tackled.

APPENDIX A

LIST OF RESULTS

The input data and algorithm used to obtain results from section(5.2) to section(5.3) are mentioned in this appendix.

All the input data that will be needed to produce these results are mentioned below,

- Time steps (Δt) - Three different sizes of time steps are used,

$$\Delta t_1 = 0.01s, \quad \Delta t_2 = 0.05s \text{ and } \quad \Delta t_3 = 0.1s$$

- Total runtime (T) - All the runtime will be mentioned. The number of step (k_f) will then be mentioned as ratio of T and Δt, i.e., $k_f = \frac{T}{\Delta t_1}$.

$$T_1 = 3s, \quad T_2 = 4s, \quad T_3 = 5s \text{ and } \quad T_4 = 6s$$

- State transition matrices ($\mathbf{\Phi}$) - Three different state transition matrices are used,

$$\mathbf{\Phi}_1 = \begin{bmatrix} 1 & 0.01 \\ -0.1 & 0.95 \end{bmatrix}, \quad \mathbf{\Phi}_2 = \begin{bmatrix} 1 & 0.1 \\ -1 & 0.5 \end{bmatrix} \text{ and } \quad \mathbf{\Phi}_3 = \begin{bmatrix} \mathbf{I}_3 & 0.05\mathbf{I}_3 \\ -0.5\mathbf{I}_3 & 0.7\mathbf{I}_3 \end{bmatrix}$$

- Input matrix ($\mathbf{\Gamma}$) - Three different input matrices are used,

$$\mathbf{\Gamma}_1 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}, \quad \mathbf{\Gamma}_2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \text{ and } \quad \mathbf{\Gamma}_3 = \begin{bmatrix} 0 \\ 0.05\mathbf{I}_3 \end{bmatrix}$$

- Initial constraints ($\mathcal{X}_0$) - All the initial constraints are mentioned below,

$$\mathcal{X}_0 = \left\{ \mathbf{x} \in \mathbb{R}^n : \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\}$$

$$\Delta\mathcal{X}_0 = \left\{ \mathbf{x} \in \mathbb{R}^n : \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix} \right\}$$

$$\xi(\mathbf{P}_{0s}, 0) = \left\{ \mathbf{e}_0 \mid \mathbf{e}_0^\intercal \begin{bmatrix} 100\mathbf{I}_3 & \mathbf{0}_{3,3} \\ \mathbf{0}_{3,3} & 25\mathbf{I}_3 \end{bmatrix} \mathbf{e}_0 \leq 1 \right\}$$

$$\xi(\mathbf{P}_{0b}, 0) = \left\{ \mathbf{e}_0 \mid \mathbf{e}_0^\intercal \begin{bmatrix} 1.56\mathbf{I}_3 & \mathbf{0}_{3,3} \\ \mathbf{0}_{3,3} & 0.39\mathbf{I}_3 \end{bmatrix} \mathbf{e}_0 \leq 1 \right\}$$

- Control input constraints ($\mathcal{U}$) - All the control input constraints are mentioned below,

$$\mathcal{U}_1 = \{u_k \in \mathbb{R} : \ |u_k| \leq u_{max,1} = 0\}$$

$$\mathcal{U}_2 = \{u_k \in \mathbb{R} : \ |u_k| \leq u_{max,2} = 5\}$$

$$\mathcal{U}_3 = \{u_k \in \mathbb{R} : \ |u_k| \leq u_{max,3} = 15\}$$

$$\mathcal{U}_4 = \{u_k \in \mathbb{R} : \ |u_k| \leq u_{max,3} = 25\}$$

- Lipschitz constant (ℓ) - All the value of Lipschitz constant are mentioned below,

$$\ell_1 = 5, \quad \ell_2 = 10, \quad \ell_3 = 15 \text{ and } \quad \ell_4 = 20$$

- Constraints for nonlinearity as disturbance ($\mathcal{W}$) - These constraints are mentioned below,

$$\mathcal{W}_1 = \left\{ w \in \mathbb{R} : \begin{bmatrix} 0 \\ -10 \end{bmatrix} \le \begin{bmatrix} 0 \\ w \end{bmatrix} \le \begin{bmatrix} 0 \\ 10 \end{bmatrix} \right\}$$

$$\mathcal{W}_2 = \left\{ w \in \mathbb{R} : \begin{bmatrix} 0 \\ -20 \end{bmatrix} \le \begin{bmatrix} 0 \\ w \end{bmatrix} \le \begin{bmatrix} 0 \\ 20 \end{bmatrix} \right\}$$

Remark A.1. $\mathcal{U}_{a,+\ell}$ will be used to denote control input constraints, whose $u_{max} = u_{max,a} + \ell$.

List of results:

1. Figure (5.2) is obtained from algorithm (4.2.2).

   Data : $\mathcal{X}_0,\ \mathcal{U}_4,\ \mathbf{\Phi}_2,\ \mathbf{\Gamma}_2,\ k_f = \frac{T_1}{\Delta t_3}$.

2. Figure (5.3) is obtained from algorithm (4.2.2).

   Data : $\mathcal{X}_0,\ \mathcal{U}_4,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_1}{\Delta t_1}$.

3. Figure (5.4) is obtained from algorithm (4.2.2).

   Data : $\mathcal{X}_0,\ \mathcal{U}_2,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_1}{\Delta t_1}$.

4. Figure (5.5) is obtained from algorithm (4.2.2).

   Data : $\mathcal{X}_0,\ \mathcal{U}_4,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_1}{\Delta t_1}$.

5. Figure (5.6) is obtained from algorithm (4.1.1).

   Data : $\mathcal{X}_0,\ \mathcal{U}_1,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1$.

6. Figure (5.7) is obtained from algorithm (4.1.1).

   Data : $\mathcal{X}_0,\ \mathcal{U}_4,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1$.

7. - Figure (5.9)(a) is obtained from algorithm (Linear - 4.2.2, Perturbed - 4.3.1, Upper - 4.2.2).

     Linear Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_1}{\Delta t_1}$.

Perturbed Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ \ell_3 \sin\Delta t_1,\ k_f = \frac{T_1}{\Delta t_1}$.

Upper Data : $\mathcal{X}_0,\ \mathcal{U}_{3,+\ell_3},\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_!}{\Delta t_1}$.

- Figure (5.9)(b) is obtained from algorithm (Linear - 4.2.2, Perturbed - 4.3.1, Upper - 4.2.2).

  Linear Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_2}{\Delta t_1}$.

  Perturbed Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ \ell_3 \sin\Delta t_1,\ k_f = \frac{T_2}{\Delta t_1}$.

  Upper Data : $\mathcal{X}_0,\ \mathcal{U}_{3,+\ell_3},\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_2}{\Delta t_1}$.

- Figure (5.9)(c) is obtained from algorithm (Linear - 4.2.2, Perturbed - 4.3.1, Upper - 4.2.2).

  Linear Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_3}{\Delta t_1}$.

  Perturbed Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ \ell_3 \sin\Delta t_1,\ k_f = \frac{T_3}{\Delta t_1}$.

  Upper Data : $\mathcal{X}_0,\ \mathcal{U}_{3,+\ell_3},\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_3}{\Delta t_1}$.

- Figure (5.9)(d) is obtained from algorithm (Linear - 4.2.2, Perturbed - 4.3.1, Upper - 4.2.2).

  Linear Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_4}{\Delta t_1}$.

  Perturbed Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ \ell_3 \sin\Delta t_1,\ k_f = \frac{T_4}{\Delta t_1}$.

  Upper Data : $\mathcal{X}_0,\ \mathcal{U}_{3,+\ell_3},\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_4}{\Delta t_1}$.

8. Figure (5.10) is obtained from algorithm (Linear - 4.2.2, Total - 4.3.2, Upper - 4.2.2).

   Linear Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_1}{\Delta t_1}$.

   Total Data : $\mathcal{X}_0,\ \Delta\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ \ell_3 \sin\Delta t_1,\ k_f = \frac{T_1}{\Delta t_1}$.

   Upper Data : $\mathcal{X}_0,\ \mathcal{U}_{3,+\ell_3},\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_1}{\Delta t_1}$.

9. - Figure (5.11)(a) is obtained from algorithm (Linear - 4.2.2, Perturbed - 4.3.1, Total - 4.3.2, Upper - 4.2.2).

     Linear Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_1}{\Delta t_1}$.

Perturbed Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ \ell_3\sin\Delta t_1,\ k_f = \frac{T_1}{\Delta t_1}$.

Total Data : $\mathcal{X}_0,\ \Delta\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ \ell_3\sin\Delta t_1,\ k_f = \frac{T_1}{\Delta t_1}$.

Upper Data : $\mathcal{X}_0,\ \mathcal{U}_{3,+\ell_3},\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_1}{\Delta t_1}$.

- Figure (5.11)(b) is obtained from algorithm (Linear - 4.2.2, Perturbed - 4.3.1, Total - 4.3.2, Upper - 4.2.2).

  Linear Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_2}{\Delta t_1}$.

  Perturbed Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ \ell_3\sin\Delta t_1,\ k_f = \frac{T_2}{\Delta t_1}$.

  Total Data : $\mathcal{X}_0,\ \Delta\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ \ell_3\sin\Delta t_1,\ k_f = \frac{T_2}{\Delta t_1}$.

  Upper Data : $\mathcal{X}_0,\ \mathcal{U}_{3,+\ell_3},\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_2}{\Delta t_1}$.

- Figure (5.11)(c) is obtained from algorithm (Linear - 4.2.2, Perturbed - 4.3.1, Total - 4.3.2, Upper - 4.2.2).

  Linear Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_3}{\Delta t_1}$.

  Perturbed Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ \ell_3\sin\Delta t_1,\ k_f = \frac{T_3}{\Delta t_1}$.

  Total Data : $\mathcal{X}_0,\ \Delta\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ \ell_3\sin\Delta t_1,\ k_f = \frac{T_3}{\Delta t_1}$.

  Upper Data : $\mathcal{X}_0,\ \mathcal{U}_{3,+\ell_3},\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_3}{\Delta t_1}$.

- Figure (5.11)(d) is obtained from algorithm (Linear - 4.2.2, Perturbed - 4.3.1, Total - 4.3.2, Upper - 4.2.2).

  Linear Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_4}{\Delta t_1}$.

  Perturbed Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ \ell_3\sin\Delta t_1,\ k_f = \frac{T_4}{\Delta t_1}$.

  Total Data : $\mathcal{X}_0,\ \Delta\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ \ell_3\sin\Delta t_1,\ k_f = \frac{T_4}{\Delta t_1}$.

  Upper Data : $\mathcal{X}_0,\ \mathcal{U}_{3,+\ell_3},\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_4}{\Delta t_1}$.

10.

- Figure (5.12)(a) is obtained from algorithm (Linear - 4.2.2, Total - 4.3.2, Upper - 4.2.2).

  Linear Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_1}{\Delta t_1}$.

Total Data : $\mathcal{X}_0,\ \Delta\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ \ell_1 \sin\Delta t_1,\ k_f = \frac{T_1}{\Delta t_1}.$

Upper Data : $\mathcal{X}_0,\ \mathcal{U}_{3,+\ell_1},\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_1}{\Delta t_1}.$

- Figure (5.12)(b) is obtained from algorithm (Linear - 4.2.2, Total - 4.3.2, Upper - 4.2.2).

  Linear Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_1}{\Delta t_1}.$

  Total Data : $\mathcal{X}_0,\ \Delta\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ \ell_2 \sin\Delta t_1,\ k_f = \frac{T_1}{\Delta t_1}.$

  Upper Data : $\mathcal{X}_0,\ \mathcal{U}_{3,+\ell_2},\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_1}{\Delta t_1}.$

- Figure (5.12)(c) is obtained from algorithm (Linear - 4.2.2, Total - 4.3.2, Upper - 4.2.2).

  Linear Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_1}{\Delta t_1}.$

  Total Data : $\mathcal{X}_0,\ \Delta\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ \ell_3 \sin\Delta t_1,\ k_f = \frac{T_1}{\Delta t_1}.$

  Upper Data : $\mathcal{X}_0,\ \mathcal{U}_{3,+\ell_3},\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_1}{\Delta t_1}.$

- Figure (5.12)(d) is obtained from algorithm (Linear - 4.2.2, Total - 4.3.2, Upper - 4.2.2).

  Linear Data : $\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_1}{\Delta t_1}.$

  Total Data : $\mathcal{X}_0,\ \Delta\mathcal{X}_0,\ \mathcal{U}_3,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ \ell_4 \sin\Delta t_1,\ k_f = \frac{T_1}{\Delta t_1}.$

  Upper Data : $\mathcal{X}_0,\ \mathcal{U}_{3,+\ell_4},\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1,\ k_f = \frac{T_1}{\Delta t_1}.$

11.
- Figure (5.13)(a) is obtained from algorithm (4.3.3).

  Data : $\mathcal{X}_0,\ \mathcal{U}_1,\ \mathcal{W}_1,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1.$

- Figure (5.13)(b) is obtained from algorithm (4.3.3).

  Data : $\mathcal{X}_0,\ \mathcal{U}_1,\ \mathcal{W}_2,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1.$

- Figure (5.13)(c) is obtained from algorithm (4.3.3).

  Data : $\mathcal{X}_0,\ \mathcal{U}_4,\ \mathcal{W}_1,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1.$

- Figure (5.13)(d) is obtained from algorithm (4.3.3).

  Data : $\mathcal{X}_0,\ \mathcal{U}_4,\ \mathcal{W}_2,\ \mathbf{\Phi}_1,\ \mathbf{\Gamma}_1.$

12. Figure (5.14) is obtained from algorithm (4.4.1).

    Data : $\xi(\mathbf{P}_{0s}, 0),\ \mathbf{\Phi}_3,\ \mathbf{\Gamma}_3,\ \Delta = 0,\ \bar{s}_\phi = 0,\ k_f = \frac{T_3}{\Delta t_2},\ \mathbf{p}_r,\ \dot{\mathbf{p}}_r.$

13. Figure (5.15) is obtained from algorithm (4.4.1).

    Data : $\xi(\mathbf{P}_{0b}, 0),\ \mathbf{\Phi}_3,\ \mathbf{\Gamma}_3,\ \Delta = 0,\ \bar{s}_\phi = 0,\ k_f = \frac{T_3}{\Delta t_2},\ \mathbf{p}_r,\ \dot{\mathbf{p}}_r.$

14. Figure (5.17) is obtained from algorithm (4.4.1).

    Data : $\xi(\mathbf{P}_{0s}, 0),\ \mathbf{\Phi}_3,\ \mathbf{\Gamma}_3,\ \Delta = 2.85,\ \bar{s}_\phi = 0.052,\ k_f = \frac{T_3}{\Delta t_2},\ \mathbf{p}_r,\ \dot{\mathbf{p}}_r.$

15. Figure (5.18) is obtained from algorithm (4.4.1).

    Data : $\xi(\mathbf{P}_{0s}, 0),\ \mathbf{\Phi}_3,\ \mathbf{\Gamma}_3,\ \Delta = 1.1,\ \bar{s}_\phi = 0.052,\ k_f = \frac{T_3}{\Delta t_2},\ \mathbf{p}_r,\ \dot{\mathbf{p}}_r.$

16. Figure (5.20) is obtained from algorithm (4.4.1).

    Data : $\xi(\mathbf{P}_{0b}, 0),\ \mathbf{\Phi}_3,\ \mathbf{\Gamma}_3,\ \Delta = 2.85,\ \bar{s}_\phi = 0.052,\ k_f = \frac{T_3}{\Delta t_2},\ \mathbf{p}_r,\ \dot{\mathbf{p}}_r.$

17. Figure (5.21) is obtained from algorithm (4.4.1).

    Data : $\xi(\mathbf{P}_{0b}, 0),\ \mathbf{\Phi}_3,\ \mathbf{\Gamma}_3,\ \Delta = 1.1,\ \bar{s}_\phi = 0.052,\ k_f = \frac{T_3}{\Delta t_2},\ \mathbf{p}_r,\ \dot{\mathbf{p}}_r.$

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BIOGRAPHICAL STATEMENT

Prabhjeet Singh Arora was born in Indore, India, in 1993. He received his B.Tech. degree from Indian Institute of Technology (Indian School of Mines), Dhanbad, in 2015. He worked on regenerative braking systems for his undergraduate project at the alma mater. He worked on stacker and reclaimer power system for his internship at Eaton corporation, in India. In 2016, he moved to USA to pursue M.S. in Mechanical Engineering at University of Texas at Arlington. His research interest include optimal control and nonlinear control. His hobbies include playing chess and tennis.