A STUDY ON THE ROTATIONAL b-FAMILY OF EQUATIONS

by

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To my parents Nezihe and Yaşar,

And my sisters Ebru, Eda, Ece

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ABSTRACT

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In this thesis, we study a mathematical model of long-crested water waves propagating in one direction with the effect of Earth's rotation near the equator by following the formal asymptotic procedures. Firstly, we derive a new model equation called the rotational b-family of equations by using the Camassa-Holm approximation of the two-dimensional incompressible and irrotational Euler equations. Secondly, we establish that the local well-posedness of the Cauchy problem for the rotational b-family of equations on the Sobolev space H^s, for $s > 3/2$. In addition, we study the effects of the Coriolis force and nonlocal higher nonlinearities on blow-up criteria and wave-breaking phenomena.

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CHAPTER 1

INTRODUCTION

In this thesis, we study the modelling and analysis of the rotational b-family of equations, such as local well-posedness, blow-up criteria, and wave breaking criteria. The early studies about shallow water equations, b-family of equations and rotational b-family of equations are reviewed in this chapter.

1.1 Early Studies on Shallow Water Equations

The studies on shallow water waves have been studied in the areas of geophysical fluid dynamics, physics and applied mathematics to interpret the wave behavior on the geophysical flows. Firstly, in the history of shallow water wave theory, John Scott Russell (1808-1882) explored a phenomena that he called as the wave of translation and explained in his words [39, 40]:

“I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped-not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."

After his observation, he also did an experiment to replicate his observation by building a tank and trying to repeat the solitary waves in the tank. He received successful results at the end of the experiment. However, the importance of his phenomena was not understood until the studies of D. J. Korteweg and G. de Vries in 1895. They analyzed the following equation

$$u_t + u_x + \frac{3}{2}uu_x + \frac{1}{6}u_{xxx} = 0,$$

which is called as the Korteweg-de Vries (KdV) equation [32] . For the KdV equation, they also derived the traveling wave solution, which confirmed "the Wave of Translation" of Russell. Later, it is called as one soliton solution.

The KdV equation and many other shallow water models known as approximations to the full Euler dynamics are only valid in the weakly nonlinear regime [32]. On the other hand, this is not enough to satisfy some physical phenomena such as wave breaking, which means that the wave remains bounded while its slope becomes unbounded in finite time [43], waves of maximum height [1, 42]. They need a transition to full nonlinearity. This leads researchers to investigate new models for nonlinear shallow water waves [43].

To describe the long-wave regime, the positive parameters ε and μ are defined by respectively the amplitude parameter and the shallowness parameter as

$$\varepsilon = \frac{a}{h_0}, \qquad \mu = \frac{h_0^2}{\lambda^2},$$

where long wavelength is λ, small amplitude is a and mean level of water surface is h_0. Considering the Boussinesq regime $\varepsilon = O(\mu)$ as $\mu \ll 1$, asymptotic approximations for the unidirectional solutions of the irrotational two-dimensional water waves problem is obtained by the KdV model [3, 13]. However, for more accurate asymptotic approximations for these types waves which have more nonlinear behavior than dispersive, larger values of ε are considered, which is called the Camassa-Holm (CH) scaling [11], $\varepsilon = O(\sqrt{\mu})$ as $\mu \ll 1$. Stronger nonlinear effects are obtained with CH scaling. That means the presence of breaking waves could be investigated with a higher nonlinearity [11].

Camassa and Holm derived a new completely integrable dispersive shallow water equation [4]

$$u_t - u_{txx} + \kappa u_x + 3uu_x = 2u_x u_{xxx} + uu_{xxx},$$

where u is the fluid velocity and κ is a constant related to the critical shallow water wave speed. They derived the equation by using Hamiltonian methods in [4]. In addition, by using the asymptotic expansion at linear order for unidirectional shallow water waves, the integrable third-order KdV equation was obtained in [18]. On the other hand, a family of shallow water wave equations was derived at quadratic order in this asymptotic expansion and these equations are asymptotically equivalent to each other under a group of nonlinear, non-local, normal-form transformations introduced by Kodama in combination with the application of the Helmholtz-operator [18].

The b-family of equations is described by the following family of 1+1 evolutionary equations which is one dimensional nonlinear waves in fluids [29]

$$m_t + \underbrace{um_x}_{\text{convection}} + \underbrace{bu_x m}_{\text{stretching}} = 0, \quad with \quad u = g * m, \tag{1.1}$$

where $u(t, x)$ is denoted as fluid velocity on the real line and vanishing at spatial infinity and the $*$ denotes the convolution, yielding

$$u(x) = \int_{-\infty}^{\infty} g(x-y)m(y)dy,$$

which relates the velocity u to the momentum density m by integration against the kernel $g(x)$ over the real line. The kernel function $g(x)$ is chosen as $g(x) = \frac{1}{2}e^{-|x|}$ which implies that $m = u - u_{xx}$. The kernel g and the dimensionless constant b which is the ratio of stretching to convection represents the family of equations (1.1). The traveling wave shape and length scale for (1.1) are established by the function $g(x)$, while a balance or bifurcation parameter for the nonlinear solution behavior is given by the constant b.

In the equation (1.1), the quadratic terms show the balance, in fluid convection between nonlinear steeping and amplification owing to b-dimensional stretching. For $b \neq -1$ the equation, (1.1) can be derived as the family of asymptotically analogous shallow water wave equations that appears at quadratic-order accuracy by a proper Kodama transformation [16, 18].

In addition, the local well-posedness of b-family of equations is established by using Kato's semi group theory in [30]. Also, Escher and Yin obtained the local well-posedness of b-family of equations to get a precise blow-up scenario, to prove that the equation has strong and finite-time blow-up solutions[21]. Degasperis, Holm, and Hone[17] showed that the b-family of equations have the peakon solutions

$$u(x,t) = ce^{-|x-ct|}, c > 0, \tag{1.2}$$

besides, they have multipeakon solutions

$$u(x,t) = \sum_{j=1}^{N} p_k(t)e^{-|x-q_j|}.$$

For any b, the quantities p_j and q_j are not canonical variables but satisfy the dynamical system

$$\dot{p_j} = -(b-1)\frac{\partial G_N}{\partial q_j}, \qquad \dot{q_j} = \frac{\partial G_N}{\partial p_j},$$

where the overdot denotes the t-derivative and the generating function G_N is given by

$$G_N = \frac{1}{2}\sum_{j,k=1}^{N} p_j p_k(t) e^{-|q_j - q_k|}.$$

For each $b \neq 0$, (1.1) has at least three conserved quantities as follows:

$$E_1(u) = \int_{\mathbb{R}} m dx,$$
$$E_2(u) = \int_{\mathbb{R}} m^{1/b} dx,$$
$$E_3(u) = \int_{\mathbb{R}} m^{-1/b}\left(\frac{m_x^2}{b^2 m^2} + 1\right) dx.$$

On the other hand, the equation (1.1) is completely integrable if $b = 2$ or $b = 3$. The integrability is proved by using the method of asymptotic integrability [16].

When $b = 2$, the equation (1.1) is called the Camassa-Holm equation in the form

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xxx} + uu_{xxx}, \quad t > 0, \quad x \in \mathbb{R}, \tag{1.3}$$

which is the model of the unidirectional propagation of shallow water waves over a flat bottom [4, 19]. Here $u(t, x)$ is the fluid velocity at time t in the spatial x direction. (1.3) has bi-Hamiltonian structure [23, 41] and is completely integrable [4, 10]. It has also solitons as the KdV equation, while the CH equation yields permanent and breaking waves, and has peaked solitons of the form (1.2) [4, 5, 8, 36]. The Cauchy problem for (1.3) is studied thoroughly. Local well-posedness for the Camassa-Holm equation with the initial data $u_0 \in H^s(\mathbb{R})$, $s > 3/2$ is shown in [7, 33, 38].

When $b = 3$, (1.1) is called the Degasperis-Procesi (DP) equation given by

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xxx} + uu_{xxx}, \quad t > 0, \quad x \in \mathbb{R}. \tag{1.4}$$

The formal integrability of the Degasperis-Procesi equation (1.4) is proved in [17] by setting up a Lax pair. In addition, bi-Hamiltonian structure and an infinite number of conserved quantities for (1.4) are shown in [17]. On the other hand, similar to the Camassa- Holm equation, local well-posedness of the Degasperis-Procesi equation is proved in [44], for the initial data $u_0 \in H^s(\mathbb{R})$, $s > 3/2$.

The Degasperis-Procesi equation has many properties which are similar to the Camassa-Holm equation such as global strong solutions shown in [9, 7] for the CH equation and in [20] for the DP equation, finite time blow-up solutions, peakon solutions $u_c(t,x) = ce^{-|x-ct|}$ for $c > 0$ [17]. In addition, the Degasperis-Procesi equation has shock peakons [34].

In the next section, some details of the new model equation are given.

1.2 Rotational b-Family of Equations

The rotational b-family (R-b-family) of equations is a model equation with the Coriolis effect from the incompressible and irrotational two-dimensional shallow water in the equatorial region in the following form

$$\begin{aligned} u_t - \beta\mu u_{xxt} + cu_x + (b+1)\alpha\varepsilon u u_x - \beta_0\mu u_{xxx} + \omega_1\varepsilon^2 u^2 u_x + \omega_2\varepsilon^3 u^3 u_x \\ = \alpha\beta\varepsilon\mu(bu_x u_{xx} + u u_{xxx}), \end{aligned} \tag{1.5}$$

where the constant rotational frequency due to the Coriolis effect is defined as the parameter Ω and the wave speed is $c := \sqrt{1+\Omega^2} - \Omega$. The other constants in (1.5) are defined by

$\alpha := \frac{3c^2}{(b+1)(1+c^2)}$, $\beta_0 := \frac{c(3c^4+(5b+8)c^2-(b+1))}{9b(c^2+1)^2}$, $\beta := \frac{(b+1)(3c^4+8c^2-1)}{9b(c^2+1)^2}$,
$\omega_1 := \frac{-3c(c^2-1)(c^2-2)}{2(1+c^2)^3}$, and $\omega_2 := \frac{(c^2-2)(c^2-1)^2(8c^2-1)}{2(1+c^2)^5}$.

The horizontal velocity field at height z_0 is represented by the solution u of (1.5) , and after the re-scaling, it is required that $0 \le z_0 \le 1$, where

$$z_0 = \sqrt{\frac{b-1}{b} - \frac{2}{3}\frac{1}{(c^2+1)} + \frac{(b-2)c^2+2b+4}{3b(c^2+1)^2}}. \tag{1.6}$$

As the constant β has to be greater than 0 , it must be the case

$$0 \leq \Omega < \sqrt{\frac{1}{6}(1+2\sqrt{19})} \approx 1.273.$$

Let $b \geq \frac{10}{11}$ and $c_0 = \sqrt{\frac{\sqrt{19}-4}{3}}$,

$$\sqrt{\frac{11b-10}{12b}} = z_0(b,1) = \inf_{c_0<c\leq 1} z_0(c) \leq z_0 \leq \sup_{c_0<c\leq 1} z_0(c) < 0.984.$$

There are two special cases for R-b-family equation, which corresponds to some well-known equations for some different b values. The first case, $b = 2$ is corresponding to Rotational Camassa-Holm (R-CH) equation:

$$\begin{cases} c = 1,\ \beta = \frac{5}{12},\ \beta_0 = \frac{1}{4} \\ \omega_1 = 0,\ \omega_2 = 0,\ \alpha = \frac{1}{2} \end{cases} ; \Omega = 0. \tag{1.7}$$

Also, if $\Omega = 0$, $z_0 = \frac{1}{\sqrt{2}}$ which corresponds to the height of the case of the classical CH equation.

The second case, $b = 3$ is corresponding to the Rotational Degasperis-Procesi(R-DP) equation:

$$\begin{cases} c = 1,\ \beta = \frac{10}{27},\ \beta_0 = \frac{11}{54} \\ \omega_1 = 0,\ \omega_2 = 0,\ \ \alpha = \frac{3}{8} \end{cases} ; \Omega = 0. \tag{1.8}$$

In addition, if $\Omega = 0$, $z_0 = \frac{\sqrt{23}}{6}$ which corresponds to the height of the case of classical DP equation

To derive the R-b-family of equations model in (1.5), we refer the reader to the paper [27] where the classical CH equation was derived. The R-b-family equation in (1.5) is established by showing that after a double asymptotic expansion with respect to ε and μ, the free surface $\eta = \eta(\tau, \xi)$ under the field variable (η, ξ) defined in (2.2) in 2D Euler's dynamics (2.3) (see Section 2), is governed by the equation

$$2(\Omega + c)\eta_\tau + 3c^2\eta\eta_\xi + \frac{c^2}{3}\mu\eta_{\xi\xi\xi} + A_1\varepsilon\eta^2\eta_\xi + A_2\varepsilon^2\eta^3\eta_\xi + A_5\varepsilon^3\eta^4\eta_\xi$$
$$= \varepsilon\mu\Big[A_3\eta_\xi\eta_{\xi\xi} + A_4\eta\eta_{\xi\xi\xi}\Big] + O(\varepsilon^4, \mu^2),$$

where the constants

$$A_1 := \frac{3c^2(c^2-2)}{(c^2+1)^2}, \quad A_2 := -\frac{c^2(2-c^2)(c^6-7c^4+5c^2-5)}{(c^2+1)^4},$$

$$A_3 := \frac{-c^2(9c^4+16c^2-2)}{3(c^2+1)^2}, \quad A_4 := \frac{-c^2(3c^4+8c^2-1)}{3(c^2+1)^2},$$

$$A_5 := \frac{c^2(c^2-2)(3c^{10}+228c^8-540c^6-180c^4-13c^2+42)}{12(c^2+1)^6}.$$

The free surface η with respect to the horizontal component of the velocity u at $z = z_0$ under the CH regime $\varepsilon = O(\sqrt{\mu})$ as $\mu \to 0$ is also given by

$$\eta = \frac{1}{c}u + \gamma_1\varepsilon u^2 + \gamma_2\varepsilon^2 u^3 + \gamma_3\varepsilon^3 u^4 + \gamma_4\varepsilon\mu u_{\xi\xi} + O(\varepsilon^4, \mu^2),$$

where the constants in the expression are given by

$$\gamma_1 = \frac{2-c^2}{2c^2(c^2+1)}, \quad \gamma_2 = \frac{(c^2-1)(c^2-2)(2c^2+1)}{2c^3(c^2+1)^3}$$

$$\gamma_3 = -\frac{(c^2-1)^2(c^2-2)(21c^4+16c^2+4)}{8c^4(c^2+1)^5}, \quad \gamma_4 = \frac{z_0^2}{2c} - \frac{3c^2+1}{6c(c^2+1)} = \frac{-(3c^4+6c^2-5)}{12c(c^2+1)^2}$$

(here the height parameter z_0 is determined by (1.6)).

The equation (1.5) can be rewritten by defining $m := (1 - \beta\mu\partial_x^2)u$, in terms of the evolution of the momentum density m, namely,

$$\partial_t m + \alpha\varepsilon(um_x + bmu_x) + cu_x - \beta_0\mu u_{xxx} + \omega_1\varepsilon^2u^2u_x + \omega_2\varepsilon^3u^3u_x = 0. \qquad (1.9)$$

In the case that the Coriolis effect vanishes ($\Omega = 0$) for $b = 2$, the coefficients in the higher-power nonlinearities $\omega_1 = 0$ and $\omega_2 = 0$. Using the scaling transformation

$u(t,x) \mapsto \alpha\varepsilon u(\sqrt{\beta\mu}\; t, \sqrt{\beta\mu}\; x)$ and then the Galilean transformation $u(t,x) \mapsto u(t, x - \frac{3}{4}t) + \frac{1}{4}$, the R-CH equation (1.9) is then reduced to the classical CH equation (1.3).

Note that the R-b-family equation (1.9) has the following conserved quantity

$$I(u) := \int_{\mathbb{R}} u\, dx. \tag{1.10}$$

As a special case $b = 2$, the equation has three conserved quantities as follows

$$I(u) = \int_{\mathbb{R}} u dx, \quad E(u) = \frac{1}{2}\int_{\mathbb{R}} \left(u^2 + u_x^2\right) dx,$$

and

$$F(u) = \frac{1}{2}\int_{\mathbb{R}} \left(cu^2 + u^3 + \frac{\beta_0}{\beta}u_x^2 + \frac{\omega_1}{6\alpha^2}u^4 + \frac{\omega_2}{10\alpha^3}u^5 + uu_x^2\right) dx.$$

In this thesis, due to the Coriolis effect, we obtain higher power nonlinear terms from the derivation of the rotational b-family of equations given in next chapter. Then, some intriguing inferences can be made for the fluid motion, especially breaking waves and the permanent waves. In addition, we analyze the effects of the Coriolis force with the Earth rotation on the appearance of the wave-breaking phenomena. Local well-posedness, blow-up and wave breaking criteria are proved for this equation in chapter 3. In chapter 4, wave breaking phenomena is investigated for the rotational Camassa-Holm equation.

CHAPTER 2

THE ROTATIONAL b-FAMILY OF EQUATIONS

2.1 Derivation of the Rotational b-Family of Equations Model

The derivation of b-Family of equations model with the Coriolis effect is given in this section. To establish this model equation, incompressible and inviscid with a constant density ρ and no surface tension is considered for water flow. Also the interface between the air and the water is a free surface. Then such a motion of water flow occupying a domain D_t in $\mathbb{R}^3$ under the influence of the gravity g and the Coriolis force due to the Earth's rotation can be described by the Euler equations [24], *viz.*,[37]

$$\begin{cases} \vec{u}_t + (\vec{u} \cdot \nabla)\, \vec{u} + 2\vec{\Omega} \times \vec{u} = -\frac{1}{\rho}\nabla P + \vec{g}, \quad x \in D_t, \\ \nabla \cdot \vec{u} = 0, \quad x \in D_t, \\ \vec{u}|_{t=0} = \vec{u_0}, \quad x \in D_0, \end{cases}$$

where $\vec{u} = (u, v, w)^T$ is the fluid velocity, $P(t, x, y, z)$ is the pressure in the fluid, $\vec{g} = (0, 0, -g)^T$ with $g \approx 9.8m/s^2$ the constant gravitational acceleration at the Earth's surface, and $\vec{\Omega} = (0,\, \Omega_0 \cos\phi,\, \Omega_0 \sin\phi)^T$, with the rotational frequency $\Omega_0 \approx 73 \cdot 10^{-6}$rad/s and the local latitude ϕ, is the angular velocity vector which is directed along the axis of rotation of the rotating reference frame.

The origin of the rotating reference frame is adopted at a point on the Earth's surface with the x-axis, the y-axis and z-axis respectively chosen horizontally eastward, northward and upward. Also $D_t = \{(x, y, z) : 0 < z < h_0 + \eta(t, x, y)\}$ is defined where h_0 is the typical depth of the water and $\eta(t, x, y)$ measures the deviation from the average level. Under the f-plane approximation ($\sin\phi \approx 0,\ \phi \ll 1$), the motion of inviscid irrotational fluid near the Equator in the region $0 < z < h_0 + \eta(t, x, y)$ with a constant density ρ is described by the Euler equations [12, 24] in the form

$$\begin{cases} u_t + uu_x + vu_y + wu_z + 2\Omega_0 w = -\frac{1}{\rho}P_x, \\ v_t + uv_x + vv_y + wv_z = -\frac{1}{\rho}P_y, \\ w_t + uw_x + vw_y + ww_z - 2\Omega_0 u = -\frac{1}{\rho}P_z - g, \end{cases}$$

the incompressibility of the fluid

$$u_x + v_y + w_z = 0,$$

and the irrotational condition

$$(w_y - v_z, u_z - w_x, v_x - u_y)^T = (0, 0, 0)^T.$$

The pressure is written as

$$P(t, x, z) = P_a + \rho g(h_0 - z) + p(t, x, y, z),$$

where P_a is the constant atmosphere pressure, and p is a pressure variable measuring the hydrostatic pressure distribution.

When $P = P_a$ is the dynamic condition on the surface $z = h_0 + \eta$, pressure p is

$$p = \rho g \eta.$$

Besides, the kinematic condition on the surface $z = h_0 + \eta(t, x, y)$ is given by

$$w = \eta_t + u\eta_x + v\eta_y, .$$

Lastly, considering "no-flow" condition at the flat bottom $z = 0$, that is,

$$w|_{z=0} = 0.$$

The two-dimensional flow is considered which is moving in the east-west direction along the Equator. It means that $v \equiv 0$ over the flow which is independent of the y-coordinate. Then the irrotational condition can be restated as $u_z - w_x = 0$. Moreover, the dimensionless quantities are given

$$x \mapsto \lambda x, \quad z \mapsto h_0 z, \quad \eta \mapsto a\eta, \quad t \mapsto \frac{\lambda}{\sqrt{gh_0}} t,$$

which implies

$$u \mapsto \sqrt{gh_0} u, \quad w \mapsto \sqrt{\mu g h_0} w, \quad p \mapsto \rho g h_0 p.$$

Also considering the effect of the Earth rotation, we describe

$$\Omega = \sqrt{\frac{h_0}{g}} \Omega_0.$$

additionally, as $\varepsilon \to 0$,

$$u \mapsto 0, \quad w \mapsto 0, \quad p \mapsto 0,$$

that is, u, w and p are proportional to the wave amplitude so that a scaling is required as

$$u \mapsto \varepsilon u, \quad w \mapsto \varepsilon w, \quad p \mapsto \varepsilon p.$$

As a result the governing equations turn into

$$\begin{cases} u_t + \varepsilon(uu_x + wu_z) + 2\Omega w = -p_x & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ \mu\{w_t + \varepsilon(uw_x + ww_z)\} - 2\Omega u = -p_z & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ u_x + w_z = 0 & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ u_z - \mu w_x = 0 & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ p = \eta & \text{on} \quad z = 1 + \varepsilon\eta(t,x), \\ w = \eta_t + \varepsilon u\eta_x & \text{on} \quad z = 1 + \varepsilon\eta(t,x), \\ w = 0 & \text{on} \quad z = 0. \end{cases} \tag{2.1}$$

We begin with setting up a proper scale and a double asymptotic expansion to obtain equations in groups with respect to ε and μ independent on each other, where $\varepsilon,\ \mu \ll 1$, to derive the rotational b-family of equations for shallow water waves. The proper far field variable is defined in terms of ε

$$\xi = \varepsilon^{1/2}(x - ct), \quad \tau = \varepsilon^{3/2}t, \quad w = \sqrt{\varepsilon}\, W \tag{2.2}$$

where c is the group speed of water waves [27, 28].

Applying this transformations to the governing equations (2.1), we get

$$\begin{cases} -cu_\xi + \varepsilon(u_\tau + uu_\xi + Wu_z) + 2\Omega W = -p_\xi & \text{in} \quad 0 < z < 1 + \varepsilon\eta, \\ \varepsilon\mu\{-cW_\xi + \varepsilon(W_\tau + uW_\xi + WW_z)\} - 2\Omega u = -p_z & \text{in} \quad 0 < z < 1 + \varepsilon\eta, \\ u_\xi + W_z = 0 & \text{in} \quad 0 < z < 1 + \varepsilon\eta, \\ u_z - \varepsilon\mu W_\xi = 0 & \text{in} \quad 0 < z < 1 + \varepsilon\eta, \\ p = \eta & \text{on} \quad z = 1 + \varepsilon\eta, \\ W = -c\eta_\xi + \varepsilon(\eta_\tau + u\eta_\xi) & \text{on} \quad z = 1 + \varepsilon\eta, \\ W = 0 & \text{on} \quad z = 0. \end{cases} \tag{2.3}$$

To find a solution for the system (2.3), a double asymptotic expansion is given as

$$q \sim \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \varepsilon^n \mu^m q_{nm}$$

as $\varepsilon \to 0, \mu \to 0$, where the each function q_{nm} provides the far field conditions $q_{nm} \to 0$ as $|\xi| \to \infty$ for every $n,\ m = 0, 1, 2, 3, ...$. This expansion is considered for the scale functions u, W, p, and η.

Now, every coefficients of the order $O(\varepsilon^i, \mu^j)$ $(i,\ j = 0, 1, 2, 3, ...)$ are analyzed by applying the asymptotic expansions of u, W, p, η into (2.3).

Considering the order $O(\varepsilon^0, \mu^0)$ terms of (2.3),

$$f(z) = f(1) + \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!} f^{(n)}(1) \tag{2.4}$$

is derived from the Taylor expension that

$$
\begin{cases}
-cu_{00,\xi} + 2\Omega W_{00} = -p_{00,\xi} & \text{in} \quad 0 < z < 1, \\
2\Omega u_{00} = p_{00,z} & \text{in} \quad 0 < z < 1, \\
u_{00,\xi} + W_{00,z} = 0 & \text{in} \quad 0 < z < 1, \\
u_{00,z} = 0 & \text{in} \quad 0 < z < 1, \\
p_{00} = \eta_{00} & \text{on} \quad z = 1, \\
W_{00} = -c\eta_{00,\xi} & \text{on} \quad z = 1, \\
W_{00} = 0 & \text{on} \quad z = 0,
\end{cases}
\tag{2.5}
$$

where the expression $u_{00,\xi}$ is the derivation of u_{00} with respect to ξ.

First of all, from the fourth equation of the (2.5), we see that u_{00} is independent of z, which yields $u_{00} = u_{00}(\tau, \xi)$.

Nextly, from the third equation in (2.5) and the boundary condition of W on $z = 0$, we obtain

$$
W_{00} = W_{00}|_{z=0} + \int_0^z W_{00,z'} dz' = -\int_0^z u_{00,\xi} \, dz' = -z u_{00,\xi}, \tag{2.6}
$$

Additionally, considering the boundary condition of W on $z = 1$, it gives us the following

$$
u_{00,\xi}(\tau, \xi) = c\eta_{00,\xi}(\tau, \xi). \tag{2.7}
$$

As a result, the following equations are obtained from (2.5)

$$
u_{00}(\tau, \xi) = c\eta_{00}(\tau, \xi), \quad W_{00} = -cz\eta_{00,\xi}, \tag{2.8}
$$

where use has been made of the far field conditions u_{00}, $\eta_{00} \to 0$ as $|\xi| \to \infty$.

In addition, integrating the second equation of (2.5), we get

$$
p_{00} = p_{00}|_{z=1} + \int_1^z p_{00,z'} \, dz' = \eta_{00} + 2\Omega \int_1^z u_{00} \, dz' = \eta_{00} + 2\Omega(z-1)u_{00}. \tag{2.9}
$$

Then from (2.9) and (2.7), we can have

$$p_{00,\xi} = \left(\frac{1}{c} + 2\Omega(z-1)\right) u_{00,\xi}. \tag{2.10}$$

To conclude, the first equation of (2.5),(2.10) and (2.6) provide us the following

$$(c^2 + 2\Omega c - 1)u_{00,\xi} = 0, \tag{2.11}$$

that means

$$c^2 + 2\Omega c - 1 = 0, \tag{2.12}$$

if u_{00} is assumed a non-trivial velocity. Consequently, if we suppose the waves flow to the right direction ($c > 0$),

$$c = \sqrt{1+\Omega^2} - \Omega \tag{2.13}$$

is obtained.

Following the similar way to get results for the order $O(\varepsilon^0, \mu^1)$, $O(\varepsilon^2, \mu^0)$, $O(\varepsilon^1, \mu^1)$,$O(\varepsilon^3, \mu^0)$, $O(\varepsilon^4, \mu^0)$, and $O(\varepsilon^2, \mu^1)$terms of (2.3) respectively, we have

$$u_{01} = c\eta_{01} = c\eta_{01}(\tau, \xi), \tag{2.14}$$

$$\begin{aligned} u_{20} = u_{20}(\tau,\xi) &= c\eta_{20} - 2(c+c_1)\eta_{00}\eta_{10} - \frac{2c_1 - 3\Omega}{3(c+\Omega)}(c+c_1)\eta_{00}^3, \\ u_{11} = u_{11}(\tau,\xi) &= \left(\frac{c}{6} - \frac{2c_1}{9} - \frac{c}{2}z^2\right)\eta_{00,\xi\xi} + c\eta_{11} - 2(c+c_1)\eta_{00}\eta_{01}, \end{aligned} \tag{2.15}$$

$$\begin{aligned} u_{30} = u_{30}(\tau,\xi) =& c\eta_{30} - 2(c+c_1)(\eta_{00}\eta_{20}) - (c+c_1)(\eta_{10}^2) - \frac{2c_1 - 3\Omega}{\Omega + c}(c+c_1)(\eta_{00}^2\eta_{10}) \\ & - \frac{(64cc_1 + 24c_1^2 + 45c^2 + 24\Omega^2 - 3)}{24(c+\Omega)^2}(c+c_1)(\eta_{00}^4), \end{aligned} \tag{2.16}$$

$$\begin{aligned} &2(c+\Omega)\eta_{30,\tau} + 3c^2(\eta_{00}\eta_{30} + \eta_{10}\eta_{20})_\xi - 2(3c+2c_1)(c+c_1)(\eta_{00}^2\eta_{20} + \eta_{00}\eta_{10}^2)_\xi \\ &- \frac{(64cc_1 + 24c_1^2 + 45c^2 - 15)}{3(c+\Omega)}(c+c_1)(\eta_{00}^3\eta_{10})_\xi - B_2(\eta_{00}^5)_\xi = 0, \end{aligned} \tag{2.17}$$

$$
\begin{aligned}
&2(\Omega+c)\eta_{11,\tau}+3c^2(\eta_{00}\eta_{11}+\eta_{10}\eta_{01})_\xi-2(c+c_1)(3c+2c_1)(\eta_{00}^2\eta_{01})_\xi+\frac{c^2}{3}\eta_{10,\xi\xi\xi}\\
&-\left(\frac{c^2}{6}+\frac{10cc_1}{9}+\frac{2c_1^2}{9}\right)(\eta_{00,\xi}^2)_\xi-\left(\frac{c^2}{3}+\frac{20cc_1}{9}+\frac{8c_1^2}{9}\right)(\eta_{00}\eta_{00,\xi\xi})_\xi=0,
\end{aligned}
\tag{2.18}
$$

with

$$
c_1:=-\frac{3c^2}{4(\Omega+c)}=-\frac{3c^3}{2(c^2+1)}, \tag{2.19}
$$

$$
\begin{aligned}
B_1:=&\frac{(c+c_1)^2(82cc_1+36c_1^2+45c^2-18\Omega c_1-27\Omega c-15)}{3(\Omega+c)^2}\\
&+\frac{c_1(c+c_1)(64cc_1+24c_1^2+45c^2+24\Omega^2-3)}{3(\Omega+c)^2},
\end{aligned}
$$

$$
\begin{aligned}
B_2:=&\frac{1}{5}B_1-\frac{(c+c_1)^2(2c_1-3\Omega)}{3(\Omega+c)}+\frac{2c(c+c_1)(64cc_1+24c_1^2+45c^2+24\Omega^2-3)}{12(\Omega+c)^2}\\
=&\frac{c^2(2-c^2)(3c^{10}+228c^8-540c^6-180c^4-13c^2+42)}{60(c^2+1)^6}.
\end{aligned}
$$

Details for the each order can be found in next chapter. After analyzing all orders, we consider η as the following

$$
\eta:=\eta_{00}+\varepsilon\eta_{10}+\varepsilon^2\eta_{20}+\varepsilon^3\eta_{30}+\mu\eta_{01}+\varepsilon\mu\eta_{11}+O(\varepsilon^4,\mu^2). \tag{2.20}
$$

Multiplying the equations (2.38), (2.50), (2.61), (2.71), (2.80), and (2.90) by 1, ε, μ, ε^2, ε^3, and $\varepsilon\mu$, respectively, and considering (2.20), we obtain the equation of η up to the order $O(\varepsilon^4,\mu^2)$ that

$$
\begin{aligned}
&2(\Omega+c)\eta_\tau+3c^2\eta\eta_\xi+\frac{c^2}{3}\mu\eta_{\xi\xi\xi}+\varepsilon A_1\eta^2\eta_\xi+\varepsilon^2A_2\eta^3\eta_\xi+A_0\varepsilon^3\eta^4\eta_\xi\\
&=\varepsilon\mu\Big(A_3\eta_\xi\eta_{\xi\xi}+A_4\eta\eta_{\xi\xi\xi}\Big)+O(\varepsilon^4,\mu^2),
\end{aligned}
\tag{2.21}
$$

where $c_1=-\frac{3c^3}{2(c^2+1)}$ is defined in (2.39),

$$
A_0=-5B_2:=\frac{c^2(c^2-2)(3c^{10}+228c^8-540c^6-180c^4-13c^2+42)}{12(c^2+1)^6},
$$

$$A_1 := -2(3c+2c_1)(c+c_1) = \frac{3c^2(c^2-2)}{(c^2+1)^2}$$

$$A_2 := -\frac{(64cc_1+24c_1^2+45c^2-15)}{3(c+\Omega)}(c+c_1) = -\frac{c^2(2-c^2)(c^6-7c^4+5c^2-5)}{(c^2+1)^4},$$

$$A_3 := \frac{2c^2}{3}+\frac{40cc_1}{9}+\frac{4c_1^2}{3} = \frac{-c^2(9c^4+16c^2-2)}{3(c^2+1)^2}, \quad A_4 := \frac{c^2}{3}+\frac{20cc_1}{9}+\frac{8c_1^2}{9} = \frac{-c^2(3c^4+8c^2-1)}{3(c^2+1)^2}.$$

Furthermore, we have the followings from analyzing all orders

$$u_{00} = c\eta_{00},$$

$$u_{10} = c\eta_{10} - (c_1+c)\eta_{00}^2,$$

$$u_{01} = c\eta_{01},$$

$$u_{11} = c\eta_{11} - 2(c_1+c)\eta_{00}\eta_{01} + \left(\frac{c}{6} - \frac{2c_1}{9} - \frac{cz^2}{2}\right)\eta_{00,\xi\xi},$$

$$u_{20} = c\eta_{20} - 2(c+c_1)(\eta_{00}\eta_{10}) - \frac{2c_1-3\Omega}{3(c+\Omega)}(c+c_1)(\eta_{00}^3),$$

$$u_{30} = c\eta_{30} - 2(c+c_1)(\eta_{00}\eta_{20}) - (c+c_1)(\eta_{10}^2) - \frac{2c_1-3\Omega}{\Omega+c}(c+c_1)(\eta_{00}^2\eta_{10}) - \frac{(64cc_1+24c_1^2+45c^2+24\Omega^2-3)}{24(c+\Omega)^2}(c+c_1)(\eta_{00}^4).$$

Then we obtain

$$\eta_{00} = \frac{1}{c}u_{00},\ \eta_{10} = \frac{1}{c}u_{10} + \gamma_1 u_{00}^2,\ \eta_{01} = \frac{1}{c}u_{01},\ \eta_{20} = \frac{1}{c}u_{20} + 2\gamma_1 u_{00}u_{10} + \gamma_2 u_{00}^3,$$

$$\eta_{30} = \frac{1}{c}u_{30} + \gamma_1 u_{10}^2 + 2\gamma_1 u_{00}u_{20} + 3\gamma_2 u_{00}^2 u_{10} + \gamma_3 u_{00}^4,$$

$$\eta_{11} = \frac{1}{c}u_{11} + 2\gamma_1 u_{00}u_{01} + \gamma_4 u_{00,\xi\xi},$$

where

$$\gamma_1 \stackrel{\text{def}}{=} \frac{c_1+c}{c^3},$$

$$\gamma_2 \stackrel{\text{def}}{=} \frac{2(c+c_1)^2}{c^5} + \frac{(2c_1-3\Omega)(c+c_1)}{3c^4(c+\Omega)},$$

$$\gamma_3 \stackrel{\text{def}}{=} \frac{5(c+c_1)^3}{c^7} + \frac{5(2c_1-3\Omega)(c+c_1)^2}{3c^6(c+\Omega)} + \frac{(64cc_1+24c_1^2+45c^2+24\Omega^2-3)}{24c^5(c+\Omega)^2}(c+c_1),$$

$$\gamma_4 \stackrel{\text{def}}{=} -\left(\frac{1}{6c} - \frac{2c_1}{9c^2} - \frac{z^2}{2c}\right),$$

or it is the same,

$$\begin{aligned}
&\gamma_1 = \frac{2-c^2}{2c^2(c^2+1)}, \quad \gamma_2 = \frac{(c^2-1)(c^2-2)(2c^2+1)}{2c^3(c^2+1)^3}, \\
&\gamma_3 = -\frac{(c^2-1)^2(c^2-2)(21c^4+16c^2+4)}{8c^4(c^2+1)^5}, \quad \gamma_4 = \frac{z^2}{2c} - \frac{3c^2+1}{6c(c^2+1)}.
\end{aligned} \tag{2.22}$$

Therefore, we can rewrite η with respect to u,

$$\begin{aligned}
\eta &= \eta_{00} + \varepsilon\eta_{10} + \varepsilon^2\eta_{20} + \mu\eta_{01} + \varepsilon^3\eta_{30} + \varepsilon\mu\eta_{11} + O(\varepsilon^4, \mu^2) \\
&= \frac{1}{c}u_{00} + \varepsilon\left(\frac{1}{c}u_{10} + \gamma_1 u_{00}^2\right) + \varepsilon^2\left(\frac{1}{c}u_{20} + 2\gamma_1 u_{00}u_{10} + \gamma_2 u_{00}^3\right) \\
&+ \mu\frac{1}{c}u_{01} + \varepsilon\mu\left(\frac{1}{c}u_{11} + 2\gamma_1 u_{00}u_{01} + \gamma_4 u_{00,\xi\xi}\right) \\
&+ \varepsilon^3\left(\frac{1}{c}u_{30} + \gamma_1 u_{10}^2 + 2\gamma_1 u_{00}u_{20} + 3\gamma_2 u_{00}^2 u_{10} + \gamma_3 u_{00}^4\right) + O(\varepsilon^4, \mu^2).
\end{aligned}$$

and consider

$$u = u_{00} + \varepsilon u_{10} + \varepsilon^2 u_{20} + \mu u_{01} + \varepsilon^3 u_{30} + \varepsilon\mu u_{11} + O(\varepsilon^4, \mu^2),$$

so we obtain

$$\eta = \frac{1}{c}u + \gamma_1\varepsilon u^2 + \gamma_2\varepsilon^2 u^3 + \gamma_3\varepsilon^3 u^4 + \gamma_4\varepsilon\mu u_{\xi\xi} + O(\varepsilon^4, \mu^2), \tag{2.23}$$

where γ_i $(i = 1, 2, 3, 4)$ are defined in (2.22) and the parameter $z \in [0, 1]$.

The equation (2.23) gives us a result that the free surface η and the horizontal velocity u do not have the relation with Coriolis effect. Also, it states that we can derive other water wave models ,like the classical KdV equation, the BBM equation. and the (improved) Boussinesq equation, from relation (2.23) in the KdV regime $\varepsilon = O(\mu)$. Now, we find the each term of the equation (2.21) with respect to u by using (2.23), so we have

$$\begin{aligned}
2(\Omega + c)\eta_\tau =& \frac{2(\Omega + c)}{c}u_\tau + \frac{2(\Omega + c)(c_1 + c)}{c^3}\varepsilon(u^2)_\tau + 2(\Omega + c)\gamma_2\varepsilon^2(u^3)_\tau \\
&+ 2(\Omega + c)\gamma_3\varepsilon^3(u^4)_\tau + 2(\Omega + c)\gamma_4\varepsilon\mu u_{\tau\xi\xi} + O(\varepsilon^4, \mu^2),
\end{aligned} \tag{2.24}$$

as $\varepsilon, \mu \to 0$

$$
\begin{aligned}
3c^2\eta\eta_\xi &= \frac{3c^2}{2}\left((\frac{1}{c}u + \frac{c_1+c}{c^3}\varepsilon u^2 + \gamma_2\varepsilon^2 u^3 + \gamma_3\varepsilon^3 u^4)^2 + \gamma_4\varepsilon\mu u u_{\xi\xi} \right)_\xi + O(\varepsilon^4, \mu^2) \\
&= \frac{3c^2}{2}\left(\frac{1}{c^2}u^2 + \frac{2(c_1+c)}{c^4}\varepsilon u^3 + (\frac{(c_1+c)^2}{c^6} + \frac{2}{c}\gamma_2)\varepsilon^2 u^4 + \frac{2}{c}\gamma_4\mu\varepsilon u u_{\xi\xi} \right. \\
&\qquad \left. + (\frac{2}{c}\gamma_3 + \frac{2(c_1+c)}{c^3}\gamma_2)\varepsilon^3 u^5 \right)_\xi + O(\varepsilon^4, \mu^2),
\end{aligned}
$$

$$
\begin{aligned}
\frac{c^2}{3}\mu\eta_{\xi\xi\xi} &= \frac{c^2}{3}\mu(\frac{1}{c}u + \frac{c_1+c}{c^3}\varepsilon u^2)_{\xi\xi\xi} + O(\varepsilon^4, \mu^2), \\
\varepsilon\mu\left(A_3\eta_\xi\eta_{\xi\xi} + A_4\eta\eta_{\xi\xi\xi} \right) &= \varepsilon\mu\left(\frac{A_3}{c^2}u_\xi u_{\xi\xi} + \frac{A_4}{c^2}u u_{\xi\xi\xi} \right) + O(\varepsilon^4, \mu^2), \\
A_1\varepsilon\eta^2\eta_\xi &= \frac{A_1}{3}\varepsilon\left[\frac{1}{c^3}u^3 + \frac{3(c_1+c)}{c^5}\varepsilon u^4 + (\frac{3(c_1+c)^2}{c^7} + \frac{3}{c^2}\gamma_2)\varepsilon^2 u^5 \right]_\xi + O(\varepsilon^4, \mu^2), \\
A_2\varepsilon^2\eta^3\eta_\xi &= \frac{A_2}{4c^4}\varepsilon^2(u^4)_\xi + \frac{A_2(c_1+c)}{c^6}\varepsilon^3(u^5)_\xi + O(\varepsilon^4, \mu^2),
\end{aligned}
$$

and

$$
-5B_2\varepsilon^3\eta^4\eta_\xi = -\frac{B_2}{c^5}\varepsilon^3(u^5)_\xi + O(\varepsilon^4, \mu^2).
$$

Therefore, the equation (2.21) turns out

$$
\begin{aligned}
u_\tau &+ \frac{2(c_1+c)}{c^2}\varepsilon u u_\tau + 3\gamma_2 c\varepsilon^2 u^2 u_\tau + \gamma_4 c\varepsilon\mu u_{\tau\xi\xi} + 4\gamma_3 c\varepsilon^3 u^3 u_\tau + \frac{3c}{2(\Omega+c)}u u_\xi \\
&+ \frac{cA_5}{2(\Omega+c)}\varepsilon^2 u^3 u_\xi + \frac{cA_6}{2(\Omega+c)}\varepsilon u^2 u_\xi + \frac{c^2}{6(\Omega+c)}\mu u_{\xi\xi\xi} + \frac{cA_7}{2(\Omega+c)}\varepsilon^3 u^4 u_\xi \\
&+ (\frac{cA_8}{2(\Omega+c)}u_\xi u_{\xi\xi} + \frac{cA_9}{2(\Omega+c)}u u_{\xi\xi\xi})\varepsilon\mu = O(\varepsilon^4, \varepsilon^2\mu, \mu^2),
\end{aligned}
\tag{2.25}
$$

where

$$
\begin{aligned}
A_5 &:= \frac{6(c_1+c)^2}{c^4} + 12c\gamma_2 + \frac{4A_1(c_1+c)}{c^5}\varepsilon^2 + \frac{A_2}{c^4}, \\
A_6 &:= \frac{9(c_1+c)}{c^2} + \frac{A_1}{c^3}, \\
A_8 &:= 3c\gamma_4 + \frac{2(c_1+c)}{c} - \frac{A_3}{c^2}, \\
A_9 &:= 3c\gamma_4 + \frac{2(c_1+c)}{3c} - \frac{A_4}{c^2},
\end{aligned}
$$

$$A_7 := 5\Big[\frac{3}{2}c^2(\frac{2}{c}\gamma_3 + \frac{2(c_1+c)}{c^3}\gamma_2) + \frac{A_1}{3}(\frac{3}{c^7}(c_1+c)^2 + \frac{3}{c^2}\gamma_2) + \frac{A_2(c_1+c)}{c^6} - \frac{B_2}{c^5}\Big].$$

Multiplying the equation (2.25) by εu, we have

$$\begin{aligned}\varepsilon u u_\tau \; &= -\varepsilon u\Big(\frac{2(c_1+c)}{c^2}\varepsilon u u_\tau + 3\gamma_2 c\varepsilon^2 u^2 u_\tau + \frac{3c}{2(\Omega+c)}uu_\xi + \frac{cA_5}{2(\Omega+c)}\varepsilon^2 u^3 u_\xi \\ &\quad + \frac{cA_6}{2(\Omega+c)}\varepsilon u^2 u_\xi + \frac{c^2}{6(\Omega+c)}\mu u_{\xi\xi\xi}\Big) + O(\varepsilon^4, \varepsilon^2\mu, \mu^2),\end{aligned}$$

which implies

$$\begin{aligned}\varepsilon u\Big(1 + \frac{2(c_1+c)}{c^2}\varepsilon u \; + 3\gamma_2 c\varepsilon^2 u^2\Big)u_\tau &= -\varepsilon u\Big(\frac{3c}{2(\Omega+c)}uu_\xi + \frac{cA_5}{2(\Omega+c)}\varepsilon^2 u^3 u_\xi \\ &\quad + \frac{cA_6}{2(\Omega+c)}\varepsilon u^2 u_\xi + \frac{c^2}{6(\Omega+c)}\mu u_{\xi\xi\xi}\Big) + O(\varepsilon^4, \varepsilon^2\mu, \mu^2).\end{aligned}$$

From the last equation, we obtain

$$\begin{aligned}\varepsilon u u_\tau \; &= -\varepsilon u\Big[1 - (\frac{2(c_1+c)}{c^2}\varepsilon u + 3\gamma_2 c\varepsilon^2 u^2) + (\frac{2(c_1+c)}{c^2}\varepsilon u)^2\Big]\Big[\frac{3c}{2(\Omega+c)}uu_\xi \\ &\quad + \frac{cA_5}{2(\Omega+c)}\varepsilon^2 u^3 u_\xi + \frac{cA_6}{2(\Omega+c)}\varepsilon u^2 u_\xi + \frac{c^2}{6(\Omega+c)}\mu u_{\xi\xi\xi}\Big] + O(\varepsilon^4, \mu^2),\end{aligned}$$

and then

$$\begin{aligned}\varepsilon u u_\tau \; &= -\varepsilon u\Big[\frac{3c}{2(\Omega+c)}uu_\xi + \frac{c^2}{6(\Omega+c)}\mu u_{\xi\xi\xi} + \frac{c^2A_6 - 6(c_1+c)}{2c(\Omega+c)}\varepsilon u^2 u_\xi \\ &\quad + \frac{c^2A_5 - 2A_6(c_1+c) + 3c^2(\frac{4(c_1+c)^2}{c^4} - 3\gamma_2 c)}{2c(\Omega+c)}\varepsilon^2 u^3 u_\xi\Big] + O(\varepsilon^4, \mu^2),\end{aligned} \tag{2.26}$$

$$\begin{aligned}\varepsilon^2 u^2 u_\tau &= -\varepsilon^2 u^2\Big[\frac{3c}{2(\Omega+c)}uu_\xi + \frac{c^2A_6 - 6(c_1+c)}{2c(\Omega+c)}\varepsilon u^2 u_\xi\Big] + O(\varepsilon^4, \varepsilon^2\mu, \mu^2),\\ \varepsilon^3 u^3 u_\tau &= -\frac{3c}{2(\Omega+c)}\varepsilon^3 u^4 u_\xi + O(\varepsilon^4, \mu^2), \quad \varepsilon\mu u_{\tau\xi\xi} = -\frac{3c}{2(\Omega+c)}\varepsilon\mu(uu_\xi)_{\xi\xi} + O(\varepsilon^4, \mu^2)\end{aligned} \tag{2.27}$$

In (2.27), we use the decompositon of the term $\varepsilon\mu u_{\tau\xi\xi}$, which is $\varepsilon\mu(1-\nu)u_{\tau\xi\xi} + \varepsilon\mu\nu u_{\tau\xi\xi}$ for some constant ν (to be determined later), as follows

$$\varepsilon\mu u_{\tau\xi\xi} = \varepsilon\mu(1-\nu)u_{\tau\xi\xi} - \frac{3c\nu}{2(\Omega+c)}\varepsilon\mu(uu_\xi)_{\xi\xi} + O(\varepsilon^4, \mu^2). \tag{2.28}$$

Considering (2.26)-(2.28), we can rewrite (2.25)

$$\begin{aligned} u_\tau \; &+ c\gamma_4(1-\nu)\mu\varepsilon u_{\tau\xi\xi} + \frac{3c}{2(\Omega+c)}uu_\xi + \frac{c^2}{6(\Omega+c)}\mu u_{\xi\xi\xi} - \frac{9c^2\gamma_2}{2(\Omega+c)}\varepsilon^2 u^3 u_\xi \\ &- \frac{3c^2\gamma_4\nu}{2(\Omega+c)}\mu\varepsilon(uu_\xi)_{\xi\xi} + \frac{2(c_1+c)}{c^2}\varepsilon\Big[\frac{3c}{2(\Omega+c)}u^2u_\xi + \frac{c^2}{6(\Omega+c)}\mu uu_{\xi\xi\xi} \\ &+ \frac{c^2A_6 - 6(c_1+c)}{2c(\Omega+c)}\varepsilon u^3 u_\xi\Big] + \frac{cA_5}{2(\Omega+c)}\varepsilon^2 u^3 u_\xi + \frac{cA_6}{2(\Omega+c)}\varepsilon u^2 u_\xi \\ &+ \mu\varepsilon\Big(\frac{cA_8}{2(\Omega+c)}u_\xi u_{\xi\xi} + \frac{cA_9}{2(\Omega+c)}uu_{\xi\xi\xi}\Big) + A_{10}\varepsilon^3u^4u_\xi = O(\varepsilon^4,\mu^2), \end{aligned}$$

where

$$\begin{aligned} A_{10} := \frac{cA_7}{2(\Omega+c)} &- \frac{(c_1+c)\Big(c^2A_5 - 2A_6(c_1+c) + 3c^2\big(\frac{4(c_1+c)^2}{c^4} - 3\gamma_2 c\big)\Big)}{c^3(\Omega+c)} \\ &- \frac{3\gamma_2(c^2A_6 - 6(c_1+c)) + 12c^2\gamma_3}{2(\Omega+c)}, \end{aligned}$$

which implies

$$\begin{aligned} u_\tau \; &+ \frac{3c^2}{c^2+1}uu_\xi + \frac{c^3}{3(c^2+1)}\mu u_{\xi\xi\xi} + c\gamma_4(1-\nu)\mu\varepsilon u_{\tau\xi\xi} + A_{11}\varepsilon u^2 u_\xi \\ &+ A_{12}\varepsilon^2u^3u_\xi + A_{10}\varepsilon^3u^4u_\xi + \mu\varepsilon\Big[A_{13}uu_{\xi\xi\xi} + A_{14}u_\xi u_{\xi\xi}\Big] = O(\varepsilon^4,\varepsilon^2\mu,\mu^2). \end{aligned} \tag{2.29}$$

where

$$A_{11} := \frac{c^2A_6 - 6(c_1+c)}{2c(\Omega+c)} = \frac{-3c(c^2-1)(c^2-2)}{2(c^2+1)^3},$$

$$A_{12} := \frac{cA_5}{2(\Omega+c)} - \frac{9c^2\gamma_2}{2(\Omega+c)} - \frac{2(c_1+c)}{c^2}\frac{c^2A_6 - 6(c_1+c)}{2c(\Omega+c)} = \frac{(c^2-1)^2(c^2-2)(8c^2-1)}{2(c^2+1)^5},$$

$$A_{13} := \frac{cA_9}{2(\Omega+c)} - \frac{3c^2\gamma_4\nu}{2(\Omega+c)} - \frac{c_1+c}{3(\Omega+c)} = \frac{3c^3\gamma_4}{(c^2+1)}(1-\nu) + \frac{c^2(3c^4+8c^2-1)}{3(c^2+1)^3},$$

$$A_{14} := \frac{cA_8}{2(\Omega+c)} - \frac{9c^2\gamma_4\nu}{2(\Omega+c)} = \frac{3c^3}{(c^2+1)}\gamma_4(1-3\nu) + \frac{c^2(6c^4+19c^2+4)}{3(c^2+1)^3}.$$

Now, we need to use the transformation $x = \varepsilon^{-\frac{1}{2}}\xi + c\varepsilon^{-\frac{3}{2}}\tau, \quad t = \varepsilon^{-\frac{3}{2}}\tau$ to go back to the original variables, and have

$$\frac{\partial}{\partial\xi} = \varepsilon^{-\frac{1}{2}}\partial_x, \quad \frac{\partial}{\partial\tau} = \varepsilon^{-\frac{3}{2}}(c\partial_x + \partial_t).$$

Applying this transformation to the equation (2.29), we obtain

$$u_t + cu_x + \frac{3c^2}{c^2+1}\varepsilon uu_x + A_{11}\varepsilon^2 u^2 u_x + A_{12}\varepsilon^3 u^3 u_x + c\gamma_4(1-\nu)\mu u_{txx}$$
$$+ \Big(\frac{c^3}{3(c^2+1)} - c^2\gamma_4(1-\nu)\Big)\mu u_{xxx} + \mu\varepsilon\Big(A_{13}uu_{xxx} + A_{14}u_x u_{xx}\Big) = O(\varepsilon^4, \mu^2).$$

In order to get the rotational b-family of equations, we need

$$A_{14} = bA_{13} = \frac{3bc^2}{(b+1)(c^2+1)}c\gamma_4(1-\nu)$$

which turns out

$$\frac{bc^3}{(c^2+1)}\gamma_4 = \frac{-c^2(3c^4+(8-b)c^2-(2b+1))}{6(c^2+1)^3} \tag{2.30}$$

and then

$$\frac{3bc^2}{(b+1)(c^2+1)}c\gamma_4(1-\nu) = bA_{13} = A_{14} = \frac{-c^2(3c^4+8c^2-1)}{3(c^2+1)^3}.$$

Therefore, it enables us to derive the rotational b-family of equations in the form

$$u_t - \beta\mu u_{xxt} + cu_x + (b+1)\alpha\varepsilon uu_x - \beta_0\mu u_{xxx} + \omega_1\varepsilon^2 u^2 u_x + \omega_2\varepsilon^3 u^3 u_x$$
$$= \alpha\beta\varepsilon\mu(bu_x u_{xx} + uu_{xxx}).$$

Combining (2.30) and (2.22), it is found that the height parameter z in γ_4 may take the value

$$z_0 = \left(\frac{b-1}{b} - \frac{2}{3}\frac{1}{(c^2+1)} + \frac{(b-2)c^2+2b+4}{3b(c^2+1)^2}\right)^{1/2}. \tag{2.31}$$

2.2 Details for Derivation of The Rotational b-Family of Equations

In this section the details for the asymptotic expansions of u, W, p, η are given considering the vanishing orders $O(\varepsilon^1, \mu^0)$, $O(\varepsilon^0, \mu^1)$, $O(\varepsilon^2, \mu^0)$, $O(\varepsilon^1, \mu^1)$, $O(\varepsilon^3, \mu^0)$, $O(\varepsilon^4, \mu^0)$, $O(\varepsilon^2, \mu^1)$ and $O(\varepsilon^4, \mu^2)$.

Considering the order $O(\varepsilon^1, \mu^0)$ terms of (2.3), we obtain from the second equation in (2.33) and the Taylor expansion

$$f(z) = f(1) + \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!} f^{(n)}(1) \tag{2.32}$$

that

$$\begin{cases} -cu_{10,\xi} + u_{00,\tau} + u_{00}u_{00,\xi} + 2\Omega W_{10} = -p_{10,\xi} & \text{in} \quad 0 < z < 1, \\ 2\Omega u_{10} = p_{10,z} & \text{in} \quad 0 < z < 1, \\ u_{10,\xi} + W_{10,z} = 0 & \text{in} \quad 0 < z < 1, \\ u_{10,z} = 0 & \text{in} \quad 0 < z < 1, \\ p_{10} + p_{00,z}\eta_{00} = \eta_{10} & \text{on} \quad z = 1, \\ W_{10} + \eta_{00}W_{00,z} = -c\eta_{10,\xi} + \eta_{00,\tau} + u_{00}\eta_{00,\xi} & \text{on} \quad z = 1, \\ W_{10} = 0 & \text{on} \quad z = 0. \end{cases} \tag{2.33}$$

Similar to the order $O(\varepsilon^0, \mu^0)$, we have u_{10} as a function independent of z, which is $u_{10} = u_{10}(\tau, \xi)$. Also, from the third equation in (2.33) and the boundary conditions of W on $z = 0$ and $z = 1$, we obtain

$$W_{10} = W_{10}|_{z=0} + \int_0^z W_{10,z'} dz' = -zu_{10,\xi} \tag{2.34}$$

and

$$W_{10}|_{z=1} = -c\eta_{10,\xi} + \eta_{00,\tau} + (u_{00}\eta_{00})_\xi. \tag{2.35}$$

Considering the third equation in (2.5) and (2.8), (2.35) turns out

$$u_{10,\xi} = c\eta_{10,\xi} - \eta_{00,\tau} - (u_{00}\eta_{00})_\xi, \tag{2.36}$$

and then

$$W_{10} = z(\eta_{00,\tau} + 2c\eta_{00}\eta_{00,\xi} - c\eta_{10,\xi}).$$

To solve the second equation in (2.33), we consider (2.8) and fourth equation in (2.33). Then, it follows that

$$p_{10} = p_{10}|_{z=1} + \int_1^z p_{10,z'} dz' = \eta_{10} - 2\Omega u_{00}\eta_{00} + 2\Omega(z-1)u_{10},$$

and

$$p_{10,\xi} = \eta_{10,\xi} - 2\Omega(u_{00}\eta_{00})_\xi + 2\Omega(z-1)u_{10,\xi}. \tag{2.37}$$

Now, from the first equation in (2.33), (2.34), and (2.8) we get

$$-p_{10,\xi} = -cu_{10,\xi} + c\eta_{00,\tau} + c^2\eta_{00}\eta_{00,\xi} - 2\Omega z u_{10,\xi},$$

and then applying (2.37) and (2.36) to the this equation, we have

$$\begin{aligned} 0 =& -(c+2\Omega)u_{10,\xi} + \eta_{10,\xi} + c\eta_{00,\tau} + c^2\eta_{00}\eta_{00,\xi} - 2\Omega(u_{00}\eta_{00})_\xi \\ =& c(u_{00}\eta_{00})_\xi - (c^2 + 2\Omega c - 1)\eta_{10,\xi} + 2(c+\Omega)\eta_{00,\tau} + c^2\eta_{00}\eta_{00,\xi}. \end{aligned}$$

Since we have (2.8) and (2.12), the equation (2.2) implies

$$2(\Omega + c)\eta_{00,\tau} + 3c^2\eta_{00}\eta_{00,\xi} = 0. \tag{2.38}$$

Let define

$$c_1 := -\frac{3c^2}{4(\Omega + c)} = -\frac{3c^3}{2(c^2+1)}. \tag{2.39}$$

Then (2.38) can be written as

$$\eta_{00,\tau} = c_1(\eta_{00}^2)_\xi. \tag{2.40}$$

If we put (2.36) and (2.40) together, we have

$$u_{10,\xi} = \big(c\eta_{10} - (c + c_1)\eta_{00}^2\big)_\xi. \tag{2.41}$$

As a result, from the far field conditions u_{10}, $\eta_{00}, \eta_{10} \to 0$ as $|\xi| \to \infty$, the following equation is obtained

$$u_{10} = c\eta_{10} - (c + c_1)\eta_{00}^2. \tag{2.42}$$

Also, thanks to (2.40), we have

$$u_{10,\tau} = c\eta_{10,\tau} - 4(c + c_1)c_1\eta_{00}^2\eta_{00,\xi}. \tag{2.43}$$

For the order $O(\varepsilon^0, \mu^1)$, the terms of (2.3) are obtained from the second equation in (2.5) and the Taylor expansion (2.32) that

$$\begin{cases} -cu_{01,\xi} + 2\Omega W_{01} = -p_{01,\xi} & \text{in} \quad 0 < z < 1, \\ 2\Omega u_{01} = p_{01,z} & \text{in} \quad 0 < z < 1, \\ u_{01,\xi} + W_{01,z} = 0 & \text{in} \quad 0 < z < 1, \\ u_{01,z} = 0 & \text{in} \quad 0 < z < 1, \\ p_{01} = \eta_{01} & \text{on} \quad z = 1, \\ W_{01} = -c\eta_{01,\xi} & \text{on} \quad z = 1, \\ W_{01} = 0 & \text{on} \quad z = 0. \end{cases} \tag{2.44}$$

The following results may be easily obtained from the (2.44) using similar methods from the previous orders

$$u_{01} = c\eta_{01} = c\eta_{01}(\tau, \xi),\ W_{01} = -cz\eta_{01,\xi},\ p_{01} = [2\Omega c(z-1) + 1]\eta_{01}. \tag{2.45}$$

For the order $O(\varepsilon^2, \mu^0)$, the terms of (2.3) are obtained from the Taylor expansion (2.32) that

$$\begin{cases} -cu_{20,\xi} + u_{10,\tau} + (u_{00}u_{10})_\xi + 2\Omega W_{20} = -p_{20,\xi} & \text{in} \quad 0 < z < 1, \\ -2\Omega u_{20} = -p_{20,z} & \text{in} \quad 0 < z < 1, \\ u_{20,\xi} + W_{20,z} = 0 & \text{in} \quad 0 < z < 1, \\ u_{20,z} = 0 & \text{in} \quad 0 < z < 1, \\ p_{20} + \eta_{00}p_{10,z} + \eta_{10}p_{00,z} = \eta_{20} & \text{on} \quad z = 1, \\ W_{20} + \eta_{00}W_{10,z} + \eta_{10}W_{00,z} & \\ \qquad = -c\eta_{20,\xi} + \eta_{10,\tau} + u_{00}\eta_{10,\xi} + u_{10}\eta_{00,\xi} & \text{on} \quad z = 1, \\ W_{20} = 0 & \text{on} \quad z = 0. \end{cases} \tag{2.46}$$

Similar to the previous orders, u_{20} is independent of z, which is $u_{20} = u_{20}(\tau, \xi)$. Also, from the third equation in (2.46) and the boundary condition of W_{20} at $z = 0$, we obtain

$$W_{20} = -zu_{20,\xi}. \tag{2.47}$$

Considering the boundary condition of W_{20} at $z = 1$ and (2.47) with the equations of $W_{00,z}$ and $W_{10,z}$, we have

$$u_{20,\xi} = c\eta_{20,\xi} - \eta_{10,\tau} - (u_{00}\eta_{10} + u_{10}\eta_{00})_\xi,$$

which follows from (2.42) and (2.8)

$$u_{20,\xi} = c\eta_{20,\xi} - \eta_{10,\tau} - 2c(\eta_{00}\eta_{10})_\xi + (c + c_1)(\eta_{00}^3)_\xi. \tag{2.48}$$

Moreover, integrating the second equation in (2.46) and applying the boundary condition of p_{20} at $z = 1$, we obtain

$$\begin{aligned} p_{20} = p_{20}|_{z=1} + \int_1^z p_{20,z'}\, dz' &= \eta_{20} - (\eta_{00}p_{10,z} + \eta_{10}p_{00,z}) + 2\Omega \int_1^z u_{20}\, dz' \\ &= \eta_{20} - 2\Omega(\eta_{00}u_{10} + \eta_{10}u_{00}) + 2\Omega(z - 1)u_{20}. \end{aligned}$$

As taking derivative of the equation with respect to ξ, it follows that

$$p_{20,\xi} = \eta_{20,\xi} - 2\Omega(\eta_{00}u_{10} + \eta_{10}u_{00})_\xi + 2\Omega(z-1)u_{20,\xi}, \tag{2.49}$$

and combining with the first equation in (2.46), we have

$$\eta_{20,\xi} - 2\Omega(\eta_{00}u_{10} + \eta_{10}u_{00})_\xi - (c+2\Omega)u_{20,\xi} + u_{10,\tau} + (u_{00}u_{10})_\xi = 0.$$

From (2.7), (2.42), and (2.43), we get

$$2(c+\Omega)\eta_{10,\tau} + 3c^2(\eta_{00}\eta_{10})_\xi - (2c + \frac{4}{3}c_1)(c+c_1)(\eta_{00}^3)_\xi = 0. \tag{2.50}$$

It implies that

$$\eta_{10,\tau} = 2c_1(\eta_{00}\eta_{10})_\xi + \frac{2c_1+3c}{3(c+\Omega)}(c+c_1)(\eta_{00}^3)_\xi. \tag{2.51}$$

Then, the equation (2.48) turns out

$$u_{20,\xi} = c\eta_{20,\xi} - 2(c+c_1)(\eta_{00}\eta_{10})_\xi - \frac{2c_1 - 3\Omega}{3(c+\Omega)}(c+c_1)(\eta_{00}^3)_\xi.$$

As a result, from the far field conditions η_{00}, η_{10}, $\eta_{20} \to 0$ as $|\xi| \to \infty$, the following equation is obtained

$$u_{20} = c\eta_{20} - 2(c+c_1)\eta_{00}\eta_{10} - \frac{2c_1 - 3\Omega}{3(c+\Omega)}(c+c_1)\eta_{00}^3. \tag{2.52}$$

Also, from (2.40) and (2.51), we have

$$u_{20,\tau} = c\eta_{20,\tau} - 4(c+c_1)c_1(\eta_{00}^2\eta_{10})_\xi - \frac{8cc_1 + 4c_1^2 + \frac{21}{4}c^2}{2(c+\Omega)}(c+c_1)(\eta_{00}^4)_\xi. \tag{2.53}$$

For the order $O(\varepsilon^1, \mu^1)$, the terms of (2.3) are obtained from the Taylor expansion (2.32) that

$$\begin{cases} -cu_{11,\xi} + u_{01,\tau} + u_{00}u_{01,\xi} + u_{10}u_{00,\xi} + W_{00}u_{01,z} \\ \qquad\qquad + W_{10}u_{00,z} + 2\Omega W_{11} = -p_{11,\xi} & \text{in} \quad 0 < z < 1, \\ -cW_{00,\xi} - 2\Omega u_{11} = -p_{11,z} & \text{in} \quad 0 < z < 1, \\ u_{11,\xi} + W_{11,z} = 0 & \text{in} \quad 0 < z < 1, \\ u_{11,z} - W_{00,\xi} = 0 & \text{in} \quad 0 < z < 1, \\ p_{11} = \eta_{11} - (\eta_{00}p_{01,z} + \eta_{01}p_{00,z}) & \text{on} \quad z = 1, \\ W_{11} + W_{00,z}\eta_{01} + W_{01,z}\eta_{00} \\ \qquad = -c\eta_{11,\xi} + \eta_{01,\tau} + u_{00}\eta_{01,\xi} + u_{01}\eta_{00,\xi} & \text{on} \quad z = 1, \\ W_{11} = 0 & \text{on} \quad z = 0. \end{cases} \tag{2.54}$$

From (2.8) and the fourth equation of (2.54), we get

$$u_{11,z} = -cz\eta_{00,\xi\xi}.$$

Then we have

$$u_{11} = -\frac{c}{2}z^2\eta_{00,\xi\xi} + \Phi_{11}(\tau, \xi) \tag{2.55}$$

for any smooth function $\Phi_{11}(\tau, \xi)$ which is independent of z. Applying integration to the third equation in (2.54) with the boundary condition $W_{11}|_{z=0} = 0$, the following equation is obtained

$$W_{11} = W_{11}|_{z=0} + \int_0^z W_{11,z'}\, dz' = \frac{c}{6}z^3\eta_{00,\xi\xi\xi} - z\partial_\xi\Phi_{11}(\tau, \xi). \tag{2.56}$$

Thanks to the equations of $W_{00,z}$ and $W_{01,z}$, and the boundary condition of W_{11} on $\{z = 1\}$, we have

$$-\partial_\xi\Phi_{11}(\tau, \xi) = -\frac{c}{6}\eta_{00,\xi\xi\xi} + (u_{00}\eta_{01} + \eta_{00}u_{01})_\xi - c\eta_{11,\xi} + \eta_{01,\tau}. \tag{2.57}$$

Taking account of (2.56) and (2.57), it follows

$$W_{11} = \frac{c}{6} z(z^2-1)\eta_{00,\xi\xi\xi} + z\Big(- c\eta_{11,\xi} + \eta_{01,\tau} + (u_{00}\eta_{01} + \eta_{00}u_{01})_\xi \Big). \tag{2.58}$$

Due to (2.8), (2.45), (2.55), and the boundary condition of p_{11} in (2.54), we deduce from the second equation of (2.54) that

$$\begin{aligned} p_{11} &= p_{11}|_{z=1} + \int_1^z p_{11,z'}\, dz' = p_{11}|_{z=1} + \int_1^z (cW_{00,\xi} + 2\Omega u_{11})\, dz' \\ &= \eta_{11} - 2\Omega(u_{00}\eta_{01} + \eta_{00}u_{01}) - \Big(\frac{c^2}{2}(z^2-1) + \frac{\Omega c}{3}(z^3-1)\Big)\eta_{00,\xi\xi} + 2\Omega(z-1)\Phi_{11}, \end{aligned}$$

which implies

$$\begin{aligned} p_{11,\xi} = \eta_{11,\xi} - 2\Omega(u_{00}\eta_{01} + \eta_{00}u_{01})_\xi - \Big(\frac{c^2}{2}(z^2-1) + \frac{\Omega c}{3}(z^3-1)\Big)\eta_{00,\xi\xi\xi} \\ + 2\Omega(z-1)\partial_\xi\Phi_{11}. \end{aligned} \tag{2.59}$$

Combining (2.59) and the first equation in (2.54), it follows from (2.8), (2.45), and (2.55) that

$$\begin{aligned} & - cu_{11,\xi} + c\eta_{01,\tau} + c^2(\eta_{00}\eta_{01})_\xi + 2\Omega W_{11} + \eta_{11,\xi} - 4\Omega c(\eta_{00}\eta_{01})_\xi \\ & - \Big(\frac{c^2}{2}(z^2-1) + \frac{\Omega c}{3}(z^3-1)\Big)\eta_{00,\xi\xi\xi} + 2\Omega(z-1)\partial_\xi\Phi_{11} = 0. \end{aligned} \tag{2.60}$$

Substituting (2.55) and (2.57) into (2.60), we obtain

$$2(\Omega + c)\eta_{01,\tau} + 3c^2(\eta_{00}\eta_{01})_\xi + \frac{c^2}{3}\eta_{00,\xi\xi\xi} = 0, \tag{2.61}$$

that is,

$$\eta_{01,\tau} = 2c_1(\eta_{00}\eta_{01})_\xi + \frac{2c_1}{9}\eta_{00,\xi\xi\xi}, \tag{2.62}$$

which, together with (2.57), (2.58), and (2.55), leads to

$$-\partial_\xi\Phi_{11}(\tau,\xi) = (\frac{2c_1}{9} - \frac{c}{6})\eta_{00,\xi\xi\xi} + 2(c+c_1)(\eta_{00}\eta_{01})_\xi - c\eta_{11,\xi},$$

and then

$$W_{11} = \Big(\frac{2c_1}{9} + \frac{c}{6}(z^2-1)\Big)\, z\,\eta_{00,\xi\xi\xi} + 2(c+c_1)\, z\,(\eta_{00}\eta_{01})_\xi - c\, z\,\eta_{11,\xi}$$

and

$$u_{11} = \left(\frac{c}{6} - \frac{2c_1}{9} - \frac{c}{2}z^2\right)\eta_{00,\xi\xi} + c\eta_{11} - 2(c+c_1)\eta_{00}\eta_{01}, \tag{2.63}$$

where use has been made by the far field conditions u_{11}, $\eta_{00,\xi\xi}$, η_{00}, η_{01}, $\eta_{11} \to 0$ as $|\xi| \to \infty$.

Thanks to (2.40) and (2.62), we obtain

$$\begin{aligned} u_{11,\tau} =& c\eta_{11,\tau} + \left(\frac{cc_1}{6} - \frac{2c_1^2}{9} - \frac{cc_1}{2}z^2\right)(\eta_{00}^2)_{\xi\xi\xi} \\ &- 2(c+c_1)\left(2c_1(\eta_{00}^2\eta_{01})_\xi + \frac{2c_1}{9}\eta_{00}\eta_{00,\xi\xi\xi}\right). \end{aligned} \tag{2.64}$$

For the order $O(\varepsilon^3, \mu^0)$, the terms of (2.3) are obtained from the Taylor expansion (2.32) that

$$\begin{cases} -cu_{30,\xi} + u_{20,\tau} + (u_{00}u_{20} + \frac{1}{2}u_{10}^2)_\xi + 2\Omega W_{30} = -p_{30,\xi} & \text{in} \quad 0<z<1, \\ -2\Omega u_{30} = -p_{30,z} & \text{in} \quad 0<z<1, \\ u_{30,\xi} + W_{30,z} = 0 & \text{in} \quad 0<z<1, \\ u_{30,z} = 0 & \text{in} \quad 0<z<1, \\ p_{30} + \eta_{00}p_{20,z} + \eta_{10}p_{10,z} + \eta_{20}p_{00,z} = \eta_{30} & \text{on} \quad z=1, \\ W_{30} + \eta_{00}W_{20,z} + \eta_{10}W_{10,z} + \eta_{20}W_{00,z} & \\ \qquad = -c\eta_{30,\xi} + \eta_{20,\tau} + u_{00}\eta_{20,\xi} + u_{10}\eta_{10,\xi} + u_{20}\eta_{00,\xi} & \text{on} \quad z=1, \\ W_{30} = 0 & \text{on} \quad z=0. \end{cases} \tag{2.65}$$

Similarly, it follows that u_{30} is independent of z,which is $u_{30} = u_{30}(\tau,\xi)$ from the fourth equation in (2.65). Also, from the third equation in (2.65) and the boundary condition of W_{30} at $z=0$, we obtain

$$W_{30} = -zu_{30,\xi}. \tag{2.66}$$

Considering the boundary condition of W_{20} at $z=1$ and (2.47), we have

$$u_{30,\xi} = c\eta_{30,\xi} - \eta_{20,\tau} - (u_{00}\eta_{20} + u_{10}\eta_{10} + u_{20}\eta_{00})_\xi. \tag{2.67}$$

In addition, integrating the second equation in (2.65) and applying the boundary condition of p_{30} at $z = 1$, we have

$$\begin{aligned}
p_{30} &= p_{30}|_{z=1} + \int_1^z p_{30,z'}\, dz' \\
&= \eta_{30} - (\eta_{00}p_{20,z} + \eta_{10}p_{10,z} + \eta_{20}p_{00,z}) + 2\Omega \int_1^z u_{30}\, dz' \\
&= \eta_{30} - 2\Omega(u_{00}\eta_{20} + u_{10}\eta_{10} + u_{20}\eta_{00}) + 2\Omega(z-1)u_{30}.
\end{aligned}$$

As taking derivative of the equation with respect to ξ, it follows that

$$p_{30,\xi} = \eta_{30,\xi} - 2\Omega(u_{00}\eta_{20} + u_{10}\eta_{10} + u_{20}\eta_{00})_\xi + 2\Omega(z-1)u_{30,\xi}, \tag{2.68}$$

and from the first equation in (2.65), we get

$$-p_{30,\xi} = -cu_{30,\xi} + u_{20,\tau} + (u_{00}u_{20} + \frac{1}{2}u_{10}^2)_\xi - 2\Omega z u_{30,\xi}. \tag{2.69}$$

Therefore, from (2.68) and (2.69), it follows that

$$0 = \eta_{30,\xi} - 2\Omega(u_{00}\eta_{20} + u_{10}\eta_{10} + u_{20}\eta_{00})_\xi - (c+2\Omega)u_{30,\xi} + u_{20,\tau} + (u_{00}u_{20} + \frac{1}{2}u_{10}^2)_\xi. \tag{2.70}$$

Applying (2.67) and (2.53) to (2.70), we have

$$\begin{aligned}
2(c+\Omega)\eta_{20,\tau} + 3c^2(\eta_{00}\eta_{20})_\xi + \frac{3c^2}{2}(\eta_{10}^2)_\xi - 2(2c_1+3c)(c+c_1)(\eta_{00}^2\eta_{10})_\xi \\
- \frac{(64cc_1 + 24c_1^2 + 45c^2 - 15)}{12(c+\Omega)}(c+c_1)(\eta_{00}^4)_\xi = 0,
\end{aligned} \tag{2.71}$$

which is equivalent to

$$\begin{aligned}
\eta_{20,\tau} = 2c_1(\eta_{00}\eta_{20})_\xi + c_1(\eta_{10}^2)_\xi + \frac{2c_1+3c}{\Omega+c}(c+c_1)(\eta_{00}^2\eta_{10})_\xi \\
+ \frac{(64cc_1 + 24c_1^2 + 45c^2 - 15)}{24(c+\Omega)^2}(c+c_1)(\eta_{00}^4)_\xi.
\end{aligned} \tag{2.72}$$

Then substituting (2.72) into (2.67), we obtain Thanks to (2.67) again, we have

$$\begin{aligned}
u_{30,\xi} =& c\eta_{30,\xi} - 2(c+c_1)(\eta_{00}\eta_{20})_\xi - (c+c_1)(\eta_{10}^2)_\xi - \frac{2c_1 - 3\Omega}{\Omega+c}(c+c_1)(\eta_{00}^2\eta_{10})_\xi \\
&- \frac{(64cc_1 + 24c_1^2 + 45c^2 + 24\Omega^2 - 3)}{24(c+\Omega)^2}(c+c_1)(\eta_{00}^4)_\xi,
\end{aligned}$$

which follows

$$
\begin{aligned}
u_{30} =& c\eta_{30} - 2(c+c_1)(\eta_{00}\eta_{20}) - (c+c_1)(\eta_{10}^2) - \frac{2c_1 - 3\Omega}{\Omega + c}(c+c_1)(\eta_{00}^2\eta_{10}) \\
&- \frac{(64cc_1 + 24c_1^2 + 45c^2 + 24\Omega^2 - 3)}{24(c+\Omega)^2}(c+c_1)(\eta_{00}^4).
\end{aligned}
\tag{2.73}
$$

Taking account of (2.40), (2.51), and (2.72), it turns into

$$
\begin{aligned}
u_{30,\tau} =& c\eta_{30,\tau} - \frac{2(3c^2 + 5cc_1 + 4c_1^2 - 3\Omega c_1)}{\Omega + c}(c+c_1)(\eta_{00}^3\eta_{10})_\xi \\
&- 4c_1(c+c_1)(\eta_{00}\eta_{10}^2)_\xi - 4c_1(c+c_1)(\eta_{00}^2\eta_{20})_\xi - B_1\eta_{00}^4\eta_{00,\xi},
\end{aligned}
\tag{2.74}
$$

where

$$
\begin{aligned}
B_1 \stackrel{\text{def}}{=} & \frac{(c+c_1)^2(82cc_1 + 36c_1^2 + 45c^2 - 18\Omega c_1 - 27\Omega c - 15)}{3(\Omega+c)^2} \\
&+ \frac{c_1(c+c_1)(64cc_1 + 24c_1^2 + 45c^2 + 24\Omega^2 - 3)}{3(\Omega+c)^2}.
\end{aligned}
$$

For order $O(\varepsilon^4, \mu^0)$, the terms of (2.3) are obtained from the Taylor expansion (2.32) that

$$
\begin{cases}
-cu_{40,\xi} + u_{30,\tau} + (u_{00}u_{30} + u_{10}u_{20})_\xi + 2\Omega W_{40} = -p_{40,\xi} & \text{in} \quad 0 < z < 1, \\
-2\Omega u_{40} = -p_{40,z} & \text{in} \quad 0 < z < 1, \\
u_{40,\xi} + W_{40,z} = 0 & \text{in} \quad 0 < z < 1, \\
u_{40,z} = 0 & \text{in} \quad 0 < z < 1, \\
p_{40} + \eta_{00}p_{30,z} + \eta_{10}p_{20,z} + \eta_{20}p_{10,z} + \eta_{30}p_{00,z} = \eta_{40} & \text{on} \quad z = 1, \\
W_{40} + \eta_{00}W_{30,z} + \eta_{10}W_{20,z} + \eta_{20}W_{10,z} + \eta_{30}W_{00,z} & \\
\quad = -c\eta_{40,\xi} + \eta_{30,\tau} + u_{00}\eta_{30,\xi} + u_{10}\eta_{20,\xi} + u_{20}\eta_{10,\xi} + u_{30}\eta_{00,\xi} & \text{on} \quad z = 1, \\
W_{40} = 0 & \text{on} \quad z = 0.
\end{cases}
\tag{2.75}
$$

Similarly to the previous orders, it follows that u_{40} is independent of z, which is $u_{40} = u_{40}(\tau, \xi)$, from the fourth equation in (2.75). Also, from the third equation in (2.75) and the boundary condition of W_{40} at $z = 0$, we have

$$W_{40} = -z u_{40,\xi}. \tag{2.76}$$

Taking account of the boundary condition of W_{40} at $z = 1$, (2.76), (2.66), (2.47), (2.34) and (2.6), we obtain

$$u_{40,\xi} = c\eta_{40,\xi} - \eta_{30,\tau} - (u_{00}\eta_{30} + u_{10}\eta_{20} + u_{20}\eta_{10} + u_{30}\eta_{00})_\xi, \tag{2.77}$$

Moreover, integrating the second equation in (2.75) and considering the boundary condition of p_{30} at $z = 1$, we have

$$\begin{aligned}
p_{40} &= p_{40}|_{z=1} + \int_1^z p_{40,z'}\, dz' \\
&= \eta_{40} - (\eta_{00}p_{30,z} + \eta_{10}p_{20,z} + \eta_{20}p_{10,z} + \eta_{30}p_{00,z}) + 2\Omega \int_1^z u_{40}\, dz' \\
&= \eta_{40} - 2\Omega(u_{00}\eta_{30} + u_{10}\eta_{20} + u_{20}\eta_{10} + u_{30}\eta_{00}) + 2\Omega(z-1)u_{40}.
\end{aligned}$$

Taking derivative of the equation with respect to ξ, we get

$$p_{40,\xi} = -\eta_{40,\xi} - 2\Omega(u_{00}\eta_{30} + u_{10}\eta_{20} + u_{20}\eta_{10} + u_{30}\eta_{00})_\xi + 2\Omega(z-1)u_{40,\xi}. \tag{2.78}$$

and then from the first equation in (2.75), we obtain

$$-p_{40,\xi} = -cu_{40,\xi} + u_{30,\tau} + (u_{00}u_{30} + u_{10}u_{20})_\xi + 2\Omega W_{40}.$$

Therefore, from (2.76) and (2.78), they turn into

$$\begin{aligned}
0 = &-(c + 2\Omega)u_{40,\xi} + u_{30,\tau} + (u_{00}u_{30} + u_{10}u_{20})_\xi \\
&+ \eta_{40,\xi} - 2\Omega(u_{00}\eta_{30} + u_{10}\eta_{20} + u_{20}\eta_{10} + u_{30}\eta_{00})_\xi.
\end{aligned} \tag{2.79}$$

Applying (2.77) and (2.74) to (2.79), we have

$$\begin{aligned}
&2(c+\Omega)\eta_{30,\tau} + 3c^2(\eta_{00}\eta_{30} + \eta_{10}\eta_{20})_\xi - 2(3c + 2c_1)(c + c_1)(\eta_{00}^2\eta_{20} + \eta_{00}\eta_{10}^2)_\xi \\
&\quad - \frac{(64cc_1 + 24c_1^2 + 45c^2 - 15)}{3(c+\Omega)}(c + c_1)(\eta_{00}^3\eta_{10})_\xi - B_2(\eta_{00}^5)_\xi = 0,
\end{aligned} \tag{2.80}$$

where

$$B_2 \overset{\text{def}}{=} \frac{1}{5}B_1 - \frac{(c+c_1)^2(2c_1-3\Omega)}{3(\Omega+c)} + \frac{2c(c+c_1)(64cc_1+24c_1^2+45c^2+24\Omega^2-3)}{12(\Omega+c)^2}$$
$$= \frac{c^2(2-c^2)(3c^{10}+228c^8-540c^6-180c^4-13c^2+42)}{60(c^2+1)^6}.$$

Similarly, For the order $O(\varepsilon^2, \mu^1)$ the terms in (2.3) are obtained as the following,

$$\begin{cases} -cu_{21,\xi} + u_{11,\tau} + (u_{00}u_{11} + u_{10}u_{01})_\xi + W_{00}u_{11,z} + 2\Omega W_{21} = -p_{21,\xi} & \text{in} \quad 0<z<1, \\ -cW_{10,\xi} + W_{00,\tau} + u_{00}W_{00,\xi} + W_{00}W_{00,z} - 2\Omega u_{21} = -p_{21,z} & \text{in} \quad 0<z<1, \\ u_{21,\xi} + W_{21,z} = 0 & \text{in} \quad 0<z<1, \\ u_{21,z} - W_{10,\xi} = 0 & \text{in} \quad 0<z<1, \\ p_{21} + \eta_{10}p_{01,z} + \eta_{01}p_{10,z} + \eta_{00}p_{11,z} + \eta_{11}p_{00,z} = \eta_{21} & \text{on} \quad z=1, \\ W_{21} + \eta_{10}W_{01,z} + \eta_{01}W_{10,z} + \eta_{00}W_{11,z} + \eta_{11}W_{00,z} & \\ = -c\eta_{21,\xi} + \eta_{11,\tau} + u_{00}\eta_{11,\xi} + u_{11}\eta_{00,\xi} + u_{10}\eta_{01,\xi} + u_{01}\eta_{10,\xi} & \text{on} \quad z=1, \\ W_{21} = 0 & \text{on} \quad z=0. \end{cases} \tag{2.81}$$

Applying (2.34) and(2.41) to the fourth equation in (2.81), it follows

$$u_{21,z} = W_{10,\xi} = z\bigg(2(c+c_1)(\eta_{00,\xi}^2 + \eta_{00}\eta_{00,\xi\xi}) - c\eta_{10,\xi\xi}\bigg),$$

which turns into

$$u_{21} = \frac{z^2}{2}\bigg(2(c+c_1)(\eta_{00,\xi}^2 + \eta_{00}\eta_{00,\xi\xi}) - c\eta_{10,\xi\xi}\bigg) + \Phi_{21}(\tau,\xi) = \frac{z^2}{2}H_1 + \Phi_{21}(\tau,\xi)$$

for any smooth function $\Phi_{21}(\tau,\xi)$ independent of z, where we describe

$$H_1 \overset{\text{def}}{=} 2(c+c_1)(\eta_{00,\xi}^2 + \eta_{00}\eta_{00,\xi\xi}) - c\eta_{10,\xi\xi}.$$

Then, we get

$$u_{21,\xi} = \frac{z^2}{2}H_{1,\xi} + \partial_\xi \Phi_{21}(\tau,\xi).$$

Also, from the third equation in (2.81) and the boundary condition of W_{21} on $\{z = 0\}$, we obtain

$$W_{21} = W_{21}|_{z=0} + \int_0^z W_{21,z'}\, dz' = -\int_0^z u_{21,\xi}\, dz' = -\frac{z^3}{6} H_{1,\xi} - z\partial_\xi \Phi_{21}(\tau, \xi).$$

Considering the third equation of the orders $O(\varepsilon^0, \mu^1)$, $O(\varepsilon^1, \mu^0)$, $O(\varepsilon^1, \mu^1)$, $O(\varepsilon^0, \mu^0)$ with the boundary condition of W_{21} on $\{z = 1\}$, it follows

$$\begin{aligned} -\frac{1}{6} H_{1,\xi} - \partial_\xi \Phi_{21}(\tau, \xi) &= -c\eta_{21,\xi} + \eta_{11,\tau} + (u_{00}\eta_{11} + u_{11}\eta_{00} + u_{10}\eta_{01} + u_{01}\eta_{10})_\xi|_{z=1} \\ &= -c\eta_{21,\xi} + \eta_{11,\tau} + H_{2,\xi}|_{z=1}, \end{aligned}$$

where H_2 is defined as

$$H_2 \stackrel{\text{def}}{=} u_{00}\eta_{11} + u_{11}\eta_{00} + u_{10}\eta_{01} + u_{01}\eta_{10}.$$

Then we have

$$\partial_\xi \Phi_{21}(\tau, \xi) = c\eta_{21,\xi} - \eta_{11,\tau} - \frac{1}{6} H_{1,\xi} - H_{2,\xi}|_{z=1}. \tag{2.82}$$

This gives us

$$u_{21,\xi} = c\eta_{21,\xi} - \eta_{11,\tau} + (\frac{z^2}{2} - \frac{1}{6}) H_{1,\xi} - H_{2,\xi}|_{z=1} \tag{2.83}$$

and

$$W_{21} = \frac{z(1 - z^2)}{6} H_{1,\xi} - cz\eta_{21,\xi} + z\eta_{11,\tau} + z(H_{2,\xi}|_{z=1}). \tag{2.84}$$

Substituting the expressions of $W_{00,\tau}$, u_{00}, $W_{00,\xi}$, W_{00}, $W_{00,z}$, and $W_{10,\xi}$ into On the other hand, using the results from the previous orders, the second equation in (2.81) turns into

$$p_{21,z} = 2\Omega u_{21} - c^2 z\eta_{10,\xi\xi} + c(c + 4c_1) z\eta_{00,\xi}^2 + c(3c + 4c_1) z\eta_{00}\eta_{00,\xi\xi}. \tag{2.85}$$

From the boundary condition of p_{21} on $z = 1$, we get

$$p_{21}|_{z=1} = \eta_{21} + c^2\eta_{00}\eta_{00,\xi\xi} - 2\Omega H_2|_{z=1},$$

Also, integrating (2.85), it follows

$$\begin{aligned} p_{21} &= p_{21}|_{z=1} + \int_1^z p_{21,z'}\,dz' \\ &= \eta_{21} - 2\Omega H_2|_{z=1} + 2\Omega\int_1^z u_{21}\,dz' - \frac{c^2}{2}(z^2-1)\eta_{10,\xi\xi} \\ &\quad + \frac{c(c+4c_1)}{2}(z^2-1)\eta_{00,\xi}^2 + \left(c^2 + \frac{c(3c+4c_1)}{2}(z^2-1)\right)\eta_{00}\eta_{00,\xi\xi}. \end{aligned} \tag{2.86}$$

Then we have

$$\begin{aligned} p_{21,\xi} &= \eta_{21,\xi} - 2\Omega H_{2,\xi}|_{z=1} + 2\Omega\int_1^z u_{21,\xi}\,dz' - \frac{c^2}{2}(z^2-1)\eta_{10,\xi\xi\xi} \\ &\quad + \frac{c(c+4c_1)}{2}(z^2-1)(\eta_{00,\xi}^2)_\xi + \left(c^2 + \frac{c(3c+4c_1)}{2}(z^2-1)\right)(\eta_{00}\eta_{00,\xi\xi})_\xi \\ &= -2\Omega z H_{2,\xi}|_{z=1} + 2\Omega(z-1)\Big(c\eta_{21,\xi} - \eta_{11,\tau}\Big) + \frac{z(z^2-1)}{6}H_{1,\xi} - \frac{c^2}{2}(z^2-1)\eta_{10,\xi\xi\xi} \\ &\quad + \eta_{21,\xi} + \frac{c(c+4c_1)}{2}(z^2-1)(\eta_{00,\xi}^2)_\xi + \left(c^2 + \frac{c(3c+4c_1)}{2}(z^2-1)\right)(\eta_{00}\eta_{00,\xi\xi})_\xi. \end{aligned} \tag{2.87}$$

From the first equation in (2.81), (2.84), and (2.8), we obtain

$$\begin{aligned} -p_{21,\xi} =& - cu_{21,\xi} + u_{11,\tau} + (u_{00}u_{11} + u_{10}u_{01})_\xi + c^2z^2\eta_{00,\xi}\eta_{00,\xi\xi} \\ & + \frac{\Omega}{3}z(1-z^2)H_{1,\xi} - 2\Omega cz\eta_{21,\xi} + 2\Omega z\eta_{11,\tau} + 2\Omega zH_{2,\xi}|_{z=1}. \end{aligned} \tag{2.88}$$

Taking account of (2.88) and (2.86), we have

$$\begin{aligned} 0 =& -cu_{21,\xi} + u_{11,\tau} + (u_{00}u_{11} + u_{10}u_{01})_\xi + \left(\frac{c^2}{2}z^2 + \frac{c(c+4c_1)}{2}(z^2-1)\right)(\eta_{00,\xi}^2)_\xi \\ & + \frac{\Omega}{3}z(1-z^2)H_{1,\xi} + (1-2\Omega c)\eta_{21,\xi} + 2\Omega\eta_{11,\tau} + \frac{z(z^2-1)}{6}H_{1,\xi} - \frac{c^2}{2}(z^2-1)\eta_{10,\xi\xi\xi} \\ & + \left(c^2 + \frac{c(3c+4c_1)}{2}(z^2-1)\right)(\eta_{00}\eta_{00,\xi\xi})_\xi. \end{aligned} \tag{2.89}$$

Notice that

$$
\begin{aligned}
&(u_{01}u_{10} + u_{00}u_{11})_\xi \\
&= c^2(\eta_{01}\eta_{10} + \eta_{00}\eta_{11})_\xi + \left(\frac{c^2}{6} - \frac{2cc_1}{9} - \frac{c^2z^2}{2}\right)(\eta_{00}\eta_{00,\xi\xi})_\xi - 3c(c+c_1)(\eta_{00}^2\eta_{01})_\xi
\end{aligned}
$$

and

$$
cH_{2,\xi}|_{z=1} = 2c^2(\eta_{01}\eta_{10} + \eta_{00}\eta_{11})_\xi - \left(\frac{c^2}{3} + \frac{2cc_1}{9}\right)(\eta_{00}\eta_{00,\xi\xi})_\xi - 3c(c+c_1)(\eta_{00}^2\eta_{01})_\xi.
$$

We substitute (2.83) and (2.64) into (2.89) to get

$$
\begin{aligned}
&2(\Omega + c)\eta_{11,\tau} + 3c^2(\eta_{00}\eta_{11} + \eta_{10}\eta_{01})_\xi - 2(c+c_1)(3c+2c_1)(\eta_{00}^2\eta_{01})_\xi + \frac{c^2}{3}\eta_{10,\xi\xi\xi} \\
&- \left(\frac{c^2}{6} + \frac{10cc_1}{9} + \frac{2c_1^2}{9}\right)(\eta_{00,\xi}^2)_\xi - \left(\frac{c^2}{3} + \frac{20cc_1}{9} + \frac{8c_1^2}{9}\right)(\eta_{00}\eta_{00,\xi\xi})_\xi = 0.
\end{aligned}
\tag{2.90}
$$

CHAPTER 3

LOCAL WELL POSEDNESS OF ROTATIONAL b-FAMILY OF EQUATIONS

In this section, the local well-posedness for the R-b-family equations is investigated.

Consider the R-b-family equation (1.5) in terms of the evolution of m, namely, the equation (1.9). Applying the transformation $u_{\varepsilon,\mu}(t,x) = \alpha\varepsilon u(\sqrt{\beta\mu}\,t, \sqrt{\beta\mu}\,x)$ to (1.9), we know that $u_{\varepsilon,\mu}(t,x)$ solves

$$u_t - u_{xxt} + cu_x + (b+1)uu_x - \frac{\beta_0}{\beta}u_{xxx} + \frac{\omega_1}{\alpha^2}u^2u_x + \frac{\omega_2}{\alpha^3}u^3u_x = bu_xu_{xx} + uu_{xxx}, \tag{3.1}$$

and its corresponding one conserved quantity denoted by $I(u)$ is as follows

$$I(u) = \int_{\mathbb{R}} u\,dx.$$

As a special case $b = 2$, the equation has three conserved quantities as follows:

$$I(u) = \int_{\mathbb{R}} udx, \quad E(u) = \frac{1}{2}\int_{\mathbb{R}} u^2 + u_x^2 dx,$$

and

$$F(u) = \frac{1}{2}\int_{\mathbb{R}} cu^2 + u^3 + \frac{\beta_0}{\beta}u_x^2 + \frac{\omega_1}{6\alpha^2}u^4 + \frac{\omega_2}{10\alpha^3}u^5 + uu_x^2 dx.$$

Also, we have two more forms of equations,

$$\begin{cases} m_t + um_x + bu_xm + cu_x - \frac{\beta_0}{\beta}u_{xxx} + \frac{\omega_1}{\alpha^2}u^2u_x + \frac{\omega_2}{\alpha^3}u^3u_x = 0, \\ m = u - u_{xx}, \end{cases} \tag{3.2}$$

and

$$u_t + uu_x + \frac{\beta_0}{\beta}u_x + p * \partial_x\left\{\left(c - \frac{\beta_0}{\beta}\right)u - \frac{(b-3)}{2}(u_x)^2 + \frac{b}{2}u^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right\} = 0, \tag{3.3}$$

where $p = \frac{1}{2}e^{-|x|}$.

3.1 Preliminaries

In the Lebesgue space $L^p(\mathbb{R})$ the norm is defined as $\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{1/p}$. For the space $L^\infty(\mathbb{R})$ consisting of all essentially bounded functions, the norm is given by $\|f\|_\infty = \inf_{\mu(E)=0} \sup_{x\in\mathbb{R}\setminus E} |f(x)|$, where f is Lebesgue measurable functions. For a function f in the classical Sobolev spaces $H^s(\mathbb{R})$ $(s \geq 0)$ the norm is denoted by $\|f\|_{H^s}$. We denote $p(x) = \frac{1}{2}e^{-|x|}$ the fundamental solution of $1 - \partial_x^2$ on $\mathbb{R}$, and define the two convolution operators $p_+,\ p_-$ as

$$p_+ * f(x) = \frac{e^{-x}}{2}\int_{-\infty}^{x} e^y f(y)dy$$
$$p_- * f(x) = \frac{e^{x}}{2}\int_{x}^{\infty} e^{-y} f(y)dy.$$

Then we have the relations $p = p_+ + p_-, \quad p_x = p_- - p_+$.

Definition 3.1.1 ([15])**.** Let $s \in \mathbb{R}$. A tempered distribution u belongs to $H^s(\mathbb{R}^N)$ if $\widehat{u} \in L^2_{loc}(\mathbb{R}^N)$ and

$$\|u\|_{H^s} := \left(\int_{\mathbb{R}^N} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}} < \infty.$$

It is classical that H^s endowed with the norm $\|\cdot\|_{H^s}$ is a Banach space

Lemma 3.1.1 ([31])**.** *[Commutator Estimate] Let Λ be a operator defined as $\Lambda = (1 - {\partial_x}^2)^{1/2}$. If f and g are smooth enough, then*

$$\|[\Lambda^s, f]g\|_{L^2} \leq C(\|f\|_{H^s}\|g\|_{L^\infty} + \|\partial_x f\|_{L^\infty}\|g\|_{H^{s-1}}), \tag{3.4}$$

for all $s > \frac{3}{2}$ and $C > 0$.

Lemma 3.1.2. *[22] Let $f(t)$ and $g(t)$ be two positive function on $[0, T]$. If the differential inequality*

$$\frac{d}{dt}(h^2)(t) \leq f(t)h^2(t) + g(t)h(t),$$

for a nonnegative absolutely continuous function $h(\cdot)$ on $[0,T]$ holds for almost everywhere on $[0,T]$, then

$$h(t) \le e^{\frac{1}{2}\int_0^t f(\tau)d\tau}\left[h(0) + \frac{1}{2}\int_0^t g(\tau)d\tau\right]. \tag{3.5}$$

3.2 Local Well Posedness

In this section, we introduce the local well-posedness result of the following Cauchy problem, which is similar to the approach in [14] with some adjustments

$$\begin{cases} u_t - u_{xxt} + cu_x + (b+1)uu_x - \frac{\beta_0}{\beta}u_{xxx} + \frac{\omega_1}{\alpha^2}u^2u_x + \frac{\omega_2}{\alpha^3}u^3u_x = bu_xu_{xx} + uu_{xxx}, \\ u|_{t=0} = u_0. \end{cases} \tag{3.6}$$

Theorem 3.2.1 (Local well-posedness)**.** *Let $s > \frac{3}{2}$, $u_0 \in H^s(\mathbb{R})$. Then, there exist the existence time $T > 0$ and a unique solution $u \in C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; H^{s-1}(\mathbb{R}))$ to the Cauchy problem.*

Proof. **Step 1:** First of all, the Cauchy problem (3.6) can be written as

$$\begin{cases} \partial_t u + u\partial_x u + \left[\frac{\beta_0}{\beta} + \left(c - \frac{\beta_0}{\beta}\right)p*\right]\partial_x u \\ \qquad\qquad = -p*\partial_x\left\{\frac{3-b}{2}u_x^2 + \frac{b}{2}u^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right\}, \\ u|_{t=0} = u_0. \end{cases}$$

Then, we establish a sequence of approximation solutions $\{u^{(n)}\}_{n\in\mathbb{N}}$ as follow. Let $u^{(0)}$=0,$\forall n \in \mathbb{N}$

$$\begin{cases} \partial_t u^{(n+1)} + u^{(n)}\partial_x u^{(n+1)} + \left[\frac{\beta_0}{\beta} + \left(c - \frac{\beta_0}{\beta}\right)p*\right]\partial_x u^{(n+1)} \\ \qquad\qquad = -p*\partial_x\left\{\frac{3-b}{2}\left[u^{(n)}\right]^2_x + \frac{b}{2}\left[u^{(n)}\right]^2 + \frac{\omega_1}{3\alpha^2}\left[u^{(n)}\right]^3 + \frac{\omega_2}{4\alpha^3}\left[u^{(n)}\right]^4\right\}, \\ u^{(n+1)}|_{t=0} = S_{n+1}u_0, \end{cases} \tag{3.7}$$

where

$$\widehat{S_{n+1}u_0}(\xi) = 1_{|\xi|<2\times 2^{n+2}}(\xi)\hat{u_0}(\xi).$$

By the theory of the linear evolution, there is a unique smooth solution $u^{(n+1)}$ of (3.7) and $\forall n \in \mathbb{N}$, $u^{(n+1)} \in C^1((R); H^\infty(\mathbb{R}))$. Next we will show that the sequence $\{u^{(n)}\}_{n\in\mathbb{N}}$ converges. To prove the convergence, we will prove that $\{u^{(n)}\}_{n\in\mathbb{N}}$ is uniformly bounded and a Cauchy sequence, since the Sobolev space is compact.

Step 2: In this step, we show that the sequence $\{u^{(n)}\}_{n\in\mathbb{N}}$ is uniformly bounded in some Sobolev space. Consider the operator

$$\Lambda^s = (1 - \partial_x{}^2)^{s/2}.$$

Applying the operator Λ^s to the (3.7), we get

$$\begin{aligned}\partial_t \Lambda^s u^{(n+1)} &= -\Lambda^s \left(u^{(n)}\partial_x u^{(n+1)}\right) - \frac{\beta_0}{\beta}\Lambda^s \partial_x u^{(n+1)} - \left(c - \frac{\beta_0}{\beta}\right)\Lambda^s \left(p * \partial_x u^{(n+1)}\right) \\ &\quad - \Lambda^s \left(p * \partial_x \left\{\frac{3-b}{2}\left[u^{(n)}\right]^2{}_x + \frac{b}{2}\left[u^{(n)}\right]^2 + \frac{\omega_1}{3\alpha^2}\left[u^{(n)}\right]^3 + \frac{\omega_2}{4\alpha^3}\left[u^{(n)}\right]^4\right\}\right).\end{aligned}$$

Multiply the equation by $\Lambda^s u^{(n+1)}$ and integrate on $\mathbb{R}$, then we have

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial_t}\left\|\Lambda^s u^{(n+1)}\right\|_{L^2}^2 = &-\left\langle \Lambda^s \left(u^{(n)}\partial_x u^{(n+1)}\right), \Lambda^s u^{(n+1)}\right\rangle - \frac{\beta_0}{\beta}\left\langle \Lambda^s \partial_x u^{(n+1)}, \Lambda^s u^{(n+1)}\right\rangle \\ &- \left(c - \frac{\beta_0}{\beta}\right)\left\langle \Lambda^{(s-2)}\partial_x u^{(n+1)}, \Lambda^s u^{(n+1)}\right\rangle \\ &+ \left(\frac{b-3}{2}\right)\left\langle \Lambda^{(s-2)}\partial_x \left(\left[u^{(n)}\right]^2{}_x\right), \Lambda^s u^{(n+1)}\right\rangle \\ &- \frac{b}{2}\left\langle \Lambda^{(s-2)}\partial_x \left[u^{(n)}\right]^2, \Lambda^s u^{(n+1)}\right\rangle - \frac{\omega_1}{3\alpha^2}\left\langle \Lambda^{(s-2)}\partial_x \left[u^{(n)}\right]^3, \Lambda^s u^{(n+1)}\right\rangle \\ &- \frac{\omega_2}{4\alpha^3}\left\langle \Lambda^{(s-2)}\partial_x \left[u^{(n)}\right]^4, \Lambda^s u^{(n+1)}\right\rangle.\end{aligned} \tag{3.8}$$

Since Λ^s is a symmetric operator and ∂_x is skew-symmetric, we have

$$\left\langle \Lambda^s \partial_x u^{(n+1)}, \Lambda^s u^{(n+1)}\right\rangle = \left\langle \Lambda^{(s-2)}\partial_x u^{(n+1)}, \Lambda^s u^{(n+1)}\right\rangle = 0,$$

such that (3.8) becomes

$$
\begin{aligned}
\frac{\partial}{\partial_t}\left\|\Lambda^s u^{(n+1)}\right\|_{L^2}^2 = -2\left\langle \Lambda^s\left(u^{(n)}\partial_x u^{(n+1)}\right), \Lambda^s u^{(n+1)}\right\rangle + (b-3)\left\langle \Lambda^{(s-2)}\partial_x\left(\left[u^{(n)}\right]^2_x\right), \Lambda^s u^{(n+1)}\right\rangle \\
- b\left\langle \Lambda^{(s-2)}\partial_x\left[u^{(n)}\right]^2, \Lambda^s u^{(n+1)}\right\rangle - \frac{2\omega_1}{3\alpha^2}\left\langle \Lambda^{(s-2)}\partial_x\left[u^{(n)}\right]^3, \Lambda^s u^{(n+1)}\right\rangle \\
- \frac{\omega_2}{2\alpha^3}\left\langle \Lambda^{(s-2)}\partial_x\left[u^{(n)}\right]^4, \Lambda^s u^{(n+1)}\right\rangle \\
= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
\tag{3.9}
$$

Part I_1 : For all constant coefficient skew-symmetric differential polynomial P, and f, g smooth enough, a commutator process is

$$
\Lambda^s(fPg) = fP\Lambda^s g + [\Lambda^s, f]Pg. \tag{3.10}
$$

Then, taking P as ∂_x, $f = u^{(n)}$, and $g = u^{(n+1)}$, we have

$$
\Lambda^s(u^{(n)}\partial_x u^{(n+1)}) = u^{(n)}\partial_x\Lambda^s u^{(n+1)} + [\Lambda^s, u^{(n)}]\partial_x u^{(n+1)}.
$$

So, I_1 becomes

$$
\begin{aligned}
I_1 = \left\langle \Lambda^s(u^{(n)}\partial_x u^{(n+1)}), \Lambda^s u^{(n+1)}\right\rangle = \int_{\mathbb{R}} \Lambda^s(u^{(n)}\partial_x u^{(n+1)})\Lambda^s u^{(n+1)}\,dx \\
= \int_{\mathbb{R}} u^{(n)}\partial_x\Lambda^s u^{(n+1)}\Lambda^s u^{(n+1)}\,dx + \int_{\mathbb{R}}[\Lambda^s, u^{(n)}]\partial_x u^{(n+1)}\Lambda^s u^{(n+1)}\,dx \\
= I_{11} + I_{12}.
\end{aligned}
$$

Using integration by parts,

$$
\begin{aligned}
|I_{11}| &= \frac{1}{2}\left|\int_{\mathbb{R}} u^{(n)}\partial_x\left(\Lambda^s u^{(n+1)}\right)^2\,dx\right| \\
&= \frac{1}{2}\left|\int_{\mathbb{R}} u_x^{(n)}\left(\Lambda^s u^{(n+1)}\right)^2\,dx\right| \\
&\le \frac{1}{2}\|u_x^{(n)}\|_{L^\infty}\|u^{(n+1)}\|_{H^s}^2.
\end{aligned}
$$

Moreover, commutator estimate and Sobolev embedding theorem ($H^{s-1}(R) \hookrightarrow L^\infty(R)$)give that for all $s > \frac{1}{2}$, and some $C > 0$,

$$\begin{aligned} |I_{11}| &\le C\|u^{(n)}{}_x\|_{L^\infty}\|u^{(n+1)}\|_{H^s}^2 \\ &\le C\|u^{(n)}\|_{H^s}\|u^{(n+1)}\|_{H^s}^2. \end{aligned} \tag{3.11}$$

On the other hand, by Hölder's inequality

$$\begin{aligned} |I_{12}| &= \left| \int_{\mathbb{R}} [\Lambda^s, u^{(n+1)}]\partial_x u^{(n+1)} \Lambda^s u^{(n+1)}\, dx \right| \\ &\le \|\Lambda^s u^{(n+1)}\|_{L^2} \|[\Lambda^s, u^{(n)}]\partial_x u^{(n+1)}\|_{L^2}. \end{aligned}$$

Applying the commutator estimate on $\|[\Lambda^s, u^{(n)}]\partial_x u^{(n+1)}\|_{L^2}$ and by Sobolev embedding theorem, there exists $C' > 0$ such that

$$\begin{aligned} |I_{12}| &\le C'\|\Lambda^s u^{(n+1)}\|_{L^2} \left(\|u^{(n)}\|_{H^s}\|u^{(n+1)}{}_x\|_{L^\infty} + \|u_x^{(n)}\|_{L^\infty}\|u_x^{(n+1)}\|_{H^{s-1}} \right) \\ &\le C'\|u^{(n+1)}\|_{H^s} \left(\|u^{(n)}\|_{H^s}\|u^{(n+1)}\|_{H^s} + \|u^{(n)}\|_{H^s}\|u^{(n+1)}\|_{H^s} \right) \\ &\le C'\|u^{(n)}\|_{H^s}\|u^{(n+1)}\|_{H^s}^2 \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) , there exists $C_1 > 0$

$$|I_1| = |I_{11} + I_{12}| \le |I_{11}| + |I_{12}| \le C_1\|u^{(n)}\|_{H^s}\|u^{(n+1)}\|_{H^s}^2. \tag{3.13}$$

Part I_2 : There exists $C_2 > 0$, such that

$$\begin{aligned} |I_2| &= \left| (b-3) \left\langle \Lambda^{s-2}\partial_x (u^{(n)}{}_x)^2, \Lambda^s u^{(n+1)} \right\rangle \right| \\ &\le C_2|b-3|\|\Lambda^{s-2}\partial_x(u^{(n)}{}_x)^2\|_{L^2}\|\Lambda^s u^{(n+1)}\|_{L^2} \\ &\le C_2|b-3|\|\partial_x(u^{(n)}{}_x)^2\|_{H^{s-2}}\|u^{(n+1)}\|_{H^s} \\ &\le C_2|b-3|\|(u^{(n)}{}_x)^2\|_{H^{s-1}}\|u^{(n+1)}\|_{H^s} \\ &\le C_2|b-3|\|u^{(n)}{}_x\|_{H^{s-1}}^2\|u^{(n+1)}\|_{H^s} \\ &\le C_2|b-3|\|u^{(n)}\|_{H^s}^2\|u^{(n+1)}\|_{H^s}. \end{aligned} \tag{3.14}$$

Part I_3 : There exists $C_3 > 0$ such that

$$\begin{aligned}
|I_3| &= \left| b \left\langle \Lambda^{s-2}\partial_x [u^{(n)}]^2, \Lambda^s u^{(n+1)} \right\rangle \right| \\
&\le C_3 b \|\Lambda^{s-2}\partial_x [u^{(n)}]^2\|_{L^2} \|\Lambda^s u^{(n+1)}\|_{L^2} \\
&\le C_3 b \|\partial_x [u^{(n)}]^2\|_{H^{s-2}} \|u^{(n+1)}\|_{H^s} \\
&\le C_3 b \|u^{(n)}\|^2_{H^{s-1}} \|u^{(n+1)}\|_{H^s} \\
&\le C_3 b \|u^{(n)}\|^2_{H^s} \|u^{(n+1)}\|_{H^s}.
\end{aligned} \tag{3.15}$$

Part I_4 : There exists $C_4 > 0$ such that

$$\begin{aligned}
|I_4| &= \left| \frac{2\omega_1}{3\alpha^2} \left\langle \Lambda^{s-2}\partial_x [u^{(n)}]^3, \Lambda^s u^{(n+1)} \right\rangle \right| \\
&\le C_4' \|\Lambda^{s-2}\partial_x [u^{(n)}]^3\|_{L^2} \|\Lambda^s u^{(n+1)}\|_{L^2} \\
&\le C_4' \|\partial_x [u^{(n)}]^3\|_{H^{s-2}} \|u^{(n+1)}\|_{H^s} \\
&\le C_4' \|u^{(n)}\|^3_{H^{s-1}} \|u^{(n+1)}\|_{H^s} \\
&\le C_4' \|u^{(n)}\|^3_{H^s} \|u^{(n+1)}\|_{H^s}.
\end{aligned} \tag{3.16}$$

Part I_5 : There exists $C_5 > 0$ such that

$$\begin{aligned}
|I_5| &= \left| \frac{\omega_2}{2\alpha^3} \left\langle \Lambda^{s-2}\partial_x [u^{(n)}]^4, \Lambda^s u^{(n+1)} \right\rangle \right| \\
&\le C_5' \|\Lambda^{s-2}\partial_x [u^{(n)}]^4\|_{L^2} \|\Lambda^s u^{(n+1)}\|_{L^2} \\
&\le C_5' \|\partial_x [u^{(n)}]^4\|_{H^{s-2}} \|u^{(n+1)}\|_{H^s} \\
&\le C_5' \|u^{(n)}\|^4_{H^{s-1}} \|u^{(n+1)}\|_{H^s} \\
&\le C_5' \|u^{(n)}\|^4_{H^s} \|u^{(n+1)}\|_{H^s}.
\end{aligned} \tag{3.17}$$

From (3.13) - (3.17), we get

$$\begin{aligned}
\frac{\partial}{\partial_t} \|u^{(n+1)}\|^2_{H^s} &\le C_1 \|u^{(n)}\|_{H^s} \|u^{(n+1)}\|^2_{H^s} + C_2 |b-3| \|u^{(n)}\|^2_{H^s} \|u^{(n+1)}\|_{H^s} \\
&\quad + C_3 b \|u^{(n)}\|^2_{H^s} \|u^{(n+1)}\|_{H^s} + C_4' \|u^{(n)}\|^3_{H^s} \|u^{(n+1)}\|_{H^s} + C_5' \|u^{(n)}\|^4_{H^s} \|u^{(n+1)}\|_{H^s}.
\end{aligned} \tag{3.18}$$

From (3.18) and by redefining C_1 and C_2, we have

$$\frac{\partial}{\partial_t}\|u^{(n+1)}\|_{H^s}^2 \leq C_1\|u^{(n)}\|_{H^s}\|u^{(n+1)}\|_{H^s}^2+C_2\left(\|u^{(n)}\|_{H^s}^2+\|u^{(n)}\|_{H^s}^3+\|u^{(n)}\|_{H^s}^4\right)\|u^{(n+1)}\|_{H^s}.$$

Then, taking C_0=max$\{C_1, C_2\}$, we obtain

$$\frac{\partial}{\partial_t}\|u^{(n+1)}\|_{H^s}^2 \leq C_0\left(\|u^{(n)}\|_{H^s}\|u^{(n+1)}\|_{H^s}^2+\left(\|u^{(n)}\|_{H^s}^2+\|u^{(n)}\|_{H^s}^3+\|u^{(n)}\|_{H^s}^4\right)\|u^{(n+1)}\|_{H^s}\right). \tag{3.19}$$

Without lost of generality, suppose $C_0 \geq 2$. Let's fix a $T > 0$ so that

$$C_0^5 max\{\|u_0\|^3, 1\}T < 1. \tag{3.20}$$

Now, we claim that for all $n \in \mathbb{N}$

$$\|u^{(n)}\|_{H^s} \leq C_0\|u_0\|_{H^s}, \qquad \forall t \in [0, T]. \tag{3.21}$$

We prove this claim by using an inductive argument. For n=0, from (3.7) and $u^{(0)} = 0$, we have

$$\partial_t u^{(1)} + \left[\frac{\beta_0}{\beta} + \left(c - \frac{\beta_0}{\beta}\right)p*\right]\partial_x u^{(1)} = 0,$$

which implies that

$$\frac{d}{dt}\|u^{(1)}\|_{H^s}^2 = 0$$

by (3.7) and $u^{(1)}|_{t=0} = S_1 u_0$, then

$$u^{(1)} = S_1 u_0, \qquad \forall t \in \mathbb{R}.$$

Therefore, we get

$$\|u^{(1)}\|_{H^s} = \|S_1 u_0\|_{H^s} \leq \|u_0\|_{H^s} \leq C_0\|u_0\|_{H^s}.$$

For fixed $n \in \mathbb{N}$, we assume

$$\sup_{0\leq t\leq T}\|u^{(n)}\|_{H^s} \leq C_0\|u_0\|_{H^s}. \tag{3.22}$$

Applying Gronwall's inequality to (3.19),

$$\|u^{(n+1)}\|_{H^s} \le e^{\frac{1}{2}C_0\int_0^t \|u^{(n)}\|_{H^s}d\tau}\left(\|u^{(n+1)}(0)\|_{H^s} + \frac{1}{2}C_0\int_0^t \left(\|u^{(n)}\|_{H^s}^2 + \|u^{(n)}\|_{H^s}^3 + \|u^{(n)}\|_{H^s}^4\right)d\tau\right). \tag{3.23}$$

Now, if we consider our assumption (3.22), then we have

$$\|u^{(n+1)}\|_{H^s} \le e^{\frac{C_0^2}{2}\|u_0\|_{H^s}T}\left(\|u^{(n+1)}(0)\|_{H^s} + \frac{1}{2}\left(C_0^3\|u_0\|_{H^s}^2 + C_0^4\|u_0\|_{H^s}^3 + C_0^5\|u_0\|_{H^s}^4\right)T\right).$$

Also, we have

$$\|u^{(n+1)}(0)\|_{H^s} = \|S_{n+1}u_0\|_{H^s} \le \|u_0\|_{H^s},$$

so

$$\begin{aligned}\|u^{(n+1)}\|_{H^s} &\le e^{\frac{C_0^2}{2}\|u_0\|_{H^s}T}\left(\|u_0\|_{H^s} + \frac{1}{2}\left(C_0^3\|u_0\|_{H^s}^2 + C_0^4\|u_0\|_{H^s}^3 + C_0^5\|u_0\|_{H^s}^4\right)T\right)\\ &= e^{\frac{C_0^2}{2}\|u_0\|_{H^s}T}\left[1 + \frac{T}{2}\left(C_0^3\|u_0\|_{H^s} + C_0^4\|u_0\|_{H^s}^2 + C_0^5\|u_0\|_{H^s}^3\right)\right]\|u_0\|_{H^s}.\end{aligned}$$

From our assumption (3.20) and $C_0 \ge 2$, we have

$$C_0^2\|u_0\|_{H^s}T \le C_0^2 max\{\|u_0\|^3, 1\}T < \frac{1}{C_0^3} \le \frac{1}{8},$$

$$C_0^3\|u_0\|_{H^s}T \le C_0^3 max\{\|u_0\|^3, 1\}T < \frac{1}{C_0^2} \le \frac{1}{4},$$

and

$$C_0^4\|u_0\|_{H^s}^2T \le C_0^4 max\{\|u_0\|^3, 1\}T < \frac{1}{C_0} \le \frac{1}{2}.$$

Thus, we obtain

$$\|u^{(n+1)}\|_{H^s} \le \frac{15}{8}e^{\frac{1}{16}}\|u_0\|_{H^s} \le 2\|u_0\|_{H^s} \le C_0\|u_0\|_{H^s}$$

which implies that the claim (3.21) holds for $\forall n \in \mathbb{N}$. Therefore, the sequence $\left\{u^{(n)}\right\}_{n\in\mathbb{N}}$ is uniformly bounded in $C\left([0,T];H^s\right)$ and then $\left\{u^{(n)}\partial_x u^{(n+1)}\right\}_{n\in\mathbb{N}}$ is uniformly bounded in $C\left([0,T];H^{s-1}\right)$ where we use the fact that

$$\begin{aligned}\|u^{(n)}\partial_x u^{(n+1)}\|_{H^{s-1}} &\le C\|u^{(n)}\|_{H^{s-1}}\|\partial_x u^{(n+1)}\|_{H^{s-1}}\\ &\le C\|u^{(n)}\|_{H^{s-1}}\|u^{(n+1)}\|_{H^s}\\ &\le C{C_0}^2\|u_0\|_{H^s}^2.\end{aligned}$$

Thanks to the equation (3.7) equation , we obtain that $\partial_t u^{(n+1)}$ is uniformly bounded in $C\left([0,T];H^{s-1}\right)$. Therefore we deduce that $\left\{u^{(n)}\right\}_{n\in\mathbb{N}}^{\infty}$ is uniformly bounded in $C([0,T];H^s)\cap C^1([0,T];H^{s-1})$.

Step 3: In this step, we will prove that $\left\{u^{(n)}\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C\left([0,T];H^{s-1}\right)$. In fact, from (3.7), we have for all $n,m\in\mathbb{N}$

$$\begin{aligned}
&\partial_t u^{(m+n+1)} + u^{(m+n)}\partial_x u^{(m+n+1)} + \left[\frac{\beta_0}{\beta} + \left(c-\frac{\beta_0}{\beta}\right)\left(1-\partial_x^2\right)^{-1}\right]\partial_x u^{(m+n+1)} \\
&= -\left(1-\partial_x^2\right)^{-1}\partial_x\left\{\left(\frac{3-b}{2}\right)\partial_x\left[u^{(m+n)}\right]^2 + \frac{b}{2}\left[u^{(m+n)}\right]^2 + \frac{\omega_1}{3\alpha^2}\left[u^{(m+n)}\right]^3 + \frac{\omega_2}{4\alpha^3}\left[u^{(m+n)}\right]^4\right\},
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
&\partial_t u^{(n+1)} + u^{(n)}\partial_x u^{(n+1)} + \left[\frac{\beta_0}{\beta} + \left(c-\frac{\beta_0}{\beta}\right)\left(1-\partial_x^2\right)^{-1}\right]\partial_x u^{(n+1)} \\
&= -\left(1-\partial_x^2\right)^{-1}\partial_x\left\{\left(\frac{3-b}{2}\right)\partial_x\left[u^{(n)}\right]^2 + \frac{b}{2}\left[u^{(n)}\right]^2 + \frac{\omega_1}{3\alpha^2}\left[u^{(n)}\right]^3 + \frac{\omega_2}{4\alpha^3}\left[u^{(n)}\right]^4\right\}.
\end{aligned} \tag{3.25}$$

From (3.24) and (3.25), we get

$$\begin{aligned}
&\partial_t\left(u^{(m+n+1)} - u^{(n+1)}\right) + \left[\frac{\beta_0}{\beta} + \left(c-\frac{\beta_0}{\beta}\right)\left(1-\partial_x^2\right)^{-1}\right]\partial_x\left(u^{(m+n+1)} - u^{(n+1)}\right) \\
&= -u^{(m+n)}\partial_x\left(u^{(m+n+1)} - u^{(n+1)}\right) - \left(u^{(m+n)} - u^{(n)}\right)\partial_x u^{(n+1)} \\
&- \left(1-\partial_x^2\right)^{-1}\partial_x\left\{\left(\frac{3-b}{2}\right)\left(\partial_x\left[u^{(m+n)}\right]^2 - \partial_x\left[u^{(n)}\right]^2\right) + \frac{b}{2}\left(\left[u^{(m+n)}\right]^2 - \left[u^{(n)}\right]^2\right)\right. \\
&\left. + \frac{\omega_1}{3\alpha^2}\left(\left[u^{(m+n)}\right]^3 - \left[u^{(n)}\right]^3\right) + \frac{\omega_2}{4\alpha^3}\left(\left[u^{(m+n)}\right]^4 - \left[u^{(n)}\right]^4\right)\right\}.
\end{aligned} \tag{3.26}$$

Then, applying Λ^{s-1} operator to the equation, we obtain the following

$$
\begin{aligned}
&\partial_t \Lambda^{s-1}\left(u^{(m+n+1)}-u^{(n+1)}\right)+\left[\frac{\beta_0}{\beta}+\left(c-\frac{\beta_0}{\beta}\right)\left(1-\partial_x^2\right)^{-1}\right]\partial_x \Lambda^{s-1}\left(u^{(m+n+1)}-u^{(n+1)}\right)\\
&=-\Lambda^{s-1}\left\{u^{(m+n)}\partial_x\left(u^{(m+n+1)}-u^{(n+1)}\right)\right\}-\Lambda^{s-1}\left\{\left(u^{(m+n)}-u^{(n)}\right)\partial_x u^{(n+1)}\right\}\\
&-\Lambda^{s-1}\left(1-\partial_x^2\right)^{-1}\partial_x\left\{\left(\frac{3-b}{2}\right)\left(\partial_x u^{(m+n)}-\partial_x u^{(n)}\right)\left(\partial_x u^{(m+n)}+\partial_x u^{(n)}\right)\right.\\
&+\frac{b}{2}\left(u^{(m+n)}-u^{(n)}\right)\left(u^{(m+n)}+u^{(n)}\right)\\
&+\frac{\omega_1}{3\alpha^2}\left(u^{(m+n)}-u^{(n)}\right)\left(\left[u^{(m+n)}\right]^2+u^{(m+n)}u^{(n)}+\left[u^{(n)}\right]^2\right)+\\
&\left.\frac{\omega_2}{4\alpha^3}\left(u^{(m+n)}-u^{(n)}\right)\left(\left[u^{(m+n)}\right]^3+\left[u^{(m+n)}\right]^2u^{(n)}+u^{(m+n)}\left[u^{(n)}\right]^2+\left[u^{(n)}\right]^3\right)\right\}.
\end{aligned}
\tag{3.27}
$$

If we multiply this equation by $\Lambda^{s-1}\left(u^{(m+n+1)}-u^{(n+1)}\right)$ and take an integration on $\mathbb{R}$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u^{(m+n+1)}-u^{(n+1)}\|_{H^{s-1}}{}^2\\
&=\int_{\mathbb{R}}\Lambda^{s-1}\left[u^{(m+n)}\partial_x\left(u^{(m+n+1)}-u^{(n+1)}\right)\right]\Lambda^{s-1}\left(u^{(m+n+1)}-u^{(n+1)}\right)dx\\
&-\int_{\mathbb{R}}\Lambda^{s-1}\left[\left(u^{(m+n)}-u^{(n)}\right)\partial_x u^{(n+1)}\right]\Lambda^{s-1}\left(u^{(m+n+1)}-u^{(n+1)}\right)dx\\
&-\int_{\mathbb{R}}\Lambda^{s-3}\partial_x\left\{\left(\frac{3-b}{2}\right)\left(\partial_x u^{(m+n)}-\partial_x u^{(n)}\right)\left(\partial_x u^{(m+n)}+\partial_x u^{(n)}\right)\right.\\
&+\left(u^{(m+n)}-u^{(n)}\right)\left[\frac{b}{2}\left(u^{(m+n)}+u^{(n)}\right)+\frac{\omega_1}{3\alpha^2}\left(\left[u^{(m+n)}\right]^2+u^{(m+n)}u^{(n)}+\left[u^{(n)}\right]^2\right)\right.\\
&\left.\left.+\frac{\omega_2}{4\alpha^3}\left(\left[u^{(m+n)}\right]^3+\left[u^{(m+n)}\right]^2u^{(n)}+u^{(m+n)}\left[u^{(n)}\right]^2+\left[u^{(n)}\right]^3\right)\right]\right\}\Lambda^{s-1}\left(u^{(m+n+1)}-u^{(n+1)}\right)\\
&=I_1+I_2+I_3.
\end{aligned}
\tag{3.28}
$$

Apply commutator estimate to I_1, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \Lambda^{s-1}\left[u^{(m+n)}\partial_x\left(u^{(m+n+1)}-u^{(n+1)}\right)\right]\Lambda^{s-1}\left(u^{(m+n+1)}-u^{(n+1)}\right)dx\\
&=\int_{\mathbb{R}} u^{(m+n)}\partial_x\Lambda^{s-1}\left(u^{(m+n+1)}-u^{(n+1)}\right)\left(u^{(m+n+1)}-u^{(n+1)}\right)dx\\
&+\int_{\mathbb{R}}\left[\Lambda^{s-1},u^{(m+n)}\right]\partial_x\left(u^{(m+n+1)}-u^{(n+1)}\right)\Lambda^{s-1}\left(u^{(m+n+1)}-u^{(n+1)}\right)dx\\
&=I_{11}+I_{12}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
|I_{11}| &\le \frac{1}{2}\|u^{(m+n)}{}_x\|_{L^\infty}\|\Lambda^{s-1}\left(u^{(m+n+1)}-u^{(n+1)}\right)\|_{L^2}^2\\
&\le \frac{1}{2}\|u^{(m+n)}{}_x\|_{L^\infty}\|\left(u^{(m+n+1)}-u^{(n+1)}\right)\|_{H^{s-1}}^2 \qquad (3.29)\\
&\le C\|u^{(m+n)}\|_{H^s}\|\left(u^{(m+n+1)}-u^{(n+1)}\right)\|_{H^{s-1}}^2
\end{aligned}$$

and by Banach algebra

$$\begin{aligned}
|I_{12}| &\le \|u^{(m+n+1)}-u^{(n+1)}\|_{H^{s-1}}\|\left[\Lambda^{s-1},u^{(m+n)}\right]\partial_x\left(u^{(m+n+1)}-u^{(n+1)}\right)\|_{L^2}\\
&\le C\|u^{(m+n+1)}-u^{(n+1)}\|_{H^{s-1}}\|u^{(m+n)}{}_x\|_{H^{s-1}}\|\partial_x\left(u^{(m+n+1)}-u^{(n+1)}\right)\|_{H^{s-2}} \qquad (3.30)\\
&\le C\|u^{(m+n)}\|_{H^s}\|u^{(m+n+1)}-u^{(n+1)}\|_{H^{s-1}}^2.
\end{aligned}$$

From (3.29) and (3.30), we get

$$\begin{aligned}
|I_{11}|+|I_{12}| &\le C\|u^{(m+n)}\|_{H^s}\|u^{(m+n+1)}-u^{(n+1)}\|_{H^{s-1}}^2\\
&\le C\|u_0\|_{H^s}\|u^{(m+n+1)}-u^{(n+1)}\|_{H^{s-1}}^2.
\end{aligned}$$

Also, applying commutator estimate to I_2, we have

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}}\Lambda^{s-1}\left[\left(u^{(m+n)}-u^{(n)}\right)\partial_x u^{(n+1)}\right]\Lambda^{s-1}\left(u^{(m+n+1)}-u^{(n+1)}\right)dx\\
&=\int_{\mathbb{R}}\left(u^{(m+n)}-u^{(n)}\right)\partial_x\Lambda^{s-1}u^{(n+1)}\Lambda^{s-1}\left(u^{(m+n+1)}-u^{(n+1)}\right)dx\\
&+\int_{\mathbb{R}}\left[\Lambda^{s-1},u^{(m+n)}-u^{(n)}\right]\partial_x u^{(n+1)}\Lambda^{s-1}\left(u^{(m+n+1)}-u^{(n+1)}\right)dx\\
&=I_{21}+I_{22}.
\end{aligned}$$

Therefore, we obtain

$$
\begin{aligned}
|I_{21}| &\leq \|u^{(m+n)} - u^{(n)}\|_{L^2} \|\partial_x \Lambda^{s-1} u^{(n+1)}\|_{L^\infty} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&\leq \|u^{(m+n)} - u^{(n)}\|_{L^2} \|u^{(n+1)}\|_{H^s} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}},
\end{aligned}
\tag{3.31}
$$

and

$$
\begin{aligned}
|I_{22}| &\leq \| \left[\Lambda^{s-1}, u^{(m+n)} - u^{(n)}\right] \partial_x u^{(n+1)}\|_{L^2} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&\leq C' \|\partial_x \left(u^{(m+n)} - u^{(n)}\right)\|_{H^{s-1}} \|\partial_x u^{(n+1)}\|_{H^{s-2}} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&\leq C' \|u^{(m+n)} - u^{(n)}\|_{H^s} \|u^{(n+1)}\|_{H^{s-1}} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&\leq C' \|u^{(m+n)} - u^{(n)}\|_{H^s} \|u^{(n+1)}\|_{H^s} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}.
\end{aligned}
\tag{3.32}
$$

From (3.31) and (3.32), we have

$$
\begin{aligned}
|I_{21}| + |I_{22}| &\leq C \|u^{(n+1)}\|_{H^s} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \|u^{(m+n)} - u^{(n)}\|_{H^s} \\
&\leq \|u_0\|_{H^s} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \|u^{(m+n)} - u^{(n)}\|_{H^s}.
\end{aligned}
$$

In addition, we get

$$
\begin{aligned}
|I_3| \leq \Bigg\| &\partial_x \Bigg\{ \left(\frac{3-b}{2}\right) \left(\partial_x u^{(m+n)} - \partial_x u^{(n)}\right) \left(\partial_x u^{(m+n)} + \partial_x u^{(n)}\right) \\
&+ \left(u^{(m+n)} - u^{(n)}\right) \Bigg[\frac{b}{2} \left(u^{(m+n)} + u^{(n)}\right) + \frac{\omega_1}{3\alpha^2} \left(\left[u^{(m+n)}\right]^2 + u^{(m+n)} u^{(n)} + \left[u^{(n)}\right]^2 \right) \\
&+ \frac{\omega_2}{4\alpha^3} \left(\left[u^{(m+n)}\right]^3 + \left[u^{(m+n)}\right]^2 u^{(n)} + u^{(m+n)} \left[u^{(n)}\right]^2 \right. \\
&\left. + \left[u^{(n)}\right]^3 \right) \Bigg] \Bigg\} \Bigg\|_{H^{s-3}} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}},
\end{aligned}
\tag{3.33}
$$

so

$$
\begin{aligned}
|I_3| \leq \Bigg\| &\Bigg\{ \left(\frac{3-b}{2}\right) \left(\partial_x u^{(m+n)} - \partial_x u^{(n)}\right) \left(\partial_x u^{(m+n)} + \partial_x u^{(n)}\right) \\
&+ \left(u^{(m+n)} - u^{(n)}\right) \Bigg[\frac{b}{2} \left(u^{(m+n)} + u^{(n)}\right) + \frac{\omega_1}{3\alpha^2} \left(\left[u^{(m+n)}\right]^2 + u^{(m+n)} u^{(n)} + \left[u^{(n)}\right]^2 \right) \\
&+ \frac{\omega_2}{4\alpha^3} \left(\left[u^{(m+n)}\right]^3 + \left[u^{(m+n)}\right]^2 u^{(n)} + u^{(m+n)} \left[u^{(n)}\right]^2 \right. \\
&\left. + \left[u^{(n)}\right]^3 \right) \Bigg] \Bigg\} \Bigg\|_{H^{s-2}} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}.
\end{aligned}
$$

To simplify this inequality, we obtain

$$\Big\|\Big\{\left(\frac{3-b}{2}\right)\left(\partial_x u^{(m+n)}-\partial_x u^{(n)}\right)\left(\partial_x u^{(m+n)}+\partial_x u^{(n)}\right)$$
$$+\left(u^{(m+n)}-u^{(n)}\right)\left[\frac{b}{2}\left(u^{(m+n)}+u^{(n)}\right)+\frac{\omega_1}{3\alpha^2}\left(\left[u^{(m+n)}\right]^2+u^{(m+n)}u^{(n)}+\left[u^{(n)}\right]^2\right)\right.$$
$$\left.+\frac{\omega_2}{4\alpha^3}\left(\left[u^{(m+n)}\right]^3+\left[u^{(m+n)}\right]^2u^{(n)}+u^{(m+n)}\left[u^{(n)}\right]^2+\left[u^{(n)}\right]^3\right)\right]\Big\}\Big\|_{H^{s-2}}$$

$$\leq\left(\frac{3-b}{2}\right)\|\partial_x\left(u^{(m+n)}-u^{(n)}\right)\|_{H^{s-2}}\|\partial_x\left(u^{(m+n)}+u^{(n)}\right)\|_{H^{s-2}}$$
$$+C\|u^{(m+n)}-u^{(n)}\|_{H^{s-2}}\left\{\|u^{(m+n)}\|_{H^{s-2}}+\|u^{(n)}\|_{H^{s-2}}+\|u^{(m+n)}\|^2_{H^{s-2}}+\|u^{(n)}\|^2_{H^{s-2}}\right.$$
$$+\|u^{(m+n)}\|_{H^{s-2}}\|u^{(n)}\|_{H^{s-2}}+\|u^{(m+n)}\|^3_{H^{s-2}}+\|u^{(m+n)}\|^2_{H^{s-2}}\|u^{(n)}\|_{H^{s-2}}$$
$$\left.+\|u^{(m+n)}\|_{H^{s-2}}\|u^{(n)}\|^2_{H^{s-2}}+\|u^{(n)}\|^3_{H^{s-2}}\right\}$$

$$\leq\left(\frac{3-b}{2}\right)\|u^{(m+n)}-u^{(n)}\|_{H^{s-1}}\|u^{(m+n)}+u^{(n)}\|_{H^{s-1}}$$
$$+C\|u^{(m+n)}-u^{(n)}\|_{H^{s-1}}\left\{\|u^{(m+n)}\|_{H^{s-1}}+\|u^{(n)}\|_{H^{s-1}}+\|u^{(m+n)}\|^2_{H^{s-1}}+\|u^{(n)}\|^2_{H^{s-1}}\right.$$
$$+\|u^{(m+n)}\|_{H^{s-1}}\|u^{(n)}\|_{H^{s-1}}+\|u^{(m+n)}\|^3_{H^{s-1}}+\|u^{(m+n)}\|^2_{H^{s-1}}\|u^{(n)}\|_{H^{s-1}}$$
$$\left.+\|u^{(m+n)}\|_{H^{s-1}}\|u^{(n)}\|^2_{H^{s-1}}+\|u^{(n)}\|^3_{H^{s-1}}\right\}$$

$$\leq C'\|u^{(m+n)}-u^{(n)}\|_{H^{s-1}}\left\{\|u^{(m+n)}\|_{H^{s-1}}+\|u^{(n)}\|_{H^{s-1}}+\|u^{(m+n)}\|^2_{H^{s-1}}+\|u^{(n)}\|^2_{H^{s-1}}\right.$$
$$+\|u^{(m+n)}\|_{H^{s-1}}\|u^{(n)}\|_{H^{s-1}}+\|u^{(m+n)}\|^3_{H^{s-1}}+\|u^{(m+n)}\|^2_{H^{s-1}}\|u^{(n)}\|_{H^{s-1}}$$
$$\left.+\|u^{(m+n)}\|_{H^{s-1}}\|u^{(n)}\|^2_{H^{s-1}}+\|u^{(n)}\|^3_{H^{s-1}}\right\},$$

where we use $\|\partial_x u^{(n)}\|_{H^{s-1}} \leq \|u^{(n)}\|_{H^{s-1}}$ and $\|\partial_x u^{(n)}\|_{H^{1/2}\cap L^\infty} \leq \|u^{(n)}\|_{H^s}$, for all $s > 3/2$. By using $\|u^{(n)}\|_{H^{s-1}} < C\|u_0\|_{H^{s-1}}$,

$$\begin{aligned}
&\Bigg\| \Bigg\{ \left(\frac{3-b}{2}\right) \left(\partial_x u^{(m+n)} - \partial_x u^{(n)}\right) \left(\partial_x u^{(m+n)} + \partial_x u^{(n)}\right) \\
&\left(u^{(m+n)} - u^{(n)}\right) \left[\frac{b}{2}\left(u^{(m+n)} + u^{(n)}\right) + \frac{\omega_1}{3\alpha^2}\left(\left[u^{(m+n)}\right]^2 + u^{(m+n)}u^{(n)} + \left[u^{(n)}\right]^2\right)\right. \\
&\left.+\frac{\omega_2}{4\alpha^3}\left(\left[u^{(m+n)}\right]^3 + \left[u^{(m+n)}\right]^2 u^{(n)} + u^{(m+n)}\left[u^{(n)}\right]^2 + \left[u^{(n)}\right]^3\right)\right]\Bigg\}\Bigg\|_{H^{s-2}} \\
&\leq C'\|u^{(m+n)} - u^{(n)}\|_{H^{s-1}}\left(\|u_0\|_{H^{s-1}} + \|u_0\|^2_{H^{s-1}} + \|u_0\|^3_{H^{s-1}}\right).
\end{aligned} \tag{3.34}$$

Combining (3.33) and (3.34), we have

$$\begin{aligned}
|I_3| &\leq C'\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}\|u^{(m+n)} - u^{(n)}\|_{H^{s-1}}\left(\|u_0\|_{H^{s-1}} + \|u_0\|^2_{H^{s-1}} + \|u_0\|^3_{H^{s-1}}\right) \\
&\leq C'\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}\|u^{(m+n)} - u^{(n)}\|_{H^s}\left(\|u_0\|_{H^s} + \|u_0\|^2_{H^s} + \|u_0\|^3_{H^s}\right).
\end{aligned} \tag{3.35}$$

Considering all results from (3.28) to (3.35), we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}{}^2 &\leq C\left\{\|u^{(m+n+1)} - u^{(n+1)}\|^2_{H^{s-1}}\|u_0\|_{H^s}\right. \\
&\quad +\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}\|u^{(m+n)} - u^{(n)}\|_{H^s}\|u_0\|_{H^s} \\
&\left.+\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}\|u^{(m+n)} - u^{(n)}\|_{H^s}\left(\|u_0\|_{H^s} + \|u_0\|^2_{H^s} + \|u_0\|^3_{H^s}\right)\right\} \\
&\leq C\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}\left[\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} +\right. \\
&\left.\|u^{(m+n)} - u^{(n)}\|_{H^s}\right]\left(\|u_0\|_{H^s} + \|u_0\|^2_{H^s} + \|u_0\|^3_{H^s}\right).
\end{aligned}$$

So, we can have

$$\frac{d}{dt}\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \leq C'\left(\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} + \|u^{(m+n)} - u^{(n)}\|_{H^s}\right).$$

Thanks to the Gronwall's inequality, we have

$$\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \leq e^{\int_0^t C'ds}\left(\|u_0{}^{(m+n+1)} - u_0{}^{(n+1)}\|_{H^{s-1}} + \int_0^t \|u^{(m+n)} - u^{(n)}\|_{H^s}ds\right).$$

Then, we obtain

$$\|u^{(m+n+1)}-u^{(n+1)}\|_{H^{s-1}} \leq C_T \left(\|{u_0}^{(m+n+1)} - {u_0}^{(n+1)}\|_{H^{s-1}} + \int_0^t \|u^{(m+n)} - u^{(n)}\|_{H^s} d\tau \right). \tag{3.36}$$

Notice that

$${u_0}^{(m+n+1)} - {u_0}^{(n+1)} = S_{m+n+1}u_0 - S_{n+1}u_0 = 1_{2\times 2^{n+2}\leq|\xi|\leq 2\times 2^{m+n+2}}(\xi)u_0(\xi).$$

Then, we have

$$\begin{aligned}
\|{u_0}^{(m+n+1)} - {u_0}^{(n+1)}\|_{H^{s-1}}^2 &= \int_{\mathbb{R}} \left(1+|\xi|^2\right)^{s-1} |\widehat{u_0}(\xi)|^2 1_{2^{n+3}\leq|\xi|\leq 2^{m+n+3}} d\xi \\
&= \int_{2^{n+3}\leq|\xi|\leq 2^{m+n+3}} \left(1+|\xi|^2\right)^{s-1} |\widehat{u_0}(\xi)|^2 d\xi \\
&\leq 2^{-2(n+3)} \int_{2^{n+3}\leq|\xi|\leq 2^{m+n+3}} |\xi|^2 \left(1+|\xi|^2\right)^{s-1} |\widehat{u_0}(\xi)|^2 d\xi \\
&\leq 2^{-2n} \frac{1}{8^2} \|u_0\|_{H^s}^2,
\end{aligned}$$

which implies that

$$\|{u_0}^{(m+n+1)} - {u_0}^{(n+1)}\|_{H^{s-1}} \leq \frac{1}{2} 2^{-n} \|u_0\|_{H^s}.$$

Therefore, from (3.36) we get that $\forall t \in [0, T]$

$$\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \leq C_T \left(2^{-n} + \int_0^t \|u^{(m+n)} - u^{(n)}\|_{H^s} d\tau \right).$$

Applying induction argument, we can prove that for all $t \in [0, T]$

$$\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \leq \sum_{k=0}^{n} C_T 2^{-(n-k)} (C_T t)^k \frac{1}{k!} + {C_T}^{n+1} \frac{T^{n+1}}{(n+1)!} \|u^{(m)} - u^{(0)}\|_{L_T^\infty H^{s-1}}.$$

Due to the fact that $\|u^{(m+n)} - u^{(0)}\|_{L_T^\infty H^s} \leq \|u^{(m)}\|_{L_T^\infty H^s} \leq C_0 \|u_0\|_{H^s}$ implies that for all $t \in [0, T]$, $\forall m, n \in \mathbb{N}$ and some ${C_T}'$ (independent of m,n)

$$\begin{aligned}
\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} &\leq 2^{-n} \left(\sum_{k=0}^{n} C_T \frac{(2C_T T)^k}{k!} + C_0 \|u_0\|_{H^s} \frac{(2C_T T)^{n+1}}{(n+1)!} \right) \\
&\leq {C_T}' 2^{-n}.
\end{aligned} \tag{3.37}$$

Therefore, $\{u^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];H^{s-1})$. Since the Banach space is complete $C([0,T];H^{s-1})$, we get a limit $u \in C([0,T];H^{s-1})$ such that $u^{(n)} \to u$ in $C([0,T];H^{s-1})$.

Step 4: (Passing to the limit) Since $\{u^{(n)}\}_{n\in\mathbb{N}}^{\infty}$ is uniformly bounded in $C([0,T];H^s)\cap C^1([0,T];H^{s-1})$ and using interpolation inequality, we have for any $s' \in (s-1, s')$,

$$
\begin{aligned}
\|u^{(m+n+1)} - u^{(n)}\|_{H^{s'}} &\le C\|u^{(m+n+1)} - u^{(n)}\|_{H^{s-1}}^{s-s'}\|u^{(m+n+1)} - u^{(n)}\|_{H^s}^{1+s'-s} \\
&\le C\|u^{(m+n+1)} - u^{(n)}\|_{H^{s-1}}^{s-s'}\left(\|u^{(m+n+1)}\|_{H^s} + \|u^{(n)}\|_{H^s}\right)^{1+s'-s} \\
&\le C\|u^{(m+n+1)} - u^{(n)}\|_{H^{s-1}}^{s-s'}\|u_0\|_{H^s}^{1+s'-s} \\
&\le C'\|u^{(m+n+1)} - u^{(n)}\|_{H^{s-1}}^{s-s'}
\end{aligned},
$$

which along with (3.37) implies that $\{u^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C\left([0,T];H^{s'}\right)$, for all $s' \in (s-1,s)$. By the uniqueness of the limit, we deduce that $u^{(n)} \to u$ in $C\left([0,T];H^{s'}\right)$ which yields that $u^{(n)}\partial_x u^{(n+1)} \to u\partial_x u$ in $C\left([0,T];H^{s'-1}\right)$, where $s' > \frac{3}{2}$. Therefore, we have

$$
\left[\frac{\beta_0}{\beta} + \left(c - \frac{\beta_0}{\beta}\right)p*\right]\partial_x u^{(n+1)} \to \left[\frac{\beta_0}{\beta} + \left(c - \frac{\beta_0}{\beta}\right)p*\right]\partial_x u
$$

in $C\left([0,T];H^{s'-1}\right)$, where $s' > \frac{3}{2}$;

$$
p*\partial_x\left\{\frac{b}{2}\left[u^{(n)}\right]^2 + \frac{\omega_1}{3\alpha^2}\left[u^{(n)}\right]^3 + \frac{\omega_2}{4\alpha^3}\left[u^{(n)}\right]^4\right\} \to p*\partial_x\left\{\frac{b}{2}u^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right\}
$$

in $C\left([0,T];H^{s'+1}\right)$ where $s' > -\frac{1}{2}$; and

$$
p*\partial_x\left(\frac{3-b}{2}\right)\partial_x\left[u^{(n)}\right]^2 \to p*\partial_x\left(\frac{3-b}{2}\right)\partial_x u^2
$$

in $C\left([0,T];H^{s'}\right)$ where $s' > \frac{1}{2}$. Then, from (3.25), it implies that $\{\partial_t u^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C\left([0,T];H^{s'-1}\right)$, for any $s' > \frac{3}{2}$ and there exists a $v \in C\left([0,T];H^{s'-1}\right)$ such that $\partial_t u^{(n)} \to v$ in $C\left([0,T];H^{s'-1}\right)$, for all $\frac{3}{2} < s' < s$.

On the other hand, since $u^{(n)}$ converges to u in $C\left([0,T];H^{s'}\right)$ and $\left\{u^{(n)}\right\}_{n\in\mathbb{N}}^{\infty}$ is uniformly bounded in $C([0,T];H^s)\cap C^1([0,T];H^{s-1})$ we conclude that $\partial_t u^{(n)}\to\partial_t u$ in the sense of distribution which along with $\exists v$ in $C\left([0,T];H^{s'-1}\right)$ such that $\partial_t u^{(n)}\to v$ in $C\left([0,T];H^{s'-1}\right), \forall s'\in(\frac{3}{2},s)$ implies that $v=\partial_t u$ and $\partial_t u^{(n)}\to\partial_t u$ in $C\left([0,T];H^{s'-1}\right)$ for any $s'\in(\frac{3}{2},s)$.

Therefore, it follows that

$$\begin{cases} \partial_t u + u\partial_x u + \left[\frac{\beta_0}{\beta} + \left(c-\frac{\beta_0}{\beta}\right)p*\right]\partial_x u \\ \qquad\qquad\qquad = -p*\partial_x\left\{\left(\frac{3-b}{2}\right)u_x^2 + \frac{b}{2}u^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right\}, \\ u|_{t=0} = u_0, \end{cases} \tag{3.38}$$

in $C\left([0,T];H^{s'-1}\right), \forall\frac{3}{2}<s'<s$ and $u\in C\left([0,T];H^{s'}\right)\cap C^1\left([0,T];H^{s'-1}\right)$.

Moreover, by the Banach-Alaoglu Theorem, since $\left\{u^{(n)}\right\}_{n\in\mathbb{N}}^{\infty}$ is uniformly bounded in $C([0,T];H^s)\cap C^1([0,T];H^{s-1})$, there is a subsequence $\left\{u^{(n_j)}\right\}_{j\in\mathbb{N}}$ of $\left\{u^{(n)}\right\}_{n\in\mathbb{N}}$ such that

$$u^{(n_j)} \overset{weakly}{\rightharpoonup} u^*$$

in $L^2([0,T];H^s)$. Also, thanks to $L^\infty([0,T];H^s)\hookrightarrow L^2([0,T];H^s)$, for $\forall t\in[0,T]$;

$$\begin{aligned} u^{(n_j)} &\overset{weakly}{\rightharpoonup} u^* \quad in \quad H^s \\ \partial_t u^{(n_j)} &\overset{weakly}{\rightharpoonup} \partial_t u \quad in \quad H^{s-1} \end{aligned} \tag{3.39}$$

which along with the fact that $u^{(n)}\to u$ in $C\left([0,T];H^{s'}\right)$ implies

$$u = u^* \in L^\infty([0,T];H^s)\cap Lip\left([0,T];H^{s-1}\right).$$

Next, we claim that $u\in C_w([0,T];H^s)$ that means for all $\varphi\in H^{-s}$, $[u(t),\varphi]_{H^s\times H^{-s}}$ is continuous with respect to $t\in[0,T]$.

In fact, $\forall\varphi\in H^{-s'}$, for $s'\in(\frac{3}{2},s)$, $u^{(n_j)}\rightharpoonup u$ in $C\left([0,T];H^{s'}\right)$. Then,

$$[u^{n_j}(t),\varphi]_{H^{s'}\times H^{-s'}} \to [u(t),\varphi]_{H^{s'}\times H^{-s'}} \quad \text{uniformly on} \quad [0,T].$$

While $H^{-s'}$ is dense in H^{-s}, for $s' < s$, we get from last result that for all $\psi \in H^{-s}$, $[u^{n_j}(t), \psi]_{H^s \times H^{-s}} \to [u(t), \psi]_{H^s \times H^{-s}}$ is uniformly on [0,T]. Since $[u^{n_j}(t), \psi]_{H^s \times H^{-s}} \in C([0,T])$ and it is uniformly continuous, we obtain that its limit $[u(t), \psi]_{H^s \times H^{-s}}$ is uniformly continuous on $[0,T]$ which implies that the claim holds.

Up to a subsequence, we get from (3.39) that for fixed $t \in [0,T]$,

$$\limsup_{n\to\infty} \|u^{(n)}(t)\|_{H^s} \geq \|u(t)\|_{H^s}$$

which implies that

$$\begin{aligned} \|u(t)\|_{H^s} &\leq \lim_{n\to\infty} \sup \|u^{(n)}(t)\|_{H^s} \\ &\leq \|u_0\|_{H^s} e^{\frac{C_0{}^2}{2}\|u_0\|_{H^s} t}. \end{aligned} \tag{3.40}$$

Hence, we have

$$\limsup_{t\to 0^+} \|u^{(n)}(t)\|_{H^s} \leq \|u_0\|_{H^s}. \tag{3.41}$$

On the other hand, from $u \in C_w([0,T];H^s)$ and the fact that

$$\|f\|_{H^s} = \sup_{\|\psi\|_{H^{-s}} \leq 1} |\langle f, \psi\rangle_{H^s \times H^{-s}}|,$$

we get

$$\liminf_{t\to 0^+} \|u(t)\|_{H^s} \geq \|u_0\|_{H^s}$$

which along with (3.41) implies that

$$\lim_{t\to 0^+} \|u(t)\|_{H^s} = \|u_0\|_{H^s}$$

i.e. $\|u(t)\|_{H^s}$ is strongly right continuous at t=0. Similarly we may get that $\|u(t)\|_{H^s}$ is strongly left continuous at t=0 and so $\|u(t)\|_{H^s}$ is strongly continuous at t=0. It remains to prove continuity of $\|u(t)\|_{H^s}$ at times other than the initial time.For this, let's first prove the uniqueness of the solution $u \in C([0,T];H^{s'}) \cap L^\infty([0,T];H^s)$ with $\partial_t u \in C([0,T];H^{s'-1}) \cap L^\infty([0,T];H^s)$

Step 5: (Uniqueness)

It is easy to prove the uniqueness of the solution of the Rotational b-family of equations which process is similar to the one in Step 3.

In fact, assume that $u, v \in C([0,T]; H^{s'}) \cap L^{\infty}([0,T]; H^{s})$ with $\partial_t u, \partial_t v \in C([0,T]; H^{s'-1}) \cap L^{\infty}([0,T]; H^{s-1})$ where $\frac{3}{2} < s' < s$ and $u|_{t=0} = v|_{t=0}$ solve the Rotational b-family of equations, then we have

$$\partial_t (u-v) + u\partial_x(u-v) + (u-v)\partial_x v + \left[\frac{\beta_0}{\beta} + \left(c - \frac{\beta_0}{\beta}\right) p*\right] \partial_x(u-v)$$
$$= -p * \partial_x \left\{ \left(\frac{3-b}{2}\right) ((u_x^2 - v_x^2) + \frac{b}{2}(u^2 - v^2) + \frac{\omega_1}{3\alpha^2}(u^3 - v^3) + \frac{\omega_2}{4\alpha^3}(u^4 - v^4) \right\}$$

Let $u - v = r$, so

$$\partial_t r + u\partial_x r + r\partial_x v + \left[\frac{\beta_0}{\beta} + \left(c - \frac{\beta_0}{\beta}\right) p*\right] \partial_x r$$
$$= -p * \partial_x \left\{ \left(\frac{3-b}{2}\right) r_x(u_x + v_x) + \frac{b}{2} r(u+v) + \frac{\omega_1}{3\alpha^2} r(u^2 + uv + v^2) + \frac{\omega_2}{4\alpha^3} r(u+v)(u^2+v^2) \right\}.$$

Apply $\Lambda^{s'-1}$ operator , multiply by $\Lambda^{s'-1}r$ and integrate over $\mathbb{R}$, we have

$$\langle \Lambda^{s'-1}\partial_t r, \Lambda^{s'-1} r\rangle + \langle \Lambda^{s'-1} u\partial_x r, \Lambda^{s'-1} r\rangle + \langle \Lambda^{s'-1} r\partial_x v, \Lambda^{s'-1} r\rangle + \left[\frac{\beta_0}{\beta} + \left(c - \frac{\beta_0}{\beta}\right) p*\right] \langle \Lambda^{s'-1}\partial_x r, \Lambda^{s'-1} r\rangle$$
$$= -\langle \Lambda^{s'-3}\partial_x \left\{ \left(\frac{3-b}{2}\right) r_x(u_x + v_x) + \frac{b}{2} r(u+v) + \frac{\omega_1}{3\alpha^2} r(u^2 + uv + v^2) + \frac{\omega_2}{4\alpha^3} r(u+v)(u^2+v^2) \right\}, \Lambda^{s'-1} r\rangle,$$

which is equivalent to

$$\frac{d}{dt}\|\Lambda^{s'-1} r\|_{L^2}^2 = -2\langle \Lambda^{s'-1} u\partial_x r, \Lambda^{s'-1} r\rangle - 2\langle \Lambda^{s'-1} r\partial_x v, \Lambda^{s'-1} r\rangle - 2\left[\frac{\beta_0}{\beta} + \left(c - \frac{\beta_0}{\beta}\right) p*\right] \langle \Lambda^{s'-1}\partial_x r, \Lambda^{s'-1} r\rangle$$
$$+ (b-3)\langle \Lambda^{s'-3}\partial_x(r_x(u_x+v_x)), \Lambda^{s'-1} r\rangle - b\langle \Lambda^{s'-3}\partial_x(r(u+v)), \Lambda^{s'-1} r\rangle$$
$$- \frac{2\omega_1}{3\alpha^2}\langle \Lambda^{s'-3}\partial_x(r(u^2+uv+v^2)), \Lambda^{s'-1} r\rangle - \frac{\omega_2}{2\alpha^3}\langle \Lambda^{s'-3}\partial_x(r(u+v)(u^2+v^2)), \Lambda^{s'-1} r\rangle$$
$$= -2\langle \Lambda^{s'-1} u\partial_x r, \Lambda^{s'-1} r\rangle - 2\langle \Lambda^{s'-1} r\partial_x v, \Lambda^{s'-1} r\rangle$$
$$+ (b-3)\langle \Lambda^{s'-3}\partial_x(r_x(u_x+v_x)), \Lambda^{s'-1} r\rangle - b\langle \Lambda^{s'-3}\partial_x(r(u+v)), \Lambda^{s'-1} r\rangle$$
$$- \frac{2\omega_1}{3\alpha^2}\langle \Lambda^{s'-3}\partial_x(r(u^2+uv+v^2)), \Lambda^{s'-1} r\rangle - \frac{\omega_2}{2\alpha^3}\langle \Lambda^{s'-3}\partial_x(r(u+v)(u^2+v^2)), \Lambda^{s'-1} r\rangle$$
$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.$$

Then, we obtain each part such as

$$
\begin{aligned}
|I_1| &= |-2\langle \Lambda^{s'-1} u\partial_x r, \Lambda^{s'-1} r\rangle| \\
&= |2\langle u\partial_x \Lambda^{s'-1} r + [\Lambda^{s'-1}, u]\partial_x r, \Lambda^{s'-1} r\rangle| \\
&= |2\langle u\partial_x \Lambda^{s'-1} r, \Lambda^{s'-1} r\rangle + \langle [\Lambda^{s'-1}, u]\partial_x r, \Lambda^{s'-1} r\rangle| \\
&= |2\int_{\mathbb{R}} u\partial_x \Lambda^{s'-1} r \Lambda^{s'-1} r dx + \int_{\mathbb{R}} [\Lambda^{s'-1}, u]\partial_x r \Lambda^{s'-1} r dx| \\
&\le 2\|u_x\|_{L^\infty} \|r\|^2_{H^{s'-1}} + \|\Lambda^{s'-1} r\|_{L^2} \|[\Lambda^{s'-1}, u]\partial_x r\|_{L^2} \\
&\le C\|u\|_{H^{s'}} \|r\|^2_{H^{s'-1}} + \|r\|_{H^{s'-1}} \left(\|u\|_{H^{s'-1}} \|r_x\|_{L^\infty} + \|u_x\|_{L^\infty} \|r\|_{H^{s'-1}}\right) \\
&\le C\|u\|_{H^{s'}} \|r\|^2_{H^{s'-1}} + \|r\|_{H^{s'-1}} \left(\|u\|_{H^{s'-1}} \|r\|_{H^{s'-1}} + \|u\|_{H^{s'}} \|r\|_{H^{s'-1}}\right) \\
&\le C\|u\|_{H^{s'}} \|r\|^2_{H^{s'-1}},
\end{aligned}
$$

$$
\begin{aligned}
|I_2| &= 2|\int_{\mathbb{R}} \Lambda^{s'-1}(r\partial_x v) \Lambda^{s'-1} r dx| \\
&\le C\|r\partial_x v\|_{H^{s'-1}} \|r\|_{H^{s'-1}} \\
&\le C\|v\|_{H^{s'}} \|r\|^2_{H^{s'-1}},
\end{aligned}
$$

$$
\begin{aligned}
|I_3| &= |b-3||\int_{\mathbb{R}} \Lambda^{s'-3}\partial_x(r_x(u_x+v_x))\Lambda^{s'-1} r dx \\
&\le C|b-3|\|\partial_x(r_x(u_x+v_x))\|_{H^{s'-3}} \|r\|_{H^{s'-1}} \\
&\le C\|(r_x(u_x+v_x))\|_{H^{s'-2}} \|r\|_{H^{s'-1}} \\
&\le C\|u+v\|_{H^{s'-1}} \|r\|^2_{H^{s'-1}},
\end{aligned}
$$

$$
\begin{aligned}
|I_4| &= b|\int_{\mathbb{R}} \Lambda^{s'-3}\partial_x(r(u+v))\Lambda^{s'-1} r dx| \\
&\le C\|(r(u+v))\|_{H^{s'-2}} \|r\|_{H^{s'-1}} \\
&\le C\|u+v\|_{H^{s'-1}} \|r\|^2_{H^{s'-1}},
\end{aligned}
$$

$$
\begin{aligned}
|I_5| &= |\frac{2\omega_1}{3\alpha^2}\int_{\mathbb{R}} \Lambda^{s'-3}\partial_x(r(u^2+uv+v^2))\Lambda^{s'-1} r dx| \\
&\le C\|r\|_{H^{s'-2}} \|u^2+uv+v^2\|_{H^{s'-2}} \|r\|_{H^{s'-1}} \\
&\le C\|r\|^2_{H^{s'-1}} \left(\|u\|^2_{H^{s'-1}} + \|u\|_{H^{s'-1}} \|v\|_{H^{s'-1}} + \|v\|^2_{H^{s'-1}}\right),
\end{aligned}
$$

$$|I_6| = |\frac{\omega_2}{2\alpha^3}\int_{\mathbb{R}} \Lambda^{s'-3}\partial_x(r(u+v)(u^2+v^2))\Lambda^{s'-1} r dx|$$
$$\leq C\|r\|^2_{H^{s'-1}}\|u+v\|_{H^{s'-2}}\|u^2+v^2\|_{H^{s'-2}}$$
$$\leq C\|r\|^2_{H^{s'-1}}\left(\|u\|^3_{H^{s'-1}} + \|u\|_{H^{s'-1}}\|v\|_{H^{s'-1}}{}^2 + \|u\|_{H^{s'-1}}{}^2\|v\|_{H^{s'-1}} + \|v\|^3_{H^{s'-1}}\right).$$

From I_1 toI_6, we get

$$\frac{d}{dt}\|r\|^2_{H^{s'-1}} \leq C\|r\|^2_{H^{s'-1}}\left[\|u\|_{H^{s'}} + \|v\|_{H^{s'}} + \|u\|^2_{H^{s'}} + \|u\|_{H^{s'}}\|v\|_{H^{s'}}\right.$$
$$\left. + \|v\|^2_{H^{s'}} + \|u\|^3_{H^{s'}} + \|v\|^3_{H^{s'}} + \|u\|_{H^{s'}}\|v\|^2_{H^{s'}} + \|u\|^2_{H^{s'}}\|v\|_{H^{s'}}\right].$$

When Gronwall's inequality is applied to the last inequality, we conclude that $\forall t \in [0,t]$,

$$\|u-v\|_{H^{s'-1}} \equiv 0$$

which implies $u \equiv v$, $\forall x \in \mathbb{R}, t \in [0,T]$.

Step 6: (Conclusion)

Back to the proof of the fact that $u \in C([0,T]; H^s)$, we have known that $\|u(t)\|_{H^s}$ is continuous at t=0. Then $\forall T_0 \in [0,T]$ and the solution $u(\cdot,T_0)$ at the fixed time T_0 $u(\cdot,T_0) \equiv {u_0}^{T_0} \in H^s((R)$ and from (3.40) we obtain

$$\|{u_0}^{T_0}\|_{H^s} \leq \|u_0\|_{H^s} e^{\frac{{C_0}^2}{2}\|u_0\|_{H^s} T_0}$$

So we take ${u_0}^{T_0}$ as initial data and construct a forward and backward -in-time solution by solving (3.7). We obtain approximation solution $u^{(n)}_{T_0}(t)$ and its limit $u_{T_0}(t)$ which solves the Rotational b-family of equations with initial data

$$u_{T_0}(t)|_{t=0} = {u_0}^{T_0} \equiv u(\cdot, T_0)$$

with $u_{T_0} \in C([0,T_1]; H^{s'}) \cap L^\infty([0,T_1]; H^s)$ and $\partial_t u_{T_0} \in C([0,T_1]; H^{s'-1}) \cap L^\infty([0,T_1]; H^s)$ for some positive time $T_1 > 0$ and then $\|u_{T_0}(t)\|_{H^s}$ is continuous at t=0.

By the uniqueness, we get that

$$u_{T_0}(t) = u(t+T_0) \quad on \quad [T_0 - T_1, T_0 + T_1]$$

which implies that $u(t)$ is continuous at $t = T_0$. Therefore, we obtain that $u \in C([0,T]; H^s)$.

$\square$

Return to the original Rotational b-family of equations and define

$$\|f\|^2_{X^{s+1}_\mu} \stackrel{\text{def}}{=} \|f\|^2_{H^s} + \mu\,\beta\|f_x\|^2_{H^s},$$

where $\mu > 0, \beta > 0$. For some $\mu_0 > 0$ and $M > 0$, we define the Camassa-Holm regime $\mathcal{P}_{\mu_0,M} := \{(\varepsilon,\mu) : 0 < \mu \le \mu_0, 0 < \varepsilon \le M\sqrt{\mu}\}$. Then, we have the following corollary.

Corollary 3.2.2. *Let $u_0 \in H^{s+1}(\mathbb{R})$,$\mu_0 > 0$ and $M > 0$, $s > \frac{3}{2}$. Then, there exist $T > 0$ and a unique family of solutions $(u_{\varepsilon,\mu})|_{(\varepsilon,\mu)} \in \mathcal{P}_{\mu_0,M}$ in $C([0,\frac{T}{\varepsilon}]; X^{s+1}(\mathbb{R})) \cap C^1([0,\frac{T}{\varepsilon}; X^s(\mathbb{R})])$ to the Cauchy problem*

$$\begin{cases} \partial_t u - \beta\mu\partial_t u_{xx} + cu_x + (b+1)\alpha\varepsilon u u_x - \beta_0\mu u_{xxx} + \\ \qquad\qquad \omega_1\varepsilon^2u^2u_x + \omega_2\varepsilon^3u^3u_x = \alpha\beta\varepsilon\mu(bu_xu_{xx} + uu_{xxx}), \\ u|_{t=0} = u_0. \end{cases}$$

CHAPTER 4

WAVE BREAKING PHENOMENA

In this chapter, we investigated effects of higher nonlinear terms on finite-time blow-up solutions and wave breaking phenomena is investigated for the rotational b-family of equations. Since the rotational b-family of equations does not have enough conserved quantities, it leads us to find different approach.

4.1 Finite-time blow-up solutions

The blow-up criterion and wave-breaking criterion for the rotational b-family of equations have been proved in this section.

Theorem 4.1.1. *(Blow-up criterion) Let $u_0 \in H^s$, $s > \frac{3}{2}$ and u be the corresponding solution to* (3.6)*. Assume $T^*_{u_0}$ is the maximal time of existence. Then,*

$$T^*_{u_0} < \infty \Longrightarrow \int_0^{T^*_{u_0}} \|\partial_x u(\tau)\|_{L^\infty} d\tau = +\infty. \tag{4.1}$$

Approach of the proof for this theorem base on the following propositions.

Proposition 4.1.2. *[14, 15] Support that $s > -\frac{d}{2}$. Let v be a vector field such that ∇v belongs to $L^1([0,1]; H^{s-1})$ if $s > 1 + \frac{d}{2}$ or to $L^1([0,1]; H^{\frac{d}{2}} \cap L^\infty)$ otherwise. Suppose also that $f_0 \in H^s$, $F \in L^1([0,1]; H^s)$ and that $f \in L^\infty([0,T]; S')$ solves the $d-$ dimensional linear transport equations*

$$\begin{cases} \partial_t f + v \cdot \nabla f = F \\ f|_{t=0} = f_0 \end{cases}$$

Then $f \in C([0,T]; H^s)$. More precisely, there exists a constant C depending only on s, p and d, and such that the following statements hold:

(1) If $s \neq 1 + \frac{d}{2}$

$$\|f\|_{H^s} \leq \|f_0\|_{H^s} + \int_0^t \|F(\tau)\|_{H^s} d\tau + C \int_0^t V'(\tau) \|f(\tau)\|_{H^s} d\tau, \tag{4.2}$$

or hence,

$$\|f\|_{H^s} \leq e^{CV(t)} \left(\|f_0\|_{H^s} + \int_0^t e^{-CV(\tau)} \|F(\tau)\|_{H^s} d\tau \right) \tag{4.3}$$

with $V(t) = \int_0^t \|\nabla v(\tau)\|_{H^{\frac{d}{2}} \cap L^\infty} d\tau$ *if* $s < 1 + \frac{d}{2}$ *and* $V(t) = \int_0^t \|\nabla v(\tau)\|_{H^{s-1}} d\tau$ *else.*

(2) If $f = v$*, then for all* $s > 0$ *the estimates* (4.2) *and* (4.3) *hold with* $V(t) = \int_0^t \|\partial_x v(\tau)\|_{L^\infty} d\tau$.

Proposition 4.1.3. *[6]* $(1-D$ *Moser-type estimates). The following estimates hold,*

(1) For $s \geq 0$,

$$\|fg\|_{H^s(\mathbb{R})} \leq C \left(\|f\|_{H^s(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})} \|g\|_{H^s(\mathbb{R})} \right), \tag{4.4}$$

(2) For $s > 0$,

$$\|f \partial_x g\|_{H^s(\mathbb{R})} \leq C \left(\|f\|_{H^{s+1}(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})} \|\partial_x g\|_{H^s(\mathbb{R})} \right),$$

(3) For $s_1 \leq \frac{1}{2}$, $s_2 > \frac{1}{2}$ *and* $s_1 + s_2 > 0$,

$$\|fg\|_{H^{s_1}(\mathbb{R})} \leq C \|f\|_{H^{s_1}(\mathbb{R})} \|g\|_{H^{s_2}(\mathbb{R})}.$$

Proposition 4.1.4. *([14, 15]) Let* $m \in \mathbb{R}$ *and* f *be an* S^m*-multiplier (that is,*$f : \mathbb{R}^d \to \mathbb{R}$ *is smooth and satisfies that for all multi-index* α*, there exists a constant* C_α *such that* $\forall \xi \in \mathbb{R}^d$, $|\partial^\alpha f(\xi)| \leq C_\alpha (1+|\xi|)^{m-|\alpha|}$*). Then for all* $s \in \mathbb{R}$ *and* $1 \leq p, r \leq \infty$*, the operator* $f(D)$ *is continuous from* $B^s_{p,r}$ *to* $B^{s-m}_{p,r}$.

The proof of the Theorem 4.1.1 will be given as follows.

Proof. First of all, we apply the translation $u(t,x) \to u(t, x - \frac{\beta_0}{\beta} t)$ to the equation (3.3)

$$u_t + u u_x = -p_x * \left\{ \left(c - \frac{\beta_0}{\beta} \right) u - \frac{b-3}{2} (u_x)^2 + \frac{b}{2} u^2 + \frac{\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{4\alpha^3} u^4 \right\}. \tag{4.5}$$

Also applying Λ^s operator to the equation

$$\partial_t \Lambda^s u + \Lambda^s (u \partial_x u) = \Lambda^s \left(-p_x * \left\{ \left(c - \frac{\beta_0}{\beta} \right) u - \frac{b-3}{2}(u_x)^2 + \frac{b}{2}u^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \right\} \right). \tag{4.6}$$

Then taking the inner product between the equation (4.6) and $\Lambda^s u$ in L^2, we have

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s u\|_{L^2}^2 = -\frac{1}{2}\int_{\mathbb{R}} u \partial_x (\Lambda^s u)^2 dx - \int_{\mathbb{R}} [\Lambda^s, u]\partial_x u \Lambda^s u dx$$
$$+ \int_{\mathbb{R}} \Lambda^s \left(-p_x * \left\{ \left(c - \frac{\beta_0}{\beta} \right) u - \frac{b-3}{2}(u_x)^2 + \frac{b}{2}u^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \right\} \right) \Lambda^s u dx$$

$$\leq \|\partial_x u\|_{L^\infty} \|\Lambda^s u\|_{L^2}^2 + \|[\Lambda^s, u]\partial_x u\|_{L^2} \|\Lambda^s u\|_{L^2}$$
$$+ \|\Lambda^s \left(-p_x * \left\{ \left(c - \frac{\beta_0}{\beta} \right) u - \frac{b-3}{2}(u_x)^2 + \frac{b}{2}u^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \right\} \right) \|_{L^2} \|\Lambda^s u\|_{L^s}.$$

By using commutator estimate for $s > 0$, we get

$$\|[\Lambda^s, u]\partial_x u\|_{L^2} \leq C(\|u\|_{H^s}\|\partial_x u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}\|\partial_x u\|_{H^{s-1}}) \leq C\|\partial_x u\|_{L^\infty}\|u\|_{H^s}.$$

Then, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^s}^2 \leq C \left(\|\partial_x u\|_{L^\infty} \|u\|_{H^s} \right.$$
$$\left. + \|p_x * \left\{ \left(c - \frac{\beta_0}{\beta} \right) u - \frac{b-3}{2}(u_x)^2 + \frac{b}{2}u^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \right\} \|_{H^s} \right) \|u\|_{H^s}. \tag{4.7}$$

Taking integration of the equation from 0 to t, we get

$$\|u\|_{H^s} \leq \|u_0\|_{H^s} + \int_0^t \|u(\tau)\|_{H^s} \|\partial_x u(\tau)\|_{L^\infty} d\tau + C \int_0^t \|p_x * \left\{ \left(c - \frac{\beta_0}{\beta} \right) u \right.$$
$$\left. - \frac{b-3}{2}(u_x)^2 + \frac{b}{2}u^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \right\} \|_{H^s} d\tau.$$

Thanks to the Moser-type estimate (4.4) and the Proposition 4.1.4, we obtain

$$\|p_x * \left\{ \left(c - \frac{\beta_0}{\beta} \right) u - \frac{b-3}{2}(u_x)^2 + \frac{b}{2}u^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \right\} \|_{H^s} \leq C \left(\|u\|_{H^{s-1}} \right.$$
$$\left. + \|u_x\|_{H^{s-1}} \|u_x\|_{L^\infty} + \|u\|_{H^{s-1}} \|u\|_{L^\infty} + \|u\|_{H^{s-1}} \|u\|_{L^\infty}^2 + \|u\|_{H^{s-1}} \|u\|_{L^\infty}^3 \right).$$

Therefore, we have

$$\|u\|_{H^s} \leq \|u_0\|_{H^s} + C\int_0^t \|u(\tau)\|_{H^s}\left(1 + \|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|u\|_{L^\infty}^3\right) d\tau.$$

Applying the Gronwall's inequality, it follows

$$\|u\|_{H^s} \leq \|u_0\|_{H^s} e^{C\int_0^t \left(1+\|u\|_{L^\infty}+\|u_x\|_{L^\infty}+\|u\|_{L^\infty}^2+\|u\|_{L^\infty}^3\right)d\tau}. \tag{4.8}$$

On the other hand, if we multiply (3.2) by u and integrate over $\mathbb{R}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}} um_t dx + \int_{\mathbb{R}} u^2 m_x dx + b\int_{\mathbb{R}} uu_x m dx + c\int_{\mathbb{R}} cuu_x dx - \frac{\beta_0}{\beta}\int_{\mathbb{R}} uu_{xxx} dx \\ + \frac{\omega_1}{\alpha^2}\int_{\mathbb{R}} u^3 u_x dx + \frac{\omega_2}{\alpha^3}\int_{\mathbb{R}} u^4 u_x dx = 0.\end{aligned}$$

Then, we have

$$\int_{\mathbb{R}} um_t dx + \int_{\mathbb{R}} u^2 m_x dx + b\int_{\mathbb{R}} uu_x m dx = 0.$$

Note that, by applying integration by parts, we have

$$\int_{\mathbb{R}} um_t dx = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (u^2 + (u_x)^2) dx,$$

and

$$\begin{aligned}\int_{\mathbb{R}} u^2 m_x dx + b\int_{\mathbb{R}} uu_x m dx &= \int_{\mathbb{R}} (u^2 u_x - u^2 u_{xxx}) dx + b\int_{\mathbb{R}} (u^2 u_x - uu_x u_{xx}) dx \\ &= (2-b)\int_{\mathbb{R}} uu_x u_{xx} dx \\ &= \frac{b-2}{2}\int_{\mathbb{R}} (u_x)^3 dx.\end{aligned}$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (u^2 + (u_x)^2) dx = \frac{2-b}{2}\int_{\mathbb{R}} u_x (u_x)^2 dx. \tag{4.9}$$

Then, it follows

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} (u^2 + (u_x)^2) dx &\leq |b-2|\|u_x\|_{L^\infty}\int_{\mathbb{R}} (u_x)^2 dx \\ &\leq |b-2|\|u_x\|_{L^\infty}\int_{\mathbb{R}} (u^2 + (u_x)^2) dx,\end{aligned}$$

$$\frac{d}{dt}\|u\|_{H^1}^2 \leq |b-2|\|u_x\|_{L^\infty}\|u\|_{H^1}^2.$$

Thanks to the Gronwall inequality, we get

$$\|u\|_{H^1} \leq e^{|b-2|\int_0^T \|u_x\|_{L^\infty} dt}\|u_0\|_{H^1}.$$

By using the following Sobolev inequality

$$\|u\|_{L^\infty} \leq \frac{\|u\|_{H_1}}{\sqrt{2}},$$

we have

$$\|u\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|u\|_{H^1} \leq \frac{1}{\sqrt{2}} e^{|b-2|\int_0^T \|u_x\|_{L^\infty} dt}\|u_0\|_{H^1}. \tag{4.10}$$

Then if the maximal existence time $T_{u_0}^* < \infty$ satisfies $\int_0^{T_{u_0}^*} \|\partial_x u(\tau)\|_{L^\infty} d\tau < \infty$, we have from (4.8) and (4.10)

$$\|u\|_{L^\infty} \leq C(T),$$

and

$$\|u\|_{H^s} \leq \|u_0\|_{H^s} e^{C\int_0^t C(T) d\tau} < \infty,$$

where $C(T)$ is any positive constant depending on T. Therefore, we conclude

$$\limsup_{t \to T_{u_0}^*} \|u(t)\|_{H^s} < \infty,$$

which contradicts the assumption on the maximal existence time $T_{u_0}^* < \infty$. This completes the proof of Theorem 4.1.1 □

Theorem 4.1.5. *(Wave-breaking criterion) Assume* $u_0 \in H^s(\mathbb{R})$ *with* $s > \frac{3}{2}$. *Let* $T_{u_0}^* > 0$ *be maximal existence time of the corresponding solution* $u(t,x)$ *to the system* (3.6). *Then the corresponding solution breaks down in the following two cases,*

(i) for $b \geq 2$, *if and only if*

$$\liminf_{t \uparrow T_{u_0}^*} \inf_{x \in \mathbb{R}} u_x(t,x) = -\infty, \tag{4.11}$$

(ii) for $b \leq 1$, if and only if

$$\limsup_{t\uparrow T^*_{u_0}} \sup_{x\in\mathbb{R}} u_x(t,x) = +\infty. \tag{4.12}$$

Proof. We have the following equation by differentiating the equation (4.5) with respect to x and considering $\partial_x^2 p * f = p * f - f$

$$\begin{aligned} u_{tx} + {u_x}^2 + uu_{xx} &= \left(c - \frac{\beta_0}{\beta}\right) u - \frac{b-3}{2}(u_x)^2 + \frac{b}{2}u^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \\ &\quad - p * \left\{ \left(c - \frac{\beta_0}{\beta}\right) u - \frac{b-3}{2}(u_x)^2 + \frac{b}{2}u^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \right\}. \end{aligned} \tag{4.13}$$

Also, we consider the following differential equation

$$\begin{cases} \frac{\partial q}{\partial t} = u(t,q), & 0 < t < T, \\ q(0,x) = x, & x \in \mathbb{R}, \end{cases} \tag{4.14}$$

where $u \in C^1([0,T), H^{s-1})$ is the solution to equation (4.5) with initial data $u_0 \in H^s$, for $s > 3/2$. Differentiating (4.14) with respect to x, we obtain

$$\begin{aligned} \frac{d}{dt}u_x(t,q(t,x)) &= u_{xt}(t,q(t,x)) + u_{xx}(t,q(t,x))q_t(t,x) \\ &= (u_{tx} + uu_{xx})(t,q(t,x)). \end{aligned} \tag{4.15}$$

From (4.13) and (4.15), we achieve

$$\begin{aligned} \frac{d}{dt}u_x(t,q(t,x)) &= -\frac{b-1}{2}{u_x}^2 + \left(c - \frac{\beta_0}{\beta}\right) u + \frac{b}{2}u^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \\ &\quad - p * \left\{ \left(c - \frac{\beta_0}{\beta}\right) u - \frac{b-3}{2}(u_x)^2 + \frac{b}{2}u^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \right\}. \end{aligned} \tag{4.16}$$

Considering (4.9) and if $b \geq 2$, we have

$$\begin{aligned} \frac{d}{dt}\|u\|_{H^1}^2 &\leq (b-2)(-\inf_{x\in\mathbb{R}} u_x) \int_{\mathbb{R}} {u_x}^2 dx \\ &\leq (b-2)(-\inf_{x\in\mathbb{R}} u_x) \int_{\mathbb{R}} u^2 + {u_x}^2 dx \\ &= (b-2)(-\inf_{x\in\mathbb{R}} u_x)\|u\|_{H^1}^2. \end{aligned}$$

Applying Gronwall inequality, we get

$$\|u\|_{H^1} \leq e^{(b-2)\int_0^T(-\inf_{x\in\mathbb{R}} u_x)dt}\|u_0\|_{H^1}.$$

Assume that $T^*_{u_0} < \infty$ and there exists $M > 0$ such that

$$u_x(t,x) \geq -M, \forall (t,x) \in [0, T^*_{u_0}) \times \mathbb{R}.$$

Then, we have

$$\|u\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|u\|_{H^1} \leq e^{(b-2)MT^*_{u_0}}\|u_0\|_{H^1} \leq C.$$

where $C > 0$ is a constant, depending on $T^*_{u_0}$, M and $\|u_0\|_{H^1}$. Considering the following inequalities such as ${u_x}^2 \geq 0$, $p * u^2 \geq 0$, $\|p*u\|_{L^\infty} \leq \|p\|_{L^1}\|u\|_{L^\infty} \leq e^{(b-2)MT^*_{u_0}}\|u_0\|_{H^1}$,

$$\begin{aligned}
&\|p * {u_x}^2\|_{L^\infty} \leq \|p\|_{L^1}\|u_x\|^2_{L^2} \leq e^{2(b-2)MT^*_{u_0}}\|u_0\|^2_{H^1},\\
&\|p * u^2\|_{L^\infty} \leq \|p\|_{L^1}\|u\|^2_{L^2} \leq e^{2(b-2)MT^*_{u_0}}\|u_0\|^2_{H^1},\\
&\|p * u^3\|_{L^\infty} \leq \|p\|_{L^1}\|u\|^3_{L^\infty} \leq e^{3(b-2)MT^*_{u_0}}\|u_0\|^3_{H^1},\\
&\|p * u^4\|_{L^\infty} \leq \|p\|_{L^1}\|u\|^4_{L^2} \leq e^{4(b-2)MT^*_{u_0}}\|u_0\|^4_{H^1},
\end{aligned}$$

then from (4.16), we have

$$\begin{aligned}
\frac{d}{dt}u_x(t,q(t,x)) \leq C' \Big(&e^{(b-2)MT^*_{u_0}}\|u_0\|_{H^1} + e^{2(b-2)MT^*_{u_0}}\|u_0\|^2_{H^1}\\
&+e^{3(b-2)MT^*_{u_0}}\|u_0\|^3_{H^1} + e^{4(b-2)MT^*_{u_0}}\|u_0\|^4_{H^1}\Big)\\
&\leq C(T^*_{u_0}, M).
\end{aligned}$$

where C' is a positive constant. Integrating the inequality with respect to $t < T^*_{u_0}$ on $[0,t]$, we obtain

$$u_x(t,q(t,x)) \leq u_x(0) + C(T^*_{u_0}, M)t.$$

Then, for $\forall t \in [0, T^*_{u_0})$, we have

$$\sup_{x\in\mathbb{R}} u_x(t,x) \leq \|\partial_x u_0\|_{L^\infty} + Ct \leq \|u_0\|_{H^1} + Ct,$$

which together with

$$\inf_{x\in\mathbb{R}} u_x \geq -M, \quad \forall (t,x) \in [0, T^*_{u_0}) \times \mathbb{R}$$

and $T^*_{u_0} < \infty$ implies that

$$\int_0^{T^*_{u_0}} \|\partial u_x(\tau)\|_{L^\infty} d\tau < \infty.$$

This contradicts Theorem 4.1.1. On the contrary, by using the Sobolev embedding theorem, $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, (with $s > \frac{1}{2}$) if

$$\liminf_{t\uparrow T^*_{u_0}} \inf_{x\in\mathbb{R}} u_x(t,x) = -\infty$$

holds, then the solution u blows up in finite time,

On the other hand, suppose that $\sup_{x\in\mathbb{R}} u_x \geq 0$ if $b \leq 1$, we have

$$\begin{aligned}
\frac{d}{dt}\|u\|^2_{H^1} &\leq (2-b)(\sup_{x\in\mathbb{R}} u_x) \int_{\mathbb{R}} {u_x}^2 dx \\
&\leq (2-b)(\sup_{x\in\mathbb{R}} u_x) \int_{\mathbb{R}} u^2 + {u_x}^2 dx \\
&\leq (2-b)(\sup_{x\in\mathbb{R}} u_x)\|u\|^2_{H^1}.
\end{aligned}$$

Applying Gronwall inequality, it follows

$$\|u\|_{H^1} \leq e^{(2-b)\int_0^T \left(\sup_{x\in\mathbb{R}} u_x\right) dt} \|u_0\|_{H^1}$$

Assume that $T^*_{u_0} < \infty$ and there exists $M > 0$ such that

$$u_x(t,x) < M, \forall (t,x) \in [0, T^*_{u_0}) \times \mathbb{R}.$$

Then, we have

$$\|u\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|u\|_{H^1} \leq e^{(2-b)MT^*_{u_0}} \|u_0\|_{H^1} \leq C,$$

where $C > 0$ is a constant, depending on $T^*_{u_0}$, M and $\|u_0\|_{H^1}$. Considering the following inequalities such as ${u_x}^2 \geq 0$, $p * u^2 \geq 0$, $\|p * u\|_{L^\infty} \leq \|p\|_{L^1}\|u\|_{L^\infty} \leq e^{(2-b)MT^*_{u_0}}\|u_0\|_{H^1}$,

$$
\begin{aligned}
\|p * {u_x}^2\|_{L^\infty} &\leq \|p\|_{L^1}\|u_x\|^2_{L^2} \leq e^{2(2-b)MT^*_{u_0}}\|u_0\|^2_{H^1}, \\
\|p * u^2\|_{L^\infty} &\leq \|p\|_{L^1}\|u\|^2_{L^2} \leq e^{2(2-b)MT^*_{u_0}}\|u_0\|^2_{H^1}, \\
\|p * u^3\|_{L^\infty} &\leq \|p\|_{L^1}\|u\|^3_{L^\infty} \leq e^{3(2-b)MT^*_{u_0}}\|u_0\|^3_{H^1}, \\
\|p * u^4\|_{L^\infty} &\leq \|p\|_{L^1}\|u\|^4_{L^2} \leq e^{4(2-b)MT^*_{u_0}}\|u_0\|^4_{H^1},
\end{aligned}
$$

then from (4.16), we have

$$
\begin{aligned}
\frac{d}{dt}u_x(t, q(t,x)) \geq -C' \Big(& e^{(2-b)MT^*_{u_0}}\|u_0\|_{H^1} + e^{2(2-b)MT^*_{u_0}}\|u_0\|^2_{H^1} \\
& + e^{3(2-b)MT^*_{u_0}}\|u_0\|^3_{H^1} + e^{4(2-b)MT^*_{u_0}}\|u_0\|^4_{H^1} \Big) \\
\geq & -C(T^*_{u_0}, M),
\end{aligned}
$$

where C' is a positive constant. Applying absolute value to this inequality, we get

$$
-C(T^*_{u_0}, M) \leq \frac{d}{dt}u_x(t, q(t,x)) \leq C(T^*_{u_0}, M).
$$

So, by integrating the inequality with respect to $t < T^*_{u_0}$ on $[0, t]$, we obtain

$$
-C(T^*_{u_0}, M)t + u_x(0) \leq \inf_{x \in \mathbb{R}} u_x(t, q(t,x)).
$$

Then, $\forall t \in [0, T^*_{u_0})$, we have

$$
-C(T^*_{u_0}, M)t - \|u_0\|_{H^1} \leq \inf_{x \in \mathbb{R}} u_x(t, q(t,x)),
$$

which together with

$$
\sup_{x \in \mathbb{R}} u_x \leq M, \quad \forall (t,x) \in [0, T^*_{u_0}) \times \mathbb{R},
$$

and $T^*_{u_0} < \infty$ implies that

$$
\int_0^{T^*_{u_0}} \|\partial u_x(\tau)\|_{L^\infty} d\tau < \infty.
$$

This contradicts Theorem 4.1.1. On the contrary, by using the Sobolev embedding theorem, $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, (with $s > \frac{1}{2}$) if

$$\limsup_{t \uparrow T^*_{u_0}} \sup_{x \in \mathbb{R}} u_x(t,x) = +\infty$$

holds, then the solution u blows up in finite time, which completes proof.

□

4.2 Wave-Breaking Phenomena

In this section, we consider the special case of the rotational b-family of equations which is called Rotational Camassa-Holm equation ($b = 2$)

$$u_t + uu_x + \frac{\beta_0}{\beta} u_x + p_x * \left\{ \left(c - \frac{\beta_0}{\beta} \right) u + \frac{1}{2}(u_x)^2 + u^2 + \frac{\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{4\alpha^3} u^4 \right\} = 0, \quad (4.17)$$

where $p = \frac{1}{2} e^{-|x|}$. We state the following theorem to find the wave breaking data for the R-CH.

Theorem 4.2.1. *Assume that* $u \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$, $s > 3/2$ *is a solution of the Cauchy problem* (3.6) *for* $b = 2$ *with the initial value* $u_0 \in H^s(\mathbb{R})$). *Suppose that*

$$m_0 < \min \{A, B\},$$

where

$$A = -2 \left(\frac{|\gamma|}{2} + \sqrt{\frac{\gamma^2}{4} + \frac{1}{8} \|u_0\|^2_{H^1} + \frac{1}{2} {C_0}^2} \right),$$

$$B = - \left(\frac{|\gamma|}{2} + \sqrt{\frac{\gamma^2}{4} + 2 \left(\frac{|\gamma|}{2} M_0 + \frac{1}{4} \|u_0\|^2_{H^1} + {C_0}^2 \right)} \right),$$

$$\gamma = c - \frac{\beta_0}{\beta},$$

and $C_0 > 0$ *is defined by*

$$C_0^2 = \frac{2|\omega_1|}{3\alpha^2} E_0^{3/2} + \frac{|\omega_2|}{2\alpha^3} E_0^2,$$

and

$$E_0 = \frac{1}{2}\int_{\mathbb{R}} (u_0^2 + (\partial_x u_0)^2) dx.$$

The corresponding solution of (3.6) *for* $b = 2$ *breaks down in finite time* T_0 *with an estimate as*

$$T_0 \le t^* = \frac{4m_0^2 + 4|\gamma| m_0 - 2\|u_0\|_{H^1}^2 - 2C_0^2}{(m_0^2 + 2|\gamma| m_0 - \frac{1}{2}\|u_0\|_{H^1}^2 - 2C_0^2)\sqrt{m_0^2 - |\gamma|(M_0 - m_0) - \frac{1}{2}\|u_0\|_{H^1}^2 - 2C_0^2}}.$$

We will give the proof for this theorem by using the following proposition and lemma.

Proposition 4.2.2. *([35]) Let* $m(t) = \inf_{x\in\mathbb{R}} \{u_x(t,x)\}$ *and* $M(t) = \sup_{x\in\mathbb{R}} \{u_x(t,x)\}$ *be two continuous and almost everywhere differentiable defined in* $t \in [0,T)$ $T \le \infty$ *with satisfying*

$$\begin{cases} \frac{dm}{dt} \le -a' m^2(t) + b'[M(t) - m(t)] + c', \\ \frac{dM}{dt} \le -a' M^2(t) + b'[M(t) - m(t)] + c', \end{cases} ; a.e. \quad in \quad t \in [0,T),$$

where a' *is a positive constant,* b' *and* c' *are non-negative constants, and* $M(t)$ *is a nonnegative function of* t. *Suppose that the initial data* $m_0 = m(0)$ *and* $M_0 = M(0)$ *satisfy*

$$m_0 < \min\left\{ -\frac{1}{a'}(b' + \sqrt{(b')^2 + a'c'}), -\frac{1}{2a'}(b' + \sqrt{(b')^2 + 4a'(b'M_0 + c')}) \right\}.$$

Then $m(t)$ *is monotonically decreasing and breaks down in the finite time* T_0 *with*

$$T_0 \le t^* = \frac{a'm_0^2 + b'm_0 - c}{a'(a'm_0^2 + 2b'm_0 - c')\sqrt{a'(a'm_0^2 - b'(M_0 - m_0) - c')}}$$

in the sense that

$$\liminf_{t\to T_0^-} m(t) = -\infty.$$

In the case of $T_0 = t^*$, *the wave-breaking rate can be estimated by*

$$m(t) \le \frac{a'm_0^2 + b'm_0 - c'}{a'm_0^2 + 2b'm_0 - c'} \frac{1}{t - t^*}.$$

Furthermore, if $m(t)$ *is bounded below by some negative constant* m_l, *i.e.* $m(t) \ge m_l$, *then* $M(t)$ *is bounded by*

$$M(t) \le \max\left\{ M_0, \frac{b' + \sqrt{b'^2 + 4a'(c' - b'm_l)}}{2a'} \right\}.$$

Lemma 4.2.3. *([8] Let* $T > 0$ *and* $v \in C^1([0,T); H^2(\mathbb{R}))$. *Then for every* $t \in [0,T)$ *there exists at least one point* $\xi \in \mathbb{R}$ *with*

$$m(t) := \inf_{x\in\mathbb{R}} [v_x(t,x)] = v_x(t, \xi(t)), \tag{4.18}$$

and the function m *is almost everywhere differentiable on* $(0,T)$ *with*

$$\frac{dm}{dt}(t) = v_{tx}(t, \xi(t)) \quad a.e. \quad on \quad (0,T). \tag{4.19}$$

Now, we can prove the Theorem 4.2.1.

Proof. First, we apply the translation $u(t,x) \to u(t, x - \frac{\beta_0}{\beta}t)$ to the equation (4.17). Then we get the equation in the form, as follows

$$u_t + uu_x + p_x * \left\{ \left(c - \frac{\beta_0}{\beta} \right) u + \frac{1}{2}(u_x)^2 + u^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \right\} = 0. \tag{4.20}$$

Differentiating the equation with respect to x, it yields

$$u_{xt} + uu_{xx} + u_x^2 + \gamma\partial_x p * (u_x) + p * \left(u^2 + \frac{1}{2}(u_x)^2 \right) - \left(u^2 + \frac{1}{2}(u_x)^2 \right)$$
$$+ p * \left(\frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \right) - \left(\frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \right) = 0,$$

where $\gamma = \left(c - \frac{\beta_0}{\beta} \right)$. Using the Lemma 3.1 in [2], we get the following

$$p * (u^2 + \frac{1}{2}u_x^2) \ge \frac{1}{2}u^2.$$

On the other hand, we have

$$\Big|\Big(\frac{\omega_1}{3\alpha^2}u^3+\frac{\omega_2}{4\alpha^3}u^4\Big)-p*\Big(\frac{\omega_1}{3\alpha^2}u^3+\frac{\omega_2}{4\alpha^3}u^4\Big)\Big| \le \frac{|\omega_1|}{3\alpha^2}\|u\|_{L^\infty}^3+\frac{|\omega_2|}{4\alpha^3}\|u\|_{L^\infty}^4 \\ +\frac{1}{2}\frac{|\omega_1|}{3\alpha^2}\|u\|_{L^\infty}\|u\|_{L^2}^2+\frac{1}{2}\frac{|\omega_2|}{4\alpha^3}\|u\|_{L^\infty}^2\|u\|_{L^2}^2 \\ \le \frac{2|\omega_1|}{3\alpha^2}E_0^{\frac{3}{2}}+\frac{|\omega_2|}{2\alpha^3}E_0^2=C_0^2>0.$$

Then, we obtain

$$\begin{aligned} u_{xt}+uu_{xx} &\le -\frac{1}{2}u_x^2-\gamma\partial_x p*(u_x)+\frac{1}{2}u^2+C_0^2 \\ &\le -\frac{1}{2}u_x^2-\gamma\partial_x p*(u_x)+\frac{1}{4}\|u_0\|_{H^1}^2+C_0^2. \end{aligned} \tag{4.21}$$

Now, let estimate $\partial_x p*(u_x)$, first of all define for $t\in[0,T)$,

$$\begin{aligned} m(t) &:= \inf_{\mathbb{R}}[u_x(t,x)] = u_x(t,\xi_1(t)), \\ M(t) &:= \sup_{\mathbb{R}}[u_x(t,x)] = u_x(t,\xi_2(t)), \end{aligned} \tag{4.22}$$

where $\xi_i, i=1,2$, are some points in $\mathbb{R}$; see *Theorem* 2.1 in [8] for the existence of $\xi_1(t),\xi_2(t)$. By Lebesgue's dominated convergence theorem, we have that

$$\lim_{n\to\infty}\int_{-n}^{n}p(\eta)u_{xx}(t,\xi(t)-\eta)d\eta=\int_{\mathbb{R}}p(\eta)u_{xx}(t,\xi(t)-\eta)d\eta. \tag{4.23}$$

If $[a,b]\subset\mathbb{R}$ is an interval where p is monotone and $f:[a,b]\to\mathbb{R}$ is continuous, by the second mean-value theorem there is some $c\in[a,b]$ such that

$$\int_a^b p(x)f(x)dx=p(a)\int_a^c f(x)dx+p(b)\int_c^b f(x)dx. \tag{4.24}$$

Therefore, we get for $n\ge 1$ and i=1,2, points $c\in[-n,0], d\in[0,n]$ such that

$$\begin{aligned} \int_{-n}^{0}\frac{1}{2}e^{\eta}u_{xx}(t,\xi_i(t)-\eta)d\eta = &-\frac{1}{2}e^{-n}[u_x(t,\xi_i(t)-c)-u_x(t,\xi_i(t)+n)] \\ &-\frac{1}{2}[u_x(t,\xi_i(t))-u_x(t,\xi_i(t)-c)], \end{aligned} \tag{4.25}$$

$$\begin{aligned} \int_{0}^{n}\frac{1}{2}e^{-\eta}u_{xx}(t,\xi_i(t)-\eta)d\eta = &-\frac{1}{2}e^{-n}[u_x(t,\xi_i(t)-n)-u_x(t,\xi_i(t)-d)] \\ &-\frac{1}{2}[u_x(t,\xi_i(t)-d)-u_x(t,\xi_i(t))] \end{aligned} \tag{4.26}$$

respectively. Recalling definition of $m(t)$ and $M(t)$, we deduce by adding (4.25) and (4.26) that for $n \geq 1$

$$\begin{aligned}\int_{-n}^{n} \frac{1}{2} e^{\eta} u_{xx}(t, \xi_i(t) - \eta) d\eta = &-\frac{1}{2} e^{-n} [u_x(t, \xi_i(t) - c) - u_x(t, \xi_i(t) + n) + u_x(t, \xi_i(t) - n) \\ &- u_x(t, \xi_i(t) - d)] - \frac{1}{2} [u_x(t, \xi_i(t) - d) - u_x(t, \xi_i(t) - c)] \\ &\leq \frac{M(t) - m(t)}{2} + e^{-n} [M(t) - m(t)].\end{aligned}$$

Thus,on account of (4.23), we obtain the estimate

$$\begin{aligned}\partial_x p * (u_x) = p * (u_{xx}) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-\xi|} u_{xx}(t, \xi) d\xi \\ = \int_{\mathbb{R}} \frac{1}{2} e^{\eta} u_{xx}(t, \xi_i(t) - \eta) d\eta \leq \frac{M(t) - m(t)}{2}, t \in (0, T), \text{for } i = 1, 2.\end{aligned}$$

Then it follows from (4.21) and by Lemma (4.2.3), $u_{xx}(t, \xi(t)) = 0$ since $m(t)$ is minimum for $u_x(t, \cdot) \in C^2$ that $m(t)$ and $M(t)$ defined by (4.22) satisfy

$$\begin{aligned}\frac{dm}{dt} \leq -\frac{1}{2} m^2 + \frac{|\gamma|}{2} (M - m) + \frac{1}{4} \|u_0\|_{H^1}^2 + C_0^2, \\ \frac{dM}{dt} \leq -\frac{1}{2} M^2 + \frac{|\gamma|}{2} (M - m) + \frac{1}{4} \|u_0\|_{H^1}^2 + C_0^2.\end{aligned}$$

Applying Proposition 4.2.2, we complete the proof. □

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BIOGRAPHICAL STATEMENT

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