BASES OF INFINITE-DIMENSIONAL REPRESENTATIONS OF ORTHOSYMPLECTIC LIE SUPERALGEBRAS

by

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ABSTRACT

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We provide explicit bases of representations of the Lie superalgebra $\mathfrak{osp}(1|2n)$ obtained by taking tensor products of infinite-dimensional representation and the standard representation. This infinite-dimensional representation is the space of polynomials $\mathbb{C}[x_1, \ldots, x_n]$. Also, we provide a new differential operator realization of $\mathfrak{osp}(1|2n)$ in terms of differential operators of n commuting variables $x_1, \ldots, x_n$ and $2n$ anti-commuting variables $\xi_1, \ldots, \xi_{2n}$.

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CHAPTER 1

Introduction

The current work is an investigation concerning the infinite-dimensional super representation theory of the family of complex orthosymplectic Lie superalgebras $\mathfrak{osp}(1|2n, \mathbb{C})$. The results relate broadly to the algebraic corners of supermathematics, which has both mathematical and physical foundations.

1.1 History of Supermathematics

The popularization of supermathematics is largely due to the writings of Felix Berezin. Posthumously, Berezin's manuscripts were translated into English from Russian and his colleagues edited [3], in which there is an extensive account of supermathematics with the mathematician in mind. The mathematical emphasis is important as supermathematics is a term credited to physicists and much of the motivation for the theory stems from physical problems. Algebraically, the formulation of supermathematics involves heavily the study of objects in the symmetric monoidal category of $\mathbb{Z}_2$-graded vector spaces adhering to a twisted braiding. That is, the category with super vector spaces as objects and parity-preserving linear maps as morphisms has a braiding on the tensor product related to the Koszul sign rule such that exchanging odd elements results in negation. Super vector spaces underlie (supercommutative and associative) superalgebras. As in the classical case, an important example of superalgebras arises from considering functions on a space, one now having both commutative and anti-commutative coordinates or values in a Grassmann algebra as described in [2]. A sheaf-theoretic viewpoint uses these superfunctions to define supermanifolds and Lie supergroups. Appropriately, associated to a Lie supergroup is a Lie superalgebra, however the present discussion will align with a far more algebraic approach to Lie superalgebras and superalgebras.

1.2 Superalgebras

In 1941, Whitehead defined a product on the graded homotopy groups of a pointed topological space, the first non-trivial example of a Lie superalgebra. Whitehead's work in algebraic topology [41] was known to mathematicians who defined new graded geo-algebraic structures, such as $\mathbb{Z}_2$-graded algebras, or superalgebras to physicists, and their modules. A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ would come to be a $\mathbb{Z}_2$-graded vector space, even (espectively, odd) elements found in $g_{\bar{0}}$ (respectively, $\mathfrak{g}_{\bar{1}}$), with a parity-respecting bilinear multiplication $[\cdot, \cdot]$, the Lie superbracket, that satisfies $[x, [x, x]] = 0$ for all $x \in \mathfrak{g}_{\bar{1}}$ and induces a symmetric intertwining map $\mathfrak{g}_{\bar{1}} \otimes \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ of $(\mathfrak{g}_{\bar{0}}, [\cdot, \cdot]\big|_{\mathfrak{g}_{\bar{0}}})$-modules [5]. Researchers of the 1960s and 1970s furthered the systematic study of Lie superalgebras with a view for supermanifolds and the need for physicists to extend the classical symmetry formulation behind Wigner's Nobel Prize to one incorporating bosonic ("even") and fermionic ("odd") particles simultaneously—the famed supersymmetry (SUSY) [see 12, 44]. As mentioned above, supermathematics has become synonymous with the study of $\mathbb{Z}_2$-graded structures and spaces with Grassmann-valued coordinates. Now the algebraic development of supermathematics in terms of Lie superalgebras has become its own source of motivation—even providing a dictionary [15]—beginning with the classification of simple finite-dimensional Lie superalgebras over algebraically closed fields of characteristic zero in [23].

1.3 Orthosymplectic Lie Superalgebras

Specifically, the Lie superalgebra $\mathfrak{osp}(1|2n, \mathbb{R})$ is of great importance to superconformal theories; see chapters 6 and 7 of [11] for an early survey of physical applications. The natural purusit to describe simple objects in a module category of $\mathfrak{osp}(1|2n, \mathbb{C})$, and of other classical Lie superalgebras defined in [24], was considered by Dimitrov, Mathieu, Penkov in [9]. Gorelik and Grantcharov completed the classification began in [9] by publishing [14] then [18], the former with Ferguson, who classified the simple bounded highest-weight modules of $\mathfrak{osp}(1|2n, \mathbb{C})$ in his dissertation [13]. Primitive vectors in certain tensor product representations of $\mathfrak{osp}(1|2n, \mathbb{C})$ were essential to the completion of [13]. Coulembier also paid close attention to primitive vectors in [7] to determine whether tensor products of certain irreducible highest-weight $\mathfrak{osp}(2m + 1|2n, \mathbb{R})$-representations were completely reducible; a series of papers by Coulembier and co-authors [6, 7, 8] motivated inspecting the reducibility of certain $\mathfrak{osp}(1|2n, \mathbb{R})$-modules to establishing a Clifford analysis on supermanifolds. The case of $m = 0$ over $\mathbb{C}$ is considered herein.

1.4 Representation Theory of $\mathfrak{osp}(1|2n)$

This dissertation focuses on the infinite-dimensional (super) representation theory of the complex orthosymplectic Lie superalgebra $B(0|n)$ [23].[1] In detail, the present exposition recalls necessary definitions and concepts of multilinear algebra in Chapter 2, of algebras in Chapters 3, of superalgebras in Chapter 4, and of orthosymplectic Lie superalgebras in Chapter 5. Now a classical pursuit in representation theory is to consider tensor products of infinite-dimensional representations with finite-dimensional representations as in [4, 25, 43]. Chapter 6 analyzes the tensor product

[1] The series of Lie superalgebras $B(m|n)$ over a field $\mathbb{K}$ of characteristic zero is the series of orthosymplectic algebras $\mathfrak{osp}(2m + 1|2n, \mathbb{K})$.

$\mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}^{1|2n}$ of an infinite-dimensional $\mathfrak{osp}(1|2n)$-representation $\mathbb{C}[x_1, \ldots, x_n]$ with the standard $\mathfrak{osp}(1|2n)$-representation $\mathbb{C}^{1|2n}$. Primitive vectors play a key role in the first new result of decomposing $\mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}^{1|2n}$ as an $\mathfrak{osp}(1|2n)$-representation with explicit descriptions of the $\mathfrak{osp}(1|2n)$-subrepresentations, including bases. Lastly, Chapter 7 provides a map of associative superalgebras between the universal enveloping Lie superalgebra $\mathcal{U}(\mathfrak{osp}(1|2n))$ and (super) differential operators inside the endomorphism superalgebra of $\mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}[\xi_1, \ldots, \xi_{2n}]$, where the x_i commute and the ξ_j anti-commute.

1.5 Notation & Conventions

Many of the symbols and notation of the early chapters will be familiar or quickly digestible to the casual reader of mathematical writing; however, it is best to lay out the following conventions explicitly:

blackboard bold	$\mathbb{N}$	natural numbers
	$\mathbb{N}_0$	natural numbers with zero
	$\mathbb{Z}$	integers
	$\mathbb{Q}$	rationals
	$\mathbb{R}$	reals
	$\mathbb{C}$	complex numbers
Greek letters	α, Ξ, Υ	scalars
	$\beta,\ \theta$	forms/maps
Roman letters	V, V^*	vector spaces and dual vector spaces
	$A,\ A_0 \bigoplus A_1$	associative algebras and superalgebras
Fraktur	$\mathfrak{g},\ \mathfrak{g}_0 \bigoplus \mathfrak{g}_1$	Lie algebras and Lie superalgebras

A familiarity with the standard topics of a first-year (linear) algebra course is assumed from the interested reader. Specifically, the reader should be comfortable with the content found in [1, 38]. A two-semester sequence of graduate algebra is also recommended.

CHAPTER 2

Groups, Rings, Vector Spaces

This chapter serves to recall some definitions found in [22] and other fundamental treatments of modern algebra. The reader familiar with groups, rings, and vectors spaces may refer to this chapter for notation and verification of definitions at play. The end of the chapter provides some standard references to definitions and examples found in any text on multilinear algebra. Some standard proofs are provided.

2.1 Basic Definitions

Definition 2.1.1. A *monoid* M is a set of composable transformations of a space. Axiomatically, a monoid $\langle M, \circ, e\rangle$ is a nonempty set M with an associative binary operation $\circ$ called multiplication and a distinguished element e serving as a multiplicative identity. We often write ab for $a \circ b$ and 1 for e.

Definition 2.1.2. The *opposite monoid* $M^{\text{op}} = \langle M, \circ^*, e\rangle$ with $a \circ^* b = ba$, for all $a, b \in M$. The identity $\mathbb{1}_M : \langle M, \circ, e\rangle \to \langle M, \circ^*, e\rangle$ is only a map of monoids if $\circ$ is commutative.

Example 2.1.3.

the set of natural numbers with zero under addition	$\langle \mathbb{N}_0, +, 0 \rangle$
the set of natural numbers under multiplication	$\langle \mathbb{N}, \cdot, 1 \rangle$
the set of integers under multiplication	$\langle \mathbb{Z}, \cdot, 1 \rangle$
the set of integers under addition	$\langle \mathbb{Z}, +, 0 \rangle$
the set of rational numbers under addition	$\langle \mathbb{Q}, +, 0 \rangle$
the set of real numbers under addition	$\langle \mathbb{R}, +, 0 \rangle$
the set of complex numbers under addition	$\langle \mathbb{C}, +, 0 \rangle$

Example 2.1.4. For any set X, the set $Fun(X)$ of functions $f : X \to X$ is a monoid under function composition. The identity function serves as 1.

Definition 2.1.5. A monoid G where every element has a multiplicative inverse is called a *group*, which is a certain set of symmetries of an object in ambient space.

Example 2.1.6. The set S_Ω of permutations on a fixed set Ω is the *symmetric group*.

Example 2.1.7. Let G^X be the set of all functions from a fixed nonempty set X to a group G. Then G^X is a group under pointwise products.

Definition 2.1.8. An *abelian group* is a group G satisfying $x \circ y = y \circ x$ for all elements x and y of the group. Then the operation $\circ$ is often referred to as the addition $+$ of the group with additive identity 0.

Example 2.1.9. Abelian groups are found in the last five rows of Example 2.1.3.

Definition 2.1.10. A *ring* R is a set of structure-preserving maps on an abelian group. Formally, a unital ring or ring with unity $\langle R, +, \circ, 0, 1 \rangle$ is both an abelian group $\langle R, +, 0 \rangle$ and a monoid $\langle R, \circ, 1 \rangle$ in which the operations are compatible via left (and right) distributivity of left (and right) multiplication over addition.

Like mosquitoes, rings without unity do exist; unlike mosquitoes, many mathematicians find it easy to ignore them.

Example 2.1.11. With the usual addition and multiplication, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are commutative rings, which means the multiplication is commutative.

Example 2.1.12. With the same operations as Example 2.1.11, the set of even integers $2\mathbb{Z}$ forms a commutative ring without unity, while the set of odd integers $2\mathbb{Z}+1$ fails to satisfy closure of addition. Oddly enough, $\langle 2\mathbb{Z}+1, \cdot, 1\rangle$ is a commutative monoid under the usual multiplication of integers.

Remark 2.1.13. We will require a map of unital rings to map unity to unity.

Example 2.1.14. Given a group G and a ring R, we denote the *group ring* of G over R by $R[G]$. We may view $R[G]$ as the set of functions $\{f : G \to R \mid f \text{ has finite support}\}$. The group ring $R[G]$ is commutative if and only if R is commutative and G is abelian. In particular, for all $f : G \to R$ and $g : G \to R$ in the group ring $R[G]$, we define the product fg by

$$fg(z) = \sum_{\substack{xy=z \\ x,y\in G}} f(x)g(y) = \sum_{x\in G} f(x)g(x^{-1}z).$$

Example 2.1.15. Let M be an abelian group with more than one element. Let $\mathrm{Fun}(M) = M^M$. As function composition distributes over function addition, Example 2.1.4 and Example 2.1.7 define $\mathrm{Fun}(M)$ as a noncommutative unital ring. The *ring* $\mathrm{End}(M)$ *of (group) endomorphisms* is a subring of $\mathrm{Fun}(M)$.

Example 2.1.16. The ring $\mathrm{Mat}(n, R)$ consists of $n \times n$ matrices with entries from a ring R under matrix addition and matrix multiplication. Unless $n = 1$, $\mathrm{Mat}(n, R)$ is a noncommutative ring.

Definition 2.1.17. *Division rings* are rings where all nonzero elements have a multiplicative inverse.

Example 2.1.18. A *field*, such as $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, is a commutative division ring.

Definition 2.1.19. An abelian group M upon which each element r of a ring R acts as a structure-preserving map ϕ_r in a compatible manner is called a left module over R or *left R-module* for short. Compatibility means $\phi_{rs}(m) = \phi_r(\phi_s(m))$, written $(rs) \cdot m = r \cdot (s \cdot m)$, $\forall m \in M$. Even simpler, we write rm for $r \cdot m$.

Definition 2.1.20. A *right R-module M* is exactly a left R^{op}-module. The (right) action $m \cdot^* r$ of R on M is defined through the (left) action $r \cdot m$ of R^{op} on M.

Intuitively, we gain some data about a choice object by what it does to something else, usually a vector space. We represent our object of note as transformations of a familiar space—a representation space—and representation theory is the systematic investigation into these representations. Modules are the beginning of our interest in representation theory.

Definition 2.1.21. Modules over a division ring D are called *D-vector spaces* with D as the ground ring. Modules over commutative rings R are both left and right R-modules as $R \cong R^{\text{op}}$ via the identity map.

Example 2.1.22. The real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$ are naturally $\mathbb{R}$-vector spaces, while the multiplication of $\mathbb{C}$ does not define a $\mathbb{C}$-vector space structure on the reals. More generally, every division ring D is a $Z(D)$-vector space over its center $Z(D) := \{z \in D \mid za = az,\ \forall a \in D\}$.

Example 2.1.23. The group $\text{End}_D(V)$ consists of D-linear maps on a D-vector space V. The *dual space* V^* of V is the set $\{T : V \to D \mid T \text{ is a } D\text{-linear map}\}$. Both $\text{End}_D(V)$ and V^* are D-subspaces inside D-vector spaces $\text{Fun}(V)$ and D^V, respectively.

We will deal primarily with vector spaces over the complex numbers $\mathbb{C}$. Moreover, the ground field is assumed to be $\mathbb{C}$, unless otherwise stated, and we will suppress the ground field from notation when context affords us brevity.

All the usual notions of substructures and quotient structures are standard.

2.2 Classical Examples

A broad introduction to Lie algebras involves the areas of differential geometry, Lie groups, differential equations, functional analysis, and many more subfields of mathematics along with particle physics. In contrast, this work serves as a narrower road and quicker path from linear algebra to the main results of my dissertation. Still, this section may lend both background and motivation into the study of complex orthosymplectic Lie superalgebras, briefly, sets of linear transformations on $\mathbb{C}^{m+2n}$ compatible with a geometry expressed algebraically through certain maps from $\mathbb{C}^{m+2n}$ to $\mathbb{C}$.

In a real and concrete case, we consider $\mathbb{R}^2$ with the dot product, $v \cdot w = |v||w|\cos\theta$, as two-dimensional Euclidean space and an example of an inner product space. Then the geometry of $\mathbb{R}^2$ (or of the underlying affine space) is revealed through distance and angular measurement defined through the dot product. Intuitively, we see that rotations around a fixed point preserve the dot product while adhering to the linear structure on the space, and we have the luxury of fixing a basis in order to represent these transformations as 2×2 matrices. These rotations form a group called the special orthogonal group of $\mathbb{R}^2$. Moreover, this type of set-up finds generalizations in many other vector spaces. One can explore groups of linear maps that respect both the vector space structure and the geometry controlled by a form, as defined below.

2.2.1 Forms on a Vector Space

Many algebraic objects described herein are related to maps on vector spaces.

Definition 2.2.1. A *multilinear n-form* on a vector space V is a map

$$f : \underbrace{V \times V \times \cdots \times V}_{n \text{ times}} \longrightarrow \mathbb{C}$$

such that f is linear in each entry. That is, for each i,

$$f(v_1, v_2, \ldots, x_i + y_i, v_{i+1}, \ldots, v_n) = f(v_1, v_2, \ldots, x_i, v_{i+1}, v_{i+1}, \ldots, v_n) + f(v_1, v_2, \ldots, y_i, v_{i+1}, v_{i+1}, \ldots, v_n)$$

and

$$f(v_1, v_2, \ldots, \alpha x_i, v_{i+1}, \ldots, v_n) = \alpha f(v_1, v_2, \ldots, x_i, v_{i+1}, \ldots, v_n),$$

with all arguments taken from V and α a complex number. More generally, we can defined mixed forms on a finite collection of vector spaces of varying dimensions or on vector spaces of the same dimension. The definition above also describes a *multilinear map* where any arbitrary vector space may take the place of the ground field as the codomain.

For $n = 1, 2, 3$, we have *linear functionals*, *bilinear forms*, and *trilinear forms*, respectively. Forms are assumed to be n-linear. For a vector space V with a bilinear form β, it should be clear that $\beta(0, x) = 0 = \beta(x, 0)$, for all x in V.

Recall, $\mathbb{C}^n$ is isomorphic to any complex vector space V of dimension n, and the existence of a linear space isomorphism gives a notion of equivalency amongst spaces. Now let us use the term *bilinear space* to describe a vector space with a bilinear form (V, β_V) and define a map of bilinear spaces $f : (V, \beta_V) \to (W, \beta_W)$ to be a linear map $f : V \to W$ such that $\beta_W(f(x), f(y)) = \beta_V(x, y)$ for all x, y in V. If f is bijective, as well, then we call f an *isometry* and say the bilinear spaces are equivalent. We may also say the forms themselves are equivalent.

Now the set $\mathrm{Bil}(V)$ of all bilinear forms on a fixed vector space V also has the structure of a vector space. Both the study of these spaces of forms and the classification of equivalent forms on V are well-established directions of research along with interest in a particular space with a given form. As much as linear algebra is a life blood for all of mathematics, the ubiquitous nature of forms is apparent. Number theoretic questions motivate the classification of equivalent bilinear forms defined on a vector space over a finite field. The study of real Hilbert spaces is the study of complete normed vector spaces in which an inner product, a positive-definite symmetric bilinear form, induces the norm. The complex case for Hilbert spaces requires sesquilinearity instead of bilinearity. Definitions and examples of the terms used in this paragraph can be found in any text on functional analysis such as [35] or [26]. We provide some of the definitions here.

Definition 2.2.2. A bilinear form $\beta : V \times V \to \mathbb{C}$ is called a *symmetric bilinear form* if $\beta(x, y) = \beta(y, x)$, an *alternating bilinear form* if $\beta(x, x) = 0$, and a *skew-symmetric bilinear form* if $\beta(x, y) = -\beta(y, x)$, for all $x, y \in V$. Each of these type of forms is an example of a *reflexive bilinear form* defined by the condition $\beta(x, y) = 0$ implies $\beta(y, x) = 0$.

Theorem 2.2.3. *Alternating bilinear forms are equivalent to skew-symmetric bilinear forms.*

Proof. We reemphasize that the ground field is $\mathbb{C}$. An alternating bilinear form β and vectors x and y give $0 = \beta(x + y, x + y) = \beta(x, x) + \beta(x, y) + \beta(y, x) + \beta(y, y)$. Thus, $\beta(x, y) = -\beta(y, x)$. In the other direction, a skew-symmetry β implies $2\beta(x, x) = 0$. Since the characteristic of $\mathbb{C}$ is 0 we have $\beta(x, x) = 0$. □

The proof shows that in any characteristic an alternating bilinear form is skew-symmetric, and skew-symmetry implies alternativity over fields with characteristic different from 2. Then the study of bilinear forms on a fixed space V, in some sense, breaks down to the study of symmetric and skew-symmetric forms on V as any bilinear form β can be expressed as the sum of a symmetric bilinear form β_{sym} and a skew-symmetric bilinear form β_{skew}:

$$\beta_{sym}(x,y) = \frac{\beta(x,y)+\beta(y,x)}{2}, \tag{2.1}$$

$$\beta_{skew}(x,y) = \frac{\beta(x,y)-\beta(y,x)}{2}. \tag{2.2}$$

Definition 2.2.4. A reflexive bilinear form β is called a *nondegenerate bilinear form* if its *(left) radical*, denoted here as $\mathrm{Rad}_\beta = \{x \in V \mid \beta(x,y) = 0, \ \forall y \in V\}$, contains only the zero vector.

Bilinear forms are assumed to be reflexive from now on, but nondegeneracy is explicitly stated in each case.

Definition 2.2.5. For a subspace $W \subset V$ and β a bilinear form on V, denote by $W^\perp$ the subspace termed the *orthogonal complement* of W with respect to β. The orthogonal complement of W is defined to be the set of all vectors in V orthogonal (precisely, β-orthogonal) to every vector in W:

$$W^\perp = \{v \in V \mid \beta(v,w) = 0, \ \forall w \in W\}.$$

In other words, nondegeneracy amounts to the zero vector being the unique vector orthogonal to all other vectors, i.e., $V^\perp = \{0\}$.

For the rest of this section vector spaces will have finite dimension.

Definition 2.2.6. A *symplectic bilinear form* is a nondegenerate alternating bilinear form. A pair (V,β) consisting of a vector space with a symplecitc bilinear form is called a *symplectic vector space* or just a symplectic space. If the form is possibly degenerate, then we may speak of an *alternating space.*

Definition 2.2.7. A symmetric bilinear form β on V defines a unique *binary quadratic form* $\omega : V \to \mathbb{C}$ by $\omega(x) = \beta(x,x)$. We will call a pair $(V,\beta) = (V,\omega)$ an orthogonal space or *quadratic space* when β is a (not necessarily nondegenerate) symmetric bilinear form, equivalently, when ω is a (not necessarily nonsingular) quadratic form associated to β as explained below. The form ω has the following properties, $\forall \alpha \in \mathbb{C},\ \forall x, y \in V$:

(2.3) $\quad \omega(\alpha x) = \alpha^2 \omega(x)$

(2.4) $\quad f(x, y) = \omega(x + y) - \omega(x) - \omega(y)$ is a symmmetric bilinear form

(2.5) $\quad \beta(x, y) = \frac{1}{2} f(x, y)$ is the symmetric bilinear form such that$\beta(x, x) = \omega(x)$

Thus, β is nondegenerate if and only if ω is nonsingular. Take the previous statement as a definition.

Here are some results concerning symplectic spaces and quadratic spaces that may support the results on orthosymplectic spaces found later.

Theorem 2.2.8. *If* (V, β) *is a symplectic space with* $\dim(V) = n < \infty$*, then* n *is even.*

Lemma 2.2.9. *Let* V *be a finite-dimensional vector space with nondegenerate bilinear form* β*. If* $W \subset V$ *is a subspace such that* β *restricted to* $W \times W$ *is also nondegenerate, then* $V = W \oplus W^{\perp}$*.*

Proof of Lemma 2.2.9. First note that the map $\theta : V \to W^*$ given by $v \mapsto \beta_v = \beta(v, -)\big|_W$ is a linear map. As such, $\dim(V) = \dim(\text{Im } \theta) + \dim(\text{Ker } \theta)$. Since Ker $\theta = \{v \in V \mid \beta(v, w) = 0,\ \forall w \in W\} = W^{\perp}$, we have the inequality $\dim(V) \leq \dim(W^*) + \dim(W^{\perp}) = \dim(W) + \dim(W^{\perp})$.

Now it remains to show $W \cap W^{\perp} = \{0\}$ as $\dim(W \oplus W^{\perp}) = \dim(V)$ would follow: $\dim(V) \leq \dim(W) + \dim(W^{\perp}) = \dim(W + W^{\perp}) + \dim(W \cap W^{\perp})$ and $W + W^{\perp}$ is a subspace of V.

Write $\beta\big|_W$ for $\beta\big|_{W \times W}$. The nondegenracy of $\beta\big|_W$ implies that the only element in W orthogonal to all elements of W is the zero vector. Consequently, if $v \in W \cap W^{\perp}$, then $v = 0$. $\square$

Remark 2.2.10. Under the conditions of Lemma 2.2.9, we not only conclude V decomposes as a vector space but, additionally, as a symplectic space or quadratic space.

Proof of Theorem 2.2.8. The proof here is one of strong induction on the dimension n of V. In the base case, we note the zero space is a symplectic space given any bilinear form; on the other hand, a nondegenerate alternating form cannot exist on a one-dimensional vector space. Turning to the inductive hypothesis, we assume that any symplectic space of dimension $k < n$ has even dimension. Now a symplectic space (V, β) of dimension n has a basis, say, $B = \{x_1, x_2, \ldots, x_n\}$. nondegenracy implies that for some $i \geq 2$ there exists $\alpha \in \mathbb{C}\backslash\{0\}$ such that $\beta(x_1, \alpha x_i) = 1$.
Let $y_1 = \alpha x_i$ and $W = \mathbb{C}x_1 \oplus \mathbb{C}y_1$. Inherently, $\beta\big|_W$ is an alternating bilinear form, and recalling $\beta\big|_W(x_1, y_1) = \beta(x_1, y_1) = 1$, we see $\beta\big|_W$ is nondegenerate. In particular, $w \in W$ is orthogonal to all members of W and must satisfy

$$w = \Xi x_1 + \Upsilon y_1, \quad \Xi, \Upsilon \in \mathbb{C},$$

$$\beta(w, x_1) = 0, \text{ and}$$

$$\beta(w, y_1) = 0;$$

thus, $w = 0$. By Lemma 2.2.9, $V = W \oplus W^\perp$, with $\dim(V) = \dim(W) + \dim(W^\perp)$, i.e., $n = 2 + k$. We show $(W^\perp, \beta\big|_{W^\perp})$ is a symplectic space of dimension $k = n - 2 < n$ to apply the induction hypothesis and conclude k is even. As bilinearity and alternativity are inherited, we focus on exhibiting the nondegeneracy of $\beta\big|_{W^\perp}$: Assuming a vector v is in the radical of $\beta\big|_{W^\perp}$ amounts to supposing v is orthogonal to all of W and all of $W^\perp$, the sum of which is V. The nondegeneracy of β forces v to be the zero vector as desired. We conclude that $(W^\perp, \beta\big|_{W^\perp})$ is a symplectic space of even dimension k, i.e., V is of even dimension $n = k + 2$. The theorem holds in light of the base case. □

Corollary 2.2.11. *Given a non-zero symplectic space* (V, β)*, there exists a basis*

$$B = \{x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_m\}$$

such that

$$\beta(x_i, y_j) = \delta_{ij}, \quad \beta(x_i, x_j) = \beta(y_i, y_j) = 0, \ 1 \leq i, j \leq m.$$

We will call such a basis a Darboux basis of V*.*

Remark 2.2.12. An analogous result holds for quadratic spaces. Mainly, there exists a basis $B = \{x_1, x_2, \dots, x_n\}$ such that $\beta(x_i, x_j) = \delta_{ij}$. Such a basis is called an *orthogonal basis* of V.

The theory of bilinear forms is presented in full generality in [27]. For any complex V of fixed dimension, it is known that all nondegenerate symmetric forms are equivalent, i.e., there is one equivalence class of quadratic spaces for a fixed dimension n. Over any field with characteristic different from two, there is one equivalence class of symplectic spaces of dimension $2n$.

2.2.2 Classical Groups

We turn our attention to the group of all invertible linear transformations on $V \cong \mathbb{C}^n$ and the subgroups whose elements are isometries relative to some form β. A choice of basis leads to the expression of these groups as matrix groups.

Definition 2.2.13. The *general linear group* of V is denoted as $\mathrm{GL}(V)$, the subset of $\mathrm{End}(V)$ preserving linearly independent subsets, a basis in particular, of V. Equivalently, $\mathrm{GL}(V)$ is the automorphism group $\mathrm{Aut}(V)$. We let $\mathrm{Aut}(\beta)$ be the set of *isometries* or the set of linear maps preserving a nondegenerate bilinear form β. That is, $\beta(Tx, Ty) = \beta(x, y)$ for all vectors x and y and T in $\mathrm{GL}(V)$.

Proposition 2.2.14. *If* $\dim(V) = n < \infty$*, then* $\mathrm{GL}(V) \cong \mathrm{GL}(n)$*, the general linear group of degree* n *over* $\mathbb{C}$*, which are the invertible* $n \times n$ *matrices.*

Proof. Choose a basis $B = \{x_1, x_2, \dots, x_n\}$ of V. Then for $T : V \to V$ a linear map on V, we have $T(x_j) = \sum_{i=1}^{n} \alpha_{ij} x_i$. Define the matrix $A_{T,B} =$

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix}.$$

It can be shown that the map $T \mapsto A_{T,B}$ is a ring isomorphism from $\mathrm{End}(V)$ onto $\mathrm{Mat}(n) = \mathrm{Mat}(n, \mathbb{C})$ as the a_{ij} uniquely determine T given the basis B. It follows that the group of units in each ring are isomorphic groups: $\mathrm{GL}(V) \cong \mathrm{GL}(n, \mathbb{C})$. □

Remark 2.2.15 (non-canonical isomorphism). The isomorphisms in the proof rely on a choice of basis for V. Once a basis $B = \{x_1, x_2, \dots, x_n\}$ is fixed, we exploit the coordinate expressions of v and maintain

$$T(v) = T(\sum_{i=1}^{n} \alpha_i x_i) = A_{T,B} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

Often the basis and transformation are understood from context and one writes $T(v) = Av,\ \forall v \in V$, where the v on the right-hand side of the equation is the coordinate vector of v with respect to the basis B.

With Proposition 2.2.14, we suppose bilinear forms themselves can be expressed as matrices.

Theorem 2.2.16. *Given a basis $B = \{x_1, x_2, \dots, x_n\}$ of V, a nondegenerate bilinear form $\beta : V \times V \to \mathbb{C}$ is uniquely associated to the matrix*
$Mat(\beta)$, where

$$\mathrm{Mat}(\beta)_{ij} = \beta(x_i, x_j),$$

$$so\ that\ \beta(v,w) = v^T$$

$$Mat(\beta)w.$$

Any invertible skew-symmetric matrix of the appropriate size can encode a symplectic form β on some non-zero $2n$-dimensional space V. Due to Corollary 2.2.11, there always exists a basis such that a symplectic form is associated to the matrix

$$J_{2n}^{skew} = \overset{2n\times 2n}{\left[\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right]},\ \dim(V) = 2n,\ \ I_n \text{ the } n \times n \text{ identity matrix.}$$

Analogously, any nonsingular symmetric matrix encodes a nondegenerate symmetric bilinear form β relative to some basis of V. Moreover, for any dimension n, remark 2.2.12 implies there exists an orthogonal basis of V such that I_n is associated to β. We may also find a basis such that the matrix J_{2n}^{sym} is associated to β when the dimension of V is even. When the dimension of V is odd we may associate the matrix J_{2n+1}^{sym} to β. We define

$$J_{2n}^{sym} = \left[\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right], \dim(V) = 2n, \text{ and}$$

$$J_{2n+1}^{sym} = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0 & I_n \\ \hline 0 & I_n & 0 \end{array}\right], \dim(V) = 2n+1.$$

Symmetry implies $J_1^{sym} = 1$ viewed as a 1×1 matrix.

We can further ask how to express elements of $\text{Aut}(\beta)$ in terms of matrices. A detailed construction of these matrix groups and coordinate-free realizations is found in [17].

Here we present a table featuring a selection of the so-called classical groups over $\mathbb{C}$ viewed as both subgroups of linear maps contained in $\mathrm{GL}(\mathbb{C}^n)$ and as matrix groups contained in $\mathrm{GL}(n) = \mathrm{GL}(n, \mathbb{C})$:

Name	**Notation**		**Defining Condition**	
Special linear group	$\mathrm{SL}(\mathbb{C}^n)$	$\mathrm{SL}(n)$	Preserves oriented volume	$\det(A) = 1$
Symplectic group	$\mathrm{Sp}(\mathbb{C}^{2n})$	$\mathrm{Sp}(2n)$	Preserves symplectic form	$A^T J_{2n}^{skew} A = J_{2n}^{skew}$
Orthogonal group	$\mathrm{O}(\mathbb{C}^n)$	$\mathrm{O}(n)$	Preserves nonsingular quadratic form	$A^T J_n^{sym} A = J_n^{sym}$
Special orthogonal group	$\mathrm{SO}(\mathbb{C}^n)$	$\mathrm{SO}(n)$	Preserves nonsingular quadratic form and oriented volume	$A^T J_n^{sym} A = J_n^{sym}$ $\det(A) = 1$

CHAPTER 3

Algebras

A peculiar case: There is a word derived from the Arabic for "reunion of broken parts" whose mathematical usage is attributed to Muhammed ibn Musa al-Khwarizmi. This word algebra is found in many a course title from grammar/secondary school's Algebras I & II to College Algebra, from Linear Algebra to Abstract Algebra. With so many appearances one hesitantly stumbles upon the question titling Section 1 of [37]: "What is Algebra?" The answer therein describes a broader view of scientific pursuit as we attempt to measure the world around us. The same text shows the term algebra also has place amongst the definitions of Chapter 2 as a set with certain structure, as an object describing transformation of space. Indeed, an algebra over a commutative ring is essentially a module with its own multiplication compatible with the action of the ring, i.e., $r(mn) = (m)rn = rm(n)$, for r a ring element and m, n module elements. The multiplication can be non-associative, associative, commutative, carry a multiplicative identity, etc., giving rise to non-associative algebras, associative algebras, commutative algebras, unital algebras, etc., respectively. A first walk through a graduate algebra course may even describe an algebra via maps of rings to the surprise of the student fixed on set-axiomatic descriptions of algebraic objects. From a categorical point of view, groups, rings, vector spaces, and many more structures are well-defined and well-explained through maps, the objects and arrows between them [as in, e.g., 33]. In this chapter we shall define specific algebras of interest and hint at the functorial nature involved in constructing algebras from vector spaces. Beyond here, the reader is encouraged to explore the mathematical object, scientific subfield of study, and philosophical perspective that is algebra.

First we make clear the definition of an associative unital algebra over $\mathbb{C}$ and a map of such algebras.

Definition 3.0.1. Suppose $f : \mathbb{C} \to A$ is a map of commutative unital rings such that the image of f is contained in $Z(A)$. Then we say A is a $\mathbb{C}$-algebra by which we mean an *associative unital complex algebra*.

The condition that $f(\mathbb{C}) \subset Z(A)$, recalling example 2.1.22, implies A is a vector space over $\mathbb{C}$. Naturally, a map of algebras is required to be both a linear map and a map of rings.

There are various associative unital algebras associated with a vector space V.

3.1 Algebra of Endomorphisms and Free Algebras

In Chapter 2 we introduced the ring $\mathrm{End}(V)$ while considering the abelian group structure on V. The vector space $\mathrm{End}(V)$ of linear maps also has an associative product through function composition with $\mathbb{1}_V$ as the unit. That means we can view $\mathrm{End}(V)$ as an associative unital algebra. Following the proof of Proposition 2.2.14, we have $\mathrm{End}(V)$ is isomorphic to $\mathrm{Mat}(n)$ as algebras and $\dim(\mathrm{End}(V)) = n^2$ when V has finite dimension n.

Remark 3.1.1. The set $\mathrm{End}(V)$ will continue to be a primary example to exhibit various algebraic structures. Already we have seen $\mathrm{End}(V)$ as a group, ring, vector space, and algebra. Any associative algebra is certainly a group, ring, and vector space, so there are no surprises here. Furthermore, an abstract algebra A acts on a vector space V through algebra maps $A \to \mathrm{End}(V)$.

Let V be the infinite-dimensional vector space of complex polynomials $\mathbb{C}[x_1, \ldots, x_n]$ in n indeterminates. We will be interested in an algebra A such that V is an A-

representation; equivalently, we will seek certain maps $A \to \mathrm{End}(V)$ such that the elements of A act on polynomials, a welcome sight to mathematicians. Some particular algebras A with V as a representation space arise from quotients of free algebras generated by elements from $\mathrm{End}(V)$. The ideals in the quotients will reflect the relations on the elements present in the endomorphism ring. We give the definitions needed below.

For any set X we define the free algebra $\mathbb{C}\langle X\rangle$ on X.

Definition 3.1.2. The *free algebra* $\mathbb{C}\langle X\rangle$ *on* X consists of all finite sums of words $w = \alpha w_1^{k_1} \cdots w_m^{k_m},\ \alpha \in \mathbb{C},\ m \in \mathbb{N},\ w_i \in X,\ k_i \in \mathbb{N},\ 1 \le i \le m$. By convention, the empty word $(\alpha = 1,\ m = 0)$ is the multiplicative identity.

Definition 3.1.3. If $\mathbb{C}\langle X\rangle = \mathbb{C}\langle Y\rangle$ for a finite set Y, then $\mathbb{C}\langle X\rangle = \mathbb{C}\langle Y\rangle$ is called a *finitely-generated (free) algebra.*

The construction above will be of particular use when X is a basis of a finite-dimensional vector space V.

Remark 3.1.4. An ideal J of an associative algebra A is taken to be an ideal of the ring structure of A.

Again consider $V = \mathbb{C}[x_1, \ldots, x_n]$. Consider the endomorphisms $P_i : f \mapsto x_i f$ and $Q_i : f \mapsto \frac{\partial f}{\partial x_i},\ 1 \le i \le n$. Let $X = \{P_i, Q_i \mid 1 \le i \le n\} \subset \mathrm{End}(V)$. Then we have the following relations on X

$$P_iP_j - P_jP_i = 0,\ 1 \le i, j \le n$$

$$Q_iQ_j - Q_jQ_i = 0,\ 1 \le i, j \le n$$

$$Q_iP_j - P_jQ_i = \delta_{ij}\mathbb{1}_V,\ 1 \le i, j \le n.$$

Then $\mathcal{D}(n)$ is the n^{th} *Weyl algebra of polynomial differential operators* on $\mathbb{C}[x_1, \ldots, x_n]$ as the quotient $\mathbb{C}\langle\{P_i, Q_i \mid 1 \le i \le n\}\rangle / J$, where J is the ideal generated by elements of the form $P_iP_j - P_jP_i,\ Q_iQ_j - Q_jQ_i,\ Q_iP_j - P_jQ_i - \mathbb{1}_{\mathbb{C}[x_1,\ldots,x_n]}$. Sometimes, $\mathcal{D}(n)$ is known as the $n$the Weyl algebra.

By a slight abuse of notation we will use x_i for both the indeterminate in $\mathbb{C}[x_1, \ldots, x_n]$ and endomorphism P_i. Also, ∂_{x_i} will be used to mean Q_i, and we may avoid new notation for elements of quotients or subscripts for identity elements when context is clear.

3.2 Tensor Algebras and Quotients

Let V and W be finite-dimensional complex vector spaces. We want to define a multiplication to combine elements of V and W which keeps intact the action of the original spaces. For example, we expect that if $\otimes$ is a desirable product, then $2(a \otimes b) = 2a \otimes b = a \otimes 2b = 4a \otimes \frac{1}{2}b$. More generally, if $a, c \in V,\ b, d \in W,\ \alpha \in \mathbb{C}$, then we ask that

$$\alpha(a \otimes b) = \alpha a \otimes b = a \otimes \alpha b, \tag{3.1}$$

$$(a + c) \otimes b = a \otimes b + c \otimes b, \tag{3.2}$$

$$a \otimes (b + d) = a \otimes b + a \otimes d. \tag{3.3}$$

The collection of the products should be a vector space: So we define the tensor product of V with W as the vector space $V \otimes W$ as a quotient of the free abelian group on the set of ordered pairs $V \times W$. Thus, $V \otimes W$ is a vector space of equivalence classes (similar to L^p spaces). In more technicality, let $S = V \times W = \{(a, b) \mid a \in V,\ b \in W\}$. The *free abelian group* $F(S)$ of S consists of all finite sums with summands from S and the relation $(x + y) - (y + x) = 0$, for all elements x and y in S. Now, in order to have the desired properties mentioned above, we form the quotient group ${}^{F(S)}\!/\!_{\langle\sim\rangle}$ using relations induced by eqs. (3.1) to (3.3): here the notation $\langle\sim\rangle$ stands for the group generated by elements in the form of the relations.

The details of the construction using noncommutative rings and modules are in [Chapter 4 of 22, Section 5]. As the factors of the tensor product do not have to be the same vector space we can form the tensor products $(V \otimes V) \otimes V$ and $V \otimes (V \otimes V)$.

Proposition 3.2.1. *Let U, V, W be vector spaces. The vector spae $(U \otimes V) \otimes W$ is isomorphic to $U \otimes (V \otimes W)$. Thus, we write $U \otimes V \otimes W$ without concern for the parentheses, i.e., the tensor product is associative.*

Proposition 3.2.2. *If B_V is basis for V and B_W a basis for W, then $\{b_V \otimes b_W \mid b_V \in B_V, b_W \in B_W\}$ is a basis for $V \otimes W$.*

With Proposition 3.2.1 and Proposition 3.2.2 we are ready to define the tensor algebra on V.

Definition 3.2.3. The k^{th} *tensor power of V* is

$$T^k(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{\text{k factors}},$$

with $T^0(V) = \mathbb{C}$ and $T^1(V) = V$. Elements of the vector space $T^k(V)$ are called *k-tensors.*

Definition 3.2.4. The *tensor algebra $T(V)$ on V* is the associative algebra

$$T(V) = \bigoplus_{k=0}^{\infty} T^k(V),$$

with associative product $T^m(V) \otimes T^n(V) \to T^{m+n}(V)$ defined on m-tensors and n-tensors by $(v_{i_1} \otimes \cdots \otimes v_{i_m}) \otimes (v_{j_1} \otimes \cdots \otimes v_{j_n}) = v_{i_1} \otimes \cdots \otimes v_{i_m} \otimes v_{j_1} \otimes \cdots \otimes v_{j_n}$, and extended by linearity.

Another formulation of $T(V)$ is as the algebra of complex polynomials with basis elements of V serving as noncommuting indeterminates.

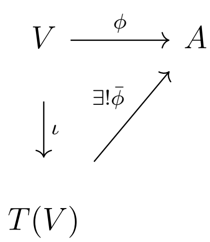
Proposition 3.2.5. *The tensor algebra on $T(V)$ is identified with the free algebra $\mathbb{C}\langle B \rangle$, where B is any basis of V.*

Remark 3.2.6. We will write products as either $a \otimes b$ or ab depending on emphasis.

The algebra $T(V)$ is a coordinate-free realization of the free algebra associated with V.

As mentioned, maps help define many algebraic objects. Universal properties can also speak to the uniqueness of objects. In particular, we say the the tensor algebra of V is a unique (up to isomorphism) associative algebra by the following proposition:

Proposition 3.2.7 (Universal property of $T(V)$)**.** *Recall $T^1(V) = V$ in the construction of Definition 3.2.4. Let $\iota : V \to T(V)$ be the natural inclusion of the vector space V into $T(V)$. Then the tensor algebra as the pair $(T(V), \iota)$ satisfies the following universal property: For any linear map $\phi : V \to A$ to an associative algebra A, there exists a unique map of associative algebras $\bar{\phi} : T(V) \to A$ such that $\bar{\phi}\iota = \phi$. That is, the following diagram is a commutative diagram:*



Remark 3.2.8. The preceding proposition can be used to prove Proposition 3.2.5.

Not only do we associate an algebra $T(V)$ having a universal property with a vector space V, we provide an algebra for alternating spaces (V, β) and an algebra for quadratic spaces (V, ω), as well.

3.2.1 Weyl and Symmetric Algebras

Let (V, β) be an alternating space. Define $J(\beta)$ to be the two-sided ideal of $T(V)$ generated by elements $v \otimes w - w \otimes v - \beta(w, v)$ for all $w, v \in V$. We form the algebra whose underlying vector space is the quotient $\mathcal{D}(V)_\beta = {T(V)}/{J(B)}$.

Definition 3.2.9. In the case β is a symplectic form, Corollary 2.2.11 tells us that the dimension of V determines β. With the dimension of V equal to $2n$, we define the n^{th} *Weyl algebra* $\mathcal{D}(n) = \mathcal{D}(V)_\beta$.

Definition 3.2.10. When β is null we define the *symmetric algebra* $\mathcal{S}(V) = \mathcal{D}(V)_\beta$.

Remark 3.2.11. In light of Proposition 3.2.5, the algebra of polynomial differential operators $\mathcal{D}(n)$ and the n^{th} Weyl algebra $\mathcal{D}(n)$ are isomorphic. We use $\mathcal{D}(n)$ for both. Furthermore, we view $\mathcal{S}(V)$ as polynomials in V and $\mathcal{S}(V^*)$ as polynomials on V.

Remark 3.2.12. Each of $\mathcal{D}(n)$ and $\mathcal{S}(V)$ are associative unital algebras with universal properties analogous to Proposition 3.2.7.

3.2.2 Clifford and Exterior Algebras

Let (V, ω) be a quadratic space. Define $J(\omega)$ to be the two-sided ideal of $T(V)$ generated by elements $v \otimes w + w \otimes v - \omega(w, v)$ for all $w, v \in V$. We form the algebra whose underlying vector space is the quotient $\mathcal{C}(V, \beta) = T(V)/J(\omega)$.

Definition 3.2.13. We call $\mathcal{C}(V, \omega)$ the Clifford algebra of V associated with ω. If ω is nonsingular, then we choose an orthogonal basis and speak of the *Clifford algebra* $Cl(V)$ of V.

Definition 3.2.14. We define the *exterior algebra* $\bigwedge(V)$ of V when ω is null.

Again, Proposition 3.2.5 gives us an isomorphism of algebras. We recognize $\bigwedge V = \bigwedge(V)$ as anti-commuting polynomials $\mathbb{C}[\xi_1, \ldots, \xi_n]$ where $\xi_i^2 = 0$. The standard way to express products in $\bigwedge(V)$ is with the anti-commuting wedge: $a \wedge b = -b \wedge a$ with $a \wedge a = 0$.

Remark 3.2.15. Each of $Cl(V)$ and $\bigwedge(V)$ are associative unital algebras with universal properties analogous to Proposition 3.2.7.

3.3 Non-associative Algebras & Universal Enveloping Algebras

Non-associative algebras do exist! We give a complex vector space $\mathfrak{g}$ a product generalizing the properties of the pair $(\mathbb{R}^3, \times)$, real 3-space with the cross product.

Definition 3.3.1. A Lie algebra is a pair $(\mathfrak{g}, [\ ,\ ])$ such that $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear product (called the *Lie bracket*) on $\mathfrak{g}$ and the following properties hold for all vectors $x, y, z \in \mathfrak{g}$.

$$[x, y] = -[y, x]. \tag{3.4}$$

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]. \tag{3.5}$$

Remark 3.3.2. Abstract Lie algebras can be defined over commutative rings. Also, requiring alternativity, $[x, x] = 0,\ \forall x \in \mathfrak{g}$, replaces eq. (3.4). Over $\mathbb{C}$ there is no difference in assumptions.

Texts such as [17, 39] introduce Lie algebras through the connection with Lie groups and the geometry of matrix groups. We take the more algebraic approach of [10, 21] and give some standard examples before providing the "superized" definitions in Chapter 4 needed for the main results.

3.3.1 Lie algebras

The set $\mathrm{End}(V)$ serves as an example of a Lie algebra when paired with the commutator as the bracket. It will be given a particular name and feature prominently in the representation theoretic aspects of this work.

Remark 3.3.3. Any associative algebra with the commutator is a Lie algebra. Note that the associativity of the multiplication in $\mathrm{End}(V)$ is lacking in the Lie bracket that is the commutator.

Definition 3.3.4. The *general linear Lie algebra* $\mathfrak{gl}(V)$ is the pair $(\mathrm{End}(V), [\ ,\ ])$ where $[X, Y] = XY - YX$ is the commutator on endomorphisms X, Y.

Definition 3.3.5. The *general linear Lie algebra* $\mathfrak{gl}(n)$ *of rank* n is the pair $(\mathrm{Mat}(n), [\ ,\ ])$ where $[A, B] = AB - BA$ is the commutator on $n \times n$ matrices A, B.

As might be expected, $\mathfrak{gl}(V)$ and $\mathfrak{gl}(n)$ are isomorphic Lie algebras when $\dim(V) = n < \infty$. We say two Lie algebras are isomorphic as Lie algebras if there exists an invertible linear map of the underlying vector spaces that respects the bracket of each Lie algebra. We clarify below:

Definition 3.3.6. Let $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\ ,\ ]_{\mathfrak{h}})$ be two Lie algebras. A linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is a *homomorphism of Lie algebras* if $\phi([x, y]_{\mathfrak{g}}) = [\phi(x), \phi(y)]_{\mathfrak{h}},\ \forall x, y \in \mathfrak{g}$.

Lie algebras are also algebraic objects whose elements can act on vector spaces as linear transformations. Precisely, the most important map of Lie algebras will be those of the form $\mathfrak{g} \to \mathrm{End}(V)$ for some Lie algebra $\mathfrak{g}$ and some vector space V serving as a representation space.

We continue with a list of definitions before a table of important examples.

Definition 3.3.7. A *Lie subalgebra* of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{h}$ is closed under the bracket: $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. Without hesitation, we will refer to Lie subalgebras as subalgebras.

Subalgebras of $\mathfrak{gl}(n)$ are called *linear Lie algebras.* The following is a list of linear Lie algebras.

Example 3.3.8.

the set of $n \times n$ matrices with trace equal to zero	$\mathfrak{sl}(n)$
the set of all scalar multiples of I_n	$\mathbb{C}I_n = \mathfrak{s}(n)$
the set of diagonal $n \times n$ matrices	$\mathfrak{d}(n)$
the set of upper triangular $n \times n$ matrices	$\mathfrak{t}(n)$
the set of strictly upper triangular $n \times n$ matrices	$\mathfrak{n}(n)$

Definition 3.3.9. An *ideal of a Lie algebra* is a subalgebra $\mathfrak{i} \subset \mathfrak{g}$ such that $[\mathfrak{i}, \mathfrak{g}] \subset \mathfrak{g}$.

Example 3.3.10. Recall: $\text{tr}\,(AB) = \text{tr}\,(BA)$ for any matrices A and B of size $m \times n$ and $n \times m$, respectively. From here we see $\mathfrak{sl}(n)$ is an ideal of $\mathfrak{gl}(n)$.

Definition 3.3.11. The *center of a Lie algebra* is the ideal $Z(\mathfrak{g}) \subset \mathfrak{g}$ of elements which give a trivial bracket when paired with any other element:
$Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0, \ \forall y \in \mathfrak{g}\}$. We call $\mathfrak{g}$ an *abelian Lie algebra* when $Z(\mathfrak{g}) = \mathfrak{g}$.

Example 3.3.12. The center of $\mathfrak{gl}(n)$ is $\mathfrak{s}(n)$.

A non-abelian Lie algebra is called a *simple Lie algebra* if its only ideals are itself and the zero subspace. A *semisimple Lie algebra* is the direct sum of simple algebras where the bracket is trivial on distinct summands; hence, the algebras in the summands are simple ideals.

Remark 3.3.13. Clearly, $\mathfrak{gl}(n)$ is not simple, nor is it semisimple. Instead, $\mathfrak{gl}(n)$ is a *reductive Lie algebra* as the direct sum of a semisimple ideal $\mathfrak{sl}(n)$ and its center $\mathfrak{s}(n)$.

We emphasize that in $\mathfrak{gl}(n)$ the center is the set of all elements commuting with all others. This terminology of commuting elements is used also with abstract Lie algebras.

Definition 3.3.14. Let $\mathfrak{g}$ be a Lie algebra. The *normalizer of a subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ is the subalgebra $N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{h}] \subset \mathfrak{h}\}$.

A subalgebra is *self-normalizing* if $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

Example 3.3.15. The diagonal matrices $\mathfrak{d}(n)$ is a key example of a self-normalizing algebra.

Definition 3.3.16. The *derived algebra of a Lie algebra* $\mathfrak{g}$ is the ideal $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ and defined to be the set $\text{span}(\{[x, y] \mid x, y \ \in \mathfrak{g}\})$. In light of remark 3.3.13, $[\mathfrak{gl}(n), \mathfrak{gl}(n)] = \mathfrak{sl}(n)$.

The *derived series of a Lie algebra* $\mathfrak{g}$ is the sequence of ideals $\mathfrak{g}^{(0)} = \mathfrak{g}, \mathfrak{g}^{(1)} = [\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}], \ldots, \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}]$. The *lower central series of a Lie algebra* $\mathfrak{g}$ is the sequence of ideals $\mathfrak{g}^0 = \mathfrak{g}, \mathfrak{g}^1 = [\mathfrak{g}^0, \mathfrak{g}^0], \ldots, \mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]$.

Definition 3.3.17. A *solvable Lie algebra* is a Lie algebra $\mathfrak{g}$ such that $g^{(k)} = 0$ for some $k \in \mathbb{N}_0$. We say $\mathfrak{g}$ is a *nilpotent Lie algebra* if $\mathfrak{g}^k = 0$ for some $k \in \mathbb{N}_0$. Nilpotent algebras are solvable.

Example 3.3.18. The Lie algebras $\mathfrak{t}(n)$ and $\mathfrak{n}(n)$ are solvable. Indeed, $\mathfrak{n}(n)$ is nilpotent.

The diagonal matrices $\mathfrak{d}(n)$ is a Lie algebra that is both nilpotent and self-normalizing within the parent algebra $\mathfrak{gl}(n)$. Such a Lie algebra is called a *Cartan algebra*. When Cartan algebras exist, they play an important role in the representation theory of Lie algebras.

We now describe some families of Lie algebras akin to the classical groups of *Chapter* 2. They are subalgebras of $\mathfrak{sl}(n)$ and are simple in most cases. Together with $\mathfrak{sl}(n)$ they form four infinite series of linear Lie algebras called classical Lie algebras.

Remark 3.3.19. An explanation of the correspondence between so-called connected Lie groups and linear Lie algebras through the exponential map and 1-parameter subgroups is carefully detailed in [34, 36].

Remark 3.3.20. We can choose a Cartan algebra $\mathfrak{h}_\mathfrak{g}$ of each Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(m)$ below as $\mathfrak{h}_\mathfrak{g} = \mathfrak{d}(m) \cap \mathfrak{g}$.

Definition 3.3.21. The *symplectic Lie algebra* $\mathfrak{sp}(2n) \subset \mathfrak{gl}(2n)$ is the set of $2n \times 2n$ matrices M such that $J_{2n}^{skew} M + M^T J_{2n}^{skew} = 0$. We can express each element M of $\mathfrak{sp}(2n)$ as a block matrix with $n \times n$ matrices A, B, C:

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & -A^T \end{array}\right]$$

where B and C are symmetric matrices.

Definition 3.3.22. The *orthogonal Lie algebra* $\mathfrak{so}(2n) \subset \mathfrak{gl}(2n)$ is the set of $2n \times 2n$ matrices M such that $J_{2n}^{sym} M + M^T J_{2n}^{sym} = 0$. We can express each element of $\mathfrak{so}(2n)$ as a block matrix with $n \times n$ matrices A, B, C:

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & -A^T \end{array}\right]$$

where B and C are skew-symmetric matrices.

Definition 3.3.23. The *orthogonal Lie algebra* $\mathfrak{so}(2n+1) \subset \mathfrak{gl}(n+1)$ is the set of $(2n+1) \times (2n+1)$ matrices M such that $J_{2n+1}^{sym} M + M^T J_{2n+1}^{sym} = 0$. For completeness, $\mathfrak{so}(1) = 0$. We can express each element M of $\mathfrak{so}(2n+1)$ as a block matrix with $n \times n$ matrices A, B, C and $1 \times n$ row vectors r, s:

$$M = \left[\begin{array}{c|c|c} 0 & r & s \\ \hline -s^t & A & B \\ \hline -r^T & C & D \end{array}\right],$$

where B and C are skew-symmetric matrices

3.3.2 Universal Enveloping Algebras & Representation Theory

Earlier we associated an associative unital algebra to a vector space carrying a particular form by constructing a quotient of the tensor algebra. Now we construct an associative algebra $\mathcal{U}(\mathfrak{g})$ affiliated with the generally non-associative Lie algebra $\mathfrak{g}$. The result of this process connects the representation theory of Lie algebras to the representation theory of associative algebras. In particular, if V is a $\mathcal{U}(g)$-representation, then V is a $\mathfrak{g}$-representation; the converse is true, as well.

A definition of $\mathfrak{g}$-representation is given here for thoroughness.

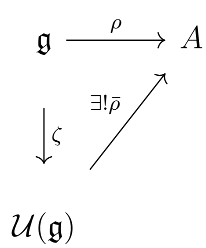
Definition 3.3.24. A *representation of a Lie algebra* $\mathfrak{g}$ is a Lie algebra map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. We can also speak of the pair (ρ, V) or V by itself as a $\mathfrak{g}$-representation.

Example 3.3.25. Let V be a vector space of dimension n. If $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$, then the inclusion map ι: $\mathfrak{g} \to \mathfrak{gl}(V)$ is the *standard representation of* $\mathfrak{g}$. Normally, we refer to the vector space $\mathbb{C}^n$ as the standard representation $\mathfrak{g}$.

Definition 3.3.26. Given a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ with some abstract Lie bracket, form the quotient $T(V)/I$, where I is the ideal generated by elements of the form $x \otimes y - y \otimes x - [x, y]$. In this case, we use the term *universal enveloping algebra* of $\mathfrak{g}$ for the quotient $\mathcal{U}(\mathfrak{g}) = T(V)/I$.

By the observation made in remark 3.3.3, we view $\mathcal{U}(\mathfrak{g})$ as a Lie algebra with the commutator as the Lie bracket. The relations of the embedded Lie algebra $\mathfrak{g}$ remain and are key in justifying the use of "universal" as seen in the following proposition.

Proposition 3.3.27. *The universal enveloping algebra is the pair $(\mathcal{U}(\mathfrak{g}), \zeta)$ satisfying the following universal property: For any Lie algebra map $\rho : \mathfrak{g} \to A$ to an associative algebra A, there exists a unique map of associative algebras $\bar{\rho} : \mathcal{U}(\mathfrak{g}) \to A$ such that $\bar{\rho}\zeta = \rho$. That is, the following diagram is a commutative diagram:*



The map ζ is the composition of the natural projection $T(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ with the inclusion ι from Proposition 3.2.7.

Certainly, ρ above could be any $\mathfrak{g}$-representation. Alternatively, a map of algebras $\mathcal{U}(\mathfrak{g}) \to \mathrm{End}(V)$ restricts to a bracket-preserving map $\mathfrak{g} \to \mathrm{End}(V)$ (using the identification induced by ι). Then we have evidence to support the representation-theoretic claims of the paragraph introducing this subsection.

The representations of a Lie algebra are also called $\mathfrak{g}$-modules, which seems to make sense when considering $\mathcal{U}(\mathfrak{g})$ is a ring and we now equate (left) modules of $\mathcal{U}(\mathfrak{g})$ with representations of $\mathfrak{g}$. Formally, a $\mathfrak{g}$-module is a vector space carrying a $\mathfrak{g}$-action as described below:

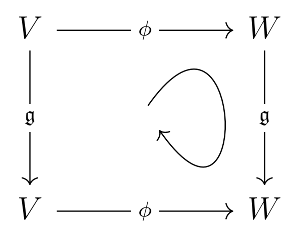
Definition 3.3.28. Let V be a vector space and $\mathfrak{g}$ a Lie algebra. Then V is a *module of* $\mathfrak{g}$, more commonly, a $\mathfrak{g}$-module, when there is a well-defined $\mathfrak{g}$-action $\mathfrak{g} \times V \to V$, $(x, v) \to x.v$, obeying

$$x.(v + w) = x.v + x.w, \;\; x \in \mathfrak{g}, \;\; v, w \in V \tag{3.6}$$

$$(x + y).(v) = x.v + y.v, \;\; x, y \in \mathfrak{g}, v \in V \tag{3.7}$$

$$[x, y].v = x.(y.v) - y.(x.v), \;\; x, y \in \mathfrak{g}, v \in V. \tag{3.8}$$

Definition 3.3.29. Give a Lie algebra $\mathfrak{g}$, a *map of* $\mathfrak{g}$*-modules* is called an *intertwining map*. That is, if V and W are $\mathfrak{g}$-modules, then a linear map $\phi : V \to W$ is an intertwining map if we have the following commutative diagram:



,

where $\mathfrak{g}$ on the left is for the $\mathfrak{g}$-action on V and $\mathfrak{g}$ on the right is for the $\mathfrak{g}$-action on W.

Example 3.3.30. A Lie algebra $\mathfrak{g}$ with any subalgebra $\mathfrak{h}$ is always an $\mathfrak{h}$-module when $x.y$ is defined to be $[x, y]$, $\forall x \in \mathfrak{h}$, $\forall y \in \mathfrak{g}$. In particular, $\mathfrak{g}$ is a $\mathfrak{g}$-module. What is described is the *adjoint representation* given by $\mathrm{ad} : x \mapsto [x, \cdot] \in \mathrm{End}(\mathfrak{g})$. The image $\mathrm{ad}(x)$ is called the *adjoint map* of x.

We give a detailed example of an intertwining map using $\mathfrak{g} = \mathfrak{sl}(2)$, as the concept will be crucial to future results.

Example 3.3.31. Let

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Using linear algebra arguments one can see that $\mathfrak{g} = \mathfrak{sl}(2) = \text{span}(\{e, f, h\})$. Now $\mathfrak{g}$ acts on $V = \mathbb{C}^2$ by matrix multiplication via the standard representation. We also have a $\mathfrak{g}$-action on the space of homogeneous bivariate polynomials of degree t, say, $P_t = \mathbb{C}_t[x, y]$, by differential operators. Exactly, $e \cdot p(x, y) = x\frac{\partial p}{\partial y}$, $f \cdot p(x, y) = y\frac{\partial p}{\partial x}$, $h \cdot p(x, y) = x\frac{\partial p}{\partial x} - y\frac{\partial p}{\partial y}$. Let us take $t = 1$. Then V and P_1 are isomorphic as $\mathfrak{g}$-modules via the intertwining isomorphism $\phi : V \to W$ defined on basis elements by

$$\phi\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = x, \quad \phi\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = y.$$

Each of the definitions below have an analogous description via maps of Lie algebras as in definition 3.3.24.

Definition 3.3.32. A subspace W of a $\mathfrak{g}$-module V is called a *submodule of* $\mathfrak{g}$, $\mathfrak{g}$-submodule of V, if W is invariant under the action, i.e., $x.w \in W$, $\forall w \in W$. If $\mathfrak{g}$ is understood in context, then we find it appropriate to call W a submodule of V.

Definition 3.3.33. A non-zero $\mathfrak{g}$-module V is a *simple module* or *irreducible module* if its only submodules are 0 and V. We say a $\mathfrak{g}$-module V is a *reducible module* if it has a proper non-zero submodule and a *completely reducible module* is one that is semisimple in the sense that V is the sum of simple modules. A reducible module may be an *indecomposable module*, one that is not the direct sum of proper submodules of any sort.

We end this subsection on Lie algebras with two satisfying results—one on the classification of simple Lie algebras and the other related to the representation theory of semisimple Lie algebras—before heading towards super algebras and super representation theory.

Proposition 3.3.34 (Killing-Cartan classification of finite-dimensional simple Lie algebras)**.** *By definition, a finite-dimensional semisimple Lie algebra* $\mathfrak{g}$ *is the direct sum* $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ *of simple Lie algebras* $\mathfrak{g}_i$, $1 \leq i \leq k$. *It can be shown such a decomposition is unique up to reordering of summands. Moreover, the Killing-Cartan classification states that*

$$\mathfrak{g}_i \in \{\mathfrak{sl}(n+1) \mid n \geq 1\} \cup \{\mathfrak{so}(2n+1) \mid n \geq 2\} \cup \{\mathfrak{sp}(2n) \mid n \geq 3\} \\ \cup \{\mathfrak{so}(2n) \mid n \geq 4\} \cup \{\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2\}.$$

Or, in tabular form, the simple finite-dimensional (complex) Lie algebras are the following:

Series (rank $=$ ***n)***	***Lie Algebra***
$A_n,\ n \geq 1$	*special linear algebra* $\mathfrak{sl}(n+1)$
$B_n,\ n \geq 2$	*(special) orthogonal algebra* $\mathfrak{so}(2n+1)$
$C_n,\ n \geq 3$	*symplectic algebra* $\mathfrak{sp}(2n)$
$D_n,\ n \geq 4$	*(special) orthogonal algebra* $\mathfrak{so}(2n)$
$E_n,\ n = 6, 7, 8$	*the exceptional Lie algebras* $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$
$F_n = F_4$	*the exceptional Lie algebras* $\mathfrak{f}_4$
$G_n = G_2$	*the exceptional Lie algebras* $\mathfrak{g}_2$

Proposition 3.3.35 (Weyl's theorem on complete reducibility). *Let* $\mathfrak{g}$ *be a finite-dimensional semisimple Lie algebra. Then every non-trivial finite-dimensional* $\mathfrak{g}$*-module is completely reducible.*

The infinite-dimensional representation theory of Lie algebras does not have such a classification as the results of Killing and Cartan provide. In the world of Lie superalgebras, even the status of being finite-dimensional and simple does not allow for a broad analogue to Weyl's theorem. Fortunately, there is a Lie superalgebra amongst the orthosymplectic Lie superalgebras which has very nice properties, including a Weyl theorem parallel. The infinite-dimensional representation theory is wide open, however. The following chapters introduce Lie superalgebras, define a certain orthosymplectic Lie superalgebra $\mathfrak{g}$, and explores explicit descriptions of certain infinite-dimensional $\mathfrak{g}$-representations, including a new representation space of polynomials in both commuting and anti-commuting variables.

CHAPTER 4

Superalgebras

While the goal of this work can be expressed in purely algebraic terms, and we do tread an algebraic road, an exploration in algebra allows a picturesque view of geometry, as well. The geometric connections are broad in application and conception. One bridge between algebra and geometry is found in the aptly named area of algebraic geometry, which relies on the tools of commutative algebra. Whereas the algebras herein can be both noncommutative and non-associative, previous chapters described vector spaces possessing some kind of geometry (certain bilinear spaces) affiliated with Clifford and Weyl algebras, exterior and symmetric algebras, and classical Lie algebras, few of which are commutative. Symplectic geometry begins with a symplectic form as an element of the exterior algebra and involves (formal) Weyl algebras as another example of the algebra-geometry link. Still, there are other concepts of geometry, for example, noncommutative geometry [29] and supergeometry [30], arising from beautiful mathematics and physical theories. These less-classical geometries rely on the foundations of Hopf algebras [see 42] and the so-called quantum groups [29, again], superalgebras, and Lie superalgebras. This chapter serves to develop a background in superalgebras and Lie superalgebras and the prerequisite super Linear algebra to define the extra structure on the underlying vector spaces. Here we encounter $\mathbb{Z}_2$-graded vector spaces, often called super vector spaces[1] or super spaces. The foundations of $\mathbb{Z}_2$-graded vector spaces and general G-graded algebraic structures, where G a

[1] Technically, the symmetric monoidal category of super vector spaces has a different braiding than that of the symmetric monoidal categroy of $\mathbb{Z}_2$-graded vector spaces.

monoid, preferably a commutative ring, is in [2, 28, 40]. Standard introductions to Lie superalgebras include [5, 32]. We follow a similar course.

Throughout, we set $\mathbb{Z}_2$ to be $\mathbb{Z}/2\mathbb{Z}$ and denote the elements of $\mathbb{Z}_2$ by $\bar{0}$ and $\bar{1}$.

4.1 Super Linear Algebra

Definition 4.1.1. Let V be a vector space. We say V is a *super vector space* if a choice of subspaces $V_{\bar{0}}$ and $V_{\bar{1}}$ is made such that $V = V_{\bar{0}} \oplus V_{\bar{1}}$. We say V is graded by $\mathbb{Z}_2$ or has a $\mathbb{Z}_2$*-grading*.

We freely use the term "super space" to refer to the now-defined "super vector space". Note that a super space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is not the same super space as $V' = V'_{\bar{0}} \oplus V'_{\bar{1}}$, where $V'_{\bar{0}} = V_{\bar{1}}$ and $V'_{\bar{1}} = V_{\bar{0}}$. The definition of super space depends on a choice[2].

Definition 4.1.2. For a super space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, call $V_{\bar{0}}$ the *even part of* V, the vector subspace of *even elements*; likewise, call $V_{\bar{1}}$ the *odd part of* V, the vector subspace of *odd elements*.

Example 4.1.3. The vector space $\mathrm{Bil}(V)$ of bilinear forms on a fixed space V decomposes into the direct sum of the even space of symmetric bilinear forms and the odd space of skew-symmetric forms.

Remark 4.1.4. As a

Example 4.1.5. The vector space $Fun(\mathbb{R})$ decomposes into subspaces comprising even functions and odd functions, respectively.

The dimension $\dim(V)$ of a super space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is the sum $d = m + n$, where $\dim(V_{\bar{0}}) = m$ and $\dim(V_{\bar{1}}) = n$. It is common to write $\dim(V) = (m|n)$. A basis $\{x_i; x_k\}$ for a super space is sometimes written with even elements before a

[2]Equivalently, a choice of $\mathbb{Z}_2$-grading is defining a representation of the group $\mathbb{Z}_2$. The underlying vector spaces may be the same, though the actions may differ.

semi-colon, with $i \in I_{\bar{0}}$, $k \in I_{\bar{1}}$, for some indexing set $I = I_{\bar{0}} \sqcup I_{\bar{1}}$. We will often suppress the semi-colon.

An important example of a super vector space is $\mathbb{C}^{m|n} = \mathbb{C}^m \oplus \mathbb{C}^n$.

Definition 4.1.6. Let V be a super space. Elements of $V_{\bar{0}} \sqcup V_{\bar{1}}$ are called *homogeneous elements* of V.

Remark 4.1.7. Definition 4.1.6 provides a natural *parity map* $p : V_{\bar{0}} \sqcup V_{\bar{1}} \to \mathbb{Z}_2$ (sometimes $|\cdot| : V_{\bar{0}} \sqcup V_{\bar{1}} \to \mathbb{Z}_2$) from homogeneous vectors of V to $\mathbb{Z}_2$. Indeed, the use of $p(x)$ implies x is homogeneous, while a homogeneous element $x_{\bar{i}}$ may carry a subscript for emphasis.

More generally, we can defined the degree of a homogeneous element in a G-*graded space*, where G is a monoid. For example, $V = \bigoplus_{i \in \mathbb{N}_0} V_i$, with $V_i = \mathrm{span}_{\mathbb{C}}(\{x^i\})$ is the $\mathbb{N}$-*graded vector space* we recognize as the complex polynomials in one indeterminate. The *degree map* deg is defined on homogeneous elements by $deg(x) = i$ for $x \in V_i$. A $\mathbb{Z}$-graded space of V is quickly made by setting $V_i = 0$, whenever $i < 0$.

Remark 4.1.8. Let $V = \bigoplus_{i \in \mathbb{Z}} V_i$ be a $\mathbb{Z}$-graded vector space. Then we have at least one natural $\mathbb{Z}_2$-grading from the $\mathbb{Z}$-grading by making $V_{\bar{0}} = \bigoplus_{i \in 2\mathbb{Z}} V_i$ and $V_{\bar{1}} = \bigoplus_{i \in 2\mathbb{Z}+1} V_i$.

Taking zero to be simultaneously even with parity $\bar{0}$ and odd with parity $\bar{1}$ has no adverse effects; similarly, zero can have any degree, and sometimes it is not defined.

Definition 4.1.9. A *map of super spaces* $\phi : V \to W$ is a *parity-preserving map*. In particular, for every $x_{\bar{0}} + x_{\bar{1}} = x \in V$, $\phi(x) = \phi(x_{\bar{0}}) + \phi(x_{\bar{1}})$, with $\phi(x_{\bar{i}}) \in W_{\bar{i}}$, $\bar{i} \in \mathbb{Z}_2$.

A *parity-reversing map* ϕ is one where $\phi(x_{\bar{i}}) \in W_{\overline{i+1}}$, for all homogeneous $x_{\bar{i}}$.

Proposition 4.1.10. *If V and W are super spaces and $\phi : V \to W$ is linear, then there exists a unique pair $(\phi_{\bar{0}}, \phi_{\bar{1}})$ of linear maps such that $\phi = \phi_{\bar{0}} + \phi_{\bar{1}}$, the map $\phi_{\bar{0}}$ is parity-preserving, and the map $\phi_{\bar{1}}$ is parity-reversing.*

Proof. Suppose $\phi : V \to W$ is a linear map with $V = V_{\bar{0}} + V_{\bar{1}}$ and $W = W_{\bar{0}} + W_{\bar{1}}$. Choose a basis $B_V = \{x_i; x_k\}$ for V and a basis $B_W = \{y_j; y_l\}$ for W, with $x \in B$. If $\phi(x) = \sum_{j=1}^{m} \alpha_j y_j + \sum_{l=1}^{n} \lambda_l y_l$, then define $\phi_{\bar{0}}$ and $\phi_{\bar{1}}$ by

$$\phi(x)_{\bar{0}} = \begin{cases} \sum_{j=1}^{m} \alpha_j y_j, & \text{if } x \text{ is even} \\ \sum_{l=1}^{n} \lambda_l y_l, & \text{if } x \text{ is odd} \end{cases}$$

$$\phi(x)_{\bar{1}} = \begin{cases} \sum_{l=1}^{n} \lambda_l y_l, & \text{if } x \text{ is even} \\ \sum_{j=1}^{m} \alpha_j y_j, & \text{if } x \text{ is odd.} \end{cases}$$

Extend by linearity. □

Corollary 4.1.11. *For a super space V, the vector space* End(V) *is a super space, as well.*

Remark 4.1.12. We will rely on Corollary 4.1.11 as, again, End(V) plays a key role in representation theory. Note that the space End$(V)_{\bar{0}}$ consists of exactly the maps ϕ on the super space V.

Definition 4.1.13. A *super subspace* V of $W = W_{\bar{0}} \oplus W_{\bar{1}}$ is a super space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ such that $V_{\bar{i}} \subset W_{\bar{i}}$, $\bar{i} \in \mathbb{Z}_2$.

4.2 Lie Superalgebras

Many of the definitions below will be near-familiar as analogues of those found in Section 3.3.1.

Definition 4.2.1. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a superspace. A Lie superalgebra is a pair $(\mathfrak{g}, [\ ,\ ])$ such that $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear product (called the *Lie superbracket*) on $\mathfrak{g}$ and the following properties hold for all homogeneous vectors $x, y, z \in \mathfrak{g}$.

$$[\mathfrak{g}_{\bar{i}}, \mathfrak{g}_{\bar{j}}] \subset \mathfrak{g}_{\bar{i}+\bar{j}},\ \forall \bar{i}, \bar{j} \in \mathbb{Z}_2 \tag{4.1}$$

$$[x, y] = -(-1)^{p(x)p(y)}[y, x] \tag{4.2}$$

$$[x, [y, z]] = [[x, y], z] + (-1)^{p(x)p(y)}[y, [x, z]]. \tag{4.3}$$

Equation (4.1) means the Lie superbracket is a *parity-respecting product*; Equation (4.2) is the equation defining *super skew-symmetry*; and, Equation (4.3) is the super Jacobi identity.

Remark 4.2.2. It should be clear that any vector space $\mathfrak{g}$ has a trivial decomposition $\mathfrak{g} = \mathfrak{g} \oplus \{0\}$ as a super space. Moreover, a trivial Lie (super)bracket exists in all cases, and, in the case $\mathfrak{g}$ is a Lie algebra, then $\mathfrak{g}$ has the structure of a Lie superalgebra using the original Lie bracket.

Definition 4.2.3. *Lie subsuperalgebra*s $\mathfrak{h}$ are super subspaces of a Lie superalgebra $\mathfrak{g}$ such that $\mathfrak{h}$ is closed under the Lie superbracket.

Definition 4.2.4. A *map of Lie superalgebras* $\phi : \mathfrak{g} \to \mathfrak{h}$ is parity-preserving map that is linear and bracket-preserving as in definition 3.3.1. We define isomorphisms similarly.

Definition 4.2.5. Let X be a subset of a Lie superalgebra $\mathfrak{g}$. The *Lie subsuperalgebra* $\langle X \rangle_{\mathfrak{g}}$ *generated by* X is the span of all Lie superbrackets

$$\langle X \rangle_{\mathfrak{g}} = \left\{ \sum_{i=1}^{n} \alpha_i y_i \;\middle|\; y_i = [x_1, [x_2, ...[x_k, x_{k+1}]...]],\ x_i \in X,\ \alpha_i \in \mathbb{C},\ k, n \in \mathbb{N} \right\}.$$

We say that a set X *generates a Lie superalgebra* $\mathfrak{g}$ if $\langle X \rangle_{\mathfrak{g}} = \mathfrak{g}$. If X is finite then $\mathfrak{g}$ is a *finitely-generated Lie superalgebra.* The empty set generates the zero Lie superalgebra.

Proposition 4.2.6. *Let $\mathfrak{g} = \langle X \rangle_{\mathfrak{g}}$ and $\mathfrak{h}$ be Lie superalgebras with a map of superspaces $\phi : \mathfrak{g} \to \mathfrak{h}$. Then ϕ is a map of Lie superalgebras if ϕ preserves the brackets on generators $x \in X$. That is, $\phi([x,y]_{\mathfrak{g}}) = [\phi(x), \phi(y)]_{\mathfrak{h}}$ for all generators $x, y \in X$.*

4.2.1 The general linear Lie superalgebra

Let V be a super space. The ubiquity of $\mathrm{End}(V)$ emerges once more.

Definition 4.2.7. The *general linear Lie superalgebra* $\mathfrak{gl}(V)$ is the pair $(\mathrm{End}(V), [\ ,\ ])$ where $[X, Y] = (-1)^{p(X)p(Y)}XY - YX$ is the supercommutator on homogeneous endomorphisms X, Y.

If a vector space V has dimension $m+n$, then we know $\mathrm{End}(V)$ and $\mathrm{End}(m+n)$ are isomorphic vector spaces. For similar reasons, the Lie superalgebra $\mathrm{End}(V)$ is isomorphic to $\mathfrak{gl}(m|n)$, which is defined below.

Definition 4.2.8. The *Lie superalgebra* $\mathfrak{gl}(m|n)$ consists of block matrices

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A \in \mathrm{Mat}(m)$ and $D \in \mathrm{Mat}(n)$. The *even matrices* are those block matrices X with $B = C = 0$, while the *odd matrices* have $A = D = 0$.

The even part $\mathfrak{gl}(m|n)_{\bar{0}}$ of $\mathfrak{gl}(m|n)$ is isomorphic to the direct sum $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$ of two general linear Lie algebras.

Remark 4.2.9. The even part $\mathfrak{g}_{\bar{0}}$ of a Lie superalgebra $\mathfrak{g}$ is a Lie algebra with Lie bracket taken to be the superbracket restricted to $\mathfrak{g}_{\bar{0}}$.

The Lie superalgebra analogue of $\mathfrak{sl}(n)$ is the Lie superalgebra $\mathfrak{sl}(m|n)$ consisting of all block matrices $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in $\mathfrak{gl}(m|n)$ such that $\mathrm{str}\,(X) = 0$, where the *supertrace* $\mathrm{str}\,(X)$ of X is equal to $\mathrm{tr}\,(A) - \mathrm{tr}\,(D)$.

For matrix Lie superalgebras, we can choose a subset $B \subset \{E_{ij}\}$ of elementary matrices as a basis.

Proposition 4.2.10. *The Lie superbracket is defined on basis elements of* $\mathfrak{gl}(m|n)$ *as follows:*

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - (-1)^{p(E_{ij})p(E_{kl})}\delta_{il}E_{kj}.$$

4.2.2 Modules of Lie superalgebras

Representations of a Lie algebra $\mathfrak{g}$ and $\mathfrak{g}$-modules are equivalent. The same holds for Lie superalgebras, and we will move to using the term "module" more so than "representation" for the sake of reading consistency. In this subsection, we provide a few definitions of supermodules. The reader is encouraged to employ a super version of the module theory, including examples, found in Section 3.3.2.

Definition 4.2.11. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a super vector space and $\mathfrak{g}$ a Lie superalgebra. Then V is a *supermodule of* $\mathfrak{g}$, more succinctly, a $\mathfrak{g}$-supermodule, when there is a well-defined action $\mathfrak{g} \times V \to V$, $(x, v) \to x.v$, obeying

$$p(x.v) = p(x) + p(v) = \bar{i} + \bar{j},\ x \in g_{\bar{i}}, v \in V_{\bar{j}} \tag{4.4}$$

$$x.(v + w) = x.v + x.w,\ x \in \mathfrak{g},\ v, w \in V \tag{4.5}$$

$$(x + y).(v) = x.v + y.v,\ x, y \in \mathfrak{g}, v \in V \tag{4.6}$$

$$[x, y].v = x.(y.v) - (-1)^{p(x)p(y)}y.(x.v),\ x, yx \in g_{\bar{i}}, v \in V_{\bar{j}} \tag{4.7}$$

Remark 4.2.12. Equation (4.4) ensures that the action of $\mathfrak{g}$ on V is compatible with the $\mathbb{Z}_2$-gradings of $\mathfrak{g}$ and V.

Clearly, quotients of super spaces by super subspaces are again superspaces.

Definition 4.2.13. Let $\mathfrak{g}$ be a Lie superalgebra with supermodules $W \subset V$. The *quotient module* V/W is the $\mathfrak{g}$-supermodule defined by $x \cdot (v + W) = x.v + W,\ \forall v \in V,\ \forall x \in \mathfrak{g}$.

4.3 Associative superalgebras

As in Chapter 3, we now establish a correspondence between the non-associative Lie superalgebra $\mathfrak{g}$ and an associative (super)algebra $\mathcal{U}(\mathfrak{g})$. This universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ will have a $\mathbb{Z}_2$-grading but does not require construction via a graded tensor product of super vector spaces. Still, we provide a definition of a super tensor product. We also relay a specific version of the celebrated PBW theorem which provides a basis for $\mathcal{U}(\mathfrak{g})$ when $\mathfrak{g}$ has finite dimension. Lastly, the universal enveloping algebra plays a part in determining the simplicity of a super module.

Definition 4.3.1. A complex *associative unital superalgebra* is a super vector space $A = A_{\bar{0}} \oplus A_{\bar{1}}$, with a bilinear *parity-respecting product* $(a, b) \mapsto ab \in A$, for all $a, b \in A$. That is, $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is an associative unital algebra with $A_{\bar{i}} A_{\bar{j}} = A_{\overline{i+j}}$, for $\bar{i}, \bar{j} \in \mathbb{Z}_2$.

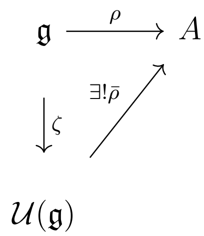
Remark 4.3.2. We have the super version of remark 3.3.3. Namely, any associative superalgebra with the supercommutator, $[x, y]{=}xy - (-1)^{p(x)p(y)}yx$, for all homogeneous elements x and y, is a Lie superalgebra.

Definition 4.3.3. Let A and B be associative superalgebras. A map $\phi : A \to B$ is a *map of associative superalgebras* if ϕ is a map of super spaces such that $\phi(xy) = \phi(x)\phi(y)$, for all $x, y \in A$.

4.3.1 Universal enveloping algebras of Lie superalgebras

We appeal to a superization of *Proposition* 3.3.27.

Definition 4.3.4. The *universal enveloping algebra of a Lie superalgebra* $\mathfrak{g}$ is the pair $(\mathcal{U}(\mathfrak{g}), \zeta)$ satisfying the following *universial property*: For any Lie superalgebra map $\rho : \mathfrak{g} \to A$ to an associative superalgebra A, there exists a unique map of associative superalgebras $\bar{\rho} : \mathcal{U}(\mathfrak{g}) \to A$ such that $\bar{\rho}\zeta = \rho$. That is, the following diagram is a commutative diagram:



The map ζ is the composition of the natural projection $T(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ with the inclusion ι from Proposition 3.2.7.

We use a variation of the PBW theorem as given in [Chapter 1 of 5, Section 5].

Proposition 4.3.5 (Poincaré-Birkhoff-Witt (PBW) Theorem)**.** *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional Lie superalgebra. Let $B_{\mathfrak{g}_{\bar{0}}} = \{x_1, \ldots, x_k\}$ be a basis for the even space and $B_{\mathfrak{g}_{\bar{1}}} = \{y_1, \ldots, y_l\}$ a basis for the odd space. Then the set*

$$\{x_1^{r_1} \cdots x_1^{r_k} y_1^{s_1} \cdots y_l^{s_l} \mid r_i \in \mathbb{N}_0,\ s_j \in \{0, 1\}\}$$

is a $\mathcal{U}(\mathfrak{g})$-basis.

The following proposition provides a key mechanism for proving that super modules are simple and relies on universal enveloping algebras of Lie superalgebras.

Proposition 4.3.6. *Let $\mathfrak{g}$ be a Lie superalgebra with universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and let M be a $\mathfrak{g}$-module. Assume that there is $u \in M$ such that*

(i) for every $w \in M$ there exists $X \in \mathcal{U}(\mathfrak{g})$ such that $w = X(u)$;

(ii) for every $v \in M$ there exists $Y \in \mathcal{U}(\mathfrak{g})$ such that $u = Y(v)$.

Then M is a simple module.

Proof. Suppose that K is a non-trivial submodule of M. It then suffices to show that $K = M$. Let $v \in K \setminus \{0\}$. Then $u = Y(v)$ for some $Y \in \mathcal{U}(\mathfrak{g})$. Therefore $u \in K$. But then for any $w \in M$ there exists $X \in \mathcal{U}(\mathfrak{g})$ such that $w = X(u)$. Thus $w \in K$ which implies that $K = M$. $\square$

Interestingly enough, we did not define the "super tensor algebra" in order to define the universal enveloping algebra of a Lie superalgebra. Indeed, the tensor algebra as normally constructed will be graded in the manner of remark 4.1.8 (and as the Weyl superalgebra in the next subsection). Nevertheless, we define a particular graded tensor product here.

Definition 4.3.7. The *super tensor product* $V \otimes W$ on super spaces V and W is the super space $V \otimes W = (V \otimes W)_{\bar{0}} \oplus (V \otimes W)_{\bar{1}}$, using the tensor product of vector spaces, where

$$(V \otimes W)_{\bar{0}} = (V_{\bar{0}} \otimes W_{\bar{0}}) \oplus (V_{\bar{1}} \otimes W_{\bar{1}}) \tag{4.8}$$

$$(V \otimes W)_{\bar{1}} = (V_{\bar{0}} \otimes W_{\bar{1}}) \oplus (V_{\bar{1}} \otimes W_{\bar{0}}). \tag{4.9}$$

In particular, $p(a \otimes b) = p(a) + p(b)$, for all tensors $a \otimes b$.

Remark 4.3.8. We could define a $\mathbb{Z}_2$-graded tensor product on $\mathbb{Z}_2$-graded vector spaces in the exact same way. The subtlety[3] appears in the tensor product of superalgebras versus the tensor product of $\mathbb{Z}_2$-graded algebras. A tensor product of superalgebras is again a superalgebra with a product extended from this definition on homogeneous tensors: $(a \otimes b)(c \otimes d) = (-1)^{p(b)p(c)} ac \otimes bd$. No such power of -1 is required in the non-super, but $\mathbb{Z}_2$-graded, case.

4.4 Weyl superalgebras

We close the chapter describing the n^{th} Weyl superalgebra.

Recall the n^{th} Weyl algebra $\mathcal{D}(n)$ of Chapter 3. In particular, refer to section 3.1 to define $\mathcal{D}(n)$ as the algebra of polynomial differential operators whose underlying vector space is the span of monomials

$$\text{span}(\{x_{i_1} \cdots x_{i_k} \partial_{x_{j_1}} \cdots \partial_{x_{j_l}} \mid 1 < i_1, \ldots, i_k, j_1, \ldots, j_l \leq n,\ k, l \in \mathbb{N}_0\}) \subset \text{End}(\mathbb{C}[x_1, \ldots, x_n]).$$

[3] The categorical approach/difference was hinted at in a footnote of this chapter's introduction

To define parity on $\mathcal{D}(n)$, we establish a $\mathbb{Z}$-grading of $\mathcal{D}(n)$ as follows. Set $deg(x_i) = 1$ and $\deg(\partial_{x_i}) = -1$ before extending the degree map on $\mathcal{D}(n)$ multiplicatively. Namely,

$$deg(x_{i_1} \cdots x_{i_k} \partial_{x_{j_1}} \cdots \partial_{x_{j_l}}) = \sum_{r=1}^{k} i_p - \sum_{q=1}^{l} j_q.$$

Now $\mathcal{D}(n)$ is an associative unital superalgebra by remark 4.1.8. We will use $\mathcal{D}(n)$ to construct $\mathfrak{g}$-modules of certain Lie superalgebras introduced in the next chapter.

CHAPTER 5

Orthosymplectic Lie Superalgebras

As with Lie algebras, there is a nice classification of simple Lie superalgebras over $\mathbb{C}$ [23]. One class of simple Lie superalgebras, and an example of the so-called classical Lie superalgebra [24], are the *orthosymplectic Lie superalgebras*, which we will denote by $\mathfrak{osp}(m|2n)$. [1] In this chapter we first introduce the necessary definitions for orthosymplectic Lie superalgebras and then focus on the case $\mathfrak{osp}(1|2n)$. This chapter gives a similar background to that found in [14] before new results are given here in Chapter 6 and in Chapter 7.

5.1 The Lie Superalgebra $\mathfrak{osp}(1|2n)$

Definition 5.1.1. Let $\mathfrak{g}$ be a Lie superalgebra. A bilinear form B on $\mathfrak{g}$ is said to be *supersymmetric* if $\beta(x,y) = (-1)^{p(x)p(y)}\beta(y,x)$ for all $x, y \in g$.

Definition 5.1.2. An *even form* β on a Lie superalgebra $\mathfrak{g}$ is a form satisfying $\beta(x,y) = 0$ whenever $p(x) \neq p(y)$ for all homogeneous elements x and y in $\mathfrak{g}$.

Given an even nondegenerate supersymmetric bilinear form β, the *orthosymplectic Lie superalgebra* is defined as the Lie superalgebra of linear maps preserving the pair (V, β) where V is a finite-dimensional super space. Perhaps a more concrete description follows when β, relative to the standard basis, is associated to the matrix $s = \begin{bmatrix} G & 0 \\ 0 & H \end{bmatrix}$, where $G = J_{2n}^{sym}$ or J_{2n+1}^{sym} and $H = J_{2n}^{skew}$.

[1] The orthosymplectic Lie superalgebras comprise the three families $B(m|n) = \mathfrak{osp}(2m+1|n),\ m,n \geq 0$, $C(n) = \mathfrak{osp}(2|2n-2),\ n \geq 2$, and $D(m|n) = \mathfrak{osp}(2m|2n),\ m \geq 2, n \geq 1$.

Definition 5.1.3. Define the *orthosymplectic Lie superalgbera* $\mathfrak{osp}(m|2n)$ as a super analogue of a linear Lie algebra. That is, $\mathfrak{osp}(1|2n)$ is the Lie subsuperalgeba

$$\left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{gl}(m|2n) \mid A^tG + GA = B^tG - HC = D^tH + HD = 0 \right\}$$

of $\mathfrak{gl}(m|2n)$.

We note that for $\mathfrak{g} = \mathfrak{osp}(m|2n)$, we have that $\mathfrak{g}_{\bar{0}} \cong \mathfrak{so}(m) \bigoplus \mathfrak{sp}(2n)$. Most of the remaining discussion will focus on the case $m = 1$ when $\mathfrak{osp}(1|2n)_{\bar{0}} \cong \mathfrak{sp}(2n)$.

Again, the only orthosymplectic Lie superalgebra of concern will be $\mathfrak{osp}(1|2n)$:

$$\mathfrak{osp}(1|2n) := \left\{ \begin{bmatrix} 0 & W \\ U & Y \end{bmatrix} \in \mathfrak{gl}(1|2n) \mid Y \in \mathfrak{sp}(2n), W, U \in \mathbb{C}^{2n} \right\}. \tag{5.1}$$

Here W is realized as a row vector formed from $, W_2 \in \mathbb{C}^n$ and U is a column vector formed from W_2^t and $-W_1^t$.

Remark 5.1.4. From (5.1), we can easily compute the dimension of $\mathfrak{osp}(1|2n)$ as follows:

$$\dim(\mathfrak{osp}(1|2n)) = \dim(\mathfrak{sp}(2n)) + 2n = 2n^2 + 3n.$$

In particular, $\dim(\mathfrak{osp}(1|2)) = 5$ and $\dim(\mathfrak{osp}(1|4)) = 14$.

The definitions of Cartan subalgebras has a natural extension to the super case when considering $\mathfrak{osp}(1|2n)$. From now on we fix

$$\mathfrak{h} = \{a_1E_{11} + a_2E_{22} + ... + a_nE_{n,n} - a_1E_{n+1,n+1} - ... - a_nE_{2n,2n} | a_i \in \mathbb{C}\}$$

to be standard Cartan subalgebra of $\mathfrak{g}$.

The modules under consideration in the last two chapters will be weight modules.

Definition 5.1.5. An $\mathfrak{osp}(1|2n)$-module M is a *weight module* if $M = \bigoplus_{\lambda\in\mathfrak{h}^*} M^\lambda$, and $\dim M^\lambda < \infty$. Here, $M^\lambda = \{v \in M | hv = \lambda(h)v, \text{ for all } h \in \mathfrak{h}\}$ is the λ*–weight space of* M and $\dim M^\lambda$ is the *weight multiplicity of* λ.

We note here that *for the remainder of the dissertation,* $\mathfrak{g} = \mathfrak{osp}(1|2n)$ *unless otherwise specified.*

Definition 5.1.6. Let M be a weight $\mathfrak{g}$-module.

(i) We say M is a module of *bounded multiplicities or bounded* $\mathfrak{g}$*-module* if there exists $k \in \mathbb{N}$ such that $dim\ M^\lambda \leq k$ for all weights λ of M. The minimal such k is the *degree of* M. Note that a module of degree 1 is completely pointed.

(ii) If $v \in M^\lambda$ we say that the weight of v is λ and write $\lambda = \mathrm{wt}(v)$.

5.1.1 Root System of $\mathfrak{osp}(1|2n)$

The definition of a root system of a Lie algebra is in [21]. We extend this definition specifically for $\mathfrak{osp}(1|2n)$ as follows.We first define $\delta_i \in \mathfrak{h}^*$ by the map

$$\delta_i(a_1, a_2, ..., a_n, -a_1, -a_2, ..., a_n) = a_i,$$

for $i = 1, 2, ..., n$.

Definition 5.1.7. Let $\alpha \in \mathfrak{h}^*$, with $\alpha \neq 0$, then α is a *root of* $\mathfrak{osp}(1|2n)$ if $\mathfrak{g}^\alpha \neq 0$, where $\mathfrak{g}^\alpha = \{x \in \mathfrak{osp}(1|2n) | [h, x] = \alpha(h)x, \text{ for all } h \in \mathfrak{h}\}$. The set of all roots of $\mathfrak{osp}(1|2n)$ will be denoted by Δ. The *even roots* are $\Delta_{\bar{0}} = \{\alpha \in \Delta | \mathfrak{g}^\alpha \cap \mathfrak{g}_{\bar{0}}\}$ and *the odd roots* are $\Delta_{\bar{1}} = \{\alpha \in \Delta | \mathfrak{g}^\alpha \cap \mathfrak{g}_{\bar{1}}\}$.

The roots of $\mathfrak{g}$ are listed in the following proposition.

Proposition 5.1.8. $\Delta_{\bar{0}} = \{\pm\delta_i \pm \delta_j; \pm 2\delta_i\}$ *and* $\Delta_{\bar{1}} = \{\pm\delta_i\}$ *for* $i = 1, 2, 3, ..., n$, $i \neq j$.

Note that, in future, when we identify $\mathfrak{sp}(2n)$ as the even part of $\mathfrak{osp}(1|2n)$, we will use δ_i's instead of ϵ_i's for the roots.

Definition 5.1.9. Let $\mathfrak{g} = \mathfrak{osp}(1|2n)$ and Δ be the set of roots of $(\mathfrak{g}, \mathfrak{h})$. A subset Π of Δ is a *base of* Δ if:

(i) Π is a basis of $\mathfrak{h}^*$,

(ii) Each $\beta \in \Delta$ can be written as $\beta = \Sigma k_\alpha \alpha$, with $\alpha \in \Pi$ such that the coefficients k_α are either all nonnegative or all nonpositive.

If all k_α are nonnegative then β is a *positive root*. Otherwise we call β a *negative root*. The sets of positive and negative roots are denoted by Δ^+ and Δ^-, respectively.

We fix the following bases of the root system of $\mathfrak{osp}(1|2n)$

$$\Pi_{\mathfrak{osp}} = \{-\delta_1, \delta_1 - \delta_2, ..., \delta_{n-1} - \delta_n\}.$$

We should note that our choice of base of Δ is different from the one in [32]. Once the choice of a base of Δ is fixed, we can identify the positive and negative roots as follows:

$$\Delta^+ = \{-\delta_i, -\delta_i - \delta_j, \delta_i - \delta_j, -2\delta_i \mid 1 \le i < j \le n\}$$
$$\Delta^- = -\Delta^+$$

Definition 5.1.10. Let $\mathfrak{g} = \mathfrak{osp}(1|2n)$ and let Δ^+ be the set of positive roots associated with $\Pi_{\mathfrak{osp}}$. Then the subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ with $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha$ is the *Borel subalgebra of* $\mathfrak{g}$ *corresponding to* Π.

5.1.2 Root Vectors of $\mathfrak{osp}(1|2n)$

Let $\mathfrak{g} = \mathfrak{osp}(1|2n)$. In this section we fix elements X_α in $\mathfrak{g}^\alpha$ for all $\alpha \in \Delta$.

First, we list the even elements X_α, namely, those in $\mathfrak{osp}(1|2n)_{\bar{0}} \cong \mathfrak{sp}(2n)$:

$$X_{\delta_i - \delta_j} = \begin{bmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{bmatrix}, X_{2\delta_i} = \begin{bmatrix} 0 & E_{ii} \\ 0 & 0 \end{bmatrix}, X_{-2\delta_i} = \begin{bmatrix} 0 & 0 \\ E_{ii} & 0 \end{bmatrix},$$

$$X_{\delta_i + \delta_j} = \begin{bmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{bmatrix}, X_{-\delta_i - \delta_j} = \begin{bmatrix} 0 & 0 \\ E_{ij} + E_{ji} & 0 \end{bmatrix}$$

The odd elements $X_{\pm\delta_i}$ are of the form $\begin{bmatrix} 0 & W_1 & W_2 \\ W_2^t & 0 & 0 \\ -W_1^t & 0 & 0 \end{bmatrix}$ where

- $W_1 = (0, ..., 1, ..., 0)$ (1 on the ith position) and $W_2 = (0, 0, ..., 0)$ for $X_{-\delta_i}$;
- $W_1 = (0, 0, ..., 0)$ and $W_2 = (0, ..., 1, ..., 0)$ (1 on the ith position) for X_{δ_i}.

Finally we fix the following elements in $\mathfrak{h}$:
$h_{\delta_i-\delta_j} = \begin{bmatrix} E_{ii} - E_{jj} & 0 \\ 0 & -E_{ii} + E_{jj} \end{bmatrix}$, $h_{2\delta_i} = \begin{bmatrix} E_{ii} & 0 \\ 0 & -E_{ii} \end{bmatrix}$.
Note that $\{h_{2\delta_1}, h_{\delta_1-\delta_2}, ..., h_{\delta_{n-1}-\delta_m}\}$ forms a basis of $\mathfrak{h}$.

5.1.3 Root Vector Relations for $\mathfrak{osp}(1|2n)$

It will be convenient to have root vector relations for $[X_\alpha, X_\beta]$ available for later use. They are grouped in the order odd-odd, even-odd, and even-even based on the parity of the root vectors X_α, X_β. Note that if the sum $\alpha + \beta$ of the roots α, β is not a root, then the corresponding Lie superbracket is zero.

The odd-odd relations are symmetric.

$$[X_{\pm\delta_i}, X_{\pm\delta_i}] = \pm 2X_{\pm 2\delta_i}$$

$$[X_{\pm\delta_i}, X_{\pm\delta_{i+1}}] = \pm X_{\pm\delta_i \pm\delta_{i+1}}$$

$$[X_{\delta_i}, X_{-\delta_i}] = h_{2\delta_i}$$

$$[X_{\pm\delta_i}, X_{\mp\delta_{i+1}}] = \pm X_{\pm\delta_i \mp\delta_{i+1}}$$

The even-odd relations.

$$[X_{2\delta_i}, X_{-\delta_i}] = -X_{\delta_i}$$

$$[X_{\delta_i-\delta_{i+1}}, X_{\delta_{i+1}}] = X_{\delta_i}$$

$$[X_{\delta_i-\delta_{i+1}}, X_{-\delta_i}] = -X_{-\delta_{i+1}}$$

$$[X_{-2\delta_i}, X_{\delta_i}] = -X_{-\delta_i}$$

$$[X_{-\delta_i-\delta_{i+1}}, X_{\delta_i}] = -X_{-\delta_{i+1}}$$

$$[X_{-\delta_i-\delta_{i+1}}, X_{\delta_{i+1}}] = -X_{-\delta_i}$$

$$[X_{\delta_i+\delta_{i+1}}, X_{-\delta_i}] = -X_{\delta_{i+1}}$$

$$[X_{\delta_i+\delta_{i+1}}, X_{-\delta_{i+1}}] = -X_{\delta_i}$$

$$[X_{\delta_{i+1}+\delta_i}, X_{\delta_i}] = X_{\delta_{i+1}}$$

$$[X_{\delta_{i+1}-\delta_i}, X_{-\delta_{i+1}}] = -X_{-\delta_i}$$

The even-even relations are antisymmetric.

$$[X_{2\delta_i}, X_{-2\delta_i}] = h_{2\delta_i}$$

$$[X_{2\delta_i}, X_{\delta_{i+1}-\delta_i}] = -X_{\delta_i+\delta_{i+1}}$$

$$[X_{2\delta_{i+1}}, X_{\delta_i-\delta_{i+1}}] = -X_{\delta_i+\delta_{i+1}}$$

$$[X_{-2\delta_{i+1}}, X_{\delta_{i+1}-\delta_i}] = X_{-\delta_i-\delta_{i+1}}$$

$$[X_{\delta_{i+1}-\delta_i}, X_{\delta_i-\delta_{i+1}}] = -h_{\delta_i-\delta_{i+1}}$$

$$[X_{-2\delta_i}, X_{\delta_i-\delta_{i+1}}] = -X_{\delta_i-\delta_{i+1}}$$

The following computations will be referred to later in the document:

$$[X_{\delta_1}, X_{-\delta_1}] = h_{2\delta_1} \tag{5.2}$$

$$[X_{-\delta_1}, X_{2\delta_1}] = X_{\delta_1} \tag{5.3}$$

$$[X_{\delta_k-\delta_{k+1}}, X_{\delta_1}] = 0 \tag{5.4}$$

Corollary 5.1.11. *The Lie superalgebra* $\mathfrak{osp}(1|2n)$ *is generated by the set* $X = \{X_{\delta_j}, X_{-\delta_j} \mid 1 \leq n\}$ *of odd root vectors.*

In fact a stronger results holds, as the following results shows. For the proof, see for example [16].

Proposition 5.1.12. *The Lie superalgebra* $\mathfrak{osp}(1|2n)$ *is generated by* $X_{\delta_j}, X_{-\delta_j}$, $j = 1, ..., n$, *subject to the relations*

$$[[X_{\xi\delta_j}, X_{\eta\delta_k}], X_{\epsilon\delta_l}] = (\epsilon - \xi)\delta_{jl}X_{\eta\delta_k} + (\epsilon - \eta)\delta_{kl}X_{\xi\delta_j}, \tag{5.5}$$

where $\xi, \eta, \delta \in \{-1, 1\}$.

Remark 5.1.13. We will identify $\mathfrak{h}^*$ with $\mathbb{C}^n$ via the vector space isomorphism $\mathfrak{h}^* \mapsto \mathbb{C}^n$, $\lambda \mapsto (\lambda(h_{2\delta_1}), ..., \lambda(h_{2\delta_n}))$. Note that $\lambda = \sum_{i=1}^n \lambda(h_{2i})\delta_i$.

5.1.4 $\mathbb{C}^{1|2n}$ as an $\mathfrak{osp}(1|2n)$-module

We introduce the $\mathbb{C}^{1|2n}$ as an $\mathfrak{osp}(1|2n)$-module here and compute the action of certain roots vectors of $\mathfrak{osp}(1|2n)$ on $\mathbb{C}^{1|2n}$. This module plays an important role in Chapter 6.

Denote the standard basis of $\mathbb{C}^{1|2n}$ by $\{v_0, v_1, ..., v_n, v_{n+1}, ..., v_{2n}\}$, with v_0 being even and the remaining elements odd. The action of the basis root vectors $X_{-\delta_1}$ and $X_{\delta_i-\delta_{i+1}}$, $i \geq 1$, on $\mathbb{C}^{1|2n}$ is given by matrix multiplication. For completeness, we write down the action of $X_{\pm\delta_i}$, $X_{\delta_i-\delta_{i+1}}$:

$$X_{-\delta_i}(v_j) = \begin{cases} -v_{n+i} \text{ if } j = 0 \\ v_0 \text{ if } j = i \\ 0 \text{ otherwise} \end{cases}$$

$$X_{\delta_i}(v_j) = \begin{cases} -v_i \text{ if } j = 0 \\ v_0 \text{ if } j = n+i \\ 0 \text{ otherwise} \end{cases} \qquad X_{\delta_i-\delta_{i+1}}(v_j) = \begin{cases} v_i \text{ if } j = i+1 \\ -v_{n+i+1} \text{ if } j = n+i \\ 0 \text{ otherwise.} \end{cases}$$

5.2 Super Module Theory

We have already refrained from using super module and speak only of modules; however, some of the definitions and examples below are specific to the modules of Lie superalgebras.

5.2.1 Highest Weight Modules and Primitive Vectors

Definition 5.2.1. Let M be a $\mathfrak{g}$-module, then $0 \neq v \in M$ is a *primitive vector* if $xv = 0$ for all $x \in \mathfrak{n}^+$. Equivalently, v is a primitive vector if $X_\alpha v = 0$ for $\alpha = -\delta_1, \delta_1 - \delta_2, ..., \delta_{n-1} - \delta_n$.

Example 5.2.2. Let $n = 2$ and $M = \mathbb{C}^{1|4}$. In this case, we have

$$X_{-\delta_1}(v_4) = X_{\delta_2-\delta_1}(v_4) = 0,$$

thus v_4 is a primitive vector of M of weight $\mathrm{wt}(v_4) = (0, -1)$.

5.2.2 Tensor products of weight modules

The action of $\mathfrak{g}$ on the tensor product $V \otimes W$ of two $\mathfrak{g}$-modules V, W is given by the formula

$$x(v \otimes w) = xv \otimes w + (-1)^{|x||v|} v \otimes xw \text{ with } x \in \mathcal{U}(\mathfrak{g}) \text{ and } v \in V, w \in W. \quad (5.6)$$

Remark 5.2.3. Given an elementary tensor $u = v \otimes w$ of weight vectors in a tensor product module $V \otimes W$ of weight modules V and W, the weight of u is the sum of the weights of v and w. And if v and w are highest weight vectors, then u is a primitive vector.

We begin with a proposition defining a homomorphism from $\mathfrak{U}(\mathfrak{osp}(1|2n))$ to $\mathcal{D}(n)$. We will use this homomorphism to define an $\mathfrak{osp}(1|2n)$-module structure on every $\mathcal{D}(n)$-module.

5.2.3 Weyl superalgebra homomorphism

The following proposition has been useful in studying the representation theory of $\mathfrak{osp}(1|2n)$. See [31] and [13].

Proposition 5.2.4. *The following correspondence defines a map* $\phi : \mathfrak{U}(\mathfrak{osp}(1|2n)) \to \mathcal{W}_n$ *of associative superalgebras:*

$$\begin{aligned} X_{\delta_i-\delta_j} &\longmapsto x_i\partial_j; i \neq j; \\ X_{2\delta_i} &\longmapsto \frac{1}{2}x_i^2; \\ X_{-2\delta_i} &\longmapsto \frac{-1}{2}\partial_i^2; \\ X_{\delta_i+\delta_j} &\longmapsto x_i x_j; \\ X_{-\delta_i-\delta_j} &\longmapsto -\partial_i\partial_j; \\ h_{\delta_i-\delta_j} &\longmapsto x_i\partial_i - x_j\partial_j; \\ h_{2\delta_i} &\longmapsto x_i\partial_i + \frac{1}{2}; \\ X_{\delta_i} &\longmapsto \frac{1}{\sqrt{2}}x_i; \\ X_{-\delta_i} &\longmapsto \frac{1}{\sqrt{2}}\partial_i. \end{aligned}$$

Proposition 5.2.4 implies that the complex polynomials $\mathbb{C}[x_1, \ldots, x_n]$ has the structure of an $\mathfrak{osp}(1|2n)$-module through ϕ.

5.2.4 Primitive Vectors of Tensor Products of $\mathfrak{osp}(1|2n)$-modules

Recall the definition of $M = \mathcal{F}(0)$ and $M^+ = \mathcal{F}(0)^+$ (Definition **??**). We begin by considering the tensor products of M^+ with $\mathbb{C}^{1|2n}$. Recall that the action of $\mathfrak{osp}(1|2n)$ on $M^+ \otimes V$ is given by $X_\alpha(f \otimes v) = X_\alpha(f) \otimes v + (-1)^{|X_\alpha||f|} f \otimes X_\alpha(v)$ with $f \in M^+$ and $v \in \mathbb{C}^{1|2n}$ (see 5.6).

Proposition 5.2.5. *[13] Let $V = \mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}^{1|2n}$. Then any primitive vector in V is a linear combination of w_1 and w_2 defined as follows:*

$$w_1 := 1 \otimes v_{2n},\ w_2 := 1 \otimes v_0 + \sqrt{2}\sum_{i=1}^{n} x_i \otimes v_{n+i}.$$

We will see that w_1 and w_2 generate submodules of V whose direct sum is V.

CHAPTER 6

Bases of a Direct Sum and Formulas

In Chapter 5, we defined the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ and described the $\mathfrak{osp}(1|2n)$-action on the tensor product $\mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}^{1|2n}$ of $\mathfrak{osp}(1|2n)$-modules, where $\mathfrak{osp}(1|2n)$ acts via differential operators on the complex polynomials $\mathbb{C}[x_1, \ldots, x_n]$ (Proposition 5.2.4) and $\mathbb{C}^{1|2n}$ is the standard module. We now move to the first main result of this study in infinite-dimensional orthosymplectic module theory. Let $V = \mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}^{1|2n}$. Here we exhibit two simple submodules of V such that their direct sum is V. By intertwining operators on V, we present a new module structure on the tensor product $\mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}^{2n}$ of an $\mathfrak{osp}(1|2n)$-module with the standard module of $\mathfrak{gl}(2n)$. This is the beginning of determining new infinite-dimensional modules of the Lie superalgebra $\mathfrak{osp}(1|2n)$ which are tensor products of our so-called Weyl module with certain $\mathfrak{gl}(2n)$-modules, for example, $\bigwedge \mathbb{C}^{2n}$.

We rely on the following notation and conventions for the penultimate and final chapters.

6.1 Notation

Fix the ground field $\mathbb{C}$ with $n \in \mathbb{N}$. We establish the following conventions:

$$\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, x_2, \ldots, x_n],$$

$$\mathbf{x}^{\mathbf{k}} = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}, \ \ \mathbf{k} \in \mathbb{Z}^n_{\geq 0},$$

$$v_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ 1 \text{ in the } (j+1)^{st} \text{ entry},\ 0 \le j \le 2n,$$

$$Y_{\mathbf{k}} = \mathbf{x}^{\mathbf{k}} \otimes v_0,\ \mathbf{k} \in \mathbb{Z}^n_{\ge 0},\ \text{and}$$

$$Z_{\mathbf{k},i} = \mathbf{x}^{\mathbf{k}} \otimes v_i,\ 1 \le i \le 2n,\ \mathbf{k} \in \mathbb{Z}^n_{\ge 0}.$$

We emphasize that certain elements of $\mathbb{C}^n$ are given special notation as roots of $\mathfrak{osp}(1|2n)$. Precisely, $\delta_j = (\underbrace{0, \dots, 0, 1, 0, \dots, 0}_{1 \text{ in } j^{th} \text{ position}})$ and $\nu_j = \sum\limits_{i=1}^{j} \delta_i$.

6.2 First Main Result

We briefly restate some key facts about $\mathfrak{g}$ before introducing the operators important to the first main goal in the decomposition of an infinite-dimensional $\mathfrak{g}$-module.

The Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(1|2n)$ is the set of linear maps on $\mathbb{C}^{1|2n}$ preserving [as in 32, Appendix A.2.28] a nondegenerate supersymmetric even bilinear form. Also, $\mathfrak{g}$ is generated by odd root vectors $\{X_{\delta_j}, X_{-\delta_j} \mid 1 \le n\}$ and acts on the tensor product module $V = \mathbb{C}[x_1, x_2, \dots, x_n] \otimes \mathbb{C}^{1|2n}$. The vector $w = w_2 = 1 \otimes v_0 + \sqrt{2} \sum\limits_{i=1}^{n} x_i \otimes v_{n+i}$ is a primitive vector of V; thus, $X_{-\delta_i}(w) = 0, \quad 1 \le i \le n$.

6.2.1 The Super Subspaces V^0 and V^+

In our goal to decompose $V = \mathbb{C}[\mathbf{x}] \otimes \mathbb{C}^{1|2n}$ into $\mathfrak{osp}(1|2n)$-submodules we note a ready-made vector space decomposition.

Definition 6.2.1. Fix $V^0 = \mathbb{C}[\mathbf{x}] \otimes \mathbb{C}v_0$ with basis $\{Y_{\mathbf{k}} \mid \mathbf{k} \in \mathbb{Z}^n_{\ge 0}\}$.

Definition 6.2.2. Fix $V^+ = \mathbb{C}[\mathbf{x}] \otimes (\bigoplus\limits_{i>0} \mathbb{C}v_i)$ with basis $\{Z_{\mathbf{k},i} \mid \mathbf{k} \in \mathbb{Z}_{\ge 0},\ 1 \le i \le 2n\ \}$.

Remark 6.2.3. We can make V^0 a super space by choosing a grading based on polynomial degree similar to remark 4.1.8 (and the preceding comments). We must pay attention to the parity of a tensor. Mainly, $V^0_{\bar{i}} = \mathbb{C}[\mathbf{x}]_{\bar{i}} \otimes v_0$. Likewise, take V^+ to be the super space with $V^+_{\bar{i}} = \mathbb{C}[\mathbf{x}]_{\overline{i+1}} \otimes (\bigoplus_{i>0} \mathbb{C}v_i)$.

It is clear that $V = V^0 \oplus V^+$ as vector spaces. Remark 6.2.3 says that V^0 and V^+ are super subspaces. Truly, Remark 6.2.3 is not needed in lieu of definition 4.3.7, but concrete computations and constructions are helpful to the super mathematician.

We now introduce important operators to the story and consider their images on V^0 and V^+.

6.2.2 Defining New Operators on V

The following operators are essential in finding submodules of V. Define the operators Γ^- and Γ^+ on V as

$$\Gamma^- = \mathbb{1} \otimes \mathbb{1} + \sqrt{2}\sum_{i=1}^{n} \partial_{x_i} \otimes X_{\delta_i} - \sqrt{2}\sum_{i=1}^{n} x_i \otimes X_{-\delta_i},$$

$$\Gamma^+ = \mathbb{1} \otimes \mathbb{1} - \sqrt{2}\sum_{i=1}^{n} \partial_{x_i} \otimes X_{\delta_i} + \sqrt{2}\sum_{i=1}^{n} x_i \otimes X_{-\delta_i}.$$

It is important to note $\Gamma^-(Y_0) = w$. The status of w as a primitive vector of V implies $\mathfrak{n}^-(\Gamma^-(Y_0))$ is a submodule of V. The Lie algebra $\mathfrak{n}^-$ is generated by odd negative root vectors $\{X_{\delta_j}, X_{-\delta_j} \mid 1 \leq n\}$. The action of the odd negative root vectors justifies the definition of Γ^- and influences the determination of Γ^+.

6.2.3 Obtaining Submodules of V through New Operators

The generators of $\mathfrak{osp}(1|2n)$ act invariantly on $\Gamma^-(V^0)$ and $\Gamma^+(V^+)$.

Lemma 6.2.4. *For* $1 \leq j \leq n$,

$$X_{\delta_j}\Gamma^-(Y_{\mathbf{k}}) = \Gamma^-(-\frac{1}{\sqrt{2}}Y_{\mathbf{k}+\delta_j}).$$

Proof. The proof serves as computational reference:

$$X_{\delta_j}\Gamma^-(Y_{\mathbf{k}}) = X_{\delta_j}\Gamma^-(\mathbf{x}^{\mathbf{k}} \otimes v_0)$$

$$= X_{\delta_j}(\mathbf{x}^{\mathbf{k}} \otimes v_0 + (-1)^{|\mathbf{k}|}\sqrt{2}\sum_{i=1}^{n} k_i \mathbf{x}^{\mathbf{k}-\delta_i} \otimes v_i$$

$$+ (-1)^{|\mathbf{k}|}\sqrt{2}\sum_{i=1}^{n} \mathbf{x}^{\mathbf{k}+\delta_i} \otimes v_{n+i})$$

$$= -\frac{1}{\sqrt{2}}\mathbf{x}^{\mathbf{k}+\delta_j} \otimes v_0 + (-1)^{|\mathbf{k}|}\mathbf{x}^{\mathbf{k}} \otimes v_j$$

$$+ (-1)^{|\mathbf{k}|}\sum_{i=1}^{n} k_i \mathbf{x}^{\mathbf{k}-\delta_i+\delta_j} \otimes v_i$$

$$+ (-1)^{|\mathbf{k}|}\sum_{i=1}^{n} \mathbf{x}^{\mathbf{k}+\delta_i+\delta_j} \otimes v_{n+i}, \text{ whereas}$$

$$\Gamma^-(Y_{\mathbf{k}+\delta_j}) = \Gamma^-(\mathbf{x}^{\mathbf{k}+\delta_j} \otimes v_0)$$

$$= \mathbf{x}^{\mathbf{k}+\delta_j} \otimes v_0 + (-1)^{|\mathbf{k}+1|}\sqrt{2}\sum_{i=1}^{n}(k_i + \delta_{ij})\mathbf{x}^{\mathbf{k}-\delta_i+\delta_j} \otimes v_i$$

$$+ (-1)^{|\mathbf{k}+1|}\sqrt{2}\sum_{i=1}^{n} \mathbf{x}^{\mathbf{k}+\delta_i+\delta_j} \otimes v_{n+i}.$$

□

Lemma 6.2.5. *For* $1 \le j \le n$, $\mathbf{k} \ne \mathbf{0}$,

$$X_{-\delta_j}\Gamma^-(Y_{\mathbf{k}}) = \Gamma^-(-\frac{1}{\sqrt{2}}k_j Y_{\mathbf{k}-\delta_j}).$$

Proof. The proof serves as computational reference:

$$X_{-\delta_j}\Gamma^-(Y_{\mathbf{k}}) = X_{-\delta_j}\Gamma^-(\mathbf{x}^{\mathbf{k}} \otimes v_0)$$

$$= X_{-\delta_j}(\mathbf{x}^{\mathbf{k}} \otimes v_0 + (-1)^{|\mathbf{k}|}\sqrt{2}\sum_{i=1}^{n} k_i \mathbf{x}^{\mathbf{k}-\delta_i} \otimes v_i$$

$$+ (-1)^{|\mathbf{k}|}\sqrt{2}\sum_{i=1}^{n} \mathbf{x}^{\mathbf{k}+\delta_i} \otimes v_{n+i})$$

$$= -\frac{1}{\sqrt{2}}k_j \mathbf{x}^{\mathbf{k}-\delta_j} \otimes v_0 - (-1)^{|\mathbf{k}|}\mathbf{x}^{\mathbf{k}} \otimes v_{n+j}$$

$$+ (-1)^{|\mathbf{k}|}\sum_{i=1}^{n} k_i(k_j - \delta_{ij})\mathbf{x}^{\mathbf{k}-\delta_i-\delta_j} \otimes v_i$$

$$+ (-1)^{|\mathbf{k}|}\sum_{i=1}^{n}(k_j+\delta_{ij})\mathbf{x}^{\mathbf{k}+\delta_i-\delta_j}\otimes v_{n+i}, \text{ whereas}$$

$$\begin{aligned}\Gamma^-(Y_{\mathbf{k}-\delta_j}) &= \Gamma^-(\mathbf{x}^{\mathbf{k}-\delta_j}\otimes v_0)\\ &= \mathbf{x}^{\mathbf{k}-\delta_j}\otimes v_0 + (-1)^{|\mathbf{k}-1|}\sqrt{2}\sum_{i=1}^{n}(k_i-\delta_{ij})\mathbf{x}^{\mathbf{k}-\delta_i+\delta_j}\otimes v_i\\ &+ (-1)^{|\mathbf{k}-1|}\sqrt{2}\sum_{i=1}^{n}\mathbf{x}^{\mathbf{k}+\delta_i-\delta_j}\otimes v_{n+i}.\end{aligned}$$

□

Lemma 6.2.6. *For $1 \le j \le n$,*

$$X_{\delta_j}\Gamma^+(Z_{\mathbf{k},i}) = \Gamma^+(\frac{1}{\sqrt{2}}Z_{\mathbf{k}+\delta_j,i} - \sqrt{2}Z_{\mathbf{k}+\delta_i,j})$$

and

$$X_{\delta_j}\Gamma^+(Z_{\mathbf{k},n+i}) = \Gamma^+(\frac{1}{\sqrt{2}}Z_{\mathbf{k}+\delta_j,n+i} + \sqrt{2}k_iZ_{\mathbf{k}-\delta_i,j})$$

Proof. The proof serves as computational reference:

$$\begin{aligned}X_{\delta_j}\Gamma^+(Z_{\mathbf{k},i}) &= X_{\delta_j}\Gamma^+(\mathbf{x}^{\mathbf{k}}\otimes v_i)\\ &= X_{\delta_j}(\mathbf{x}^{\mathbf{k}}\otimes v_i + (-1)^{|\mathbf{k}|}\sqrt{2}\mathbf{x}^{\mathbf{k}+\delta_i}\otimes v_0)\\ &= \frac{1}{\sqrt{2}}\mathbf{x}^{\mathbf{k}+\delta_j}\otimes v_i + (-1)^{|\mathbf{k}|}\mathbf{x}^{\mathbf{k}+\delta_i+\delta_j}\otimes v_0 - \sqrt{2}\mathbf{x}^{\mathbf{k}+\delta_i}\otimes v_j \qquad \text{and}\end{aligned}$$

$$\begin{aligned}X_{\delta_j}\Gamma^+(Z_{\mathbf{k},n+i}) &= X_{\delta_j}\Gamma^+(\mathbf{x}^{\mathbf{k}}\otimes v_{n+i})\\ &= X_{\delta_j}(\mathbf{x}^{\mathbf{k}}\otimes v_{n+i} - (-1)^{|\mathbf{k}|}\sqrt{2}k_i\mathbf{x}^{\mathbf{k}-\delta_i}\otimes v_0)\\ &= \frac{1}{\sqrt{2}}\mathbf{x}^{\mathbf{k}+\delta_j}\otimes v_{n+i} + \delta_{ij}(-1)^{|\mathbf{k}|}\mathbf{x}^{\mathbf{k}}\otimes v_0\\ &- (-1)^{|\mathbf{k}|}k_i\mathbf{x}^{\mathbf{k}-\delta_i+\delta_j}\otimes v_0 + \sqrt{2}k_i\mathbf{x}^{\mathbf{k}-\delta_i}\otimes v_j, \text{ whereas}\end{aligned}$$

$$\Gamma^+(Z_{\mathbf{k}+\delta_j,i} - Z_{\mathbf{k}+\delta_i,j}) = \mathbf{x}^{\mathbf{k}+\delta_j}\otimes v_i - (-1)^{|\mathbf{k}|}\sqrt{2}\mathbf{x}^{\mathbf{k}+\delta_i+\delta_j}\otimes v_0$$

$$- \mathbf{x}^{\mathbf{k}+\delta_i} \otimes v_j + (-1)^{|\mathbf{k}|}\sqrt{2}\mathbf{x}^{\mathbf{k}+\delta_i+\delta_j} \otimes v_0 \qquad \text{and}$$

$$\Gamma^+(Z_{\mathbf{k}+\delta_j,n+i} + Z_{\mathbf{k}-\delta_i,j}) = \mathbf{x}^{\mathbf{k}+\delta_j} \otimes v_{n+i} + (-1)^{|\mathbf{k}|}\sqrt{2}(k_i + \delta_{ij})\mathbf{x}^{\mathbf{k}-\delta_i+\delta_j} \otimes v_0$$

$$+ \mathbf{x}^{\mathbf{k}-\delta_i} \otimes v_j - (-1)^{|\mathbf{k}|}\sqrt{2}\mathbf{x}^{\mathbf{k}-\delta_i+\delta_j} \otimes v_0$$

□

Lemma 6.2.7. *For* $1 \leq j \leq n$*,*

$$X_{-\delta_j}\Gamma^+(Z_{\mathbf{k},i}) = \Gamma^+(\frac{1}{\sqrt{2}}k_j Z_{\mathbf{k}-\delta_j,i} + \sqrt{2}Z_{\mathbf{k}+\delta_i,n+j})$$

and

$$X_{-\delta_j}\Gamma^+(Z_{\mathbf{k},n+i}) = \Gamma^+(\frac{1}{\sqrt{2}}k_j Z_{\mathbf{k}-\delta_j,n+i} - \sqrt{2}k_i Z_{\mathbf{k}-\delta_i,n+j})$$

Proof. The proof serves as computational reference:

$$\begin{aligned} X_{-\delta_j}\Gamma^+(Z_{\mathbf{k},i}) &= X_{-\delta_j}\Gamma^+(\mathbf{x}^{\mathbf{k}} \otimes v_i) \\ &= X_{-\delta_j}(\mathbf{x}^{\mathbf{k}} \otimes v_i + (-1)^{|\mathbf{k}|}\sqrt{2}\mathbf{x}^{\mathbf{k}+\delta_i} \otimes v_0) \\ &= \frac{1}{\sqrt{2}}k_j\mathbf{x}^{\mathbf{k}-\delta_j} \otimes v_i + \delta_{ij}(-1)^{|\mathbf{k}|}\mathbf{x}^{\mathbf{k}} \otimes v_0 \\ &\quad + (-1)^{|\mathbf{k}|}(k_j + \delta_{ij})\mathbf{x}^{\mathbf{k}+\delta_i-\delta_j} \otimes v_0 + \sqrt{2}\mathbf{x}^{\mathbf{k}+\delta_i} \otimes v_{n+j} \qquad \text{and} \end{aligned}$$

$$\begin{aligned} X_{-\delta_j}\Gamma^+(Z_{\mathbf{k},n+i}) &= X_{-\delta_j}\Gamma^+(\mathbf{x}^{\mathbf{k}} \otimes v_{n+i}) \\ &= X_{-\delta_j}(\mathbf{x}^{\mathbf{k}} \otimes v_{n+i} - (-1)^{|\mathbf{k}|}\sqrt{2}k_i\mathbf{x}^{\mathbf{k}-\delta_i} \otimes v_0) \\ &= \frac{1}{\sqrt{2}}k_j\mathbf{x}^{\mathbf{k}-\delta_j} \otimes v_{n+i} \\ &\quad - (-1)^{|\mathbf{k}|}k_i(k_j - \delta_{ij})\mathbf{x}^{\mathbf{k}-\delta_i-\delta_j} \otimes v_0 - \sqrt{2}k_i\mathbf{x}^{\mathbf{k}-\delta_i} \otimes v_{n+j}, \ \text{whereas} \end{aligned}$$

$$\begin{aligned} \Gamma^+(Z_{\mathbf{k}-\delta_j,i} - Z_{\mathbf{k}+\delta_i,n+j}) &= \mathbf{x}^{\mathbf{k}+\delta_j} \otimes v_i - (-1)^{|\mathbf{k}|}\sqrt{2}\mathbf{x}^{\mathbf{k}+\delta_i+\delta_j} \otimes v_0 \\ &\quad + \mathbf{x}^{\mathbf{k}+\delta_i} \otimes v_j - (-1)^{|\mathbf{k}|}\sqrt{2}\mathbf{x}^{\mathbf{k}+\delta_i+\delta_j} \otimes v_0 \qquad \text{and} \end{aligned}$$

$$\begin{aligned} \Gamma^+(Z_{\mathbf{k}-\delta_j,n+i} + Z_{\mathbf{k}-\delta_i,n+j}) &= \mathbf{x}^{\mathbf{k}+\delta_j} \otimes v_i + (-1)^{|\mathbf{k}|}\sqrt{2}k_i\mathbf{x}^{\mathbf{k}+\delta_j-\delta_i} \otimes v_0 + \delta_{ij}(-1)^{|\mathbf{k}|}\sqrt{2}\mathbf{x}^{\mathbf{k}+\delta_j-\delta_i} \otimes v_0 \\ &\quad + \mathbf{x}^{\mathbf{k}-\delta_i} \otimes v_j - (-1)^{|\mathbf{k}|}\sqrt{2}\mathbf{x}^{\mathbf{k}-\delta_i+\delta_j} \otimes v_0. \end{aligned}$$

□

Definition 6.2.8. The super space M^0 is the image $\Gamma^-(V^0)$ of V^0 under Γ^-.

Definition 6.2.9. The super space M^+ is the image $\Gamma^-(V^+)$ of V^+ under Γ^+.

Theorem 6.2.10. *The super spaces M^0 and M^+ are $\mathfrak{g}$-submodules of V.*

Proof. Recall that the set $\{X_{\pm\delta_i} \mid 1 \leq i \leq n\}$ generates $\mathfrak{g}$. Apply Lemma 6.2.5, Lemma 6.2.7, Lemma 6.2.4, and Lemma 6.2.6. □

6.2.4 Main Theorem I

The next result is one of the main results of this dissertation. We present a decomposition of the infinite-dimensional $\mathfrak{g}$-module $V = \mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}^{1|2n} = \mathbb{C}[\mathbf{x}] \otimes \mathbb{C}^{1|2n}$ into the direct sum $V = M^0 \oplus M^+$ of two simple infinite-dimensional $\mathfrak{g}$-submodules; we determine a $\mathfrak{g}$-module structure on the infinite-dimensional superspace $\mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}^{2n}$, the tensor product of a $\mathfrak{g}$-module with the standard module for $\mathfrak{gl}(2n)$; moreover, we give formulas for the action of $\mathfrak{g}$ on $V^0 \cong \mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}$ and on $V^+ \cong \mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}^{2n}$, respectively.

Theorem 6.2.11.

(i) The operators $\Gamma^-, \Gamma^+ : V \to V$ are automorphisms of (super) vector spaces.

(ii) As $\mathfrak{g}$-modules, $V = M^0 \oplus M^+$.

(iii) The following formulas hold on V^0 with $\Gamma^-\big|_{V^0}$ as an intertwining isomorphism:

$$(\Gamma^-)^{-1} X_{\pm\delta_j} \Gamma^- = -X^{(1)}_{\pm j}, \ 1 \leq j \leq n;$$

$X^{(1)}_{\pm j}$ are defined by

$$X^{(1)}_{j} = X_{\delta_j} \otimes \mathbb{1},$$

$$X^{(1)}_{-j} = X_{\delta_{-j}} \otimes \mathbb{1}.$$

(iv) The following formulas hold on V^+ with $\Gamma^+\big|_{V^+}$ as an intertwining isomorphism:

$$(\Gamma^+)^{-1}X_{\pm\delta_j}\Gamma^+ = \Psi_{\pm j},\ 1 \leq j \leq n;$$

$\Psi_{\pm j}$ are defined by

$$\Psi_j = X_{\delta_j} \otimes \mathbb{1} - \sqrt{2}\sum_{l=1}^{n} x_l \otimes E_{jl} + \sqrt{2}\sum_{l=1}^{n} \partial_{x_l} \otimes E_{j,n+l},$$

$$\Psi_{-j} = X_{-\delta_j} \otimes \mathbb{1} + \sqrt{2}\sum_{l=1}^{n} x_l \otimes E_{n+j,l} - \sqrt{2}\sum_{l=1}^{n} \partial_{x_l} \otimes E_{n+j,n+l};$$

and, $E_{pq}v_r = \delta_{qr}v_p$, as linear maps on $\mathbb{C}^{2n}$.

Proof. The $\mathfrak{g}$-module V is a bounded weight module [14, Theorem 6.4.15, Lemma 7.1.9], and it is clear that Γ^- preserves weight spaces. Thus, it suffices to show Γ^- maps the weight spaces injectively. Now any vector v of V is the unique sum of finite weight vectors, and each weight vector of weight λ_l belongs to a finite-dimensional weight space with basis elements of the form $Y_{\mathbf{k}}$ and $Z_{\mathbf{k},j}$. Thus, let v be in V. For some natural number m we can decompose v as

$$v = \sum_{l=1}^{m}\left(\alpha_{0,\lambda_l}\mathbf{x}^{\mathbf{k}_{\lambda_l}} \otimes v_0 + \sum_{j=1}^{n}\alpha_{j,\lambda_l}\mathbf{x}^{\mathbf{k}_{\lambda_l}-\delta_j} \otimes v_j + \sum_{j=1}^{n}\alpha_{n+j,\lambda_l}\mathbf{x}^{\mathbf{k}_{\lambda_l}+\delta_j} \otimes v_{n+j}\right).$$

So, keeping sums separate for emphasis and clarity,

$$\begin{aligned}\Gamma^-(v) = &\sum_{l=1}^{m}\left(\alpha_{0,\lambda_l}\mathbf{x}^{\mathbf{k}_{\lambda_l}} \otimes v_0 + \sum_{j=1}^{n}\alpha_{j,\lambda_l}\mathbf{x}^{\mathbf{k}_{\lambda_l}-\delta_j} \otimes v_j + \sum_{j=1}^{n}\alpha_{n+j,\lambda_l}\mathbf{x}^{\mathbf{k}_{\lambda_l}+\delta_j} \otimes v_{n+j}\right)\\ &+\sum_{l=1}^{m}\left((-1)^{|\mathbf{k}_{\lambda_l}|}\sqrt{2}\sum_{j=1}^{n}\alpha_{0,\lambda_l}k_{j,\lambda_l}\mathbf{x}^{\mathbf{k}_{\lambda_l}-\delta_j} \otimes v_j + (-1)^{|\mathbf{k}_{\lambda_l}|}\sqrt{2}\sum_{j=1}^{n}\alpha_{0,\lambda_l}\mathbf{x}^{\mathbf{k}_{\lambda_l}+\delta_j} \otimes v_{n+j}\right)\\ &+\sum_{l=1}^{m}\left((-1)^{|\mathbf{k}_{\lambda_l}|}\sqrt{2}\sum_{j=1}^{n}\alpha_{j,\lambda_l}\mathbf{x}^{\mathbf{k}_{\lambda_l}} \otimes v_0 - (-1)^{|\mathbf{k}_{\lambda_l}|}\sqrt{2}\sum_{j=1}^{n}\alpha_{n+j,\lambda_l}(k_{j,\lambda_l}+1)\mathbf{x}^{\mathbf{k}_{\lambda_l}} \otimes v_0\right).\end{aligned}$$

If we suppose $\Gamma^-(v) = 0$, then, collecting terms and observing linear independence, we have

$$\alpha_{j,\lambda_l} = (-1)^{|\mathbf{k}_{\lambda_l}|}\sqrt{2}\alpha_{0,\lambda_l}k_{j,\lambda_l},\ 1 \leq j \leq n \tag{6.1}$$

$$\alpha_{n+j,\lambda_l} = (-1)^{|\mathbf{k}_{\lambda_l}|}\sqrt{2}\alpha_{0,\lambda_l},\ 1 \leq j \leq n. \tag{6.2}$$

We see eqs. (6.1) and (6.2) imply $\alpha_{0,\lambda_l} = 0$ for $1 \le l \le m$. That is, $v = 0$, and Γ^- is injective. Similarly, Γ^+ is injective. Moreover, Γ^- and Γ^+ are automorphisms of V as to be shown. The rest of the proof of Theorem 6.2.11 follows from Theorem 6.2.10 and the lemmas in its proof. □

6.3 Mixed Images

Here we use the term "action" loosely. For completeness, we show the result of the generators of $\mathfrak{g}$ acting on $\Gamma^-(V^+)$ and acting on $\Gamma^+(V^-)$, neither of which are invariant subspaces under the action.

Define $S = \sum_{l=1}^{n} X_{\delta_l} \otimes X_{-\delta_l} - \sum_{l=1}^{n} X_{-\delta_l} \otimes X_{\delta_l}$ to have $\Gamma^- = 1 - 2S$ and $\Gamma^+ = 1 + 2S$. And as in Theorem 6.2.11, we write

$$S = \sum_{l=1}^{n} X_l \otimes X_{-l} - \sum_{l=1}^{n} X_{-l} \otimes X_l, \tag{6.3}$$

$$X_{\pm i}^{(1)} = X_{\pm\delta_i} \otimes \mathbb{1}, \tag{6.4}$$

$$X_{\pm i}^{(2)} = \mathbb{1} \otimes X_{\pm\delta_i}. \tag{6.5}$$

Definition 6.3.1. Define the following operators on V:

(6.6)

$$F_i = X_i^{(1)} - 2X_i^{(2)} S - \frac{8}{2n+1} X_i^{(1)} S + \frac{16}{2n+1} S^2 X_i^{(1)} + \frac{2}{2n+1} X_2^{(2)} - \frac{4}{2n+1} S X_i^{(2)}$$

(6.7)

$$G_i = X_{-i}^{(1)} - 2X_{-i}^{(2)} S + \frac{8}{2n+1} S X_{-i}^{(1)} + \frac{16}{2n+1} S^2 X_{-i}^{(1)} - \frac{2}{2n+1} X_{-i}^{(2)} + \frac{4}{2n+1} S X_{-i}^{(2)}.$$

Theorem 6.3.2. *The following formulas hold on V^+ for $1 \le i \le n$:*

$$(\Gamma^-)^{-1} X_{\delta_i} \Gamma^- = F_i$$

.

Proof. For $1 \le i, j \le n$, we have

$$\begin{aligned}\Gamma^- F_i(\mathbf{x}^{\mathbf{k}} \otimes v_j) = \Gamma^- \Bigg(&\frac{1}{\sqrt{2}} \mathbf{x}^{\mathbf{k}+\delta_i} \otimes v_j + \sqrt{2} \mathbf{x}^{\mathbf{k}+\delta_j} \otimes v_i \\ &- \frac{4}{2n+1} \Bigg[(-1)^{|\mathbf{k}|} \mathbf{x}^{\mathbf{k}+\delta_i+\delta_j} \otimes v_0 - \sqrt{2} \sum_{l=1}^{n} \mathbf{x}^{\mathbf{k}+\delta_i+\delta_j+\delta_l} \otimes v_{n+l} \\ &-\sqrt{2} \sum_{l=1}^{n} (k_l + \delta_{il} + \delta_{jl}) \mathbf{x}^{\mathbf{k}+\delta_i+\delta_j-\delta_l} \otimes v_l \Bigg] \Bigg) \\ = &\frac{1}{\sqrt{2}} \mathbf{x}^{\mathbf{k}+\delta_i} \otimes v_j - (-1)^{|\mathbf{k}|} \mathbf{x}^{\mathbf{k}+\delta_i+\delta_j} \otimes v_0 + \sqrt{2} \mathbf{x}^{\mathbf{k}+\delta_j} \otimes v_i\end{aligned}$$

and

$$\begin{aligned}\Gamma^- F_i(\mathbf{x}^{\mathbf{k}} \otimes v_{n+j}) = \Gamma^- \Bigg(&\frac{1}{\sqrt{2}} \mathbf{x}^{\mathbf{k}+\delta_i} \otimes v_{n+j} - \sqrt{2} k_j \mathbf{x}^{\mathbf{k}-\delta_j} \otimes v_i \\ &+ \frac{4k_j}{2n+1} \Bigg[(-1)^{|\mathbf{k}|} \mathbf{x}^{\mathbf{k}+\delta_i-\delta_j} \otimes v_0 - \sqrt{2} \sum_{l=1}^{n} \mathbf{x}^{\mathbf{k}+\delta_i-\delta_j+\delta_l} \otimes v_{n+l} \\ &-\sqrt{2} \sum_{l=1}^{n} (k_l + \delta_{il} + \delta_{jl}) \mathbf{x}^{\mathbf{k}+\delta_i-\delta_j-\delta_l} \otimes v_l \Bigg] \\ &+\delta_{ij} \left(\frac{2}{2n+1} \left[(-1)^{|\mathbf{k}|} \mathbf{x}^{\mathbf{k}} \otimes v_0 - \sqrt{2} \sum_{l=1}^{n} \mathbf{x}^{\mathbf{k}+\delta_l} \otimes v_{n+l} - \sqrt{2} \sum_{l=1}^{n} k_l \mathbf{x}^{\mathbf{k}-\delta_l} \otimes v_l \right] \right) \Bigg) \\ = &\frac{1}{\sqrt{2}} \mathbf{x}^{\mathbf{k}+\delta_i} \otimes v_{n+j} + (-1)^{|\mathbf{k}|} k_j \mathbf{x}^{\mathbf{k}+\delta_i-\delta_j} \otimes v_0 \\ &+ \delta_{ij} (-1)^{|\mathbf{k}|} \mathbf{x}^{\mathbf{k}} \otimes v_0 - \sqrt{2} k_j \mathbf{x}^{\mathbf{k}-\delta_j} \otimes v_i.\end{aligned}$$

Compare:

$$\begin{aligned}X_{\delta_i} \Gamma^- (\mathbf{x}^{\mathbf{k}} \otimes v_j) &= X_{\delta_i} (\mathbf{x}^{\mathbf{k}} \otimes v_j - \sqrt{2} (-1)^{|\mathbf{k}|} \mathbf{x}^{\mathbf{k}+\delta_j} \otimes v_0) \\ &= \frac{1}{\sqrt{2}} \mathbf{x}^{\mathbf{k}+\delta_i} \otimes v_j - (-1)^{|\mathbf{k}|} \mathbf{x}^{\mathbf{k}+\delta_i+\delta_j} \otimes v_0 + \sqrt{2} \mathbf{x}^{\mathbf{k}+\delta_j} \otimes v_i\end{aligned}$$

and

$$\begin{aligned}X_{\delta_i} \Gamma^- (\mathbf{x}^{\mathbf{k}} \otimes v_{n+j}) &= X_{\delta_i} (\mathbf{x}^{\mathbf{k}} \otimes v_{n+j} + \sqrt{2} (-1)^{|\mathbf{k}|} k_j \mathbf{x}^{\mathbf{k}-\delta_j} \otimes v_0) \\ &= \frac{1}{\sqrt{2}} \mathbf{x}^{\mathbf{k}+\delta_i} \otimes v_{n+j} + \delta_{ij} (-1)^{|\mathbf{k}|} \mathbf{x}^{\mathbf{k}} \otimes v_0 \\ &\quad + (-1)^{|\mathbf{k}|} k_j \mathbf{x}^{\mathbf{k}+\delta_i-\delta_j} \otimes v_0 - \sqrt{2} k_j \mathbf{x}^{\mathbf{k}-\delta_j} \otimes v_i.\end{aligned}$$

□

Theorem 6.3.3. *The following formulas hold on V^+ for $1 \leq i \leq n$:*

$$(\Gamma^-)^{-1} X_{-\delta_i} \Gamma^- = G_i$$

.

Proof. The proof is computationally similar to that of Theorem 6.3.2. □

CHAPTER 7

New Differential Operator Realizations

A pursuit to generalize the results of Chapter 6 is in order. Here we determine an $\mathfrak{osp}(1|2n)$-module structure on $W = \mathbb{C}[x_1, x_2, \dots, x_n] \otimes \mathbb{C}[\xi_1, \xi_2, \dots, \xi_{2n}]$, with the second factor being the complex polynomials in $2n$ anti-commuting indeterminates. Like the case of the Weyl map (see Proposition 5.2.4), we use a map from the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ to an associative superalgebra consisting of differential operators sitting inside the larger endomorphism algebra $\mathrm{End}(W)$. This is the beginning of more general results which we omit to err on the side of self-contained exposition.

7.1 New "Psi" Operators

From here on, we use $\mathbb{C}[\boldsymbol{\xi}]$ for $\mathbb{C}[\xi_1, \xi_2, \dots, \xi_{2n}]$ so that $W = \mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\boldsymbol{\xi}]$.

Recall the operators $\Psi_{\pm i}$ were defined in Chapter 6 as

$$\Psi_j = X_{\delta_j} \otimes \mathbb{1} - \sqrt{2} \sum_{l=1}^{n} x_l \otimes E_{jl} + \sqrt{2} \sum_{l=1}^{n} \partial_{x_l} \otimes E_{j,n+l},$$

$$\Psi_{-j} = X_{-\delta_j} \otimes \mathbb{1} + \sqrt{2} \sum_{l=1}^{n} x_l \otimes E_{n+j,l} - \sqrt{2} \sum_{l=1}^{n} \partial_{x_l} \otimes E_{n+j,n+l}.$$

We redefine the $\Psi_{\pm j}$ to be operators on W.

Definition 7.1.1. For $1 \le i \le n$, define

$$\Psi_i = \frac{1}{\sqrt{2}} x_i \otimes \mathbb{1} - \sqrt{2} \sum_{l=1}^{n} x_l \otimes \xi_i \partial_{\xi_l} + \sqrt{2} \sum_{l=1}^{n} \partial_{x_l} \otimes \xi_i \partial_{\xi_{l+n}}$$

$$\Psi_{-i} = \frac{1}{\sqrt{2}} \partial_{x_i} \otimes \mathbb{1} + \sqrt{2} \sum_{l=1}^{n} x_l \otimes \xi_{i+n} \partial_{\xi_l} - \sqrt{2} \sum_{l=1}^{n} \partial_{x_l} \otimes \xi_{i+n} \partial_{\xi_{l+n}}.$$

The following table provides important relations for the goal of this chapter, establishing a morphism of associative superalgebras in order to present new infinite-dimensional modules of $\mathfrak{g} = \mathfrak{osp}(1|2n)$.

Table 7.1. Relations for differential operators

commuting operators	$x_i x_j = x_j x_i$
	$\partial_{x_i}\partial_{x_j} = \partial_{x_j}\partial_{x_i}$
	$\partial_{x_i} x_j = x_j \partial_{x_i} + \delta_{ij}$
anti-commuting operators	$\xi_i \xi_j = -\xi_j \xi_i$
	$\partial_{\xi_i}\partial_{\xi_j} = -\partial_{\xi_j}\partial_{\xi_i}$
	$\partial_{\xi_i}\xi_j = -\xi_j\partial_{\xi_i} + \delta_{ij}$

7.2 Second Main Result

We show the map

$$\Psi : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(W]),\ X_{\pm\delta_j} \mapsto \Psi_{\pm j},$$

is a morphism of superalgebras by appealing to the universal property of $\mathcal{U}(\mathfrak{g})$.

7.2.1 Relations on the New "Psi"

Lemma 7.2.1. *The operators Ψ_j, Ψ_{-j}, $j = 1, \ldots, n$, satisfy the relations in (eq.* (5.5)*). In particular,*

$$[[\Psi_p, \Psi_q], \Psi_{-q}] = -\Psi_p \tag{7.1}$$

$$[[\Psi_p, \Psi_p], \Psi_{-p}] = -2\Psi_p \tag{7.2}$$

Proof. We prove in details the two identities listed in the statement of the lemma. The remaining identities are proven in an analogous way using the formulas for $[\Psi_p, \Psi_{-q}]$ and $[\Psi_{-p}, \Psi_{-q}]$ that are provided at the end of this proof.

$$
\begin{aligned}
&[\Psi_p, \Psi_q] \\
&= \Psi_p\Psi_q + \Psi_q\Psi_p \\
&= (\frac{1}{\sqrt{2}}x_p \otimes \mathbb{1} - \sqrt{2}\sum_{l=1}^{n} x_l \otimes \xi_p\partial_{\xi_l} + \sqrt{2}\sum_{l=1}^{n} \partial_{x_l} \otimes \xi_p\partial_{\xi_{l+n}})(\frac{1}{\sqrt{2}}x_q \otimes \mathbb{1} \\
&\quad - \sqrt{2}\sum_{l=1}^{n} x_l \otimes \xi_q\partial_{\xi_l} + \sqrt{2}\sum_{l=1}^{n} \partial_{x_l} \otimes \xi_q\partial_{\xi_{l+n}}) + (\frac{1}{\sqrt{2}}x_q \otimes \mathbb{1} - \sqrt{2}\sum_{l=1}^{n} x_l \otimes \xi_q\partial_{\xi_l} \\
&\quad + \sqrt{2}\sum_{l=1}^{n} \partial_{x_l} \otimes \xi_q\partial_{\xi_{l+n}})(\frac{1}{\sqrt{2}}x_p \otimes \mathbb{1} - \sqrt{2}\sum_{l=1}^{n} x_l \otimes \xi_p\partial_{\xi_l} + \sqrt{2}\sum_{l=1}^{n} \partial_{x_l} \otimes \xi_p\partial_{\xi_{l+n}}) \\
&= \frac{1}{2}x_px_q \otimes \mathbb{1} - \sum_{l=1}^{n} x_px_l \otimes \xi_q\partial_{\xi_l} + \sum_{l=1}^{n} x_p\partial_{x_l} \otimes \xi_q\partial_{\xi_{l+n}} \\
&\quad + \frac{1}{2}x_qx_p \otimes \mathbb{1} - \sum_{l=1}^{n} x_qx_l \otimes \xi_p\partial_{\xi_l} + \sum_{l=1}^{n} x_q\partial_{x_l} \otimes \xi_p\partial_{\xi_{l+n}} \\
&\quad - \sum_{l=1}^{n} x_lx_q \otimes \xi_p\partial_{\xi_l} + 2\sum_{k=1}^{n}\sum_{l=1}^{n} x_kx_l \otimes \xi_p\partial_{\xi_k}\xi_q\partial_{\xi_l} - 2\sum_{k=1}^{n}\sum_{l=1}^{n} x_k\partial_{x_l} \otimes \xi_p\partial_{\xi_k}\xi_q\partial_{\xi_{l+n}} \\
&\quad - \sum_{l=1}^{n} x_lx_p \otimes \xi_q\partial_{\xi_l} + 2\sum_{k=1}^{n}\sum_{l=1}^{n} x_kx_l \otimes \xi_q\partial_{\xi_k}\xi_p\partial_{\xi_l} - 2\sum_{k=1}^{n}\sum_{l=1}^{n} x_k\partial_{x_l} \otimes \xi_q\partial_{\xi_k}\xi_p\partial_{\xi_{l+n}} \\
&\quad + \sum_{l=1}^{n} \partial_{x_l}x_q \otimes \xi_p\partial_{\xi_{l+n}} - 2\sum_{k=1}^{n}\sum_{l=1}^{n} \partial_{x_k}x_l \otimes \xi_p\partial_{\xi_{k+n}}\xi_q\partial_{\xi_l} + 2\sum_{k=1}^{n}\sum_{l=1}^{n} \partial_{x_k}\partial_{x_l} \otimes \xi_p\partial_{\xi_{k+n}}\xi_q\partial_{\xi_{l+n}} \\
&\quad + \sum_{l=1}^{n} \partial_{x_l}x_p \otimes \xi_q\partial_{\xi_{l+n}} - 2\sum_{k=1}^{n}\sum_{l=1}^{n} \partial_{x_k}x_l \otimes \xi_q\partial_{\xi_{k+n}}\xi_p\partial_{\xi_l} + 2\sum_{k=1}^{n}\sum_{l=1}^{n} \partial_{x_k}\partial_{x_l} \otimes \xi_q\partial_{\xi_{k+n}}\xi_p\partial_{\xi_{l+n}} \\
&= x_px_q \otimes \mathbb{1} - 2\sum_{l=1}^{n} x_px_l \otimes \xi_q\partial_{\xi_l} - 2\sum_{l=1}^{n} x_qx_l \otimes \xi_p\partial_{\xi_l} \\
&\quad + 2\sum_{l=1}^{n} x_p\partial_{x_l} \otimes \xi_q\partial_{\xi_{l+n}} + 2\sum_{l=1}^{n} x_q\partial_{x_l} \otimes \xi_p\partial_{\xi_{l+n}} + \mathbb{1} \otimes \xi_p\partial_{\xi_{q+n}} + \mathbb{1} \otimes \xi_q\partial_{\xi_{p+n}} \\
&\quad + 2\sum_{l=1}^{n} x_px_l \otimes \xi_q\partial_{\xi_l} + 2\sum_{l=1}^{n} x_qx_l \otimes \xi_p\partial_{\xi_l} - 2\sum_{l=1}^{n} x_p\partial_{x_l} \otimes \xi_q\partial_{\xi_{l+n}} - 2\sum_{l=1}^{n} x_q\partial_{x_l} \otimes \xi_p\partial_{\xi_{l+n}} \\
&= x_px_q \otimes \mathbb{1} + \mathbb{1} \otimes \xi_p\partial_{\xi_{q+n}} + \mathbb{1} \otimes \xi_q\partial_{\xi_{p+n}};
\end{aligned}
$$

$$
\begin{aligned}
&[[\Psi_p,\Psi_q],\Psi_{-q}]\\
&=(x_px_q\otimes\mathbb{1}+\mathbb{1}\otimes\xi_p\partial_{\xi_{q+n}}+\mathbb{1}\otimes\xi_q\partial_{\xi_{p+n}})(\frac{1}{\sqrt{2}}\partial_{x_q}\otimes\mathbb{1}+\sqrt{2}\sum_{l=1}^{n}x_l\otimes\xi_{q+n}\partial_{\xi_l}\\
&\quad-\sqrt{2}\sum_{l=1}^{n}\partial_{x_l}\otimes\xi_{q+n}\partial_{\xi_{l+n}})-(\frac{1}{\sqrt{2}}\partial_{x_q}\otimes\mathbb{1}+\sqrt{2}\sum_{l=1}^{n}x_l\otimes\xi_{q+n}\partial_{\xi_l}\\
&\quad-\sqrt{2}\sum_{l=1}^{n}\partial_{x_l}\otimes\xi_{q+n}\partial_{\xi_{l+n}})(x_px_q\otimes\mathbb{1}+\mathbb{1}\otimes\xi_p\partial_{\xi_{q+n}}+\mathbb{1}\otimes\xi_q\partial_{\xi_{p+n}})\\
&=\frac{1}{\sqrt{2}}x_px_q\partial_{x_q}\otimes\mathbb{1}+\sqrt{2}\sum_{l=1}^{n}x_px_qx_l\otimes\xi_{q+n}\partial_{\xi_l}-\sqrt{2}\sum_{l=1}^{n}x_px_q\partial_{x_l}\otimes\xi_{q+n}\partial_{\xi_{l+n}}\\
&\quad+\frac{1}{\sqrt{2}}\partial_{x_q}\otimes\xi_p\partial_{\xi_{q+n}}+\sqrt{2}\sum_{l=1}^{n}x_l\otimes\xi_p\partial_{\xi_{q+n}}\xi_{q+n}\partial_{\xi_l}-\sqrt{2}\sum_{l=1}^{n}\partial_{x_l}\otimes\xi_p\partial_{\xi_{q+n}}\xi_{q+n}\partial_{\xi_{l+n}}\\
&\quad+\frac{1}{\sqrt{2}}\partial_{x_q}\otimes\xi_q\partial_{\xi_{p+n}}+\sqrt{2}\sum_{l=1}^{n}x_l\otimes\xi_q\partial_{\xi_{p+n}}\xi_{q+n}\partial_{\xi_l}-\sqrt{2}\sum_{l=1}^{n}\partial_{x_l}\otimes\xi_q\partial_{\xi_{p+n}}\xi_{q+n}\partial_{\xi_{l+n}}\\
&\quad-\frac{1}{\sqrt{2}}\partial_{x_q}x_px_q\otimes\mathbb{1}-\frac{1}{\sqrt{2}}\partial_{x_q}\otimes\xi_p\partial_{\xi_{q+n}}-\frac{1}{\sqrt{2}}\partial_{x_q}\otimes\xi_q\partial_{\xi_{p+n}}\\
&\quad-\sqrt{2}\sum_{l=1}^{n}x_lx_px_q\otimes\xi_{q+n}\partial_{\xi_l}-\sqrt{2}\sum_{l=1}^{n}x_l\otimes\xi_{q+n}\partial_{\xi_l}\xi_p\partial_{\xi_{q+n}}-\sqrt{2}\sum_{l=1}^{n}x_l\otimes\xi_{q+n}\partial_{\xi_l}\xi_q\partial_{\xi_{p+n}}\\
&\quad+\sqrt{2}\sum_{l=1}^{n}\partial_{x_l}x_px_q\otimes\xi_{q+n}\partial_{\xi_{l+n}}+\sqrt{2}\sum_{l=1}^{n}\partial_{x_l}\otimes\xi_{q+n}\partial_{\xi_{l+n}}\xi_p\partial_{\xi_{q+n}}+\sqrt{2}\sum_{l=1}^{n}\partial_{x_l}\otimes\xi_{q+n}\partial_{\xi_{l+n}}\xi_q\partial_{\xi_{p+n}}\\
&=\ -\frac{1}{\sqrt{2}}x_p\otimes\mathbb{1}+\sqrt{2}x_p\otimes\xi_{q+n}\partial_{\xi_{q+n}}+\sqrt{2}x_q\otimes\xi_{q+n}\partial_{\xi_{p+n}}\\
&\quad-\sqrt{2}x_p\otimes\xi_{q+n}\partial_{\xi_{q+n}}+\sqrt{2}\sum_{l=1}^{n}x_l\otimes\xi_p\partial_{\xi_l}-\sqrt{2}\sum_{l=1}^{n}\partial_{x_l}\otimes\xi_p\partial_{\xi_{l+n}}\\
&\quad-\sqrt{2}x_q\otimes\xi_{q+n}\partial_{\xi_{p+n}}\\
&=-\Psi_p,
\end{aligned}
$$

$$
\begin{aligned}
&[\Psi_p, \Psi_p] \\
&= 2\Psi_p\Psi_p \\
&= 2(\frac{1}{\sqrt{2}}x_p \otimes \mathbb{1} - \sqrt{2}\sum_{l=1}^{n} x_l \otimes \xi_p\partial_{\xi_l} + \sqrt{2}\sum_{l=1}^{n} \partial_{x_l} \otimes \xi_p\partial_{\xi_{l+n}})(\frac{1}{\sqrt{2}}x_p \otimes \mathbb{1} \\
&\quad - \sqrt{2}\sum_{l=1}^{n} x_l \otimes \xi_p\partial_{\xi_l} + \sqrt{2}\sum_{l=1}^{n} \partial_{x_l} \otimes \xi_p\partial_{\xi_{l+n}}) \\
&= x_p^2 \otimes \mathbb{1} - 2\sum_{l=1}^{n} x_p x_l \otimes \xi_p\partial_{\xi_l} + 2\sum_{l=1}^{n} x_p\partial_{x_l} \otimes \xi_p\partial_{\xi_{l+n}} \\
&\quad - 2\sum_{l=1}^{n} x_l x_p \otimes \xi_p\partial_{\xi_l} + 4\sum_{k=1}^{n}\sum_{l=1}^{n} x_k x_l \otimes \xi_p\partial_{\xi_k}\xi_p\partial_{\xi_l} - 4\sum_{k=1}^{n}\sum_{l=1}^{n} x_k\partial_{x_l} \otimes \xi_p\partial_{\xi_k}\xi_p\partial_{\xi_{l+n}} \\
&\quad + 2\sum_{l=1}^{n} \partial_{x_l} x_p \otimes \xi_p\partial_{\xi_{l+n}} - 4\sum_{k=1}^{n}\sum_{l=1}^{n} \partial_{x_k} x_l \otimes \xi_p\partial_{\xi_{k+n}}\xi_p\partial_{\xi_l} + 4\sum_{k=1}^{n}\sum_{l=1}^{n} \partial_{x_k}\partial_{x_l} \otimes \xi_p\partial_{\xi_{k+n}}\xi_p\partial_{\xi_{l+n}}) \\
&= x_p^2 \otimes \mathbb{1} - 4\sum_{l=1}^{n} x_p x_l \otimes \xi_p\partial_{\xi_l} + 4\sum_{l=1}^{n} x_p\partial_{x_l} \otimes \xi_p\partial_{\xi_{l+n}} + \mathbb{2} \otimes \xi_p\partial_{\xi_{p+n}} \\
&\quad + 4\sum_{l=1}^{n} x_p x_l \otimes \xi_p\partial_{\xi_l} - 2\sum_{l=1}^{n} x_p\partial_{x_l} \otimes \xi_p\partial_{\xi_{l+n}} \\
&= x_p^2 \otimes \mathbb{1} + \mathbb{2} \otimes \xi_p\partial_{\xi_{p+n}};
\end{aligned}
$$

$$
\begin{aligned}
&[[\Psi_p, \Psi_p], \Psi_{-p}] \\
&= (x_p^2 \otimes \mathbb{1} + \mathbb{2} \otimes \xi_p \partial_{\xi_{p+n}})(\frac{1}{\sqrt{2}} \partial_{x_p} \otimes \mathbb{1} + \sqrt{2} \sum_{l=1}^{n} x_l \otimes \xi_{p+n} \partial_{\xi_l} - \sqrt{2} \sum_{l=1}^{n} \partial_{x_l} \otimes \xi_{p+n} \partial_{\xi_{l+n}}) \\
&\quad - (\frac{1}{\sqrt{2}} \partial_{x_p} \otimes \mathbb{1} + \sqrt{2} \sum_{l=1}^{n} x_l \otimes \xi_{p+n} \partial_{\xi_l} - \sqrt{2} \sum_{l=1}^{n} \partial_{x_l} \otimes \xi_{p+n} \partial_{\xi_{l+n}})(x_p^2 \otimes \mathbb{1} + \mathbb{2} \otimes \xi_p \partial_{\xi_{p+n}}) \\
&= \frac{1}{\sqrt{2}} x_p^2 \partial_{x_p} \otimes \mathbb{1} + \sqrt{2} \sum_{l=1}^{n} x_p^2 x_l \otimes \xi_{p+n} \partial_{\xi_l} - \sqrt{2} \sum_{l=1}^{n} x_p^2 \partial_{x_l} \otimes \xi_{p+n} \partial_{\xi_{l+n}} \\
&\quad + \sqrt{2} \partial_{x_p} \otimes \xi_p \partial_{\xi_{p+n}} + 2\sqrt{2} \sum_{l=1}^{n} x_l \otimes \xi_p \partial_{\xi_{p+n}} \xi_{p+n} \partial_{\xi_l} - 2\sqrt{2} \sum_{l=1}^{n} \partial_{x_l} \otimes \xi_p \partial_{\xi_{p+n}} \xi_{p+n} \partial_{\xi_{l+n}} \\
&\quad - \frac{1}{\sqrt{2}} \partial_{x_p} x_p^2 \otimes \mathbb{1} - \sqrt{2} \partial_{x_p} \otimes \xi_p \partial_{\xi_{p+n}} \\
&\quad - \sqrt{2} \sum_{l=1}^{n} x_l x_p^2 \otimes \xi_{p+n} \partial_{\xi_l} - 2\sqrt{2} \sum_{l=1}^{n} x_l \otimes \xi_{p+n} \partial_{\xi_l} \xi_p \partial_{\xi_{p+n}} \\
&\quad + \sqrt{2} \sum_{l=1}^{n} \partial_{x_l} x_p^2 \otimes \xi_{p+n} \partial_{\xi_{l+n}} + 2\sqrt{2} \sum_{l=1}^{n} \partial_{x_l} \otimes \xi_{p+n} \partial_{\xi_{l+n}} \xi_p \partial_{\xi_{p+n}} \\
&= -2x_p \otimes \mathbb{1} + 2\sqrt{2} x_p \otimes \xi_{p+n} \partial_{\xi_{p+n}} - 2\sqrt{2} x_p \otimes \xi_{p+n} \partial_{\xi_{p+n}} \\
&\quad + 2\sqrt{2} \sum_{l=1}^{n} x_p \otimes \xi_p \partial_{\xi_l} - 2\sqrt{2} \sum_{l=1}^{n} \partial_{x_p} \otimes \xi_p \partial_{\xi_{l+n}} \\
&= -2\Psi_p.
\end{aligned}
$$

$$
\begin{aligned}
&[\Psi_p, \Psi_{-q}] \\
&= \Psi_p\Psi_{-q} + \Psi_{-q}\Psi_p \\
&= (\frac{1}{\sqrt{2}}x_p \otimes \mathbb{1} - \sqrt{2}\sum_{l=1}^{n} x_l \otimes \xi_p\partial_{\xi_l} + \sqrt{2}\sum_{l=1}^{n} \partial_{x_l} \otimes \xi_p\partial_{\xi_{l+n}})(\frac{1}{\sqrt{2}}\partial_{x_q} \otimes \mathbb{1} \\
&\quad + \sqrt{2}\sum_{l=1}^{n} x_l \otimes \xi_{q+n}\partial_{\xi_l} - \sqrt{2}\sum_{l=1}^{n} \partial_{x_l} \otimes \xi_{q+n}\partial_{\xi_{l+n}}) \\
&\quad + (\frac{1}{\sqrt{2}}\partial_{x_q} \otimes \mathbb{1} + \sqrt{2}\sum_{l=1}^{n} x_l \otimes \xi_{q+n}\partial_{\xi_l} - \sqrt{2}\sum_{l=1}^{n} \partial_{x_l} \otimes \xi_{q+n}\partial_{\xi_{l+n}})(\frac{1}{\sqrt{2}}x_p \otimes \mathbb{1} \\
&\quad - \sqrt{2}\sum_{l=1}^{n} x_l \otimes \xi_p\partial_{\xi_l} + \sqrt{2}\sum_{l=1}^{n} \partial_{x_l} \otimes \xi_p\partial_{\xi_{l+n}}) \\
&= \frac{1}{2}x_p\partial_{x_q} \otimes \mathbb{1} + \sum_{l=1}^{n} x_p x_l \otimes \xi_{q+n}\partial_{\xi_l} - \sum_{l=1}^{n} x_p\partial_{x_l} \otimes \xi_{q+n}\partial_{\xi_{l+n}} \\
&\quad - \sum_{l=1}^{n} x_l\partial_{x_q} \otimes \xi_p\partial_{\xi_l} - 2\sum_{k=1}^{n}\sum_{l=1}^{n} x_k x_l \otimes \xi_p\partial_{\xi_k}\xi_{q+n}\partial_{\xi_l} + 2\sum_{k=1}^{n}\sum_{l=1}^{n} x_k\partial_{x_l} \otimes \xi_p\partial_{\xi_k}\xi_{q+n}\partial_{\xi_{l+n}} \\
&\quad + \sum_{l=1}^{n} \partial_{x_l}\partial_{x_q} \otimes \xi_p\partial_{\xi_{l+n}} + 2\sum_{k=1}^{n}\sum_{l=1}^{n} \partial_{x_k}x_l \otimes \xi_p\partial_{\xi_{k+n}}\xi_{q+n}\partial_{\xi_l} - 2\sum_{k=1}^{n}\sum_{l=1}^{n} \partial_{x_k}\partial_{x_l} \otimes \xi_p\partial_{\xi_{k+n}}\xi_{q+n}\partial_{\xi_{l+n}} \\
&\quad + \frac{1}{2}\partial_{x_q}x_p \otimes \mathbb{1} - \sum_{l=1}^{n} \partial_{x_q}x_l \otimes \xi_p\partial_{\xi_l} + \sum_{l=1}^{n} \partial_{x_q}\partial_{x_l} \otimes \xi_p\partial_{\xi_{l+n}} \\
&\quad + \sum_{l=1}^{n} x_l x_p \otimes \xi_{q+n}\partial_{\xi_l} - 2\sum_{k=1}^{n}\sum_{l=1}^{n} x_k x_l \otimes \xi_{q+n}\partial_{\xi_k}\xi_p\partial_{\xi_l} + 2\sum_{k=1}^{n}\sum_{l=1}^{n} x_k\partial_{x_l} \otimes \xi_{q+n}\partial_{\xi_k}\xi_p\partial_{\xi_{l+n}} \\
&\quad - \sum_{l=1}^{n} \partial_{x_l}x_p \otimes \xi_{q+n}\partial_{\xi_{l+n}} + 2\sum_{k=1}^{n}\sum_{l=1}^{n} \partial_{x_k}x_l \otimes \xi_{q+n}\partial_{\xi_{k+n}}\xi_p\partial_{\xi_l} - 2\sum_{k=1}^{n}\sum_{l=1}^{n} \partial_{x_k}\partial_{x_l} \otimes \xi_{q+n}\partial_{\xi_{k+n}}\xi_p\partial_{\xi_{l+n}} \\
&= x_p\partial_{x_q} \otimes \mathbb{1} + \frac{\delta_{pq}}{2} \otimes \mathbb{1} + 2\sum_{l=1}^{n} x_p x_l \otimes \xi_{q+n}\partial_{\xi_l} - 2\sum_{l=1}^{n} x_p\partial_{x_l} \otimes \xi_{q+n}\partial_{\xi_{l+n}} - \mathbb{1} \otimes \xi_{q+n}\partial_{\xi_{p+n}} \\
&\quad - 2\sum_{l=1}^{n} x_l\partial_{x_q} \otimes \xi_p\partial_{\xi_l} - \mathbb{1} \otimes \xi_p\partial_{\xi_q} - 2\sum_{l=1}^{n} x_p x_l \otimes \xi_{q+n}\partial_{\xi_l} + 2\sum_{l=1}^{n} x_p\partial_{x_l} \otimes \xi_{q+n}\partial_{\xi_{l+n}} \\
&\quad + 2\sum_{l=1}^{n} \partial_{x_q}\partial_{x_l} \otimes \xi_p\partial_{\xi_{l+n}} + 2\sum_{l=1}^{n} \partial_{x_q}x_l \otimes \xi_p\partial_{\xi_l} - 2\sum_{l=1}^{n} \partial_{x_q}\partial_{x_l} \otimes \xi_p\partial_{\xi_{l+n}} \\
&= x_p\partial_{x_q} + \frac{\delta_{pq}}{2} \otimes \mathbb{1} + \mathbb{1} \otimes \xi_p\partial_{\xi_q} - \mathbb{1} \otimes \xi_{q+n}\partial_{\xi_{p+n}};
\end{aligned}
$$

$$
\begin{aligned}
&[\Psi_{-p}, \Psi_{-q}] \\
&= \Psi_{-p}\Psi_{-q} + \Psi_{-q}\Psi_{-p} \\
&= (\frac{1}{\sqrt{2}}\partial_{x_p} \otimes \mathbb{1} + \sqrt{2}\sum_{l=1}^{n} x_l \otimes \xi_{p+n}\partial_{\xi_l} - \sqrt{2}\sum_{l=1}^{n} \partial_{x_l} \otimes \xi_{p+n}\partial_{\xi_{l+n}})(\frac{1}{\sqrt{2}}\partial_{x_q} \otimes \mathbb{1} \\
&\quad + \sqrt{2}\sum_{l=1}^{n} x_l \otimes \xi_{q+n}\partial_{\xi_l} - \sqrt{2}\sum_{l=1}^{n} \partial_{x_l} \otimes \xi_{q+n}\partial_{\xi_{l+n}}) \\
&\quad + (\frac{1}{\sqrt{2}}\partial_{x_q} \otimes \mathbb{1} + \sqrt{2}\sum_{l=1}^{n} x_l \otimes \xi_{q+n}\partial_{\xi_l} - \sqrt{2}\sum_{l=1}^{n} \partial_{x_l} \otimes \xi_{q+n}\partial_{\xi_{l+n}})(\frac{1}{\sqrt{2}}\partial_{x_p} \otimes \mathbb{1} \\
&\quad + \sqrt{2}\sum_{l=1}^{n} x_l \otimes \xi_{p+n}\partial_{\xi_l} + \sqrt{2}\sum_{l=1}^{n} \partial_{x_l} \otimes \xi_{p+n}\partial_{\xi_{l+n}}) \\
&= \frac{1}{2}\partial_{x_p}\partial_{x_q} \otimes \mathbb{1} + \sum_{l=1}^{n} \partial_{x_p} x_l \otimes \xi_{q+n}\partial_{\xi_l} - \sum_{l=1}^{n} \partial_{x_p}\partial_{x_l} \otimes \xi_{q+n}\partial_{\xi_{l+n}} \\
&\quad + \sum_{l=1}^{n} x_l\partial_{x_q} \otimes \xi_{p+n}\partial_{\xi_l} + 2\sum_{k=1}^{n}\sum_{l=1}^{n} x_k x_l \otimes \xi_{p+n}\partial_{\xi_k}\xi_{q+n}\partial_{\xi_l} - 2\sum_{k=1}^{n}\sum_{l=1}^{n} x_k\partial_{x_l} \otimes \xi_{p+n}\partial_{\xi_k}\xi_{q+n}\partial_{\xi_{l+n}} \\
&\quad - \sum_{l=1}^{n} \partial_{x_l}\partial_{x_q} \otimes \xi_{p+n}\partial_{\xi_{l+n}} - 2\sum_{k=1}^{n}\sum_{l=1}^{n} \partial_{x_k} x_l \otimes \xi_{p+n}\partial_{\xi_{k+n}}\xi_{q+n}\partial_{\xi_l} \\
&\quad + 2\sum_{k=1}^{n}\sum_{l=1}^{n} \partial_{x_k}\partial_{x_l} \otimes \xi_{p+n}\partial_{\xi_{k+n}}\xi_{q+n}\partial_{\xi_{l+n}} \\
&\quad + \frac{1}{2}\partial_{x_q}\partial_{x_p} \otimes \mathbb{1} + \sum_{l=1}^{n} \partial_{x_q} x_l \otimes \xi_{p+n}\partial_{\xi_l} - \sum_{l=1}^{n} \partial_{x_q}\partial_{x_l} \otimes \xi_{p+n}\partial_{\xi_{l+n}} \\
&\quad + \sum_{l=1}^{n} x_l\partial_{x_p} \otimes \xi_{q+n}\partial_{\xi_l} + 2\sum_{k=1}^{n}\sum_{l=1}^{n} x_k x_l \otimes \xi_{q+n}\partial_{\xi_k}\xi_{p+n}\partial_{\xi_l} - 2\sum_{k=1}^{n}\sum_{l=1}^{n} x_k\partial_{x_l} \otimes \xi_{q+n}\partial_{\xi_k}\xi_{p+n}\partial_{\xi_{l+n}} \\
&\quad - \sum_{l=1}^{n} \partial_{x_l}\partial_{x_p} \otimes \xi_{q+n}\partial_{\xi_{l+n}} - 2\sum_{k=1}^{n}\sum_{l=1}^{n} \partial_{x_k} x_l \otimes \xi_{q+n}\partial_{\xi_{k+n}}\xi_{p+n}\partial_{\xi_l} \\
&\quad + 2\sum_{k=1}^{n}\sum_{l=1}^{n} \partial_{x_k}\partial_{x_l} \otimes \xi_{q+n}\partial_{\xi_{k+n}}\xi_{p+n}\partial_{\xi_{l+n}} \\
&= \partial_{x_p}\partial_{x_q} \otimes \mathbb{1} - \mathbb{1} \otimes \xi_{p+n}\partial_{\xi_q} - \mathbb{1} \otimes \xi_{q+n}\partial_{\xi_p}.
\end{aligned}
$$

□

Denote by $\Lambda(\boldsymbol{\xi}) = \Lambda(\xi_1, \ldots, \xi_{2n})$ the superalgebra generated by the odd variables ξ_i subject to $\xi_i\xi_j + \xi_j\xi_i = 0$.

7.2.2 Main Theorem II

Theorem 7.2.2. *The following correspondence defines a homomorphism* $\Psi : \mathfrak{U}(\mathfrak{osp}(1|2n)) \to \mathrm{End}(\mathbb{C}[\mathbf{x}] \otimes \Lambda(\boldsymbol{\xi}))$ *of associative superalgebras with identity:*

$$\begin{aligned}
X_{\delta_p-\delta_q} &\longmapsto x_p\partial_{x_q}\mathbb{1} + \mathbb{1}\otimes\xi_p\partial_{\xi_q} - \mathbb{1}\otimes\xi_{q+n}\partial_{\xi_{p+n}};\\
X_{2\delta_p} &\longmapsto \frac{1}{2}x_p^2\otimes\mathbb{1} + \mathbb{1}\otimes\xi_p\partial_{\xi_{p+n}};\\
X_{-2\delta_p} &\longmapsto -\frac{1}{2}\partial_{x_p}^2\otimes\mathbb{1} - \mathbb{1}\otimes\xi_{p+n}\partial_{\xi_p};\\
X_{\delta_p+\delta_q} &\longmapsto x_px_q\otimes\mathbb{1} + \mathbb{1}\otimes\xi_p\partial_{\xi_{q+n}} + \mathbb{1}\otimes\xi_q\partial_{\xi_{p+n}};\\
X_{-\delta_p-\delta_q} &\longmapsto -\partial_{x_p}\partial_{x_q}\otimes\mathbb{1} + \mathbb{1}\otimes\xi_{p+n}\partial_{\xi_q} + \mathbb{1}\otimes\xi_{q+n}\partial_{\xi_p};\\
h_{\delta_p-\delta_q} &\longmapsto x_p\partial_{x_q} - x_q\partial_{x_p}\otimes\mathbb{1} + \mathbb{1}\otimes\xi_p\partial_{\xi_p} - \xi_q\partial_{\xi_q} + \mathbb{1}\otimes\xi_{q+n}\partial_{\xi_{q+n}} - \xi_{p+n}\partial_{\xi_{p+n}};\\
h_{2\delta_p} &\longmapsto x_p\partial_{x_p} + \frac{1}{2}\otimes\mathbb{1} + \mathbb{1}\otimes\xi_p\partial_{\xi_p} - \mathbb{1}\otimes\xi_{p+n}\partial_{\xi_{p+n}};\\
X_{\delta_p} &\longmapsto \frac{1}{\sqrt{2}}x_p\otimes\mathbb{1} - \sqrt{2}\sum_{l=1}^{n}x_l\otimes\xi_p\partial_{\xi_l} + \sqrt{2}\sum_{l=1}^{n}\partial_{x_l}\otimes\xi_p\partial_{\xi_{l+n}};\\
X_{-\delta_p} &\longmapsto \frac{1}{\sqrt{2}}\partial_{x_p}\otimes\mathbb{1} + \sqrt{2}\sum_{l=1}^{n}x_l\otimes\xi_{p+n}\partial_{\xi_l} - \sqrt{2}\sum_{l=1}^{n}\partial_{x_l}\otimes\xi_{p+n}\partial_{\xi_{l+n}}.
\end{aligned}$$

Proof. The proof follows from Lemma 7.2.1 and the identities provided in its proof. □

Denote by $\mathcal{D}(n|2n)^+$ the associative subalgebra of $\mathrm{End}(\mathbb{C}[\mathbf{x}] \otimes \Lambda[\boldsymbol{\xi}])$ generated by $x_i \otimes 1$, $\partial_{x_i} \otimes 1$, $1 \otimes \xi_j$, $1 \otimes \partial_{\xi_j}$, $i = 1, \ldots, n$, $j = 1, \ldots, 2n$. Then the proceeding theorem implies that $\Psi : \mathfrak{U}(\mathfrak{osp}(1|2n)) \to \mathcal{D}(n|2n)^+$ is a homomorphism of associative superalgebras. Hence, any module of the algebra $\mathcal{D}(n|2n)^+$ has a $\mathfrak{g}$-module structure through Ψ. Let $\Lambda(\xi_1, \ldots, \xi_{2n})_d$ stand for the space spanned by the degree d monomials $\xi_{i_1}\xi_{i_2}\ldots\xi_{i_d}$, $i_t \neq i_s$. Then we have the following.

Corollary 7.2.3. *The super spaces* $\mathbb{C}[\mathbf{x}] \otimes \Lambda[\boldsymbol{\xi}]$ *and* $\mathbb{C}[\mathbf{x}] \otimes \Lambda[\boldsymbol{\xi}]_d$ *have structures of* $\mathfrak{osp}(1|2n)$*-modules.*

The methods of Chapter 6 are now in play.

7.2.3 New Idea

The preceding results can be re-framed without tensor products and using non-conventional relations on the even and odd variables.

Remark 7.2.4. We can consider different grading on $\mathcal{D}(n|2n)^+$ and define the variable ξ_i and their derivatives to be even. In fact, using the ideas of [20], we have a more invariant description. Namely, if we first fix n odd variables x_i and $2n$ even variables ξ_p, satisfying $x_i x_j - x_j x_i = x_i \xi_p + \xi_p x_i = \xi_p \xi_q + \xi_q \xi_p = 0$. Then we introduce derivations ∂_{x_i} and ∂_{ξ_j} and relations between the derivations that are analogous t those between the variables. The resulting superalgebra is denoted by $A^+_{n|2n}$ and we obtain a homomorphism $\mathcal{U}(\mathfrak{osp}(1|2n)) \to A^+_{n|2n}$.

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