Modeling an M/M/1 Queue with unreliable service and a working vacation

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Submitted in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

The University of Texas at Arlington

May, 2019

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Abstract

We define the new term 'unreliable service' where the service itself is unreliable (i.e. may fail). We discuss how this differs from the current literature, and give examples showing just how common this phenomena is in many real-world scenarios. We first consider the classic M/M/1 queue with unreliable service and find some striking similarities with the well studied M/M/1 derivation. Next, we consider the M/M/1 queue with unreliable service and a working vacation. In each of these cases, surprising explicit results are found including positive recurrence conditions, the stationary queue length distribution, and a decomposition of both the queue length and waiting time. We also purpose a number of ideas for future research based on this newly defined phenomenon.

Acknowledgements

I would like to thanks my advisor Professor Andrzej Korzeniowski for his expertise and vast knowledge of queue theory and stochastic processes, the history of these topics, and major contributors therein. In addition, I would like to extend my gratitude towards my committee members, Drs. Suvra Pal, Shan Sun-Mitchell, and Jonghyun Yun.

I would also like to acknowledge my mother, Brenda; father, Ken; grandmother, Brenda; step-grandfather, James; aunt, Amy; uncle, Brad; brothers Michael, Aaron; and step-brother David for their ongoing love and support from the very beginning–and I mean the *very* beginning.

Lastly, I would like to thank all my former professors, instructors, classmates, tutors, colleagues, and others from the math department to the cafeteria at various institutions who encouraged me to pursue a Ph.D. in Mathematics. In particular, I would like to thank Dr. Paula Wilhite from Northeast Texas Community College who first suggested that I earn my Ph.D. as a mere Calculus I undergraduate, then after graduating with my Associates degree, persisted at every opportunity by asking if I had enrolled in a doctoral program yet. Her constant support, encouragement and countless office hours will never be forgotten.

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Chapter 1

Introduction

Our interest in Queue Theory has been primarily motivated by its application to computer networks. That is, when a system is attached to a network via a network interface, one understands there is at least one queue governing that interface's egress data flow. These queues tend to exist at the MAC (Media Access Control) level and are typically FIFO (First In, First Out) queues whose job is simply to 'buffer' the flow of outgoing traffic to a level that is sustainable by the interface's data rate. For example, imagine watching a movie, we understand that the movie 'streams' to our TV; that is, we watch it **as** it transfers to our TV over time. What stops the movie from instantly transferring? Well, the data rate of our Home internet service does! If we tried to transfer the entire movie over our internet connection instantly, we would have many, many lost data packets as a result. Therefore, there are queues governing each and every individual link that make up the path from the servers to our TV. These queues control the flow of data along their respective parts, the slowest part of this path is typically our Home internet connection, sometimes called the 'last mile' link. For more information on application of Queue Theory in a FIFO network environment for both egress and ingress data, see [1].

Another motivation for our work comes from the application of Queue Theory to a wireless network interface. What makes this case unique is partly due to the physical RF (Radio Frequency) medium being shared. To see how this plays a role, consider a fiber-optic data link which uses light transmitted down a special cable to transmit digital 0s and 1s extremely quickly. The physical medium used here is the light spectrum within the cable which is isolated from the outside world–in other words, the medium is dedicated (i.e. not shared).

However, wireless networks are unique for reasons beyond the fact that the physical medium is shared. Consider a network interface delivered by a Cable TV provider. This network has a shared physical medium (the 75Ω coaxial cable), but all members of this network speak the same digital language, namely the backwards compatible DOCSIS 3.x, 2.x, or 1.x standard. This is crucial because it means that all members on this shared network are aware of one another and can coordinate by some scheduling scheme in an effort to not interfere with each other.

Additionally, we would like to consider wireless networks in a congested environment. These networks are again unique because the physical medium is shared by devices that do not necessarily speak the same language. This leads to both on-channel and co-channel interference that cannot be avoided. This interference necessitates data re-transmissions which come at the cost of data throughput. It should be noted that ieee 802.11x networks can reduce on-channel collisions by using the CTS/RTS mechanism (collisions are defined to be two 802.11x devices on the same channel transmitting simultaneously). For details on CTS/RTS, its implementation, ideal use cases, and effectiveness see [2].

Let us consider the queue governing a wireless network interface, we define the following terms.

- The service is the transfer of data to the other end of the link.
- The server is the transmitter.
- The server rate is the interface egress data rate.
- The customers are relatively small snippets of data called frames.

If we wish to model this queue, we must somehow account for the presence of interference and the consequential re-transmission event. That is, we must include the possibility that the service can fail <u>after</u> the service time has elapsed and <u>without</u> a faulty server. Surprisingly, this concept does not appear in the literature, so we coin the term 'unreliable service' to denote this phenomenon, define it rigorously, and begin the analysis of models subject to it, namely.

- Chapter 2: The M/M/1 with unreliable service.
  - We find and the prove positive recurrence condition.
  - We give the stationary distribution explicitly.
  - We prove the decomposition for the stationary number of customers in the system, N, into the sum of two independent geometric random variables.
  - We prove the decomposition for the stationary waiting time W to be the sum of two independent exponential random variables.
  - We explicitly give the $E(.)$ and $Var(.)$ for both N and W.
  - Lastly, we recover a number of more familiar models as special cases.

- Chapter 3: The M/M/1 with unreliable server and a working vacation.
  - We again find and the prove positive recurrence condition, noting it to be the same as that of Chapter 2.
  - We again give the stationary distribution explicitly.
  - We find and prove the decomposition for N, the stationary number of customers in the system, to be the sum of four independent geometric random variables and a finitely valued generalized random variable.
  - We again explicitly give the $E(.)$ and $Var(.)$ for both N and W.
  - We recover Chapter 2 as a special case of this model.
  - Lastly, we plot $E(W)$ for various choices of the success and failure rates.
- Chapter 4: Future Work.
  - We discuss ideas for future research motivated by the application of this newly defined concept of 'unreliable service.'

Chapter 2

M/M/1 model with unreliable service

2.1 Introduction

We first consider an M/M/1 model with unreliable service but with a fixed service rate. This straightforward extension of the classical M/M/1 queue will play an important role in establishing the mechanisms which will allow further generalizations later on.

We introduce a new model of ’unreliable service’ whose key element is based on the fact that the server may not always complete its service successfully. While service failure has been studied extensively in the literature, our model is different in that the failure is not due to the server itself by means of a ’breakdown,’ nor is it due to the customer leaving the queue during the service time. Rather, the success or failure of a job is due to external forces and entirely random. Furthermore, neither the customer nor server know whether a job has failed or was successful until after the job’s service time has been completed. The application of such a queue can come from many different areas and fields – all that is necessary is for some sort of quality check to be performed after service. This quality check would look at some set of measurements with certain thresholds and would conclude that the service was either successful or not.

Another key aspect to our model is that it will preserve the FCFS (First Come First Serve) discipline structure of the queue. Namely, when a customer’s service fails, the customer does not lose its place in the queue and the service is repeated until it is successful. We will approach our analytical study of this new model in a similar fashion to that done by Xu, Xiuli and Tian, Naishuo [3]. It should be noted that one can construct an M/PH/1 queue with similar properties. However, such a model will impose an additional, undesirable restriction: $\mu \geq \beta_1 + \beta_2$. See below for definitions of these parameters and page 58 of [4] for the details.

2.2 Definitions

We begin by defining our process, state space, and parameters.

Definition 2.2.1. Let $\{N(t) \mid t \geq 0\}$ be the number of customers in the queue at time t, and

$$S(t) = \begin{cases} 1 & \text{immediately after service is rendered} \\ 0 & \text{otherwise} \end{cases}$$

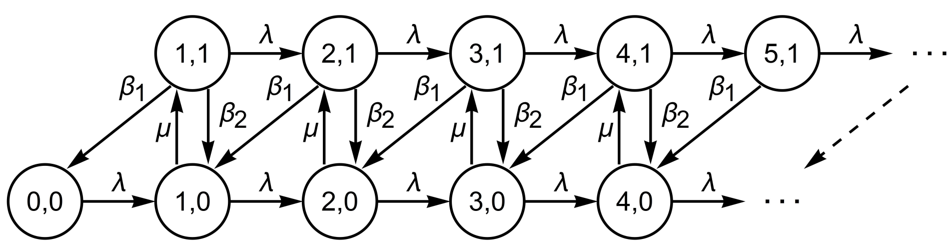
Then $\{(N(t), S(t)) \mid t \geq 0\}$ is a Markov process on the state space:
$\Omega = \{(0,0)\} \cup \{(k,s) \mid k \in \mathbb{N}, s \in \{0,1\}\}$

Define the following parameters:

- λ : the rate of the Poisson arrivals process.
- μ : the rate of service, successful or not.
- β_1 : the rate of a successful service.
- β_2 : the rate of a failed service.

To visualize such a Markovian process, it is helpful to construct the state transition rate diagram.

FIGURE 2.1: Markovian state transition rate diagram.



Formally, we define a 'successful service' to be a transition from $(n,1) \longrightarrow (n-1,0)$, which is represented in the state transition diagram as having rate β_1. Similarly, we define a 'failed service' to be a transition from $(n,1) \longrightarrow (n,0)$ with transition rate β_2. Accordingly, we can compute the probabilities of a 'successful' or 'failed' service explicitly by considering the transition probabilities of the embedded Markov Chain.

$$
\begin{aligned}
p_s &= \frac{\beta_1}{\beta_1+\beta_2+\lambda}\sum_{i=0}^{\infty}\left(\frac{\lambda}{\beta_1+\beta_2+\lambda}\right)^i & p_f &= \frac{\beta_2}{\beta_1+\beta_2+\lambda}\sum_{i=0}^{\infty}\left(\frac{\lambda}{\beta_1+\beta_2+\lambda}\right)^i \\
p_s &= \frac{\beta_1}{\beta_1+\beta_2+\lambda}\left(\frac{1}{1-\frac{\lambda}{\beta_1+\beta_2+\lambda}}\right) & p_f &= \frac{\beta_2}{\beta_1+\beta_2+\lambda}\left(\frac{1}{1-\frac{\lambda}{\beta_1+\beta_2+\lambda}}\right) \\
p_s &= \frac{\beta_1}{\beta_1+\beta_2+\lambda}\left(\frac{1}{\frac{\beta_1+\beta_2}{\beta_1+\beta_2+\lambda}}\right) & p_f &= \frac{\beta_2}{\beta_1+\beta_2+\lambda}\left(\frac{1}{\frac{\beta_1+\beta_2}{\beta_1+\beta_2+\lambda}}\right) \\
p_s &= \frac{\beta_1}{\beta_1+\beta_2+\lambda}\left(\frac{\beta_1+\beta_2+\lambda}{\beta_1+\beta_2}\right) & p_f &= \frac{\beta_2}{\beta_1+\beta_2+\lambda}\left(\frac{\beta_1+\beta_2+\lambda}{\beta_1+\beta_2}\right) \\
p_s &= \frac{\beta_1}{\beta_1+\beta_2} & p_f &= \frac{\beta_2}{\beta_1+\beta_2}
\end{aligned}
\tag{2.1}
$$

From here, we can list the countable state space in lexicographical order; formally defined below.

Definition 2.2.2. Lexicographical Ordering
We say $(k_1, s_1) < (k_2, s_2)$ if and only if $k_1 < k_2$ or $(k_1 = k_2$ and $s_1 < s_2)$

Using this re-ordering convention (see [5], pg. 353), we can write $\Omega = \{(0,0),(1,0),(1,1),(2,0),(2,1),\dots\}$ and define the corresponding infinitesimal matrix Q.

2.3 Infinitesimal Matrix Q

$$
\mathbf{Q} = \begin{bmatrix} \hat{\mathbf{A}} & \hat{\mathbf{C}} & 0 & 0 & 0 & \dots \\ \hat{\mathbf{B}} & \mathbf{A} & \mathbf{C} & 0 & 0 & \dots \\ 0 & \mathbf{B} & \mathbf{A} & \mathbf{C} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{bmatrix} \tag{2.2}
$$

where

$$
\hat{\mathbf{A}} = \begin{bmatrix} -\lambda \end{bmatrix} \qquad \hat{\mathbf{B}} = \begin{bmatrix} 0 \\ \beta_1 \end{bmatrix} \qquad \hat{\mathbf{C}} = \begin{bmatrix} \lambda & 0 \end{bmatrix}
$$

$$
\mathbf{A} = \begin{bmatrix} -(\lambda+\mu) & \mu \\ \beta_2 & -(\beta_1+\beta_2+\lambda) \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ \beta_1 & 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}
$$

2.4 Positive Recurrence

Since the matrix Q has a block-tridiagonal structure, we have a QBD (Quasi Birth Death) Markovian process. Accordingly, we apply Theorem 1.5.1 from Neuts [6] to prove a lemma that will be used to show positive recurrence and find the stationary distribution explicitly. To this end, we need the following lemma.

Lemma 2.4.1. *The irreducible, block-tridiagonal Markov process with infinitesimal matrix* $\mathbf{Q}$ *is positive recurrent if and only if:*

- *the minimal non-negative solution* $\mathbf{R}$ *of quadratic matrix equation:*

$$\mathbf{R^2B} + \mathbf{RA} + \mathbf{C} = \mathbf{0} \tag{2.3}$$

  has $sp(\mathbf{R}) < 1$*, and*

- *there exists a positive vector* $(\mathbf{x_0}, \mathbf{x_1})$ *such that* $(\mathbf{x_0}, \mathbf{x_1})\mathbf{B}[\mathbf{R}] = 0$ *where:* $\mathbf{B}[\mathbf{R}] = \begin{bmatrix} \hat{\mathbf{A}} & \hat{\mathbf{C}} \\ \hat{\mathbf{B}} & \mathbf{A} + \mathbf{RB} \end{bmatrix}$*, and* $(\mathbf{x_0}, \mathbf{x_1})$ *is normalized by* $\mathbf{x_0 e} + \mathbf{x_1}(\mathbf{I} - \mathbf{R})^{-1}\mathbf{e} = \mathbf{1}$

The stationary distribution satisfying: $\begin{cases} \boldsymbol{\pi Q} = \mathbf{0} \\ \boldsymbol{\pi e} = 1 \end{cases}$ *is given by:* $\boldsymbol{\pi_k} = \begin{cases} \mathbf{x_0} & \text{if } k = 0 \\ \mathbf{x_1} & \text{if } k = 1 \\ \mathbf{x_1 R}^{k-1} & \text{if } k \geq 2 \end{cases}$

Our lemma, unlike Theorem 1.5.1 in Neuts [6], is stated in terms of the infinitesimal matrix $\mathbf{Q}$ rather than a Markov chain transition probability matrix.

Proof. Let $\mathbf{P} = \mathbf{I} + \tau^{-1}\mathbf{Q}$, where $\tau = -\min\{\text{diag}(\hat{\mathbf{A}}) \cup \text{diag}(\mathbf{A})\} > 0$. Then we have:

$$\mathbf{P} = \begin{bmatrix} \hat{\mathbf{A}}' & \hat{\mathbf{C}}' & 0 & 0 & 0 & \dots \\ \hat{\mathbf{B}}' & \mathbf{A}' & \mathbf{C}' & 0 & 0 & \dots \\ 0 & \mathbf{B}' & \mathbf{A}' & \mathbf{C}' & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{I} + \tau^{-1}\hat{\mathbf{A}} & \tau^{-1}\hat{\mathbf{C}} & 0 & 0 & 0 & \dots \\ \tau^{-1}\hat{\mathbf{B}} & \mathbf{I} + \tau^{-1}\mathbf{A} & \tau^{-1}\mathbf{C} & 0 & 0 & \dots \\ 0 & \tau^{-1}\mathbf{B} & \mathbf{I} + \tau^{-1}\mathbf{A} & \tau^{-1}\mathbf{C} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{bmatrix} \tag{2.4}$$

Since $\mathbf{Pe} = (\mathbf{I} + \tau^{-1}\mathbf{Q})\mathbf{e} = \mathbf{Ie} = \mathbf{e}$
$\Longrightarrow \mathbf{P}$ is a stochastic probability matrix of a discrete time Markov chain.

Moreover, if $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi} \Longrightarrow \boldsymbol{\pi}(\mathbf{I} + \tau^{-1}\mathbf{Q}) = \boldsymbol{\pi} \Longrightarrow \boldsymbol{\pi}\mathbf{Q} = 0$
and $\boldsymbol{\pi}\mathbf{Q} = 0 \Longrightarrow \boldsymbol{\pi} + \tau^{-1}\boldsymbol{\pi}\mathbf{Q} = \boldsymbol{\pi} \Longrightarrow \boldsymbol{\pi}(\mathbf{I} + \tau^{-1}\mathbf{Q}) = \boldsymbol{\pi} \Longrightarrow \boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$
$\Longrightarrow \boldsymbol{\pi}\mathbf{Q} = 0 \Longleftrightarrow \boldsymbol{\pi}\mathbf{P} = 1$

Theorem 1.5.1 from [6] states that $\mathbf{P}$, and consequently $\mathbf{Q}$, is positive recurrent if and only if:

- the minimal non-negative solution $\mathbf{R}$ of quadratic matrix equation:

$$\mathbf{R^2B}' + \mathbf{RA}' + \mathbf{C}' = \mathbf{R} \tag{2.5}$$

  has $\text{sp}(\mathbf{R}) < 1$, and

- there exists a positive vector $(\mathbf{x_0}, \mathbf{x_1})$ such that $(\mathbf{x_0}, \mathbf{x_1})\mathbf{B}'[\mathbf{R}] = (\mathbf{x_0}, \mathbf{x_1})$ where: $\mathbf{B}'[\mathbf{R}] = \begin{bmatrix} \hat{\mathbf{A}}' & \hat{\mathbf{C}}' \\ \hat{\mathbf{B}}' & \mathbf{A}' + \mathbf{R}\mathbf{B}' \end{bmatrix}$, and $(\mathbf{x_0}, \mathbf{x_1})$ is normalized by $\mathbf{x_0}\mathbf{e} + \mathbf{x_1}(\mathbf{I} - \mathbf{R})^{-1}\mathbf{e} = \mathbf{1}$

The stationary distribution satisfying: $\begin{cases} \boldsymbol{\pi P} = \boldsymbol{\pi} \\ \boldsymbol{\pi e} = 1 \end{cases}$ is given by: $\boldsymbol{\pi_k} = \begin{cases} \mathbf{x_0} & \text{if } k = 0 \\ \mathbf{x_1} & \text{if } k = 1 \\ \mathbf{x_1}\mathbf{R}^{k-1} & \text{if } k \geq 2 \end{cases}$

To finish our proof, we must restate the conditions on $\mathbf{P}$ in terms of conditions on $\mathbf{Q}$:

$$\begin{aligned} \mathbf{R^2B}' + \mathbf{RA}' + \mathbf{C}' = \mathbf{R} &\iff \tau^{-1}\mathbf{R^2B} + \mathbf{R}(\mathbf{I} + \tau^{-1}\mathbf{A}) + \tau^{-1}\mathbf{C} = \mathbf{R} \\ &\iff \tau^{-1}\mathbf{R^2B} + \tau^{-1}\mathbf{RA} + \tau^{-1}\mathbf{C} = \mathbf{0} \\ &\iff \mathbf{R^2B} + \mathbf{RA} + \mathbf{C} = \mathbf{0} \end{aligned}$$

and

$$\begin{aligned} (\mathbf{x_0}, \mathbf{x_1})\mathbf{B}'[\mathbf{R}] = (\mathbf{x_0}, \mathbf{x_1}) &\iff (\mathbf{x_0}, \mathbf{x_1}) \begin{bmatrix} \hat{\mathbf{A}}' & \hat{\mathbf{C}}' \\ \hat{\mathbf{B}}' & \mathbf{A}' + \mathbf{RB}' \end{bmatrix} = (\mathbf{x_0}, \mathbf{x_1}) \\ &\iff (\mathbf{x_0}, \mathbf{x_1}) \begin{bmatrix} \mathbf{I} + \tau^{-1}\hat{\mathbf{A}} & \tau^{-1}\hat{\mathbf{C}} \\ \tau^{-1}\hat{\mathbf{B}} & \mathbf{I} + \tau^{-1}(\mathbf{A} + \mathbf{RB}) \end{bmatrix} = (\mathbf{x_0}, \mathbf{x_1}) \\ &\iff (\mathbf{x_0}, \mathbf{x_1})(\mathbf{I} + \tau^{-1}\mathbf{B}[\mathbf{R}]) = (\mathbf{x_0}, \mathbf{x_1}) \\ &\iff (\mathbf{x_0}, \mathbf{x_1})\mathbf{B}[\mathbf{R}] = \mathbf{0} \end{aligned}$$

□

2.5 The Quadratic Matrix Equation

Thanks to Lemma 2.4.1, we seek the minimal non-negative solution $\mathbf{R}$ to the quadratic matrix equation:

$$\mathbf{R^2B} + \mathbf{RA} + \mathbf{C} = \mathbf{0} \tag{2.6}$$

There are many methods for solving such equations in the literature. Some are numerical in nature (see [7] and [8]), others are analytical for particular cases (see [9]). However, pure analytical methods are generally preferred to numerical ones when they are feasible. In our case, we employ the direct method whereby we solve the system of equations generated by equating the matrices entry by entry.

Let $\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \implies$ (2.3) can be restated as the following system:

$$\begin{cases} \lambda - (\lambda + \mu)r_{11} + (r_{11}r_{12} + r_{12}r_{22})\,\beta_1 + r_{12}\beta_2 = 0 \\ \mu r_{11} - r_{12}\,(\lambda + \beta_1 + \beta_2) = 0 \\ -(\lambda + \mu)r_{21} + (r_{12}r_{21} + r_{22}^2)\,\beta_1 + r_{22}\beta_2 = 0 \\ \lambda + \mu r_{21} - r_{22}\,(\lambda + \beta_1 + \beta_2) = 0 \end{cases} \tag{2.7}$$

The analytical minimal non-negative solution to (2.7) is given by:

$$\begin{cases} r_{11} = \frac{\lambda(\lambda+\beta_1+\beta_2)}{\mu\beta_1} \\ r_{12} = \frac{\lambda}{\beta_1} \\ r_{21} = \frac{\lambda(\lambda+\beta_2)}{\mu\beta_1} \\ r_{22} = \frac{\lambda}{\beta_1} \end{cases} \implies \mathbf{R} = \begin{bmatrix} \frac{\lambda(\lambda+\beta_1+\beta_2)}{\mu\beta_1} & \frac{\lambda}{\beta_1} \\ \frac{\lambda(\lambda+\beta_2)}{\mu\beta_1} & \frac{\lambda}{\beta_1} \end{bmatrix} = \frac{\lambda}{\beta_1} \begin{bmatrix} \frac{\lambda+\beta_1+\beta_2}{\mu} & 1 \\ \frac{\lambda+\beta_2}{\mu} & 1 \end{bmatrix} \tag{2.8}$$

2.6 The Spectral Radius of R

At this point, we can compute the spectral radius of **R** explicitly and construct a more readily verifiable sufficient condition under which our model will be positive recurrent.

Corollary. *By Lemma 2.4.1, the infinitesimal matrix* **Q** *given in equation* (2.2) *is positive recurrent if and only if:* $\beta_1(\mu-\lambda)-\lambda(\mu+\beta_2) > 0$.

Proof. We compute the spectral radius of **R** by solving the scalar quadratic equation generated by $\det(\mathbf{R}-\rho_i\mathbf{I}) = 0$, yielding that ρ_i satisfies the following quadratic equation:

$$\mu\beta_1\rho_i^2 - \lambda(\lambda+\mu+\beta_1+\beta_2)\rho_i + \lambda^2 = 0 \tag{2.9}$$

$$\implies \rho_i = \frac{\lambda\left(\lambda+\mu+\beta_1+\beta_2+(-1)^i\sqrt{(\lambda+\mu+\beta_1+\beta_2)^2-4\mu\beta_1}\right)}{2\mu\beta_1}, \qquad i = 0,1$$

It is clear by inspection that the largest of these eigenvalues in (2.9) will contain the positive radical. Thus, by Lemma 2.4.1, **Q** is positive recurrent if and only if:

$$\rho_0 = \frac{\lambda\left(\lambda+\mu+\beta_1+\beta_2+\sqrt{(\lambda+\mu+\beta_1+\beta_2)^2-4\mu\beta_1}\right)}{2\mu\beta_1} < 1 \tag{2.10}$$

$$\iff \lambda+\mu+\beta_1+\beta_2+\sqrt{(\lambda+\mu+\beta_1+\beta_2)^2-4\mu\beta_1} < \frac{2\mu\beta_1}{\lambda}$$

$$\iff \sqrt{(\lambda+\mu+\beta_1+\beta_2)^2-4\mu\beta_1} < \frac{2\mu\beta_1}{\lambda} - (\lambda+\mu+\beta_1+\beta_2)$$

$$\iff (\lambda+\mu+\beta_1+\beta_2)^2-4\mu\beta_1 < \frac{4\mu^2\beta_1^2}{\lambda^2} - \frac{4\mu\beta_1(\lambda+\mu+\beta_1+\beta_2)}{\lambda} + (\lambda+\mu+\beta_1+\beta_2)^2$$

$$\iff -4\mu\beta_1 < \frac{4\mu^2\beta_1^2}{\lambda^2} - \frac{4\mu\beta_1(\lambda+\mu+\beta_1+\beta_2)}{\lambda}$$

$$\iff -4\mu\beta_1\lambda^2 < 4\mu^2\beta_1^2 - 4\mu\beta_1\lambda(\lambda+\mu+\beta_1+\beta_2)$$

$$\iff 0 < 4\mu\beta_1\lambda^2 + 4\mu^2\beta_1^2 - 4\lambda\mu\beta_1(\lambda+\mu+\beta_1+\beta_2)$$

$$\iff 0 < \lambda^2 + \mu\beta_1 - \lambda(\lambda+\mu+\beta_1+\beta_2)$$

$$\iff 0 < \mu\beta_1 - \lambda(\mu+\beta_1+\beta_2)$$

$$\iff \beta_1(\mu-\lambda) - \lambda(\mu+\beta_2) > 0$$

□

2.7 The Stationary Distribution

2.7.1 The Explicit form of $\mathbf{R}^k$

Proposition 2.7.1. *Using the scalar-factored form of* $\mathbf{R}$ *in (2.8), we find:*

$$\mathbf{R}^k = \begin{bmatrix} \frac{(\beta_1\rho_0-\lambda)\rho_0^k+\rho_1^k(\lambda-\beta_1\rho_1)}{\beta_1(\rho_0-\rho_1)} & \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)} \\ \frac{\lambda(\lambda+\beta_2)\left(\rho_0^k-\rho_1^k\right)}{\mu\beta_1(\rho_0-\rho_1)} & \frac{(\lambda-\beta_1\rho_1)\rho_0^k+\rho_1^k(\beta_1\rho_0-\lambda)}{\beta_1(\rho_0-\rho_1)} \end{bmatrix} \tag{2.11}$$

Proof. By Mathematical Induction, we will show this is true for $k=1$, assume it is true for arbitrary k, then show it is true for $k+1$.

<u>k=1</u>

$$\begin{aligned}
\mathbf{R}^1 &= \begin{bmatrix} \frac{(\beta_1\rho_0-\lambda)\rho_0+\rho_1(\lambda-\beta_1\rho_1)}{\beta_1(\rho_0-\rho_1)} & \frac{\lambda(\rho_0-\rho_1)}{\beta_1(\rho_0-\rho_1)} \\ \frac{\lambda(\lambda+\beta_2)(\rho_0-\rho_1)}{\mu\beta_1(\rho_0-\rho_1)} & \frac{(\lambda-\beta_1\rho_1)\rho_0+\rho_1(\beta_1\rho_0-\lambda)}{\beta_1(\rho_0-\rho_1)} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\beta_1\rho_0^2-\lambda\rho_0+\lambda\rho_1-\beta_1\rho_1^2}{\beta_1(\rho_0-\rho_1)} & \frac{\lambda}{\beta_1} \\ \frac{\lambda(\lambda+\beta_2)}{\mu\beta_1} & \frac{\lambda\rho_0-\beta_1\rho_1\rho_0+\beta_1\rho_1\rho_0-\lambda\rho_1}{\beta_1(\rho_0-\rho_1)} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\beta_1\left(\rho_0^2-\rho_1^2\right)-\lambda\rho_0+\lambda\rho_1}{\beta_1(\rho_0-\rho_1)} & \frac{\lambda}{\beta_1} \\ \frac{\lambda(\lambda+\beta_2)}{\mu\beta_1} & \frac{\lambda\rho_0-\lambda\rho_1}{\beta_1(\rho_0-\rho_1)} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\beta_1(\rho_0-\rho_1)(\rho_0+\rho_1)-\lambda(\rho_0-\rho_1)}{\beta_1(\rho_0-\rho_1)} & \frac{\lambda}{\beta_1} \\ \frac{\lambda(\lambda+\beta_2)}{\mu\beta_1} & \frac{\lambda(\rho_0-\rho_1)}{\beta_1(\rho_0-\rho_1)} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\beta_1(\rho_0+\rho_1)-\lambda}{\beta_1} & \frac{\lambda}{\beta_1} \\ \frac{\lambda(\lambda+\beta_2)}{\mu\beta_1} & \frac{\lambda}{\beta_1} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\lambda(\lambda+\beta_1+\beta_2)}{\mu\beta_1} & \frac{\lambda}{\beta_1} \\ \frac{\lambda(\lambda+\beta_2)}{\mu\beta_1} & \frac{\lambda}{\beta_1} \end{bmatrix}, \quad \text{where } \rho_0+\rho_1 = \frac{\lambda(\lambda+\mu+\beta_1+\beta_2)}{\mu\beta_1}
\end{aligned}$$

<u>k+1</u>

$$\begin{aligned}
\mathbf{R}^k\mathbf{R} &= \begin{bmatrix} \frac{(\beta_1\rho_0-\lambda)\rho_0^k+\rho_1^k(\lambda-\beta_1\rho_1)}{\beta_1(\rho_0-\rho_1)} & \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)} \\ \frac{\lambda(\lambda+\beta_2)\left(\rho_0^k-\rho_1^k\right)}{\mu\beta_1(\rho_0-\rho_1)} & \frac{(\lambda-\beta_1\rho_1)\rho_0^k+\rho_1^k(\beta_1\rho_0-\lambda)}{\beta_1(\rho_0-\rho_1)} \end{bmatrix} \begin{bmatrix} \frac{\lambda(\lambda+\beta_1+\beta_2)}{\mu\beta_1} & \frac{\lambda}{\beta_1} \\ \frac{\lambda(\lambda+\beta_2)}{\mu\beta_1} & \frac{\lambda}{\beta_1} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\lambda(\lambda+\beta_1+\beta_2)\left((\beta_1\rho_0-\lambda)\rho_0^k+\rho_1^k(\lambda-\beta_1\rho_1)\right)+\lambda^2(\lambda+\beta_2)\left(\rho_0^k-\rho_1^k\right)}{\mu\beta_1^2(\rho_0-\rho_1)} & \frac{\lambda}{\beta_1}\left(\frac{(\beta_1\rho_0-\lambda)\rho_0^k+\rho_1^k(\lambda-\beta_1\rho_1)+\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)}\right) \\ \frac{\lambda(\lambda+\beta_2)\left(\lambda(\lambda+\beta_1+\beta_2)\left(\rho_0^k-\rho_1^k\right)+(\lambda-\beta_1\rho_1)\mu\rho_0^k+\mu\rho_1^k(\beta_1\rho_0-\lambda)\right)}{\mu^2\beta_1^2(\rho_0-\rho_1)} & \frac{\lambda}{\beta_1}\left(\frac{\lambda(\lambda+\beta_2)\left(\rho_0^k-\rho_1^k\right)+(\lambda-\beta_1\rho_1)\mu\rho_0^k+\mu\rho_1^k(\beta_1\rho_0-\lambda)}{\mu\beta_1(\rho_0-\rho_1)}\right) \end{bmatrix} \\
&= \begin{bmatrix} \frac{\lambda(\lambda+\beta_1+\beta_2)\left(\beta_1\left(\rho_0^{k+1}-\rho_1^{k+1}\right)-\lambda\left(\rho_0^k-\rho_1^k\right)\right)+\lambda^2(\lambda+\beta_2)\left(\rho_0^k-\rho_1^k\right)}{\mu\beta_1^2(\rho_0-\rho_1)} & \frac{\lambda}{\beta_1}\left(\frac{\beta_1\rho_0^{k+1}-\beta_1\rho_1^{k+1}}{\beta_1(\rho_0-\rho_1)}\right) \\ \frac{\lambda(\lambda+\beta_2)\left(\lambda(\lambda+\mu+\beta_1+\beta_2)\left(\rho_0^k-\rho_1^k\right)-\mu\beta_1\rho_1\rho_0^k+\mu\beta_1\rho_1^k\rho_0\right)}{\mu^2\beta_1^2(\rho_0-\rho_1)} & \frac{\lambda}{\beta_1}\left(\frac{\lambda(\lambda+\mu+\beta_2)\left(\rho_0^k-\rho_1^k\right)-\mu\beta_1\rho_1\rho_0^k+\mu\beta_1\rho_1^k\rho_0}{\mu\beta_1(\rho_0-\rho_1)}\right) \end{bmatrix} \\
&= \begin{bmatrix} \frac{\lambda(\lambda+\beta_1+\beta_2)\left(\rho_0^{k+1}-\rho_1^{k+1}\right)-\lambda^2\left(\rho_0^k-\rho_1^k\right)}{\mu\beta_1(\rho_0-\rho_1)} & \frac{\lambda\left(\rho_0^{k+1}-\rho_1^{k+1}\right)}{\beta_1(\rho_0-\rho_1)} \\ \frac{\lambda(\lambda+\beta_2)\left((\rho_0+\rho_1)\left(\rho_0^k-\rho_1^k\right)-\rho_1\rho_0^k+\rho_1^k\rho_0\right)}{\mu\beta_1(\rho_0-\rho_1)} & \frac{\lambda}{\beta_1}\left(\frac{\lambda(\lambda+\mu+\beta_1+\beta_2)\left(\rho_0^k-\rho_1^k\right)-\lambda\beta_1\left(\rho_0^k-\rho_1^k\right)-\mu\beta_1\rho_1\rho_0^k+\mu\beta_1\rho_1^k\rho_0}{\mu\beta_1(\rho_0-\rho_1)}\right) \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} \frac{\mu\beta_1(\rho_0+\rho_1)\left(\rho_0^{k+1}-\rho_1^{k+1}\right)-\lambda\mu\left(\rho_0^{k+1}-\rho_1^{k+1}\right)-\mu\beta_1\rho_0\rho_1\left(\rho_0^{k}-\rho_1^{k}\right)}{\mu\beta_1(\rho_0-\rho_1)} & \frac{\lambda\left(\rho_0^{k+1}-\rho_1^{k+1}\right)}{\beta_1(\rho_0-\rho_1)} \\ \frac{\lambda(\lambda+\beta_2)\left(\rho_0^{k+1}-\rho_1^{k+1}\right)}{\mu\beta_1(\rho_0-\rho_1)} & \frac{\lambda}{\beta_1}\left(\frac{\mu\beta_1\left(\rho_0^{k+1}-\rho_1^{k+1}\right)-\lambda\beta_1\left(\rho_0^{k}-\rho_1^{k}\right)}{\mu\beta_1(\rho_0-\rho_1)}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\beta_1\left(\rho_0^{k+2}-\rho_1^{k+2}\right)-\lambda\left(\rho_0^{k+1}-\rho_1^{k+1}\right)}{\beta_1(\rho_0-\rho_1)} & \frac{\lambda\left(\rho_0^{k+1}-\rho_1^{k+1}\right)}{\beta_1(\rho_0-\rho_1)} \\ \frac{\lambda(\lambda+\beta_2)\left(\rho_0^{k+1}-\rho_1^{k+1}\right)}{\mu\beta_1(\rho_0-\rho_1)} & \frac{\lambda\left(\rho_0^{k+1}-\rho_1^{k+1}\right)-\beta_1\rho_0\rho_1\left(\rho_0^{k}-\rho_1^{k}\right)}{\beta_1(\rho_0-\rho_1)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(\beta_1\rho_0-\lambda)\rho_0^{k+1}+\rho_1^{k+1}(\lambda-\beta_1\rho_1)}{\beta_1(\rho_0-\rho_1)} & \frac{\lambda\left(\rho_0^{k+1}-\rho_1^{k+1}\right)}{\beta_1(\rho_0-\rho_1)} \\ \frac{\lambda(\lambda+\beta_2)\left(\rho_0^{k+1}-\rho_1^{k+1}\right)}{\mu\beta_1(\rho_0-\rho_1)} & \frac{(\lambda-\beta_1\rho_1)\rho_0^{k+1}+\rho_1^{k+1}(\beta_1\rho_0-\lambda)}{\beta_1(\rho_0-\rho_1)} \end{bmatrix} = \mathbf{R}^{k+1}$$

Remark. Two substitutions were needed in this derivation, namely: $\rho_0 + \rho_1 = \frac{\lambda(\lambda+\mu+\beta_1+\beta_2)}{\mu\beta_1}$ and $\rho_0\rho_1 = \frac{\lambda^2}{\mu\beta_1}$. These can easily be verified from (2.9).

□

2.7.2 The Initial terms of π

Next we turn our attention to computing $\mathbf{B}[\mathbf{R}]$, and a positive vector $(x_0, \mathbf{x_1})$, such that $(x_0, \mathbf{x_1})\mathbf{B}[\mathbf{R}] = 0$:

$$(x_0, \mathbf{x_1})\mathbf{B}[\mathbf{R}] = (x_0, \mathbf{x_1}) \begin{bmatrix} \hat{\mathbf{A}} & \hat{\mathbf{C}} \\ \hat{\mathbf{B}} & \mathbf{A} + \mathbf{R}\mathbf{B} \end{bmatrix} = (x_0, \mathbf{x_1}) \begin{bmatrix} -\lambda & \lambda & 0 \\ 0 & -\mu & \mu \\ \beta_1 & \lambda+\beta_2 & -(\lambda+\beta_1+\beta_2) \end{bmatrix} = \mathbf{0} \tag{2.12}$$

$$\implies (x_0, \mathbf{x_1}) = \left(1, \left(\tfrac{\lambda(\beta_1+\beta_2+\lambda)}{\beta_1\mu}, \tfrac{\lambda}{\beta_1}\right)\right)$$

We now seek to normalize the solution in order to generate the first three terms of π:

$$K(x_0 + \mathbf{x_1}(\mathbf{I}-\mathbf{R})^{-1}\mathbf{e}) = 1 \implies K = \frac{\beta_1(\mu-\lambda) - \lambda\,(\beta_2+\mu)}{\beta_1\mu} = (1-\rho_0)(1-\rho_1) \tag{2.13}$$

$$\implies (\pi_{00}, \pi_{10}, \pi_{11}) = K(x_0, \mathbf{x_1}) \implies \begin{cases} \pi_{00} = \frac{\beta_1(\mu-\lambda)-\lambda(\beta_2+\mu)}{\beta_1\mu} = K \\ \pi_{10} = \frac{\lambda(\beta_1+\beta_2+\lambda)(\beta_1(\mu-\lambda)-\lambda(\beta_2+\mu))}{\beta_1^2\mu^2} = \frac{\lambda K(\beta_1+\beta_2+\lambda)}{\beta_1\mu} \\ \pi_{11} = \frac{\lambda(\beta_1(\mu-\lambda)-\lambda(\beta_2+\mu))}{\beta_1^2\mu} = \frac{\lambda K}{\beta_1} \end{cases}$$

Remark. We observe that the condition given by the Corollary to Lemma 2.4.1 for positive recurrence: $\beta_1(\mu-\lambda) - \lambda(\mu+\beta_2) > 0$ is equivalent to $K > 0$.

2.7.3 The Remaining Terms of $\boldsymbol{\pi}$

Proposition 2.7.2. *The remaining elements* $\{(\pi_{k0}, \pi_{k1}) \mid k \geq 2\}$ *of our stationary distribution satisfying* $(\pi_{k0}, \pi_{k1}) = (\pi_{10}, \pi_{11})\mathbf{R}^{k-1}$ *and* $\pi_{00} + \sum_{k=1}^{\infty}(\pi_{k0} + \pi_{k1}) = 1$ *are given by:*

$$\begin{cases} \pi_{k0} = K\left(\frac{\rho_0^{k+1}-\rho_1^{k+1}}{\rho_0-\rho_1} - \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)}\right) \\ \pi_{k1} = \frac{\lambda K\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)} \end{cases} \tag{2.14}$$

Proof. To motivate the proof, we begin by noting that:

$$\begin{aligned}(\pi_{k0}, \pi_{k1}) = (\pi_{10}, \pi_{11})\mathbf{R}^{k-1} &\iff (\pi_{k0}, \pi_{k1}) = (\pi_{10}, \pi_{11})\mathbf{R}^{k-2}\mathbf{R} \\ &\iff (\pi_{k0}, \pi_{k1}) = (\pi_{k-1,0}, \pi_{k-1,1})\mathbf{R}\end{aligned}$$

and consider:

$$\begin{aligned}
(\pi_{k-1,0}, \pi_{k-1,1})\mathbf{R} &= \frac{\lambda K}{\beta_1}\left(\frac{\rho_0^k-\rho_1^k}{\rho_0-\rho_1} - \frac{\lambda\left(\rho_0^{k-1}-\rho_1^{k-1}\right)}{\beta_1(\rho_0-\rho_1)}, \frac{\lambda\left(\rho_0^{k-1}-\rho_1^{k-1}\right)}{\beta_1(\rho_0-\rho_1)}\right)\begin{bmatrix}\frac{\lambda+\beta_1+\beta_2}{\mu} & 1 \\ \frac{\lambda+\beta_2}{\mu} & 1\end{bmatrix} \\
&= \frac{\lambda K}{\beta_1}\left(\frac{\lambda+\beta_1+\beta_2}{\mu}\frac{\rho_0^k-\rho_1^k}{\rho_0-\rho_1} - \frac{\lambda+\beta_1+\beta_2}{\mu}\frac{\lambda\left(\rho_0^{k-1}-\rho_1^{k-1}\right)}{\beta_1(\rho_0-\rho_1)} + \frac{\lambda+\beta_2}{\mu}\frac{\lambda\left(\rho_0^{k-1}-\rho_1^{k-1}\right)}{\beta_1(\rho_0-\rho_1)},\right. \\
&\qquad \left.\frac{\rho_0^k-\rho_1^k}{\rho_0-\rho_1} - \frac{\lambda\left(\rho_0^{k-1}-\rho_1^{k-1}\right)}{\beta_1(\rho_0-\rho_1)} + \frac{\lambda\left(\rho_0^{k-1}-\rho_1^{k-1}\right)}{\beta_1(\rho_0-\rho_1)}\right) \\
&= \frac{\lambda K}{\beta_1}\left(\frac{\lambda+\beta_1+\beta_2}{\mu}\frac{\rho_0^k-\rho_1^k}{\rho_0-\rho_1} - \frac{\beta_1}{\mu}\frac{\lambda\left(\rho_0^{k-1}-\rho_1^{k-1}\right)}{\beta_1(\rho_0-\rho_1)}, \frac{\rho_0^k-\rho_1^k}{\rho_0-\rho_1}\right) \\
&= K\left(\frac{\lambda(\lambda+\beta_1+\beta_2)}{\mu\beta_1}\frac{\rho_0^k-\rho_1^k}{\rho_0-\rho_1} - \frac{\lambda^2\left(\rho_0^{k-1}-\rho_1^{k-1}\right)}{\mu\beta_1(\rho_0-\rho_1)}, \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)}\right) \\
&= K\left(\frac{\lambda(\lambda+\mu+\beta_1+\beta_2)}{\mu\beta_1}\frac{\rho_0^k-\rho_1^k}{\rho_0-\rho_1} - \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)} - \frac{\lambda^2\left(\rho_0^{k-1}-\rho_1^{k-1}\right)}{\mu\beta_1(\rho_0-\rho_1)}, \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)}\right) \\
&= K\left(\frac{(\rho_0+\rho_1)\left(\rho_0^k-\rho_1^k\right)}{\rho_0-\rho_1} - \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)} - \frac{\rho_0\rho_1\left(\rho_0^{k-1}-\rho_1^{k-1}\right)}{(\rho_0-\rho_1)}, \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)}\right) \\
&= K\left(\frac{\rho_0^{k+1}-\rho_1^{k+1}}{\rho_0-\rho_1} - \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)}, \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)}\right) = (\pi_{k0}, \pi_{k1})
\end{aligned}$$

We have:

$$\begin{aligned}
\pi_{00} + \sum_{k=1}^{\infty}(\pi_{k0} + \pi_{k1}) &= \sum_{k=0}^{\infty}(\pi_{k0} + \pi_{k1}), \quad \text{where (2.14) is valid for all } k \in \mathbb{N} \cup \{0\} \\
&= K\sum_{k=0}^{\infty}\left(\frac{\rho_0^{k+1}-\rho_1^{k+1}}{\rho_0-\rho_1} - \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)} + \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)}\right) \\
&= K\sum_{k=0}^{\infty}\left(\frac{\rho_0^{k+1}-\rho_1^{k+1}}{\rho_0-\rho_1}\right) \\
&= \frac{K}{\rho_0-\rho_1}\sum_{k=0}^{\infty}\rho_0^{k+1} - \sum_{k=0}^{\infty}\rho_1^{k+1} \\
&= \frac{K}{\rho_0-\rho_1}\left(\frac{\rho_0}{1-\rho_0} - \frac{\rho_1}{1-\rho_1}\right) \\
&= \frac{K}{\rho_0-\rho_1}\left(\frac{\rho_0(1-\rho_1)-\rho_1(1-\rho_0)}{(1-\rho_0)(1-\rho_1)}\right)
\end{aligned}$$

$$= \frac{K}{\rho_0-\rho_1}\left(\frac{\rho_0-\rho_1}{(1-\rho_0)(1-\rho_1)}\right)$$
$$= \frac{K}{(1-\rho_0)(1-\rho_1)} = 1, \quad \text{since } K = (1-\rho_0)(1-\rho_1).$$
□

2.8 Decomposition

2.8.1 Decomposition of N

Theorem 2.8.1. *If $K > 0$, then the stationary number of customers in steady-state, N, can be decomposed into the sum of two independent geometric random variables parameterized by $1-\rho_0$ and $1-\rho_1$. Namely: $N = X_0 + X_1$, where:*
$X_0 \sim$ Geometric$(1-\rho_0)$, and $X_1 \sim$ Geometric$(1-\rho_1)$.

Proof. With our stationary distribution explicitly found, we compute the stationary queue-length probability generating function (P.G.F.) defined by:

$$\mathcal{G}_{\mathcal{N}}(z) = \sum_{k=0}^{\infty} P(N=k)z^k \tag{2.15}$$

$$\begin{aligned}
\implies \mathcal{G}_{\mathcal{N}}(z) = \sum_{k=0}^{\infty} P(N=k)z^k &= \sum_{k=0}^{\infty} P\Big((N=k\cap S=0)\cup(N=k\cap S=1)\Big)z^k \\
&= \sum_{k=0}^{\infty}\Big(P(N=k\cap S=0)+P(N=k\cap S=1)\Big)z^k \\
&= \sum_{k=0}^{\infty}\Big(\pi_{k0}+\pi_{k1}\Big)z^k \\
&= K\sum_{k=0}^{\infty}\left(\frac{\rho_0^{k+1}-\rho_1^{k+1}}{\rho_0-\rho_1} - \frac{\lambda\big(\rho_0^k-\rho_1^k\big)}{\beta_1(\rho_0-\rho_1)} + \frac{\lambda\big(\rho_0^k-\rho_1^k\big)}{\beta_1(\rho_0-\rho_1)}\right)z^k \\
&= K\sum_{k=0}^{\infty}\left(\frac{\rho_0^{k+1}-\rho_1^{k+1}}{\rho_0-\rho_1}\right)z^k \\
&= \frac{K}{\rho_0-\rho_1}\sum_{k=0}^{\infty}(\rho_0^{k+1}-\rho_1^{k+1})z^k \\
&= \frac{K}{\rho_0-\rho_1}\left(\rho_0\sum_{k=0}^{\infty}(\rho_0 z)^k - \rho_1\sum_{k=0}^{\infty}(\rho_1 z)^k\right) \\
&= \frac{K}{\rho_0-\rho_1}\left(\frac{\rho_0}{1-\rho_0 z} - \frac{\rho_1}{1-\rho_1 z}\right) \\
&= \frac{K}{\rho_0-\rho_1}\left(\frac{\rho_0(1-\rho_1 z)-\rho_1(1-\rho_0 z)}{(1-\rho_0 z)(1-\rho_1 z)}\right) \\
&= \frac{K}{\rho_0-\rho_1}\left(\frac{\rho_0-\rho_1}{(1-\rho_0 z)(1-\rho_1 z)}\right) \\
&= \frac{K}{(1-\rho_0 z)(1-\rho_1 z)} \\
&= \frac{1-\rho_0}{1-\rho_0 z}\,\frac{1-\rho_1}{1-\rho_1 z} = \mathcal{G}_{X_0}(z)\mathcal{G}_{X_1}(z)
\end{aligned}$$

$\implies N = X_0 + X_1$, where X_0 and X_1 are independent,
$X_0 \sim \text{Geometric}(1-\rho_0)$, and
$X_1 \sim \text{Geometric}(1-\rho_1)$.
□

2.8.2 Little's Distributional Law

Since we have the P.G.F. of N, we may now employ Little's Distributional Law, named after John Little for his work in [10], proved in general by Keilson, J and Servi, LD [11].

Theorem 2.8.2. *Little's Distributional Law*
Let N be the stationary number of customers in a steady-state queue where the arrivals come according to a Poisson stream with rate λ. Let W be the stationary waiting time. Let $\mathcal{W}^*(s)$ denote the L.S.T. (Laplace Stieltjes Transform) of W. Then:

$$\mathcal{G}_N(z) = \mathcal{W}^*((1-z)\lambda) \tag{2.16}$$

Proof. While we will refer the reader to [11] for details, we give an elementary direct proof in line with our notation. Given the definition of the P.G.F. of N, we rewrite $P(N=k)$ via total probability and obtain:

$$\mathcal{G}_N(z) = \sum_{k=0}^{\infty} P(N=k)z^k = \sum_{k=0}^{\infty} z^k \int_{-\infty}^{\infty} P(N=k \mid W=t)dF_W = \sum_{k=0}^{\infty} z^k \int_{-\infty}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}dF_W$$
$$= \int_{-\infty}^{\infty} e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t z)^k}{k!}dF_W = \int_{-\infty}^{\infty} e^{-\lambda t}e^{\lambda t z}dF_W = \int_{-\infty}^{\infty} e^{-\lambda t(1-z)}dF_W = \mathcal{W}^*((1-z)\lambda)$$

□

2.8.3 Decomposition of W

Theorem 2.8.3. *If $K > 0$, then the stationary waiting time W for customers in the steady-state queue of length N can be decomposed into the sum of two independent exponential random variables parameterized by $\frac{\lambda(1-\rho_0)}{\rho_0}$ and $\frac{\lambda(1-\rho_1)}{\rho_1}$. Namely: $W = Y_0 + Y_1$, where:*
$Y_0 \sim Exponential\Big(\frac{\lambda(1-\rho_0)}{\rho_0}\Big)$, and $Y_1 \sim Exponential\Big(\frac{\lambda(1-\rho_1)}{\rho_1}\Big)$.

Proof. Using Theorem 2.8.2, we can find the L.S.T. of W explicitly:

$$\begin{aligned}
\mathcal{W}^*(s) = \mathcal{G}_N\Big(1-\frac{s}{\lambda}\Big) &= \frac{K}{(1-\rho_0(1-\frac{s}{\lambda}))(1-\rho_1(1-\frac{s}{\lambda}))} \\
&= \frac{K\lambda^2}{(\lambda-\rho_0(\lambda-s))(\lambda-\rho_1(\lambda-s))} \\
&= \frac{K\lambda^2}{(\lambda(1-\rho_0)+\rho_0 s)(\lambda(1-\rho_1)+\rho_1 s)} \\
&= \frac{\frac{K\lambda^2}{\rho_0\rho_1}}{(\frac{\lambda(1-\rho_0)}{\rho_0}+s)(\frac{\lambda(1-\rho_1)}{\rho_1}+s)} \\
&= \frac{\frac{\lambda(1-\rho_0)}{\rho_0}\frac{\lambda(1-\rho_1)}{\rho_1}}{(\frac{\lambda(1-\rho_0)}{\rho_0}+s)(\frac{\lambda(1-\rho_1)}{\rho_1}+s)} \\
&= \frac{\frac{\lambda(1-\rho_0)}{\rho_0}}{(\frac{\lambda(1-\rho_0)}{\rho_0}+s)}\frac{\frac{\lambda(1-\rho_1)}{\rho_1}}{(\frac{\lambda(1-\rho_1)}{\rho_1}+s)} \\
&\implies W = Y_0 + Y_1, \text{ where } Y_0 \text{ and } Y_1 \text{ are independent,} \\
&\qquad Y_0 \sim \text{Exponential}\Big(\frac{\lambda(1-\rho_0)}{\rho_0}\Big), \text{ and } Y_1 \sim \text{Exponential}\Big(\frac{\lambda(1-\rho_1)}{\rho_1}\Big).
\end{aligned}$$

□

2.9 Analytical Results

	Steady-State # of Customers: N	Steady-State Waiting Time: W
	$\mathcal{G}_N(z) = \frac{1-\rho_0}{1-\rho_0 z}\frac{1-\rho_1}{1-\rho_1 z}$	$\mathcal{W}^*(s) = \frac{\frac{\lambda(1-\rho_0)}{\rho_0}}{(\frac{\lambda(1-\rho_0)}{\rho_0}+s)}\frac{\frac{\lambda(1-\rho_1)}{\rho_1}}{(\frac{\lambda(1-\rho_1)}{\rho_1}+s)}$
E(.)	$\frac{\rho_0}{1-\rho_0} + \frac{\rho_1}{1-\rho_1}$	$\frac{\rho_0}{\lambda(1-\rho_0)} + \frac{\rho_1}{\lambda(1-\rho_1)}$
Var(.)	$\frac{\rho_0}{(1-\rho_0)^2} + \frac{\rho_1}{(1-\rho_1)^2}$	$\frac{\rho_0^2}{\lambda^2(1-\rho_0)^2} + \frac{\rho_1^2}{\lambda^2(1-\rho_1)^2}$

Table 2.1: Analytical Results

2.10 Special cases

We can now recover the stationary behavior of the following known queue types as particular cases from our model as follows:

- $\beta_1 \to \infty$ with $0 \leq \beta_2 < \infty$ results in the classical $M/M/1$ queue.
- $0 < \beta_1 < \infty$ with $\beta_2 = 0$ and $\mu = \beta_1$ results in an $M/E_2/1$ queue, where E_2 refers to an 'Erlang' service time distribution with shape 2 and rate μ.
- $0 < \beta_1 < \infty$ with $\beta_2 = 0$ and $\mu > \beta_1$ results in an $M/HE/1$ queue, where HE refers to a hypoexponential service time distribution $\sim f(t) = \frac{\mu\beta_1(e^{-\beta_1 t}-e^{-\mu t})}{\mu-\beta_1}$ [12].

2.10.1 M/M/1

Proposition 2.10.1.
If $0 \leq \beta_2 < \infty$ and $\beta_1 \to \infty$, our model recovers the stationary behavior of the classical M/M/1.

Proof.
We begin by computing the eigenvalues of $\mathbf{R}$ under these conditions.

$$\lim_{\beta_1\to\infty} \rho_0 = \lim_{\beta_1\to\infty} \frac{\lambda\left(\lambda+\mu+\beta_1+\beta_2+\sqrt{(\lambda+\mu+\beta_1+\beta_2)^2-4\mu\beta_1}\right)}{2\mu\beta_1} = \frac{\lambda}{\mu} = \rho < 1 \Longrightarrow \text{Positive Recurrent}$$

$$\lim_{\beta_1\to\infty} \rho_1 = \lim_{\beta_1\to\infty} \frac{\lambda\left(\lambda+\mu+\beta_1+\beta_2-\sqrt{(\lambda+\mu+\beta_1+\beta_2)^2-4\mu\beta_1}\right)}{2\mu\beta_1} = 0$$

We now consider our P.G.F. with the appropriate substitutions.

$$\mathcal{G}_N(z) = \frac{1-\rho_0}{1-\rho_0 z}\frac{1-\rho_1}{1-\rho_1 z} = \frac{1-\rho}{1-\rho z}$$

Since the P.G.F. of N matches that of the classical M/M/1 queue given on page 32 of [13], we conclude that the stationary queue lengths are equivalent in distribution. □

2.10.2 M/E_2/1

Proposition 2.10.2.

If $0 < \beta_1 < \infty$, $\beta_2 = 0$ and $\mu = \beta_1$, our model recovers the stationary behavior of an $M/E_2/1$ queue, where E_2 refers to an 'Erlang' service time distribution with shape 2 and rate μ.

Proof.

We begin, again, by computing the eigenvalues of $\mathbf{R}$ under these conditions and obtain:

$$\rho_0 = \frac{\lambda\left(\lambda+\mu+\beta_1+\beta_2+\sqrt{(\lambda+\mu+\beta_1+\beta_2)^2-4\mu\beta_1}\right)}{2\mu\beta_1} = \frac{\lambda\left(\lambda+2\mu+\sqrt{(\lambda+2\mu)^2-4\mu^2}\right)}{2\mu^2} = \frac{\lambda\left(\lambda+2\mu+\sqrt{\lambda(\lambda+4\mu)}\right)}{2\mu^2}$$

We also note that $\frac{\lambda\left(\lambda+2\mu+\sqrt{\lambda(\lambda+4\mu)}\right)}{2\mu^2} < 1 \iff \frac{2\lambda}{\mu} < 1$

Therefore, let $\rho = \frac{2\lambda}{\mu}$ and we have $\rho_0 < 1 \iff \rho < 1 \implies$ Positive Recurrent

We must now consider our L.S.T. $\mathcal{W}^*(s)$:

$$\begin{aligned}
\mathcal{W}^*(s) = \mathcal{G}_N\left(1-\tfrac{s}{\lambda}\right) &= \frac{K}{(1-\rho_0(1-\frac{s}{\lambda}))(1-\rho_1(1-\frac{s}{\lambda}))} \\
&= \frac{K\lambda^2}{(\lambda-\rho_0(\lambda-s))(\lambda-\rho_1(\lambda-s))} \\
&= \frac{K\lambda^2}{(\lambda(1-\rho_0)+\rho_0 s)(\lambda(1-\rho_1)+\rho_1 s)} \\
&= \frac{K\lambda^2}{\lambda^2(1-\rho_0)(1-\rho_1)+\lambda\rho_1(1-\rho_0)s+\lambda\rho_0(1-\rho_1)s+\rho_0\rho_1 s^2} \\
&= \frac{K\lambda^2}{\lambda^2(1-\rho_0)(1-\rho_1)+\lambda\rho_1 s-\lambda\rho_0\rho_1 s+\lambda\rho_0 s-\lambda\rho_0\rho_1 s+\rho_0\rho_1 s^2} \\
&= \frac{K\lambda^2}{\lambda^2(1-\rho_0)(1-\rho_1)+\lambda s(\rho_0+\rho_1)+\rho_0\rho_1(s^2-2\lambda s)}
\end{aligned}$$

We now need some substitutions: namely:

$$K = (1-\rho_0)(1-\rho_1) = \frac{\beta_1(\mu-\lambda)-\lambda(\beta_2+\mu)}{\beta_1\mu} = \frac{\mu(\mu-\lambda)-\lambda(\mu)}{\mu^2} = \frac{\mu^2-2\lambda\mu}{\mu^2}$$

$$\rho_0\rho_1 = \frac{\lambda^2}{\mu\beta_1} = \frac{\lambda^2}{\mu^2}$$

$$\begin{aligned}
\rho_0+\rho_1 &= \frac{\lambda\left(\lambda+\mu+\beta_1+\beta_2+\sqrt{(\lambda+\mu+\beta_1+\beta_2)^2-4\mu\beta_1}\right)}{2\mu\beta_1} + \frac{\lambda\left(\lambda+\mu+\beta_1+\beta_2-\sqrt{(\lambda+\mu+\beta_1+\beta_2)^2-4\mu\beta_1}\right)}{2\mu\beta_1} \\
&= \frac{2\lambda(\lambda+\mu+\beta_1+\beta_2)}{2\mu\beta_1} \\
&= \frac{\lambda^2+2\lambda\mu}{\mu^2}
\end{aligned}$$

Proceeding where we left off, we have:

$$\begin{aligned}
\mathcal{W}^*(s) = \mathcal{G}_N\left(1-\tfrac{s}{\lambda}\right) &= \frac{\frac{\lambda^2\mu^2-2\lambda^3\mu}{\mu^2}}{\lambda^2\frac{\mu^2-2\lambda\mu}{\mu^2}+\lambda s\frac{\lambda^2+2\lambda\mu}{\mu^2}+\frac{\lambda^2}{\mu^2}(s^2-2\lambda s)} \\
&= \frac{\mu^2-2\lambda\mu}{\mu^2-2\lambda\mu+s(\lambda+2\mu)+(s^2-2\lambda s)} \\
&= \frac{\mu^2-2\lambda\mu}{\mu^2-2\lambda\mu+\lambda s+2\mu s+s^2-2\lambda s} \\
&= \frac{\mu^2 s(1-\rho)}{\mu^2 s-2\lambda\mu s+2\mu s^2+s^3-\lambda s^2} \\
&= \frac{\mu^2 s(1-\rho)}{-\lambda s^2-2\lambda\mu s-\lambda\mu^2+\lambda\mu^2+2\mu s^2+s^3+\mu^2 s} \\
&= \frac{\mu^2 s(1-\rho)}{-\lambda(s+\mu)^2+s^3+2\mu s^2+\mu^2 s+\lambda\mu^2} \\
&= \frac{\mu^2 s(1-\rho)}{-\lambda(s+\mu)^2+s(s+\mu)^2+\lambda\mu^2} \\
&= \frac{s(1-\rho)\left(\frac{\mu}{s+\mu}\right)^2}{s-\lambda+\lambda\left(\frac{\mu}{s+\mu}\right)^2}
\end{aligned}$$

which matches what is given on page 85 of [14].

Therefore, we conclude that the stationary queue length is equivalent in distribution to an M/E_2/1. □

2.10.3 M/HE/1

Proposition 2.10.3.
If $0 < \beta_1 < \infty$, $\beta_2 = 0$ and $\mu > \beta_1$ we have the stationary behavior of an $M/HE/1$ queue, where HE refers to a hypoexponential service time $\sim f(t) = \frac{\mu\beta_1(e^{-\beta_1 t} - e^{-\mu t})}{\mu - \beta_1}$ (see [12]).

Proof.
We once more compute the eigenvalues of **R** under these given conditions and obtain:
$\rho_0 = \frac{\lambda\left(\lambda+\mu+\beta_1+\beta_2+\sqrt{(\lambda+\mu+\beta_1+\beta_2)^2-4\mu\beta_1}\right)}{2\mu\beta_1} = \frac{\lambda\left(\lambda+\mu+\beta_1+\sqrt{(\lambda+\mu+\beta_1)^2-4\mu\beta_1}\right)}{2\mu\beta_1}$
We again note that $\frac{\lambda\left(\lambda+\mu+\beta_1+\sqrt{(\lambda+\mu+\beta_1)^2-4\mu\beta_1}\right)}{2\mu\beta_1} < 1 \iff \frac{\lambda(\mu+\beta_1)}{\mu\beta_1} < 1$
Therefore, for notational convenience, we will let $\rho = \frac{\lambda(\mu+\beta_1)}{\mu\beta_1}$
and we have $\rho_0 < 1 \iff \rho < 1 \implies$ Positive Recurrent
We must now consider our L.S.T. $\mathcal{W}^*(s)$:

$$\begin{aligned}
\mathcal{W}^*(s) = \mathcal{G}_N\left(1-\tfrac{s}{\lambda}\right) &= \frac{K}{(1-\rho_0(1-\frac{s}{\lambda}))(1-\rho_1(1-\frac{s}{\lambda}))} \\
&= \frac{K\lambda^2}{(\lambda-\rho_0(\lambda-s))(\lambda-\rho_1(\lambda-s))} \\
&= \frac{K\lambda^2}{(\lambda(1-\rho_0)+\rho_0 s)(\lambda(1-\rho_1)+\rho_1 s)} \\
&= \frac{K\lambda^2}{\lambda^2(1-\rho_0)(1-\rho_1)+\lambda\rho_1(1-\rho_0)s+\lambda\rho_0(1-\rho_1)s+\rho_0\rho_1 s^2} \\
&= \frac{K\lambda^2}{\lambda^2(1-\rho_0)(1-\rho_1)+\lambda\rho_1 s-\lambda\rho_0\rho_1 s+\lambda\rho_0 s-\lambda\rho_0\rho_1 s+\rho_0\rho_1 s^2} \\
&= \frac{K\lambda^2}{\lambda^2(1-\rho_0)(1-\rho_1)+\lambda s(\rho_0+\rho_1)+\rho_0\rho_1(s^2-2\lambda s)}
\end{aligned}$$

We now need some substitutions: namely:

$$K = (1-\rho_0)(1-\rho_1) = \frac{\beta_1(\mu-\lambda)-\lambda(\beta_2+\mu)}{\beta_1\mu} = \frac{\beta_1(\mu-\lambda)-\lambda\mu}{\beta_1\mu} = 1-\frac{\lambda(\mu+\beta_1)}{\mu\beta_1} = 1-\rho$$

$$\rho_0\rho_1 = \frac{\lambda^2}{\mu\beta_1}$$

$$\begin{aligned}
\rho_0+\rho_1 &= \frac{\lambda\left(\lambda+\mu+\beta_1+\beta_2+\sqrt{(\lambda+\mu+\beta_1+\beta_2)^2-4\mu\beta_1}\right)}{2\mu\beta_1} + \frac{\lambda\left(\lambda+\mu+\beta_1+\beta_2-\sqrt{(\lambda+\mu+\beta_1+\beta_2)^2-4\mu\beta_1}\right)}{2\mu\beta_1} \\
&= \frac{2\lambda(\lambda+\mu+\beta_1+\beta_2)}{2\mu\beta_1} = \frac{\lambda^2+\lambda(\mu+\beta_1)}{\mu\beta_1} = \frac{\lambda^2}{\mu\beta_1}+\rho
\end{aligned}$$

Proceeding where we left off, we have:

$$\begin{aligned}
\mathcal{W}^*(s) = \mathcal{G}_N\left(1-\tfrac{s}{\lambda}\right) &= \frac{\lambda^2(1-\rho)}{\lambda^2(1-\rho)+\lambda s(\frac{\lambda^2}{\mu\beta_1}+\rho)+\frac{\lambda^2}{\mu\beta_1}(s^2-2\lambda s)} \\
&= \frac{\lambda^2(1-\rho)}{\lambda^2(1-\rho)+\lambda s(\rho-\frac{\lambda^2}{\mu\beta_1})+\frac{\lambda^2}{\mu\beta_1}s^2} \\
&= \frac{(1-\rho)\mu\beta_1}{(\mu\beta_1-\lambda(\mu+\beta_1)+s(\mu+\beta_1-\lambda)+s^2)} \\
&= \frac{s(1-\rho)\mu\beta_1}{s((s+\mu)(s+\beta_1)-\lambda(\mu+\beta_1)-s\lambda)} \\
&= \frac{s(1-\rho)\mu\beta_1}{s(s+\mu)(s+\beta_1)-s\lambda(\mu+\beta_1)-s^2\lambda} \\
&= \frac{s(1-\rho)\mu\beta_1}{s(s+\mu)(s+\beta_1)-\lambda(s+\mu)(s+\beta_1)+\lambda\mu\beta_1} \\
&= \frac{s(1-\rho)\left(\frac{\mu}{s+\mu}\right)\left(\frac{\beta_1}{s+\beta_1}\right)}{s-\lambda+\lambda\left(\frac{\mu}{s+\mu}\right)\left(\frac{\beta_1}{s+\beta_1}\right)}
\end{aligned}$$

Since our "waiting time" includes the customer's service time, this matches what is given on by J.W. Cohen on page 255 of [15]. Therefore, we conclude that the stationary queue length is equivalent in distribution to an M/HE/1. □

2.10.4 Instantaneous Success / Failure

We can also study somewhat familiar type of queue as follows:

Proposition 2.10.4.
If $\beta_1 \to \infty$ with $\beta_1 = \gamma\beta_2$, we have a queue with instantaneous 'success' or 'failure' after the service time has elapsed, where the probability of a successful service is $p_s = \frac{\gamma}{1+\gamma}$ and likewise for failure, we have $p_f = \frac{1}{1+\gamma}$. This queue has a stationary queue length which is equivalent to an M/M/1 queue with service time ∼Exponential(μp_s).

Proof.
Computing the eigenvalues of $\mathbf{R}$ under these conditions yields:
$\lim_{\beta_1=\gamma\beta_2\to\infty} \rho_0 = \lim_{\beta_1=\gamma\beta_2\to\infty} \frac{\lambda\left(\lambda+\mu+\beta_1+\beta_2+\sqrt{(\lambda+\mu+\beta_1+\beta_2)^2-4\mu\beta_1}\right)}{2\mu\beta_1} = \frac{\lambda(1+\gamma)}{\mu\gamma} = \frac{\lambda}{\mu p_s} = \rho$
$\lim_{\beta_1=\gamma\beta_2\to\infty} \rho_1 = \lim_{\beta_1=\gamma\beta_2\to\infty} \frac{\lambda\left(\lambda+\mu+\beta_1+\beta_2-\sqrt{(\lambda+\mu+\beta_1+\beta_2)^2-4\mu\beta_1}\right)}{2\mu\beta_1} = 0$
$\rho < 1 \iff$ Positive Recurrent

We can now compute our P.G.F. with the appropriate substitutions.
$\mathcal{G}_N(z) = \frac{1-\rho_0}{1-\rho_0 z}\frac{1-\rho_1}{1-\rho_1 z} = \frac{1-\rho}{1-\rho z}$

This shows that the P.G.F. of N matches that of the classical M/M/1 queue with arrival rate μp_s (see [13], pg. 32), we thus conclude that the stationary queue lengths are equivalent in distribution. □

Chapter 3

M/M/1 model with unreliable service and a working vacation

3.1 Introduction

Within the literature, a vacation queue is typically defined as a queue with multiple service rates governed by a policy that reduces the service rate when the number of customers is below certain threshold. For an example of a vacation queue within the M/G/1 framework, see [16]. We extend our M/M/1 model with unreliable service by including two service rates. The use of multiple service rates is important since the customer service time depends not only on the customer, but on the state of the server at the time of service as well. We noted in Chapter 2 that we could construct an M/PH/1 queue with similar behavior and identical stationary distribution if we imposed the undesirable restriction $\mu \geq \beta_1 + \beta_2$. However, since the service time now depends on more than the customer alone, this is no longer possible within the M/PH/1 framework irregardless of any restrictions that might be imposed.

We adopt assumptions and terminology from Chapter 2. Namely, service failure is not due to the server as it would be in breakdown models, nor due to the customer as it would be in some interruption models. Customer's do not leave the queue–that is we preserve the FCFS (First Come First Service) service discipline. This differentiates our model from retrial queues in the literature. We again consider service failures to be due to external, random forces and repeat a customer's service until it has been completed successfully. Furthermore, neither the server nor customer know whether the service was successful until the service time has been completed, at which time we envision a 'quality check' to take place which determines if the service was a success or failure.

3.2 Definitions

We define our process, state space, and parameters as follows:

Definition 3.2.1. Let $\{N(t) \mid t \geq 0\}$ be the number of customers in the queue at time t,

$$J(t) = \begin{cases} 0 & \text{the server is on working vacation} \\ 1 & \text{the server is in a busy state} \end{cases}$$

and

$$S(t) = \begin{cases} 1 & \text{immediately after service is rendered} \\ 0 & \text{otherwise} \end{cases}$$

Then $\{(N(t), J(t), S(t)) \mid t \geq 0\}$ is a Markov process on the state space:
$\Omega = \{(0,0,0)\} \cup \{(k,j,s) \mid k \in \mathbb{N},\ j,s \in \{0,1\}\}$

Define the following parameters:

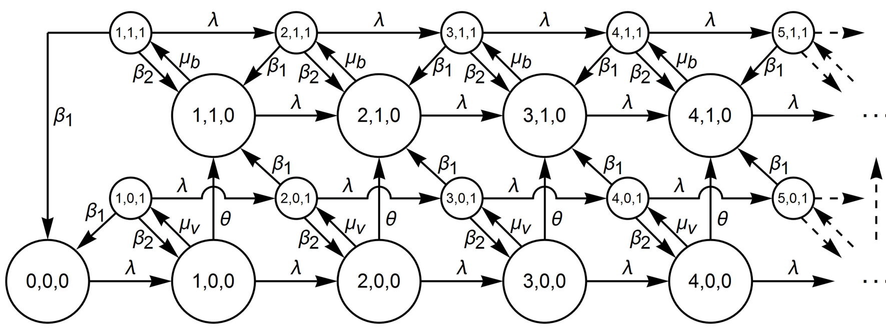
- λ : the rate of the Poisson arrivals process.
- μ_b : the rate of service when the server is busy.
- μ_v : the rate of service when the server is on vacation.
- β_1 : the rate of a successful service.
- β_2 : the rate of a failed service.
- θ : vacation duration is exponentially distributed with rate α.

Definition 3.2.2. We define the vacation policy:

- When the server becomes idle (i.e. $N(t) = 0$), the server goes on a working vacation; by this we mean that customers arriving while the server is on vacation get served at a reduced rate $\mu_v < \mu_b$.
- When the server is not idle (i.e. $N(t) \neq 0$), a vacationing server begins a working vacation duration that is exponentially distributed with rate θ, after which it begins a busy period and operates at rate μ_b until the server becomes idle again, renewing the process.
- If a customer is served successfully while the server is on a working vacation and there are additional customers waiting in the queue, the server then immediately ends its vacation and enters into a busy state until the queue is emptied.

To help visualize this 3-dimensional Markovian process in 2-dimensions, we visualize the state transition rates by a 2D diagram.

FIGURE 3.1: 3D Markovian state transition rates diagram.



We define a 'successful service' similarly to that done in Chapter 2 to be a transition from $(n, j, 1) \longrightarrow (n-1, 1, 0)$ or $(n, j, 1) \longrightarrow (n-1, 0, 0)$, which is represented in the state transition diagram as having rate β_1. Accordingly, we will define a 'failed service' to be a transition from $(n, j, 1) \longrightarrow (n, j, 0)$ with transition rate β_2. We will compute the probabilities of a 'successful' or 'failed' service in an explicit manner by considering the transition probabilities of the embedded Markov Chain and will note a striking similarity with the results from Chapter 2.

Let $E_s = \{\text{a customer was served successfully}\}$

$S_v = \{\text{the server is on a working vacation}\}$

$S_b = \{\text{the server is busy}\}$

$$p_s = P(E_s \cap S_v) + P(E_s \cap S_b) = P(E_s|S_v)P(S_v) + P(E_s|S_b)P(S_b)$$

$$p_s = \frac{\beta_1}{\beta_1+\beta_2+\lambda}\sum_{i=0}^{\infty}\left(\frac{\lambda}{\beta_1+\beta_2+\lambda}\right)^i P(S_v) + \frac{\beta_1}{\beta_1+\beta_2+\lambda}\sum_{i=0}^{\infty}\left(\frac{\lambda}{\beta_1+\beta_2+\lambda}\right)^i P(S_b)$$

$$p_s = \frac{\beta_1}{\beta_1+\beta_2+\lambda}\sum_{i=0}^{\infty}\left(\frac{\lambda}{\beta_1+\beta_2+\lambda}\right)^i \big(P(S_v) + P(S_b)\big)$$

$$p_s = \frac{\beta_1}{\beta_1+\beta_2+\lambda}\left(\frac{1}{1-\frac{\lambda}{\beta_1+\beta_2+\lambda}}\right)$$

$$p_s = \frac{\beta_1}{\beta_1+\beta_2+\lambda}\left(\frac{1}{\frac{\beta_1+\beta_2}{\beta_1+\beta_2+\lambda}}\right)$$

$$p_s = \frac{\beta_1}{\beta_1+\beta_2+\lambda}\left(\frac{\beta_1+\beta_2+\lambda}{\beta_1+\beta_2}\right)$$

$$p_s = \frac{\beta_1}{\beta_1+\beta_2}$$

From here, we can list the countable state space in lexicographical order; formally defined for triplets below.

Definition 3.2.3. Lexicographical Ordering
We say $(k_1, j_1, s_1) < (k_2, j_2, s_2)$ if and only if $k_1 \frown j_1 \frown s_1 < k_2 \frown j_2 \frown s_2$, where $\frown$ denotes concatenation (see [17]). For example, $7 \frown 0 \frown 1 = 701$.

It should be noted that this equivalent definition for lexicographical ordering can easily be extended to n-tuples. Using this re-ordering convention we can write:
$\Omega = \{(0,0,0), (1,0,0), (1,0,1), (1,1,0), (1,1,1), \dots\}$ and define the corresponding infinitesimal matrix Q.

3.3 Infinitesimal Matrix Q

$$\mathbf{Q} = \begin{bmatrix} \hat{\mathbf{A}} & \hat{\mathbf{C}} & 0 & 0 & 0 & \dots \\ \hat{\mathbf{B}} & \mathbf{A} & \mathbf{C} & 0 & 0 & \dots \\ 0 & \mathbf{B} & \mathbf{A} & \mathbf{C} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{bmatrix} \tag{3.1}$$

where

$$\hat{\mathbf{A}} = \begin{bmatrix} -\lambda \end{bmatrix} \qquad \hat{\mathbf{B}} = \begin{bmatrix} 0 \\ \beta_1 \\ 0 \\ \beta_1 \end{bmatrix} \qquad \hat{\mathbf{C}} = \begin{bmatrix} \lambda & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -(\lambda + \mu_v + \theta) & \mu_v & \theta & 0 \\ \beta_2 & -(\lambda + \beta_1 + \beta_2) & 0 & 0 \\ 0 & 0 & -(\lambda + \mu_b) & \mu_b \\ 0 & 0 & \beta_2 & -(\lambda + \beta_1 + \beta_2) \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

3.4 The Quadratic Matrix Equation

Using Lemma 2.4.1, we seek the minimal non-negative solution $\mathbf{R}$ to the quadratic matrix equation:

$$\mathbf{R^2B} + \mathbf{RA} + \mathbf{C} = \mathbf{0} \tag{3.2}$$

We will again employ the direct method whereby we solve the system of equations generated by equating the matrices entry by entry.

Let $\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{bmatrix} \Longrightarrow$ (3.2) can be restated as the following system:

$$\begin{cases} r_{11}\left(\theta+\beta_1\left(r_{12}+r_{14}\right)\right)+\beta_1 r_{12}\left(r_{22}+r_{24}\right)+r_{14}\left(\beta_2+\beta_1\left(r_{42}+r_{44}\right)\right)=r_{13}\left(\mu_b+\lambda-\beta_1\left(r_{32}+r_{34}\right)\right) \\ r_{21}\left(\theta+\beta_1\left(r_{12}+r_{14}\right)\right)+\beta_1 r_{22}\left(r_{22}+r_{24}\right)+r_{24}\left(\beta_2+\beta_1\left(r_{42}+r_{44}\right)\right)=r_{23}\left(\mu_b+\lambda-\beta_1\left(r_{32}+r_{34}\right)\right) \\ \lambda+r_{31}\left(\theta+\beta_1\left(r_{12}+r_{14}\right)\right)+\beta_1\left(r_{22}+r_{24}\right)r_{32}+r_{34}\left(\beta_2+\beta_1\left(r_{42}+r_{44}\right)\right)=r_{33}\left(\mu_b+\lambda-\beta_1\left(r_{32}+r_{34}\right)\right) \\ r_{41}\left(\theta+\beta_1\left(r_{12}+r_{14}\right)\right)+\beta_1\left(r_{22}+r_{24}\right)r_{42}+r_{44}\left(\beta_2+\beta_1\left(r_{42}+r_{44}\right)\right)=r_{43}\left(\mu_b+\lambda-\beta_1\left(r_{32}+r_{34}\right)\right) \\ r_{11}\left(\theta+\lambda+\mu_v\right)-\beta_2 r_{12}-\lambda=0 \\ r_{12}\left(\beta_1+\beta_2+\lambda\right)-r_{11}\mu_v=0 \\ r_{14}\left(\lambda+\beta_1+\beta_2\right)-r_{13}\mu_b=0 \\ \beta_2 r_{22}-r_{21}\left(\theta+\lambda+\mu_v\right)=0 \\ r_{22}\left(\beta_1+\beta_2+\lambda\right)-\lambda-r_{21}\mu_v=0 \\ r_{24}\left(\lambda+\beta_1+\beta_2\right)-r_{23}\mu_b=0 \\ \beta_2 r_{32}-r_{31}\left(\theta+\lambda+\mu_v\right)=0 \\ r_{32}\left(\lambda+\beta_1+\beta_2\right)-r_{31}\mu_v=0 \\ r_{34}\left(\lambda+\beta_1+\beta_2\right)-r_{33}\mu_b=0 \\ \beta_2 r_{42}-r_{41}\left(\theta+\lambda+\mu_v\right)=0 \\ r_{42}\left(\beta_1+\beta_2+\lambda\right)-r_{41}\mu_v=0 \\ r_{44}\left(\lambda+\beta_1+\beta_2\right)-r_{43}\mu_b-\lambda=0 \end{cases} \tag{3.3}$$

The analytical minimal non-negative solution to (3.3) is given by:

$$\mathbf{R} = \begin{bmatrix} \frac{\lambda(\lambda+\beta_1+\beta_2)}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+(\beta_1+\lambda)\mu_v} & \frac{\lambda\mu_v}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+(\beta_1+\lambda)\mu_v} & \frac{\lambda(\beta_1+\beta_2+\lambda)((\lambda+\beta_1+\beta_2)(\theta+\lambda)+\lambda\mu_v)}{\beta_1\mu_b((\lambda+\beta_1+\beta_2)(\theta+\lambda)+(\beta_1+\lambda)\mu_v)} & \frac{\lambda((\lambda+\beta_1+\beta_2)(\theta+\lambda)+\lambda\mu_v)}{\beta_1((\lambda+\beta_1+\beta_2)(\theta+\lambda)+(\beta_1+\lambda)\mu_v)} \\ \frac{\beta_2\lambda}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+(\beta_1+\lambda)\mu_v} & \frac{\lambda(\theta+\lambda+\mu_v)}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+(\beta_1+\lambda)\mu_v} & \frac{\lambda(\beta_1+\beta_2+\lambda)((\beta_2+\lambda)(\theta+\lambda)+\lambda\mu_v)}{\beta_1\mu_b((\lambda+\beta_1+\beta_2)(\theta+\lambda)+(\beta_1+\lambda)\mu_v)} & \frac{\lambda((\beta_2+\lambda)(\theta+\lambda)+\lambda\mu_v)}{\beta_1((\lambda+\beta_1+\beta_2)(\theta+\lambda)+(\beta_1+\lambda)\mu_v)} \\ 0 & 0 & \frac{\lambda(\lambda+\beta_1+\beta_2)}{\beta_1\mu_b} & \frac{\lambda}{\beta_1} \\ 0 & 0 & \frac{\lambda(\beta_2+\lambda)}{\beta_1\mu_b} & \frac{\lambda}{\beta_1} \end{bmatrix} \tag{3.4}$$

3.5 The Spectral Radius of R

We now again compute the spectral radius of **R** explicitly and show that the sufficient condition under which our model will be positive recurrent has not changed from the case in Chapter 2.

Corollary. *By Lemma 2.4.1, the infinitesimal matrix* $\mathbf{Q}$ *given in equation* (3.1) *is positive recurrent if and only if:* $\beta_1(\mu_b - \lambda) - \lambda(\mu_b + \beta_2) > 0$.

Proof. The spectral radius of $\mathbf{R}$ will be computed by solving the scalar quadratic equations generated by $\det(\mathbf{R} - \rho_i \mathbf{I}) = 0$, yielding that ρ_i satisfies the following quadratic equations:

$$\mu_b \beta_1 \rho_i^2 - \lambda(\lambda + \mu_b + \beta_1 + \beta_2)\rho_i + \lambda^2 = 0 \tag{3.5}$$

$$\implies \rho_i = \frac{\lambda\left(\lambda+\mu_b+\beta_1+\beta_2+(-1)^i\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}\right)}{2\mu_b\beta_1}, \qquad i = 0, 1$$

$$\Big((\lambda + \beta_1 + \beta_2)(\theta + \lambda) + \mu_v(\beta_1 + \lambda) \Big)\rho_i^2 - \lambda(\beta_1 + \beta_2 + \theta + 2\lambda + \mu_v)\rho_i + \lambda^2 = 0 \tag{3.6}$$

$$\implies \rho_i = \frac{\lambda\left(\theta+2\lambda+\beta_1+\beta_2+\mu_v+(-1)^i\sqrt{(\theta-\beta_1-\beta_2+\mu_v)^2+4\beta_2\mu_v}\right)}{2((\lambda+\beta_1+\beta_2)(\theta+\lambda)+\mu_v(\beta_1+\lambda))}, \qquad i = 2, 3$$

Note, by inspection, that the largest of these eigenvalues in (3.5) and (3.6) will contain the positive radicals. Next we will show that $\rho_0 \geq \rho_2$.

Assume $\rho_0 < \rho_2$, then:

$\implies \rho_0\rho_3 < \rho_2\rho_3$

$\implies \rho_0\rho_3 < \frac{\lambda^2}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+\mu_v(\beta_1+\lambda)}$

$\implies \rho_0 \frac{\lambda\left(\theta+2\lambda+\beta_1+\beta_2+\mu_v-\sqrt{(\theta-\beta_1-\beta_2+\mu_v)^2+4\beta_2\mu_v}\right)}{2((\lambda+\beta_1+\beta_2)(\theta+\lambda)+\mu_v(\beta_1+\lambda))} < \frac{\lambda^2}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+\mu_v(\beta_1+\lambda)}$

$\implies \frac{\rho_0\left(\theta+2\lambda+\beta_1+\beta_2+\mu_v-\sqrt{(\theta-\beta_1-\beta_2+\mu_v)^2+4\beta_2\mu_v}\right)}{2} < \lambda$

$\implies \frac{\rho_0\rho_1\left(\theta+2\lambda+\beta_1+\beta_2+\mu_v-\sqrt{(\theta-\beta_1-\beta_2+\mu_v)^2+4\beta_2\mu_v}\right)}{2} < \lambda\rho_1$

$\implies \frac{\lambda^2\left(\theta+2\lambda+\beta_1+\beta_2+\mu_v-\sqrt{(\theta-\beta_1-\beta_2+\mu_v)^2+4\beta_2\mu_v}\right)}{2\mu_b\beta_1} < \lambda\rho_1$

$\implies \frac{\lambda(\theta+2\lambda+\beta_1+\beta_2+\mu_v)}{2\mu_b\beta_1} - \rho_1 < \frac{\lambda\sqrt{(\theta-\beta_1-\beta_2+\mu_v)^2+4\beta_2\mu_v}}{2\mu_b\beta_1}$

$\implies \left(\frac{\lambda(\theta+2\lambda+\beta_1+\beta_2+\mu_v)}{2\mu_b\beta_1} - \frac{\lambda\left(\lambda+\mu_b+\beta_1+\beta_2-\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}\right)}{2\mu_b\beta_1}\right)^2 < \frac{\lambda^2(\theta-\beta_1-\beta_2+\mu_v)^2+4\lambda^2\beta_2\mu_v}{4\mu_b^2\beta_1^2}$

$\implies \left(\frac{\theta+\lambda+\mu_v-\mu_b+\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}}{2\mu_b\beta_1}\right)^2 < \frac{(\theta-\beta_1-\beta_2+\mu_v)^2+4\beta_2\mu_v}{4\mu_b^2\beta_1^2}$

$\implies \frac{(\theta+\lambda+\mu_v-\mu_b)^2+2(\theta+\lambda+\mu_v-\mu_b)\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}+(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}{4\mu_b^2\beta_1^2} < \frac{(\theta-\beta_1-\beta_2+\mu_v)^2+4\beta_2\mu_v}{4\mu_b^2\beta_1^2}$

$\implies \frac{2(\theta+\lambda+\mu_v-\mu_b)\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}}{4\mu_b^2\beta_1^2} < \frac{(\theta-\beta_1-\beta_2+\mu_v)^2-(\lambda+\mu_b+\beta_1+\beta_2)^2-(\theta+\lambda+\mu_v-\mu_b)^2+4\beta_2\mu_v+4\mu_b\beta_1}{4\mu_b^2\beta_1^2}$

$\implies \frac{2(\theta+\lambda+\mu_v-\mu_b)\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}}{4\mu_b^2\beta_1^2} < \frac{(\theta-\beta_1-\beta_2+\mu_v)^2-(\theta+2\lambda+\beta_1+\beta_2+\mu_v)^2+2(\lambda+\mu_b+\beta_1+\beta_2)(\theta+\lambda+\mu_v-\mu_b)+4\beta_2\mu_v+4\mu_b\beta_1}{4\mu_b^2\beta_1^2}$

$\implies \frac{2(\theta+\lambda+\mu_v-\mu_b)\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}}{4\mu_b^2\beta_1^2} < \frac{-4(\lambda+\beta_1+\beta_2)(\theta+\lambda+\mu_v)+2(\lambda+\mu_b+\beta_1+\beta_2)(\theta+\lambda+\mu_v-\mu_b)+4\beta_2\mu_v+4\mu_b\beta_1}{4\mu_b^2\beta_1^2}$

$\implies \frac{(\theta+\lambda+\mu_v-\mu_b)\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}}{2\mu_b^2\beta_1^2} < \frac{(\mu_b-\beta_1+\beta_2-\lambda)(\theta+\lambda+\mu_v-\mu_b)-2\beta_2(\theta+\lambda)-2\lambda\mu_b}{2\mu_b^2\beta_1^2}$

$\implies \frac{(\theta+\lambda+\mu_v-\mu_b)\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}}{2\mu_b^2\beta_1^2} < \frac{(\mu_b-\beta_1+\beta_2-\lambda)(\theta+\lambda+\mu_v-\mu_b)}{2\mu_b^2\beta_1^2}$

$\Longrightarrow \sqrt{(\lambda + \mu_b + \beta_1 + \beta_2)^2 - 4\mu_b\beta_1} < (\mu_b - \beta_1 + \beta_2 - \lambda)$
$\Longrightarrow (\lambda + \mu_b + \beta_1 + \beta_2)^2 - 4\mu_b\beta_1 < (\mu_b - \beta_1 + \beta_2 - \lambda)^2$
$\Longrightarrow 4(\lambda\beta_2 + \beta_1\beta_2 + \lambda\mu_b) < 0$, which is a contradiction.

$\Longrightarrow \rho_0 \geq \rho_2$

Thus, by Lemma 2.4.1, $\mathbf{Q}$ is positive recurrent if and only if:
$\rho_0 < 1 \iff \beta_1(\mu_b - \lambda) - \lambda(\mu_b + \beta_2) > 0$. (See 2.10) □

3.6 The Stationary Distribution

3.6.1 The Explicit form of $\mathbf{R}^k$

To compute $\mathbf{R}^k$, we utilize the block upper-triangular structure of the matrix $\mathbf{R}$ given in (3.4) with the help of the following.

Lemma 3.6.1. *Given* $\mathbf{R} = \begin{bmatrix} \mathcal{A} & \mathcal{C} \\ \mathbf{0} & \mathcal{B} \end{bmatrix}$, *then* $\mathbf{R}^k = \begin{bmatrix} \mathcal{A}^k & \sum_{i=0}^{k-1} \mathcal{A}^i\mathcal{C}\mathcal{B}^{\mathbf{k-i-1}} \\ \mathbf{0} & \mathcal{B}^k \end{bmatrix}$

Proof.
We note that if $k = 1$, then $\mathbf{R}^1 = \begin{bmatrix} \mathcal{A}^1 & \sum_{i=0}^{0} \mathcal{A}^i\mathcal{C}\mathcal{B}^{\mathbf{1-i-1}} \\ \mathbf{0} & \mathcal{B}^1 \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{C} \\ \mathbf{0} & \mathcal{B} \end{bmatrix}$
Next, assume $\mathbf{R}^k = \begin{bmatrix} \mathcal{A}^k & \sum_{i=0}^{k-1} \mathcal{A}^i\mathcal{C}\mathcal{B}^{\mathbf{k-i-1}} \\ \mathbf{0} & \mathcal{B}^k \end{bmatrix}$, and write $\mathbf{R}^{k+1}$ as follows:

$$\begin{aligned}
\mathbf{R}^k\mathbf{R} &= \begin{bmatrix} \mathcal{A}^k & \sum_{i=0}^{k-1} \mathcal{A}^i\mathcal{C}\mathcal{B}^{\mathbf{k-i-1}} \\ \mathbf{0} & \mathcal{B}^k \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{A}^k & \sum_{i=0}^{k-1} \mathcal{A}^i\mathcal{C}\mathcal{B}^{\mathbf{k-i-1}} \\ \mathbf{0} & \mathcal{B}^k \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{C} \\ \mathbf{0} & \mathcal{B} \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{A}^{k+1} & \mathcal{A}^k\mathcal{C} + (\sum_{i=0}^{k-1} \mathcal{A}^i\mathcal{C}\mathcal{B}^{\mathbf{k-i-1}})\mathcal{B} \\ \mathbf{0} & \mathcal{B}^{k+1} \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{A}^{k+1} & \mathcal{A}^k\mathcal{C} + (\sum_{i=0}^{k-1} \mathcal{A}^i\mathcal{C}\mathcal{B}^{\mathbf{k-i}}) \\ \mathbf{0} & \mathcal{B}^{k+1} \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{A}^{k+1} & \sum_{i=0}^{k} \mathcal{A}^i\mathcal{C}\mathcal{B}^{\mathbf{k-i}} \\ \mathbf{0} & \mathcal{B}^{k+1} \end{bmatrix} = \mathbf{R}^{k+1}
\end{aligned}$$

□

Proposition 3.6.2. *Let* $\mathbf{R} = \begin{bmatrix} \mathcal{A} & \mathcal{C} \\ \mathbf{0} & \mathcal{B} \end{bmatrix}$ *be as in (3.4),*

where $\mathcal{A} = \begin{bmatrix} \frac{\lambda(\lambda+\beta_1+\beta_2)}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+\mu_v(\beta_1+\lambda)} & \frac{\lambda\mu_v}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+\mu_v(\beta_1+\lambda)} \\ \frac{\beta_2\lambda}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+\mu_v(\beta_1+\lambda)} & \frac{\lambda(\theta+\lambda+\mu_v)}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+\mu_v(\beta_1+\lambda)} \end{bmatrix}$

$$= \frac{\lambda}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+\mu_v(\beta_1+\lambda)} \begin{bmatrix} \lambda+\beta_1+\beta_2 & \mu_v \\ \beta_2 & \theta+\lambda+\mu_v \end{bmatrix}$$
$$= \frac{\rho_2\rho_3}{\lambda} \begin{bmatrix} \lambda+\beta_1+\beta_2 & \mu_v \\ \beta_2 & \theta+\lambda+\mu_v \end{bmatrix}, \textit{ then:}$$

$$\mathcal{A}^k = \begin{bmatrix} \frac{\rho_2\rho_3\left(\lambda\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+(\lambda+\beta_1+\beta_2)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda(\rho_3-\rho_2)} & \frac{\mu_v\rho_2\rho_3\left(\rho_3^k-\rho_2^k\right)}{\lambda(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2\rho_3\left(\rho_3^k-\rho_2^k\right)}{\lambda(\rho_3-\rho_2)} & \frac{\rho_2\rho_3\left((\theta+\lambda+\mu_v)\left(\rho_3^k-\rho_2^k\right)-\lambda\left(\rho_3^{k-1}-\rho_2^{k-1}\right)\right)}{\lambda(\rho_3-\rho_2)} \end{bmatrix}$$

Proof.
As before, we use Mathematical Induction, observe that:

$$\mathcal{A}^1 = \begin{bmatrix} \frac{\rho_2\rho_3\left(\lambda\left(\rho_2^0-\rho_3^0\right)+(\lambda+\beta_1+\beta_2)\left(\rho_3^1-\rho_2^1\right)\right)}{\lambda(\rho_3-\rho_2)} & \frac{\mu_v\rho_2\rho_3\left(\rho_3^1-\rho_2^1\right)}{\lambda(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2\rho_3\left(\rho_3^1-\rho_2^1\right)}{\lambda(\rho_3-\rho_2)} & \frac{\rho_2\rho_3\left((\theta+\lambda+\mu_v)\left(\rho_3^1-\rho_2^1\right)-\lambda\left(\rho_3^0-\rho_2^0\right)\right)}{\lambda(\rho_3-\rho_2)} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\rho_2\rho_3(\lambda+\beta_1+\beta_2)}{\lambda} & \frac{\mu_v\rho_2\rho_3}{\lambda} \\ \frac{\beta_2\rho_2\rho_3}{\lambda} & \frac{\rho_2\rho_3\left((\theta+\lambda+\mu_v)\left(\rho_3^1-\rho_2^1\right)\right)}{\lambda(\rho_3-\rho_2)} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\rho_2\rho_3(\lambda+\beta_1+\beta_2)}{\lambda} & \frac{\mu_v\rho_2\rho_3}{\lambda} \\ \frac{\beta_2\rho_2\rho_3}{\lambda} & \frac{\rho_2\rho_3(\theta+\lambda+\mu_v)}{\lambda} \end{bmatrix} = \mathcal{A}$$

Given the result for $\mathcal{A}^k$, we write $\mathcal{A}^{k+1}$:

$$\mathcal{A}^k\mathcal{A} = \begin{bmatrix} \frac{\rho_2\rho_3\left(\lambda\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+(\lambda+\beta_1+\beta_2)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda(\rho_3-\rho_2)} & \frac{\mu_v\rho_2\rho_3\left(\rho_3^k-\rho_2^k\right)}{\lambda(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2\rho_3\left(\rho_3^k-\rho_2^k\right)}{\lambda(\rho_3-\rho_2)} & \frac{\rho_2\rho_3\left((\theta+\lambda+\mu_v)\left(\rho_3^k-\rho_2^k\right)-\lambda\left(\rho_3^{k-1}-\rho_2^{k-1}\right)\right)}{\lambda(\rho_3-\rho_2)} \end{bmatrix} \begin{bmatrix} \frac{\rho_2\rho_3(\lambda+\beta_1+\beta_2)}{\lambda} & \frac{\mu_v\rho_2\rho_3}{\lambda} \\ \frac{\beta_2\rho_2\rho_3}{\lambda} & \frac{\rho_2\rho_3(\theta+\lambda+\mu_v)}{\lambda} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\rho_2^2\rho_3^2\left((\lambda+\beta_1+\beta_2)\left((\lambda+\beta_1+\beta_2)\left(\rho_3^k-\rho_2^k\right)+\lambda\left(\rho_2^{k-1}-\rho_3^{k-1}\right)\right)+\beta_2\mu_v\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} & \frac{\rho_2^2\rho_3^2\left(\mu_v\left(\lambda\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+(\lambda+\beta_1+\beta_2)\left(\rho_3^k-\rho_2^k\right)\right)+\mu_v(\theta+\lambda+\mu_v)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2^2\rho_3^2\left(\rho_3^k-\rho_2^k\right)(\lambda+\beta_1+\beta_2)+\beta_2\rho_2^2\rho_3^2\left((\theta+\lambda+\mu_v)\left(\rho_3^k-\rho_2^k\right)-\lambda\left(\rho_3^{k-1}-\rho_2^{k-1}\right)\right)}{\lambda^2(\rho_3-\rho_2)} & \frac{\mu_v\beta_2\rho_2^2\rho_3^2\left(\rho_3^k-\rho_2^k\right)+\rho_2^2\rho_3^2(\theta+\lambda+\mu_v)\left((\theta+\lambda+\mu_v)\left(\rho_3^k-\rho_2^k\right)-\lambda\left(\rho_3^{k-1}-\rho_2^{k-1}\right)\right)}{\lambda^2(\rho_3-\rho_2)} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\rho_2^2\rho_3^2\left(\lambda(\lambda+\beta_1+\beta_2)\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+\left((\lambda+\beta_1+\beta_2)^2+\beta_2\mu_v\right)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} & \frac{\mu_v\rho_2^2\rho_3^2\left(\lambda\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+(\lambda+\beta_1+\beta_2)\left(\rho_3^k-\rho_2^k\right)+(\theta+\lambda+\mu_v)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2^2\rho_3^2(\theta+2\lambda+\beta_1+\beta_2+\mu_v)\left(\rho_3^k-\rho_2^k\right)-\lambda\beta_2\rho_2^2\rho_3^2\left(\rho_3^{k-1}-\rho_2^{k-1}\right)}{\lambda^2(\rho_3-\rho_2)} & \frac{\mu_v\beta_2\rho_2^2\rho_3^2\left(\rho_3^k-\rho_2^k\right)+\rho_2\rho_3(\theta+\lambda+\mu_v)\left(\rho_2\rho_3(\theta+\lambda+\mu_v)\left(\rho_3^k-\rho_2^k\right)-\lambda\rho_2\rho_3\left(\rho_3^{k-1}-\rho_2^{k-1}\right)\right)}{\lambda^2(\rho_3-\rho_2)} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\rho_2^2\rho_3^2\left(\lambda(\lambda+\beta_1+\beta_2)\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+\left(\frac{\lambda((\lambda+\beta_1+\beta_2)(\rho_2+\rho_3)-\lambda)}{\rho_2\rho_3}\right)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} & \frac{\mu_v\rho_2^2\rho_3^2\left(\lambda\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+(\theta+2\lambda+\beta_1+\beta_2+\mu_v)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2\rho_3(\lambda(\rho_3+\rho_2))\left(\rho_3^k-\rho_2^k\right)-\lambda\beta_2\rho_2^2\rho_3^2\left(\rho_3^{k-1}-\rho_2^{k-1}\right)}{\lambda^2(\rho_3-\rho_2)} & \frac{\mu_v\beta_2\rho_2^2\rho_3^2\left(\rho_3^k-\rho_2^k\right)+\rho_2\rho_3(\theta+\lambda+\mu_v)\left(\lambda\left(\rho_3^{k+1}-\rho_2^{k+1}\right)-\rho_2\rho_3(\lambda+\beta_1+\beta_2)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\rho_2\rho_3\left(\rho_2\rho_3(\lambda+\beta_1+\beta_2)\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+((\lambda+\beta_1+\beta_2)(\rho_2+\rho_3)-\lambda)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda(\rho_3-\rho_2)} & \frac{\mu_v\rho_2^2\rho_3^2\left(\lambda\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+\frac{\lambda(\rho_3+\rho_2)}{\rho_2\rho_3}\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2\rho_3(\rho_3+\rho_2)\left(\rho_3^k-\rho_2^k\right)-\beta_2\rho_2^2\rho_3^2\left(\rho_3^{k-1}-\rho_2^{k-1}\right)}{\lambda(\rho_3-\rho_2)} & \frac{\mu_v\beta_2\rho_2^2\rho_3^2\left(\rho_3^k-\rho_2^k\right)+(\lambda(\rho_3+\rho_2)-\rho_2\rho_3(\lambda+\beta_1+\beta_2))\left(\lambda\left(\rho_3^{k+1}-\rho_2^{k+1}\right)-\rho_2\rho_3(\lambda+\beta_1+\beta_2)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\rho_2\rho_3\left(\rho_2\rho_3(\lambda+\beta_1+\beta_2)\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+(\lambda+\beta_1+\beta_2)(\rho_2+\rho_3)\left(\rho_3^k-\rho_2^k\right)+\lambda\left(\rho_2^k-\rho_3^k\right)\right)}{\lambda(\rho_3-\rho_2)} & \frac{\mu_v\rho_2\rho_3\left(\rho_2\rho_3\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+(\rho_3+\rho_2)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2\rho_3\left(\rho_3^{k+1}-\rho_2^{k+1}\right)+\beta_2\rho_2\rho_3\left(\rho_2\rho_3^k-\rho_3\rho_2^k\right)-\beta_2\rho_2^2\rho_3^2\left(\rho_3^{k-1}-\rho_2^{k-1}\right)}{\lambda(\rho_3-\rho_2)} & \frac{\lambda^2(\rho_3+\rho_2)\left(\rho_3^{k+1}-\rho_2^{k+1}\right)-\rho_2\rho_3\left(\lambda(\lambda+\beta_1+\beta_2)\left(\rho_3^{k+1}-\rho_2^{k+1}\right)+\lambda^2\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\rho_2\rho_3\left((\lambda+\beta_1+\beta_2)\left(\rho_2^k\rho_3-\rho_3^k\rho_2+(\rho_2+\rho_3)\left(\rho_3^k-\rho_2^k\right)\right)+\lambda\left(\rho_2^k-\rho_3^k\right)\right)}{\lambda(\rho_3-\rho_2)} & \frac{\mu_v\rho_2\rho_3\left(\rho_2\rho_3\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+(\rho_3+\rho_2)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2\rho_3\left(\rho_3^{k+1}-\rho_2^{k+1}\right)}{\lambda(\rho_3-\rho_2)} & \frac{\rho_2\rho_3\left(\rho_3^{k+1}-\rho_2^{k+1}\right)\left(\frac{\lambda(\rho_3+\rho_2)}{\rho_2\rho_3}-(\lambda+\beta_1+\beta_2)\right)-\lambda\rho_2\rho_3\left(\rho_3^k-\rho_2^k\right)}{\lambda(\rho_3-\rho_2)} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\rho_2\rho_3\left(\lambda\left(\rho_2^k-\rho_3^k\right)+(\lambda+\beta_1+\beta_2)\left(\rho_3^{k+1}-\rho_2^{k+1}\right)\right)}{\lambda(\rho_3-\rho_2)} & \frac{\mu_v\rho_2\rho_3\left(\rho_3^{k+1}-\rho_2^{k+1}\right)}{\lambda(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2\rho_3\left(\rho_3^{k+1}-\rho_2^{k+1}\right)}{\lambda(\rho_3-\rho_2)} & \frac{\rho_2\rho_3\left((\theta+\lambda+\mu_v)\left(\rho_3^{k+1}-\rho_2^{k+1}\right)-\lambda\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda(\rho_3-\rho_2)} \end{bmatrix} = \mathcal{A}^{k+1}$$

Remark. Three substitutions were needed in this derivation. Namely:
$(\lambda+\beta_1+\beta_2)^2+\beta_2\mu_v = \frac{\lambda((\lambda+\beta_1+\beta_2)(\rho_2+\rho_3)-\lambda)}{\rho_2\rho_3}$, and
$\theta+2\lambda+\beta_1+\beta_2+\mu_v = \frac{\lambda(\rho_3+\rho_2)}{\rho_2\rho_3}$, and
$\lambda(\rho_3+\rho_2)-\rho_2\rho_3(\lambda+\beta_1+\beta_2) = \rho_2\rho_3(\theta+\lambda+\mu_v)$.
These can readily be verified from (3.6).

□

Proposition 3.6.3. *Given the block-matrix form of* $\mathbf{R} = \begin{bmatrix} \mathcal{A} & \mathcal{C} \\ \mathbf{0} & \mathcal{B} \end{bmatrix}$, *we find by Lemma 3.6.1:*

$$\mathbf{R}^k = \begin{bmatrix} \mathcal{A}^k & \sum_{i=0}^{k-1} \mathcal{A}^i \mathcal{C} \mathcal{B}^{\mathbf{k-i-1}} \\ \mathbf{0} & \mathcal{B}^k \end{bmatrix} = \begin{bmatrix} \mathcal{A}^k & \mathcal{C}(k) \\ \mathbf{0} & \mathcal{B}^k \end{bmatrix} \tag{3.7}$$

where:

$$\mathcal{A}^k = \begin{bmatrix} \frac{\rho_2\rho_3\left(\lambda\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+(\lambda+\beta_1+\beta_2)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda(\rho_3-\rho_2)} & \frac{\mu_v\rho_2\rho_3\left(\rho_3^k-\rho_2^k\right)}{\lambda(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2\rho_3\left(\rho_3^k-\rho_2^k\right)}{\lambda(\rho_3-\rho_2)} & \frac{\rho_2\rho_3\left((\theta+\lambda+\mu_v)\left(\rho_3^k-\rho_2^k\right)-\lambda\left(\rho_3^{k-1}-\rho_2^{k-1}\right)\right)}{\lambda(\rho_3-\rho_2)} \end{bmatrix} \textit{ (by 3.6.2)}$$

$$\mathcal{B}^k = \begin{bmatrix} \frac{(\beta_1\rho_0-\lambda)\rho_0^k+\rho_1^k(\lambda-\beta_1\rho_1)}{\beta_1(\rho_0-\rho_1)} & \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)} \\ \frac{\lambda(\lambda+\beta_2)\left(\rho_0^k-\rho_1^k\right)}{\mu_b\beta_1(\rho_0-\rho_1)} & \frac{(\lambda-\beta_1\rho_1)\rho_0^k+\rho_1^k(\beta_1\rho_0-\lambda)}{\beta_1(\rho_0-\rho_1)} \end{bmatrix} \textit{ (by 2.7.1, letting } \mu=\mu_b\textit{) and if:}$$

$$\delta_1 = \frac{\rho_2\left(\lambda^2-\rho_3(\lambda(\theta+\lambda)+\beta_1\mu_v\rho_2)\right)}{\lambda\mu_b\beta_1(\rho_0-\rho_1)(\rho_2-\rho_3)},\quad \delta_2 = \frac{\rho_3\left(\lambda^2-\rho_2(\lambda(\theta+\lambda)+\beta_1\mu_v\rho_3)\right)}{\lambda\mu_b\beta_1(\rho_0-\rho_1)(\rho_2-\rho_3)},\quad \delta_3 = \frac{\rho_3\left(\lambda^2-\rho_2\left(\lambda^2+\beta_1\rho_3(\theta+\lambda+\mu_v)\right)\right)}{\lambda(\rho_0-\rho_1)(-\rho_2+\rho_3)},$$

$$\delta_4 = \frac{\rho_2\left(\lambda^2-\rho_3\left(\lambda^2+\beta_1\rho_2(\theta+\lambda+\mu_v)\right)\right)}{\lambda(\rho_0-\rho_1)(-\rho_2+\rho_3)}$$

then $\mathcal{C}(k) = \begin{bmatrix} c_{11}(k) & c_{12}(k) \\ c_{21}(k) & c_{22}(k) \end{bmatrix}$, *where:*

$$c_{11}(k) = \delta_1\left(\frac{\rho_0(\lambda-\mu_b\rho_1)\left(\rho_0^k-\rho_2^k\right)}{\rho_1(\rho_0-\rho_2)} - \frac{\rho_1(\lambda-\mu_b\rho_0)\left(\rho_1^k-\rho_2^k\right)}{\rho_0(\rho_1-\rho_2)}\right) + \delta_2\left(\frac{\rho_1(\lambda-\mu_b\rho_0)\left(\rho_1^k-\rho_3^k\right)}{\rho_0(\rho_1-\rho_3)} - \frac{\rho_0(\lambda-\mu_b\rho_1)\left(\rho_0^k-\rho_3^k\right)}{\rho_1(\rho_0-\rho_3)}\right)$$
$$c_{12}(k) = \delta_1\left(\frac{\mu_b\rho_0\left(\rho_0^k-\rho_2^k\right)}{\rho_0-\rho_2} - \frac{\mu_b\rho_1\left(\rho_1^k-\rho_2^k\right)}{\rho_1-\rho_2}\right) + \delta_2\left(\frac{\mu_b\rho_1\left(\rho_1^k-\rho_3^k\right)}{\rho_1-\rho_3} - \frac{\mu_b\rho_0\left(\rho_0^k-\rho_3^k\right)}{\rho_0-\rho_3}\right)$$
$$c_{21}(k) = \delta_3\left(\frac{\rho_0^2(\lambda-\mu_b\rho_1)\left(\rho_0^k-\rho_3^k\right)}{\lambda^2(\rho_0-\rho_3)} - \frac{\rho_1^2(\lambda-\mu_b\rho_0)\left(\rho_1^k-\rho_3^k\right)}{\lambda^2(\rho_1-\rho_3)}\right) + \delta_4\left(\frac{\rho_1^2(\lambda-\mu_b\rho_0)\left(\rho_1^k-\rho_2^k\right)}{\lambda^2(\rho_1-\rho_2)} - \frac{\rho_0^2(\lambda-\mu_b\rho_1)\left(\rho_0^k-\rho_2^k\right)}{\lambda^2(\rho_0-\rho_2)}\right)$$
$$c_{22}(k) = \delta_3\left(\frac{\rho_0\left(\rho_0^k-\rho_3^k\right)}{\beta_1(\rho_0-\rho_3)} - \frac{\rho_1\left(\rho_1^k-\rho_3^k\right)}{\beta_1(\rho_1-\rho_3)}\right) + \delta_4\left(\frac{\rho_1\left(\rho_1^k-\rho_2^k\right)}{\beta_1(\rho_1-\rho_2)} - \frac{\rho_0\left(\rho_0^k-\rho_2^k\right)}{\beta_1(\rho_0-\rho_2)}\right)$$

Proof. The above result follows from the preceding analysis, specifically Lemma 3.6.1, Proposition 3.6.2, and the observation that $\mathcal{B}$ is entry-wise identical to the matrix $\mathbf{R}$ from Chapter 2. The rest is merely the computation of $C(k) = \sum_{i=0}^{k-1} \mathcal{A}^i \mathcal{C} \mathcal{B}^{\mathbf{k-i-1}}$ which is tedious but straightforward. □

3.6.2 The Initial terms of π

Turning attention to the computation of $\mathbf{B}[\mathbf{R}] = \begin{bmatrix} \hat{\mathbf{A}} & \hat{\mathbf{C}} \\ \hat{\mathbf{B}} & \mathbf{A}+\mathbf{RB} \end{bmatrix}$, and a positive vector $(x_0, \mathbf{x_1})$, such that $(x_0, \mathbf{x_1})\mathbf{B}[\mathbf{R}] = 0$, we have:

$$(x_0, \mathbf{x_1})\mathbf{B}[\mathbf{R}] = (x_0, \mathbf{x_1}) \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 \\ 0 & -\theta-\lambda-\mu_v & \mu_v & \theta+\lambda & 0 \\ \beta_1 & \beta_2 & -\beta_1-\beta_2-\lambda & \lambda & 0 \\ 0 & 0 & 0 & -\mu_b & \mu_b \\ \beta_1 & 0 & 0 & \beta_2+\lambda & -\beta_1-\beta_2-\lambda \end{bmatrix} = \mathbf{0} \quad (3.8)$$

$$\Longrightarrow \begin{cases} x_0 = 1 \\ x_{10} = \frac{\lambda(\lambda+\beta_1+\beta_2)}{\beta_2(\theta+\lambda)+\beta_1(\theta+\lambda+\mu_v)+\lambda(\theta+\lambda+\mu_v)} \\ x_{11} = \frac{\lambda\mu_v}{\beta_2(\theta+\lambda)+\beta_1(\theta+\lambda+\mu_v)+\lambda(\theta+\lambda+\mu_v)} \\ x_{12} = \frac{\lambda(\lambda+\beta_1+\beta_2)(\beta_1(\theta+\lambda)+\beta_2(\theta+\lambda)+\lambda(\theta+\lambda+\mu_v))}{\beta_1\mu_b(\beta_2(\theta+\lambda)+\beta_1(\theta+\lambda+\mu_v)+\lambda(\theta+\lambda+\mu_v))} \\ x_{13} = \frac{\lambda(\beta_1(\theta+\lambda)+\beta_2(\theta+\lambda)+\lambda(\theta+\lambda+\mu_v))}{\beta_1(\beta_2(\theta+\lambda)+\beta_1(\theta+\lambda+\mu_v)+\lambda(\theta+\lambda+\mu_v))} \end{cases}$$

We normalize the solution in order to generate the first five terms of π:

$$K(x_0 + \boldsymbol{x_1}(\mathbf{I}-\mathbf{R})^{-1}\mathbf{e}) = 1$$

$$\Longrightarrow K = \frac{(\beta_1(\mu_b-\lambda)-\lambda(\mu_b+\beta_2))(\theta\beta_2+\beta_1(\theta+\mu_v))}{\beta_1((\theta+\lambda)(\beta_1+\beta_2)\mu_b+((\beta_1(\mu_b-\lambda)-\lambda(\mu_b+\beta_2))+\lambda\mu_b)\mu_v)} \quad (3.9)$$

$$(\pi_{00}, \pi_{10}, \pi_{11}, \pi_{12}, \pi_{13}) = K(x_0, \mathbf{x_1}) \Longrightarrow \begin{cases} \pi_{00} = K \\ \pi_{10} = \frac{K\rho_2\rho_3(\lambda+\beta_1+\beta_2)}{\lambda} \\ \pi_{11} = \frac{K\mu_v\rho_2\rho_3}{\lambda} \\ \pi_{12} = \frac{K(\lambda+\beta_1+\beta_2)\left(\lambda^2-\beta_1\mu_v\rho_2\rho_3\right)}{\lambda\beta_1\mu_b} \\ \pi_{13} = \frac{K\left(\lambda^2-\beta_1\mu_v\rho_2\rho_3\right)}{\lambda\beta_1} \end{cases}$$

Remark. We observe that the condition given by Corollary 3.5 for positive recurrence: $\beta_1(\mu_b-\lambda)-\lambda(\mu_b+\beta_2) > 0$ is equivalent to $K > 0$.

3.6.3 The Remaining Terms of $\boldsymbol{\pi}$

Proposition 3.6.4. *The remaining elements* $\{(\pi_{k0}, \pi_{k1}, \pi_{k2}, \pi_{k3}) \mid k \geq 2\}$ *of our stationary distribution satisfying* $(\pi_{k0}, \pi_{k1}, \pi_{k2}, \pi_{k3}) = (\pi_{10}, \pi_{11}, \pi_{12}, \pi_{13})\mathbf{R}^{k-1}$ *and* $\pi_{00} + \sum_{k=1}^{\infty}(\pi_{k0}+\pi_{k1}+\pi_{k2}+\pi_{k3}) = 1$ *are given by:*

$$
\begin{cases}
\pi_{k0} = \frac{K\rho_2\rho_3\left((\lambda+\beta_1+\beta_2)\left(\rho_3^k-\rho_2^k\right)-\lambda\left(\rho_3^{k-1}-\rho_2^{k-1}\right)\right)}{\lambda(\rho_3-\rho_2)} \\
\pi_{k1} = \frac{K\mu_v\rho_2\rho_3\left(\rho_2^k-\rho_3^k\right)}{\lambda(\rho_2-\rho_3)} \\
\pi_{k2} = K\left(\frac{\mu_b(\delta_1-\delta_2)\left(\beta_1\left(\rho_0^{k+1}-\rho_1^{k+1}\right)-\lambda\left(\rho_0^k-\rho_1^k\right)\right)}{\lambda} + \frac{\rho_2\rho_3(\lambda+\beta_1+\beta_2)}{\lambda}\left(\delta_1\left(\frac{\rho_0(\lambda-\mu_b\rho_1)\left(\rho_0^{k-1}-\rho_2^{k-1}\right)}{\rho_1(\rho_0-\rho_2)} - \frac{\rho_1(\lambda-\mu_b\rho_0)\left(\rho_1^{k-1}-\rho_2^{k-1}\right)}{\rho_0(\rho_1-\rho_2)}\right)\right.\right. \\
\qquad\qquad \left.+\delta_2\left(\frac{\rho_1(\lambda-\mu_b\rho_0)\left(\rho_1^{k-1}-\rho_3^{k-1}\right)}{\rho_0(\rho_1-\rho_3)} - \frac{\rho_0(\lambda-\mu_b\rho_1)\left(\rho_0^{k-1}-\rho_3^{k-1}\right)}{\rho_1(\rho_0-\rho_3)}\right)\right) \\
\qquad \left.+\frac{\mu_v\rho_2\rho_3}{\lambda^3}\left(\delta_4\left(\frac{\rho_1^2(\lambda-\mu_b\rho_0)\left(\rho_1^{k-1}-\rho_2^{k-1}\right)}{\rho_1-\rho_2} - \frac{\rho_0^2(\lambda-\mu_b\rho_1)\left(\rho_0^{k-1}-\rho_2^{k-1}\right)}{\rho_0-\rho_2}\right) + \delta_3\left(\frac{\rho_0^2(\lambda-\mu_b\rho_1)\left(\rho_0^{k-1}-\rho_3^{k-1}\right)}{\rho_0-\rho_3} - \frac{\rho_1^2(\lambda-\mu_b\rho_0)\left(\rho_1^{k-1}-\rho_3^{k-1}\right)}{\rho_1-\rho_3}\right)\right)\right) \\
\pi_{k3} = K\left(\mu_b\left(\delta_1-\delta_2\right)\left(\rho_0^k-\rho_1^k\right) + \frac{\mu_b(\lambda+\beta_1+\beta_2)\rho_2\rho_3}{\lambda}\left(\delta_1\left(\frac{\rho_0\left(\rho_0^{k-1}-\rho_2^{k-1}\right)}{\rho_0-\rho_2} - \frac{\rho_1\left(\rho_1^{k-1}-\rho_2^{k-1}\right)}{\rho_1-\rho_2}\right)\right.\right. \\
\qquad\qquad \left.+\delta_2\left(\frac{\rho_1\left(\rho_1^{k-1}-\rho_3^{k-1}\right)}{\rho_1-\rho_3} - \frac{\rho_0\left(\rho_0^{k-1}-\rho_3^{k-1}\right)}{\rho_0-\rho_3}\right)\right) \\
\qquad \left.-\frac{\mu_v\rho_2\rho_3}{\lambda\beta_1}\left(\frac{\delta_4\left((\rho_1-\rho_2)\rho_0^k-(\rho_0-\rho_2)\rho_1^k+(\rho_0-\rho_1)\rho_2^k\right)}{(\rho_0-\rho_2)(\rho_1-\rho_2)} - \frac{\delta_3\left((\rho_1-\rho_3)\rho_0^k-(\rho_0-\rho_3)\rho_1^k-(\rho_1-\rho_0)\rho_3^k\right)}{(\rho_0-\rho_3)(\rho_1-\rho_3)}\right)\right)
\end{cases}
\tag{3.10}
$$

Proof. To motivate the proof, we begin by noting that:

$$
\begin{aligned}
(\pi_{k0},\pi_{k1},\pi_{k2},\pi_{k3}) = (\pi_{10},\pi_{11},\pi_{12},\pi_{13})\mathbf{R}^{k-1} &\iff (\pi_{k0},\pi_{k1},\pi_{k2},\pi_{k3}) = (\pi_{10},\pi_{11},\pi_{12},\pi_{13})\mathbf{R}^{k-2}\mathbf{R} \\
&\iff (\pi_{k0},\pi_{k1},\pi_{k2},\pi_{k3}) = (\pi_{k-1,0},\pi_{k-1,1},\pi_{k-1,2},\pi_{k-1,3})\mathbf{R}
\end{aligned}
$$

We will use an alternate form of **R** given below. This is entry-by-entry identical to that defined by (3.4), but is merely expressed in terms of $\rho_0,\rho_1,\rho_2,\rho_3,\delta_1,\delta_2,\delta_3$ and δ_4 whenever possible.

$$
\mathbf{R} = \begin{bmatrix}
\frac{(\lambda+\beta_1+\beta_2)\rho_2\rho_3}{\lambda} & \frac{\mu_v\rho_2\rho_3}{\lambda} & (\lambda+\beta_1+\beta_2)(\delta_1-\delta_2)(\rho_0-\rho_1) & (\delta_1-\delta_2)\mu_b(\rho_0-\rho_1) \\
\frac{\beta_2\rho_2\rho_3}{\lambda} & \frac{\lambda(\rho_2+\rho_3)-(\lambda+\beta_1+\beta_2)\rho_2\rho_3}{\lambda} & \frac{(\lambda+\beta_1+\beta_2)(\delta_3-\delta_4)\rho_0(\rho_0-\rho_1)\rho_1}{\lambda^2} & \frac{(\delta_3-\delta_4)(\rho_0-\rho_1)}{\beta_1} \\
0 & 0 & \frac{\beta_1(\rho_0+\rho_1)-\lambda}{\beta_1} & \frac{\lambda}{\beta_1} \\
0 & 0 & \frac{\lambda(\rho_0+\rho_1)-(\beta_1+\mu_b)\rho_0\rho_1}{\lambda} & \frac{\lambda}{\beta_1}
\end{bmatrix}
\tag{3.11}
$$

Next, we define: $(\pi_{k-1,0},\pi_{k-1,1},\pi_{k-1,2},\pi_{k-1,3})\mathbf{R} = (a,b,c,d)$

Then:

$$
\begin{aligned}
a &= \frac{K\rho_2\rho_3\left((\lambda+\beta_1+\beta_2)\left(\rho_3^{k-1}-\rho_2^{k-1}\right)-\lambda\left(\rho_3^{k-2}-\rho_2^{k-2}\right)\right)}{\lambda(\rho_3-\rho_2)}\frac{(\lambda+\beta_1+\beta_2)\rho_2\rho_3}{\lambda} + \frac{K\mu_v\rho_2\rho_3\left(\rho_2^{k-1}-\rho_3^{k-1}\right)}{\lambda(\rho_2-\rho_3)}\frac{\beta_2\rho_2\rho_3}{\lambda} \\
&= \frac{K\rho_2^2\rho_3^2\left(\left((\lambda+\beta_1+\beta_2)^2+\beta_2\mu_v\right)\left(\rho_2^{k-1}-\rho_3^{k-1}\right)-\lambda(\lambda+\beta_1+\beta_2)\left(\rho_2^{k-2}-\rho_3^{k-2}\right)\right)}{\lambda^2(\rho_2-\rho_3)} \\
&= \frac{K\rho_2^2\rho_3^2\left(\left(\frac{\lambda((\lambda+\beta_1+\beta_2)(\rho_2+\rho_3)-\lambda)}{\rho_2\rho_3}\right)\left(\rho_2^{k-1}-\rho_3^{k-1}\right)-\lambda(\lambda+\beta_1+\beta_2)\left(\rho_2^{k-2}-\rho_3^{k-2}\right)\right)}{\lambda^2(\rho_2-\rho_3)} \\
&= \frac{K\rho_2\rho_3\left((\lambda((\lambda+\beta_1+\beta_2)(\rho_2+\rho_3)-\lambda))\left(\rho_2^{k-1}-\rho_3^{k-1}\right)-\lambda(\lambda+\beta_1+\beta_2)\rho_2\rho_3\left(\rho_2^{k-2}-\rho_3^{k-2}\right)\right)}{\lambda^2(\rho_2-\rho_3)} \\
&= \frac{K\rho_2\rho_3\left((\lambda(\lambda+\beta_1+\beta_2)(\rho_2+\rho_3)-\lambda^2)\left(\rho_2^{k-1}-\rho_3^{k-1}\right)-\lambda(\lambda+\beta_1+\beta_2)\rho_2\rho_3\left(\rho_2^{k-2}-\rho_3^{k-2}\right)\right)}{\lambda^2(\rho_2-\rho_3)} \\
&= \frac{K\rho_2\rho_3\left(\lambda(\lambda+\beta_1+\beta_2)\left(\rho_2^k-\rho_3^k\right)-\lambda^2\left(\rho_2^{k-1}-\rho_3^{k-1}\right)\right)}{\lambda^2(\rho_2-\rho_3)} \\
&= \frac{K\rho_2\rho_3\left((\lambda+\beta_1+\beta_2)\left(\rho_2^k-\rho_3^k\right)-\lambda\left(\rho_2^{k-1}-\rho_3^{k-1}\right)\right)}{\lambda(\rho_2-\rho_3)} \text{ and,}
\end{aligned}
$$

$$
b = \frac{K\rho_2\rho_3\left((\lambda+\beta_1+\beta_2)\left(\rho_3^{k-1}-\rho_2^{k-1}\right)-\lambda\left(\rho_3^{k-2}-\rho_2^{k-2}\right)\right)}{\lambda(\rho_3-\rho_2)}\frac{\mu_v\rho_2\rho_3}{\lambda} + \frac{K\mu_v\rho_2\rho_3\left(\rho_2^{k-1}-\rho_3^{k-1}\right)}{\lambda(\rho_2-\rho_3)}\frac{\lambda(\rho_2+\rho_3)-(\lambda+\beta_1+\beta_2)\rho_2\rho_3}{\lambda}
$$

$$= \frac{K\mu_v\rho_2^2\rho_3^2\left((\lambda+\beta_1+\beta_2)\left(\rho_3^{k-1}-\rho_2^{k-1}\right)-\lambda\left(\rho_3^{k-2}-\rho_2^{k-2}\right)\right)+K\mu_v\rho_2\rho_3\left(\rho_3^{k-1}-\rho_2^{k-1}\right)(\lambda(\rho_2+\rho_3)-(\lambda+\beta_1+\beta_2)\rho_2\rho_3)}{\lambda^2(\rho_3-\rho_2)}$$
$$= \frac{K\mu_v\rho_2^2\rho_3^2\left(-\lambda\left(\rho_3^{k-2}-\rho_2^{k-2}\right)\right)+K\mu_v\rho_2\rho_3\left(\rho_3^{k-1}-\rho_2^{k-1}\right)(\lambda(\rho_2+\rho_3))}{\lambda^2(\rho_3-\rho_2)}$$
$$= \frac{K\mu_v\rho_2\rho_3\left(\lambda(\rho_2+\rho_3)\left(\rho_3^{k-1}-\rho_2^{k-1}\right)-\lambda\rho_2\rho_3\left(\rho_3^{k-2}-\rho_2^{k-2}\right)\right)}{\lambda^2(\rho_3-\rho_2)}$$
$$= \frac{K\mu_v\rho_2\rho_3\left(\lambda\left(\rho_3^{k}-\rho_2^{k}\right)\right)}{\lambda^2(\rho_3-\rho_2)}$$
$$= \frac{K\mu_v\rho_2\rho_3\left(\rho_3^{k}-\rho_2^{k}\right)}{\lambda(\rho_3-\rho_2)}$$

Since the verification of c and d are similar in nature to a and b, but are too lengthy to provide the step-by-step details, they are omitted. Similarly, we have verified that $\pi_{00} + \sum_{k=1}^{\infty}(\pi_{k0} + \pi_{k1} + \pi_{k2} + \pi_{k3}) = 1$, but will omit these steps as well.

□

3.7 Decomposition

3.7.1 Decomposition of N

Theorem 3.7.1. *If $K > 0$, then the stationary number of customers in steady-state, N, can be decomposed into the sum of four independent geometric random variables and an independent finitely valued generalized random variable with an explicitly known generalized distribution. Namely: $N = X_0 + X_1 + X_2 + X_3 + X_4$, where: $X_0 \sim Geometric(1 - \rho_0)$,*
$X_1 \sim Geometric(1-\rho_1)$, $X_2 \sim Geometric(1-\rho_2)$, $X_3 \sim Geometric(1-\rho_3)$ and if:

$$K^* = \frac{K}{(1-\rho_0)(1-\rho_1)(1-\rho_2)(1-\rho_3)}$$
$$= \frac{\mu_b\left((\theta+\lambda)\beta_2+\lambda\left(\theta+\lambda+\mu_v\right)+\beta_1\left(\theta+\lambda+\mu_v\right)\right)}{(\theta+\lambda)\left(\beta_1+\beta_2\right)\mu_b+\left(\left(\beta_1\left(\mu_b-\lambda\right)-\lambda\left(\beta_2+\mu_b\right)\right)+\lambda\mu_b\right)\mu_v}$$

$$\begin{cases} \widehat{\delta_0} = 1 \\ \widehat{\delta_1} = -\frac{\lambda((\lambda+\beta_1+\beta_2)\mu_v+\mu_b(\theta+\lambda+\mu_v))}{\mu_b((\theta+\lambda)\beta_2+\lambda(\theta+\lambda+\mu_v)+\beta_1(\theta+\lambda+\mu_v))} \\ \widehat{\delta_2} = \frac{\lambda^2\mu_v}{\mu_b((\theta+\lambda)\beta_2+\lambda(\theta+\lambda+\mu_v)+\beta_1(\theta+\lambda+\mu_v))} \end{cases} \tag{3.12}$$

Then, $P(X_4 = k) = \begin{cases} K^*\widehat{\delta_0} & \text{if } k=0 \\ K^*\widehat{\delta_1} & \text{if } k=1 \\ K^*\widehat{\delta_2} & \text{if } k=2 \\ 0 & \text{otherwise} \end{cases}$

Proof.
Since all exponential rates $\lambda, \mu_b, \mu_v, \beta_1, \beta_2, \theta$ are non-negative, then by inspection we have $K^* > 0,\ \widehat{\delta_0} > 0,\ \widehat{\delta_1} < 0$, and $\widehat{\delta_2} > 0$.

We also note that:
$(1-\rho_0)(1-\rho_1) = \frac{\beta_1(\mu_b-\lambda)-\lambda(\mu_b+\beta_2)}{\beta_1\mu_b}$, and $(1-\rho_2)(1-\rho_3) = \frac{\theta\beta_2+\beta_1(\theta+\mu_v)}{(\theta+\lambda)\beta_2+\lambda(\theta+\lambda+\mu_v)+\beta_1(\theta+\lambda+\mu_v)}$
These can be verified from (3.5) and (3.6) respectively.

Therefore, using the definition of K from (3.9), we have:

$$\begin{aligned}K^* &= \frac{K}{(1-\rho_0)(1-\rho_1)(1-\rho_2)(1-\rho_3)}\\ &= \frac{(\beta_1(\mu_b-\lambda)-\lambda(\mu_b+\beta_2))(\theta\beta_2+\beta_1(\theta+\mu_v))}{\beta_1((\theta+\lambda)(\beta_1+\beta_2)\mu_b+((\beta_1(\mu_b-\lambda)-\lambda(\mu_b+\beta_2))+\lambda\mu_b)\mu_v)}\frac{1}{(1-\rho_0)(1-\rho_1)(1-\rho_2)(1-\rho_3)}\\ &= \frac{\mu_b((\theta+\lambda)\beta_2+\lambda(\theta+\lambda+\mu_v)+\beta_1(\theta+\lambda+\mu_v))}{((\theta+\lambda)(\beta_1+\beta_2)\mu_b+((\beta_1(\mu_b-\lambda)-\lambda(\beta_2+\mu_b))+\lambda\mu_b)\mu_v)}\end{aligned}$$

By definition the P.G.F. of N is given by:

$$\mathcal{G}_{\mathcal{N}}(z) = \sum_{k=0}^{\infty} P(N=k)z^k \tag{3.13}$$

$$\begin{aligned}\Longrightarrow \mathcal{G}_{\mathcal{N}}(z) &= \sum_{k=0}^{\infty} P(N=k)z^k\\ &= \pi_{00} + \sum_{k=1}^{\infty}(\pi_{k0}+\pi_{k1}+\pi_{k2}+\pi_{k3})z^k\\ &= \sum_{k=0}^{\infty}(\pi_{k0}+\pi_{k1}+\pi_{k2}+\pi_{k3})z^k\\ &\vdots\\ &= \frac{K^*(1-\rho_0)(1-\rho_1)(1-\rho_2)(1-\rho_3)}{(1-\rho_0 z)(1-\rho_1 z)(1-\rho_2 z)(1-\rho_3 z)}\left(\widehat{\delta_0}+\widehat{\delta_1}z+\widehat{\delta_2}z^2\right)\\ &= \frac{1-\rho_0}{1-\rho_0 z}\frac{1-\rho_1}{1-\rho_1 z}\frac{1-\rho_2}{1-\rho_2 z}\frac{1-\rho_3}{1-\rho_3 z}K^*\left(\widehat{\delta_0}+\widehat{\delta_1}z+\widehat{\delta_2}z^2\right)\\ &= \mathcal{G}_{X_0}(z)\mathcal{G}_{X_1}(z)\mathcal{G}_{X_2}(z)\mathcal{G}_{X_3}(z)K^*\left(\widehat{\delta_0}+\widehat{\delta_1}z+\widehat{\delta_2}z^2\right)\end{aligned}$$

Since $\mathcal{G}_{\mathcal{N}}(1) = \mathcal{G}_{X_0}(1)\mathcal{G}_{X_1}(1)\mathcal{G}_{X_2}(1)\mathcal{G}_{X_3}(1)K^*\left(\widehat{\delta_1}+\widehat{\delta_2}+\widehat{\delta_3}\right)$ and since all rates are finite
$\Longrightarrow K^*\left(\widehat{\delta_1}+\widehat{\delta_2}z+\widehat{\delta_3}z^2\right) = 1$ when $z=1$, and $\sum_{i=1}^{3}|\widehat{\delta_i}| < \infty$.
$\Longrightarrow \mathcal{G}_N(z) = \mathcal{G}_{X_0}(z)\mathcal{G}_{X_1}(z)\mathcal{G}_{X_2}(z)\mathcal{G}_{X_3}(z)\mathcal{G}_{X_4}(z)$, where $\mathcal{G}_{X_4}(z)$ is said to be the generalized generating function for the generalized random variable X_4.

Note: For reasons of length we omit the rather lengthy algebraic steps that reduced the infinite series to its simplest form. However, it is worth noting that we heavily relied upon *Mathematica*'s algebraic capabilities. Our *Mathematica* methods are included in the Apendix. □

Employing *The Fundamental Theorem of Negative Probabilities* by Ruzsa, Imre and Székely, Gábor J. [18], we are guaranteed the existence of a pair of ordinary random variables Y_0,Y_1 such that $Y_1 = Y_0 + X_4$ in distribution. Indeed, in our case we see that $Y_1 = N$ and $Y_0 = X_0 + X_1 + X_2 + X_3$ is one such pair.

3.7.2 The L.S.T. of W

We use Little's Distributional Law [11] to find the L.S.T. of the waiting time for the purposes of computing its expected value and variance.

Proposition 3.7.2. *Using Theorem 2.8.2, the L.S.T. of the stationary waiting time W for the queue with the stationary number of customers N is given by:*

$$\mathcal{W}^*(s) = \frac{\frac{\lambda(1-\rho_0)}{\rho_0}}{\left(\frac{\lambda(1-\rho_0)}{\rho_0} - s\right)} \frac{\frac{\lambda(1-\rho_1)}{\rho_1}}{\left(\frac{\lambda(1-\rho_1)}{\rho_1} - s\right)} \frac{\frac{\lambda(1-\rho_2)}{\rho_2}}{\left(\frac{\lambda(1-\rho_2)}{\rho_2} - s\right)} \frac{\frac{\lambda(1-\rho_3)}{\rho_3}}{\left(\frac{\lambda(1-\rho_3)}{\rho_3} - s\right)} K^*(\widehat{\delta}_0 + \widehat{\delta}_1 + \widehat{\delta}_2 - \left(\frac{\widehat{\delta}_1}{\lambda} + \frac{2\widehat{\delta}_2}{\lambda}\right)s + \frac{\widehat{\delta}_2}{\lambda^2}s^2)$$

Proof.

$$\begin{aligned}
\mathcal{W}^*(s) = \mathcal{G}_N\left(1 - \tfrac{s}{\lambda}\right) &= \frac{K^*(1-\rho_0)(1-\rho_1)(1-\rho_2)(1-\rho_3)(\widehat{\delta}_0+\widehat{\delta}_1(1-\frac{s}{\lambda})+\widehat{\delta}_2(1-\frac{s}{\lambda})^2)}{(1-\rho_0(1-\frac{s}{\lambda}))(1-\rho_1(1-\frac{s}{\lambda}))(1-\rho_2(1-\frac{s}{\lambda}))(1-\rho_3(1-\frac{s}{\lambda}))} \\
&= \frac{K^*(1-\rho_0)(1-\rho_1)(1-\rho_2)(1-\rho_3)(\widehat{\delta}_0+\widehat{\delta}_1(1-\frac{s}{\lambda})+\widehat{\delta}_2(1-\frac{s}{\lambda})^2)}{(1-\rho_0(1-\frac{s}{\lambda}))(1-\rho_1(1-\frac{s}{\lambda}))(1-\rho_2(1-\frac{s}{\lambda}))(1-\rho_3(1-\frac{s}{\lambda}))} \\
&= \frac{K^*\lambda^4(1-\rho_0)(1-\rho_1)(1-\rho_2)(1-\rho_3)(\widehat{\delta}_0+\widehat{\delta}_1(1-\frac{s}{\lambda})+\widehat{\delta}_2(1-\frac{s}{\lambda})^2)}{(\lambda-\rho_0(\lambda-s))(\lambda-\rho_1(\lambda-s))(\lambda-\rho_2(\lambda-s))(\lambda-\rho_3(\lambda-s))} \\
&= \frac{K^*\lambda^4(1-\rho_0)(1-\rho_1)(1-\rho_2)(1-\rho_3)(\widehat{\delta}_0+\widehat{\delta}_1(1-\frac{s}{\lambda})+\widehat{\delta}_2(1-\frac{s}{\lambda})^2)}{(\lambda(1-\rho_0)-\rho_0 s)(\lambda(1-\rho_1)-\rho_1 s)(\lambda(1-\rho_2)-\rho_2 s)(\lambda(1-\rho_3)-\rho_3 s)} \\
&= \frac{\frac{\lambda(1-\rho_0)}{\rho_0}\frac{\lambda(1-\rho_1)}{\rho_1}\frac{\lambda(1-\rho_2)}{\rho_2}\frac{\lambda(1-\rho_3)}{\rho_3}K^*(\widehat{\delta}_0+\widehat{\delta}_1(1-\frac{s}{\lambda})+\widehat{\delta}_2(1-\frac{s}{\lambda})^2)}{(\frac{\lambda(1-\rho_0)}{\rho_0}-s)(\frac{\lambda(1-\rho_1)}{\rho_1}-s)(\frac{\lambda(1-\rho_2)}{\rho_2}-s)(\frac{\lambda(1-\rho_3)}{\rho_3}-s)} \\
&= \frac{\frac{\lambda(1-\rho_0)}{\rho_0}}{(\frac{\lambda(1-\rho_0)}{\rho_0}-s)} \frac{\frac{\lambda(1-\rho_1)}{\rho_1}}{(\frac{\lambda(1-\rho_1)}{\rho_1}-s)} \frac{\frac{\lambda(1-\rho_2)}{\rho_2}}{(\frac{\lambda(1-\rho_2)}{\rho_2}-s)} \frac{\frac{\lambda(1-\rho_3)}{\rho_3}}{(\frac{\lambda(1-\rho_3)}{\rho_3}-s)} K^*(\widehat{\delta}_0 + \widehat{\delta}_1 + \widehat{\delta}_2 - \left(\frac{\widehat{\delta}_1}{\lambda} + \frac{2\widehat{\delta}_2}{\lambda}\right)s + \frac{\widehat{\delta}_2}{\lambda^2}s^2)
\end{aligned}$$

□

3.8 Results

3.8.1 Analytical Results

	Steady-State # of Customers: N
	$\mathcal{G}_N(z) = \frac{1-\rho_0}{1-\rho_0 z}\frac{1-\rho_1}{1-\rho_1 z}\frac{1-\rho_2}{1-\rho_2 z}\frac{1-\rho_3}{1-\rho_3 z}K^*\left(\widehat{\delta}_0 + \widehat{\delta}_1 z + \widehat{\delta}_2 z^2\right)$
E(.)	$\frac{\rho_0}{1-\rho_0} + \frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2} + \frac{\rho_3}{1-\rho_3} + K^*(\widehat{\delta}_1 + 2\widehat{\delta}_2)$
Var(.)	$\frac{\rho_0}{(1-\rho_0)^2} + \frac{\rho_1}{(1-\rho_1)^2} + \frac{\rho_2}{(1-\rho_2)^2} + \frac{\rho_3}{(1-\rho_3)^2} - K^*\widehat{\delta}_1(3 + K^*\widehat{\delta}_1)$

TABLE 3.1: Analytical Results on N

	Steady-State Waiting Time: W
	$\mathcal{W}^*(s) = \frac{\frac{\lambda(1-\rho_0)}{\rho_0}}{(\frac{\lambda(1-\rho_0)}{\rho_0}-s)} \frac{\frac{\lambda(1-\rho_1)}{\rho_1}}{(\frac{\lambda(1-\rho_1)}{\rho_1}-s)} \frac{\frac{\lambda(1-\rho_2)}{\rho_2}}{(\frac{\lambda(1-\rho_2)}{\rho_2}-s)} \frac{\frac{\lambda(1-\rho_3)}{\rho_3}}{(\frac{\lambda(1-\rho_3)}{\rho_3}-s)} K^*(\widehat{\delta_0}+\widehat{\delta_1}+\widehat{\delta_2}-\Big(\frac{\widehat{\delta_1}}{\lambda}+\frac{2\widehat{\delta_2}}{\lambda}\Big)s+\frac{\widehat{\delta_2}}{\lambda^2}s^2)$
E(.)	$\frac{\rho_0}{\lambda(1-\rho_0)}+\frac{\rho_1}{\lambda(1-\rho_1)}+\frac{\rho_2}{\lambda(1-\rho_2)}+\frac{\rho_3}{\lambda(1-\rho_3)}+\frac{K^*}{\lambda}(\widehat{\delta_1}+2\widehat{\delta_2})$
Var(.)	$\frac{\rho_0^2}{\lambda^2(1-\rho_0)^2}+\frac{\rho_1^2}{\lambda^2(1-\rho_1)^2}+\frac{\rho_2^2}{\lambda^2(1-\rho_2)^2}+\frac{\rho_3^2}{\lambda^2(1-\rho_3)^2}+\frac{K^*}{\lambda^2}\Big(2\widehat{\delta_2}-K^*(\widehat{\delta_1}+2\widehat{\delta_2})^2\Big)$

TABLE 3.2: Analytical Results on W

3.8.2 Numerical Results

When choosing parameters $\theta, \beta_1, \beta_2, \mu_v, \mu_b, \lambda$ to illustrate as an example, it follows that one might seek to make ρ_0, ρ_1, ρ_2, and ρ_3 given in (3.5) and (3.6) rational. However, we show that if one is interested in $E(N)$ or $E(W)$, then this in not necessary with the following Lemma.

Lemma 3.8.1. *For any $\theta, \beta_1, \beta_2, \mu_v, \mu_b, \lambda \in \mathbb{Q}$, we have $E(N) \in \mathbb{Q}$ and $E(W) \in \mathbb{Q}$.*

Proof. By Table 3.1, we have:

$$\begin{aligned} E(N) &= \frac{\rho_0}{1-\rho_0}+\frac{\rho_1}{1-\rho_1}+\frac{\rho_2}{1-\rho_2}+\frac{\rho_3}{1-\rho_3}+K^*(\widehat{\delta_1}+2\widehat{\delta_2}) \\ &= \frac{\rho_0(1-\rho_1)+\rho_1(1-\rho_0)}{(1-\rho_1)(1-\rho_0)}+\frac{\rho_2(1-\rho_3)+\rho_3(1-\rho_2)}{(1-\rho_2)(1-\rho_3)}+K^*(\widehat{\delta_1}+2\widehat{\delta_2}) \\ &= \frac{\rho_0+\rho_1-2\rho_0\rho_1}{(1-\rho_1)(1-\rho_0)}+\frac{\rho_2+\rho_3-2\rho_2\rho_3}{(1-\rho_2)(1-\rho_3)}+K^*(\widehat{\delta_1}+2\widehat{\delta_2}), \text{ where} \end{aligned}$$

$$\begin{aligned} &\rho_0\rho_1 = \frac{\lambda^2}{\mu_b\beta_1} && \rho_0+\rho_1 = \frac{\lambda(\beta_1+\beta_2+\lambda+\mu_b)}{\mu_b\beta_1} \\ &\rho_2\rho_3 = \frac{\lambda^2}{\beta_2(\theta+\lambda)+\beta_1(\theta+\lambda+\mu_v)+\lambda(\theta+\lambda+\mu_v)} && \rho_2+\rho_3 = \frac{\lambda(\beta_1+\beta_2+\theta+2\lambda+\mu_v)}{\beta_2(\theta+\lambda)+\beta_1(\theta+\lambda+\mu_v)+\lambda(\theta+\lambda+\mu_v)} \\ &(1-\rho_0)(1-\rho_1) = \frac{\beta_1(\mu_b-\lambda)-\lambda(\beta_2+\mu_b)}{\mu_b\beta_1} && (1-\rho_2)(1-\rho_3) = \frac{\theta\beta_2+\beta_1(\theta+\mu_v)}{(\theta+\lambda)\beta_2+\lambda(\theta+\lambda+\mu_v)+\beta_1(\theta+\lambda+\mu_v)} \end{aligned}$$

The above identities can be verified from (3.5) and (3.6).
$K^*, \widehat{\delta_1}$, and $\widehat{\delta_2}$ are rational and given in Theorem 3.7.1.

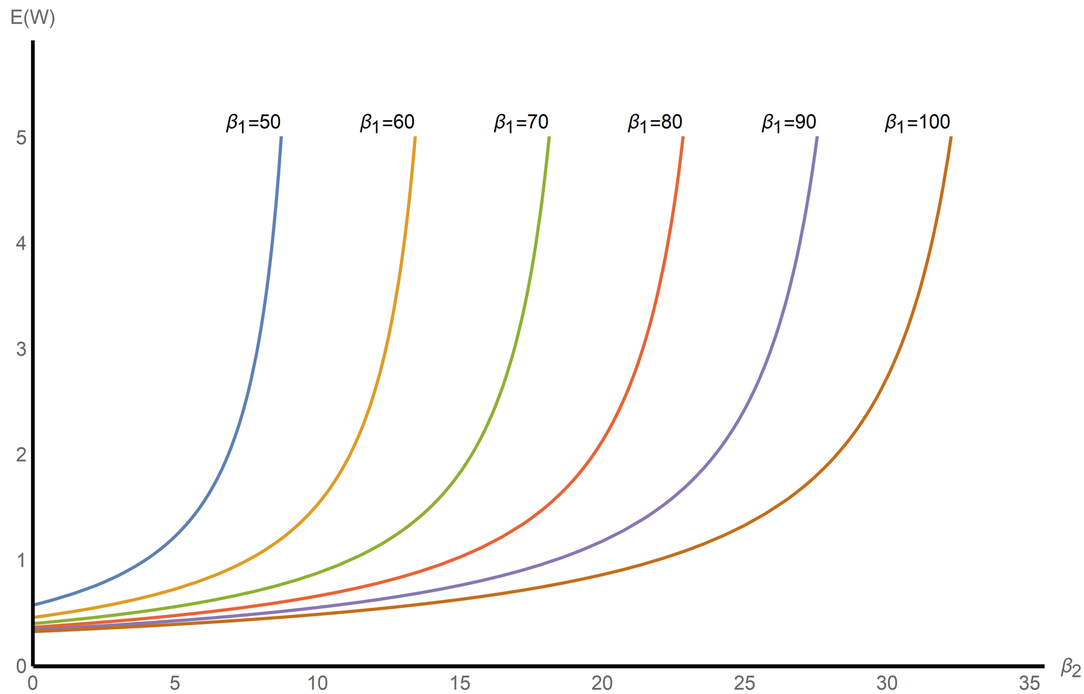
Lastly by Little's Law [10], we have $E(N) = \lambda E(W)$. □

We chose the following choice of parameters $\theta, \beta_1, \beta_2, \mu_v, \mu_b, \lambda$.

$$\begin{cases} \lambda = 10 & \theta = 5 \\ \mu_b = 15 & \mu_v = 10 \end{cases} \tag{3.14}$$

With this choice, we have: $E(W) = \frac{15\beta_1^2+38\beta_1^3-600\beta_2-90\beta_1\beta_2+57\beta_1^2\beta_2-105\beta_2^2+18\beta_1\beta_2^2-\beta_2^3}{5(\beta_1-2\beta_2-30)(3\beta_1+\beta_2)(11\beta_1+5\beta_2)}$

FIGURE 3.2: Plot of $E(W)$ as a function of β_2.



3.9 Special cases

We can now recover, as a special case, the stationary distribution of the Chapter 2 model.

Proposition 3.9.1. *Let $\theta \to \infty$ with $\mu_b = \mu_v = \mu$, then (3.10) recovers the stationary distribution of the model studied in Chapter 2. Consequently, we also get the special cases of the previous model as follows:*

- *Let $\theta \to \infty$ with $\mu_b = \mu_v = \mu$, and*

  i. $\beta_1 \to \infty$ with $0 \leq \beta_2 < \infty$ results in the classical $M/M/1$ queue.

  ii $0 < \beta_1 < \infty$ with $\beta_2 = 0$ and $\mu = \beta_1$ results in an $M/E_2/1$ queue, where E_2 refers to an 'Erlang' service time distribution with shape 2 and rate μ.

  iii $0 < \beta_1 < \infty$ with $\beta_2 = 0$ and $\mu > \beta_1$ results in an $M/HE/1$ queue, where HE refers to a hypoexponential service time distribution $\sim f(t) = \frac{\mu\beta_1(e^{-\beta_1 t} - e^{-\mu t})}{\mu - \beta_1}$ [12].

Proof. We obtain the stationary distribution from the previous model by substituting $\mu_b = \mu_v = \mu$, computing K, $\{\rho_i\}_{i=0,1,2,3}$ and $\{\delta_j\}_{j=1,2,3,4}$, and letting $\theta \to \infty$. Namely:

$$\implies \lim_{\theta\to\infty} K = \frac{\beta_1(\mu-\lambda)-\lambda(\beta_2+\mu)}{\beta_1\mu}$$

$$\implies \lim_{\theta\to\infty} \{\rho_0, \rho_1, \rho_2, \rho_3\} = \{\rho_0, \rho_1, \frac{\lambda}{\beta_1+\beta_2+\lambda}, 0\}$$

$$\implies \lim_{\theta\to\infty} \{\delta_1, \delta_2, \delta_3, \delta_4\} = \{\frac{\lambda}{\mu\beta_1(\rho_0-\rho_1)}, 0, 0, -\frac{\lambda}{\rho_0-\rho_1}\}$$

Care is needed when substituting these values into (3.10) due to indeterminate expression 0^0 arising from ρ_3^k and ρ_3^{k-1} terms, leading to three different cases.

k=0

$\boldsymbol{\pi}_k = (K, 0, 0, 0)$

$\implies \lim_{\theta\to\infty} \boldsymbol{\pi}_k = K(1, 0, 0, 0)$

k=1

$$\implies \lim_{\theta\to\infty} \boldsymbol{\pi}_k = \lim_{\theta\to\infty} \left\{ \frac{K\rho_2\rho_3(\lambda+\beta_1+\beta_2)}{\lambda}, \frac{K\mu\rho_2\rho_3}{\lambda}, \frac{K(\lambda+\beta_1+\beta_2)\left(\lambda^2-\beta_1\mu\rho_2\rho_3\right)}{\lambda\beta_1\mu}, \frac{K\left(\lambda^2-\beta_1\mu\rho_2\rho_3\right)}{\lambda\beta_1} \right\}$$
$$= K\left\{0, 0, \frac{\lambda(\lambda+\beta_1+\beta_2)}{\beta_1\mu}, \frac{\lambda}{\beta_1}\right\}$$

$k \geq 2$

$$\implies \lim_{\theta\to\infty} \boldsymbol{\pi}_k = K\left\{0, 0, \frac{\mu\delta_1\left(\beta_1\left(\rho_0^{k+1}-\rho_1^{k+1}\right)-\lambda\left(\rho_0^k-\rho_1^k\right)\right)}{\lambda}, \mu\delta_1\left(\rho_0^k-\rho_1^k\right)\right\}$$
$$= K\left\{0, 0, \frac{\beta_1\left(\rho_0^{k+1}-\rho_1^{k+1}\right)-\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)}, \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)}\right\}$$
$$= K\left\{0, 0, \frac{\rho_0^{k+1}-\rho_1^{k+1}}{\rho_0-\rho_1} - \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)}, \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)}\right\}$$

We recall that the first two entries in $\boldsymbol{\pi}_k$ are from states where $J(t) = 0$, i.e., where the server is undergoing a working vacation (see Definition 3.2.1). By taking $\theta \to \infty$, we take the expected working vacation duration to 0. Thus for $k \geq 1$, π_{k0} and π_{k1} are 0. It is still possible, however, to visit the vacation state $(0, 0, 0)$ when the queue is empty. $\square$

Chapter 4

Future Work

The following ideas for future research are intended to outline the relevance and potential of our queue model when applied to wireless networks. As we have shown in Chapters 2 and 3, we seek to construct queue models with application in mind, but do the construction and analysis in a general way. It is with this notion that we propose the following ideas for future work going forward.

4.1 Modified Working Vacations

4.1.1 A Policy-Based Modification with Application

The work of Levy and Yechiali in 1975 [16] sought to model a queue whereby the idle-time of a server, more commonly referred to as a vacation period, could be utilized for other purposes. This gave rise to a natural extension known as a *working* vacation introduced by Servi and Finn in 2002 [19] where the server is defined to be available during its vacation period, but at a reduced service rate. In either case, the event responsible for triggering a vacation period is defined by the existence of an empty queue (i.e. when the server is idle, it begins a vacation period). This policy is intuitive when using the term vacation, but a more general interpretation of a working vacation is as follows.

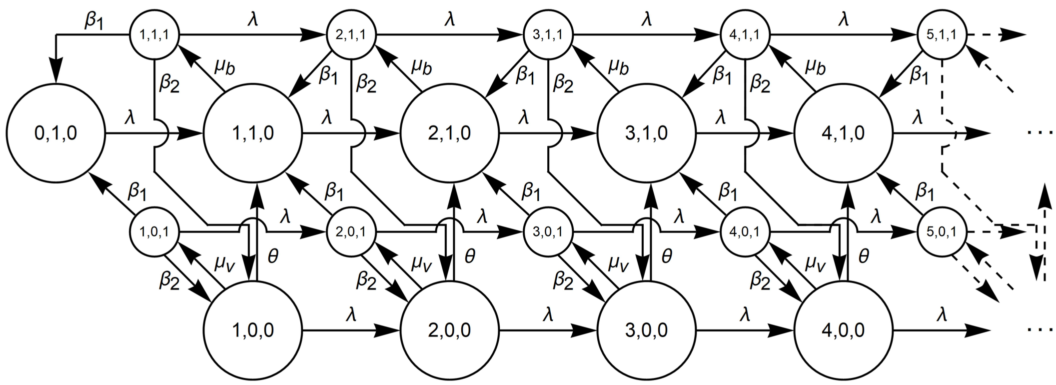
Definition 4.1.1. A working vacation is a period of time triggered by some well defined event E_1, whereby the server is still operational albeit at a reduced rate. The server returns to a normal service rate when the well defined event E_2 occurs.

Under this more general interpretation of a working vacation, we are free to define the following:

Definition 4.1.2.
Let $E_1 = \{\text{a customer's service fails}\}$
$E_2 = \{\text{a customer's service succeeds}\}$

We need only a few non-trivial modifications to the transition rates diagram from Chapter 3 to achieve this.

Figure 4.1: 3D Markovian state transition rates diagram.



These changes will of course alter our infinitesimal matrix Q, but the rest of the analysis should follow in a similar fashion to that contained in Chapter 3. One possible application of such a queue would be to an adaptive two-rate wireless network, where transmission failures (unacknowledged) trigger a reduced encoding rate to be used by the transmitter until a successful (acknowledged) transmission is observed. Likewise, a successful transmission (one that is acknowledged) would trigger the faster encoding rate to be used by the transmitter, repeating the cycle.

n Service Rates

A natural extension to the model proposed in Section 4.1.1 would be to consider n distinct types of 'vacation' periods, each with their own distinct service rates. If we then reduce the rates upon service failure and increase the rates on successful service completion, we would have a queue model potentially suitable to ieee 802.11bg type wireless networks where we have one adaptive rate carrier to encode data with. We would also want to adjust the failure rates associated with each vacation period since the probability to failure should decrease when the encoding method is slower but simpler. The Positive Recurrence condition would be of great interest since it could provide great insight into the proper use cases for various equipment based on the equipment's specifications.

4.2 Multiple Servers

4.2.1 Parallel Configuration

The M/M/c model has been studied extensively with some assumptions, see [20], [21], and [22]. However, if feasible, one might analyze this queue with c parallel servers and unreliable service in addition to perhaps n service rates, as discussed previously. Within the context of wireless communications, this can be viewed as a MIMO (Multiple In/Multiple Out) wireless configuration whereby the transmitter uses spatial multiplexing (typically orthogonally oriented antennas in a 2x2 design) to establish multiple point to point data streams on the same channel simultaneously. [23] contains some excellent ideas on how one might define the workflow of a parallel server configuration, in addition to some interesting results on the implementation of their model as a CTMP (Continuous Time Markov Process).

4.2.2 Series Configuration

Within the context of computer networking, the idea of a series n-server configuration would model the path from point A to B along n individual links. If we then add the notion of unreliable service, we begin to build the picture of a point to point data stream over a MESH network, whereby clients are connected by a whole host of access points (APs) that are then connected to one another. Client A may send data to Client B, but that data may be transmitted from one AP to another several times before reaching its final destination. Effectively, MESH networks are a distributed wireless network where the supporting network (the underlying network which connects individual access points to one another) is itself wireless. This type of queue is worthy of study based on this application alone, even though it has been largely overlooked by the literature.

In practice, we could merely add to our Chapter 2 or Chapter 3 model the additional requirement that service must succeed c times for the customer to be considered served. At each of the c times of service, separate success and failure rates could be defined to determine the 'risk' associated with that service period. The analysis should follow in a very similar manner to that of our previous work.

4.3 Service and/or Arrivals in Batches

4.3.1 Batch Service

Wireless networks are subject to an interesting conundrum, should you transmit large or small amounts of data within a single transmission?

In the case of smaller transmission, the probability of a failed transmission (i.e. the need of a retransmission) is lower, especially when using CTS/RTS (see [2]). However due to network overhead that encapsulates the data and the need for each transmission to be acknowledged, this additional reliability comes at the cost of throughput (see [24]). In the case of larger transmissions, air time can be preserved due to less overhead and the need for fewer acknowledgments, granting greater theoretical throughput, but if a transmission fails, you will need to retransmit the entire frame which can lead to *reduced* throughput in some cases.

Therefore, within the ieee 802.11x standard exists a mechanism called frame aggregation whereby 802.11x data frames can be transmitted in batches. It is the decision of the user to enable or disable frame aggregation within their devices.

In Queue Theory, we refer to this concept of 'frame aggregation' as service in 'batchs' or 'bulk' service and it is well studied (see [25], [26], [27], [28] and [29]). However, what has not been studied is the real-world trade off in a wireless environment between longer transmissions (bulk servicing) with a higher probability of failure and shorter transmissions with a lower probability of failure–this is due to the concept of unreliable service only now being well defined and studied in this paper.

4.3.2 Batch Arrivals

It seems natural that if wireless devices, specifically ieee 802.11x devices are able to transmit data in batches, or bulk, then they certainly must be able to receive data in the same way. Indeed, this is true. However a non-trivial idea must be rationalized in context before we are allowed to proceed down this seemingly logical path–that is, can we meaningfully define a queue along the receive, or ingress data path?

We tend to think of queues in a networking environment as they apply to egress, or uplink, data flows. This is because we can locally decide when to send data, but we cannot in general decide nor control when we receive or ingress data. Therefore it would seem meaningless to define a queue along the ingress data path. Indeed, some operating systems have limitations where interface queues can only be defined for egress (outgoing) data. For example, in Linux, this limitation is overcome by defining a virtual loopback interface, forwarding all ingress data to this virtual interface to be egressed out, which, by definition, then returns the data to the host. The ingress data is then queued or controlled as it 'leaves' the virtual loopback device as egress data.

While there is certainly wisdom to idea that only egress data can meaningfully be queued in the general case, TCP/IP networks are blessed by the congestion control mechanisms therein which, in the event that packets (packets are frames at a software level) are dropped or delivered out of order, can signal to the party or parties who are sending us data to slow down. In this environment, queues defined along the downlink or ingress path become meaningful because they can trigger the congestion control mechanisms of TCP/IP to alter the rate of the ingress data flow to achieve desirable effects. Therefore, it is natural that if we can meaningfully study batch or bulk service in light of its application to ieee 802.11x wireless networks, we can likewise study batch arrivals. For more details on batch or bulk arrival queues, see [28], [30] and [31].

4.3.3 Batch Service and Arrivals

Even without the above contextual insight into the workings of TCP/IP, we may consider the case of a wireless router where virtually all data leaving the device also entered the device. In this case, we may control ingress data by controlling when and how it leaves the device (i.e. batch or bulk arrivals that feed directly into batch or bulk service). See [32] and [33] for more details.

4.4 Finite Capacity

The observant reader will note that the contents of this paper have assumed an infinite capacity queue whereby arrivals are always accepted and the number of customers waiting for service may be unbounded. One area of future research would be to consider the potentially more realistic case where the queue length is bounded, i.e. the queue's capacity is finite. We would need to define a policy by which customers arriving to find the queue full will be handled.

One such possible policy would be to borrow the notion of an 'orbit' from customer retrial queues such as that done by Sherman & Kharoufeh in 2006 [34] to define a policy whereby customers who find the queue full leave, then after some possibly exponentially distributed time, return.

Another possible policy would be to drop arrivals who find the queue full. This policy is perhaps more applicable in some situations. Network queues, for example, drop data packets / frames if they do not have the capacity to hold them, however TCP/IP mechanisms guarantee that drop packets / frames will be resent by the originating source. Therefore, there is an argument to be made for both policy types in the context of computer networking.

Appendix

Mathematica Methods

Method #1

Much of this paper was possible due to the algebraic capabilities of *Mathematica*. We would like to emphasize that *Mathematica* was strictly used as a "scratch paper" environment where potential algebraic manipulations could be observed, done by hand (using a keyboard), and checked for consistency with a mere press of the Shift+Enter key combination. This greatly reduced the time necessary to complete the work, reduced the likelihood of algebraic errors, and allow us to focus our attention on the larger picture.

While we would very much like to include the *Mathematica* work explicitly, it is simply too long to feasibly fit on A4 sized paper. To give the reader an impression as to the amount of *Mathematica* work that was needed, it is contained in 38 separate *Mathematica* notebook files. Each notebook file, if printed to A4 paper by File⟶Print, contains 8-30 pages worth of trial and error work. We will, however, go over the more general ideas behind the work by illustrating an example.

Example. We would like to find an expression for $(1-\rho_0)(1-\rho_1)$, knowing that $\rho_0\rho_1 = \frac{\lambda^2}{\mu_b\beta_1}$ and $\rho_0+\rho_1 = \frac{\lambda(\beta_1+\beta_2+\lambda+\mu_b)}{\mu_b\beta_1}$.

We begin with a trivial statement in the *Mathematica* environment:

```
In[1]:= Simplify[(1-ρ0)(1-ρ1)==(1-ρ0)(1-ρ1)]

Out[1]= True
```

Next, we attempt to distribute $(1-\rho_0)(1-\rho_1)$ on the right hand side.

```
In[2]:= Simplify[(1-ρ0)(1-ρ1)==1-ρ0-ρ1+ρ0ρ1]

Out[2]= True
```

Notice that the algebra manipulation is done by hand, albeit with a keyboard. We merely ask *Mathematica* to verify our results. At this point, we'd like to factor the $-\rho_0-\rho_1$ expression since one of our identities is given as $\rho_0+\rho_1$.

In[3]:= `Simplify[(1-ρ0)(1-ρ1)==1-(ρ0-ρ1)+ρ0ρ1]`

$$\text{Simplify}[(1-\rho_0)(1-\rho_1)==1-(\rho_0-\rho_1)+\rho_0\rho_1]$$

Out[3]= $\rho_1 == 0$

At this point we see Mathematic reduced the boolean expression *lhs* == *rhs* into an equivalent expression $\rho_0 == 0$. This means that our *rhs* is *conditionally* equal to *lhs*. Since we merely wanted to factor a negative out of two terms, we should review our work to see if we made a mistake–and upon closer inspection, we see what our mistake was and correct it. Note: Ctrl+Z (the undo shortcut) helps greatly when mistakes are made!

In[4]:= $$\text{Simplify}[(1-\rho_0)(1-\rho_1)==1-(\rho_0+\rho_1)+\rho_0\rho_1]$$

Out[4]= True

We now reach a critical juncture whereby we must decide how to make our substitutions. *Mathematica*'s Simplify[] command includes the option to give identities. This is useful for our purposes.

In[5]:= $$\text{Simplify}[(1-\rho_0)(1-\rho_1)==1-(\frac{\lambda(\lambda+\beta_1+\beta_2+\mu_b)}{\mu_b\beta_1})+\rho_0\rho_1,\{\rho_0+\rho_1==\frac{\lambda(\lambda+\beta_1+\beta_2+\mu_b)}{\mu_b\beta_1}\}]$$

Out[5]= True

In[6]:= $$\text{Simplify}[(1-\rho_0)(1-\rho_1)==1-(\frac{\lambda(\lambda+\beta_1+\beta_2+\mu_b)}{\mu_b\beta_1})+\frac{\lambda^2}{\mu_b\beta_1},\{\rho_0\rho_1==\frac{\lambda^2}{\mu_b\beta_1},\rho_0+\rho_1==\frac{\lambda(\lambda+\beta_1+\beta_2+\mu_b)}{\mu_b\beta_1}\}]$$

Out[6]= True

We now see an additive cancellation of the $\frac{\lambda^2}{\mu_b\beta_1}$ term.

In[7]:= $$\text{Simplify}[(1-\rho_0)(1-\rho_1)==1-\frac{\lambda(\beta_1+\beta_2+\mu_b)}{\mu_b\beta_1},\{\rho_0\rho_1==\frac{\lambda^2}{\mu_b\beta_1},\rho_0+\rho_1==\frac{\lambda(\lambda+\beta_1+\beta_2+\mu_b)}{\mu_b\beta_1}\}]$$

Out[7]= True

Next, we add the terms on right hand side with a common denominator, then re-associate.

In[8]:= $$\text{Simplify}[(1-\rho_0)(1-\rho_1)==\frac{\beta_1(\mu_b-\lambda)-\lambda(\beta_2+\mu_b)}{\mu_b\beta_1},\{\rho_0\rho_1==\frac{\lambda^2}{\mu_b\beta_1},\rho_0+\rho_1==\frac{\lambda(\lambda+\beta_1+\beta_2+\mu_b)}{\mu_b\beta_1}\}]$$

Out[8]= True

Alas! We find an expression for $(1-\rho_0)(1-\rho_1)$! The observant reader may recall that ρ_0 and ρ_1 are conjugate pairs, so there are certainly other ways to compute an expression for $(1-\rho_0)(1-\rho_1)$. However, the purpose of this illustration was to demonstrate that the actual mathematics was first observed by human intuition then implemented by hand (with a keyboard). The results were merely verified at each step by *Mathematica*. In the context of this paper, it is clear how this environment allows for substitution ideas to be attempted on very complicated expressions in a timely manner and without error, until a desired equivalent expression is found.

Method #2

There is another method by which the computational power of *Mathematica* can be used to our advantage. This method proved to be exceptionally useful when we found the P.G.F. of N in Chapter 3 given in the proof of Theorem 3.7.1 as:

$$\mathcal{G}_{\mathcal{N}}(z) = \frac{1-\rho_0}{1-\rho_0 z}\frac{1-\rho_1}{1-\rho_1 z}\frac{1-\rho_2}{1-\rho_2 z}\frac{1-\rho_3}{1-\rho_3 z}K^*\left(\widehat{\delta_0} + \widehat{\delta_1} z + \widehat{\delta_2} z^2\right)$$

After computing the P.G.F. by definition, we found after many substitutions and simplifications that we had what appeared to be a non-zero coefficient of z^3. To be consistent with the notation of Theorem 3.7.1, lets call that coefficient $\widehat{\delta_3}$.

$$\widehat{\delta_3} = \frac{\lambda\mu_b\rho_0\rho_1\left(\left(\lambda - \left(\delta_3-\delta_4\right)\left(\rho_0-\rho_1\right)\right)\left(\delta_2\rho_2^2-\delta_1\rho_3^2\right) - \beta_1\left(\delta_2\rho_2^3-\delta_1\rho_3^3\right)\right)}{\rho_2\rho_3} + \mu_v\rho_0\rho_1\left(\delta_3\rho_2^2-\delta_4\rho_3^2\right)$$

However, we found that this coefficient of z^3 was actually 0 by using the /. operator in *Mathematica* to 'plug in' the definitions of our variables which we have stored in lists. We did this out of curiosity for all constants $\{\widehat{\delta_i}\}_{i=1}^3$. For every $i = 1, 2, 3$, we were pleasantly surprised by the results, but the result for $i = 3$ in particular was extremely useful.

In[9]:= Simplify[$\frac{\lambda\mu_b\rho_0\rho_1((\lambda-(\delta_3-\delta_4)(\rho_0-\rho_1))(\delta_2\rho_2^2-\delta_1\rho_3^2)-\beta_1(\delta_2\rho_2^3-\delta_1\rho_3^3))}{\rho_2\rho_3}$

$+\mu_v\rho_0\rho_1(\delta_3\rho_2^2-\delta_4\rho_3^2)$ /. δ_1→delta[[1]] /. δ_2→delta[[2]]

/. δ_3→delta[[3]] /. δ_4→delta[[4]] /. ρ_0→rho[[1]]

/. ρ_1→rho[[2]] /. ρ_2→rho[[3]] /. ρ_3→rho[[4]]]

Out[9]= 0

References

[1] Erac Ombati Momanyi, V. K. Oduol, and S. Musyoki. “First in first out (FIFO) and priority packet scheduling based on type of service (TOS)”. *Journal of Information Engineering and Applications* 4.7 (2014), pp. 22–36.

[2] Weinmiller et al. “Analysing the RTS/CTS mechanism in the DFWMAC media access protocol for wireless LANs” (1995).

[3] Xiuli Xu and Naishuo Tian. “Performance Analysis of an M/M/1 Working Vacation Queue with Setup Times”. *Advances in Queueing Theory and Network Applications*. Springer, 2009, pp. 65–76.

[4] G. Latouche and V. Ramaswami. *Introduction to Matrix Analytic Methods in Stochastic Modeling*. ASA-SIAM Series on Statistics and Applied Probability. Society for Industrial and Applied Mathematics, 1999.

[5] Oliver Ibe. *Markov Processes for Stochastic Modeling*. Elsevier insights. Elsevier, 2013.

[6] M. F. Neuts. *Matrix-geometric Solutions in Stochastic Models: An Algorithmic Approach*. Dover Publications, 1981.

[7] Pei-Chang Guo. “Newton-Shamanskii Method for a Quadratic Matrix Equation Arising in Quasi-Birth-Death Problems”. *East Asian Journal on Applied Mathematics* 4.4 (June 2014), pp. 386–395.

[8] Sang-Hyup Seo, Jong Hyun Seo, and Hyun-Min Kim. “Newton’s method for solving a quadratic matrix equation with special coefficient matrices”. *Journal of Physics: Conference Series* 490.1 (2014), p. 012001.

[9] I.J.B.F. Adan, J. Wessels, and W.H.M. Zijm. “Matrix-geometric analysis of the shortest queue problem with threshold jockeying”. *Operations Research Letters* 13.2 (1993), pp. 107 –112.

[10] John Little. “A Proof of the Queueing Formula L= λW”. *Operations Research* 9.3 (June 1961), pp. 383–387.

[11] J Keilson and LD Servi. “A distributional form of Little’s law”. *Operations Research Letters* 7.5 (1988), pp. 223–227.

[12] S.M. Ross. *Introduction to Probability Models*. Elsevier Science, 2006.

[13] I. Adan and J. Resing. *Queueing Theory*. Eindhoven University of Technology. Department of Mathematics and Computing Science, 2001.

[14] Oliver C. Ibe. *Fundamentals of Stochastic Networks*. John Wiley & Sons, Inc., 2011.

[15] J.W. Cohen. *The Single Server Queue*. North-Holland Series in Applied Mathematics and Mechanics. Elsevier Science, 2012.

[16] Yonatan Levy and Uri Yechiali. "Utilization of idle time in an M/G/1 queueing system". *Management Science* 22.2 (1975), pp. 202–211.

[17] W. V. Quine. "Concatenation as a Basis for Arithmetic". *The Journal of Symbolic Logic* 11.4 (1946), pp. 105–114.

[18] Imre Ruzsa and Gábor J Székely. *Algebraic probability theory*. John Wiley & Sons Inc, 1988.

[19] Leslie D. Servi and Steven G. Finn. "M/M/1 queues with working vacations (M/M/1/WV)". *Performance Evaluation* 50.1 (2002), pp. 41–52.

[20] H. A. Taha. *Operations Research An Introduction, 8th Edition*. Pearson Education, 2008, pp. 725–750.

[21] J.K. Sharma. *Operations research: Theory And Applications, Third Edition*. Macmillan India Ltd., 2006, pp. 725–750.

[22] Prem Kumar Gupta and D.S.Hira. *Operations Research, Revised Edition*. S. Chand, 2008, pp. 903–910.

[23] M. Reni Sagayaraj et al. "A Study on Queuing System with Parallel Servers". *International Journal of Science and Research* 4.1 (2015).

[24] Jyoti Kolap, Shoba Krishnan, and Ninad Shaha. "Frame aggregation mechanism for high-throughput 802.11 n wlans". *International Journal of Wireless & Mobile Networks* 4.3 (2012), p. 141.

[25] M.L. Chaudhry and J.G.C. Templeton. *A First Course in Bulk Queues*. A Wiley-interscience publication. Wiley, 1983. ISBN: 9780471862604.

[26] Norman TJ Bailey. "On queueing processes with bulk service". *Journal of the Royal Statistical Society: Series B (Methodological)* 16.1 (1954), pp. 80–87.

[27] David G Kendall. "Some problems in the theory of queues". *Journal of the Royal Statistical Society: Series B (Methodological)* 13.2 (1951), pp. 151–173.

[28] M. F. Neuts. "A general class of bulk queues with Poisson input". *The Annals of Mathematical Statistics* 38.3 (1967), pp. 759–770.

[29] NK Jaiswal. "Bulk-Service Queuing Problem". *Operations Research* 8.1 (1960), pp. 139–143.

[30] Hideaki Takagi. "Queueing analysis: a foundation of performance evaluation". *Vacation and priority systems* 1.1 (1991).

[31] Ho Woo Lee, Seung Hyun Yoon, and Soon Seok Lee. "A continuous approximation for batch arrival queues with threshold". *Computers & operations research* 23.3 (1996), pp. 299–308.

[32] Ho Woo Lee, Soon Seok Lee, and Kyung-Chul Chae. "A fixed-size batch service queue with vacations". *International Journal of Stochastic Analysis* 9.2 (1996), pp. 205–219.

[33] R Arumuganathan and S Jeyakumar. "Steady state analysis of a bulk queue with multiple vacations, setup times with N-policy and closedown times". *Applied Mathematical Modelling* 29.10 (2005), pp. 972–986.

[34] Nathan P. Sherman and Jeffrey P. Kharoufeh. "An M/M/1 retrial queue with unreliable server". *Operations Research Letters* 34 (2006), pp. 697–705.