

EXAMINING THE CONCEPT IMAGES OF FUNCTION HELD BY PRESERVICE
SECONDARY MATHEMATICS TEACHERS
WITH VARYING LEVELS OF PRIOR
MATHEMATICAL EXPERIENCES

by

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Abstract

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This multiple case study examines the function concept images of preservice secondary mathematics teachers (PSMTs) enrolled in a mathematics content course designed specifically for PSMTs at a large urban university in the southwestern United States. The primary research question explores the changes in PSMTs' function concept images when they engage with research-based explorations designed to elicit function-related cognitive conflicts. Furthermore, this study explores the extent to which there are differences in the function concept images of advanced undergraduate mathematics majors and those with the minimum prerequisite knowledge. Thematic analysis is applied to identify PSMTs function-related associations and characterize their function concept images before and after their interaction with the research-based explorations. Data analysis reveals five function concept image categories and 19 themes as well as a general shift in participants' conceptions related to the vertical line test, algebraic and graphical representations of functions, and the types of sets on which functions can be defined.

In this qualitative study, seven participants completed pre- and post-interviews, during which they answered open-ended questions designed to reveal function-related associations. Two participants started the study with the minimum prerequisite mathematical knowledge; three students completed advanced studies in undergraduate mathematics; and the mathematical backgrounds of the two remaining students fell between these two categories. Thematic analysis methods (Braun & Clarke, 2006) were used to analyze recordings, transcriptions, and student work from the interviews revealing 19 function concept image themes. Analyzing the emergence of these themes across the pre- and post-interviews did not indicate a difference in the overall concept-definition-consistency of an individual's conception based on their mathematical background; however, the majority of participants' function concept images adjusted to include functions that cannot be represented on a coordinate plane or algebraically, functions that do not "pass the vertical line test," and functions defined on non-numerical sets. These findings raise important questions regarding the types of course experiences that contributed to the shifts in participants' function concept images and the transferability of these types of experiences to other undergraduate mathematics courses.

Table of Contents

Acknowledgements	iii
Abstract	iv
List of Illustrations	x
List of Tables	xii
Chapter 1 Introduction.....	1
Chapter 2 Literature Review	4
2.1 Theory on Acquisition of the Function Concept.....	4
2.1.1 APOS Theory	4
2.1.2 Operational and Structural Theory of the Function Concept.....	6
2.1.3 Procept Theory	8
2.1.4 Covariational Reasoning	9
2.1.5 Concept Definition and Concept Image	11
2.2 Conceptions of Function	13
2.2.1 Conceptions of Function as an Entity	14
2.2.2 Conceived Characteristics of Function.....	16
Chapter 3 Methodology.....	19
3.1 Setting.....	19
3.2 Pre- and Post-Interviews	21
3.3 Student Interviews	25
3.4 Theoretical Perspective	27
3.5 Data Analysis.....	29
3.6 Validity	30
Chapter 4 Findings.....	32
4.1 Overview of Function Concept Image Themes	32

4.1.1 Functional Notation	32
4.1.2 Form Implies Function	32
4.1.3 Form Implies Equation	34
4.1.4 Numerical Domain	34
4.1.5 Non-Numerical Domain	34
4.1.6 One-to-One	35
4.1.7 Not One-to-One	35
4.1.8 Graphable	35
4.1.9 Graph Not Defining Characteristic	36
4.1.10 VLT	37
4.1.11 VLT Not Defining Characteristic	37
4.1.12 Relations on Sets	38
4.1.13 Input/Output	38
4.1.14 Univalence	38
4.1.15 Algebraic Representation	39
4.1.16 Equations are Functions	39
4.1.17 Non-Univalence for Equations	39
4.1.18 Contrived Difference from Equation	40
4.1.19 Defining Characteristics	40
4.1.20 Further Remarks on Function Concept Image Themes	40
4.2 Findings for Group A Participants	43
4.2.1 Sofia	43
4.2.2 Lee	50
4.3 Findings for Group B Participants	54
4.3.1 Jamie	54

4.3.2 Gabby	59
4.4 Findings for Group C Participants	64
4.4.1 Henry	64
4.4.2 Alan	69
4.4.3 Michael	73
4.5 Consistent, Inconsistent, and Neutral Theme Findings.....	77
Chapter 5 Discussion and Conclusions	80
5.1 Discussion of Participants Pre- and Post-interview Concept Image	
Themes within Participant Groups.....	80
5.1.1 Group A Participants	81
5.1.2 Group B Participants	90
5.1.3 Group C Participants	97
5.2 Discussion of Participants' Pre- and Post-interview Concept Image	
Themes across Participant Groups	105
5.2.1 Consistent, Inconsistent, and Neutral Theme Discussion of	
Participant Groups.....	105
5.2.2 Specific Concept Image Themes Observations across Participant	
Groups	108
5.3 Study Limitations	113
5.3.1 Study Participants.....	114
5.3.2 Task-based Interview Questions	114
5.3.3 Theme Prevalence	115
5.4 Conclusion	116
5.4.1 Implications.....	116
5.4.2 Research Questions	117

Appendix A Pre-Interview Protocol	123
Appendix B Post-Interview Protocol.....	126
Appendix C Participant Self-Report Form.....	130
Appendix D Lesson 5: What is a Function?	132
Appendix E Lesson 6: Functions and Equations	139
References.....	144
Biographical Information	151

List of Illustrations

Figure 3-1 Questions 1 and 2 of Task-based Portion of Pre- and Post-interviews	22
Figure 3-2 Question 3 of Task-based Portion of Pre- and Post-interviews	23
Figure 3-3 Question 4 of Task-based Portion of Pre- and Post-interviews	23
Figure 3-4 Questions 5, 6, and 7 of Task-based Portion of Pre- and Post-interviews	24
Figure 3-5 Question 8 of task-based portion of pre- and post-interviews	25
Figure 3-6 Overview of Coding Process	30
Figure 4-1 Categorization of Function Concept Image Themes	41
Figure 4-2 Sofia's Post-interview Example of a Function with a Non-Numerical Domain	48
Figure 4-3 Lee's Post-interview Example of a Function with a Non-Numerical Domain Graphed on a Coordinate System	53
Figure 4-4 Jamie's Pre-interview Example of a Function that is Not One-to-one	59
Figure 4-5 Gabby's Post-interview Example of a Function that cannot be Graphed	62
Figure 4-6 Gabby's Pre-interview Example of a Function that is Not One-to-one	63
Figure 4-7 Gabby's Post-interview Example of a Function that is Not One-to-one	64
Figure 4-8 Henry's Post-interview Example to Explain Univalence	69
Figure 4-9 Alan's Pre-interview Examples of Function	71
Figure 4-10 Alan's Post-interview Example of Function with a Non-Numerical Domain	72
Figure 4-11 Michael's Post-interview Example of a Function that is Not One-to-one	75
Figure 4-12 Michael's Pre-interview Example of a Function with a Non-numerical Domain	76
Figure 4-13 Michael's Post-interview Example of a Function with Non-Numerical Domain	76

Figure 4-14 Categorization of Consistent, Inconsistent, and Neutral Themes	78
Figure 5-1 Exploration 5.1 of Lesson 5	85
Figure 5-2 Student Proposed Defining Characteristics of Function from Exploration 5.1	85
Figure 5-3 Instructor Example of a Function from Students to Eye Colors	86
Figure 5-4 Exploration 6.3 of Lesson 6	89
Figure 5-5 Exploration 6.2 of Lesson 6	90
Figure 5-6 Exploration 5.3 Part 1 of Lesson 5	92
Figure 5-7 Exploration 5.4 of Lesson 5	93
Figure 5-8 Exploration 5.2 of Lesson 5	94
Figure 5-9 Exploration 5.3 Part 2 of Lesson 5	101
Figure 5-10 Comparison of Consistent, Inconsistent, and Neutral Themes by Percentage	108
Figure 5-11 Overview of Themes Common Across Participants' Pre- and Post-Interviews.....	118

List of Tables

Table 2-1 Examples of Action, Process and Object Mental Conceptions	5
Table 2-2 Levels of Covariational Reasoning	10
Table 3-1 Number of 80-minute Class Days Used in Fall 2018 for Each Lesson	21
Table 3-2 Self-Reported Participant Information	27
Table 4-1 Function Concept Image Theme Descriptions	33
Table 4-2 Themes Identified in Participants' Pre- and Post-Interviews	44
Table 4-3 Number of Consistent, Inconsistent, and Neutral Themes by Participant	79
Table 5-1 Average Number of Post-Interview Consistent, Inconsistent, and Neutral Themes by Participant Group	107

Chapter 1

Introduction

The function concept holds a central role in the study of mathematics (Dubinsky & Harel, 1992). Thus, it is tantamount that future teachers develop a robust understanding of functions and understand how to facilitate students' understanding of this concept. In particular, Oehrtman, Carlson, and Thompson (2008) report that students entering calculus require a strong understanding of function, and that calculus typically serves as a foundational gateway for the study of undergraduate mathematics and other sciences such as engineering. Although not all students will pursue fields that require knowledge of calculus or an advanced understanding of function, school mathematics standards also emphasize the importance of understanding functions (e.g. Common Core State Standards for Mathematics, 2010; National Council of Teachers of Mathematics Principles and Standards for School Mathematics, 2000; Texas Essential Knowledge and Skills, 2012). Experiences with function at the secondary level should provide students with the groundwork needed for future studies. However, various studies at the undergraduate level indicate that students, even those who wish to study mathematics or other sciences at an advanced level, do not have a strong foundation for the function concept on which to build (e. g. Carlson et al., 2008; Carlson, 1998; Vinner & Dreyfus, 1989).

The purpose of this multiple case study is to identify the extent to which there are modifications in and differences among the function conceptions of preservice secondary mathematics teachers (PSMTs). PSMTs in this study possess varying mathematical backgrounds and are enrolled in a mathematics content course for PSMTs. For this study, modifications in a PSMT's function conceptions will generally be defined as functions conceptions initially at the beginning of the course that seem to contrast with conceptions at the end of the course, and conceptions that are only apparent at the end of the course. Also, differences in function conceptions among PSMTs will be generally established using conceptions that are held by at least half of participants with similar prior mathematical experiences.

A viable explanation for why students enter into their university studies without a strong foundation for function may be grounded in their secondary mathematics teachers' knowledge of function. Researchers generally agree that teachers need a special kind of knowledge for teaching (e.g. Ball, Thames, & Phelps, 2008; Shulman, 1986). Ball et al. (2008) characterize the content knowledge unique

to teaching as specialized content knowledge. Another type of knowledge specifically needed for teaching is horizon content knowledge – the knowledge of how mathematical topics are connected across mathematics curriculum (Ball et al., 2008). These specific types of teacher subject matter knowledge influence both the content taught and how a teacher teaches the content (Even, 1993; Stein, Baxter, & Leinhardt, 1990; Watson & Harel; 2013). For example, in a study of one middle school teacher's, Mr. Gene's, understanding and teaching of function, Stein et al. (1990) found that Mr. Gene's subject matter knowledge directly affected his instructional practices. Particularly, Mr. Gene possessed a rule-based conception of function that did not include the idea of univalence – each element of the domain maps to only one element of the codomain. This resulted in a depiction of function that over emphasized function as merely an arithmetic operation; an explanation that did not provide the groundwork for future function learning; and missed opportunities to develop meaningful connections between functions and graphs. Conversely, Watson and Harel (2013) found that teachers who experience studying and using functions at a high level are able to teach functions at the secondary level in a way that lays the foundation for developing a more advanced understanding.

Since a teacher's content knowledge of function heavily influences teaching of the concept of function, possible solutions to improving the conceptual knowledge of function of students entering post-secondary studies rests in deepening preservice secondary mathematics teachers' (PSMTs) own conceptions of function. This may be achieved through a content course consisting of activities designed to elicit cognitive conflicts related to function conceptions PSMTs may hold. However, there is limited research on how such a course influences PSMTs conceptions of function. This multiple case study explores function conceptions of PSMTs and changes in their function conceptions when enrolled in a content course that uses research-based tasks and explorations aimed at deepening their understanding of topics related to function (e.g. function versus equation, graphical connections to function patterns, etc.) The primary research question examined is:

How do PSMTs' concept images of function change when they engage with research-based tasks and explorations designed to elicit cognitive conflicts related to function conceptions?

A second question examined is:

To what extent are there differences in the function concept development of PSMTs who are also advanced undergraduate mathematics majors and PSMTs with the minimum prerequisite mathematical knowledge?

Understanding these questions leads to insight on the types of experiences that deepen PSMTs conceptions of function thereby informing curriculum choices by mathematics teacher educators and program requirements determined by teacher education programs. Other studies indicate the unlikelihood of PSMTs developing a deeper, more profound understanding of concepts they will teach within their undergraduate mathematics courses (e.g. Even, 1993; Zazkis & Leiken, 2010). The findings from this study depict a similar narrative and support the recommendation by the Conference Board of Mathematical Sciences in The Mathematical Education of Teachers II (MET II) report that “prospective teachers need mathematics courses that develop a solid understanding of the mathematics they will teach” (CBMS MET II, 2012, p. 17).

Chapter 2

Literature Review

The notion of a mathematical function has a tumultuous history within the mathematics community. For centuries, mathematicians struggled to reconcile early notions of function as analytic expressions with the need to develop general theorems on large collections of relations and the need to organize results on functions (Sfard, 1991; Sierpinska, 1992). This struggle is reflected in students' learning of function, and the mathematics education community seeks to understand this phenomenon. The literature on functions naturally falls into two general categories: theoretical papers describing how individuals acquire the function concept and papers identifying particular student and teacher function conceptions. In the following sections, I will highlight the research in each of these categories.

2.1 Theory on Acquisition of the Function Concept

One goal of the mathematics education research on function is to theorize how an individual comes to learn the function concept and understandings needed for a robust conception of function. Below I present an overview of five of these theories.

2.1.1 APOS Theory

The Action, Process, Object, Schema (APOS) Theory of understanding the function concept stems from Piaget's application of the theory of reflective abstraction to functions (Dubinsky & Wilson, 2013). According to Piaget, reflective abstraction occurs when an individual forms mathematical knowledge through four mental constructions: interiorization, coordination, encapsulation, and generalization. Interiorization is the internal processes and routines a person develops to unify perceived phenomena. Developing a new process from two or more existing processes signifies the coordination component of reflective abstraction. The term encapsulation denotes the transformation of a process into a static object. Generalization transpires when an individual is able to apply an existing schema, or a coherent collection of objects and processes, to a wider array of phenomena (Dubinsky, 1991). Dubinsky (1991) also suggests the addition of a fifth mental construction, reversal, wherein an individual constructs a new process by reversing the original process.

While Piaget's work with reflective abstraction mainly concerns children's thinking, APOS Theory aims to extend the notion of reflective abstraction to areas of advanced mathematical thinking. In this

extension, Dubinsky and colleagues (Dubinsky, 1991; Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & McDonald, 2001; Dubinsky & Wilson, 2013) conceive four types of mental conceptions: action, process, object, and schema. Table 2-1 presents examples of action, process, and object views of function and coset. An action is a physical or mental transformation of objects that an individual perceives as essentially external and as requiring step-by-step instructions (Dubinsky & McDonald, 2001; Breidenbach et al., 1992). Interiorizing an action creates a process. The interiorization of an action occurs when the action is repeated, reflected upon, and can be envisioned within the mind of an individual without explicit instructions or carrying out specific steps (Dubinsky & McDonald, 2001). Hence, an individual is able to consider reversing a process as well as composing it with other processes to possibly create new processes. Encapsulation of a process into an object occurs once an individual becomes aware that the process can be transformed by an action (Breidenbach et al., 1992). Lastly, a schema for a particular mathematical concept consists of an individual's actions, processes, objects, and other schemas. This collection of mental conceptions forms a framework in an individual's mind which includes either implicit or explicit criteria for determining which contextual situations relate to the schema (Dubinsky & McDonald, 2001).

Table 2-1 Examples of Action, Process and Object Mental Conceptions

Action	Process	Object
Understanding function only as a (single) formula for computing values.	Understanding that a function receives inputs, performs operations on the inputs, and returns outputs.	Naming functions and performing operations and actions on them make sense to the individual e.g. applying linear operators on Hilbert spaces.
Understanding a coset as a set of calculations used to obtain a definite set. Consider $H = \{0, 4, 8, 12, 16\}$ a subgroup of $(Z_{20}, +_{20})$. An action perception of the coset $2 + H = \{2, 6, 10, 14, 18\}$ is viewing $2 + H$ as the addition of 2 to every element of H or as the rule of starting with 2 and adding 4.	Understanding a coset as a set formed by operating a fixed element with every element of a particular subgroup.	Naming cosets and performing operations and actions on them make sense to the individual e.g. comparing cosets for equality or applying binary operations to cosets

(Asiala, Brown, DeVries, Dubinsky, Matthews, & Thomas, 1996, p. 7-8)

Considering the function concept through the lens of APOS Theory, Dubinsky and colleagues (Asiala et al., 1996; Breidnebach et al., 1992; Dubinsky & Wilson, 2013) suggest five stages of understanding of functions: prefunction, action, process, object, and schema. An individual possesses a

prefunction conception if he or she displays little to no conception of function. For instance, examples of prefunction definitions of function include “‘I don’t know’, or ‘a mathematical equation with variables’, or ‘a mathematical statement that describes something’” (Breidenbach et al., 1992, p. 252). Since an action is a transformation of an object that requires step-by-step instructions, an individual with this conception of function needs specific instructions on how a domain element is transformed into an element of the range. Moreover, the individual must perform this transformation one element at a time such as with a formula or algorithm (Dubinsky & Wilson, 2013). Function definitions provided by individuals with an action conception focus on the substitution of numbers for variables and calculations. Furthermore, individuals with an action conception may provide definitions that are tied to an expression or equation, or they may restrict input and output objects (Breidenbach et al., 1992).

An action conception of function becomes a process conception when an individual is able to develop mental constructions pertaining to functions that generate the same transformations produced by actions and can be envisioned without actually carrying out explicit steps (Dubinsky & Wilson, 2013). Then, an individual with a process conception of function is able to perceive a function as a transformation beginning with objects, altering these objects in some way, and obtaining new objects as a result. Definitions of function by an individual with a process conception may contain references to an input, a transformation, and an output (Breidenbach et al., 1992). When an individual is able to perform actions on functions, he or she is said to have an object conception. Nyikahadzoy, Julie, Mtetwa, and Torkildsen (2008) suggest that an individual who perceives a function as a set of ordered pairs rather than a computation procedure has an object conception of function. The organization of an individual’s collection of function related process and objects in a structured manner forms a function schema. For example, forming functions into sets, introducing operations on these sets, and considering properties of these operations may all be used to construct a schema for function space. This schema for function space may then be applied to dual spaces or spaces of linear mappings (Asiala et al., 1996).

2.1.2 Operational and Structural Theory of the Function Concept

Following a historical and psychological analysis of the formation of the concept of function, Sfard (1991) introduced her operational and structural theory for the development of the conception of function. A structural conception of a mathematical notion treats it as an abstract object. With this conception, an

individual perceives the entity as a static structure and manipulates it as whole. For example, a function can be perceived as an object if it is considered as a set of ordered pairs. An operational conception views a mathematical notion as a process, algorithm, or action rather than an object. Instead of recognizing an entity as a static structure, it is viewed as a potential entity that is the result of a particular process. In the case of functions, perceiving a function as a process for moving from one system to another would be an operational conception (Sfard, 1991).

Both the structural and operational conceptions are useful and necessary within the learning and problem-solving processes. The operational conception, for instance, is essential for determining final answers to mathematical problems. However, the structural conception condenses operational information to make problem-solving into a more direct task. Sfard (1991) also theorizes that when introduced to a new mathematical idea, an operational conception usually develops first and the transition to a structural conception may take place through a three-step pattern: interiorization, condensation, and reification. Piaget defines interiorization in his theory of reflective abstraction – the internal processes an individual develops in relation to perceived phenomena on familiar objects. When a process can be considered as a self-contained whole without needing to think about component steps, it is said to have been condensed. Finally, reification is the conversion of condensed processes into an object-like entity (Sfard, 1992).

Sfard (1991) asserts that this three-step pattern in the transition from an operational to structural conception is present in the historical development of the function concept. An official notion of function first emerged at the end of the seventeenth century as a means to model physical phenomena with variable quantities. During this time, algebraic symbolism was spreading throughout mathematics, and Sfard (1991) suggests that for this reason the first definitions of functions were closely tied to algebraic manipulations on variables – an operational conception. Mathematicians struggled with the dependence of this early definition on variables. According to Sfard (1991), the development of subsequent definitions of function may then “be seen as a long sequence of strenuous, if mostly failed attempts at reification” (p. 15). The result of this struggle to transform the operational definition into a structural one was Dirichlet’s break from the traditional algebraic approach leading to Bourbaki’s definition of function – considering a function as a set of ordered pairs (Sfard, 1991).

Historically, the mathematical community wrestled with reification of the function concept. It is then logical that an individual would also struggle with reification. Sfard (1992) poses two possible sources for this difficulty. First, an individual must be willing to make concessions on their current understanding to enable reification; however, these concessions may be difficult to make. An example of this can be seen in the dissonance ancient Greek mathematicians' experienced between their understanding of numbers (as something to count with) and the idea of irrationality. For functions, the concession needed to enable reification was the algorithmic nature of function. The second possible source for the difficulty of reification is the cycle that ensues from the fact that reification must precede higher-level manipulations on a concept, but the need to perform higher-level manipulations on a concept provides the motivation for students to think structurally about a concept (Sfard, 1992).

Based on her theory that operational conceptions develop prior to structural conceptions and the struggle for reification, Sfard (1992) offers two didactic principles on how new concepts should not be taught. The first is that "new concepts should not be introduced in structural terms" as students should be given time and opportunity to move through interiorization, condensation, and reification (Sfard, 1992, p. 69). The second didactic principle states, "A structural conception should not be required as long as the student can do without it" (Sfard, 1992, p. 69). As discussed in the difficulties of reification, the need to perform higher-level manipulations on a concept motivate a structural conception. For functions at the secondary level, Sfard (1992) suggests that the need for structural conception may not occur at all. In fact, as long as the function concept only appears within basic calculus, a condensed operational conception is adequate for interacting with differentiation and integration.

2.1.3 Procept Theory

Two theories of the acquisition of the function concept have been presented so far: APOS theory and an operational-structural theory. Components of both of these theories include the idea of moving from a process or algorithmic (operational) view of a concept to perceiving the concept as a static object capable of being transformed by an action (Dubinsky, 1991; Sfard, 1991). Since both conceptions are useful in particular learning and problem-solving contexts, it suggests that a concept can be considered simultaneously as a process and an object (Sfard, 1992). Gray and Tall (1994) introduce the term "procept" as a means to explain this duality.

The way in which a concept can be viewed as both a process and object is through notation. Mathematicians are able to use notation to represent both a process and the product of a process, and they can use this notation to move flexibly back and forth between the two conceptions. For instance, the notation a/b serves as either an object or the process of dividing a by b. “The function notation $f(x) = x^2 - 3$ simultaneously tells us both how to calculate the value of the function for a particular value of x and encapsulates the complete concept of the function for a general value of x ” (Gray & Tall, 1994, p. 120). This combination of a process, an object, and a symbol that can represent the process and the object is known as an elementary procept (Gray & Tall, 1994). A procept is a collection of elementary procepts with the same object. For example, 6 is a procept because 6 represents both an object and symbol, and it consists of the process counting 6 as well as other processes such as $10 - 4$ and 3×2 (Gray & Tall, 1994).

Gray and Tall (1994) hypothesize that through proceptual thinking (the combination of conceptual and procedural thinking) a more able thinker may develop meaningful relationships between mathematical notions. On the other hand, a less able thinker, or a procedural thinker, is fixated on processes and will have more difficulty developing these same connections in order to gain new insights (Gray & Tall, 1994).

2.1.4 Covariational Reasoning

With the perspective that students must move from an action view of functions to a process view (such as in APOS Theory), Oehrtman, Carlson, and Thompson (2008) contend that a student with a process view is able to perceive the whole process as happening simultaneously to all values. Such a student is also “able to conceptually run through a continuum of input values while attending to the resulting impact on output values” (Oehrtman et al., 2008, p. 35). This coordination of two varying quantities while attending to the changes in relation to each other is defined as covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002).

Some of the earliest work on covariational reasoning as a theoretical construct started with Pat Thompson’s interest in the ways in which students perceive situations as comprised of quantities, the relationships among the quantities with varying values, and the way students understand rate of change (as cited in Thompson & Carlson, 2017). According to Thompson, covariational reasoning occurs when

an individual conceives two quantities' values varying and visualizes them varying simultaneously (as cited in Thompson & Carlson, 2017). Saldanha and Thompson (as cited in Carlson et al., 2002) also conjecture that covariational reasoning is developmental in the sense that images of covariation may be defined in a sequence of ordered levels. Extending this conjecture, Carson et al. (2002) developed a framework for analyzing covariational reasoning with levels of mental actions and levels of covariational reasoning competency.

The levels of covariational reasoning first presented by Carlson et al. (2002) have since been refined as a result of Castillo-Garsow's (as cited in Thompson & Carlson, 2017) characterizations of "discrete, chunky continuous, and smooth continuous thinking about how a quantity's value varies; research on the students' conceptions of time as a quantity; and a new understanding that conceptualizing multiplicative objects is essential to reason covariationally" (p. 435). Table 2-2 depicts Thompson and Carlson's (2017) revised levels of covariational reasoning based off Castillo-Garsow's work.

Table 2-2 Levels of Covariational Reasoning

Level	Description
Smooth continuous covariation	The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously.
Chunky continuous covariation	The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation.
Coordination of values	The person coordinates the values of one variable (x) with values in another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).
Gross coordination of values	The person forms a gross large image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values.
Precoordination of values	The person envisions two variables' values varying, but asynchronously – one variable changes, then the second variable changes, then the first one, and so on. The person does not anticipate creating pairs of values as multiplicative objects.
No coordination	The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.

(Thompson & Carlson, 2017, p. 434)

Considering the function concept, research shows that covariational reasoning is essential for illustrating and understanding the changing nature of quantities in a variety function contexts and for understanding central calculus concepts including limits, derivatives, related rates, and the Fundamental Theorem of Calculus (Carlson, Oehrtman, & Engelke, 2010; Thompson & Carlson, 2017). The development of covariational reasoning does not occur instantaneously; rather, it is a complex process that occurs over many years through interaction with tasks that encourage such thinking. However, U.S. curriculum and instruction are failing to nurture this development. An examination of 17 U.S. secondary mathematics textbooks, for example, revealed that all used a form of the Dirichlet-Bourbaki correspondence definition of function (Thompson & Carlson, 2017). Thompson and Carlson (2017) offer a covariation-based meaning of function where a “quantity” appears to be conceptualized as an object that possesses measurable attributes:

A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person’s conception, every value of one quantity determines exactly one value of the other. (p. 436)

While the correspondence definition of function does not promote a covariation view of function, the proposed meaning from Thompson and Carlson (2017) seems to diminish the arbitrary nature of functions. Conceiving a function as two quantities varying simultaneously lacks the notion that a function can describe a relationship between sets of arbitrary objects – sets that do not necessarily include quantities. For example, D_3 , the dihedral group of rigid motions on an equilateral triangle, and S_3 , the symmetric group of permutations on the set $\{1, 2, 3\}$, are isomorphic. There is then clearly a bijective function between these two groups; however, the elements of these sets would not necessarily be considered quantities. Thompson’s and Carlson’s (2017) proposed meaning also conceals the fact that the relationship itself may be arbitrary in nature – there does not have to exist a causation, rule, or underlying rationale that determines the relationship between the quantities or sets.

2.1.5 Concept Definition and Concept Image

Although mathematical concepts such as function have formal definitions, an individual may not employ the formal definition in a problem context. Instead, the individual may use examples, nonexamples, facts, and relationships associated with the concept (Vinner & Dreyfus, 1989). Vinner and colleagues (Tall & Vinner, 1981; Vinner, 1992; Vinner & Dreyfus, 1989) use the terms concept definition

and concept image to distinguish between a concept's definition and the total cognitive structure an individual develops in relation to a concept.

A concept image is the collection of memories evoked in an individual upon hearing or seeing a concept name. Specifically, a concept image consists of visual representations of the concept as well as the impressions or experiences an individual forms related to the concept (Vinner, 1991). Vinner (1991) explains, for example, that an individual's concept image for "table" may include memories of a particular table, experiences involving tables such as sitting or eating, and particular characteristics of tables like having four legs or being made of wood. In the case of functions, a concept image could include beliefs that a function is given by a rule or formula; a function is a mapping action; a function is a graph; or a function is a table of values (Tall & Vinner, 1981).

The development of a concept image of "table" or a concept image of a mathematical notion occurs over time and as a result of an individual's experiences with the concept (Tall & Vinner, 1981). Hence, an individual who has only encountered tables made of metal will probably not possess a concept image of table that includes the characteristic of wood. Similarly with mathematical concepts, an individual's concept image will form around experiences with examples and nonexamples of the concept. These examples and nonexamples an individual uses to form the concept image may not include all the mathematical objects that the concept definition encompasses (Vinner & Dreyfus, 1989). As a result, the concept image may not always be coherent, and different aspects of a concept image may conflict. Furthermore, different contextual situations may elicit different memories. The portion of the concept image that is triggered in a particular context is called the evoked concept image (Tall & Vinner, 1981).

While concept image is the collection of experiences and impressions an individual forms about a concept, a concept definition is the "form of words used to specify that concept" (Tall & Vinner, 1981, p. 152). An individual's personal concept definition may differ from the formal concept definition. The former is constructed by the individual and consists of the words used to describe the concept image. The latter is the definition accepted by the mathematical community (Tall & Vinner, 1981).

Vinner (1992) theorizes that memorization of the concept definition does not ensure an individual understands the concept; rather, he asserts that acquiring a concept actually means forming a concept image for it. The process of concept formation takes place within two different "cells" or compartments in

an individual's cognitive structure: a cell for the definition and a cell for the concept image. Formation of these cells can take place independently, or they may influence one another. However, Vinner (1991) suggests that although a definition may help form a concept image, once an image for the concept is formed, the definition becomes disposable. In fact in problem solving or task performance, it is common to not even consult the concept definition before determining the solution (Vinner, 1992). An individual's acquisition of a concept then centers on the concept image.

To better characterize the unstructured collection of notions that form an individual's concept image, Zandieh, Ellis, and Rasmussen (2017) extend the notion of concept image to include conceptual metaphors. Investigation of conceptual metaphors occurs through metaphorical expressions used in language. Considering functions and linear transformations in particular, Zandieh et al. (2017) identified three main components of students' concept images: computations, properties, and clusters of metaphorical expressions. Computations include student statements that use language characterizing functions or linear transformations as calculations or computations. A property refers to responses that "describe a characteristic of a function, transformation, or associated graph or matrix without describing the inner working of the function or transformation" (Zandieh et al., 2017, p. 27). Examples of properties include references to function as an equation; references to a mapping relationship; and references to the shape of a graph.

In the case of function and linear transformation, Zandieh et al. (2017) also identified five clusters of metaphorical expressions: input/output, traveling, morphing, mapping, and machine. The input/output cluster is characterized by the student discussing an entity going into something and another entity coming out. Traveling refers to an entity that is sent or moved to another location. Discussing an entity that changes or is morphed into another entity is identified as the morphing cluster. The mapping cluster is closely related to the Dirichlet-Bourbaki definition of function because it involves a relationship or correspondence between two entities. Finally, machine is characterized by a reference to a tool or machine that causes one entity to change into another entity (Zandieh et al., 2017).

2.2 Conceptions of Function

Another area of interest for researchers in regard to function includes the various function conceptions secondary students, university students, and teachers hold. This collection of literature can

be divided into two major categories: conceptions about the type of entity a function is, and characteristics students and teachers conceive or do not conceive about functions. Each of these categories of conceptions will be discussed in greater depth in the following sections.

2.2.1 Conceptions of Function as an Entity

One study conducted by Vinner and Dreyfus (1989) explored the types of mathematical entities college students and junior high school teachers define function to be. Six categories arose from this study: correspondence, dependence relation, rule, operation, formula, and representation.

The correspondence category includes definitions that are similar to the Dirichlet-Bourbaki definition of function – definitions that refer to a correspondence between two sets in which each element of the first set is assigned to exactly one element of the second set. In their study of 36 junior high school mathematics teachers and 271 college students, Vinner and Dreyfus (1989) found 25 teachers and 57 college students provided correspondence definitions of function. Comparing the mathematical experience of the college students in the study, they also found that the percentage of students who provided a correspondence definition increased with the level of the mathematics courses that students were taking. Furthermore, Nyikahadzoyi et al. (2008) examined the function definitions of six prospective secondary mathematics teachers who completed Calculus I, Linear Algebra, Analysis, and an introduction to proofs course. This study found that only one of the six prospective teachers provided a correspondence definition of function.

Another category identified by Vinner and Dreyfus (1989) is dependence relation. Responses coded as dependence relation consisted of those in which participants described a function as a dependence relation between two variables. Three of the 36 junior high school teachers and 78 college students provided dependence relation definitions of function. Furthermore, two prospective teachers in the study by Nyikahadzoyi et al. (2008) used the notion of a dependence relation in their definitions of function. Although functions may represent a causal process, “functions by themselves are neither true nor false descriptions of facts” (Bunge, as cited in Sierpinska, 1992, p. 56). Then, function should not merely be considered as a dependence or causal relationship, and Sierpinska (1992) indicates a fundamental act of understanding function is the ability to discriminate between functional and casual relationships.

The rule classification, as depicted by Vinner and Dreyfus (1989), is characterized by the expectation that a function is a rule that has some regularity – conversely, a correspondence may be arbitrary. In their study involving college students and junior high school teachers, Vinner and Dreyfus (1989) found that three teachers and 29 students described the definition of function as a rule. Another study of 147 high school students studying mathematics at a high level showed that 14% of the students said a function is a rule; hence, they did not consider function an arbitrary correspondence (Vinner, 1991). Nyikahadzoyi et al. (2008) also found that two of the six prospective secondary mathematics teachers in their study provided definitions of function that fell under this category.

Operation encompasses definitions that characterize function as an operation or manipulation. This consists of statements that describe function as an entity that “acts on a given number, generally by means of algebraic operations, in order to get its image” (Vinner & Dreyfus, 1989, p. 360). Examining a secondary student’s, Kasia, concept of function, Sajka (2003) found that Kasia treats function as a computational process, and that, for her, the formula encapsulates the whole function. Vinner and Dreyfus (1989) also noticed that 13 college students and one junior high school teacher in their study defined function as an operation. This characterization as an operation may be related to understanding of the equality symbol. From elementary school through high school and even college, studies show that students perceive the equality sign as an operator symbol or a ‘do something signal’ (Kieran, 1981; Knuth, Stephens, McNeil, Alibali, 2006). This perspective of the equality sign may then contribute to the idea that a function, especially a function with an algebraic expression, is an operation.

Formula is another category Vinner and Dreyfus (1989) use to classify definitions of function, and it contains responses that define function as a formula, algebraic expression, or equation. Historically, mathematicians once considered that the only relationships that could be considered functions were those with analytic expressions (Sfard, 1992; Sierpinska, 1992). This idea that a function is a formula, expression, or equation still permeates the minds of students. Sajka (2003) expresses that for a secondary student, Kasia, the concept of function is indistinguishable from the concept of the formula of a function. Thirty college students in the Vinner and Dreyfus (1989) study defined function as a formula, expression, or an equation, and Carlson (1998) determined that college algebra students consider a

function as an entity to be a formula. In addition, one prospective teacher in the Nyikahadzoyi et al. (2008) study also defined function as a formula or an equation.

The final definition category proposed by Vinner and Dreyfus (1989) is representation in which “the function is identified, in possibly a meaningless way, with one of its graphical or symbolic representations” (p. 360). For example, statements that describe the type of entity function is as a graph or something with function notation fall under the representation classification. Sajka’s (2003) study of a secondary student, Kasia, revealed that she associates a function with the process of drawing its graph. Also, in the Vinner and Dreyfus (1989) study, 24 college students and one junior high school teacher provided definitions that fell under representation.

2.2.2 Conceived Characteristics of Function

Vinner (1992) suggests that acquiring a concept means forming a concept image for it. Based on experiences students and teachers have with functions, they may associate inaccurate or incomplete characteristics with the function concept. These conceived characteristics of function include beliefs about arbitrariness; beliefs about univalence; and beliefs about functional notation.

Examining PSMTs’ understandings of the arbitrary nature of functions, Even (1993) identified that some of the prospective teachers believe that functions should be “nice”. Interviews with these prospective teachers revealed that “nice” functions include those that are smooth, continuous, and not “too weird”. Carlson (1998) and Vinner and Dreyfus (1989) report similar beliefs about continuity of functions in college students as well. These conceived characteristics of function demonstrate a lack of understanding of the arbitrary nature of function. Specifically, they indicate some prospective teachers and college students suppose functions must be graphable and that the described correspondence must have some regularity.

A limited understanding of the arbitrary nature of function is also apparent in the belief that functions are defined by known rules. Specifically, Sierpinska (1992), Carlson (1998), and Breidenbach et al. (1992) report that secondary students, successful college algebra students, and preservice teachers think a function is definable by a single algebraic formula. Although Even (1993) presents evidence that prospective teachers understand an infinite number of functions may pass through two or three given points, their justifications did not rely on the arbitrary nature of function. Instead, the preservice teachers

considered specific examples of functions saying things like, “there are infinite parabolas that would satisfy the conditions” (Even, 1993, p. 107). Meel (2003) similarly reports that prospective teachers in his study possess a rule-based interpretation of function. Furthermore, Stein et al. (1990) found that the inservice teachers in their study believed a function was an interdependent relationship between two numbers. This conceived characteristic that functions are defined by known rules demonstrates that individuals’ concept images of functions may not include arbitrary correspondences and correspondences between non-numerical sets.

Even (1993) found that PSMTs knew that the univalence characteristic distinguishes between relations that are functions and relations that are not function. This is supported by Steele, Hillen, and Smith’s (2013) finding that prospective and practicing teachers in their study could produce and identify correct examples of function even when they could not produce a correct definition of function. However, Norman (1992) noticed that secondary teachers, who were also master’s degree students, frequently tried to use the vertical line test to determine whether a given relation was a function. They even applied the vertical line test in non-Cartesian coordinate situations such as examples given in polar coordinates. Prospective teachers also tended to present students with the vertical line test as a rule to follow to get the right answer (Even, 1993). Even (1993) suggests teachers make this pedagogical choice because the prospective teachers in her study did not understand why univalence is needed.

It is also common for students to confound the univalence characteristic with the one-to-one property (Breidenbach et al., 1992; Dubinsky & Wilson, 2013). Often, students will not accept a relation as a function unless the correspondence between the sets is one-to-one. This conceived characteristic may lead to concept images that do not allow for constant functions (e.g. $f: R \rightarrow R, f(x) = 4$) to be considered functions. In fact, Meel (2003) reports that 15 of the 27 prospective teachers in his study did not accept constant functions as functions because they believed that a change in the independent variable must result in a change in the dependent variable.

Another collection of conceived function characteristics students may develop relates to their understanding of functional notation. Particularly, Vinner and Dreyfus (1989) report college students’ unfamiliarity with how function notation relates to the conceptual aspects of function. In interviews with a secondary student, Kasia, Sajka (2003) identified several conceptions Kasia developed about function

notation. These conceptions include perceiving, f , as a label for the term function that does not carry any content itself; associating a symbol such as, $f(3)$, with only the zero of a function; interpreting the symbol, $f(x)$, and the algebraic expression defining the function as the formula of the function; believing, $f(x)$, $f(y)$, and $f(x + y)$ represent three different functions; not accepting functional equations, such as $f(x + y) = f(x) + f(y)$ as equations; and confounding the distributive property with functional notation, i.e., interpreting $f(x + y)$ as $fx + fy$.

Chapter 3

Methodology

The purpose of this multiple case study is to determine how PSMTs' function concept images change when they engage with research-based explorations designed to evoke function-related cognitive conflicts, and the extent to which there are differences in the concept images of PSMTs with varying mathematical backgrounds. This study is bounded by Unit 1 of the course in which the study took place, the cases are defined by three levels of prior mathematical knowledge, and the units of analysis in each case consist of the study participants. Data analyzed in this study includes the pre- and post-interviews of each participant, the materials used in the course, classroom observation notes, and classroom videos provided context to the interviews as well. In this chapter, I will discuss the setting and background of this multiple case study, the interview protocol and data collection, the participant sampling process, the theoretical perspective I use to approach the data, the data analysis process, and the validity of this study.

3.1 Setting

This study took place at a public, urban university in the southwestern United States with an enrollment of over 42,000 undergraduate and graduate students. Of the undergraduate students enrolled at the university, 32.6% identify as Hispanic, 15.2% identify as African American, and 12.5% identify as Asian. Participants in this study include 27 PSMTs enrolled in a mathematics content course designed specifically for PSMTs offered in a Department of Mathematics, during the Fall 2018 semester. Objectives of this course include deepening and broadening function-related mathematical content knowledge from school algebra to calculus by exploring relevant topics in an inquiry-based learning situation; making connections between college mathematics and secondary school mathematics; and using reflective and collaborative learning and developing a stronger sense of professionalism and leadership. Although this course has a second-semester calculus prerequisite, students have some flexibility in their degree plan when it comes to taking this course. As a result, students may choose to enroll in this course immediately following their completion of second-semester calculus, or students may wait until after completing more upper-level mathematics courses such as Abstract Algebra I or Real Analysis I.

The course in this study, commonly referred to as Functions and Modeling or Functions in Mathematics (FM), is a part of the UTeach teacher preparation program (UTeach Institute, n.d.) curriculum. Forty-five universities across the country employ the UTeach model for preservice teachers majoring in science, mathematics, and computer science. Universities implementing this program must initially use the curriculum developed at The University of Texas at Austin. The FM course curriculum, disseminated by UTeach, uses the course manuscript Functions in Mathematics (Armendariz & Daniels, 2011), and it includes 23 lessons over three units:

- Unit 1-Functions, Rates, Patterns;
- Unit 2-Regression and Modeling; and
- Unit 3-Exploring Functions in Other Systems.

In this study, the Unit 1 materials provided by the UTeach program were supplemented and enhanced with research-based lessons developed by the *Enhancing Explorations in Functions for Preservice Secondary Mathematics Teachers* (EEFPSMT) Project, partially funded by the United States National Science Foundation. The goal of this project is to develop research-based tasks and explorations for use in mathematics courses for PSMTs, as well as develop instructor materials to assist mathematicians and instructors in implementing the materials in an inquiry-based, active learning environment (Álvarez, Jorgensen, & Rhoads, 2019). To achieve this goal, researchers followed recommendations by the Design-Based Research Collaborative (2003). Specifically, researchers engaged in the cyclic process of “design, enactment, analysis, and redesign” (Cobb et al., 2003, p. 5) by drawing on existing research and theory to develop and implement materials; collecting data and reflecting on the success of the materials; and using these findings to inform the evolution of the tasks. This process underwent three complete cycles and resulted in 11 research-based lessons consisting of multiple tasks and explorations that investigate function versus equation, graphical connections to function patterns, and other important function concepts.

Students enrolled in the Fall 2018 FM course met twice a week for 80 minutes each meeting. As the materials were designed for an inquiry-based, active learning environment, students engaged with the research-based explorations in groups of three to four. The instructor facilitated these small group interactions and occasionally initiated whole-class discussions. Overall, the class consisted of very few instances of lecture. Implementation of the 11 research-based lessons, spanned the first 10 weeks of the

15-week semester. Each of these lessons lasted on average approximately one and a half class meetings (see Table 3-1). Following each class meeting, the instructor assigned a journal prompt intended to encourage students to reflect and expand on ideas brought out by the lessons. These journals were submitted online prior to the subsequent class meeting. In addition to these daily journals, students enrolled in the FM course completed three homework assignments designed to emulate and extend the 11 research-based lessons completed in class. Following the completion of these 11 lessons, students also had an exam to assess their understanding of the topics explored in the class, journals, and homework.

Table 3-1 Number of 80-minute Class Days Used in Fall 2018 for Each Lesson

Lesson 1: Solving Problems	1.5
Lesson 2: Conic Sections	2.5
Lesson 3: Qualitative Look at Graphical Representations	3
Lesson 4: Examples of Real World Relationships Between Quantities	0.5
Lesson 5: What is a Function?	3
Lesson 6: Functions and Equations	1.5
Lesson 7: A Familiar Function from a Different Point of View	1.5
Lesson 8: Spring-Mass Motion Lab	1
Lesson 9: Sequences and Triangular Differences	1.5
Lesson 10: Functions Arising from Patterns	1.5
Lesson 11: Indistinguishable Function Transformations and Function Patterns	1

Overall, 27 of the 28 students enrolled in the Fall 2018 Functions and Modeling course consented to participate in this study. Approximately 41% of the participants self-identified as Latino, Latina, or African American, 37% self-identified Caucasian, and 22% self-identified as Asian, Multiracial, or another ethnicity. Also, 44% of the participants self-identified as male and 56% as female. Researchers collected a copy of all written work from each consenting student including lessons, homework, and exams. Additionally, I conducted pre- and post-interviews with selected students from the course.

3.2 Pre- and Post-Interviews

To identify any changes in students' concept definitions and images of function through interaction with the research-based materials, I conducted and video recorded individual, pre- and post-interviews. The research procedures were approved by a modification to the UTA IRB protocol #2016-0179.8 in August, 2018. All of the pre-interviews were conducted within the first three weeks of the

course, and the post-interviews occurred after students completed the post-assessment during the eleventh and fourteenth weeks of the 15-week semester. The pre- and post-interviews (see Appendices A and B) consist of the same open-ended, task-based questions. In addition to the task-based questions, the post-interview also included questions related to participants' perceptions of the course learning environment.

The task-based questions, developed by the EEFPSMT project, were designed to elicit students' concept definitions and concept images of function as well as students' understanding of key ideas associated with function. For instance, Question 1 asks students to define function, and Question 2 asks students what the defining characteristic(s) of function are (Figure 3-1). If participants did not express an understanding that a defining characteristic of function separates functions from other types of objects, then I explained this notion to them before asking them to list its defining characteristics in Question 2. Both of these questions are intended to draw out the words, or the concept definition, students use to express their understanding of function.

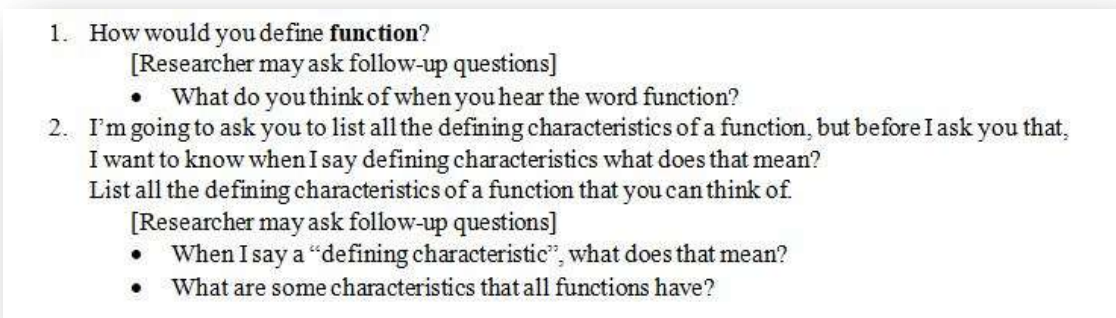
- 
1. How would you define **function**?
[Researcher may ask follow-up questions]
 - What do you think of when you hear the word function?
 2. I'm going to ask you to list all the defining characteristics of a function, but before I ask you that, I want to know when I say defining characteristics what does that mean?
List all the defining characteristics of a function that you can think of.
[Researcher may ask follow-up questions]
 - When I say a "defining characteristic", what does that mean?
 - What are some characteristics that all functions have?

Figure 3-1 Questions 1 and 2 of Task-based Portion of Pre- and Post-interviews

Question 3 (Figure 3-2) provides a window into students' concept image of function by asking students to provide three examples of functions. I also used follow up questions such as, "Could you have a function whose domain isn't the real number or some subset of the real numbers?" to try to gain a more complete picture of all the types of functions each student included in their concept image.

3. Please give three examples of functions. For each, please explain how you know it is a function
[Researcher may ask follow-up questions]
- Can you give an example of a function whose domain is not the real numbers or a subset of the real numbers?

Figure 3-2 Question 3 of Task-based Portion of Pre- and Post-interviews

The remaining questions in the task-based portion of the interview were intended to provide insight into students' understanding of ideas related to function as well as gain further understanding of their concept images. Question 4 (Figure 3-3) presents the participants with four statements and asks them to determine which represent valid mathematical definitions for function. Included within these are statements defining a function as an equation, as a graph that passes the vertical line test, and as a relation that assigns each element of the range to exactly one element of the domain (a one-to-one function).

4. Which of the following are valid mathematical definitions of *function*? For each part, explain why it is or is not a valid definition.
- A function from a set A (the domain) to a set B (the codomain) is a rule or correspondence that assigns exactly one element of the codomain to each distinct element of the domain.
 - A function is an equation that gives a particular relationship between two quantities.
 - A function is a graph that passes the vertical line test.
 - A function is a relation for which every element of the range corresponds to exactly one element of the domain.
 - A function is a relation from A to B such that each element of A is assigned to a unique element in B.
- [Researcher may ask follow-up questions]
- A valid mathematical definition can be applied to any example you call a function.

Figure 3-3 Question 4 of Task-based Portion of Pre- and Post-interviews

Then, Question 5 asks students to define equation; Question 6 asks if function and equation mean the same thing; and Question 7 asks participants if the term function is used correctly in a provided statement (Figure 3-4). This question is modeled on a released state standardized test that essentially instructs secondary students to, "Find the x-intercept and the y-intercept of the function $2x-y=8$ " (c.f. Meeks, 2012). Research on undergraduate students' function conceptions reveals a belief that functions are defined by algebraic rules or equations, and a difficulty distinguishing between equations and functions (Breidenbach et al., 1992; Carlson, 1998; Sierpinska, 1992; Vinner & Dreyfus, 1989). Then these

questions aim to reveal students' understanding of the relationship between function and equation and the extent to which their function concept image overlaps with their concept image of equation.

5. In your own words, define **equation**.
[Researcher may ask follow-up questions]
 - Can you give me some examples of equations?
 - Can you give an example of an equation that doesn't have any unknowns?
 - Some people say that anything with an equal sign is an equation. How would you respond to that?

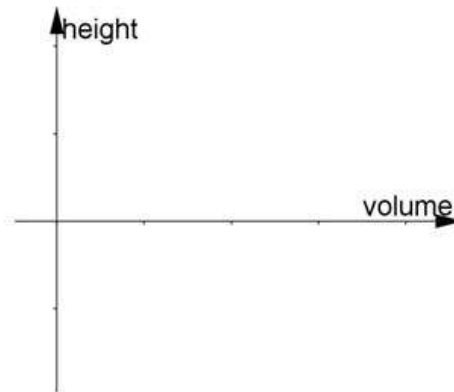
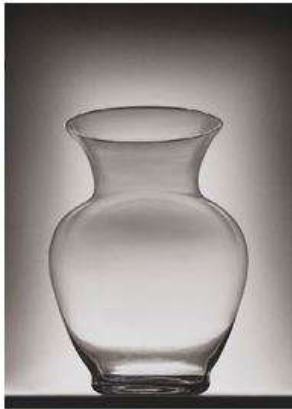
6. Do the mathematical terms *function* and *equation* mean the same thing? Explain your answer.
[Researcher may ask follow-up questions]
 - Are there functions that cannot be represented by an equation?
 - Are there equations that do not also represent a function?

7. In this example, discuss whether the word function is used correctly.
Example: "Find the x-intercept and the y-intercept of the function $2x - y = 8$."
[Researcher may ask follow-up questions]
 - This problem is stating that $2x - y = 8$ is a function. Do you agree with this?
 - When I ask you to find the x-intercept and y-intercept of the function, is that a correct use of the word function?
 - Is a function the same as the graph of a function?

Figure 3-4 Questions 5, 6, and 7 of Task-based Portion of Pre- and Post-interviews

Finally, students were given a picture of the vase and told it was being filled with water at a constant rate. They were then asked to sketch a graph of height vs. volume as the vase is being filled (Figure 3-5). This question is modeled after Carlson's (1998) and Monk's (1992) bottle tasks, and it is intended to gain some insight into students' covariational reasoning. For each of the task-based interview questions, I used follow-up questions to encourage students to explain and clarify their answers.

8. Consider the vase below being filled with water at a constant rate. Sketch a graph of height vs. volume as it is being filled. Explain your reasoning.



[Researcher may ask follow-up questions]

- Why is your graph always “rising”?
- Why does your graph have this shape?
- What happens when the curve of the vase changes?
- Is knowing the rate at which the water is filling the vase is an important factor needed to draw graph?

Figure 3-5 Question 8 of task-based portion of pre- and post-interviews

In the post-interview, I also asked participants to answer questions related to their perceptions of the class learning environment. Each participant answered four questions pertaining to their perceptions of the course:

1. What do you think about the format of this course?
2. Is this format something you would integrate into your own teaching?
3. What features of the class format or environment do you think enhance your learning?
4. What features of the class format or environment do you think detract from your learning?

I also used follow-up questions to gain further insight into students' perceptions and encourage them to extend and clarify their answers.

3.3 Student Interviews

When students consented to participate in the study, they also self-reported their personal information including their self-identified gender, their classification at the university, their ethnicity, their native language, all mathematics courses completed prior to the Fall 2018 semester, and all mathematics courses taken concurrently with the Fall 2018 FM course (see Appendix C). The information on

mathematics courses students completed prior to Fall 2018 allowed the placing of students into three general groups according to their mathematical background. Group A consists of students who completed only second- or third-semester calculus. Group B includes students who completed mathematics courses beyond third-semester calculus but not *Abstract Algebra I* or *Real Analysis I*. Finally, Group C comprises students who had completed *Abstract Algebra I* or *Real Analysis I*. Thus, students fall into distinct groups based upon their prior mathematical experience. These three groups form the three cases of this multiple case study where the participants serve as the units of analysis. Of the 27 participants in the study, researchers extended invitations to 12 students to participate in the pre-interview. These 12 students were chosen to represent the varying mathematical backgrounds of the students enrolled in the course. Ten of the 12 students who received invitations to participate in the pre-interview actually completed the pre-interview. Although all students who participated in the pre-interview were invited to return for a post-interview, only seven students chose to respond for a post-interview. Table 3-2 presents the self-reported information of the seven pre- and post-interview participants as well as their group classification.

In each of the interviews, students were provided with a pencil and interview packet that had the interview questions typed out, when pertinent, and provided ample space for students to answer the questions. I asked the students to answer the task-based interview questions in the order they are presented in the student interview protocol (see Appendices A and B) as some questions could potentially influence students' thinking on other questions. Question 1, for example, asks students to define function and Question 4 provides them with a list of possible definitions. However, I did allow students to return to questions at any point in the interview, and I provided them with the opportunity to revisit all of the task-based interview questions at the end of each interview. Each interview lasted no more than one hour, and students received \$30 as compensation for each interview completed.

To guarantee anonymity, participants were assigned a pseudonym. The video recordings of the interviews were blinded using these pseudonyms. After each round of interviews (pre- and post-interviews) was completed, the video recordings for each session were blinded, watched in slow motion, and all spoken words were transcribed. The written artefacts each student produced in an interview session were also collected, blinded using the participant's pseudonym, electronically scanned, and paired with the corresponding interview transcript.

Table 3-2 Self-Reported Participant Information

Identifier	Self-Identified Gender	Ethnicity	Native Language	Math Courses Completed Prior to Fall 2018	Math Courses Taken Concurrently with FM in Fall 2018	
Group A	Sofia	Female	Latina	English	Calculus I-III	Intro. to Proofs Intro. to Matrices & Lin. Alg.
	Lee	Male	Caucasian	English	Calculus I-II	Calculus III
Group B	Jamie	Female	Latina	Spanish	Calculus I-III Intro. to Proofs Elem. Number Theory Intro. to Matrices & Lin. Alg. Discrete Mathematics	Abstract Algebra I Real Analysis I
	Gabby	Female	Did not provide	Spanish	Calculus I-III Intro. to Proofs Statistical Inference Intro. to Matrices & Lin. Alg. Trigonometry Discrete Mathematics	Abstract Algebra I Real Analysis I Foundations of Geometry
Group C	Henry	Male	African American	French	Calculus I-III Intro. to Proofs Elem. Number Theory Intro. to Matrices & Lin. Alg. Real Analysis I Discrete Mathematics Foundations of Geometry	Abstract Algebra I Statistical Inference Multivariable Calculus
	Alan	Male	Latino	Spanish	Calculus I-III Intro. to Proofs Elem. Number Theory Intro. to Matrices & Lin. Alg. Abstract Algebra I Real Analysis I Real Analysis II Discrete Mathematics Foundations of Geometry	Statistical Inference
	Michael	Male	Latino	Spanish	Calculus I-III Intro. to Proofs Elem. Number Theory Intro. to Matrices & Lin. Alg. Abstract Algebra I Real Analysis I Foundations of Geometry	Discrete Mathematics

3.4 Theoretical Perspective

According to Vinner (1991), the process of concept formation takes place within two distinct “cells” or compartments in an individual’s cognitive structure: the cognitive cell for the definition and the cognitive cell for the concept image. Educators at the secondary and post-secondary levels may assume that the relationship between these cells is a one way process i.e. concept images are formed by and

completely controlled by the concept definition. In actuality, it is more likely that there is an ongoing interplay between these cells: the concept definition is influenced by the concept image, and the concept image can be affected by the concept definition. Vinner (1992) also asserts that once an individual forms a concept image, then the concept definition becomes disposable. Furthermore, it is common for an individual to not even consult a concept definition and instead rely on the concept image in problem solving or task performance. Then understanding of a concept hinges on an individual's concept image (Vinner, 1992). To provide a finer grained characterization of the collection of entities that form an individual's concept image, Zandieh, Ellis, and Rasmussen (2017) contend that "a person's concept image of a particular mathematical idea will likely contain a number of conceptual metaphors as well as other cognitive structures" (p. 24). Identification of conceptual metaphors is done by examining metaphorical expressions. Clusters of metaphorical expressions allow for the characterization of an individual's concept image (Zandieh et al., 2017).

These views of concept formation, understanding of a concept, and the characterization of concept image described above form the theoretical lens I ascribe to this study. Situating the study through this lens allows me to identify and group the associations a student has for the function concept including potentially conflicting associations. With this lens, I am able to examine these groupings of associations in order to make comparisons between students' concept images before and after their interaction with the 11 research-based lessons in Unit 1 as well as across students with varying levels of mathematical experience.

Identifying the groups of associations a student has for the function concept is achieved through thematic analysis. "Thematic analysis is a method for identifying, analyzing and reporting patterns (themes) within data" (Braun & Clarke, 2006). Although thematic analysis is often seen as merely a tool for other qualitative methods, Braun and Clarke (2006) contend that it is indeed a method, and they provide guidelines for conducting thematic analysis. Unlike other qualitative methods, thematic analysis offers flexibility by not subjecting researchers to a particular theoretical or epistemological position such as conversation analysis or interpretive phenomenological analysis. Thematic analysis can instead be applied across a range of theoretical and epistemological approaches and consists of six phases: familiarizing yourself with the data, generating initial codes, organizing codes into potential themes,

reviewing the themes, defining and naming the themes, and producing the report. While thematic analysis can appear similar to the grounded theory frameworks, “the goal of grounded theory analysis is to generate a plausible – and useful – theory of the phenomena that is grounded in the data” whereas thematic analysis does not necessarily attempt to produce a theory (Braun & Clarke, 2006, p.80-81).

Braun and Clarke (2006) identify two flavors of thematic analysis: inductive and theoretical (deductive) thematic analysis. In inductive thematic analysis, themes are strongly tied to the data themselves and are not driven by the researcher’s theoretical interest or preconceptions. Theoretical thematic analysis, on the other hand, is driven by the researcher’s theoretical interest and as a result provides a more detailed analysis of some particular aspect of the data (Braun & Clarke, 2006). As I am specifically trying to analyze students’ changes in their concept images of function as well as make comparisons across students, I use theoretical thematic analysis to identify the groups of associations students have for the function concept. I also analyze the data using a semantic approach only seeking to identify themes in what the participants say or write. Then, I do not look beyond these explicit communications to identify underlying assumptions or conceptualizations. The latter describes a latent approach to analyzing data (Braun & Clarke, 2006).

3.5 Data Analysis

Coding of student interview transcripts was completed using NVivo qualitative analysis software. The written works that students produced in the interviews were also uploaded into the NVivo software to provide context to the interview transcripts. Within the coding process, all interview data was analyzed without considering whether the data originated from a pre- or post-interview. To start the coding process, I identified the statements in all the interviews that reveal any function-related association. I then generated initial codes by analyzing all these function-related associations and grouping them by similar, explicit associations. By explicit associations, I mean that these groupings were formed based only on the rationale explicitly provided by a participant. I did not presuppose any rationale for an association other than what was said or written during the interviews. Following this initial coding, I reexamined every instance of a particular code across all interviews to ensure I still agreed with my initial coding of the transcript data, and I re-coded when necessary. I also conferred with another, more experienced, researcher on this initial coding. These initial, function-related association codes were then cross-

referenced in order to begin organizing them into potential function concept image themes. I then reviewed the potential themes, consulted another researcher regarding these potential themes, and reorganized them as necessary. As each participant's concept image in both interviews is incredibly multifaceted, for the scope of this study, these potential themes were narrowed based on *prevalence* across participants. Precisely, potential themes identified in at least three participants' interviews and potential themes that directly contrasted a theme associated with at least three participants constitute the narrowed potential themes. These narrowed potential themes were then reviewed, defined and named, and used to develop an impression of each student's concept image. Figure 3-6 presents an overview of this coding process. With the themes finalized and impressions of concept images formed, I compared the themes identified in each participant's pre-interview with those in their post-interview. I used this comparison to determine if students developed new or altered existing concept images. Finally, I looked for any similarities and differences in the concept images of students with varying levels of mathematical backgrounds.

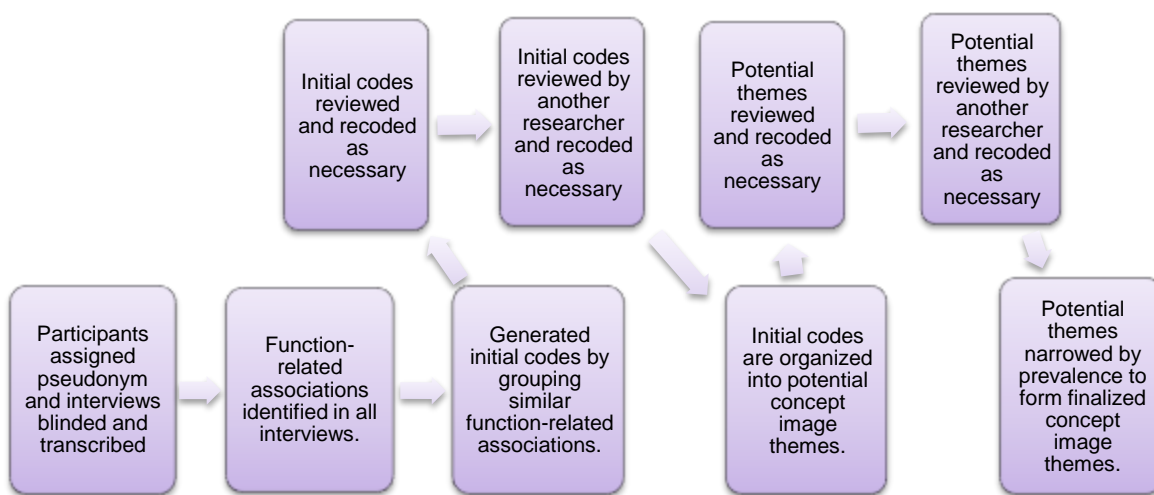


Figure 3-6 Overview of Coding Process

3.6 Validity

To assess the trustworthiness of this study, I refer to the credibility, transferability, dependability, and confirmability criteria. For the credibility, I draw on my two-year experience as a graduate research assistant on the EEFPSMT project prior to the data analysis of this study. This provided opportunities to

participate in the development of the materials, interact with and observe students engaging with the materials, interview the instructor after every class meeting, conduct task-based interviews with PSMTs, and gain experience using thematic analysis methods on these interviews. In all of this, I worked alongside more senior researchers and was able to develop these skills under their guidance. Moreover, all these experiences centered on the function conceptions of PSMTs, so I was able to refine my skills within a context consistent with that of this study. The coding process for this study also included frequent reviews by myself and periodic reviews with another, more experienced researcher. Transferability is achieved through a detailed depiction of the demographic and mathematical backgrounds of the participants, the setting, the interview protocol, and the coding process. Finally, dependability and confirmability are attained through a thorough description of the research steps in the previous sections of this chapter as well as maintaining an audit trail of my analysis.

Chapter 4

Findings

Following the transcription of the participants' pre- and post-interviews, I used thematic analysis to analyze and identify function concept image themes. Through the process described in the previous chapter, I identified 19 function concept image themes as well as five categories arising from the themes. This chapter examines each of these themes and the types of responses they encompass; an overview of the themes in each participant's interview; and an examination of each of the themes identified in each participant's interview presented by participant group. The final section of this chapter identifies the concept image themes that are consistent, inconsistent, or neutral in relation to the definitions of function and equation.

4.1 Overview of Function Concept Image Themes

A brief description of each of the 19 function concept image themes is provided in Table 4-1, and this section expands upon each of the descriptions using participant responses. Following the explanation of each theme, I present some further remarks on the themes and associated categories as well an overview of the codes that emerged in each participant's pre- and post-interview.

4.1.1 *Functional Notation*

The theme *Functional Notation* represents statements that suggest a defining characteristic of function includes a specific functional notation form such as " $f(x)$." Only statements that rely on this type of notation to decide that a relation is a function are coded as *Functional Notation*. For example, when defining function in her pre-interview, Sofia explains that "a function is more like $f(x)$ and not like a ' $y =$ ' type of thing, cause that's what we're always told in class." Gabby similarly explains in her pre-interview definition of function that functions "would be like in the form of like ' $p(x) =$ ' this." Statements such as this are taken as evidence that this specific type of notation is an aspect of a participant's function concept image.

4.1.2 *Form Implies Function*

Form Implies Function encompasses statements where a student determines that an object is a function because it is or can be written in a form such as " $y =$ ". This theme is different from merely asserting that an equation is a function or that a function has an algebraic representation. The rationale

provided must leverage a form like “ $y =$ ”. Specifically, Jaime states that an equation can be a function if it “can be solved for one variable.” This statement indicates that Jaime attributes a form such as “ $y =$ ” to functions. Michael also does this in Question 7 when he must consider whether not it is correct to use the term “function” to describe $2x - y = 8$. He concludes that it is appropriate to describe $2x - y = 8$ as a function because, “just ‘cause it’s not like, you know ‘ $y =$ ’, you can solve for it.” This rationale possibly suggests that if the object was of this form it would clearly be accepted as a function.

Table 4-1 Function Concept Image Theme Descriptions

Theme	Description
Functional Notation	Participant indicates that objects having a form similar to “ $f(x)$ ” is a defining characteristic of function.
Form Implies Function	Participant indicates that objects of the form “ $y =$ ” are functions as a result of this form.
Form Implies Equation	Participant indicates that objects of the form “ $y =$ ” are equations and not functions as a result of this form.
Numerical Domain	Participant indicates that functions are only defined on numerical sets and/or is only able to provide examples of functions defined on numerical sets.
Non-Numerical Domain	Participant indicates through general or specific examples that there exist functions defined on non-numerical sets.
One-to-One	Participant indicates that only one-to-one relations can be functions.
Not One-to-One	Participant indicates that a function does not have to be one-to-one and/or accepts a non-one-to-one relation as a function.
Graphable	Participant indicates that all functions are graphs or graphable on a coordinate system.
Graph Not Defining Characteristic	Participant indicates that not all functions are graphable on a coordinate system.
VLT	Participant indicates that all functions pass the vertical line test.
VLT Not Defining Characteristic	Participant indicates that not all functions pass the vertical line test.
Relations on Sets	Participant indicates that functions are defined on sets or specifically defined on a domain and codomain.
Input/Output	Participant indicates that a function is a relation between inputs and outputs or a relation for which the first element produces the second element.
Univalence	Participant indicates that each element of the domain corresponds to one and only one element of the codomain.
Algebraic Representation	Participant indicates that all functions are equations or that all functions can be written using an algebraic representation.
Equations are Functions	Participant indicates that all equations are functions.
Non-Univalence for Equations	Participant indicates a distinction in equations from function by the rationale that equations are not “limited” by univalence as functions are.
Contrived Difference from Equation	Participant indicates a contrived difference formed to separate the terms function and equation.
Defining Characteristics	Participant indicates that function and equation are different types of mathematical objects based on each term’s defining characteristic.

4.1.3 Form Implies Equation

Contrasting *Form Implies Function*, statements coded as *Form Implies Equation* assert that objects of the form “ $y =$ ” are in fact not functions but instead equations as a result of this form. Responses within this code, similarly to the *Form Implies Function* theme, must leverage the form, or a form similar to, “ $y =$ ” to determine an object is an equation and not a function. For example, Sofia distinguishes in her post interview that $y = 7x + 2$ is an equation and not a function because its form is a “comparison between two different variables, and therefore, that’s not what a function is.”

4.1.4 Numerical Domain

Assertions that functions must be defined on sets of numbers or that is not possible to define a function on non-numerical domains are coded as *Numerical Domain*. Henry, for example, states that, “a function will always be defined in a subset, in a subset of all real numbers.” Moreover, this theme is also characterized by the inability to provide an example of a function defined on a non-numerical domain when prompted. This is apparent in Alan’s pre-interview when he considers whether or not it is possible to have a function not defined on a numerical set. First, he suggests piecewise functions could potentially meet the criteria, and then the Dirichlet function. Upon realizing both of these are still defined on numerical sets, Alan concedes that he “can’t think of one.” These explicit statements and the inability to provide an example of such a function indicates a function concept image that only includes functions defined on numerical domains. As a result, this theme is diametrically opposed to the *Non-Numerical Domain* theme, and an individual could not exhibit both conceptions in the same interview.

4.1.5 Non-Numerical Domain

The *Non-Numerical Domain* theme – a theme that directly contrasts *Numerical Domain* – includes participant provided prompted and un-prompted examples of functions defined on a non-numerical domain. These examples may be explicit or general. For instance, Michael provides an explicit example of a function from the set containing a circle, triangle, and square to the set containing 1, 2, and 3. Sofia, on the other hand, generally explains that one could form this type of a function from a set of people to the set of eye colors. Whether the example is explicitly or generally provided, it demonstrates that mappings between non-numerical sets are included in a participant’s concept image.

4.1.6 *One-to-One*

Sometimes when describing the type of relation that qualifies as a function or the defining characteristics of function, a participant would actually describe a one-to-one function. This type of response is coded as *One-to-One*. Henry provides an example of this in his post-interview when he creates a mapping from $\{1, 2, 3, \}$ to $\{a, b, c, d\}$ and states that, “because each input should have like a unique element in [the codomain], so we, it’s not possible for us to have 1 going to a and 2 going to a .” Also, instances in which a participant does not accept a non-one-to-one correspondence as a function because an element of the range corresponds to two elements of the domain are coded as *One-to-One*. Henry depicts this type of response in his post-interview as well. In particular, he rejects $y = x^2$ as a function because “we have two input[s], 2 and -2 , that correspond to a unique output.” This theme contradicts the conceptions characterized by the *Not One-to-One* theme. Thereby, an individual could not express both conceptions within the same interview.

4.1.7 *Not One-to-One*

While *One-to-One* is characterized by indications that the one-to-one property is a defining characteristic of function, the *Not One-to-One* theme encompasses statements that imply the one-to-one property is not a defining characteristic of function. Then, these themes are antipodal. The types of responses characterized by the *Not One-to-One* theme take the form of explanations that the one-to-one property is not a requirement for functions or the acceptance of non-one-to-one correspondences as functions. For example, Jamie rejects statement (d) of Question 4 in the post-interview because she understands the statement as requiring functions to be one-to-one, and she claims that is not requirement for function. Lee’s and Sofia’s acceptance of $y = x^2$ as a function during their post-interviews is also coded as *Not One-to-One*. These, statements either direct assertions or indirect inclusions, indicate that the participant includes functions that are not one-to-one in their function concept image.

4.1.8 *Graphable*

Responses coded as *Graphable* explicitly indicate a conception that all functions are graphs or capable of being graphed on conventional coordinate system (e.g. Cartesian, polar, complex, etc.) This definition of a graph is in contrast to definitions of graph from graph theory or more abstract ways of thinking about graphs. Furthermore, the *Graphable* theme is in opposition to the *Graph Not Defining*

Characteristic theme, and both could not be exhibited in the same interview. An example of *Graphable* theme is in Lee's pre-interview assertion that a "function serves for a specific graph, it serves for a specific line." Similarly, Sofia says that a "function is like whatever this line is, like how this line goes, or how whatever is on the graph goes." Both of these statements explicitly communicate an understanding that all functions have an associated graph. However, responses that indicate all functions pass the vertical line test are not necessarily coded as *Graphable*. While it is likely participants who accept the vertical line test as a defining characteristic also only include graphable functions in their concept image, the rationale may not explicitly indicate this. Henry's acceptance, for instance, of the statement in Question 4 defining function as a graph that passes the vertical line test is not coded as *Graphable*. He states, "Well that one is true. So, the vertical line test is actually that test just to see if the vertical [line] cut[s] the function once." Although the provided statement implies all functions are graphs, his focus is on the vertical line test. In keeping with the semantic approach to the analysis, I do not make assumptions about the underlying understandings of participants, and responses such as this are not necessarily coded as *Graphable*.

4.1.9 *Graph Not Defining Characteristic*

Graph Not Defining Characteristic encompasses statements from students that contend some functions cannot be graphed, and it naturally contrasts the conceptions characterized by the *Graphable* theme. For example, Sofia directly states that "not all functions are graphable," and Alan provides an example of a function he says cannot be graphed that maps soccer players to the set of jersey numbers. Both of these responses are coded as *Graph Not Defining Characteristic* because they include direct assertions that not all functions are graphable. However, instances where a participant provides an example of a function that would not be graphable on a coordinate system but did not use the example to claim that not all functions are graphable is not coded as *Graph Not Defining Characteristic*. This situation occurs in Lee's post-interview when he provides an example he claims is a function defined on a non-numerical domain that maps a parent's eye color to the child's eye color. I recognize this is not a function that could be graphed on a coordinate system; however, Lee places his mapping on a coordinate system with a parent's eye color axis and a child's eye color axis. To limit any assumptions about participant's

conceptions, only responses that explicitly claim a function is not graphable are coded as *Graph Not Defining Characteristic*.

4.1.10 VLT

The vertical line test identifies graphs of functions on a Cartesian coordinate plane by establishing whether any vertical line would intersect the graph of the relation at more than one point. As the vertical line test applies to graphs on a Cartesian coordinate plane, it presupposes that the domain values are represented on the horizontal axis and the codomain values are represented by the vertical axis. Then the vertical line test is only applicable in the scenario that a function can be graphed under the assumptions of the Cartesian coordinate system. Indicating that the vertical line test is a defining characteristic of functions characterizes the *VLT* theme. Such indications include using the vertical line test as the primary or only rationale in determining if a relation is a function, or explicitly expressing that the vertical line test is a part of the definition of function. Alan, for instance, explains in his pre-interview that an example he provided is a function because, “The rule of thumb is if the vertical line test passes through the function more than once, then it wouldn’t be considered a function...” Since this is his primary rationale, it is coded as *VLT*. Another example of this theme occurs when Henry accepts in his pre-interview statement (c) of Question 4, “A function is a graph that passes the vertical line test,” as a valid definition of function. This theme then is diametrically opposed to the *VLT Not Defining Characteristics* theme, and an individual could not hold both these conceptions within the same interview.

4.1.11 VLT Not Defining Characteristic

In contrast to the *VLT* theme, responses coded as *VLT Not Defining Characteristic* are those that explicitly assert that the vertical line test is not part of the definition for function, or that some functions do not “pass” the vertical line test. This is seen in Sofia’s use of the vertical line test and the “one output for each input” property when deciding if a relation is a function. When asked if a function has to meet both of these criteria, Sofia explains that a function does not necessarily have to pass the vertical line test “because we said that not all functions are graphable, so I think as long as an input only has one output then you’re good...” Alan also rejects the statement in Question 4 defining function as a graph that passes the vertical line test because “you can just change the [Cartesian] coordinates... and you think it’s

a function. How would I say it? You'd be wrong to assume that." This statement indicates that Alan understands that the vertical line test is not applicable in all coordinate system scenarios.

4.1.12 Relations on Sets

As participants discussed functions, there were instances where they described or defined function as some type of relation defined on sets or, more specifically, a domain and codomain. Henry generally defines function in his pre-interview as "a relation between two, two set[s] or two group[s]." Similarly, Michael explains that a particular example is a function because "the real numbers of the first set maps to the one of the numbers from the other set." Jamie specifically discloses in the post-interview that "a function is a mapping from... our domain to our codomain." In the pre-interview, she states, "a function include[s], is composed by a domain and a range." All of these responses indicate an understanding that a function represents some sort of relationship between sets and are coded as *Relations on Sets*.

4.1.13 Input/Output

The *Input/Output* theme is characterized by references to function as a relation between inputs and outputs and descriptions of function as an action where "plugging" in a value from the domain produces an element of the codomain. For instance, Sofia's assertion that a function "describes a relationship between inputs and outputs," is coded as *Input/Output*. Sofia does not identify that these "inputs and outputs" are elements of sets, so this statement is not coded as *Relations on Sets*. Also, in explaining that $f(x) = x + 1$ is a function, Jamie expresses that "we can plug in any x ... and we're gonna obtain different y values for x ." This description of "plugging" in an element to a function to produce an output reflects the idea of an input/output relationship, so this and similar statements are also coded as *Input/Output*.

4.1.14 Univalence

Identifying or applying the univalence property of function – each element of the domain corresponds to one and only one element of the codomain – as a defining characteristic of function is coded as *Univalence*. Sofia identifies univalence as defining characteristic of function when she includes the phrase, "there is no more than one output for each input," in her definition of function. Likewise, Gabby defines function saying, "for every value in your domain, there's just one, one element in the

codomain.” An example of applying univalence as a defining characteristic occurs when Jamie determines that a particular relation is a function because “no X element is gonna be repeated.” Michael also exhibits this when he decides $f(x) = \sin(x)$ is a function because “every x you put, every value has like a specific sine value that has assigned to it, and you can’t have more than two sine values.” Each of these types of interactions depicts an understanding that univalence is a defining characteristic of function.

4.1.15 Algebraic Representation

The *Algebraic Representation* theme encompasses assertions that functions are equations and indications that all functions have an algebraic representation. In Sofia’s pre-interview, she states that “all functions can be equations.” Alan similarly suggests when answering Question 4 of his pre-interview that it is appropriate to define function as an equation, and later says that function and equation, “they mean the same thing...” Henry provides in his pre-interview a seemingly arbitrary function from the set $\{A, B, C, D\}$ to the set $\{1, 2, 3, 4\}$; however, as he explains how he formed the relation between the two sets, he says we can write $y = 2x$ to represent it. All of these examples are coded as *Algebraic Representation* because they communicate an underlying belief that all functions have some sort of algebraic representation.

4.1.16 Equations are Functions

Equations are Functions consists of statements that declare all equations are functions. Michael, for example, claims in his pre-interview, “an equation has to be a function, I’d say, for sure.” Likewise, Lee explains that, “all of equations are functions.” As both these statements directly assert that an equation is a function, they are coded as *Equations are Functions*.

4.1.17 Non-Univalence for Equations

The *Non-Univalence for Equations* theme is characterized by statements that distinguish equations from functions because equations do not have the same “limiting” property of univalence. For instance, Gabby shares, “In an equation, you can have, you have an [element of the domain] go to two different [codomain elements].” Another example of this code occurs when Henry states the equation of an ellipse is not a function “because, remember your definition of function. You say that each element of the domain should have a unique element in the codomain. With the x^2 and y^2 , we see that it will not...”

In addition, this theme includes responses that apply the vertical line test to imply the idea that equations do not need to meet the univalence requirement as functions must. This is apparent in Sofia's assertion that "equation graphs aren't limited to passing a [vertical line] test like the graphs of a function are."

4.1.18 *Contrived Difference from Equation*

At times during their interviews, some participants based their cognitive separation of function and equation on a contrived or personally-formulated difference. A contrived difference consists of distinctions an individual uses to distinguish function and equation that are not consistent with the definitions of function and equation. All such types of contrived distinctions participants formed are classified together under the *Contrived Difference from Equation* theme. For example, Sofia exhibits that she separates these terms based on the contrived difference that "an equation, it's like it's two different variables, so like a function would be just one." Jamie also claims function and equation are separate because "the goal of an equation is to find the unknown." These distinctions represent differences the participants have formulated in their own concept images, and they are not based on understandings consistent with definitions of function and equation. For this reason, these statements are coded as *Contrived Difference from Equation*.

4.1.19 *Defining Characteristics*

While *Contrived Difference from Equation* is characterized by the contrived differences participants used to identify function from equation, the *Defining Characteristics* theme represents responses that communicate a separation between function and equation based their defining characteristics. Michael exhibits this in the post-interview when he expresses that the terms are distinct because "a function is a relation between two sets, and an equation is statement about equivalence of two quantities which are fundamentally different." Similarly, Jamie identifies that "for a function, it's a relationship, and then for the equation it'll be equivalence." These statements depict that defining characteristics of function and equation are quintessentially different.

4.1.20 *Further Remarks on Function Concept Image Themes*

Although each of the themes discussed represent a distinctive component of the function concept images for participants in this study, they can be mostly grouped into overarching categories: *Structural Cue*, *Iconic Representation*, *Process Cue*, *Symbolic Cue*, and *Contrived Difference from Equation*. Figure

4-1 depicts the themes that fall under each category. Considering *Process Cue*, this category encompasses the theme *Input/Output*, and potentially other conceptions not identified in this study's data, that relate to the types of metaphorical expressions students use to describe function. In this study, the metaphorical expressions revolved around the idea that functions are some sort input and output process.

Structural Cue represents themes that communicate some underlying structure conception of function. For instance, *Numerical Domain* and *Non-Numerical Domain* indicate the type of objects a participant allows a function to be defined on. *Relations on Sets* and *Algebraic Representation* depict the types of structures that can be classified as function i.e. a relation defined on sets or a relation that can be written algebraically. Similarly, *One-to-One*, *Not One-to-One*, *Univalence*, and *Non-Univalence for Equations* communicate the ways in which objects from the first set can be related to objects from the second set. Finally, *Defining Characteristics* represents the ways in which function and equation differ structurally based on their defining characteristics.

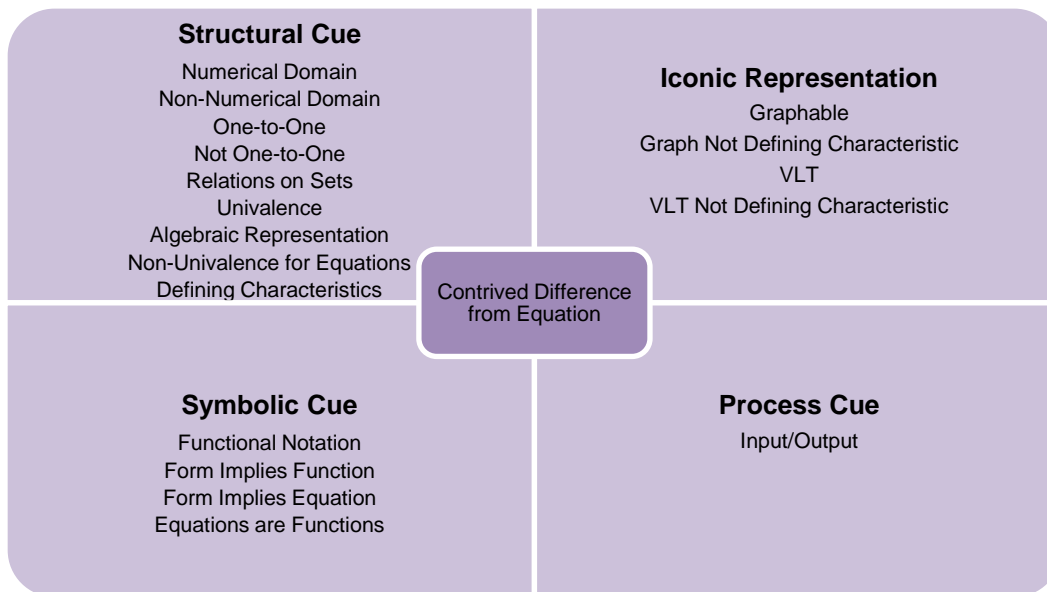


Figure 4-1 Categorization of Function Concept Image Themes

The *Iconic Representation* category includes themes where participants address whether a common visual representation is a defining characteristic of function. In particular, *Graphable* and *VLT* encompass statements that suggest all functions are graphable or that all functions pass the vertical line test – statements that suggest an iconic, visual representation characterizes functions. Conversely, *Graph*

Not Defining Characteristic and *VLT Not Defining Characteristic* communicate that these same iconic, visual representations are not characteristics of all functions.

Symbolic Cue consists of themes centered on a symbolic representation that, to the participant, indicates a mathematical object is a function or equation. Particularly, *Functional Notation*, *Form Implies Function*, and *Form Implies Equation* are all characterized by a particular symbol or combination of symbols that participants use to classify an object. *Equations are Functions* is comprised of statements that all equations are functions. This also suggests a symbolic cue in that anything written as an equation is identified as a function.

The final concept image theme and category in this study, *Contrived Difference from Equation*, is situated in the middle of the other four categories. *Contrived Difference from Equation* is a theme that consist of various types of statements that all depict a participant formulated a separation between function and equation in their concept images not based on their definitions. Then, this theme could potentially overlap with all four categories depending on the types of difference developed. For instance, Sofia's contrived difference is that equations have two different variables and functions have one. This reasoning relies on a symbolic conception of functions and equations, so it could be categorized as a *Symbolic Cue*. On the other hand, the difference Jamie contrives relates to the goal of an equation being to "find the unknown." This rationale appears to imply that function and equation are not the same types of processes because an equation is a process for finding an unknown. It is also possible that other types of contrived difference students might formulate would fall under the *Structural Cue* or *Iconic Representation* categories. For these reasons, the *Contrived Difference from Equation* category and theme is placed in the center of the other categories to depict that it potentially overlaps with all of them.

These categories could help characterize the types of components comprising an individual's function concept image, and perhaps in a study with this research focus these categories would be considered the themes and the function concept image themes in this study would be subthemes. However, the focus of this study is to identify modifications in participants' function concept images and the extent to which participants with prior mathematical backgrounds possess different function conceptions. I then refer to the concept image themes discussed in the previous sections as themes and the larger groupings of these themes as categories. While Creswell and Poth (2018) recommend no more

than six themes in a qualitative analysis study, Braun and Clarke (2006) maintain that researchers need to remain flexible in determining what constitutes as a theme in a qualitative study as “rigid rules really do not work” (p. 82). The goal of theoretical, thematic analysis is to provide a detailed analysis of a particular aspect of the data. The four categories do not present this type of detailed analysis. They also do not communicate a narrative that adequately addresses the research questions as contrasting themes like *Graphable* and *Graph Not Defining Characteristic* are both encompassed under the same category. Conversely, the 19 function concept image themes do provide a detailed window into participant’s overall function conceptions to tell the narrative of this study. For this reason, I employ the 19 function concept image themes to answer the research questions of this study. Table 4-2 presents a summary of the function concept image themes identified in each participant’s pre- and post-interview.

4.2 Findings for Group A Participants

Participants in Group A consist of students who completed only second- or third-semester calculus prior to the start of the FM course and this study. These participants include Sofia and Lee. In the following sections, I will explore the themes identified in each of their pre- and post-interviews.

4.2.1 Sofia

Prior to the start of the FM course, Sofia completed all three courses in the calculus sequence, and she concurrently enrolled in Introduction to Proofs and Introduction to Matrices and Linear Algebra. Her completion of only calculus courses classifies her as a member of Group A in this analysis. Of the 19 concept image themes discussed in this paper, 15 of these themes represent aspects of Sofia’s concept image in the pre- and post-interviews: *Functional Notation*, *Graphable*, *Graph Not Defining Characteristic*, *Not One-to-One*, *Algebraic Representation*, *Contrived Difference from Equation*, *Numerical Domain*, *Non-Numerical Domain*, *Non-Univalence for Equations*, *Univalence*, *VLT*, *VLT Not Defining Characteristic*, *Form Implies Equation*, *Form Implies Function*, and *Input/Output*.

Table 4-2 Themes Identified in Participants' Pre- and Post-Interviews

Categories	Themes	Group A				Group B				Group C					
		Sofia		Lee		Jamie		Gabby		Henry		Alan		Michael	
		Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Symbolic Cue	Functional Notation	■				■		■							
	Form Implies Function	■				■								■	
	Form Implies Equation	▲	▲				●		●		●				
	Equations are Functions			■										■	
Iconic Representation	Graphable	■		▲	▲	■				■					
	Graph Not Defining Characteristic		●				●		●		●		●	▲	▲
	VLT	■		■		■		■		▲	▲	■			
	VLT Not Defining Characteristic		●		●		●		●				●		
Process Cue	Input/Output		●		●	▲	▲	■		▲	▲	■		▲	▲
Structural Cue	Numerical Domain	■				■		■		■		■			
	Non-Numerical Domain		●	▲	▲		●		●		●		●	▲	▲
	One-to-One									▲	▲	▲	▲		
	Not One-to-One	▲	▲	▲	▲	▲	▲	▲	▲						●
	Relations on Sets					▲	▲		●	▲	▲	▲	▲	▲	▲
	Univalence		●		●	▲	▲	▲	▲	▲	▲		●	▲	▲
	Algebraic Representation	■				■		■		▲	▲	■			
	Non-Univalence for Equations	▲	▲					■			●				
	Defining Characteristics				●		●		●				●		●
Contrived	Contrived Difference from Equation		●	■		■									

■ - Themes Exclusively Present in Pre-Interview
 ● - Themes Exclusively Present in Post-Interview
 ▲ - Themes Present in Both Interviews

4.2.1.1 Themes Exclusively Present in Pre-Interview

The following themes appear exclusively in Sofia's pre-interview:

- *Algebraic Representation*,
- *Graphable*,
- *Numerical Domain*,
- *VLT*,
- *Functional Notation*, and
- *Form Implies Function*.

Specifically, when answering Question 6 (Figure 3-4), whether or not function and equation have the same meaning, Sofia states, "Functions can be equations because their graphs, the equation graphs aren't limited to passing a [vertical line] test like the graphs of a function are. So like, all functions can be equations but all equations can't be functions." This statement is coded as *Algebraic Representation* as she indicates that her concept image of function includes an idea that all functions can have algebraic representations.

Another theme that emerges only in Sofia's pre-interview is *Numerical Domain*. After providing examples of function in Question 3 (Figure 3-2), she considered if it would be possible to have a function whose domain did not consist of real numbers. Sofia answers, "No. Well not off the top of my head. But I don't think there's any." She then goes on to explain that she's thinking of the graphs that she knows, and "...if you have it on a graph, that'd have to be like real numbers... There'd have to be real numbers to graph it. And so it can't [have a domain with no real numbers]." This statement also suggests that her concept image includes the theme *Graphable*. Along with her thinking that all functions can be represented as graphs, a related theme that appears only in her pre-interview is *VLT*. While deciding which statements provided in Question 4 represent definitions of function, Sofia states, "That's one of the things we learn in, when we were taught functions, right? Is that it has to like pass that [vertical] line test."

Functional Notation and *Form Implies Function* are also only present in Sofia's pre-interview. When considering the defining characteristics of function, Sofia explains that these defining characteristics include " $f(x)$, or like something of a variable, the equal sign obviously, and then it's just got that variable in the parentheses in some form on the right side of the equation." This description of the defining characteristics she associates with function suggests a presence of the *Functional Notation* theme in her concept image. At another point in her pre-interview, Sofia considers whether or not $2x + y = 8$ should be considered a function. Within her reasoning, she rewrites this equation as $y = 2x - 8$ and says this is a function because "it passes this [vertical line] test." She is then asked if $2x + y = 8$ would be a function. Although Sofia is confident that written as $y = 2x - 8$ the equation represents a function, she expresses uncertainty over $2x + y = 8$ because "the x and y are on the same side." Sofia's acceptance of $y = 2x - 8$ as a function and her hesitancy over the format of $2x + y = 8$ indicates that the *Form Implies Function* theme is a component of her pre-interview concept image.

4.2.1.2 Themes Exclusively Present in Post-Interview

Six themes emerge exclusively in Sofia's post-interview:

- *Contrived Difference from Equation,*
- *Non-Numerical Domain,*
- *Graph Not Defining Characteristic,*
- *VLT Not Defining Characteristic,*
- *Input/Output, and*
- *Univalence.*

Contrived Differences from Equation represents aspects of participants' concept images where they formulated a perceived difference to separate the mathematical terms function and equation. In the case of Sofia's post-interview, she "used to think that functions are equations, but equations weren't functions. But then we said that was wrong in class, but [she] never got why other than the two variable thing." Sofia's

distinction based on a “two variable thing” refers back to another conversation in the post-interview where she recalls a specific class discussion concerning whether or not $f(x) = x$ is an equation. From this class discussion, Sofia remembers her classmates deciding that $f(x) = x$ is not an equation; however, Sofia understands the reason it is not an equation as because “it didn’t have the relationship with two variables.” Then Sofia’s perceived difference between function and equation is that for “an equation it’s like two different variables, so like a function would just be one.” It is also worth noting that this statement is not coded as *Algebraic Representation* because Sofia is not necessarily claiming that all functions can be represented algebraically; rather, her rationale could be centered on separating functions that can be represented algebraically from equations.

An additional theme only in Sofia’s post-interview function concept image is *Input/Output*. This theme includes descriptions of function as relation between inputs and outputs. When Sofia defines function during the interview, this is the type of language she uses. Particularly, she describes function as “a thing that represents the relationship between inputs and outputs.” Sofia also uses the input/output language when she identifies a defining characteristic of function as only having “one output for each input,” and as she explains why a given relation is or is not a function. The usage of input/output throughout the post-interview suggests that her post-interview concept image includes an idea that functions are relations between inputs and outputs.

The *Non-Numerical Domain*, *Graph Not Defining Characteristic*, and *VLT Not Defining Characteristic* themes are also exclusive to Sofia’s post-interview. In her pre-interview Sofia did not believe functions could be defined on non-numerical domains; however, when asked about this again in her post-interview, Sofia provides an example of a function whose domain consists of people and maps them to their respective eye colors (Figure 4-2). This interaction is coded as *Non-Numerical Domain*, and it

demonstrates that Sofia's concept image of function in the post-interview now includes functions defined on non-numerical sets. Moreover, the *Graph Not Defining Characteristic* and *VLT Not Defining Characteristic* themes also emerge in her post-interview. Sofia explains that $f(x) = x$ is a function because "...it only has one output for each input and it passes [the] vertical line test." When I ask if a function must have both of these components, Sofia explains, "No because we said that not all functions are graphable." This interaction indicates she believes there exist functions that are not merely graphs or graphable as well as the fact that her concept image now allows for functions that do not pass the vertical line test.

A handwritten table with two columns: 'People' and 'Eye Color'. The table lists two people, Katie and Janessa, both with the eye color 'brown'. The table is drawn with a vertical line separating the columns and horizontal lines separating the rows.

People	Eye Color
Katie	brown
Janessa	brown

Figure 4-2 Sofia's Post-interview Example of a Function with a Non-Numerical Domain

The final theme exclusive to Sofia's post-interview is *Univalence*. In the post-interview, Sofia brought up the idea of univalence at the beginning of the interview when discussing the definition of function and its defining characteristics. She describes a function as having "no more than one output for each input," and when explaining why a relationship is a function she conveys, "it only has one output for each input, and it passes the vertical [line] test." This spontaneous description of the univalence property reveals that *Univalence* is a component of her post-interview concept image.

4.2.1.3 Themes Present in Both Interviews

Finally, four themes persisted between Sofia's pre- and post-interview:

- *Not One-to-One*,
- *Non-Univalence for Equations*, and
- *Form Implies Equation*.

During the pre-interview, Sofia decides statement (d) provided in Question 4 (Figure 3-3) does not describe a function because “it says a function is a relation for which every element of the range, y , corresponds to exactly one element of domain, x . And so like, just the x^2 function, you have a negative x and a positive x , and they both can correlate to the same y , so that is wrong.” This statement is coded as *Not One-to-One* because Sofia reveals an understanding that relationships do not have to be one-to-one in order to be a function. Sofia does not have this same revelation about statement (d) in her post-interview, and she even accepts statement (d) as a definition of function; however, this may be attributed to the fact that she does not spend as much time in her post-interview reading and interpreting the statement. Sofia does provide $f(x) = x^2$ as one of her examples of function. This indicates that she still includes non-one-to-one relationships in her concept image of function.

Non-Univalence for Equations is an additional theme identified in Sofia’s pre- and post-interview. While discussing whether or not function and equation mean the same thing in the pre-interview, she explains, “Functions can be equations because their graphs, the equation graphs aren’t limited to passing a [vertical line] test like the graphs of a function are.” She similarly says in the post-interview that, “...equations aren’t limited in the same way that functions are,” because relations need, “certain characteristics in order for it to be a function.” These statements suggest that Sofia’s pre- and post-interview concept images include an understanding that not all equations are functions because the definition of equation does not include the univalence characteristic.

Another theme that persisted between Sofia’s pre- and post-interview is *Form Implies Equation*. Specifically in the pre-interview she conveys, “a function is more like

$f(x)$ and not like a 'y =' type of thing 'cause that's what we're always told in class." She says in the post-interview, "I have $y = 7x + 2$... That counts as an equation because it's got the y and the x. It's like a comparison between the two, but if you did [$f(x) = 7x + 2$] that's a function." Both these conversations indicate that Sofia's concept image includes the idea that something of the form "y=" should be considered an equation and not necessarily a function. However, her understanding of why an object of this form may not be a function appears to be more developed in her post-interview. Her rationale in the pre-interview is an echo of an external authoritative source whereas in the post-interview her reasoning rests in the idea that it shows a comparison between two variables.

4.2.2 Lee

Lee is a second year student whose mathematical background places him in Group A for this analysis. In particular, he completed the first two semesters in the three-semester calculus sequence, and he is enrolled in the third semester at the time of the FM course. Ten of the concept image themes discussed in this analysis are identified within Lee's pre- and post-interviews: *Non-Numerical Domain*, *Defining Characteristics*, *Graphable*, *Contrived Difference from Equation*, *Input/Output*, *Not One-to-One*, *VLT Not Defining Characteristic*, *Univalence*, *VLT*, and *Equations are Functions*.

4.2.2.1 Themes Exclusively Present in Pre-Interview

Of the 10 themes present in Lee's pre- and post-interview concept images, three of these themes are exclusive to his pre-interview including

- *Contrived Difference from Equation*,
- *Equations are Functions*, and
- *VLT*.

Contrived Differences from Equation and *Equations are Functions* are identifiable as he explains his understanding of functions and equations. Lee claims, "So an equation is a function. I like to use, I like to use the term that it is used for more than one function. Or it

is used in more than one scenario. So I would say [an equation is] a function usable in more than one scenario.” He then gives an example of his perceived difference between function and equation and clarifies that $y = ax^2 + bx + c$ is an equation because he “can use that for multiple parabolas to figure out [the] answer.” Conversely, $y = 2x^2 + 3x + 4$ is a function because “when we actually define what the coefficients are, it kinda turns it into a function.” This example represents Lee’s perceived difference between function and equation as well as the fact that his concept image of function includes all equations. Also, Lee accepts the statement that defines function as a graph that passes the vertical line test as a definition for function in his answer of Question 4 (Figure 3-3). He then uses the vertical line test to determine whether or not other statements provided are valid definitions of function; hence, the corresponding statements are coded as *VLT*.

4.2.2.2 Themes Exclusively Present in Post-Interview

Furthermore, four of the identified concept image themes appear only in Lee’s post-interview:

- *Defining Characteristic*,
- *Input/Output*,
- *VLT Not Defining Characteristic*, and
- *Univalence*.

The *Input/Output* and *Univalence* themes are identifiable in Lee’s definition of function and his defining characteristics. In particular, he defines function as “the relation of two ideas. And what those two ideas have to do, and how they relate to each other is that one needs to be in input and the other needs to be an output.” Lee also explains that the defining characteristic of function is that “for each input, there is one output, and it doesn’t have to be unique.” These characterizations of function reveal *Input/Output* and *Univalence* are aspects of Lee’s post-interview concept image.

When examining the statements provided in Question 4 (Figure 3-3) in the post-interview, Lee rejects the statement that defines function as graph that passes the vertical line test citing that

“...the vertical line test works if you have the independent as your x and your dependent as your y . So like if you’re talking about the coordinate axis for Cartesian stuff, it would work... but if you flip it to where now the x is the dependent, the x -axis is the dependent [and] the y -axis is the independent, that causes the vertical line test to no longer be valid.”

This implies that the *VLT Not Defining Characteristic* theme represents an aspect of his concept image. The final theme exclusive to Lee’s post-interview is *Defining Characteristics*. Specifically, he establishes function and equation are different types of mathematical objects because “we define them in two different ways. We define function as just, just relating two ideas... And we define equation as just showing equivalence of two quantities.”

4.2.2.3 Themes Present in Both Interviews

The remaining concept image themes linked to Lee’s interviews are those that are present in both his pre- and post-interviews:

- *Non-Numerical Domain,*
- *Graphable, and*
- *Not One-to-One.*

Considering the *Non-Numerical Domain* theme, he provides in both interviews an example defined on non-numerical sets. Particularly, the illustration in both interviews involves parents’ eye color and children’s eye color (Figure 4-3). Although Lee’s function concept images in the pre-and post-interview includes functions with non-numerical domains, the *Graphable* theme is also an aspect of both concept images. His pre-interview description of function and equation reveals an understanding that an equation is “a function usable in more than one scenario” i.e. $y = ax^2 + bx + c$ is an equation and $y = 2x^2 + 3x + 4$ is a function. He also explains that “the function serves for a specific

graph, it serves for a specific line.” In the post-interview, he takes his example of a function between parents’ and children’s eye colors and explains “the easiest way to kind of just show whether or not it’s a function is we usually try to graph. If we can graph it, it’s coherent, then we, then we accept it as a function most of the time.” He then situates this example on an axis system to graph it (Figure 4-3). Both of these interactions suggest an understanding that all functions are graphable.

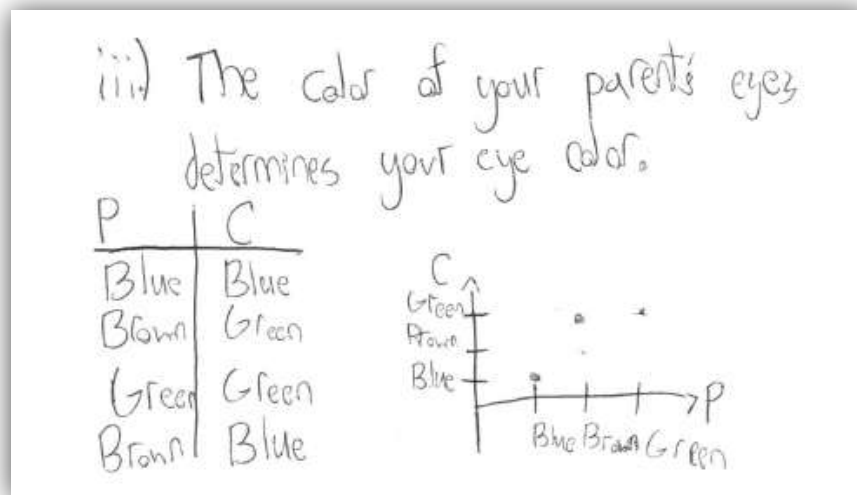


Figure 4-3 Lee’s Post-interview Example of a Function with a Non-Numerical Domain Graphed on a Coordinate System

The final theme that emerged in both of Lee’s interviews is *Not One-to-One*. In particular, he includes quadratic functions as examples in both interviews recognizing that multiple values from the domain can correspond to the same element of the codomain. Lee applies the vertical line test in the pre-interview in order to justify his claim, but he applies the univalence characteristic during the post-interview. These examples are each coded as *Not One-to-One*.

4.3 Findings for Group B Participants

Jamie and Gabby comprise the Group B participants in this study. Group B is characterized by the completion of mathematics courses beyond the three-semester calculus sequence but not *Abstract Algebra I* or *Real Analysis I* before the start of the FM course. The following sections will explore the themes from Jamie's and Gabby's interviews.

4.3.1 Jamie

Jamie is in her third year at the university, and she indicated that her native language is Spanish. Her mathematical background and the courses she took alongside the FM course places her in Group B for this analysis. Specifically, she completed the calculus sequence, Introduction to Proofs, Elementary Number Theory, Introduction to Matrices and Linear Algebra, and Discrete Mathematics all before she started the FM course. While taking FM, she concurrently enrolled in *Abstract Algebra I* and *Real Analysis I*. Of the 19 concept image themes discussed in this paper, 16 were identified within Jamie's pre- and post-interviews: *Functional Notation*, *Non-Numerical Domain*, *Defining Characteristics*, *Graphable*, *Contrived Difference from Equation*, *Relations on Sets*, *Input/Output*, *Not One-to-One*, *Algebraic Representation*, *Numerical Domain*, *Graph Not Defining Characteristic*, *VLT Not Defining Characteristic*, *Univalence*, *VLT*, *Form Implies Equation*, and *Form Implies Function*.

4.3.1.1 Themes Exclusively Present in Pre-Interview

Themes exclusive to Jamie's pre-interview include:

- *Functional Notation*,
- *Graphable*,
- *Algebraic Representation*,
- *Numerical Domain*,
- *VLT*,
- *Form Implies Function*, and
- *Contrived Difference from Equation*.

While describing the defining characteristics of function, Jamie says that a function “will look like $f(x) = y$ [sic], so it will be of that form. Um, maybe the equation might be of the form $mx + b$... I think there’s one more. I don’t know. It can be represented, I guess, in a table, or it can be represented in a graph.” Within her description of the defining characteristics of function, Jamie reveals three aspects of her concept image. The portion of this statement where she suggests that functions must be defined using ‘ $f(x)$ ’ notation is coded as *Functional Notation*. She also states that functions are capable of being graphed which is coded as *Graphable*. Finally, Jamie says that the equation might have a particular form. This suggests a belief that there exists some sort of algebraic representation for every function. Similarly when confronted with the statement in Question 4 (Figure 3-3) defining a function as “an equation that gives a particular relationships between two quantities,” Jamie accepts this as a valid mathematical definition. These statements are then coded as *Algebraic Representation*.

When asked if it possible to define a function on a domain that is not the real numbers or some subset of it, Jamie says it is not possible because, “we always graph the, it’s graph or something, in the real numbers.” This indicates *Numerical Domain* is a theme in her pre-interview concept image, and this statement is coded as *Graphable* as well. Also, the VLT theme is apparent when she answers Question 4 and accepts a statement (c) as a valid mathematical definition of function. This particular statement defines a function as a graph that passes the vertical line test, and acceptance of this as a definition of function indicates a belief that all functions pass the vertical line test. Furthermore, *Form Implies Function* is identified in Jamie’s pre-interview concept image of function. When she explains similarities and differences between function and equation in Question 5, Jamie concludes that an equation can be a function because “an equation

can be solved for one variable.” Then a representation, such as “ $y =$ ”, suggests to Jamie that a relation is a function.

The final theme that appeared only in Jamie’s pre-interview is *Contrived Difference from Equation*. During the interview, Jamie was asked to consider whether or not the mathematical terms function and equation have the same meaning. She answers that they do not have the same meaning, and she explains,

“...so a function is, like I said, a function includes, is composed by a domain and a range. So for the function of the form $f(x)$ equals some y , we need to have this x to find our y . So we need to have that input, or that x value that is gonna be plugged in, in the formula, let’s say. And then for an equation, and equation equals something, equals something, so both sides have to be equal... Equation, I guess the goal of an equation is to find the unknown value of a variable. So it’s not the same.”

Jamie follows this by identifying the difference between function and equation is that a function has “an indefinite amount of y , of x ’s for our infinite amount of y ’s. And in an equation, there’s a specific, a unique value of x .” These comments from Jamie suggest that she has contrived a separation in her concept image between function and equation.

4.3.1.2 Themes Exclusively Present in Post-Interview

In addition to the seven themes that emerged only in Jamie’s pre-interview, five themes are exclusive to her post-interview:

- *Non-Numerical Domain*,
- *Defining Characteristics*,
- *Graph Not Defining Characteristic*,
- *VLT Not Defining Characteristic*, and
- *Form Implies Equation*.

While *Numerical Domain* is a theme within her pre-interview concept image, in the post-interview Jamie is able to provide an example of a function between mothers with one child and children. This suggests that her concept image in the post-interview includes the *Non-Numerical Domain* theme. The *Defining Characteristic* theme is also identifiable

in Jamie's post-interview as she is able to articulate a difference in the defining characteristics of function and equation:

"A function is the mapping from, it just says for all x in X , there exists a y in Y , right? From our domain to our codomain. So that's why it's different. [Equation] is asserting equivalence when the other one just says that because there exists a y in my, an x in my domain, there will be a y in my codomain."

This difference between the defining characteristics Jamie identified also leads to the appearance of *Form Implies Equation* in her post-interview. Particularly when answering Question 6, she says that $y = 2x - 8$ is not a function, and the use of function in this question is "still not true until we set up $f(x) = 2x - 8$ " because it shows "a relationships between x and $f(x)$." This statement is not also coded as *Functional Notation* because Jamie is not using the notation to decide that the relation is a function. Instead, she is using the notation to represent that there is a relation that is a function.

Two other themes that appear only in Jamie's post-interview are *Graph Not Defining Characteristic* and *VLT Not Defining Characteristic*. During the pre-interview Jamie accepts the statement defining function as a graph that passes the vertical line test as a valid mathematical definition of function; however, she says in the post-interview that it "is not [a definition] because not all functions can be graphed." This indicates that Jamie's post-interview concept image allows for functions that do not pass the vertical line test as well as functions that are not graphable. Jamie's inclusion of functions that cannot be graphed in her concept image is further supported when she says, "... some functions cannot be graphed. That doesn't mean it's not a, because it's graphed, I mean because it cannot be graphed doesn't mean it's not a function."

4.3.1.3 Themes Present in Both Interviews

The remaining themes are those that emerged in both Jamie's pre- and post-interviews:

- *Relations on Sets*,
- *Input/Output*,
- *Not One-to-One*, and
- *Univalence*.

Specifically, *Relations on Sets* is apparent in both interviews when she describes function as a “mapping from... our domain to our codomain” or as “composed by a domain and a range.” Jamie also describes functions as a relation between inputs and outputs during both interviews. She says in the pre-interview that for a function, “we need to have that input, or that x value that is gonna be plugged in.” Similarly in the post-interview she explains that $f(x) = x + 1$ is a function because “we can plug in any x , and any, in the real numbers, and we’re gonna obtain different y values for every x . So for every input we get a different output.” The language used in these statements suggests the *Input/Output* theme is a component of Jamie’s pre- and post-interview concept images.

Univalence and *Not One-to-One* are also themes that persist between Jamie’s pre- and post-interviews. For instance, she explains in the pre-interview that “we cannot have two, like the same x mapping to two different y ’s because in that case it is not a function.” She makes a similar statement in the post-interview and conveys that a defining characteristic of function is that “an x cannot have two different y ’s.” These comments imply that *Univalence* is an aspect of Jamie’s concept image at the time of both the pre- and post-interview. *Not One-to-One* is the final theme that emerged in both Jamie’s pre- and post-interviews. In the pre-interview, Jamie draws the diagram in Figure 4-4 and explains it is a function “even though, um, we have the same x , uh y value for these two x ’s.” She gives another example of a function in the post-interview of a function two elements of the domain are mapped to the same element of the codomain. Jamie’s indication that these examples are functions implies that *Not One-to-One* is a component of her function concept image in both interviews.

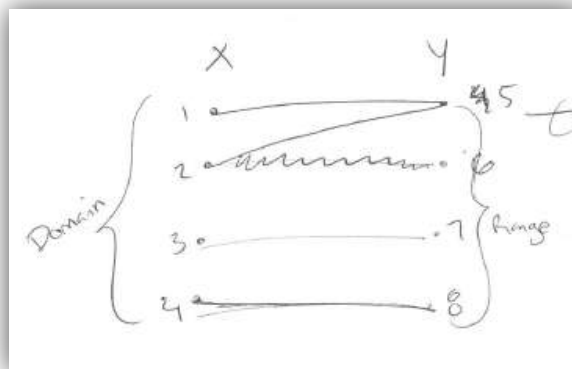


Figure 4-4 Jamie's Pre-interview Example of a Function that is Not One-to-one

4.3.2 Gabby

Gabby is another member of Group B for this analysis, and her native language is Spanish. When she started the FM course, Gabby already completed the calculus sequence, Introduction to Proofs, Statistical Inference, Introduction to Matrices and Linear Algebra, Trigonometry, and Discrete Mathematics. She was also enrolled in *Abstract Algebra I*, *Real Analysis I*, and Foundations of Geometry along with the FM course. Of the 19 concept image themes discussed in this paper, 14 are identified within Gabby's pre- and post-interviews: *Functional Notation*, *Non-numerical Domain*, *Defining Characteristic*, *One-to-One*, *Input/Output*, *Not One-to-One*, *Algebraic Representation*, *Numerical Domain*, *Relations on Sets*, *Graph Not Defining Characteristic*, *VLT Not Defining Characteristic*, *Non-Univalence for Equations*, *Univalence*, *VLT*, and *Form Implies Equation*.

4.3.2.1 Themes Exclusively Present in Pre-Interview

The themes exclusive to Gabby's pre-interview include

- *Functional Notation*,
- *Input/Output*,
- *Algebraic Representation*,
- *Non-Univalence for Equations*,

- *Numerical Domain*, and
- *VLT*.

In particular, she writes $p(x) = y^2 + b$ [sic] and explains that a defining characteristic of function is having, “An unknown variable in there.” This suggests *Algebraic Representation* is a theme in her pre-interview concept image. Gabby does, however, say that function and equation do not have the same meaning because “for a function, um, it has to pass that vertical line test, and on an equation, it doesn’t.” As Gabby distinguishes equation from function using the idea of univalence, the theme *Non-Univalence for Equations* is present in her pre-interview concept image. She also explains that another defining characteristic is that she “can put any number [for x] and [she’ll] still get an answer on this [$p(x)$] side, which implies that she views function as a relation in which the input produces an output. Hence, this statement is coded as *Input/Output*. Also while defining function, Gabby states, “A function is in the form of an expression... Like it would be like in the form of ‘ $p(x) =$ ‘ this.” This indicates *Functional Notation* is a component of Gabby’s pre-interview concept image as well.

After only providing examples of functions defined on numerical domains in the pre-interview, Gabby was asked if it would be possible to define a function on a non-numerical domain. She first suggests this could be done by introducing complex numbers. The question was rephrased to Gabby, and she was asked if it would be possible to define a function on a domain that does not include real or complex numbers. She then explains that this could be achieved by “putting a fraction on the function,” and clarifies that the domain of such a function would not be a subset of the real numbers. This interaction is coded as *Numerical Domain* because it reveals that Gabby’s pre-interview concept image may only include functions defined on numerical sets. The final theme present in her pre-interview is *VLT*. When examining the possible definitions of

function in Question 4 (Figure 3-3), Gabby accepts the statement defining function as a graph that passes the vertical line test as a valid mathematical definition. She then uses the vertical line test to decide which of the other statements provided in Question 4 are definitions of function. Then, *VLT* is a theme within her pre-interview concept image.

4.3.2.2 Themes Exclusively Present in Post-Interview

Additional to the themes exclusive to the pre-interview, five themes appeared only in Gabby's post-interview:

- *Relations on Sets*,
- *Non-Numerical Domain*,
- *Defining Characteristic*,
- *Graph Not Defining Characteristic*,
- *VLT Not Defining Characteristic*, and
- *Form Implies Equation*.

Relations on Sets emerges in the first question of her post-interview when she defines function as a “correspondence, so you have an element of the domain that goes to just one element in the codomain.” This explanation reflects a conception of function as a relation between two sets or the domain and codomain.

Although Gabby did not have any examples in her pre-interview concept image of functions defined on non-numerical domains, in the post-interview she explains that a function can be defined on “names, birthdays, eye colors, [and]... characteristics of people.” This implies that *Non-Numerical Domain* is a component of her post-interview concept image. Similarly, when considering the statement that defines function as a graph that passes the vertical line test in Question 4 (Figure 3-3), Gabby says it's not a valid definition because “you can have, uh, something like [see Figure 4-5], and you can't graph it.” This explanation reveals that both the *VLT Not Defining Characteristic* and *Graph Not Defining Characteristic* themes are present in her post-interview concept image.

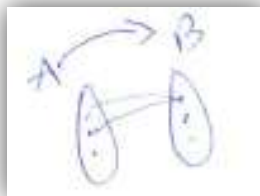


Figure 4-5 Gabby's Post-interview Example of a Function that cannot be Graphed

Gabby also identifies inherent differences in the defining characteristics of function and equation. Specifically, she says, “if you put like ‘ $f(x) =$ ’ there’s an equal sign, but that’s not an equation... ‘cause for a function it’s a relationship. And then for the equation it’ll be equivalence.” This commentary suggests that Gabby recognizes a function is intrinsically a relationship whereas an equation is an assertion of equivalence. Hence, it is coded as *Defining Characteristic*. Also, Gabby defines equation in Question 5 (Figure 3-4) as “setting up two expressions and you’re saying there’s equivalence between them, but it doesn’t have to be true.” She then says that anything with one equal sign would be an equation as long as “you don’t have like a ‘ $f(x) =$ ’ then ‘cause that’s a function.” This suggests that Gabby understands an object of the form ‘ $y =$ ’ is an equation based on its form, so *Form Implies Equation* is a theme in her post-interview concept image. However, this statement is not coded as *Functional Notation* because she goes on to explain that this notation shows “a relationship between your x and then your $f(x)$...” Then, it is not the form that implies such an object is a function but the relationship that is communicated by the notation.

4.3.2.3 Themes Present in Both Interviews

The final themes identified in this analysis are two that emerged in both Gabby’s pre- and post-interview:

- *Univalence*, and
- *Not One-to-One*.

Considering the *Univalence* theme, Gabby states in the pre-interview that “for it to be a function, you have to have one, uh, a number or a variable from A map to exactly one element of B.” She similarly explains in the post-interview that “you have an element from the domain that goes to just one element in the codomain.” It should be noted that Gabby’s discussion of univalence in the pre-interview does not occur until she reads statement (a) of Question 4 (Figure 3-3). However, Gabby applies the idea of univalence as a defining characteristic to confirm that statement (a) is a definition. She does not appear to learn any new ideas from this statement (a). Even so, this contrasts with her post-interview where the idea of univalence organically arises in her definition of function at the start of the interview. Furthermore, the first statement that reveals *Not One-to-One* is an aspect of Gabby’s pre-interview concept image is also in her discussion of Question 4. Examining statement (d), she concludes that it is not a valid definition of function because “you can have a B (see Figure 4-6) going to two different variables in A because it will still pass the vertical line test ‘cause you have one A for each, but you have uh one B for each.” Gabby also draws the relation in Figure 4-7 during the post-interviews expressing that it is a function “‘cause each element on your um left side just has one element attached to it.” Both these excerpts reveal *Not One-to-One* is a theme in her pre- and post-interview concept images.



Figure 4-6 Gabby’s Pre-interview Example of a Function that is Not One-to-one

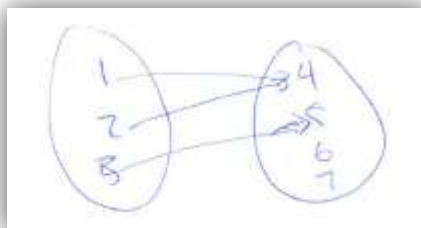


Figure 4-7 Gabby's Post-interview Example of a Function that is Not One-to-one

4.4 Findings for Group C Participants

Like Group B participants, Group C participants completed courses beyond the three-semester calculus sequence prior to the start of the FM course and this study; however, Group C participants also completed *Abstract Algebra I* or *Real Analysis I* prior to this study. Participants in this group include Henry, Alan, and Michael, and I will explore each of their interviews in the following sections.

4.4.1 Henry

Henry, a student whose native language is French, started the FM course after completing the calculus sequence, Introduction to Proofs, Elementary Number Theory, Introduction to Matrices and Linear Algebra, *Real Analysis I*, Discrete Mathematics, and Foundations of Geometry. While taking FM, Henry also took *Abstract Algebra I*, Statistical Inference, and Multivariable Calculus. This mathematical background places Henry in Group C for this analysis, and 12 of the 19 concept image themes discussed in this paper are apparent within his pre- and post-interviews: *One-to-One*, *Graphable*, *Relations on Sets*, *Input/Output*, *Algebraic Representation*, *Numerical Domain*, *Graph Not Defining Characteristic*, *Non-Numerical Domain*, *Non-Univalence for Equations*, *Univalence*, *VLT*, and *Form Implies Equation*.

4.4.1.1 Themes Exclusively Present in Pre-Interview

Two themes appear exclusively in Henry's pre-interview:

- *Numerical Domain*, and
- *Graphable*.

When he first defines function, Henry gives an example of a function from the set $\{A, B, C, D\}$ to the set $\{1, 2, 3, 4\}$. It appears at first that he is defining a function on a non-numerical domain; however, as he explains his mapping he defines the mapping by $y = 2x$ and concludes that $1 = 2A$, $2 = 2B$, and $4 = 2D$. This suggests that Henry considers $\{A, B, C, D\}$ to represent a subset of the real numbers. In fact, when asked whether or not a function could be defined on a non-numerical domain. Henry expresses, "I don't think that [is possible] because a function will be always defined in a subset, in a subset of all real numbers." This statement and the examples he provides suggest that *Numerical Domain* is a theme in Henry's pre-interview concept image.

Graphable is also a theme that emerged in Henry's pre-interview. When discussing the defining characteristics of function in Question 2 (Figure 3-1), he states that all functions have "trajectories" and explains that "we have the positive part, when the function is going up, and the negative part when the function is coming down." Henry is describing here the increasing and decreasing components of the graph of a function. This portion of the interview is coded as *Graphable* and indicates that his function concept image likely includes only functions that can be graphed.

4.4.1.2 Themes Exclusively Present in Post-Interview

Henry's concept images themes also include four themes that emerged solely in his post-interview including

- *Graph Not Defining Characteristic*,
- *Non-Numerical Domain*,
- *Non-Univalence for Equations*, and
- *Form Implies Equation*.

Considering *Graph Not Defining Characteristic* and *Non-Numerical Domain*, Henry states in the post-interview that not all functions can be graphed and even gives an example of

a function whose domain is people and therefore cannot be graphed on a coordinate plane. Also, in the interview, Henry states that “a function is an equation, an equation is not, all equations aren’t function.” He goes on to give an example of this using an ellipse and explains, “It’s not a function, first because the vertical line [test] is not true...” This conversation suggests the theme *Non-Univalence for Equations* is an aspect of Henry’s post-interview concept image. Finally, *Form Implies Equation* is a theme that appeared only in Henry’s post-interview. As he discusses whether or not calling $2x - y = 8$ a function is a correct use of the term function, Henry considers the equation $y = 2x - 8$. I ask him if it is correct to consider $y = 2x - 8$ a function. Henry answers that it is “still not true until we set up $f(x) = 2x - 8$,” because then “ $f(x) = 2x - 8$ show[s] a relationship between x and $f(x)$.” This statement is not also coded as *Functional Notation* because Henry is not suggesting that it is the $f(x)$ symbol that denotes the relation is a function, rather it is the relationship that is communicated between x and $f(x)$.

4.4.1.3 Themes Present in Both Interviews

Several concept image themes identified in Henry’s pre- and post-interviews persisted between the interviews:

- *One-to-One,*
- *Relations on Sets,*
- *Input/Output,*
- *Algebraic Representation,*
- *Univalence,* and
- *VLT.*

In both interviews, Henry defines function in such a way that implies all functions must be one-to-one. While he provides $f(x) = x^2$ as an example of function in the pre-interview, he states, “if we have two element in x having the same image y , it’s not gonna be a function.” He also seemingly becomes confused when justifying why $f(x) = x^2$ is a function and says, “Oh, no I’m not gonna talk [about] that, something else,” before

providing any explanation. However, in the post-interview he decides that $f(x) = x^2$ is not a function. Specifically, he explains,

“for each x , $f(x)$ is unique. So let’s say this function is x^2 , right? So, first I say that is a function because first vertical [line] test is good. But now, if I come back to my original definition... $f(-2) = f(2) = 4$. Negative two is different from two, but give the same number, so the definition doesn’t match. So I think that is not a function.”

These conversations in both the pre- and post-interview indicate that *One-to-One* is a theme in Henry’s function concept image.

Relations on Sets and *Input/Output* are also themes that persisted between Henry’s pre- and post-interviews. In the pre-interview, Henry defines function as, “a relation between two, two set[s] or two group[s],” and he says similarly in the post interview that, “a function is a relation between two sets such that each element in the set A has a unique element in set B.” Henry’s description of function as relation between sets suggests *Relations on Sets* is an aspect of his pre-and post-interview concept images. Furthermore, he uses language describing function as relationship between inputs and outputs in both interviews. For example, Henry says in the pre-interview that a given relation is a function “because for each element, x , because x is here and y is here, I have an image in y . So each time we have an x element, it’s possible to find y element.” He expresses in the post-interview that “each function should have a domain. And domain just like, in the domain we have the input. And the second one is the codomain, should be the output.” These statements are coded as *Input/Output*.

Moreover, *Algebraic Representation* appears in both Henry’s pre- and post-interviews. When asked in the pre-interview if function and equation have the same meaning, Henry provides an affirmative answer. This theme also emerged in a conversation where he provides an example of a function from the set $\{A, B, C, D\}$ to the set $\{1, 2, 3, 4\}$. As he explains how he chose a specific function mapping between the

sets, he says that the relation can be defined by $y = 2x$. “Um any time that we have 1, for example, any time that we have 1, 1 here corresponds to 1, so we have $2A$. So, $1 = 2A$. The same thing, $2 = 2B$.” Although Henry provides an example of a function seemingly defined without using an algebraic representation, further explanation implies an underlying assumption of a corresponding algebraic representation. He also says multiple times throughout the post-interview that “all function[s] are equation[s].” Previously in this interview, Henry defined a function with a non-numerical domain, and he was asked if this function had a corresponding equation. Henry suggests “it is difficult to set up like, an equation” for this type of function, and clarifies that “[he] was talking earlier about like numerical, like the case of a, like a function, like a numerical function. Yeah if we consider any function is an equation, but any equation is not a function.” While he claims that he was strictly referring to numerical functions in his assertion that all functions are equation, he still concludes “any function is an equation.”

Univalence and *VLT* are the remaining themes that emerged in both Henry’s interviews. Although Henry requires relations in his function concept image to be one-to-one, he also excludes relations based on the univalence property. For instance, in the pre-interview, he excludes the relation $y = \pm\sqrt{x}$ as a function of x because “a function is a relation between two group[s], and each element in the first group, it’s the domain, should have one, should have one image in the codomain, but those one[s] have two image[s].” He also presents the example in Figure 4-8 and he explains that $1 \in A$, which in the example maps to $a \in B$, could not also map to $b \in B$ “because each element in A should have a unique element in B .” These explanations show that *Univalence* is an aspect of his pre- and post-interview concept images. The *VLT* theme is also apparent in both interviews when he accepts the statement in Question 4 (Figure 3-3) using the vertical line test to define function as a valid definition.

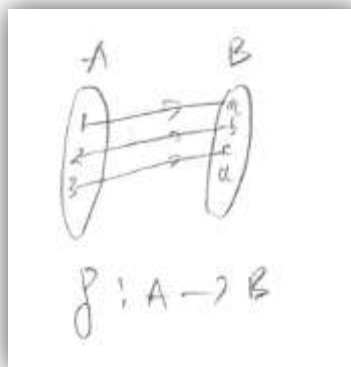


Figure 4-8 Henry's Post-interview Example to Explain Univalence

4.4.2 Alan

Alan is another student whose mathematical background places him within Group C for this analysis. He is in his fourth year at the university and his native language is Spanish. Prior to the FM course, he completed the calculus sequence, Introduction to Proofs, Elementary Number Theory, Introduction to Matrices and Linear Algebra, *Abstract Algebra I*, *Real Analysis I and II*, Discrete Mathematics, and Foundations of Geometry. Alan concurrently enrolled in Statistical Inference with the FM course. Eleven concept image themes discussed in this paper are identified in Alan's pre- and post-interviews: *Non-Numerical Domain*, *Defining Characteristic*, *One-to-One*, *Relations on Sets*, *Input/Output*, *Algebraic Representation*, *Numerical Domain*, *Graph Not Defining Characteristic*, *VLT Not Defining Characteristic*, *Univalence*, and *VLT*.

4.4.2.1 Themes Exclusively Present in Pre-Interview

Four concept image themes appeared only in Alan's pre-interview:

- *Input/Output*,
- *Algebraic Representation*,
- *Numerical Domain*, and
- *VLT*.

Particularly, when defining function in his pre-interview, Alan says that a function is “an expression for any input you get an output basically.” This language suggests that his concept image of function includes the *Input/Output* theme. *Algebraic Representation* is also apparent in Alan’s pre-interview. While identifying the statements in Question 4 (Figure 3-3) that represent valid definitions of function, Alan says that the statement defining a function as an equation that gives a particular relation between two quantities would need to be reworded. He is asked how he would reword this statement and explains, “The function is an equation, and that would be probably the part I would just keep. And then I would reword this, all this.” Deciding that it is appropriate to define a function as an equation suggests that functions in Alan’s pre-interview concept image have algebraic representations.

Furthermore, *Numerical Domain* is identified in Alan’s pre-interview when he provides examples of function. The examples he provides in Figure 4-9 are each defined on the assumed domain of all real numbers. When asked if a function could be defined on a non-numerical domain, Alan suggests $f(n) = \frac{1}{n}$ where n is an element of the set of natural numbers. The fact that this relation is still defined on a numerical set is brought to the attention of Alan, and he considers if other nonstandard functions, such as the Dirichlet or piecewise functions, are defined on non-numerical domain. Upon realizing that functions of these still have numerical domains, Alan says, “Ok I can’t think of one. Alright, I give up.” This inability to provide an example of a function with a non-numerical domain suggests that Alan’s concept image may include only those defined on numerical sets. Moreover, when Alan explains why the examples he gave are classified as functions, he reveals that, “The rule of thumb is if the vertical line test, uh, passes through the function more than once, then it wouldn’t be considered a function.” This statement is

coded as *VLT* because it indicates the vertical line test is a defining characteristic of function in Alan's concept image.

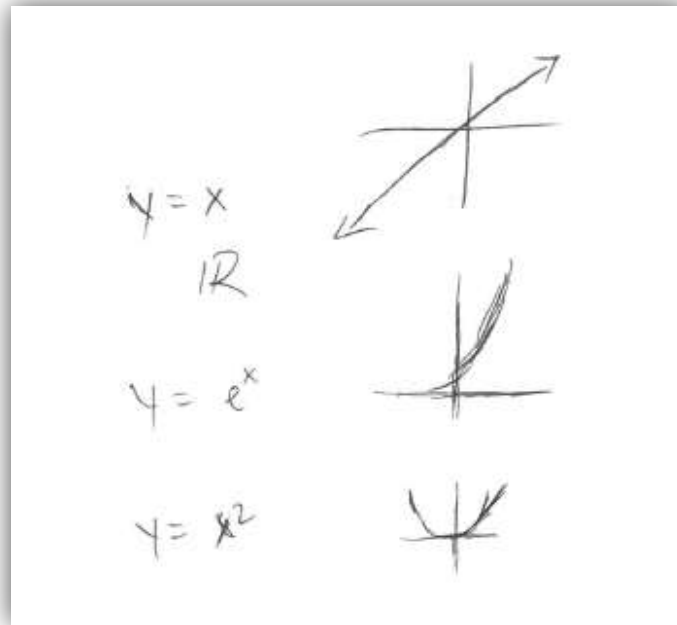


Figure 4-9 Alan's Pre-interview Examples of Function

4.4.2.2 Themes Exclusively Present in Post-Interview

There are also five themes exclusive to Alan's post interview including

- *Non-Numerical Domain,*
- *Graph Not Defining Characteristic,*
- *VLT Not Defining Characteristic,*
- *Univalence, and*
- *Defining Characteristic.*

Particularly, *Non-Numerical Domain* arises when Alan provides examples of function in the post-interview. He was unable to produce an example of a function defined on a non-numerical domain in the pre-interview; however, in the post-interview the first example he offers is a function from people to the vehicle identification number on cars (Figure 4-10), and he also provides an example of a function from soccer players on a team to their

jersey numbers. This example of a function from soccer players to jersey numbers is used later in the interview when Alan considers whether or not all functions are graphable. Alan reasons that, “not all functions are graphable... It just goes back to the, the soccer players and their jersey numbers. It just discerns some, some uh, some relationship from, from another. So how would you graph that? You can’t, right?” This statement is coded as *Graph Not Defining Characteristic*.

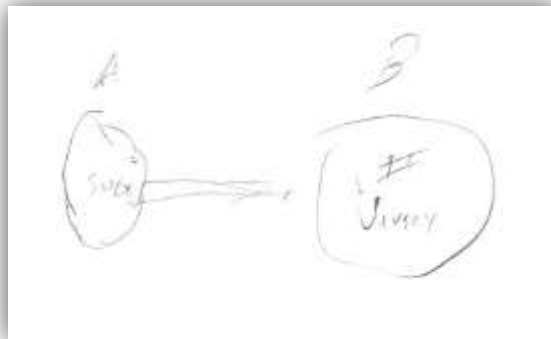


Figure 4-10 Alan's Post-interview Example of Function with a Non-Numerical Domain

When Alan discusses the vertical line test in the post-interview, he says, “there are certain, I know there are certain conditions where [the vertical line test] would fail, and if you use it, and you think it’s a function, how would I say it? You’d be wrong to assume that.” In this discussion, he reveals *VLT Not Defining Characteristic* is a theme in his post-interview concept image. *Univalence* is another theme identified when Alan defines function at the beginning of his post-interview. He explains a function is defined when “something from [one group] would be matched up to something from [another group]. Just one... One and only one, and that’s how I would say a function works. You, you attach it to one, one item from one group to another group.” In the post-interview, Alan also reports differences in the defining characteristics of function and equation:

“...a function serves a purpose like a, like a purpose, I guess in a sense that you can find, you can, you could find what the relationship between

the, why, why, why [sic] the relationship between the two sets work. Versus equation, you just assume everything is just equal.”

This suggests that he perceives function as a relationship between two sets whereas an equation asserts equivalence; hence, this is coded as *Defining Characteristic*.

4.4.2.3 Themes Present in Both Interviews

In addition to the nine themes exclusive to either Alan's pre- or post-interview, two themes persisted between the interviews:

- *Relations on Sets*, and
- *One-to-One*.

Alan says in the pre-interview that statement (b) of Question 4 (Figure 3-3) is not a valid definition of function because it needs to clarify “that the variables come from a set domain and a codomain.” As he indicates that functions are defined on sets, this statement is coded as *Relations on Sets* in the pre-interview. This theme is also identified in the post-interview when Alan explains that a specific relation is a function “because you can assign values from one set to the other...” *One-to-One* is present in Alan's pre- and post-interviews as well. In the pre-interview, he explains that a particular relation is not a function because “we have two, basically, inputs and you're getting the same result.” He similarly rejects a proposed relation in the post-interview as a function because two elements from the first set correspond to the same element of the second set. Both of these interactions are coded as *One-to-One*.

4.4.3 Michael

Michael, whose native language is Spanish, completed the calculus sequence, Introduction to Proofs, Elementary Number Theory, Introduction to Matrices and Linear Algebra, *Abstract Algebra I*, *Real Analysis I*, and Foundations of Geometry before the start of the FM course. While taking FM, Michael was also enrolled in Discrete Mathematics. This mathematical background places him within Group C for this analysis,

and 10 of the concept image themes discussed in this analysis are identified in Michael's pre- and post-interviews: *Non-numerical Domain*, *Defining Characteristics*, *Relations on Sets*, *Input/Output*, *Not One-to-One*, *Graph Not Defining Characteristic*, *VLT Not Defining Characteristic*, *Univalence*, *Form Implies Function*, and *Equations are Functions*.

4.4.3.1 Themes Exclusively Present in Pre-Interview

Of the concept image themes identified in Michael's interviews, two themes are exclusive to his pre-interview:

- *Equations are Functions*, and
- *Form Implies Function*.

Michael, in fact, defines equation as, "A function such that every element in the domain is applied the same relation." He also claims that an "equation has to be a function." These statements are coded as *Equations are Functions*. Moreover, the *Form Implies Function* theme is apparent in Michael's response to Question 6 (Figure 3-4). Particularly, Michael indicates that it is appropriate to use the term function to describe $2x - y = 8$ because "it's like, an equation which an equation is a function. So, like, just cause it's not like, you know, 'y =', you can solve for it." Although Michael thinks this is a function because it's an equation, his comment about the form and solving it for a particular variable suggests a belief that it would be evident that the equation is a function if it was written in a 'y =' form.

4.4.3.2 Themes Exclusively Present in Post-Interview

Three themes are also identified only in Michael's post-interview including

- *Defining Characteristics*,
- *Not One-to-One*, and
- *VLT Not Defining Characteristic*.

At the start of the interview, Michael defines function, and as a part of his explanation of the definition he draws the relation in Figure 4-11. He justifies this is a function because

“every element of A gets mapped to a unique element, meaning that it gets only mapped to one,” so this conversation is coded as *Not One-to-One*. Michael also rejects statement (c) in Question 4 (Figure 3-3) which defines function as a graph that passes the vertical line test. Particularly, he reveals that “the vertical line test is only like, like if it’s like $y = x$. You can have like ‘ x equals something.’ And that will be a function technically, but it wouldn’t pass the vertical line test.” Michael’s concept image then allows for functions that do not pass the vertical line test which is represented by the *VLT Not Defining Characteristic* theme. Finally, when considering whether or not the terms function and equation mean the same thing, Michael says they do not “because a function, um, is a relation between sets, and an equation is a statement about the equivalence of two quantities which are fundamentally different.” As Michael expresses that function and equations are inherently different mathematical objects, this is coded as *Defining Characteristics*.

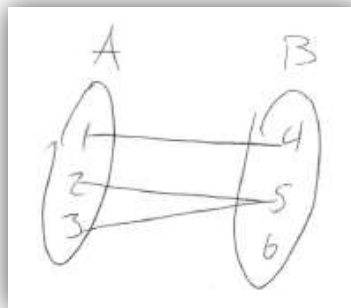


Figure 4-11 Michael’s Post-interview Example of a Function that is Not One-to-one

4.4.3.3 Themes Present in Both Interviews

The remaining themes are those that are present in both Michael’s pre- and post-interviews:

- *Non-numerical Domain,*
- *Relations on Sets,*
- *Input/Output,*

- *Graph Not Defining Characteristic*, and
- *Univalence*.

Considering the *Non-Numerical Domain* theme, Michael provides the relation in Figure 4-12 during the pre-interview and the relation in Figure 4-13 during the post-interview as examples of functions that are not defined on a numerical domain. Michael also notes in both interviews that the statement in Question 4 (Figure 3-3) which defines function as a graph that passes the vertical line test assumes the graph is a defining characteristic of a function. In the pre-interview, he conveys that a function is “not necessarily a graph.” Similarly, he rejects this statement in the post-interview in part because “it’s kind of implying that every function is a graph. Again, this one (Figure 4-13) doesn’t really have a graph.” Both of these statements indicate *Graph Not Defining Characteristic* is theme in his pre- and post-interview concept images.

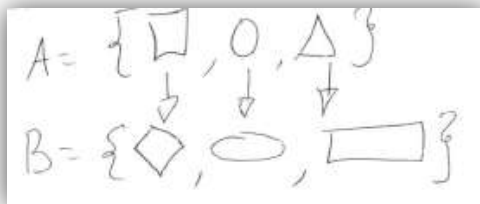


Figure 4-12 Michael's Pre-interview Example of a Function with a Non-numerical Domain

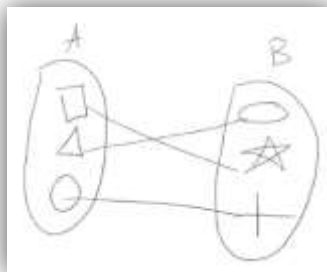


Figure 4-13 Michael's Post-interview Example of a Function with Non-Numerical Domain

Michael also describes function as a relation between sets in his pre- and post-interviews. He explains in the pre-interview that a particular relation is a function “because every x in the real numbers of the first set maps to one of the numbers from the other set.” In the post-interview, he describes functions as “relations from the set A, usually called the domain, and the set B, usually the codomain...” These statements represent codes from *Relations on Sets*. Furthermore, Michael reveals *Input/Output* is another component of his pre- and post-interview concept images. In the pre-interview, he expresses that a particular relation is a function because “you take one real number and you put it into the function, and it gives you one real number.” He reports in the post-interview that the defining characteristic of function “comes down to like the pairing. Like the input has a unique output.” This language suggests Michael perceives function as a relationship in which an input produces an output, and it indicates that *Univalence* is an aspect of his pre- and post-interview concept images.

4.5 Consistent, Inconsistent, and Neutral Theme Findings

Each of the 19 function concept image themes can be analyzed in relation to the extent to which they align with the concept definition of function and equation. I will use the term *consistent* (or *concept-definition-consistent*) to describe themes that align with these definitions; *inconsistent* (or *concept-definition-inconsistent*) to describe those that are incompatible with these definitions; and *neutral* (or *concept-definition-neutral*) to describe those that can neither be classified as consistent or inconsistent. Figure 4-14 depicts an overview of the *consistent*, *inconsistent*, and *neutral* themes.

Consistent themes are those that capture aspects of the definitions of function and equation. *VLT Not Defining Characteristic*, for example, is a consistent theme because the definition of function is not based on the vertical line test. Accordingly, eight concept image themes can be classified as consistent: *Non-Numerical Domain*, *Not One-*

to-One, Graph Not Defining Characteristic, VLT Not Defining Characteristic, Relations on Sets, Univalence, Non-Univalence for Equations, and Defining Characteristics. Similarly, inconsistent themes are those that do not coincide with the mathematical definitions. For instance, *Algebraic Representation* is an inconsistent theme because the definition of function does not imply or stipulate that all functions can be represented algebraically. There are a total of nine inconsistent themes including *Functional Notation, Form Implies Function, Numerical Domain, One-to-One, Graphable, VLT, Algebraic Representation, Equations are Functions, and Contrived Difference from Equation*.

Consistent Themes	Inconsistent Themes	Neutral Themes
<ul style="list-style-type: none"> • Non-Numerical Domain • Not One-to-One • Graph Not Defining Characteristic • VLT Not Defining Characteristic • Relations on Sets • Univalence • Non-Univalence for Equations • Defining Characteristics 	<ul style="list-style-type: none"> • Functional Notation • Form Implies Function • Numerical Domain • One-to-One • Graphable • VLT • Algebraic Representation • Equations are Functions • Contrived Difference from Equation 	<ul style="list-style-type: none"> • Form Implies Equation • Input/Output

Figure 4-14 Categorization of Consistent, Inconsistent, and Neutral Themes

The remaining two themes, *Form Implies Equation* and *Input/Output*, are neutral themes because they cannot be classified as either consistent or inconsistent. *Form Implies Equation* is characterized by indications that objects of the form “ $y =$ ” are equations and not functions because of their form. Although statements coded as *Form Implies Equation* may be related to an understanding consistent with the definition of equation (e.g. that the equal sign in this form asserts equivalence between two

quantities), they could also be the result of rote memorization and an inconsistent understanding that equations are written in the form “ $y =$ ”. *Input/Output* is likewise deemed neutral because this theme is characterized by references to function as a process in which an input produces an output. This theme does not reveal whether the participant is thinking about a relation between an originating set (domain) and a corresponding set (codomain). Moreover, it possibly suggests a “pointwise” conception of function instead of a relational conception. This lack of clarity is the reason *Input/Output* is considered a neutral theme.

With eight concept image themes classified as consistent, nine as inconsistent, and two as neutral, Table 4-3 presents an overview of the number of consistent, inconsistent, and neutral themes that emerged in each participant’s pre- and post-interview.

Table 4-3

Number of Consistent, Inconsistent, and Neutral Themes by Participant

	Group A				Group B				Group C					
	Sofia		Lee		Jamie		Gabby		Henry		Alan		Michael	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Number of Consistent Themes	2	6	2	5	3	7	3	7	2	5	1	6	4	7
Number of Inconsistent Themes	6	1	4	1	7	0	4	0	5	3	4	1	2	0
Number of Neutral Themes	1	2	0	1	1	2	1	1	1	2	1	0	1	1

Chapter 5

Discussion and Conclusions

In the previous chapter, I reported and described the concept image themes, identified in each participant's pre- and post-interview, and discussed the alignment of each theme to the definitions of function and equation. This chapter presents similarities and differences in individual participants' pre- and post-interview concept images. Specifically, I will examine themes that persisted between a participant's interviews, themes exclusive to the pre-interview that seem to contrast with themes exclusive to the post-interview, and further themes exclusive to the post-interview. This discussion will take place in tandem with a discussion highlighting similarities and differences of participants' concept images within the same participant group. I then compare the pre- and post-interview concept image themes across participant groups by examining the themes that are consistent and inconsistent with the definition of function. Finally, I will identify specific commonalities and variances across participant groups. As there are commonalities in the function concept image themes across participant groups, I do not address how these observations relate to the research literature in each of the participant group findings. Instead, I will compare the findings of other studies to these findings when I compare the pre- and post-interview concept images across participant groups.

5.1 Discussion of Participants Pre- and Post-interview Concept Image Themes within Participant Groups

The first component of this chapter examines the pre- and post-interview concept image themes of each participant to identify themes that remained throughout both interviews; themes in the post-interview that indicate a shift in function conceptions by contrasting with a pre-interview theme; and additional themes that emerged exclusively in the post-interview suggesting a change in function conceptions between the interviews.

While exploring the themes between an individual's interviews, this section will also draw comparisons to the other participants of similar mathematical backgrounds.

5.1.1 Group A Participants

Participants in Group A, Sofia and Lee, had not completed beyond third-semester calculus in their degree program prior to the start of the study. While chapter four identified the themes in each of their pre- and post-interviews, this section will explore the relationship between the pre- and post-interviews themes including themes that persist between interviews, seemingly contrasting themes between interviews, and the further themes that emerge only in the post-interviews. In both Sofia's and Lee's interviews, I observe the persistence of consistent themes; clear shifts from inconsistent themes in the pre-interviews to consistent themes in the post-interviews; and the emergence of other neutral and consistent themes in their post-interviews. Individually, Lee exhibits the persistence of an inconsistent theme and Sofia a neutral theme. An additional inconsistent theme emerges exclusively in Sofia's post-interview as well.

5.1.1.1 Persisting Themes between Interviews

Examining the themes that persisted between Sofia's and Lee's interviews, there is only one theme that they have in common, and it is a theme consistent with the concept definition of function: *Not One-to-One*. Both Sofia and Lee rely, in both their pre- and post-interviews, on an example of a function that has a graph that is a parabola to reason that a function does not have to be a one-to-one relation. Statement (d) of Question 4 (Figure 3-3) of the interviews presents participants with a statement defining function as a relationship for which every element of the range corresponds to exactly one element of the domain. In Lee's pre- and post-interviews and Sofia's pre-interview, they reason that the statement is incorrect because for "the x^2 function you have a negative x and then a positive x and they both can correlate to the same y ." Sofia actually accepts

this same statement in her post-interview as a valid definition of function; however, she seems to interpret that the statement presents the same idea as another valid definition presented in the problem. This is reflected in the fact that her only comment regarding statement (d) is to state it is a valid definition because it's "like [statement (a)] just worded differently." Earlier in the interview, Sofia does present $f(x) = x^2$ as an example of function. The persistence of this theme across the interviews suggest that this consistent conception developed in their prior mathematics courses and was not negatively affected over the course of this study.

While Sofia and Lee both have other concept image themes that persisted between their pre- and post-interviews, there is no further overlap between them. The additional themes that persisted in only Sofia's interviews include *Form Implies Equation* and *Non-Univalence for Equations* – a concept-definition-neutral and concept-definition-consistent theme respectively. Considering the *Form Implies Equation* theme, Sofia explains in the pre-interview that "a function is more like $f(x)$ and not like a ' $y =$ ' type of thing 'cause that's what we're always told in class." This reasoning is based on a recollection from an authoritative source whereas in the post-interview she asserts that $y = 7x + 2$ is an equation because "it's got the y and the x . It's like a comparison between the two..." Sofia goes on to explain that $y = 7x + 2$ is not a function because "the x and y , that's a comparison between two different variables, and therefore that's not what a function is." The rationale Sofia draws on in the post-interview is then based on a difference she's contrived between the definition of equation and function. Although Sofia arrives at the same conclusion in both interviews, it seems that over the course of her interactions with the EEFPSMT materials she started to try to distinguish function and equation for herself.

Also, *Non-Univalence for Equations* persists between Sofia's pre- and post-interviews. She explains in each interview that "equations aren't limited in the same way that functions are." This limitation in the pre-interview is described using the vertical line test i.e. "the equation graphs aren't limited to passing a test like the graphs of a function are." Whereas in the post-interview, Sofia articulates that equations are not limited as functions are to the "input-output ratio [each input corresponds to one output] thing." Both explanations reveal an understanding that equations are not subject to the univalence characteristic. However, Sofia is able to communicate this idea in her post-interview without depending on the vertical line test. She instead applies rationale more akin to the defining characteristics of function. This suggests Sofia's conception of the defining characteristics of function perhaps strengthened over the course of the study.

Two themes, in addition to *Not One-to-One*, persisted between Lee's interviews: *Non-Numerical Domain* and *Graphable*. The persistence of these two themes is particularly interesting because of the contrasting nature of the themes. Also, *Non-Numerical Domain* is consistent with the concept definition of function and *Graphable* is inconsistent. Lee provides, without prompting in both interviews, a relation between parents' and children's eye colors. The nature of this example could suggest to an individual that all functions are not graphable on a coordinate system; however, during the post-interview, Lee avoids a conflicting concept image and creates a graph for his eye-color example on a Cartesian coordinate plane. This suggests that within his interaction with the EEFPSMT materials, Lee did not delve into this conflicting concept image.

5.1.1.2 Contrasting Themes between Interviews

A commonality between Sofia's and Lee's interviews is the exclusive appearance of the *VLT* theme in their pre-interviews and the contrasting theme, *VLT Not*

Defining Characteristic, in their post-interviews. This pair of themes represents a shift from a theme inconsistent with the concept definition of function to a consistent theme. Neither Sofia nor Lee make any reference to the vertical line test in the pre-interview until it is presented as part of a possible definition for function in Question 4 (Figure 3-3); however, they both accept it as a valid definition. This is not the case in the post-interview. Instead, Sofia rejects the vertical line test as a definition for function based on the fact that “we said not all functions are graphable.” Lee dismisses this statement because “the vertical line test works if you have the independent as your x and your dependent as your y , so like if you’re talking about the coordinate axis for Cartesian stuff.” Each of these lines of reasoning can be linked to activities that took place in the EEFPSMT materials. In Exploration 5.1 of Lesson 5 (Figure 5-1), students proposed defining characteristics of function as a class and then attempted to identify any counterexamples to the statements. One such proposed defining characteristic was “passes vertical line test” (Figure 5-2). This resulted in discussions involving functions defined on non-numerical domains and not graphable on a coordinate system as well as discussions identifying that the vertical line test presumes the function follows standard Cartesian coordinate system assumptions. These discussions seem to have affected Sofia’s and Lee’s function conceptions based on their explanations in their post-interviews.

Exploration 5.1: Characteristics of Functions

What comes to mind when you hear the word *function*? Working in groups, list as many characteristics of functions as you can, based on your previous mathematical experience.

- Can the same function be represented in more than one way?
- Do all functions have a graphical representation? Why or why not?
- Are all your examples that come to mind relating two numerical sets? If so, is it necessary for the sets to be sets of numbers?

How are the various characteristics related? Which characteristics also apply to other mathematical concepts? Which are characteristics that are true of *all* functions? We call these *defining* characteristics.

Figure 5-1 Exploration 5.1 of Lesson 5

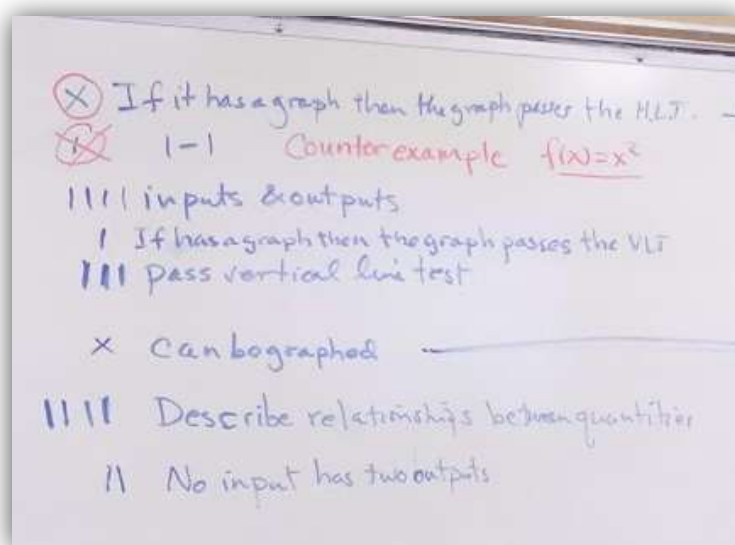


Figure 5-2 Student Proposed Defining Characteristics of Function from Exploration 5.1

Two additional sets of contrasting themes identified across Sofia's interviews include *Graphable* and *Graph Not Defining Characteristic*, and *Numerical Domain* and *Non-Numerical Domain*. Each of these sets of also depicts a concept-definition-inconsistent theme in the pre-interview contrasting with a consistent theme in the post-interview. Sofia discloses in the pre-interview that a function is "a definition of whatever like the graph is" and that functions are defined on real number domains because "if you

have it on a graph, that'd have to be like real numbers." These statements contrast with her assertions made in the post-interview that some functions do not have a graph and some functions can be defined on non-numerical domains. However, when Sofia is asked to elaborate on her statement about functions and graphs, she is unable to provide an example of such a function that is not a graphable. She is able to provide an example of a function with a non-numerical define. In fact, she refers to one she specifically remembers her instructor providing during the Exploration 5.1 of Lesson 5 (Figure 5-1) that maps people to their eye color (Figure 5-3). Clearly, the conversations sparked in the course by the EEFPSMT materials contributed to the emergence of *Non-Numerical Domain* theme in the post-interview. The fact that she is unable to produce an example of a function that is not graphable but is able to provide a function not defined on a numerical domain suggests that despite remembering the conclusion from class that not all functions are graphable, there is a disconnect between these components of her concept image.

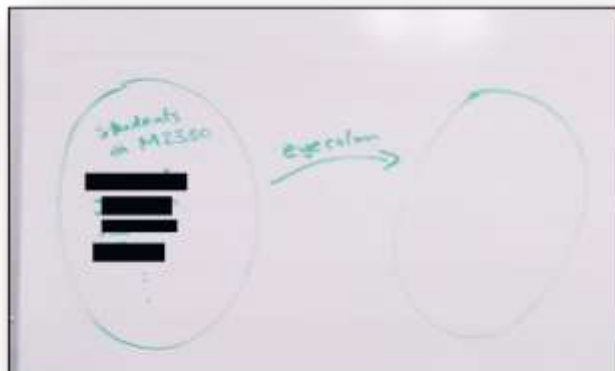


Figure 5-3 Instructor Example of a Function from Students to Eye Colors

5.1.1.3 Further Themes Exclusive to the Post-Interviews

A theme not apparent in either Sofia's or Lee's pre-interview that emerged in both of their post-interviews is *Input/Output* – a concept-definition neutral theme. Although neither of these participants gave any indication that this theme was a component of their concept image in the pre-interview, their language describing function in the post-interview is brimming with *Input/Output* language to the extent that their personal concept definitions rely on this language. This suggests that this component of their concept image developed while interacting with the EEFPSMT materials. Considering the EEFPSMT materials, this is a surprising finding as the materials never present function as this type of process or use this type of language in relation to function. The *Input/Output* theme is identified in the pre-interviews of participants who completed courses beyond the three-semester calculus sequence. Then, Sofia and Lee may have developed this type of language from their peers over the course of the study or in their other mathematics courses taken concurrently with the study.

Another theme emerging in both Sofia's and Lee's post-interviews is *Univalence*, a theme consistent with the concept definition of function. In their pre-interviews, Sofia and Lee do not indicate that univalence is a defining characteristic of function in their pre-interviews – even after they read the statements in Question 4 (Figure 3-3). They also only reference the vertical line test after they read the statement using the vertical line test to define function in Question 4. Then, they both use the vertical line test as a defining characteristic of function to decide whether the remaining statements represent valid definitions of function. Here, the vertical line test supplants any concept image they may have of univalence or other ways to determine univalence for functions that are not graphable. Conversely, the idea of univalence emerges organically in the first question of Sofia's and Lee's post-interviews. Sofia explains that “there's no more than one output

for each input,” and Lee states “you want to have one input to only one output.” Unlike their reasoning in the post-interview, these references to the univalence property do not appear to be rooted in the vertical line test. This suggests that the univalence characteristic became a more prominent component of their function concept images over the course of the study.

Contrived Difference from Equation is a concept-definition-inconsistent theme exclusive to Sofia’s post-interview. In the pre-interview, the only difference Sofia draws between the terms function and equation is that equations do not have to meet univalence requirements. This suggests that her concept image does not include any further separation between these types of mathematical objects. During the post-interview, however, she reveals a separation she perceives between function and equation based on the number of variables present. For instance, she deems $f(x) = x$ as a function and not an equation because “it [doesn’t] have relationship with two variables” and $y = x$ as an equation and not a function because it does have a relationship between two variables, “and that’s not what we decided a function was.” She states that she came to this conclusion based on whole-class discussions initiated by Exploration 6.1 of Lesson 6 (Figure 5-4). This lesson intends to highlight differences between function and equation based on their definitions by examining an equation, $y = 2x + 1$, arising from the intersection of the graphs of the functions $f(x, y) = y$ and $g(x, y) = 2x + 1$. The conclusion presented by some of Sofia’s classmates was that $y = 2x + 1$ is not a function because it arises from the intersection of the graphs of two functions with domains of \mathbb{R}^2 . During a whole-class conversation related to Exploration 6.2 of Lesson 6 (Figure 5-5) and constructed meanings of the equal sign, the students also decided that $f(x) = x^2 + 5$ did not meet the definition of equation because the equality symbol’s constructed meaning is “define” whereas in an equation its constructed meaning “equivalent”. Sofia’s response in

the post-interview suggests that she recalls the conclusion from this conversation and others in Lesson 6, but that within the conversations themselves she did not fully understand her peers' rationale.

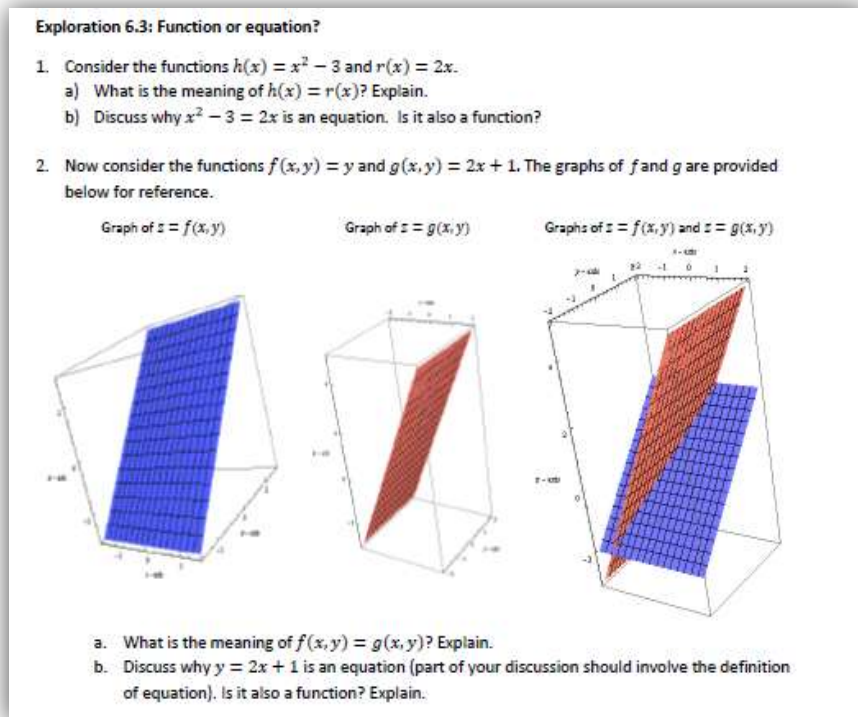


Figure 5-4 Exploration 6.3 of Lesson 6

Defining Characteristic is a theme exclusive to Lee's post-interview in addition to *Input/Output* and *Univalence*. Lee expresses in the pre-interview that an equation is "a function usable in more than one scenario" (e.g. $y = ax^2 + bx + c$ is an equation and $y = 2x^2 + 3x + 4$ is a function). However, in the post-interview, he identifies that function and equation are different mathematical objects because they are defined differently. Specifically, he draws on the different contrived meanings of the equal sign in an equation and an algebraic function. He notes that an equal sign in an equation indicates equivalence whereas in an algebraic function it defines. Student encountered this idea in

a Lesson 6 exploration in which students examine the various contrived meanings of an equal sign. This exploration was designed to help students perceive a difference in the definitions of these mathematical objects.

Exploration 6.2: Constructed meanings of the equal sign

1. The use of the equal sign evokes several constructed meanings that can cause some confusion, in particular in K-12 mathematics (e.g. Knuth, Stephens, McNeil, & Alibali, 2006; Kieran, 1981). Identify and discuss the meanings that the following uses convey:

	Meaning of "="	Give another example conveying the same meaning.
a. $3 + 5 = \underline{\quad}$		
b. $f(x) = x^2 + 5$		
c. $\frac{d}{dx}(e^{x^2} + 4x) = \underline{\quad}$		
d. $A = \pi r^2$		
e. $2 \sin x \cos x = \sin 2x$		
f. $g(x) = \cos x + x$		
g. $x^2 + 3x = x - 1$		

Figure 5-5 Exploration 6.2 of Lesson 6

5.1.2 Group B Participants

Group B participants include Jamie and Gabby. Each of these participants completed a three-semester calculus sequence as well as other required mathematics courses in their degree plan prior to the start of this study. Concurrently with the study, Jamie and Gabby were both enrolled in *Abstract Algebra I* and *Real Analysis I*. Similar to Section 5.1.1, the following section will examine relationships between pre- and post-interview concept image themes. Specifically, I will explore the relationships between themes persisted across the interviews, themes seem to contrast across interviews, and additional themes exclusive to the post-interviews. This analysis reveals the persistence of consistent themes across Jamie's and Gabby's interviews; inconsistent themes in their pre-interviews that directly contrast with post-interview consistent themes; and further neutral and consistent themes identified in their post-interviews. Jamie also exhibits the persistence of a neutral theme from her pre- to post-interview.

5.1.2.1 Persisting Themes between Interviews

Analyzing the themes in Jamie's and Gabby's interviews, there are two themes that persisted for both participants across the pre- and post-interviews: *Univalence* and *Not One-to-One*. These themes also represent themes consistent with the definition of function. During the pre-interview, Jamie introduced the idea of univalence in the first question of the interview when she defined function. Gabby, on the other hand, did not organically mention univalence. Similar to Group A participants, Sofia and Lee, Gabby makes no mention of this property until she interacts with statements provided in Question 4. Unlike Group A participants, Gabby seems to recall this property not from being reminded of the vertical line test but she appears to apply it as a defining characteristic to determine if the first statement defines function. Both Jamie and Gabby in the post-interview include the univalence property in their initial definitions of function; however, Gabby's description uses imprecise language. For example, Gabby states that "a function is [a] one-to-one correspondence, so you have an element from the domain that goes to just one element in the codomain." Follow-up questions from the interviewer reveal that Gabby's use of one-to-one actually refers to univalence. The EEFPSMT materials do not focus on injective, surjective, and bijective functions, so this is perhaps why Gabby confounds the term one-to-one with the description of univalence.

The other theme that persisted for Jamie and Gabby across interviews is *Not One-to-One*, a concept-definition-consistent theme. Statement (d) of Question 4 (Figure 3-3) proposes to define function as a relation for which every element of the range corresponds to exactly one element of the domain. Jamie and Gabby reject this as a definition of function in their pre- and post-interviews on the grounds that it only describes one-to-one functions. In fact, they seem to dismiss the statement in the post-interview almost immediately because it excludes some functions. However, in the pre-interview,

Jamie and Gabby use examples such as $f(x) = x^2$ to arrive at their conclusions, and Gabby also uses the vertical line test in her reasoning. Within the EEFPSMT materials, students completed an exploration comparing and contrasting various types of function definitions, considering the types of mental images different definitions produced, and the context a certain definition might be found (Figure 5-6). The immediate dismissal of statement (d) in the post-interview contrasted with the need to work through examples in the pre-interview suggests that Jamie and Gabby were better equipped in the post-interview to analyze the statements possibly as a result of this experience.

Exploration 5.3: Various Definitions

Part 1:
 Consider the various definitions of function given below. Each of these definitions comes from a mathematics textbook. Discuss the following questions in your group:

- Compare and contrast these definitions.
- For each definition, what mental image of function do you have after reading it?
- For what purpose would each definition be appropriate?

Textbook Definitions:

- A function is a relationship between two variables (an independent variable and a dependent variable) for which every value of the independent variable has at most one value of the dependent variable.
 From *Discovering Advanced Algebra* [High school text], Murdock, Kamischke, & Kamischke, 2010, p. 190.
- A function is a rule that assigns to each value x in a set D a unique value denoted $f(x)$. The set D is the domain of the function. The range is the set of all values of $f(x)$ produced as x varies over the entire domain.
 From *Calculus*; Briggs, Cochran, & Gillet, 2015, p. 1.
- A relation is a set of ordered pairs. A function is a relation in which, for each distinct value of the first component of the ordered pairs, there is exactly one value of the second component.
 From *College Algebra*; Lial, Hornsby, Schneider, & Daniels, 2017, p. 209.
- A function is a set of ordered pairs in which no first component is repeated.
 From *College Algebra*; Lial, Hornsby, Schneider, & Daniels, 2017, p. 209.
- A function is a rule or correspondence that assigns exactly one range value to each distinct domain value.
 From *College Algebra*; Lial, Hornsby, Schneider, & Daniels, 2017, p. 209.
- A function is a rule that assigns to each element of a set A a unique element of a set B (where B may or may not equal A). The set A is called the domain of the function f , the set B the codomain, and the subset of the second set B consisting of those elements that are images under the function f of some element of its domain is called the range of the function f .
 From *Mathematics for High School Teachers*; Usiskin, Peressini, Marchisotto, & Stanley, 2003, p. 68.
- A function $f: X \rightarrow Y$ is a relation R from X to Y with the property that for every $x \in X$ there is a unique $y \in Y$ such that $(x, y) \in R$, in which case we write $y = f(x)$.
 From *Real Analysis*; Folland, 1999, p. 3.
- The Cartesian product of two sets S and T , denoted $S \times T$, is the set of all ordered pairs (s, t) such that $s \in S$ and $t \in T$. For any sets X and Y , a function f from X to Y , $f: X \rightarrow Y$, is a subset f of the Cartesian product $X \times Y$ such that every $x \in X$ appears once and only once as a first element of an ordered pair (x, y) in f .
 From *Mathematics for High School Teachers*; Usiskin, Peressini, Marchisotto, & Stanley, 2003.
- A function, or map, of a set S into a set T consists of the set S , called the domain of the map, the set T , called the co-domain, and a subset α of $S \times T$ (the graph) having the following two properties:

Figure 5-6 Exploration 5.3 Part 1 of Lesson 5

Exploration 5.4: Function Identification Activity 1 (Adapted from Armendariz & Daniels, pp. 11 - 12)

In your groups, decide which of the following relations are functions. Justify your answers.

	Relation	Function? Yes or No, and Justification										
1	$y = -x^2$											
2	<table border="1"> <thead> <tr> <th>x</th> <th>y</th> </tr> </thead> <tbody> <tr> <td>2</td> <td>4</td> </tr> <tr> <td>-6</td> <td>-12</td> </tr> <tr> <td>13</td> <td>26</td> </tr> <tr> <td>-57</td> <td>-114</td> </tr> </tbody> </table>	x	y	2	4	-6	-12	13	26	-57	-114	
x	y											
2	4											
-6	-12											
13	26											
-57	-114											
3	(5, 6) (3, 2) (5, 1)											
4												
5	$y^4 = 8x^2$	Also identify: What does the graph of this relation look like?										
6												
7	Tom: Blue Jill: Brown Bill: Green Harry: Green											

Figure 5-7 Exploration 5.4 of Lesson 5

5.1.2.2 Contrasting Themes between Interviews

Jamie and Gabby share two pairs of contrasting concept image themes across their pre- and post-interviews: *Numerical Domain* and *Non-Numerical Domain*, and *VLT* and *VLT Not Defining Characteristic*. These pairs each represent a shift from a theme inconsistent with the concept definition of function to a consistent one. Examining the first pair of contrasting themes, *Numerical Domain* is exclusive to their pre-interviews and *Non-Numerical Domain* is exclusive to their post-interviews. Jamie, for instance, expresses in her pre-interview that all functions are defined on the real numbers based

on what she's encountered and "worked [with] in all [her] math classes." Gabby also indicates that the only functions she includes in her concept image are those defined on subsets of the real numbers. Conversely, in the post-interview, Jamie and Gabby both provide examples of functions defined on non-numerical sets. Gabby suggests relations involving people and their eye color or birthdays, and Jamie describes a relation between mothers with only one child and their children. All of these functions suggested by Jamie and Gabby are actually similar to examples discussed or presented within Lesson 5. In particular, the instructor presented potential a function from people to their eye color (Figure 5-3), and the instructor initiated a conversation about the restrictions one would have to make on the set of all mothers to define a function between mothers and children. Another exploration in Lesson 5 (Figure 5-8) also presents a relation between people and their birthdays. The fact that all the non-numerical function examples presented by Jamie and Gabby are also found within the EEFPSMT materials directly or are examples discussed as a result of the materials suggest that these interactions contributed to this shift in their concept images.

Exploration 5.2: Examples of Functions

- a. Working in groups, use the examples below and your prior knowledge about functions to determine which of those relationships are *functions*. Justify your answers, being certain to explain which set you are considering as the input.
 - i. From Exploration 3.1: The relation between Secret Agent Cody's distance from the safety line (d_1) and the log on his pedometer (d_2).
 - ii. From Exploration 4.1 #3: The relation between the orbital period of a planet and its mean distance from the sun.
 - iii. The relation between the daily high temperature in Fahrenheit and the same day's high temperature in Celsius.
 - iv. The relation that pairs each university student's age (in whole years) with his/ her university ID number.
 - v. The relation between the diameter of a circle and its circumference.
 - vi. The relation between each person and his/ her birthday.
 - vii. The relation between finishing time at the 2017 Boston marathon and the persons who finish at that time.
- b. Your homework was to bring examples of relationships between quantities in the real world. Repeat part (a) with the examples you brought to class.
- c. In what contexts or situations might functions be more useful than other relations?

Figure 5-8 Exploration 5.2 of Lesson 5

Furthermore, *VLT* is exclusive to both Jamie's and Gabby's pre-interviews and *VLT Not Defining Characteristic* is exclusive to their post-interviews. Similar to participants in Group A, Jamie and Gabby made no mention of the vertical line test until encountering it as a potential definition for function in Question 4, but they both confirmed that it was a valid definition of function. Also like the Group A participants, Jamie and Gabby dismiss the statement that uses the vertical line test as a valid definition of function in the post-interview. Jamie states that there exist some functions that do not pass the vertical line test. This relates to an exploration in Lesson 5 where students are given the graph that could be interpreted as arising from the function $f(y) = y^2$ (Figure 5-7) – an example of a function that does not pass the vertical line test. While examining this problem, students began to wrestle with the idea that the vertical line test did not work in all scenarios. Gabby suggests it's not a valid definition because some functions are not graphable. This can be linked to various examples of functions in Lesson 5 defined on non-numerical domains that inspired conversations about whether or not all function could be graphed on a coordinate system.

One more set of contrasting themes between Jamie's interviews is *Graphable*, a concept-definition-inconsistent theme exclusive to her pre-interview, and *Graph Not Defining Characteristic*, a concept-definition-consistent theme exclusive to her post-interview. Specifically, Jamie indicates in the pre-interview that the ability to be represented as a graph is a defining characteristic of function. This contrasts with her statements in the post-interview expressing that "some functions cannot be graphed." As with Gabby's reasoning in the post-interview that the vertical line test cannot define function because not all functions can be graphed, Jamie's reasoning can also be linked to the conversations sparked by relations in Lesson 5 defined on non-numerical domains.

5.1.2.3 Further Themes Exclusive to the Post-Interviews

Two other themes not present in their pre-interviews, one neutral and one consistent with the concept definition of function, emerged in both Jamie's and Gabby's post-interviews: *Form Implies Equation* and *Defining Characteristics*. Examining *Form Implies Equation*, Jamie concludes the mathematical statement $2x - y = 8$ is an equation and not a function – even if the statement is written as $y = 2x - 8$. She asserts that because the statement is not $f(x) = 2x - 8$ then “[the equal sign]’s just assuming that this [left] side equals this [right] side, so that will make it an equation.” Similarly, Gabby explains also explains that any mathematical object with “one equal sign... and if you don't have like ‘ $f(x) =$ ’” will be an equation. According to her criteria, Gabby would then accept objects of the form “ $y =$ ” as an equation based on their form. Jamie and Gabby each reference the constructed meaning of the equal sign in an equation versus function. This relates to an exploration in Lesson 6 (Figure 5-5) where students investigated these various constructed meanings. Furthermore, their explanations allude to Exploration 6.3 of Lesson 6 (Figure 5-4) where students decided $y = 2x + 1$ would not be considered a function. In this exploration, students explore an equation, $y = 2x + 1$, arising from the intersection of two planes which inspired conversations about the fact that $y = 2x + 1$ generates all the solutions (x, y) for which the planes intersect and that functions must be clearly defined on domains and codomains. It appears that Jamie's and Gabby's explorations of these ideas in Lesson 6 influenced their responses to this question.

In the pre-interviews, Jamie and Gabby both indicate that all functions could be represented algebraically, and Gabby only used the univalence property to distinguish functions from equations while Jamie contrived a difference between function and equation. Conversely, *Defining Characteristics*, a concept-definition-consistent theme, surfaced in both their post-interviews. Both Jamie and Gabby identify that based on its

definition, an equation “shows equivalence between two terms,” and this is not how a function is defined. This distinction between function and equation based on the type of mathematical objects they define can also be linked back to Exploration 6.3 of Lesson 6 (Figure 5-4). Specifically, this exploration sparked conversations in which students examined the context of the statement $y = 2x + 1$ and the differences in the defining characteristics of function and equation. Jamie’s and Gabby’s responses in the post-interview suggest that these conversations influenced their concept images.

Relations on Sets is an additional consistent theme that is apparent only in Gabby’s post-interview. This theme is consistent with the concept definition of function. While she defines function as a “form of expression that you use to solve for an equation,” in the pre-interview, her post-interview definition is a “correspondence so you have a[n] element from the domain that goes to just one element in the codomain.” This idea that a function needs to be clearly defined on a domain is emphasized in Exploration 5.4 of Lesson 5 (Figure 5-7). In this exploration, students are provided with various relations and asked to identify which are also functions. They are also asked to justify their rationale. It is this justification of their rationale that encouraged students to consider the assumptions they might make about the domain and codomain when identifying a relation as a function, and how these assumptions affect the ability to determine if a relation is a function. The examination of these assumptions seems to have influenced the defining characteristics of function in Jamie’s concept image.

5.1.3 Group C Participants

Henry, Alan, and Michael comprise Group C participants. In addition to completing a three-semester calculus sequence and various mathematics courses in their degree programs, all of these participants completed a course in *Abstract Algebra I* or *Real Analysis I* prior to the start of this study. As in Sections 5.1.1 and 5.1.2, the

following section explores relationships in the pre- and post-interview themes that persist across the interviews, themes that seem to contrast across interviews, and additional themes that appear in the post-interview. Across all three participants, I observe the persistence of consistent themes; an inconsistent, pre-interview theme that contrasts with a consistent, post-interview theme for Henry and Alan; and the emergence of an additional consistent themes in Michael's and Alan's post-interviews. I also identify the persistence of a neutral theme in Henry's and Michael's interviews, and inconsistent themes in Henry's and Alan's interviews. Furthermore, another neutral theme is exclusive to Henry's post-interview.

5.1.3.1 Persisting Themes between Interviews

Several persisting themes overlap between two or more members of Group C. One theme, in particular a concept-definition-consistent theme, persists for all three participants: *Relations on Sets*. Henry, Alan, and Michael all emphasize in their pre- and post-interviews that a function is a relation between sets. Furthermore, Alan indicates in his pre-interview and Henry and Michael in their post-interviews that these sets are the domain and codomain. This suggests that this consistent structural conception of function developed in their prior mathematical courses, and it does not seem that their interactions with the EEFPSMT materials diminished this conception. *Input/Output* and *Univalence*, a concept-definition-neutral and concept-definition-consistent theme, persist across Henry's and Michael's interviews as well. Particularly, Michael depicts the univalence property in both of his interviews using input and output terminology such as "the input has a unique output". Henry, on the other hand, describes the input-output and univalence ideas in the pre-interview using " x " and " y " elements such as "each element of x has only one element in y " and "each time we have an x element it's possible to find a y element." In the post-interview, he expresses both these ideas using language related to sets. For

instance, he says that a function has a domain and “in the domain we have the input, and the second one is the codomain, should be the output” and “each element in the set A has a unique element in set B.” Although Henry does not specifically use input and output language to illustrate univalence as Michael does, Henry expresses the input-output and univalence ideas using similar language within his pre- and post-interviews. The EEFPSMT materials do not apply this type of language to function, and the fact that it emerged in two of the participants’ pre- and post-interviews suggests this is a function conception formed in their studies of mathematics prior to this course and carried through this course.

The *One-to-One* theme, a theme inconsistent with the definition of function, persisted between Henry’s and Alan’s interviews as well. They both rejected a relation as a function based on the fact that two elements of the domain corresponded to the same element of the codomain. While the EEFPMST materials do not focus heavily on one-to-one functions, within Lesson 5, students compare and contrast various definitions of function from mathematics textbooks. This examination inspired conversations over the word choices “unique,” “exactly one”, and “at most one” which highlighted possible confusions with one-to-one students might form based on these word choices. Students also completed a homework problem where they defined surjective, injective, and bijective, and they provided an example of a function that is injective and not surjective as well as an example of a function that is surjective and not injective. In the post-interview, however, Henry still seems to hold some misconceptions over the word “unique” and how it is used in the function definition as he wants to reject $y = x^2$ as a function because 2 and -2 do not correspond to a unique output in the codomain in the sense that they correspond to the same value. Alan also still excludes a relation in the post-interview as a function based on the fact that it was not one-to-one. The persistence of this theme

suggests that Henry and Alan did not thoroughly grapple with the difference between univalence and the one-to-one property during the course of this study nor did their exposure to *Real Analysis I* seem to have influenced their reasoning.

Two additional concept-definition-inconsistent themes persisted across Henry's interviews: *Algebraic Representation* and *VLT*. He proclaims in the post-interview that "all function[s] are equation[s]," and that function and equation have the same meaning in the pre-interview. Although the EEFPSMT materials contained various explorations highlighting the differences between function and equation as well as several examples and in-class conversations depicting that some functions cannot be represented algebraically, Henry's concept image was unaffected. That is, there is no apparent change regarding this component of Henry's concept image. This is in contrast to Sofia, Jamie, Gabby, and Alan who exhibited the *Algebraic Representation* theme in the pre-interview but not in their post-interviews. Similarly, despite explicit conversations and examples in the materials – including Exploration 5.3 (Figure 5-9) where participants examined a student attempt to define function using the vertical line test – the *VLT* theme persists across both interviews. Henry accepts the vertical line test as a valid definition of function in the pre-interview, and in the post-interview, he seems to use the vertical line test as the final rationale to determine whether or not a relation is also a function. For example, as the *Univalence* and *One-to-One* themes also persisted across his interviews, Henry is unsure in the interview if $y = x^2$ is a function because it does meet the univalence property but it's not one-to-one. His final conclusion is that this is a function because "for the vertical test, it's good." The persistence of *Algebraic Representation* and *VLT* could be a result of the "rule of four," often taught in secondary mathematics courses. This "rule of four" emphasizes representing mathematical concepts, like function, numerically, graphically, symbolically, and verbally, and it has been touted by

the National Council of Teachers of Mathematics (2000) (as cited in Thompson & Chappell, 2007). However, teaching the “rule of four” as representations associated with function presupposes that all functions can be represented in each of these ways, and it may inadvertently limit the function concept image of an individual.

Exploration 5.3: Various Definitions

Part 2:
Suppose you asked high school students to define *function*, and they provided the following. For each, discuss the following questions in your group:

- Is the student’s attempt a valid definition that works for *all* functions?
- What does the student’s attempt indicate about his or her thinking and experience with functions?

Student Attempts to Define Function:

- A function passes the vertical line test.
- A function is an equation with two variables (usually called x and y).
- In a function, no x -value is repeated.
- A function is a rule where each input goes to a unique output.
- A function is a relationship between two variables.

Figure 5-9 Exploration 5.3 Part 2 of Lesson 5

Non-Numerical Domain and *Graph Not Defining Characteristic* are also concept-definition-consistent themes that persist between Michael’s pre- and post-interviews. He, in contrast to all other participants interviewed, provides examples in both interviews of relations between sets of shapes to show that a function can be defined on non-numerical sets. Also, Michael rejects in both interviews statement (c) of Question 4 (Figure 3-3) which defines function as a graph that passes the vertical line test. He claims in the pre-interview that a function is “not necessarily a graph” and in the post-interview he recognizes that the statement is “kind of implying that every function is a graph.” These similar examples of functions on non-numerical domains and explanations that functions are not always graphable suggest that, prior to the start of the course and the

study, Michael included functions defined on non-numerical domains and not graphable in his concept image. Also, this conception was not adversely affected by his interaction with the materials. These conceptions plausibly developed during his previous mathematics courses and remained after his experiences with the EEFPSMT materials.

5.1.3.2 Contrasting Themes between Interviews

Examining the pre- and post-interview concept image themes of Henry, Alan, and Michael, Michael's interviews do not exhibit any themes exclusive to the pre-interview that seem to contrast directly with themes exclusive to his post-interview. Henry and Alan, however, share *Numerical Domain* and *Non-Numerical Domain* as a pair of contrasting themes across their interviews where a concept-definition-inconsistent theme shifts to a concept-definition-consistent theme. Neither Henry nor Alan is able to provide an example in the pre-interview of a function defined on a non-numerical domain. Henry immediately concluded that all functions are defined "in the subset of all real numbers" when asked if a function could be defined on a set other than a subset of the real numbers. Alan, on the other hand, initially attempted to devise a function not defined on a subset of the real numbers and suggested using a piecewise function such as the Dirichlet function. Upon realizing this too is defined on a subset of the real numbers, Alan stated that he was unable to think of such a function. This contrasts with the examples Henry and Alan provide in the post-interview of functions defined on non-numerical domains such as letters, people, and cars. Although these examples are not directly from the EEFPSMT materials for examples discussed as a whole class, their emergence still implies that their concept images grew to encompass functions defined on non-numerical sets over the course of the study.

Another pair of contrasting themes between Henry's pre- and post-interviews is *Graphable* and *Graph Not Defining Characteristic*. Henry seems to only include functions

that can be graphed in his pre-interview concept image of function, but this is not the case in the post-interview. Although the example he uses to cite his explanation that all functions cannot be graphed does not come explicitly from the EEFPSMT materials, this contrast between his pre- and post-interview suggests Henry came to include these types of functions in his concept image during the course of this study. The contrasting themes *VLT* and *VLT Not Defining Characteristic* in Alan's pre- and post-interviews also suggest a change in the types of functions he includes in his function concept image. Alan, in the pre-interview, spontaneously offers the vertical line test as a "rule of thumb" for determining if a relation is a function. However, he says in the post-interview that there are conditions where the vertical line test is not applicable.

5.1.3.3 Further Themes Exclusive to the Post-Interviews

Additional themes emerged exclusively in Henry's, Alan's, and Michael's post-interview. In particular, *Defining Characteristics*, a theme consistent with the concept definition of function, is present in Alan's and Michael's post-interviews. Both participants identify that a primary difference in the definitions of function and equation is that a function is a relation between sets and an equation is not. This is a concept that was emphasized in the EEFPSMT materials. Specifically, in Exploration 5.4 of Lesson 5 (Figure 5-7), students were provided with a table of relations and considered which would represent functions. These examples did not explicitly define the domain and codomain of the relations. This sparked conversations about the assumptions made when they determine whether or not a relation was a function as well as the importance of explicitly defining the domain and codomain of a function. It appears that these conversations influenced Alan's and Michael's understandings of the relationship between function and equation.

Form Implies Equation and *Non-Univalence for Equations* are a concept-definition-neutral and concept-definition-consistent theme exclusive to Henry's post-interview. In the pre-interview, Henry does not provide much indication that he sees function any differently from equation. He says multiple times that a function is an equation or a function is "given to be an equation." He also claims that $y = 2x - 8$ is a function because there is a "kind of relation between x and y ." A relation between x and y is how Henry defines a function in the pre-interview. Ironically, in the post-interview, he determines this same example is not a function because it "show[s] a relationship between x and y ." This is a result of the fact that he defines equation in the post-interview as a relation between two quantities. This suggests that, unlike in the pre-interview, he interprets this form as depicting a relation between two quantities and that the defining characteristics he attributes to function and equation changed over the time he interacted with the EEFPSMT materials. Henry also indicates in the post-interview that equation differs from function because equations do not have to meet the univalence criteria. While Henry does not imply this idea in the pre-interview, it is uncertain if this is not a component of his concept image or if it simply did not appear in his pre-interview concept image.

Two other concept-definition-consistent themes exclusive to Michael's post-interview are *Not One-to-One* and *VLT Not Defining Characteristic*. Interestingly, the themes that contrast with *Not One-to-One* and *VLT Not Defining Characteristic* (*One-to-One* and *VLT*) are not identified in his pre-interview. It is not clear then if these concept image themes in the post-interview simply did not emerge in the pre-interview or if these ideas were developed over the course of the study.

5.2 Discussion of Participants' Pre- and Post-interview Concept Image Themes across Participant Groups

In the previous section, I examined similarities and differences between each participants' pre- and post-interviews and between participants in the same mathematical background group. The following sections will focus on similarities and differences across participant groups. Specifically, I examine the existence themes consistent, inconsistent and neutral with the concept definition across participant groups, and then discuss explicit commonalities in the concept image themes between participant groups.

5.2.1 *Consistent, Inconsistent, and Neutral Theme Discussion of Participant Groups*

In Chapter 4 Section 4.5 , I discussed the classification of each of the 19 function concept image themes as *consistent*, *inconsistent*, or *neutral* in regards to the concept definitions of function and equation (Figure 4-1). Additionally, I identified the number of concept-definition-consistent, concept-definition-inconsistent, and concept-definition-neutral themes across each participant's pre- and post-interview (Table 4-3). The following section examines patterns within these numbers between participant groups.

Reviewing Table 4-3, an obvious pattern is not apparent within the varying mathematical backgrounds of participants and the number of pre-interview themes consistent and inconsistent with the concept definition. Michael is the only participant whose number of consistent themes in the pre-interview exceeds his number of inconsistent themes. While Michael is a participant who completed advanced mathematical courses including *Real Analysis I* and *Abstract Algebra I* prior to the start of this study, the other participants in Group C, Henry and Alan, exhibited a number of pre-interview consistent and inconsistent themes similar to the participants in Group A and Group B. In fact, all other participants demonstrated evidence in the pre-interview of at least four inconsistent themes and at most three consistent themes. According to Sfard's

(1992) historical analysis of the development of function, this greater number of inconsistent conceptions in the pre-interview may be a natural reflection of the mathematical community's struggle to view function as an abstract object. The lack of patterns in the participant groups' pre-interview consistent and inconsistent themes suggests that advanced mathematical courses may not advance an overall function conception consistent with the concept definition, and several inconsistent conceptions may develop or persist throughout these studies. This coincides with Monk's (1994) finding that conceptions needed for teaching secondary mathematics are not developed in a way that affects student achievement after five undergraduate mathematics courses.

Moreover, there is not a discernible difference in the number of post-interview concept-definition-consistent and concept-definition-inconsistent themes by participant group. Henry, a Group C participant, displays three post-interview themes inconsistent with the concept definition while every other participant exhibits at most one. Considering consistent post-interview themes, between five and seven consistent themes emerge across all three groups, but as with the inconsistent themes, there is no overt pattern within the groups. Specifically, participants in Group B and Group C had the most consistent themes – Jamie, Gabby, and Michael each exhibited seven themes – and participants in Group A and Group C had the least number of consistent post-interview themes – Lee and Henry each exhibited five themes. Table 5-1 also shows that participant groups have a similar average number of post-interview consistent, inconsistent, and neutral themes. This absence of patterns between participant groups' number of post-interview consistent and inconsistent themes suggests that their interaction with the EEFPSMT materials and not prior or concurrent mathematical experiences may be what contributed to their function conceptions identified in the post-interview.

Table 5-1

Average Number of Post-Interview Consistent, Inconsistent, and Neutral Themes by Participant Group

	Group A	Group B	Group C
Average* Number of Consistent Themes	6	7	6
Average* Number of Inconsistent Themes	1	0	1
Average* Number of Neutral Themes	2	2	1

*Average is rounded to the nearest whole number

One pattern observable in Table 4-3 is that the majority of themes identified in the pre-interview for six of the seven participants are concept-definition-inconsistent, and the majority of themes in the post-interview for all participants are concept-definition-consistent. Figure 5-10 depicts a comparison by percentage of consistent, inconsistent, and neutral themes across all participants' pre- and post-interviews. In general, the number of consistent themes for all participants increased from the pre-interview to the post-interview, and the number of inconsistent themes decreased.

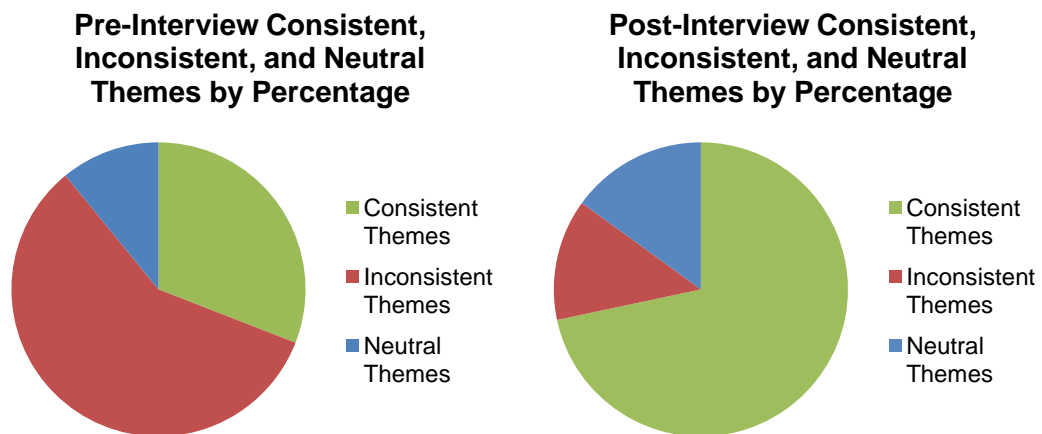


Figure 5-10 Comparison of Consistent, Inconsistent, and Neutral Themes by Percentage

The number of inconsistent themes decreased by at least two themes for each participant, and Jamie exhibited a decrease of seven themes. Also, each participant evinced at least three more consistent themes in the post-interview than in the pre-interview. Even for Michael, the only participant with more consistent than inconsistent themes in the pre-interview, this increase in consistent themes and decrease of inconsistent themes is evident between his pre- and post-interviews. This suggests that participants' experiences with the EEFPSMT materials may have contributed to the prominence of consistent themes over inconsistent themes in their post-interview function conceptions. Since students were concurrently enrolled in other mathematics courses, it may be that the coupling of EEFPSMT experiences with those courses affected this change. However, the results of these courses, in absence of FM course with the EEFPSMT materials, can be observed in Group C's pre-interviews where two of the three participants still exhibited more inconsistent than consistent function conceptions.

5.2.2 Specific Concept Image Themes Observations across Participant Groups

In addition to the concept-definition-consistent, concept-definition-inconsistent, and concept-definition-neutral themes findings discussed, Table 4-2 also reveals specific commonalities and differences across participant groups. Two such similarities center on the themes *Numerical Domain* and *Non-Numerical Domain*. Five participants in the pre-interview indicated that all functions are defined on numerical domains – a conception inconsistent with the definition of function. Similarly, Stein et al. (1990) report that teachers may believe that a function is an interdependent relationship between two numbers i.e. functions are defined on numerical sets. In the pre-interview of this study, the PSMTs with this belief represent members of all three mathematical background participant groups. The only two students who suggested that functions could be defined on non-numerical sets include a member from Group A and a member from Group C. As

the inconsistent theme *Numerical Domain* emerged within all three participant groups' pre-interviews and the only students who suggested otherwise represent the least and most prior mathematical experience, the mathematics classes taken as part of the PSMTs degree program may not promote the inclusion of non-numerical functions in their concept image. However in the post interview, all seven participants suggest that functions can be defined on non-numerical sets, and participants typically cited examples within the course materials or specific examples discussed in class. This suggests that the PSMTs' experiences with the EEFPSMT course materials, no matter their prior mathematical experiences, may have contributed to this change in their function concept images.

The *VLT* theme appears to be common among all three participant groups' pre-interviews as well. Sofia, Lee, Jamie, Gabby, Alan, and Henry convey in the pre-interview that all functions will pass the vertical line test. Since these PSMTs represent members of all three participant groups, this finding may indicate that PSMTs do not confront this conception related to the vertical line test in their university mathematics courses. In fact, research literature depicts that preservice and practicing mathematics teachers believe and teach that the vertical line test is a defining characteristic of function (Even, 1993; Norman, 1992). This contrasts with the results of the post-interview where six participants divulge that the vertical line test is not a defining characteristic of function because there exist some functions that are not graphed on a Cartesian coordinate system. Only Henry in the post-interview continued to suggest that the vertical line test is a defining characteristic of function. It is a potential then that the EEFPSMT course materials influenced this aspect of participants' function concept image across diverse mathematical backgrounds.

Graphable is identified in the pre-interviews of Sofia, Lee, Jamie, and Alan, participants in all three participant groups. This is not surprising considering Even's (1993) and Norman's (1992) finding that practicing teachers use and teach the vertical line test as a defining characteristic of function which presupposes that all functions are graphable on a Cartesian coordinate system. Vinner and Dreyfus (1989) similarly report that undergraduate students identify and define function as "A graph that can be described mathematically" (p. 360). While Jamie and Alan only represent one member of Group B and Group C respectively, other participants in these groups, Gabby and Henry, did not indicate an understanding in the pre-interview that some functions are not graphable on coordinate systems. Gabby and Henry did reveal in their pre-interviews that all functions must pass the vertical line test. The inclusion of the *VLT* theme in their pre-interview concept image suggests that *Graphable* may be a component of Gabby's and Henry's concept images that the pre-interview did not reveal. Conversely, Sofia, Jamie, Gabby, Henry, and Alan all explicitly express in their post-interview that there exist some functions that cannot be graphed on a coordinate system. The fact that the *Graph Not Defining Characteristic* theme was not identified in any of these participants' pre-interviews but emerged in their post-interviews indicates that this function conception may have developed throughout their interactions with these materials.

Another theme that emerges in the pre-interviews of all three participant groups is *Algebraic Representation*. Sofia, Jamie, Gabby, Alan, and Henry each suggest that all functions can be represented algebraically, and the varied mathematical backgrounds of these PSMTs suggest that despite advanced mathematics courses, this conception of function may persist throughout undergraduate mathematics studies. The emergence and persistence of this function conception through undergraduate studies is supported by findings from Carlson (1998), Vinner and Dreyfus (1989), Meel (2003), Even (1993),

and Marmur and Zazkis (2019). Particularly, these studies report that college students, prospective teachers, and practicing teachers believe all functions can be defined algebraically or by a single formula. The varied mathematical backgrounds of the PSMTs in this study as well as other studies indicate that other undergraduate mathematics courses may not address this function conception in a way that alters students' function concept images.

Defining Characteristics is a theme common across the post-interviews of participants representing all three mathematical background groups: Lee, Jamie, Gabby, Alan, and Michael. This theme is characterized by statements that differentiate function and equation based on their defining characteristics, and it is not a theme that emerged in any participant's pre-interview. Distinguishing function and equation in this way contrasts with Carlson's (1998) finding that second-semester calculus students experienced difficulty differentiating function and equation at all. Since this theme is apparent in five participants' post-interviews and the participants have varying mathematics backgrounds and experiences concurrent to this study, the emergence of this theme may be attributed to experiences within the course related to the EEFPSMT materials. Specifically, in Exploration 6.3 (Figure 5-4) of Lesson 6, students investigated whether the equation $y = 2x + 1$ arising from the intersection of the two planes defined by $f(x, y) = y$ and $g(x, y) = 2x + 1$ would also be a function. This exploration generated rich discussions about the defining characteristics of function and equation as well as their differences. Specifically, that functions are defined as relations on sets and the equal sign in an equation asserts equivalence between two quantities. It appears then that conversations brought out by this exploration and others contributed to the separation participants formed between function and equation in their concept images.

Although several similarities in the concept image themes are observed between participant groups, there are a few differences. One distinction between participant groups concerns the *One-to-One* and *Not One-to-One* themes. Every Group A and Group B participant identified in both interviews that relations do not need the one-to-one property in order to be a function. However, two of the three Group C participants, Henry and Alan, maintained that a relation should be one-to-one in order to be a function. The identification of this conception aligns with reports in the literature that students and PSMTs may confound the univalence property with the one-to-one characteristic (Breidenbach et al., 1992; Dubinsky & Wilson, 2013; Meel, 2003). It is surprising though that this conflation is present in the concept images of two participants who completed upper level courses such as *Real Analysis I* and not in the participants with less mathematical experience. However, Melhuish, Lew, Hicks, and Kandasamy (2109) also report that one of the six *Abstract Algebra* students in their study had not distinguished the univalence characteristic from the one-to-one property. Perhaps this conflation of univalence with the one-to-one property developed in Henry's and Alan's advanced mathematics courses, or perhaps this conception merely persisted throughout all of their undergraduate studies. In either case, the fact that the *One-to-One* theme persisted in both of their interviews suggests that within this study the participants did delve into a cognitive conflict related to this conception.

Three other differences in the concept image themes across participant groups relate to the themes *Relations on Sets*, *Input/Output*, and *Univalence*. Each of these themes only emerged in Group B and Group C participants' pre-interviews. As neither Group A participant indicated any of these conceptions in their pre-interviews, then perhaps these conceptions of function developed for the Group B and Group C participants in other areas of their undergraduate studies. Examining the post-interviews,

there is only one participant who exhibited the *Relations on Sets* theme in this interview that did not also reveal it in the pre-interview. Overall though, only Group B and Group C participants manifested this theme in either their pre- or post-interview. Perhaps the development of this function conception is supported by mathematics courses beyond the three-semester calculus sequence. The *Input/Output* and *Univalence* themes, however, did emerge in both Group A participants' post-interviews. Similarly, Zandieh et al. (2017) and Carlson (1998) found that undergraduate students view function as an input-output process, and Even (1993) reports that only about half of the PSMTs in her study include the univalence characteristic in their definitions of function. While the *Input/Output* theme is not entirely consistent with the formal definitions of function, the univalence characteristic certainly is. This suggests that function conceptions Group B and Group C participants established in their mathematics studies were developed by Group A participants with less mathematical experience during the course of this study.

5.3 Study Limitations

Limitations of this study are related to study participants, the task-based interview questions, and the identification of themes by prevalence.

5.3.1 Study Participants

Initially, I extended interview invitations to 12 students with hopes of recruiting participants of different genders for each of the mathematical background groups. Not all invited students chose to participate in the interviews, and between the pre-interview and the post-interview, I collected data on a total of seven participants' pre- and post-interview concept images. As only just over half of the initially invited students participated in both interviews, there was no gender diversity in Group B and Group C. Participants in Group B consisted of females while participants in Group C consisted of only males. In addition, the only participants who identified as native English speakers

were the Group A participants. I was then unable to capture or speculate on whether the course experiences and concept images would be different for males in Group B, females in Group C, native English speakers in Groups B and C, or for non-native English speakers in Group A. Although Group A did consist of a male and female participant, the small sample size did not lend itself to hypothesizing about the results in relation to gender differences. Overall, the lack of diversity and small sample sizes led to the inability to conjecture on any potential differences in participant concept images based on gender or native language.

5.3.2 Task-based Interview Questions

The task-based interview questions reveal much about the participants' concept images; however, they do not reveal whether or not covariation is a component of participants' function concept image. Covariational reasoning and a covariation view of function is a prominent component of current function research as these are important ideas for calculus (Carlson, Oehrtman, & Engelke, 2010; Thompson & Carlson, 2017). The EEFPSMT materials recognize this and include explorations on covariational reasoning. Question 8, the vase filling problem, of the task-based interview questions aims to examine participants' covariational reasoning, and the data could reveal findings similar to Thompson's and Carlson's (2017) levels of variational reasoning. Yet, this question does not disclose any indication that covariational reasoning is a component of participants' function concept images, and analyzing students' levels of variational reasoning is beyond the scope of this study. Also, no other task-based interview question reveals a covariation view of function is a component of any of the participants' function concept images.

While the idea of covarying quantities is paramount for understanding central ideas in calculus, the concept of covariation depends on functions defined from the set of

real numbers to the set of real numbers. Hence, thinking about two quantities covarying precludes functions from abstract sets to other abstract sets. Materials developed by the EEFPSMT project aim to enhance overall function conceptions of PSMTs which includes an understanding that functions can be arbitrary relations defined on arbitrary sets. Then, the EEFPSMT materials did not fixate on producing in participants a covariation view of function. As a result, the interview questions did not reveal this idea of function in participants' interviews nor did the questions intentionally try to lead participants to exhibit this particular function conception.

5.3.3 Theme Prevalence

Another limitation in this study arises from the way I identified the themes in this study. As I described in Chapter 4, the 19 themes might be thought of as subthemes and the five categories as the themes, but based upon my research questions that was not a salient way to tell the narrative. In addition, the themes were narrowed by prevalence. This narrowing by prevalence is explained in Chapter 3 as themes emerging in an interview of at least three participants' and additional themes that contrast directly with them. Recall though that an individual's function concept image consists of all their experiences, examples, nonexamples, and impressions related to function. Of course, every individual's concept image is then incredibly complex. Although the described narrowing of themes is acceptable under thematic analysis methodology (Braun & Clarke, 2006), it naturally will exclude function conceptions unique to one or two participants. There then may be interesting function conceptions unique to an individual that are not explored in this study.

5.4 Conclusion

Following the analysis of the coding themes and comparing the results between and among the participant groups, a couple hypotheses arose around the development of

function concept images and using the EEFPSMT materials in an undergraduate mathematics course. I will present these hypotheses and address the research questions that guided this study.

5.4.1 Implications

Analyzing the concept image themes across participant groups, students with differing mathematical backgrounds seemed to carry into this class beliefs that all functions are defined on numerical domains, are graphable, meet the requirements of the vertical line test, and can be represented algebraically. This suggests that undergraduate mathematics courses including advanced courses such as *Abstract Algebra I* and *Real Analysis I* may not affect these particular conceptions of function. However, I observed the appearance of seemingly contrasting concept image themes in these same participants' post-interviews. Although the participants had varying prior and concurrent mathematical experiences, the commonality is the interactions all the participants shared in relation to the EEFPSMT materials. I hypothesize these contrasting themes may have developed as a result of students' course interactions with the research-based explorations designed in the EEFPSMT materials to elicit function-related cognitive conflicts.

Furthermore, I suggest these conceptions that seemingly developed during participants' interaction with the EEFPSMT materials are not dependent on the fact that participants are PSMTs. In fact, researchers designed the explorations based on literature-reported function conceptions of PSMTs, practicing teachers, and undergraduate students. This is supported by the fact that participants who engaged with the EEFPSMT materials represented varying levels of the undergraduate experiences – some participants starting their program and others preparing for their student-teaching placement. Yet, when comparing the pre- and post-interview concept image themes

between participant groups, I observed no apparent difference in the concept-definition-consistency of participants' function concept image. I also identified the emergence of several consistent themes in the post-interviews across participant groups that were not common in participants' pre-interviews: *Non-Numerical Domain*, *VLT Not Defining Characteristic*, *Graph Not Defining Characteristic*, and *Defining Characteristics*. I propose that the explorations and tasks within the EEFPSMT materials could be implemented in a range of undergraduate mathematics courses, outside of those specifically for PSMTs, and potentially aid the development of a function concept image consistent with the definition.

5.4.2 Research Questions

The first research question I address asks how PSMTs' concept images of function change when they engage with research-based tasks and explorations designed to elicit cognitive conflicts related to function conceptions. While each individual's function concept image is incredibly complex and each participant exhibited specific concept image shifts, I observed similarities across the pre- and post-interviews of participants with varying mathematical backgrounds. *Numerical Domain*, *VLT*, *Graphable*, and *Algebraic Representation* are themes common across the pre-interviews of participants (Figure 5-11). These findings are consistent with other studies involving undergraduate students and prospective and practicing secondary mathematics teachers (e.g. Carlson, 1998; Even, 1993; Norman, 1992; Stein et al., 1990; Vinner & Dreyfus, 1989). However, these conceptions are not common among participants' post-interviews (Figure 5-11). Instead, I identified *Non-Numerical Domain*, *VLT Not Defining Characteristic*, *Graph Not Defining Characteristic*, and *Defining Characteristics* which contrast with many of conceptions reported in the pre-interview and related literature. This suggests a general

change in participants' function conceptions related to the types of sets functions are defined on, the vertical line test, graphs, and equations.

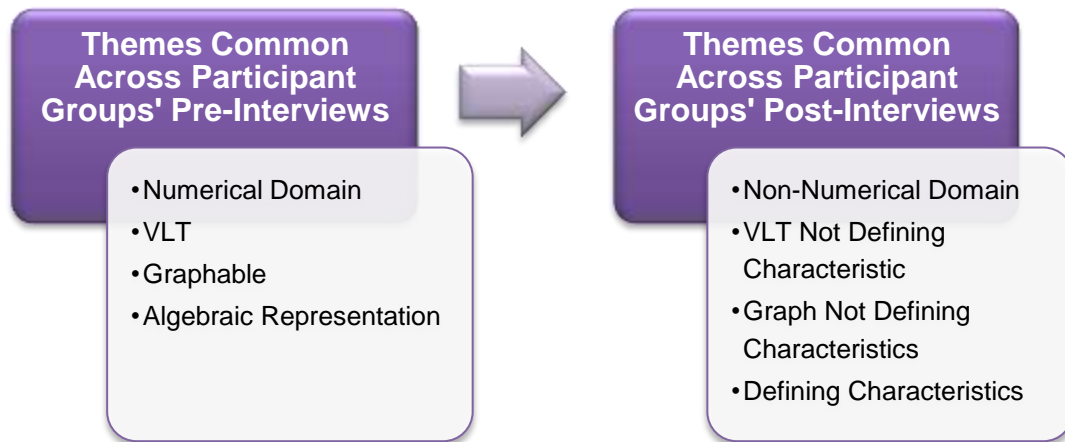


Figure 5-11 Overview of Themes Common across Participants' Pre- and Post-Interviews

The second research question I address focuses on the differences in the function concept images developed by PSMTs who have completed upper-level mathematics courses and PSMTs who have only taken undergraduate calculus courses. Overall, there is no apparent difference across participant groups in the number of pre- or post-interview consistent and inconsistent function concept image themes. All but one participant actually exhibited more inconsistent conceptions in the pre-interview than consistent conceptions. The number of inconsistent conceptions held by students with several advanced mathematics courses and the findings that teachers' conceptions influence their teachings (e.g. Even, 1993; Stein et al., 1990; Watson & Harel; 2013) may indicate why Monk's (1994) findings that courses beyond the first five undergraduate mathematics courses make the most impact on student achievement since more

advanced courses did not seem to influence acquisition of a concept-definition-consistent concept image for these participants. These findings also contrast with Carlson (1998) report that full, function concept development “appears to evolve over a period of years” (p. 143). In this study, all participants exhibited an increase in the number consistent themes, and the number of consistent themes surpassed the number of inconsistent themes in the post-interview over the course of a single semester. This suggests that participants with various mathematical backgrounds were able to release and adjust inconsistent function conceptions to form an overall function concept image more consistent with the definitions of function and equation.

Furthermore, there appears to be some sort of leveling effect in the function conceptions of participants regardless of their mathematical backgrounds as all participants exhibited a similar number of consistent, inconsistent, and neutral themes in the post-interview. Since a commonality in all participants’ experiences between the pre- and post-interview is their interaction with the EEFPSMT materials, these interactions may be what contributed to this observed leveling. This finding may also indicate why Monk (1994) reports that additional mathematics education courses for PSMTs had a positive effect on student achievement when more than five mathematics courses did not, and why the MET II report (2012) recommends mathematics courses for prospective teachers that develop a solid understanding of the mathematics they will teach.

Although there is no apparent difference in the overall consistency of participants’ function conceptions with the definition of function within the pre- or post-interviews, there are some specific differences in the actual themes across participant groups. One identifiable difference is that the univalence property is not identifiable in the concept images of participants who, at the start of the study, only completed courses in the calculus sequence. Vinner and Dreyfus (1989) similarly report that the majority of first-

year college students did not include univalence in their definitions of function. Group A participants were not first-year college students. However, if univalence is not a conception brought in by first-year college students, it seems that their experiences in calculus may not help them develop this conception. Of the five participants who entered the study with mathematical experience beyond a three-semester calculus sequence, four participants revealed that univalence is a component of their pre-interview concept image. This suggests that students may not incorporate univalence as a defining characteristic into their function concept images until sometime after the completion of a calculus sequence. Carlson (1998) similarly reports that beginning graduate students possessed an understanding of major aspects of function where second-semester calculus students did not. Two courses common among all Group B and Group C participants in this study that follows the completion of the calculus sequence are Introduction to Proofs and Discrete Mathematics. Perhaps in some of the proof-writing experiences embedded in these courses, students attend to univalence as a defining characteristic of function, and this helps them incorporate univalence into their concept images.

Another observable difference between participants with varying mathematical backgrounds relates to the one-to-one property. All the participants who had not completed *Abstract Algebra I* or *Real Analysis I* at the start of the study asserted in both interviews that the one-to-one property is not a defining characteristic of function. However, two of the three participants who completed either *Abstract Algebra I* or *Real Analysis I* prior to the start of the study maintained through both interviews that the one-to-one property is a defining characteristic of function. Breidenbach et al. (1992), Dubinsky and Wilson (2013), and Meel (2003) all report a conflation between the univalence characteristic and the one-to-one property in students. It is surprising though

that this phenomenon is observed in the students with the most mathematical experience rather than those with less experience. Perhaps then, this is a conception that actually developed in these advanced mathematical courses, like *Abstract Algebra* and *Real Analysis*.

In conclusion, similar to Carlson's (1998) findings regarding the development of the function concept, it does not appear that advanced undergraduate mathematics courses cultivate a function concept image more consistent with the concept definition. Rather, I found that, for undergraduates at various stages in their degree program experiences with the researched based task and explorations in the EEFPSMT materials may promote concept-definition-consistent function conceptions and combat concept-definition-inconsistent conceptions. There is also an observable shift across all participant groups in conceptions about the defining characteristics of function including the vertical line test, numerical domains, and graphical and algebraic representations. While these identifiable changes in participants' function concept image suggest that classroom experiences positively affected their concept images, future research is needed to characterize these types of experiences as well as how to interpret whether or not a course experience contributes or detracts from students' function concept image. The similarities in these function concept image shifts among participants with varying mathematical backgrounds also warrant further investigation into the transferability of these course experiences to other undergraduate mathematics courses.

Appendix A
Pre-Interview Protocol

Thank you very much for meeting with me today! My main goal is to learn more about your thoughts on some of the mathematical ideas in this course. I am going to ask you some questions that are related to ideas we've talked about in this course. I will also ask some questions about your perception of the class learning environment. This process will take about one hour. Do you have any questions before we begin?

[Researcher will state participant's identifier for the recording]

Please use the paper to help answer and explain any of these questions. [For each question below, the researcher will use nondirective follow-up questions to ask the participants to clarify or extend their responses.]

1. How would you define **function**?

[Researcher may ask follow-up questions]

- What do you think of when you hear the word function?

2. I'm going to ask you to list all the defining characteristics of a function, but before I ask you that, I want to know when I say defining characteristics what does that mean?

List all the distinguishing characteristics of a function that you can think of.

[Researcher may ask follow-up questions]

- When I say a "defining characteristic", what does that mean?
- What are some characteristics that all functions have?

3. Please give three examples of functions. For each, please explain how you know it is a function

[Researcher may ask follow-up questions]

- Can you give an example of a function whose domain is not the real numbers or a subset of the real numbers?

4. Which of the following are valid mathematical definitions of *function*? For each part, explain why it is or is not a valid definition.

a. A function from a set A (the domain) to a set B (the codomain) is a rule or correspondence that assigns exactly one element of the codomain to each distinct element of the domain.

b. A function is an equation that gives a particular relationship between two quantities.

c. A function is a graph that passes the vertical line test.

d. A function is a relation for which every element of the range corresponds to exactly one element of the domain.

e. A function is a relation from A to B such that each element of A is assigned to a unique element in B.

[Researcher may ask follow-up questions]

- A valid mathematical definition can be applied to any example you call a function.

5. In your own words, define **equation**.

[Researcher may ask follow-up questions]

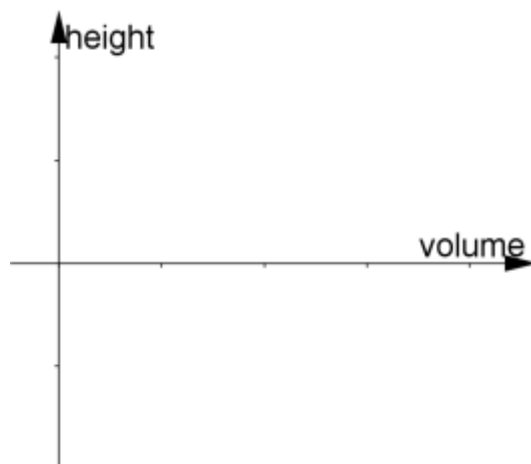
- Can you give me some examples of equations?
- Can you give an example of an equation that doesn't have any unknowns?
- Some people say that anything with an equal sign is an equation. How would you respond to that?

6. Do the mathematical terms *function* and *equation* mean the same thing? Explain your answer.

[Researcher may ask follow-up questions]

- Are there functions that cannot be represented by an equation?

- Are there equations that do not also represent a function?
7. In this example, discuss whether the word function is used correctly.
 Example: “Find the x-intercept and the y-intercept of the function $2x - y = 8$.”
 [Researcher may ask follow-up questions]
- This problem is stating that $2x - y = 8$ is a function. Do you agree with this?
 - When I ask you to find the x-intercept and y-intercept of the function, is that a correct use of the word function?
 - Is a function the same as the graph of a function?
8. Consider the vase below being filled with water at a constant rate. Sketch a graph of height vs. volume as it is being filled. Explain your reasoning.



- [Researcher may ask follow-up questions]
- Why is your graph always “rising”?
 - Why does your graph have this shape?
 - What happens when the curve of the vase changes?
 - Is knowing the rate at which the water is filling the vase is an important factor needed to draw graph?

Appendix B
Post-Interview Protocol

Thank you very much for meeting with me today! My main goal is to learn more about your thoughts on some of the mathematical ideas in this course. I am going to ask you some questions that are related to ideas we've talked about in this course. I will also ask some questions about your perception of the class learning environment. This process will take about one hour. Do you have any questions before we begin?

[Researcher will state participant's identifier for the recording]

Please use the paper to help answer and explain any of these questions. [For each question below, the researcher will use nondirective follow-up questions to ask the participants to clarify or extend their responses.]

2. How would you define **function**?

[Researcher may ask follow-up questions]

- What do you think of when you hear the word function?

3. I'm going to ask you to list all the defining characteristics of a function, but before I ask you that, I want to know when I say defining characteristics what does that mean?

List all the distinguishing characteristics of a function that you can think of.

[Researcher may ask follow-up questions]

- When I say a "defining characteristic", what does that mean?
- What are some characteristics that all functions have?

4. Please give three examples of functions. For each, please explain how you know it is a function

[Researcher may ask follow-up questions]

- Can you give an example of a function whose domain is not the real numbers or a subset of the real numbers?

5. Which of the following are valid mathematical definitions of *function*? For each part, explain why it is or is not a valid definition.

b. A function from a set A (the domain) to a set B (the codomain) is a rule or correspondence that assigns exactly one element of the codomain to each distinct element of the domain.

c. A function is an equation that gives a particular relationship between two quantities.

d. A function is a graph that passes the vertical line test.

e. A function is a relation for which every element of the range corresponds to exactly one element of the domain.

f. A function is a relation from A to B such that each element of A is assigned to a unique element in B.

[Researcher may ask follow-up questions]

- A valid mathematical definition can be applied to any example you call a function.

6. In your own words, define **equation**.

[Researcher may ask follow-up questions]

- Can you give me some examples of equations?
- Can you give an example of an equation that doesn't have any unknowns?
- Some people say that anything with an equal sign is an equation. How would you respond to that?

7. Do the mathematical terms *function* and *equation* mean the same thing? Explain your answer.

[Researcher may ask follow-up questions]

- Are there functions that cannot be represented by an equation?
- Are there equations that do not also represent a function?

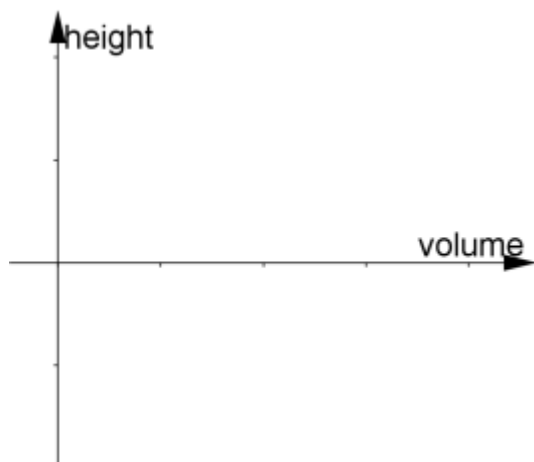
8. In this example, discuss whether the word function is used correctly.

Example: “Find the x-intercept and the y-intercept of the function $2x - y = 8$.”

[Researcher may ask follow-up questions]

- This problem is stating that $2x - y = 8$ is a function. Do you agree with this?
- When I ask you to find the x-intercept and y-intercept of the function, is that a correct use of the word function?
- Is a function the same as the graph of a function?

9. Consider the vase below being filled with water at a constant rate. Sketch a graph of height vs. volume as it is being filled. Explain your reasoning.



[Researcher may ask follow-up questions]

- Why is your graph always “rising”?
- Why does your graph have this shape?
- What happens when the curve of the vase changes?
- Is knowing the rate at which the water is filling the vase an important factor needed to draw graph?

Now I’m interested in your perception of the class learning environment in MATH 2330. [For each question below, researcher will ask questions to follow-up questions asking the participant to clarify or extend their responses.]

1. What do you think about the format of this course? Is this format something you would integrate into your own teaching?

- a. In particular, the inquiry format of the course.
 - b. Did you see connections between the homework, the in-class work, and the exam? Explain.
 - c. How has the class format influenced your ideas about persistence and problem solving?
 - d. How have the class explorations influenced your ideas about persistence and problem solving?
2. What features of the class format or environment do you think enhance your learning? Please explain.
- a. One of the things we're interested in is developing notes and lesson plans for future faculty teaching this course. With this in mind, is there anything your instructor did that enhanced your learning? Explain.
3. What features of the class format or environment do you think detract from your learning? Please explain.
- a. One of the things we're interested in is developing notes and lesson plans for future faculty teaching this course. With this in mind, is there anything your instructor did that detracted from your learning? Explain.

That's all the questions I have for today. Is there anything else you would like add about your thinking on the questions I asked or your experience in this course?

Appendix C
Participant Self-Report Form

Gender: Male Female Prefer not to report

Class: Freshman Sophomore Junior Senior Graduate

Ethnicity: Native American African American Latino/a Caucasian
 Asian Multiracial (Specify: _____)
 Other: _____

Native Language: English Spanish Other: _____

Year you graduated from high school: _____

Please mark all math courses you have completed prior to the Fall 2018 semester:

- | | |
|--|---|
| <input type="checkbox"/> CALCULUS I (MATH 1426 or equivalent) | <input type="checkbox"/> ABSTRACT ALGEBRA I (MATH 3321) |
| <input type="checkbox"/> CALCULUS II (MATH 2425 or equivalent) | <input type="checkbox"/> ABSTRACT ALGEBRA II (MATH 4321) |
| <input type="checkbox"/> CALCULUS III (MATH 2326 or equivalent) | <input type="checkbox"/> ANALYSIS I (MATH 3335) |
| <input type="checkbox"/> INTRODCUTION TO PROOFS (MATH 3300) | <input type="checkbox"/> ANALYSIS II (MATH 4335) |
| <input type="checkbox"/> STATISTICAL INFERENCE (MATH 3316) | <input type="checkbox"/> DISCRETE MATHEMATICS (MATH 3314) |
| <input type="checkbox"/> ELEMENTARY NUMBER THEORY (MATH 3307)
3301) | <input type="checkbox"/> FOUNDATIONS OF GEOMETRY (MATH
3301) |
| <input type="checkbox"/> INTRODUCTION TO MATRICES AND LINEAR ALGEBRA (MATH 3330) | |
| <input type="checkbox"/> ADVANCED MULTIVARIABLE CALCULUS (MATH 4321) | |
| <input type="checkbox"/> Other: _____ | |

Please mark all math courses you currently taking along with Functions and Modeling:

- | | |
|--|---|
| <input type="checkbox"/> CALCULUS I (MATH 1426 or equivalent) | <input type="checkbox"/> ABSTRACT ALGEBRA I (MATH 3321) |
| <input type="checkbox"/> CALCULUS II (MATH 2425 or equivalent) | <input type="checkbox"/> ABSTRACT ALGEBRA II (MATH 4321) |
| <input type="checkbox"/> CALCULUS III (MATH 2326 or equivalent) | <input type="checkbox"/> ANALYSIS I (MATH 3335) |
| <input type="checkbox"/> INTRODCUTION TO PROOFS (MATH 3300) | <input type="checkbox"/> ANALYSIS II (MATH 4335) |
| <input type="checkbox"/> STATISTICAL INFERENCE (MATH 3316) | <input type="checkbox"/> DISCRETE MATHEMATICS (MATH 3314) |
| <input type="checkbox"/> ELEMENTARY NUMBER THEORY (MATH 3307)
3301) | <input type="checkbox"/> FOUNDATIONS OF GEOMETRY (MATH
3301) |
| <input type="checkbox"/> INTRODUCTION TO MATRICES AND LINEAR ALGEBRA (MATH 3330) | |
| <input type="checkbox"/> ADVANCED MULTIVARIABLE CALCULUS (MATH 4321) | |
| <input type="checkbox"/> Other: _____ | |

Appendix D

Lesson 5: What is a Function?

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Lesson 5:

What is a function?

The concept of *function* is foundational to high school algebra as well as higher-level mathematics. However, functions are not straightforward to teach or to learn. The purpose of this lesson is to look deeper into the definition of function.

Exploration 5.1: Characteristics of Functions

What comes to mind when you hear the word *function*? Working in groups, list as many characteristics of functions as you can, based on your previous mathematical experience.

- a. Can the same function be represented in more than one way?
- b. Do all functions have a graphical representation? Why or why not?
- c. Are all your examples that come to mind relating two numerical sets? If so, is it necessary for the sets to be sets of numbers?

How are the various characteristics related? Which characteristics also apply to other mathematical concepts? Which are characteristics that are true of *all* functions? We call these *defining* characteristics.

Exploration 5.2: Examples of Functions

- a. Working in groups, use the examples below and your prior knowledge about functions to determine which of those relationships are *functions*. Justify your answers, being certain to explain which set you are considering as the input.
 - i. From Exploration 3.1: The relation between Secret Agent Cody's distance from the safety line (d_1) and the log on his pedometer (d_2).
 - ii. From Exploration 4.1 #3: The relation between the orbital period of a planet and its mean distance from the sun.
 - iii. The relation between the daily high temperature in Fahrenheit and the same day's high temperature in Celsius.
 - iv. The relation that pairs each university student's age (in whole years) with his/ her university ID number.
 - v. The relation between the diameter of a circle and its circumference.
 - vi. The relation between each person and his/ her birthday.
 - vii. The relation between finishing time at the 2017 Boston marathon and the persons who finish at that time.
- b. Your homework was to bring examples of relationships between quantities in the real world. Repeat part (a) with the examples you brought to class.
- c. In what contexts or situations might functions be more useful than other relations?

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Exploration 5.3: Various Definitions

Part 1:

Consider the various definitions of function given below. Each of these definitions comes from a mathematics textbook. Discuss the following questions in your group:

- Compare and contrast these definitions.
- For each definition, what mental image of function do you have after reading it?
- For what purpose would each definition be appropriate?

Textbook Definitions:

- A function is a relationship between two variables (an independent variable and a dependent variable) for which every value of the independent variable has at most one value of the dependent variable.
From *Discovering Advanced Algebra* [High school text]; Murdock, Kamischke, & Kamischke, 2010, p. 190.
- A function is a rule that assigns to each value x in a set D a unique value denoted $f(x)$. The set D is the domain of the function. The range is the set of all values of $f(x)$ produced as x varies over the entire domain.
From *Calculus*; Briggs, Cochran, & Gillet, 2015, p. 1.
- A relation is a set of ordered pairs. A function is a relation in which, for each distinct value of the first component of the ordered pairs, there is exactly one value of the second component.
From *College Algebra*; Lial, Hornsby, Schneider, & Daniels, 2017, p. 209.
- A function is a set of ordered pairs in which no first component is repeated.
From *College Algebra*; Lial, Hornsby, Schneider, & Daniels, 2017, p. 209.
- A function is a rule or correspondence that assigns exactly one range value to each distinct domain value.
From *College Algebra*; Lial, Hornsby, Schneider, & Daniels, 2017, p. 209.
- A function is a rule that assigns to each element of a set A a unique element of a set B (where B may or may not equal A). The set A is called the domain of the function f , the set B the codomain, and the subset of the second set B consisting of those elements that are images under the function f of some element of its domain is called the range of the function f .
From *Mathematics for High School Teachers*; Usiskin, Peressini, Marchisotto, & Stanley, 2003, p. 68.
- A function $f: X \rightarrow Y$ is a relation R from X to Y with the property that for every $x \in X$ there is a unique $y \in Y$ such that $(x, y) \in R$, in which case we write $y = f(x)$.
From *Real Analysis*; Folland, 1999, p. 3.
- The Cartesian product of two sets S and T , denoted $S \times T$, is the set of all ordered pairs (s, t) such that $s \in S$ and $t \in T$. For any sets X and Y , a function f from X to Y , $f: X \rightarrow Y$, is a subset f of the Cartesian product $X \times Y$ such that every $x \in X$ appears once and only once as a first element of an ordered pair (x, y) in f .
From *Mathematics for High School Teachers*; Usiskin, Peressini, Marchisotto, & Stanley, 2003.
- A function, or map, of a set S into a set T consists of the set S , called the domain of the map, the set T , called the co-domain, and a subset α of $S \times T$ (the graph) having the following two properties:
 - For any $s \in S$, there exists a $t \in T$ such that $(s, t) \in \alpha$.
 - If (s, t) and $(s, t') \in \alpha$ then $t = t'$
From *Basic Algebra* [Abstract Algebra text]; Jacobson, 1985, p. 5.

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Exploration 5.3: Various Definitions

Part 2:

Suppose you asked high school students to define *function*, and they provided the following. For each, discuss the following questions in your group:

- a. Is the student's attempt a valid definition that works for *all* functions?
- b. What does the student's attempt indicate about his or her thinking and experience with functions?

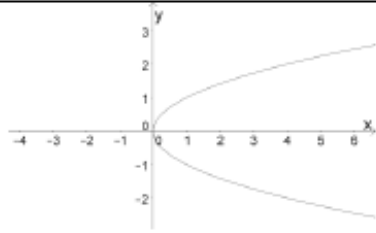
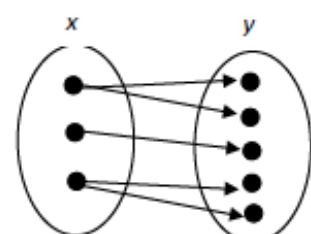
Student Attempts to Define Function:

1. A function passes the vertical line test.
2. A function is an equation with two variables (usually called x and y).
3. In a function, no x -value is repeated.
4. A function is a rule where each input goes to a unique output.
5. A function is a relationship between two variables.

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Exploration 5.4: Function Identification Activity 1 (Adapted from Armendariz & Daniels, pp. 11 - 12)

In your groups, decide which of the following relations are functions. Justify your answers.

	Relation	Function? Yes or No, and Justification										
1	$y = -x^2$											
2	<table border="1" style="margin-left: auto; margin-right: auto;"> <thead> <tr> <th>x</th> <th>y</th> </tr> </thead> <tbody> <tr> <td>2</td> <td>4</td> </tr> <tr> <td>-6</td> <td>-12</td> </tr> <tr> <td>13</td> <td>26</td> </tr> <tr> <td>-57</td> <td>-114</td> </tr> </tbody> </table>	x	y	2	4	-6	-12	13	26	-57	-114	
x	y											
2	4											
-6	-12											
13	26											
-57	-114											
3	(5, 6) (3, 2) (5, 1)											
4												
5	$y^4 = 8x^2$	Also Identify: What does the graph of this relation look like?										
6												
7	Tom: Blue Jill: Brown Bill: Green Harry: Green											

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Exploration 5.5: Function Identification Activity 2 (Adapted from Armendariz & Daniels, p. 13)

Work through the following questions in your groups.

- Let $A = \{a, b, c\}$, $B = \{4, 5, 6\}$, and $f = \{(a, 6), (b, 4), (c, 6)\}$.
 - Is f a function from A to B ? Explain.
 - If f is a function from A to B , give a relation that is not a function from A to B . If f is not a function from A to B , give a relation that is a function from A to B . Explain.
- Let $A = \{1, 2, 3\}$, $B = \{c, d, e\}$, and $g = \{(1, d), (2, c), (1, e)\}$.
 - Is g a function from A to B ? Explain.
 - If g is a function from A to B , give a relation that is not a function from A to B . If g is not a function from A to B , give a relation that is a function from A to B . Explain.
- Let M be the set of all museums, N the set of all countries, and
$$L = \{(m, n) \in M \times N \mid \text{the museum } m \text{ is in the country } n\}.$$
 - Is L a function from M to N ? Explain.
 - If L is a function from M to N , give a relation that is not a function from M to N . If L is not a function from M to N , give a relation that is a function from M to N . Explain.
- Let D be the set of all dogs, and let
$$C = \{(d, o) \in D \times D \mid \text{the dog } d \text{ is a parent of the offspring } o\}$$
 - Is C a function from D to D ? Explain.
 - If C is a function from D to D , give a relation that is not a function from D to D . If C is not a function from D to D , give a relation that is a function from D to D . Explain.

Exploration 5.6: Mapping a Map (From Armendariz & Daniels, p. 12)

Find a map of your choice. This could be a campus map, a local street map, or state map, for example. Use any the information supplied in the map or any subset of the information provided to create a function. Be sure to explain your reasoning behind the creation of your function.

Historical Note (From Armendariz & Daniels, p. 16)

Although the notion of a *function* dates back to the seventeenth century, a relation-based definition as we use today was not formulated until the beginning of the twentieth century. The concept of mathematical relations first appears in the text *Geometry*, written by Rene Descartes in 1637, and the term “function” was introduced about fifty years later by Gottfried Wilhelm Leibniz. It was Leonhard Euler, in the eighteenth century, who first used today’s notation $y = f(x)$. Finally, it was Hardy who, in 1908, defined a function as a relation between two variables x and y such that “to some values of x at any rate correspond to values of y .”

Álvarez, J.A.M., Jorgensen, T., & Rhoads, K. (2018) Enhancing Explorations in Functions for Preservice Secondary Mathematics Teachers Project, The University of Texas at Arlington.

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Principal Investigator: Dr. James A. Mendoza Álvarez; Co-PIs: Dr. Theresa Jorgensen & Dr. Kathryn Rhoads

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Appendix E

Lesson 6: Functions and Equations

Álvarez, J.A.M., Jorgensen, T., & Rhoads, K. (2018) Enhancing Explorations in Functions for Preservice Secondary Mathematics Teachers Project, The University of Texas at Arlington.

Lesson 6:

Functions and Equations

In school mathematics teaching, using vocabulary in mathematically precise ways is important because there are ambiguities in or common uses of certain terms that can cause confusion when students are confronted with the implied or informal meanings in a mathematical situation. This lesson focuses on the meaning of the term *equation* and the different meanings associated with the use of the equal sign. Carpenter, Franke, & Levi (2003) assert that a "limited conception of what the equal sign means is one of the major stumbling blocks in learning algebra" (p. 22).

Exploration 6.1: What is an equation?

1. When you hear the word equation, write down what immediately comes to your mind.
 - a. Each group member should write down, three examples of equations.
 - b. Share your examples with your group members. Create a group list of examples for which there is consensus (note that you may throw some examples out after considering duplicates or there may be examples offered that not everyone agrees are equations).

Álvarez, J.A.M., Jorgensen, T., & Rhoads, K. (2018) Enhancing Explorations in Functions for Preservice Secondary Mathematics Teachers Project, The University of Texas at Arlington.

2. Consider the following definition of *equation*: An equation is a mathematical statement that asserts the equivalence between two quantities.
 - a. Which examples from the group list generated in part 1(b) above would be considered equations according to this definition? Explain your reasoning.
 - b. According to the definition provided, is $1 + 3 = 4$ an equation? Explain why or why not.
 - c. According to the definition provided, is $1 + 3 = 4 = 11 - 7$ an equation? Explain why or why not.

Exploration 6.2: Constructed meanings of the equal sign

1. The use of the equal sign evokes several constructed meanings that can cause some confusion, in particular in K-12 mathematics (e.g. Knuth, Stephens, McNeil, & Alibali, 2006; Kieran, 1981). Identify and discuss the meanings that the following uses convey:

	Meaning of "="	Give another example conveying the same meaning.
a. $3 + 5 = \underline{\quad}$		
b. $f(x) = x^2 + 5$		
c. $\frac{d}{dx}(e^{x^2} + 4x) = \underline{\quad}$		
d. $A = \pi r^2$		
e. $2 \sin x \cos x = \sin 2x$		
f. $g(x) = \cos x + x$		
g. $x^2 + 3x = x - 1$		

2. Consider the following situations arising from students' work.
 - a. David has no problem with computations such as $3 + 5 = \underline{\quad}$, but has trouble solving $3 + 5 = 2 + \underline{\quad}$. David may have a limited understanding of the use of the equal sign. Which meaning may David be missing? How would you know?
 - b. David (from part a) is given the following word problem.

Jim has 2 apples and Mary has 3 apples. How many apples do Jim and Mary have all together? Kim comes along with 6 oranges, how many pieces of fruit do Mary, Jim, and Kim have all together?

David writes:

$$2 + 3 = 5 + 6 = \boxed{11} \text{ The answer is 11 pieces of fruit.}$$

Discuss David's work and any connections to possible issues regarding the meaning of the equal sign he displayed in part (a).

Álvarez, J.A.M., Jorgensen, T., & Rhoads, K. (2018) Enhancing Explorations in Functions for Preservice Secondary Mathematics Teachers Project, The University of Texas at Arlington.

- c. Ashton is given the following exercise: Find $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$.

Ashton shows the following work:

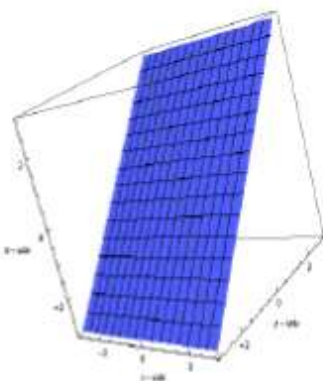
$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 = \boxed{2}$$

Comment upon Ashton's work shown. References to the definition of equation and the various meanings of the equal sign should be included in your commentary.

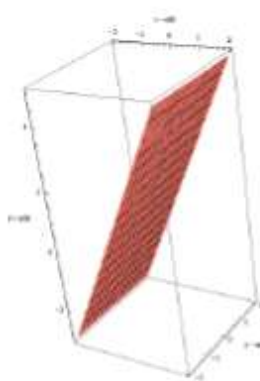
Exploration 6.3: Function or equation?

- Consider the functions $h(x) = x^2 - 3$ and $r(x) = 2x$.
 - What is the meaning of $h(x) = r(x)$? Explain.
 - Discuss why $x^2 - 3 = 2x$ is an equation. Is it also a function?
- Now consider the functions $f(x, y) = y$ and $g(x, y) = 2x + 1$. The graphs of f and g are provided below for reference.

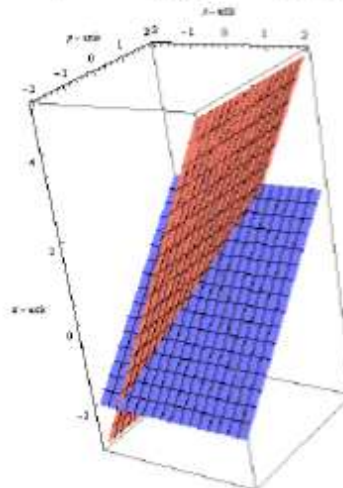
Graph of $z = f(x, y)$



Graph of $z = g(x, y)$



Graphs of $z = f(x, y)$ and $z = g(x, y)$



- What is the meaning of $f(x, y) = g(x, y)$? Explain.
- Discuss why $y = 2x + 1$ is an equation (part of your discussion should involve the definition of equation). Is it also a function? Explain.

Álvarez, J.A.M., Jorgensen, T., & Rhoads, K. (2018) Enhancing Explorations in Functions for Preservice Secondary Mathematics Teachers Project, The University of Texas at Arlington.

Exploration 6.4: Applying understanding of functions and equations to teaching

Mr. Smith is teaching a high school mathematics class and is writing a few homework and assessment questions as he plans the school year. Review Mr. Smith's questions and circle the most appropriate word(s) he should use in the question.

Consider the following:

- i. Evaluate/Simplify/Solve $f(x) = 2\sqrt{x} + 3$ when $x = 9$.
 - ii. Given functions g and h , evaluate/simplify/solve $g(x) = h(x)$.
 - iii. Evaluate/Simplify/Solve $5x + 2$ when $x = 2$.
 - iv. Evaluate/Simplify/Solve $2(x^2 + x + 1) - 5(x^2 + x) + \pi x^2$.
 - v. Evaluate/Simplify/Solve $2x^2 + 3x = 5x + 2$.
 - vi. Evaluate/Simplify/Solve $(\cos^2 x)(\sin 2x) + 2(\sin^3 x)(\cos x) - \cos x$
 - vii. Find $h(3)$ by evaluating/simplifying/solving h when $x = 3$.
- a. For which (i-vii), did your group decide there was more than one appropriate instruction? Explain.
 - b. Give an example like those above in which the term "evaluate" is used incorrectly or is problematic. Explain your reasoning.
 - c. Give an example like those above in which the term "solve" is used incorrectly or is problematic. Explain your reasoning.
 - d. Give an example like those above in which the term "simplify" is used incorrectly or is problematic. Explain your reasoning.
 - e. Create guidelines that Mr. Smith can use for determining when it is appropriate to use the instructions "solve," "evaluate," or "simplify" on his homework assignments and assessments. Your guidelines should address exercises or tasks involving functions and equations.

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