STOCHASTIC RISK MEASURES FOR THE LUNDBERG MODEL WITH REINSURANCE AND INVESTMENT

by

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ABSTRACT

STOCHASTIC RISK MEASURES FOR THE LUNDBERG MODEL WITH REINSURANCE AND INVESTMENT

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Risk measures emerge in fields such as economics, insurance, finance and are concerned with a stochastic representation of uncertainties stemming from the unpredictability of the real world events. In essence, risk analysis amounts to quantifying the chances of undesirable events and developing a model that limits the impact of potential losses. Assets and liabilities in the Insurance industry, as well as financial goals of Investment companies rely on calculating the probability that their respective portfolios satisfy the preset constraints. On the flip side, risk measures serve both industries by providing optimal strategies for minimizing losses. Our research is concerned with Distorted Risk Measures (DRMs) in stochastic optimization regarding decisions about the size of the risk exposure. We extend the classical Lundberg Risk Model to the case of periodic reinsurance with investment.

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CHAPTER 1

INTRODUCTION

In simple terms, risk is the possibility of something bad happening. It involves uncertainty about the effects/implications of an activity with respect to something that humans value (such as health, well-being, wealth, property, or the environment), often focusing on negative, undesirable consequences [30]. Risk is inherent in any enterprise and is an important factor that affects all sectors of today's economy. Key elements of risk minimization are based on quantifying the adverse effects and subsequent development of modeling through stochastic analysis. We include a sample of extensive literature on the subject such as monographs, textbooks, research articles ([1], [3], [4], [12], [13], [18]) in the REFERENCES. Our interest in mathematical risk stems from its wide-spread usage in the Insurance Industry, where stochastic model for losses dates back to the 1900s, thanks to the pioneering work by Lundberg [25]. As an integral part of risk management, reinsurance had emerged as a standard practice for strategic risk spread, i.e., ceding a portion of insurer's liability to the reinsurer for reinsurance premium. We focus on developing a strategy that not only allows the insurers to minimize their risk exposure but also maximize their profit at the same time. Our research has been motivated by the works of Golubin ([15], [16], [17]), and Cheung and Lo ([22], [23], [7] ,[8]). In a nutshell, we combined some ideas from the dynamic model of Golubin with a static Model of Cheun-Lo and developed a novel Model for periodic risk optimization. In Chapter 1, we give the background related to risk and explain the motivation for the research.

In Chapter 2, we describe our model based on the general Lundberg model, which we extended to a periodic model coupled with reinsurance. We then describe the risk process that is a subject of this study. In Chapter 3, we set up our first extended Lundberg model without investment and give solutions with regards to certain processes with independent increment. In Chapter 4, we propose our second extended Lundberg model with investment and present solutions applicable to real world problems. In Chapter 5, we extend our work to processes that no longer have independent increments and obtain results for models with or without investment. In chapter 6, we explore a new hybrid method that can be used to better assess the underline risk and pave the way for future investigation.

CHAPTER 2

MODEL FORMULATION

2.0.1 Lundberg Model

We start with the classical Lundberg model which describes a risk process of an insurance company in order to balance two opposing cash flows. The following defines the risk process (see, e.g., [17])

$$X_t^{in} = u + ct - \sum_{j=1}^{N_t} X_j \tag{2.1}$$

where $u > 0$ is the initial capital of the company, N_t is a Poisson process of claims with parameter λ that defines the number of claims for insurance payments. In other words, it is the number of claims on the interval $[0, t]$ and X_j are the claim sizes which represent the sequence of independent, identically distributed random variables, independent of N_t. The constant $c > 0$ represents the premium rate at which the insurer is being continuously paid by insured customers.

The optimal control problem for the risk in dynamical insurance models based on the Lundberg risk process has been studied in [20], [15], and [17]. In this research , we are interested in a modified version of this model which takes into account the risk associated with reinsurance.

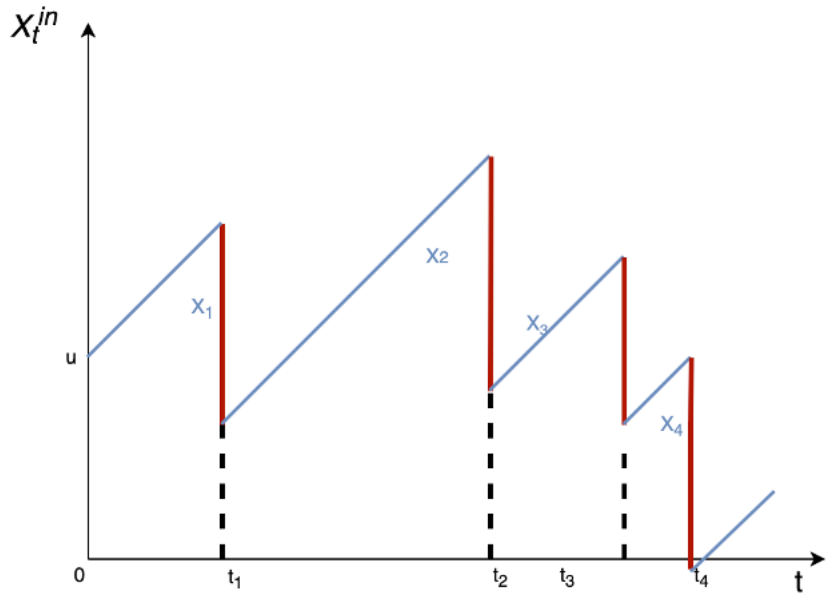


Figure 2.1. A typical sample path of the risk process.

2.0.2 The periodic Lundberg model with Reinsurance

Suppose that the interval of operation $[0, \infty)$ is partitioned into intervals of a given length T. $T > 0$ is constant that shows the duration of time intervals on which the insurer periodically chooses a strategy for reinsurance payoffs. At time $t = 0$, he chooses a reinsurance policy $I_0(Y_1)$ (amount paid by the reinsurance) and at the same time pays the reinsurer premium $P_{I_0}(Y_1)$, with $Y_i = \sum_{j=1}^{N_{t-}} X_j$. Let X_{T-}^{re} describe the risk process involving reinsurance. Then

On $[0, T)$ we obtain

$$X_{T-}^{re} = u + cT - P_{I_0}(Y_1) - \sum_{j=1}^{N_{t-}} X_j + I_0(Y_1)$$

At time T, after choosing the new reinsurance contract with paramters $(I_T(Y_2), P_{I_T}(Y_2))$ with $Y_2 = \sum_{j=N_{T+1}}^{N_{2T-}} X_j$, we obtain

$$X_T^{re} = u + cT - P_{I_0}(Y_1) - \sum_{j=1}^{N_{t-}} X_j + I_0(Y_1) - P_{I_T}(Y_2).$$

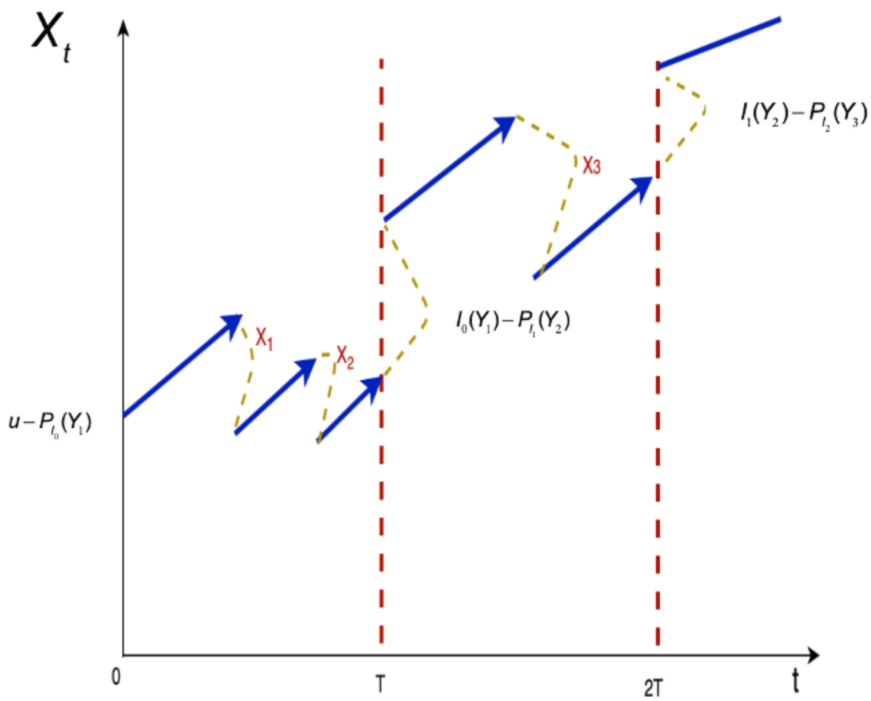


Figure 2.2. A typical sample path of the periodic risk process X_t.

Without loss of generality , let $X_t^{re} \equiv X_T$.

Notice that at the end of the first period we may consider the problem of profit maximization for the ending capital

$$X_T = (u + cT) + I_0(Y_1) - Y_1 - P_{I_0}(Y_1)$$

through some relevant utility functions.

On the other hand, by means of Distortion Risk Measures (DRM) to be defined later, we can consider the optimization problem of minimizing the risk exposure

$$R_T = Y_1 + P_{I_0}(Y_1) - (u + cT) - I_0(Y_1)$$

In this section we will focus on the insurance risk exposure.

First period: $[0, T)$, at time $t = 0$

$$(I_0(Y_1), P_{I_0}(Y_1)) \ , \ Y_1 = \sum_{j=1}^{N_{t-}} X_j$$

$$R_{T-} = Y_1 + P_{I_0}(Y_1) - (u + ct) - I_0(Y_1)$$

Second period: $[T, 2T)$; at T, $(I_T(Y_2), P_{I_t}(Y_2))$, $Y_2 = \sum_{j=N_{T+1}}^{N_{2T-}} X_j$

$$\begin{aligned} R_{T-} &= R_{T-} + P_{I_T}(Y_2) \\ &= Y_1 + P_{I_0}(Y_1) + P_{I_T}(Y_2) - (u + cT) - I_0(Y_1) \end{aligned}$$

At the end of the second period

$$\begin{aligned} R_{2T-} &= R_T + Y_2 - I_T(Y_2) - cT \\ &= Y_1 + P_{I_0}(Y_1) + P_{I_T}(Y_2) + Y_2 - I_0(Y_1) - I_T(Y_2) - (u + cT) \\ &= Y_1 + Y_2 + P_{I_0}(Y_1) + P_{I_T}(Y_2) - (I_0(Y_1) + I_T(Y_2) + u + 2cT) \end{aligned}$$

At $t = (k-1)T$, where $k = 1, 2, 3, ..,$ we consider the pair $(I_{(k-1)T}(Y_k), P_{I_{(k-1)T}}(Y_k))$ with total claim $Y_k = \sum_{j=N_{(k-1)T+1}}^{N_{kT-}} X_j$. Then the formula that defines the risk process with periodic reinsurance has the form

$$R_{nT} = \begin{cases} \sum_{k=1}^{n} Y_k + \sum_{k=1}^{n} P_{I_{(k-1)T}}(Y_k) - (\sum_{k=1}^{n} I_{(k-1)T}(Y_k) + (u + ncT)) \quad , \quad t = nT, \quad n \geq 0 \\ R_{n^t T} - c(t - n^t T) + \sum_{j=N_{n^t+1}}^{N_t} X_j \quad t \neq 0 \end{cases} \tag{2.2}$$

where $n^t = max\{n : nT < t\}$

As stated earlier, in what follows, the insurer risk exposure will be evaluated via risk measures.

2.0.3 DRM-based model

Let's first recall several definitions concerning risks measures.

Definition 2.0.1. We consider a financial position, as a non-negative random variable X representing the insurer loss. Then the Coherent Risk Measure is given by $\rho(X) : X \to [0, \infty)$ satisfies the following conditions:

1. Bounded above by the maximum loss: $\rho(X) \leq max(X)$
2. Bounded below by the mean loss: $\rho(X) \geq E(X)$
3. Scalar additive and multiplicative: $\rho(aX + b) = a\rho(X) + b$ for $a, b > 0$
4. Sub-additive : $\rho(X + Y) \leq \rho(X) + \rho(Y)$

We describe our model using the Distorted Risk Measure (DRM).

Definition 2.0.2. let $g : [0, 1] \to [0, 1]$ be a non-decreasing function with $g(0) = 0$ and $g(1) = 1$. For a non-negative random variable X, the Distorted Risk Measure (DRM) is defined by

$$\rho_g(X) := \int_0^\infty g(S_X(x))dx$$

where $S_X(x) = P(X > x)$ is the survival function of X.

DRMs satisfy (1)-(3) while (4) holds if and only if $g(x)$ is concave.

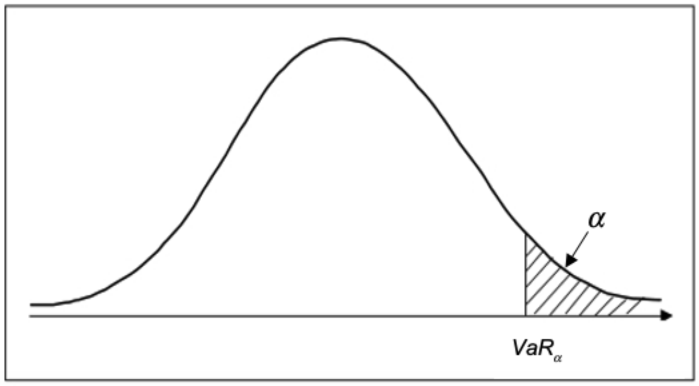


Figure 2.3. Value at Risk at the level α.

For $0 < \alpha < 1$

$$g(t) = \begin{cases} 1 & \text{if } 1-\alpha < t < 1 \\ 0 & \text{if } 0 < t < 1-\alpha \end{cases}$$

$$\begin{aligned} \rho_g(X) &:= \int_0^\infty g(S_X(x))dx \\ &:= inf\{x : P(X > x) \geq \alpha\} \\ &:= VaR_\alpha(X) \end{aligned}$$

is the classical Value at Risk, which measures the amount of asset needed to cover a possible loss at a confidence level $1 - \alpha$. In general, VaR_α is not a coherent risk measure, due to lack of sub-additivity . However, VaR_α is sub-additive (thus coherent risk measure) for a large class of elliptical distributions that include: Normal, Laplace, t-Student, Cauchy and Logistic distributions.

Example 2.0.1. Consider two loans and assume that if one lodefaults it is certain that the other loan does not default (i.e., defaults on A and B are dependent.)

D:defaulting ; D^c : non- defaulting

Table 2.1. Consider the loans separately

One year Loan ($10 millions)	**A**	**B**
P(D)	0.0125	0.0125
P(D^c)	0.9875	0.9875
Recovery $\|D$	U[0,10]	U[0,10]

Let Y_1 and Y_2 be the loss corresponding to loan A and B respectively follow $Y\ U[0, 10]$. Then

$$P(Y > 2|D) = 0.8$$

$$P(Y > \cap D) = P(Y > 2|D)P(D)$$
$$= (0.8)(0.0125) = 0.01 = \alpha$$

Therefore at the level α, the $VaR_{0.01}(Y) = VaR_{0.01}(Y_1) = VaR_{0.01}(Y_2) = 2$

Table 2.2. Consider the loans separately

Loan B — loan A	$P(D)$	$P(D^c)$
P(D)	0	0.0125
P(D^c)	0.0125	0.975

Table 2.3. Consider the above joint distribution

One year loan	**A**	**B**	$A \cup B$
P(D)	0.0125	0.0125	0.025
P(D^c)	0.9875	0.9875	0.975
Recovery $\|D$	U[0,10]	U[0,10]	U[0,10]

$$P(Y_1 + Y_2|D) = P(Y_1 + Y_2 > 6|D_A \cap D_B)$$

$$= \frac{P(Y_1+Y_2>6 \cap D_A)+P(Y_1+Y_2>6 \cap D_B)}{P(D_A)+P(D_B)}$$

$$= \frac{P(Y_1+Y_2>6|D_A)P(D_A)}{P(D_A)+P(D_B)} + \frac{P(Y_1+Y_2>6D_B)}{P(D_A)+P(D_B)}$$

$$= \frac{(0.4)(0.0125)}{2(0.0125)} + \frac{(0.4)(0.0125)}{2(0.0125)} = 0.4$$

$$P(Y_1 + Y_2 > 6 \cap D) = P(Y_1 + Y_2 > 6|D)P(D)$$

$$= (0.4)(0.025)$$

$$= 0.01 = \alpha$$

So at the level α, $VaR_\alpha(Y_1 + Y_2) = 6$ for the portfolio.

Note that $6 = VaR_\alpha(Y_1 + Y_2) > VaR_\alpha(Y_1) + VaR_\alpha(Y_2) = 4$, which shows that $VaR_\alpha(\cdot)$ is not sub-additive.

In our study, we assumed all random variables to be sufficiently integrable in the sense that their distortion risk measures are well-defined and finite.

As previously stated the insurer risk exposure on one period is given by

$$R_T^{in} = Y + P_I(Y) - (u + cT) - I(Y).$$

We now have to find a representation of the risk exposure and premiums function using DRM.

For any Distortion function g and ceding function I,

$$\rho_{g_i}(R_T^{in}) = \rho_{g_i}(Y + P_I(Y) - (u + cT) - I(Y))$$

$$= \rho_{g_i}(Y) - (u + cT) + \int_0^\infty [r(S_Y(y)) - g_i(S_Y(y))] I'(y) dy$$

and the formula for the premium is given by

$$P_I(Y) = \int_0^\infty r(S_Y(y)) I'(y) dy$$

where $r : [0, \infty] \to \mathbb{R}^+$ is a non decreasing function satisfying $r(0) = 0$ and I is restricted to the set of non-decreasing and Lipschitz functions, i.e.,

$$I = \{\text{I: } \mathbb{R}^+ \to \mathbb{R}^+ |\ \text{I}(0) = 0, 0 \leq I(x_1) - I(x_2) < x_2 - x_1 \text{ for } 0 \leq x_1 \leq x_2\}$$

In this context, we introduce an optimal reinsurance problem which consists of minimizing the insurer's total risk quantified by its distortion risk measure subject to the constraint

$$R_T^{re} = I(Y) - P_I(Y) = \int_0^\infty [g_r(S_Y(y)) - r(S_Y(y))] I'(y) dy \leq \pi$$

where π represents the reinsurer's risk tolerance and $g_I \leq g_r$.

We now have to solve the following problem

$$\begin{cases} \inf_I (\rho_{g_i}(R_T^{in})) \\ \rho_{g_r}(R_T^{re}) \leq \pi \end{cases} \Longrightarrow \begin{cases} \inf_I \{\rho_{g_i}(R_T^{in}) = \rho_{g_i}(Y + P_I(Y) - (u + cT) - I(Y))\} \\ \int_0^\infty [g_r(S_Y(y)) - r(S_Y(y))] I'(y) dy \leq \pi \end{cases}$$

$$\Longrightarrow \begin{cases} \inf_I \{\rho_{g_i}(Y) - (u + cT) + \int_0^\infty [r(S_Y(y)) - g_i(S_Y(y))] I'(y) dy \\ \int_0^\infty [g_r(S_Y(y)) - r(S_Y(y))] I'(y) dy \leq \pi \end{cases}$$

This could be simplified to

$$\Longrightarrow \begin{cases} \rho_{g_i}(Y) - (u + cT) + \inf_I \{ \int_0^\infty [r(S_Y(y)) - g_i(S_Y(y))] I'(y) dy \\ \int_0^\infty [g_r(S_Y(y)) - r(S_Y(y))] I'(y) dy \leq \pi \end{cases}$$

and reduces to the following minimization problem

$$\Longrightarrow \begin{cases} \inf_I \int_0^\infty f_1(y) I'(y) d(y) \\ \int_0^\infty f_0(y) I'(y) d(y) \leq \pi \end{cases}$$

where f_1 and f_0 are integrable functions defined on the non-negative real line such that

$$f_1(y) = r(S_Y(y)) - g_i(S_Y(y)) \text{ and } f_0(y) = g_r(S_Y(y)) - r(S_Y(y)$$

To solve this problem, we utilize the Neyman-Pearson approach introduced in [23].

Lemma 2.0.1. ***Neyman-Pearson Lemma***

Let X be a random variable with possible densities $f_0(x) = f(x|\theta_0), f_1(x) = f(x|\theta_1)$ with respect to some measure μ. Among the test function

$0 \le \varphi(x) \le 1$ *s.t.* $E_0\varphi(x) = \int \varphi(x) f_0(x) d\mu \le \alpha.$ *(type I error)*

$H_0 : f(x) = f_0(x) \qquad H_1 : f(x) = f_1(x)$

$\inf_{\varphi} E_1(1 - \varphi(x)) = \int (1 - \varphi(x)) f_1(x) d\mu$ *(type II error) is realized by*

$$\varphi(x) = \begin{cases} 1 & \text{if } f_1(x) > k f_0(x) \\ \gamma & \text{if } f_1(x) = k f_0(x) \\ 0 & \text{if } f_1(x) < k f_0(x) \end{cases} \tag{2.3}$$

for a unique constant k and $0 \le \gamma < 1$

Lemma 2.0.2. ***Generalized Neyman-Pearson Lemma***

Consider the following general minimization problem

$$g(t) = \begin{cases} \inf_{I \epsilon \mathrm{I}} \int_0^\infty f_1(y) dI(x) \\ \int_0^\infty f_0(y) dI(x) \le \pi, \pi \epsilon \mathbb{R} \end{cases}$$

where f_0 and f_1 are fixed integrable functions on $\mathbb{R}_+$.

Define a non-decreasing function $G : [-\infty, 0] \longrightarrow \mathbb{R}$ by $G(c) \triangleq \int_{\{f_1 < c f_0\}} f_0(x) dx$,

and $c \triangleq G^{-1}(\pi) = inf\{c \;\epsilon\; [-\infty, 0] | \int_{\{f_1 < c f_0\}} f_0(x) dx \ge \pi\}$

(i) $G(0) = \int\limits_{\{f_1<0\}} f(x)dx \leq \pi$, *then the optimal solution is*

$$I_*^{'}(x) = \begin{cases} 1 & \textit{if } f_1(x) < 0 \\ \gamma_* & \textit{if } f_1(x) = 0 \\ 0 & \textit{if } f_1(x) > 0 \end{cases} \tag{2.4}$$

where $\gamma_* : \mathbb{R}_+ \longrightarrow [0,1]$ *is any function s.t.*

$$\int_0^\infty f_0(x)d\mathrm{I}_* = G(0) + \int\limits_{\{f_1=0\}} f_0(x)\gamma_*(x)dx \leq \pi$$

(ii) if $G(-\infty) = \int\limits_{\{f_0<0\}} f(x)dx \leq \pi < G(0) = \int \{f_1 < 0\} f_0(x)dx$, *then the optimal solution must be in the form of*

$$I_*^{'}(x) = \begin{cases} 1 & \textit{if } f_1(x) < c^* f_0(x) \\ \gamma_* & \textit{if } f_1(x) = c^* f_0(x) \\ 0 & \textit{if } f_1(x) > c^* f_0(x) \end{cases} \tag{2.5}$$

where $\gamma_* : \mathbb{R}_+ \longrightarrow [0,1]$ *is any function s.t.*

$$\int_0^\infty f_0(x)d\mathrm{I}_* = G(c^*) + \int\limits_{\{f_1=c^*f_0\}} f_0(x)\gamma_*(x)dx = \pi$$

(iii) if $\pi < G(-\infty) = \int\limits_{\{f_0<0\}} f_0(x)dx$, *then the problem has no solution*

CHAPTER 3

LUNDBERG RISK MODEL WITHOUT INVESTMENT COMPONENT

In this section, we set up the model using VaR as our risk measure and utilize Lo's approach to solve the problem of minimizing the insurer's risk exposure with respect to the reinsurer's constraints. We take into account the insurer total liability Y, the premium received from the insurer's customers along with the premium paid to reinsurer. We also consider the indemnities paid to the insurer and assume no investment is made by the insurer.

3.1 Model set up

For fixed probability levels $0 < \alpha < 1$, $0 < \beta < 1$ chosen by the insurer and reinsurer respectively

$$\begin{cases} \inf_I VaR_\alpha\big(Y - (u - cT) + P_I(Y) - I(Y)\big) \\ VaR_\beta\big(I(Y) - P_I(Y)\big) \leq \pi \end{cases}$$

$$\iff \begin{cases} \inf_I \big(VaR_\alpha(Y) - (u - cT) + \int_0^\infty [r(S_Y(y)) - g_i(S_Y(y))]\mathrm{I}'(y)dy\big) \\ \int_0^\infty [g_r(S_Y)(y)) - r(S_Y(y))]\mathrm{I}'(y)dy \leq \pi \end{cases}$$

let $r(y) = (1 + \theta)y$; θ is the reinsurance loading factor

$$g_i(y) = \begin{cases} 1, & y > 1-\alpha \\ 0, & y < 1-\alpha \end{cases} \quad \text{and} \quad g_r(y) = \begin{cases} 1, & y > 1-\beta \\ 0, & y < 1-\beta \end{cases}$$

Then, $f_1(y) = (1 + \theta)S_Y(y) - 1_{\{S_Y(y) > 1-\alpha\}}$, and $f_0(y) = 1_{\{S_Y(y) > 1-\beta\}} - (1 + \theta)S_Y(y)$

Lemma 3.1.1. *Given n^{th} period $[(n-1)T, nT]$, let $\frac{\theta_n}{1+\theta_n} \leq \beta_n \leq \alpha_n$ where $\theta_n$is the reinsurance loading factor for the reinsurance premium $P_I(Y_1)$ and α_n, β_n are the corresponding risk levels for insurer and reinsurer $VaR_{(\alpha_n)}(\cdot)$ corresponding to claim $Y_n(\lambda_n) = \sum\limits_{j}^{N_{nT}} X_j^n$ with distribution $F_n(y)$ and X_j^n are i.i.d.*
The risk minimizing solution of

$$\begin{cases} \inf\limits_I \left\{ VaR_{\alpha_n}(Y_n + P_I(Y_n) - I(Y_n)) \right\} \\ VaR_{\beta_n}(I(Y_n) - P_I(Y_n)) \leq \pi_n \end{cases} \quad \textit{has the form}$$

$$V_n^*(Y_n, \alpha_n, \beta_n) = VaR_{\alpha_n}(Y_n) + \int\limits_0^\infty (1+\theta_n)(1-F_n(y))\mathrm{I}'(y)dy - \int\limits_0^\infty 1_{\{F_n(y)<\alpha_n\}}(y)\mathrm{I}'(y)dy$$

where $\mathrm{I}_n' = \gamma_n(y) \cdot 1_{[0,F^{-1}(\beta_n))\cup(F^{-1}(\alpha_n),\infty]}(Y)$, *for* $0 < \gamma_n = \gamma_n(\beta_n, \alpha_n, \theta_n, F_n) < 1$

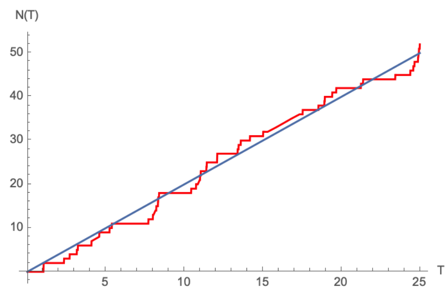
To better describe the rest of our result the following theorem is needed.

Theorem 3.1.2. ***Linderberg Central Limit theorem [4]***
Let (ξ_k) be independent with $m_k = E\xi_k$, $\sigma_k^2 = E(\xi_k - m_k)^2, s_n^2 = \sum\limits_{k=1}^n \sigma_k^2$.
Assume (L) $\lim\limits_{n\longrightarrow\infty} \sum\limits_{k=1}^n \frac{1}{s_n^2} \int_{\{|\xi_k - m_k| \geq \epsilon s_n\}} dp = 0$, $\forall \epsilon > 0$

Then (CLT) $\frac{\sum\limits_{k=1}^n \xi_k - \sum\limits_{k=1}^n m_k}{s_n} \quad \xrightarrow{D} \quad N(0,1), \quad n \longrightarrow \infty$

Proposition 3.1.3. $(\star)$*Let (ξ_k) be independent with $0 < a \leq \sigma_n^2 \leq b < \infty, n = 1, 2, ...$ for some constant a, b. Then (CLT) holds.*

Proof. it suffices to show (L) . Notice that by $(\star) s_n \longrightarrow \infty = \sqrt{\sum\limits_{k=1}^n \sigma_k^2} \geq \sqrt{na}$ and $\sigma_n < \sqrt{b}$ which implies (L) thanks to Feller $([14], p.264)$. □



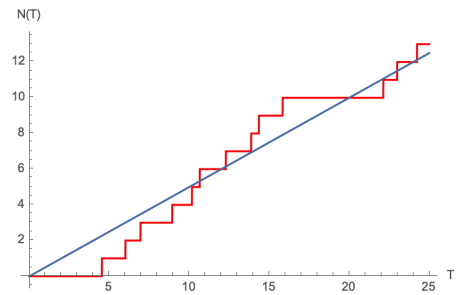


Figure 3.1. One sample path of a Poisson process with intensity 2 on the left, and intensity 0.5 on the right. The straight lines indicate the corresponding mean value functions. For $\lambda = 0.5$ jumps occur less often than for the standard homogeneous-Poisson process, whereas they occur more often when $\lambda = 2$..

3.2 Results

3.2.1 Homogeneous Compound Poisson process

The most common and best known claim arrival point process is the Homogeneous Poisson process (HPP) with stationary and independent increments and the number of claims in a given time interval governed by the Poisson law.

Definition 3.2.1. Poisson process

Let $\lambda > 0$ be fixed. The counting process $\{N(t), t\epsilon[0, \infty)\}$ is called a Poisson process with rates λ if the following conditions hold:

(1) $N(0) = 0$;

(2) $N(t)$ has independent stationary increments;

(3) The number of arrivals in any interval of length $\tau > 0$ has Poisson $(\lambda\tau)$ distribution.

If $N(t)$ is a Poisson process with rate λ, then the inter-arrival times $X_1, X_2, \cdots$ are independent and $X_i \sim$ Exponential(λt), for $i = 1, 2, 3, \cdots$.

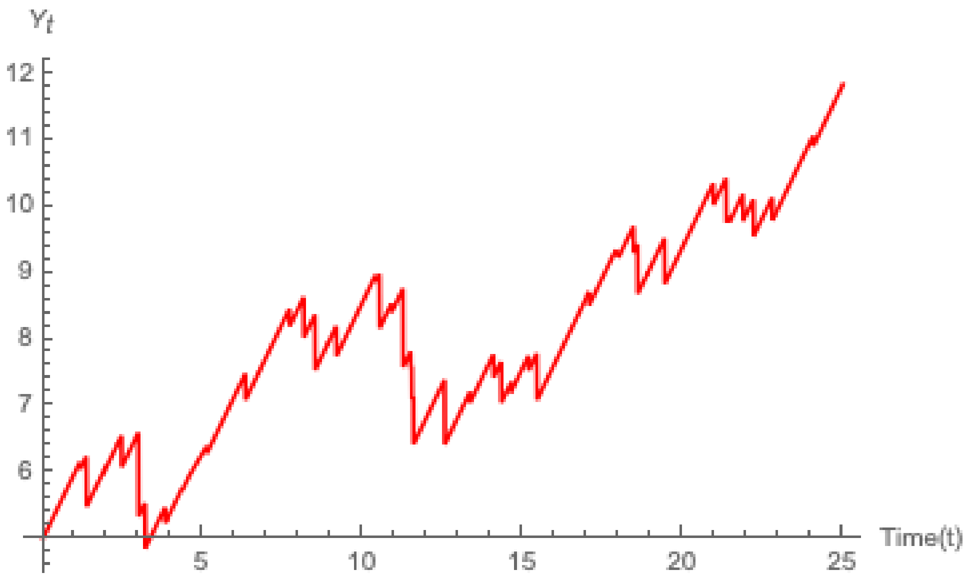


Figure 3.2. A sample path of a compound Poisson process.

Definition 3.2.2. Compound Homogeneous Poisson process

A stochastic process $\{N(t), t \geq 0\}$ is said to be a Compound Poisson process if it can be represented as $Y(t) = \sum_{i=1}^{N(t)} X_i, t \geq 0$ where $\{N(t) : t \geq 0\}$ is a counting of a Poisson process with rate λ , and $\{X_i : i \geq 1\}$ are i.i.d. that is also independent of $\{N(t), t \geq 0\}$. The expected value of a Compound Poisson process can be calculated by using Wald's identity

$$E(Y(t)) = E(X_1 + \cdots + X_{N(t)}) = E(N(t))E(X_1) \equiv E(N(t))E(X) = \lambda t E(X)$$

Making similar use of the law of total variance, the variance can be calculated as

$$Var(Y(t)) = E(Var(Y(t)|N(t))) + Var(E(Y(t)|N(t))) = \lambda t E(X^2)$$

Now based on lemma 3.1.1 the following results holds when $Y(t)$ follows a homogenous compound Poisson process.

Theorem 3.2.1. *Set $V_k^*(Y_k, \alpha_k, \beta_k) \equiv V_k^*$; for $k = 1, \cdots, n$. Then $VaR_{\alpha_1}(R_T) = V_1^* - (u + cT)$ and given then n^{th} period $[(n-1)T, nT]$, the optimal solution has the following cumulative Value at Risk at time $(n-1)T$*

$$\begin{aligned} VaR_{\alpha_n}^*(R_nT) &= \sum_{k=1}^{n-1} (Y_k - I(Y_k) + P_I(Y_k)) + V_n^* - (u + cnT) \\ &= \sum_{k=1}^{n} (Y_k - I(Y_k) + P_I(Y_k)) + VaR_{\alpha_n}(Y_n) - (Y_n) - (u + cnT) \end{aligned}$$

Furthermore

$$E\Big(VaR_{\alpha_n}^*(R_nT)\Big) = \sum_{k=1}^{n-1} E\Big(Y_k - I(Y_k) + P_I(Y_k)\Big) + V_n^* - (u + cnT)$$

and

$$P\left[VaR_{\alpha_n}^*(R_nT) > \lambda T \sum_{k=1}^{n-1} E(X^k) + \sum_{k=1}^{N_T} \Big(P_I(Y_k) - I(Y_k)\Big) \right. \\ \left. + V_n^* - (u + cnT) + \sqrt{\lambda T \sum_{k=1}^{n-1} E(X^k)^2} \cdot z_{\alpha_n}\right] \simeq \alpha_n$$

where $1 - \Phi(z_{\alpha_n}) = \alpha_n$ and $\Phi(\cdot)$ is the cdf of $N(0, 1)$

Corollary 3.2.2. *Let's consider n cycles of L periods with $(\alpha_i, \beta_i$ where $i = 1, \cdots, L$. As $n \longrightarrow \infty$, by the law of large numbers we have*

- *The expected Value at Risk per cycle*

$$\frac{E\Big(VaR_{\alpha_n}^*(R_{nLT})\Big)}{nL} \longrightarrow \sum_{i=1}^{L} \frac{E(Y_i)}{L} - \sum_{i=1}^{L} \frac{I(Y_i)}{L} - \sum_{i=1}^{L} \frac{P_I(Y_i)}{L} - cT$$

- *The expected value of the portfolio per cycle*

$$\frac{E\Big(VaR^*_{\alpha_n}(R_{nLT}\Big)}{nL} \longrightarrow \sum_{i=1}^{L}\frac{E(Y_i)}{L} - \sum_{i=1}^{L}\frac{I(Y_i)}{L} - \sum_{i=1}^{L}\frac{P_I(Y_i)}{L} - cT$$

$$\frac{E(X^*_{nLT})}{nL} \longrightarrow -\Big\{\sum_{i=1}^{L}\frac{E(Y_i)}{L} - \sum_{i=1}^{L}\frac{I(Y_i)}{L} - \sum_{i=1}^{L}\frac{P_I(Y_i)}{L} - cT\Big\}$$

$$= \sum_{i=1}^{L}\frac{I(Y_i)}{L} + cT - \sum_{i=1}^{L}\frac{E(Y_i)}{L} - \sum_{i=1}^{L}\frac{P_I(Y_i)}{L}$$

where $\sum_{i=1}^{L}\frac{I(Y_i)}{L} + cT$ *represents the Expected Income and* $\sum_{i=1}^{L}\frac{E(Y_i)}{L} + \sum_{i=1}^{L}\frac{P_I(Y_i)}{L}$ *represents the Expected Loss*

3.2.2 Non-homogeneous Compound Poisson process

Definition 3.2.3. Non-homogeneous Poisson process

Let $\lambda(t) : [0,\infty) \longrightarrow [0,\infty)$ be an integrable function. The counting process $\{N(t), t\epsilon[0,\infty)\}$ is called a non-homogeneous Poisson process with rate $\lambda(t)$ if all the following conditions hold

(1) $N(0) = 0$

(2) $N(t)$ has independent increments

(3) $\forall\ t\ \epsilon\ [0,\infty)$, we have

$P(N(t+\Delta) - N(t) = 0) = 1 - \lambda(t)\Delta + o(\Delta)$

$P(N(t+\Delta) - N(t) = 1) = \lambda(t)\Delta + o(\Delta)$

$P(N(t+\Delta) - N(t) \geq 2) = o(\Delta)$

where $o(\Delta)$ is the little-o-notation for $o(\Delta)/\Delta$ as $\Delta \to 0$.

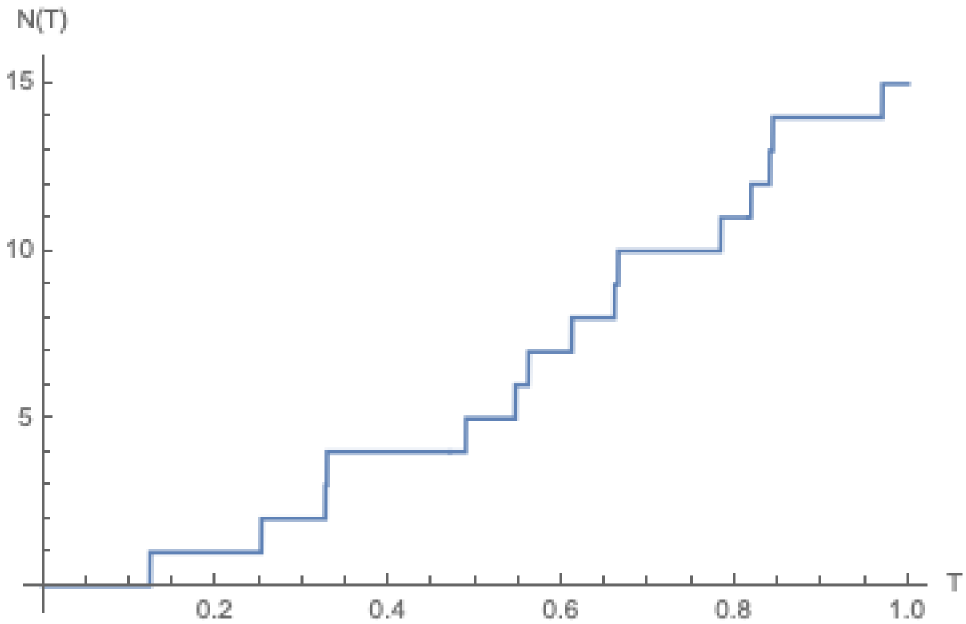


Figure 3.3. A typical sample path of a non-homogeneous Poisson process.

From those properties $N(t+\Delta) - N(t)$ is a Poisson random variable with mean

$$E[N(t+\Delta) - N(t)] = \int_t^{t+\Delta} \lambda(s)d(s),$$

which implies

$$E[N(\Delta)] = \int_0^{\Delta} \lambda(s)ds = Var[N(\Delta)]$$

Definition 3.2.4. Compound non-homogeneous Poisson process

Let's consider the compound non homogeneous Poisson process represented by: $Y(t) = \sum_{i=1}^{N(t)} X_i, t \geq 0$. Then the expected value and the variance of the compound non homogeneous Poisson process could be expressed respectively as followed :

$$E(Y(t)) = E(N)E(X) \text{ and } Var(Y(t)) = Var(N)E[(X)]^2$$

Based on lemma 3.1.1 the following results holds when $Y(t)$ is a non-homogenous compound Poisson process.

Theorem 3.2.3. *Set* $V_k^*(Y_k, \alpha_k, \beta_k) \equiv V_k^*$*; for* $k = 1, \cdots, n$*. Then* $VaR_{\alpha_1}(R_T) = V_1^* - (u + cT)$ *and given then* n^{th} *period* $[(n-1)T, nT]$*, the optimal solution has the following cumulative Value at Risk at time* $(n-1)T$

$$
\begin{aligned}
VaR^*_{\alpha_n}(R_nT) &= \sum_{k=1}^{n-1}(Y_k(\lambda_k) - I(Y_k(\lambda_k)) + P_I(Y_k(\lambda_k))) + V_n^* - (u + cnT) \\
&= \sum_{k=1}^{n}(Y_k(\lambda_k) - I(Y_k(\lambda_k)) + P_I(Y_k(\lambda_k))) + VaR_{\alpha_n}(Y_n(\lambda_k)) - (Y_n) - (u + cnT)
\end{aligned}
$$

Furthermore

$$
E\Big(VaR^*_{\alpha_n}(R_nT)\Big) = \sum_{k=1}^{n-1} E(Y_k(\lambda_k) - I(Y_k(\lambda_k)) + P_I(Y_k(\lambda_k)) + V_n^* - (u + cnT)
$$

and

$$
\begin{aligned}
P\Bigg(VaR^*_{\alpha_n}(R_nT) > \sum_{k=1}^{n-1}(\lambda_k T)E(X^k) + \sum_{k=1}^{N_T}\big(P_I(Y_k(\lambda_k)) - I(Y_k(\lambda_k))\big) \\
+ V_n^* - (u + cnT) + \sqrt{\sum_{k=1}^{n-1}(\lambda_k T)E(X^k)^2} \cdot z_{\alpha_n}\Bigg) \simeq \alpha_n
\end{aligned}
$$

where $1 - \Phi(z_{\alpha_n}) = \alpha_n$ and $\Phi(\cdot)$ is the cdf of $N(0,1)$.

Corollary 3.2.4. *Let's consider* n *cycles of* L *periods with* $(\alpha_i, \beta_i$ *where* $i = 1, \cdots, L$*. As* $n \longrightarrow \infty$*, by the law of large numbers we have*

- *The expected Value at Risk per cycle*

$$
\frac{E\Big(VaR^*_{\alpha_n}(R_{nLT})\Big)}{nL} \longrightarrow \sum_{i=1}^{L}\frac{\lambda_i T E(X^i)}{L} - \sum_{i=1}^{L}\frac{I(Y_i(\lambda_i))}{L} - \sum_{i=1}^{L}\frac{P_I(Y_i(\lambda_i))}{L} - cT
$$

- *The expected value of the portfolio per cycle*

$$\frac{E(X^*_{nLT})}{nL} \longrightarrow -\Big\{ \sum_{i=1}^{L} \frac{\lambda_i T E(X^i)}{L} - \sum_{i=1}^{L} \frac{I(Y_i(\lambda_i))}{L} - \sum_{i=1}^{L} \frac{P_I(Y_i(\lambda_i))}{L} - cT \Big\}$$

$$= \sum_{i=1}^{L} \frac{I(Y_i(\lambda_i))}{L} + cT - \sum_{i=1}^{L} \frac{\lambda_i T E(X^i)}{L} - \sum_{i=1}^{L} \frac{P_I(Y_i(\lambda_i))}{L}$$

where $\sum_{i=1}^{L} \frac{I(Y_i(\lambda_i))}{L} + cT$ *represents the Expected Income and* $\sum_{i=1}^{L} \frac{\lambda_i T E(X^i)}{L} + \sum_{i=1}^{L} \frac{P_I(Y_i(\lambda_i))}{L}$ *represents the Expected Loss.*

CHAPTER 4

LUNDBERG RISK MODEL WITH INVESTMENT COMPONENT

We focus on finding VaR for an insurer who considers investing a portion of the premium received into stocks.

Consider a scenario where an investor has an option to invest in only two different types of assets. Specifically, split the investment between risky assets such as stocks and risk-free assets such as bonds.

I-Risky Asset

A frequently used model for the dynamics of risky asset prices is the geometric Brownian motion. If S_t denotes the price of a risky asset at the time t, then S_t will follow a geometric Brownian motion if it satisfies the following Stochastic Differential Equation (SDE)

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{4.1}$$

where μ is the drift and $\sigma > 0$ is the volatility of the risky asset, with both μ and σ assumed constant. B_t is the stochastic process known as Brownian motion defined below.

Definition 4.0.1. Brownian motion B_t is a stochastic process starting at zero, i.e., $B_0 = 0$, which satisfies the following three conditions:

1. Independent increments: The random variable $B_t - B_s$ is independent of the random variable $B_u - B_v$ whenever $t > s \geq u > v \geq 0$
2. Stationary increments: The distribution of $B_{t+s} - B_s$ is independent of s
3. Normal increments: The distribution of $B_t - B_s \sim N(0, t - s)$, $t > s \geq 0$

Using the Ito formula of stochastic calculus, the explicit solution of the SDE for the geometric Brownian motion reads

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}, \qquad t \geq 0 \tag{4.2}$$

II-Risk free asset

The price of the risk-free asset at time t is denoted by M_t and satisfies the following deterministic differential equation

$$dM_t = rM_t dt \tag{4.3}$$

The parameter " r " represents the risk-free interest rate. Given $S_0 = 1$, we assume that $ES_t = e^{\mu t} > M_t = e^{rt}$ which is equivalent to $\mu > r$, and implies that the average return on stock exceeds the return on bond.

III- Combined portfolio (risky and risk-free asset)

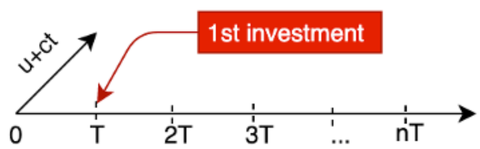
Let $A(t)$ be the amount of money available to an investor at time $t = nT$. After collection of the premium cT at the beginning of each cycle $[(n-1)T, nT)$, a portion w of the premium is invested in stock and the remaining portion $1 - w$ is accumulated without earning any interest, with $0 \leq w \leq 1$.

Our objective is to analyze the **effect** of investing a portion of the collected premium in equities (stocks) on the **insurer's risk exposure** over the time horizon $[0, nT]$.

Recall form chapter 3 the optimal solution **without investment** has the following cumulative Value at Risk at time $(n-1)T$

$$\begin{aligned} VaR_{\alpha_n}(R_{nT}) &= \sum_{k=1}^{n-1}(Y_k - I(Y_k) + P_I(Y_k)) + V_n^* - (u + cnt) \\ &= \sum_{k=1}^{n} Y_k - I(Y_k) + P_I(Y_k)) + VaR_{\alpha_n}(Y_n) - Y_n - (u + cnt) \end{aligned} \tag{4.4}$$

We begin investing upon collecting the premium cT by time T



Lemma 4.0.1. ***Difference of normal and lognormal.***

Let X, Y be independent normal and lognormal random variables respectively , and let $C > 0$ and $0 \leq w \leq 1$ be any given constant. Then,

(i) $\forall\ 0 < \alpha < 1$, $\exists$ *a unique* A *such that* $P(X - (1-w)C - CwY \geq A) = \alpha$

(ii) If $X \sim N(m_1, \sigma_1^2)$ *and* $Y \sim LN(m_2, \sigma_2^2)$ *then* A *satisfies the following equation*

$$\int_{-\infty}^{\infty}(1 - \Phi(\frac{A + C(1-w) + Cwe^{m_2+\sigma_2 z} - m_1}{\sigma_1}))\varphi(z)dz = \alpha \tag{4.5}$$

where $\Phi(z)$ *and* $\varphi(z)$ *are cumulative distribution and density of* $Z \sim N(0,1)$

(iii) A *is readily obtained via numerical integration in (4.5) by trial and error based on the fact that* $1 - \Phi(\frac{A+C(1-w)+Cwe^{m_2+\sigma_2 z}-m_1}{\sigma_1})$ *is continuous and decreasing in* A.

Remark. *When* $C = m_1$, $w = 0$ *then* $\frac{A}{\sigma_1} = Z_\alpha$ *or* $A = \sigma_1 Z_\alpha$.

Proof. To show (i) , notice that $U = X - CwY = X + V$ has density which is the convolution of $f_x(x)$ and $f_v(v) = f_y(\frac{-v}{Cw})(\frac{1}{Cw})$,$f_u(u) = f_x(x) + f_v(v)$.

Therefore

$$\begin{aligned} P(X - CwY \geq A + C(1-w)) &= P(U \geq A + C(1-w)) \\ &= \int_{A+C(-w)}^{\infty} f_u(u)du \\ &= 1 - F_u(A + C(1-w)) \\ &= \alpha \end{aligned} \tag{4.6}$$

with unique

$A = F_u^{-1}(1-\alpha) - C(1-w)$, because $F_u(\cdot)$ is strictly increasing and continuous.

To show (ii) notice that for $Z \sim N(0,1)$

$$\begin{aligned} P(X - C(1-w) - Cw \geq A) &= P(X - C(1-w) - Cwe^{m_2+\sigma_2 z} \geq A) \\ &= \int_{-\infty}^{\infty} P(X - C(1-w) - Cwe^{m_2+\sigma_2 z} \geq A \mid Z = z)\varphi(z)dz \\ &= \int_{-\infty}^{\infty} P(\frac{X - m_1}{\sigma_1} \geq \frac{A + C(1-w) + Cwe^{m_2+\sigma_2 z} - m_1}{\sigma_1})\varphi(z)dz \\ &= \int_{-\infty}^{\infty} [1 - \Phi(\frac{A + C(1-w) + Cwe^{m_2+\sigma_2 z} - m_1}{\sigma_1})]\varphi(z)dz = \alpha \end{aligned} \tag{4.7}$$

□

Lemma 4.0.2. *Let* (X_n) *be independent of* (Y_n) *where* X_n, Y_n, X, Y *are continuous random variables such that*

$F_n(x) = P(X_n \leq x) \to P(X \leq x) = F(x)$ *uniformly in* x

and

$$G_n(y) = P(Y_n \leq y) \to P(Y \leq y) = G(y) \text{ uniformly in } y$$

$$\forall \quad 0 < \alpha < 1, a \in \mathbb{R} \ \alpha = P(X_n + Y_n \geq a_n) = P(X + Y \geq a), n = 1, 2, \cdots$$

Then lim $a_n = a$

Proof. For every n

$$F_n * G_n(a_n) = P(X_n + Y_n \leq a) = F * G(a)$$

$$\text{or } \int F_n(a_n - y) dG_n(y) = \int F(a - y) dG(y)$$

Now

$$(\triangle) \quad 0 = (F_n * G_n(a_n)) - (F_n * G(a_n)) + (F_n * G(a_n) - F * G(a_n)) + (F * G(a_n) - F * G(a))$$

$$\begin{aligned} |(F_n * G_n(a_n)) - (F_n * G(a_n))| &= |G_n * F_n(a_n) - G * F_n(a_n)| \\ &= |\int (G_n(a_n - x) - G(a_n - x)) dF_n(x)| \\ &\leq \sup_x |G_n(x) - G(x)| \to 0 \end{aligned}$$

and similarly

$$\begin{aligned} |(F_n * G_n(a_n)) - (F_n * G(a_n))| &= |\int (F_n(a_n - x) - F(a_n - y)) dG(y)| \\ &\leq \sup_y |F_n(y) - F(y)| \to 0 \end{aligned}$$

Hence by $(\triangle)$ $\lim_n F * G(a_n) = F * G(a)$, since the convolution of continuous F and G is continuous and strictly increasing, it follows that $a_n \to a$. □

Theorem 4.0.3. *Set $V_k^*(Y_k, \alpha_k, \beta_k) \equiv V_k^*$ for $k = 1, ..., n$. Assume we invest in different stocks at each time iT, $i = 1, .., n-1$. Then, the formula (3.4)* ***with investment*** *becomes*

$$\begin{aligned} VaR^*_{\alpha_n}(R_{nT}) = \sum_{k=1}^{n-1}(Y_k - I(Y_k) + P_I(Y_k)) + V_n^* - [u + (n-1)(1-w)cT \\ + wcT \sum_{k=1}^{n-1} e^{(\mu_i - \frac{\sigma_i^2}{2})(n-i)T + \sigma_i(B_{nT}^{(i)} - B_{iT}^{(i)})}] \end{aligned} \tag{4.8}$$

where $\sigma_i(B_{nT}^{(i)} - B_{iT}^{(i)}) \sim N(0, \sigma_i^2)(n-1)T)$ with the Brownian motion $B^i(t)$ corresponding to the i^{th} stock and B^i are independent.

Proof. Assume we start investing on the second period given that we have collected cT on [0,T). Then we have the following recursive formulas

on $[T, 2T)$

$$\begin{aligned} VaR^*_{\alpha_2}(R_{2T}) = \sum_{k=1}^{2-1}(Y_k - I(Y_k) + P_I(Y_k)) + V_n^* \\ -[u + (1-w)cT + wcTe^{(\mu_1 - \frac{\sigma_1^2}{2})T + \sigma_1(B_{2T}^{(1)} - B_T^{(1)})}] \end{aligned}$$

on $[2T, 3T)$

$$\begin{aligned} VaR^*_{\alpha_3}(R_{3T}) = \sum_{k=1}^{3-1}(Y_k - I(Y_k) + P_I(Y_k)) + V_n^* \\ -[u + (1-w)cT + wcTe^{(\mu_1 - \frac{\sigma_1^2}{2})T + \sigma_1(B_{3T}^{(1)} - B_{2T}^{(1)})} \\ +(1-w)cT + wcTe^{(\mu_2 - \frac{\sigma_2^2}{2})T + \sigma_2(B_{3T}^{(2)} - B_{2T}^{(2)})}] \end{aligned}$$

By continuing, we arrive at the following formula

on $[(n-1)T, nT)$

$$\begin{aligned} VaR^*_{\alpha_n}(R_{nT}) = \sum_{k=1}^{n-1}(Y_k - I(Y_k) + P_I(Y_k)) + V_n^* \\ -[u + (n-1)(1-w)cT + wcT \sum_{k=1}^{n-1} e^{(\mu_i - \frac{\sigma_i^2}{2})(n-i)T + \sigma_i(B_{nT}^{(i)} - B_{iT}^{(i)})}] \end{aligned}$$

□

Theorem 4.0.4. *Let* $W = \sum_{i=1}^{n-1} e^{(\mu_i - \frac{\sigma_i^2}{2})(n-i)T + \sigma_i(B_{nT}^{(i)} - B_{iT}^{(i)})}$ *be the sum of independent lognormal. Then the central limit theorem can be used to find the insurer risk exposure after the following modifications are made.*

Fenton-Wilkinson approximation (F-W)

Let $W_i \sim LN(a_i, b_i)$ *be independent log-normally distributed, and* $W = \sum_{i=1}^{n-1} W_i$*. The distribution of* W *has no closed-form expression, but can be reasonably approximated by another log-normal distribution* $\hat{W}$ *. By Fenton-Wilkinson approximation, the parameters of* $\hat{W}$ *is obtained by matching the mean and variance of another lognormal distribution.*

Case1 *: Assume* $b = b_i, a = a_i = \mu_i - \frac{1}{2}\sigma_i^2$*, i.e. all stocks have the same drift (average rate of return) and volatility . We obtain the following mean and variance*

$$\begin{aligned} Var(\hat{W}) = \hat{b} = ln[\frac{e^{b^2} - 1}{n-1} + 1] \\ E(\hat{W}) = \hat{a} = ln[(n-1)e^a] + \frac{b^2}{2} - \frac{Var(\hat{W})}{2} \end{aligned} \tag{4.9}$$

Using (4.8) and (4.9) we obtain

$$\begin{aligned} E(VaR^*_{\alpha_n}(R_{nT})) &= \sum_{k=1}^{n-1} E(Y(\lambda_k) - I(Y(\lambda_k)) + P_I(Y(\lambda_k))) + V_n^* \\ &\quad -[u + (n-1)(1-w)cT + wcTE(\hat{W})] \\ &= \sum_{k=1}^{n-1} E(Y(\lambda_k) - I(Y(\lambda_k)) + P_I(Y(\lambda_k))) + V_n^* \\ &\quad -[u + (n-1)(1-w)cT + wcT(ln[(n-1)e^a] + \tfrac{b^2}{2} - \tfrac{Var(\hat{W})}{2})] \end{aligned}$$

and

$$Var(VaR^*_{\alpha_n}(R_{nT})) = \sum_{k=1}^{n-1} \lambda_k T E(X^{(k)})^2 + (wcT)^2 Var(\hat{W})$$

$$= \sum_{k=1}^{n-1} \lambda_k T E(X^{(k)})^2 + (wcT)^2 (ln[\tfrac{e^{b^2}-1}{n-1} + 1])$$

Case2: *Assume* $b = b_j, a_j \neq a_i, i, j = 1, ..., n-1$ *, i.e. all stocks have different drift (average rate of return) but same volatility. Then from Case 1, we can also find the corresponding mean and variance .*

$$Var(\hat{W}) = \hat{b} = ln[(e^{b^2} - 1)\frac{\sum_{j=1}^{n-1} e^{2a_j}}{(\sum_{j=1}^{n-1} e^{a_j})^2} + 1];$$

$$E(\hat{W}) = \hat{a} = ln[\sum_{j=1}^{n-1} e^{a_j}] + \tfrac{b^2}{2} - \tfrac{Var(\hat{W})}{2}$$

Using (4.8) we obtain

$$E(VaR^*_{\alpha_n}(R_{nT})) = \sum_{k=1}^{n-1} E(Y(\lambda_k) - I(Y(\lambda_k)) + P_I(Y(\lambda_k))) + V^*_n$$
$$-[u + (n-1)(1-w)cT + wcTE(\hat{W})]$$

$$= \sum_{k=1}^{n-1} E(Y(\lambda_k) - I(Y(\lambda_k)) + P_I(Y(\lambda_k))) + V^*_n$$
$$-[u + (n-1)(1-w)cT + wcT(ln[\sum_{j=1}^{n-1} e^{a_j}] + \tfrac{b^2}{2} - \tfrac{Var(\hat{W})}{2})]$$

and

$$Var(VaR^*_{\alpha_n}(R_{nT})) = \sum_{k=1}^{n-1} \lambda_k T E(X^{(k)})^2 + (wcT)^2 Var(\hat{W})$$

$$= \sum_{k=1}^{n-1} \lambda_k T E(X^{(k)})^2 + (wcT)^2 (ln[(e^{b^2}-1)\frac{\sum_{j=1}^{n-1} e^{2a_j}}{(\sum_{j=1}^{n-1} e^{a_j})^2} + 1])$$

Case3: *Assume* $a_j \neq a_i$,$b_j \neq b_i$, $i, j = 1, ..., n-1$, *i.e. all stocks have different drift (average rate of return) and volatility. Then by F-W approximation. Then from Case 1, we can also find the corresponding mean and variance .*

$$Var(\hat{W}) = \hat{b} = ln[\frac{\sum_{j=1}^{n-1} e^{2a_j+b_j^2}(e^{b^2}-1)}{(\sum_{j=1}^{n-1} e^{a_j+\frac{b_j^2}{2}})^2} + 1];$$

$$E(\hat{W}) = \hat{a} = ln[\sum_{j=1}^{n-1} e^{a_j+\frac{b_j^2}{2}}] - \frac{Var(\hat{W})}{2}$$

Using(3.8) we also get

$$E(VaR^*_{\alpha_n}(R_{nT})) = \sum_{k=1}^{n-1} E(Y(\lambda_k) - I(Y(\lambda_k)) + P_I(Y(\lambda_k))) + V^*_n$$
$$-[u + (n-1)(1-w)cT + wcTE(\hat{W})]$$

$$= \sum_{k=1}^{n-1} E(Y(\lambda_k) - I(Y(\lambda_k)) + P_I(Y(\lambda_k))) + V^*_n$$
$$-[u + (n-1)(1-w)cT + wcT(ln[\frac{\sum_{j=1}^{n-1} e^{2a_j+b_j^2}(e^{b^2}-1)}{(\sum_{j=1}^{n-1} e^{a_j+\frac{b_j^2}{2}})^2} + 1]]$$

and

$$Var(VaR^*_{\alpha_n}(R_{nT})) = \sum_{k=1}^{n-1} \lambda_k T E(X^{(k)})^2 + (wcT)^2 Var(\hat{W})$$

$$= \sum_{k=1}^{n-1} \lambda_k T E(X^{(k)})^2 + (wcT)^2 (ln[\frac{\sum_{j=1}^{n-1} e^{2a_j + b_j^2}(e^{b^2}-1)}{(\sum_{j=1}^{n-1} e^{a_j + \frac{b_j^2}{2}})^2} + 1])$$

From Linderberg central limit theorem, after the proper substitutions are made, we can find the value at risk for the insurer using the following

$$P\Bigg[VaR^*_{\alpha_n}(R_{nT}) > \sum_{k=1}^{n-1} \lambda_k T E(X^{(k)}) + \sum_{k=1}^{N_T} P_I(Y(\lambda_k)) - I(Y(\lambda_k)))$$
$$+ V_n^* - [u + (n-1)(1-w)cT + wcTE(\hat{W})] \tag{4.10}$$
$$+ z_{\alpha_n} \sqrt{\sum_{k=1}^{n-1} \lambda_k T E(X^{(k)})^2 + (wcT)^2 Var(\hat{W})} \Bigg] = \alpha_n$$

where $1 - \Phi(z_{\alpha_n}) = \alpha_n$ *and* $\Phi(\cdot)$*is the cdf of* $N(0,1)$.

Theorem 4.0.5. *Set* $V_k^*(Y_k, \alpha_k, \beta_k) \equiv V_k^*$ *for* $k = 1, ..., n$. *Assume we invest in different stocks at each time* iT,$i = 1, .., n-1$.
Consider equation (4.8) such that $\sum_{k=1}^{n-1} e^{(\mu_i - \frac{\sigma_i^2}{2})(n-i)T + \sigma_i(B_{nT}^{(i)} - B_{iT}^{(i)})}$ *is approximated by a single lognormal* $e^{\mu + \sigma z}$ *using F-W. As mentioned in chapter 3*

$$Y_k(t) = \sum_{j=N_{(k-1)T}+1}^{N_{kt}} X_j^{(k)}.$$

Denote by $m_1^{(k)} = E(X_j^k)$; $m_2^{(k)} = E(X_j^k)^2$

Then

$$E\sum_{k=1}^{n-1} Y_k(t) = \sum_{k=1}^{n-1} \lambda_k T E(X^{(k)}) = \sum_{k=1}^{n-1} \lambda_k T m_1^{(k)} = \sum_{k=1}^{n-1} m_k$$

and

$$Var\sum_{k=1}^{n-1} Y_k(t) = \sum_{k=1}^{n-1} \lambda_k T E(X^{(k)})^2 = \sum_{k=1}^{n-1} \lambda_k T m_2^{(k)}$$

which can be used along with Lemma 4.0.1 to find the insurer's value at risk.

$$\begin{aligned}
&\int_{-\infty}^{\infty} P\left(\frac{\sum_{k=1}^{n-1} Y_k - \sum_{k=1}^{n-1} m_k}{\sqrt{\sum_{k=1}{}^{n-1}\lambda_k T m_2^{(k)}}} \geq \frac{A + C(1-w) + Cwe^{\mu+\sigma z} - \sum_{k=1}^{n-1} m_k}{\sqrt{\sum_{k=1}^{n-1} \lambda_k T m_2^{(k)}}}\right)\varphi(z)dz \\
&= \int_{-\infty}^{\infty} \left[1 - \Phi\left(\frac{A + C(1-w) + Cwe^{\mu+\sigma z} - \sum_{k=1}^{n-1} m_k}{\sqrt{\sum_{k=1}^{n-1} \lambda_k T m_2^{(k)}}}\right)\right]\varphi(z)dz = \alpha
\end{aligned} \tag{4.11}$$

Example 4.0.1. Find the VaR_α

Case1: Without investment

Consider 25 independent and identically distributed random variables $X_i \sim N(0,1)$

$$P\left(\tfrac{X_1+X_2+\cdots+X_{25}-25(0)}{\sqrt{25}\cdot(1)} > z_\alpha\right) = 0.05$$

$$P\left(X_1 + X_2 + \cdots + X_{25} - 25(0) > 5z_\alpha\right) = 0.05$$

$$P\left(X_1 + X_2 + \cdots + X_{25} - 25(0) > 8.25\right) = 0.05$$

Then VaR_α without investment is 8.25.

We will now consider three cases involving investment while assuming that all stocks have the same drift (average rate of return) and volatility.
Using formula (4.10), let's find the value at risk at the level $\alpha = 0.05$. Note that A is the corresponding VaR_α to be found.

Case2: With investment and fixed average rate of return

Table 4.1. with fixed $\mu = 0.1$

Average rate of return μ	**Volatility** σ^2	**Value at Risk** VaR_α
0.1	0.5	7.92
0.1	0.2	7.8
0.1	1	8.27
0.1	0.986	8.25

Comments: As mentioned earlier, at the level $\alpha = 0.05$, the value at risk is 8.25 without investment . Now with investment, for the fixed average rate of return $\mu = 0.1$, the break-even point which is the corresponding point at which the VaR_α is the same with or without investment, is obtained when the volatility is $\sigma^2 = 0.986$. For any value of $\sigma^2 < 0.986$, the risk exposure is decreased. This implies that to lower the risk exposure, an investor should consider stocks with lower volatily. For instance, stocks with volatility 0.2 will have a risk exposure of $7.8 < 8.25$ (risk exposure without investment). On the other hand, it is essential to mention the higher the volatily, the higher the risk exposure, as it is the case for $\sigma^2 = 1$, $VaR_\alpha = 8.27$ which is higher than the Value at Risk of 8.25 corresponding to the VaR_α without investment.

Case 3: With investment with fixed volatility

Table 4.2. with fixed $\sigma^2 = 0.5$

Average rate of return μ	**Volatility** σ^2	**Value at Risk** VaR_α
0.02	0.5	8.71
0.2	0.5	6.8
0.5	0.5	2.8
0.067	0.5	8.25

Comments : While considering 8.25, which is the VaR_α without investment, our base for comparison at the level $\alpha = 0.05$, we fix our volatility σ^2 at 0.5. Then, the break-even point is obtained when the average rate of return $\mu = 0.067$. Note that the risk exposure is decreased only if the average rate of return is greater than 0.067, as it is the case for $\mu = 0.5$ and $\mu = 0.2$ whose corresponding value at risk 2.8 and 6.8 are smaller than 8.25. Nevertheless, for $\mu = 0.02$, $VaR_\alpha = 8.71 > 8.25$, which in this case implies that the smaller the average rate of return, the higher the risk exposure.

Case 4: With investment with fixed Value at Risk

Table 4.3. with fixed $VaR_\alpha = 8.25$

Average rate of return μ	**Volatility** σ^2	**Value at Risk** VaR_α
0.054	0.1	8.25
0.1	0.986	8.25
0.067	0.5	8.25
0.085	0.8	8.25

Comments: At the given level $\alpha = 0.05$, consider the fix $VaR_{\alpha} = 8.25$. Based on the investor's risk tolerance σ^2, the best stock to maintain the investor risk exposure can be found by identifying the best pair (μ, σ^2) which guarantees the break-even point as shown on the table . Based on case 3, the investor will be able to lower the risk exposure by choosing stocks with a higher rate of return.

CHAPTER 5

EXTENSION AND GENERALIZATION

In this section, we consider processes that no longer have independent increments.

5.1 Compound Mixed Poisson process

Definition 5.1.1. Mixed Poisson process

Let $\tilde{N}$ be a standard homogeneous Poisson process and $\{\mu(t), t \geq 0\}$ be the mean value function of a Poisson process on $[0, \infty)$. Let $\theta > 0$ be a random variable independent of $\tilde{N}$. Then the process $\{N(t) = \tilde{N}(\theta\mu(t)), t > 0\}$ is said to be a mixed Poisson process with mixing variable θ. We have

$$E(N(t)) = E\tilde{N}(\theta\mu(t)) = E(E[\tilde{N}(\theta\mu(t)|\theta)] = E(\theta\mu(t)) = E\theta\mu(t)$$

$$\begin{aligned} Var(N(t)) &= E[Var(N(t)|\theta)] + Var(E[N(t)|\theta]) \\ &= E[\theta\mu(t)] + Var(\theta\mu(t)) \\ &= E\theta\mu(t) + Var(\theta)(\mu(t))^2 \\ &= E(N(t)\Big(1 + \tfrac{Var(\theta)}{E\theta}\mu(t)\Big) \end{aligned}$$

It is noteworthy to mention that unlike Poisson process, the Mixed Poisson process does not only have dependent increments but, in addition, it is also over-dispersed (i.e., $Var(N(t)) > EN(t)$ for any $t > 0$ with $\mu(t) > 0$) and in general, the distribution of $N(t)$ is not Poisson.

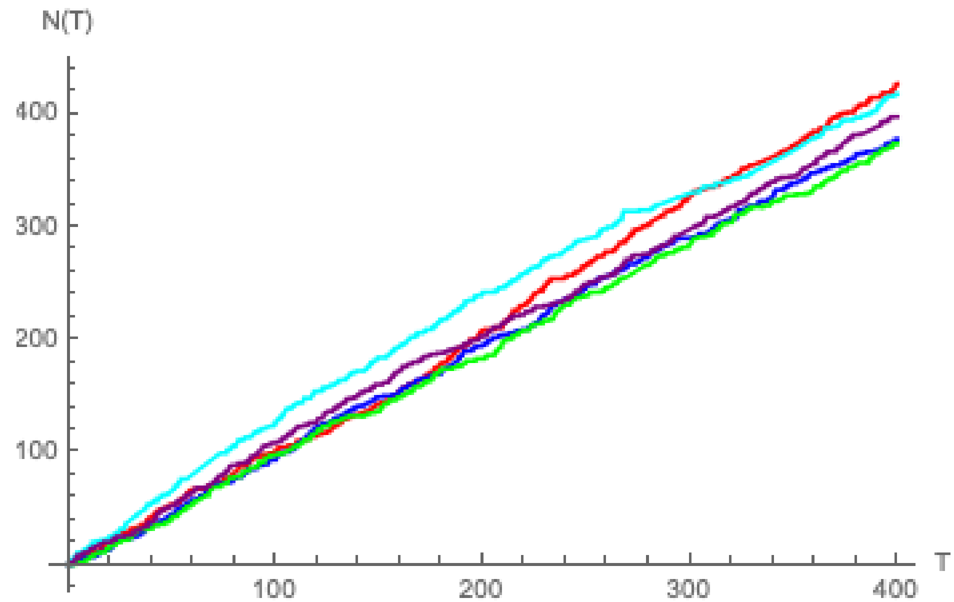


Figure 5.1. Several sample paths of a Homogeneous Poisson process.

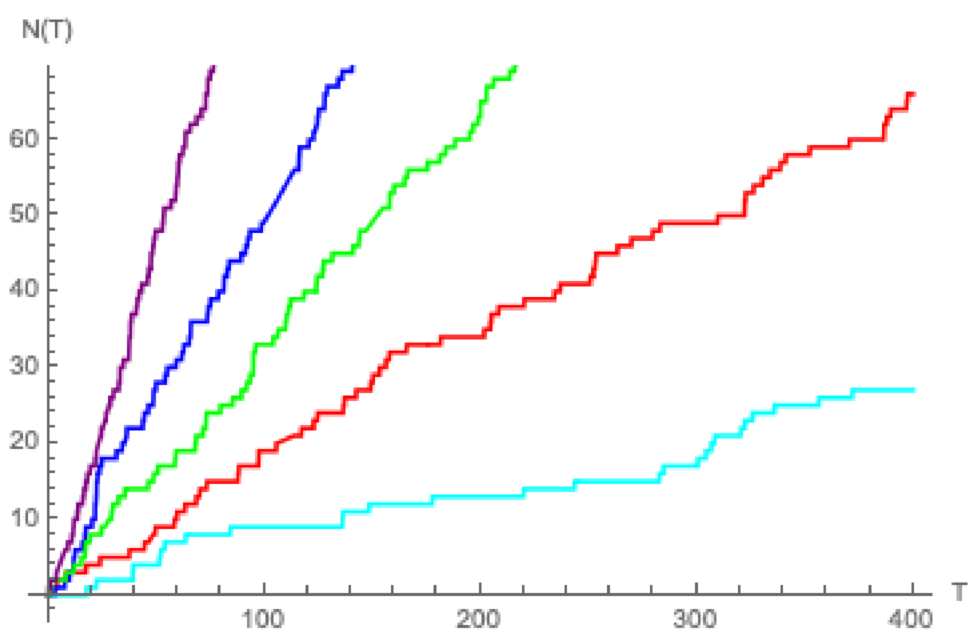


Figure 5.2. Several sample paths of a mixed Poisson process.

Notice that Figure 5.1 and Figure 5.2 represent five sample paths of a homogeneous Poisson process and mixed Poisson process respectively. In this example, the mixing variable θ is exponential with parameter 1.

Fact. Let $Y(t) = \sum_{i=1}^{N(t)} X_i, t \geq 0$, with X_i iid and independent of $N(t)$, be the compound mixed Poisson process. Then the expected value and the variance are as follows

$$E(Y(t)) = E(N(t))E(X_i) = E\theta\mu(t) \tag{5.1}$$

$$\begin{aligned} Var(Y(t)) &= Var(X_i)E(N(t)) + E(X_i)^2 Var(N(t)) \\ &= Var(X_i)E\theta\mu(t) + E(X_i)^2\big(E\theta\mu(t) + Var(\theta)(\mu(t))^2\big) \end{aligned} \tag{5.2}$$

Let's define a sequence of independent mixed Poisson processes $N^k(t)$ on $[(k-1)T, kT)$ and the corresponding compound Poisson processes

$$Y_k(t) = \sum_{j=1}^{N^k(t)} X_j^k \tag{5.3}$$

The cumulative loss reads

$$Y(t) = \sum_{j=1}^{k-1} Y_j(jT-) + Y_k(t) \tag{5.4}$$

where $(k-1)T \leq t < kT$.

Based on Lemma 3.1.1 and (5.1) – (5.4) we arrive at the following

Theorem 5.1.1. *Set* $V_k^*(Y_k, \alpha_k, \beta_k) \equiv V_k^*$*; for* $k = 1, \cdots n$*. Then* $VaR_{\alpha_1}(R_T) = V_1^* - (u + cT)$ *and given then* n^{th} *period* $[(n-1)T, nT]$*, the the optimal solution has the following cumulative Value at Risk at time* $(n-1)T$

$$\begin{aligned} VaR^*_{\alpha_n}(R_{nT}) &= \sum_{k=1}^{n-1}(Y_k) - I(Y_k) + P_I(Y_k)) + V_n^* - (u + cnT) \\ &= \sum_{k=1}^{n}(Y_k) - I(Y_k) + P_I(Y_k)) + VaR_{\alpha_n}(Y_n) - (Y_n) - (u + cnT) \end{aligned}$$

Furthermore

$$E\big(VaR^*_{\alpha_n}(R_{nT})\big) = \sum_{k=1}^{n-1} E\theta\mu(t) \cdot E(X^k) - I(Y_k) + P_I(Y_k) + V_n^* - (u + cnT)$$

and

$$P\Bigg[VaR^*_{\alpha_n}(R_{nT}) > \sum_{k=1}^{n-1} E\theta\mu(t) \cdot E(X^k) + \sum_{k=1}^{N_T} \big(P_I(Y_k) - I(Y_k)\big) + V_n^* - (u + cnT)$$
$$+ \sqrt{Var(X_i)E\theta\mu(t) + E(X_i)^2\big(E\theta\mu(t) + Var(\theta)(\mu(t))^2\big)} \cdot z_{\alpha_n}\Bigg] \simeq \alpha_n$$

where $1 - \Phi(z_{\alpha_n}) = \alpha_n$ and $\Phi(\cdot)$ is the cdf of $N(0, 1)$.

Corollary 5.1.2. *Let's consider* n *cycles of* L *periods with* $(\alpha_i, \beta_i$ *where* $i = 1, \cdot, L$*. As* $n \longrightarrow \infty$*, by the law of large numbers we have*

- *The expected Value at Risk per cycle*

$$\frac{E\big(VaR^*_{\alpha_n}(R_{nLT}\big)}{nL} \longrightarrow \sum_{i=1}^{L} \frac{E\theta\mu(t) \cdot E(X^k)}{L} - \sum_{i=1}^{L} \frac{I(Y_i)}{L} - \sum_{i=1}^{L} \frac{P_I(Y_i)}{L} - cT$$

- *The expected value of the portfolio per cycle*

$$\frac{E(X^*_{nLT})}{nL} \longrightarrow -\Big\{\sum_{i=1}^{L}\frac{E\theta\mu(t)\cdot E(X^k)}{L} - \sum_{i=1}^{L}\frac{I(Y_i)}{L} - \sum_{i=1}^{L}\frac{P_I(Y_i)}{L} - cT\Big\}$$
$$= \sum_{i=1}^{L}\frac{I(Y_i)}{L} + cT - \sum_{i=1}^{L}\frac{E\theta\mu(t)\cdot E(X^k)}{L} - \sum_{i=1}^{L}\frac{P_I(Y_i)}{L}$$

where $\sum_{i=1}^{L}\frac{I(Y_i)}{L} + cT$ *represents the Expected Income and* $\sum_{i=1}^{L}\frac{E\theta\mu(t)\cdot E(X^k)}{L} + \sum_{i=1}^{L}\frac{P_I(Y_i)}{L}$ *represents the Expected Loss.*

5.2 Renewal Process

Definition 5.2.1. Renewal Process

A renewal process $N = N(t) : t \geq 0$ is a process for which

$$N(t) = max\{n : T_n \leq t\}$$

where

$$T_0 = 0, T_n = M_1 + M_2 + \cdots + M_n \text{ for } n \geq 1$$

for i.i.d non-negative random variables M_i for all $i \geq 0$.

Definition 5.2.2. Compound Renewal Processes

Let $\{(T_1, X_1), (T_2, X_2), \cdots\}$ be a random marked point process with property that $\{T_1, T_2, \cdots\}$ is the sequence of renewal times of a renewal process, and let $\{N(t), t \geq 0\}$ be the corresponding renewal counting process. Then the stochastic process $Y(t), t \geq 0$ defined by

$$Y(t) = \begin{cases} \sum_{i=1}^{N(t)} X_i & \text{if } N(t) \geq 1 \\ 0 & \text{if } N(t) = 0 \end{cases} \tag{5.5}$$

$Y(t), t \geq 0$ is called a compound renewal process with the corresponding mean and variance

$$E(Y(t)) = E(X_i)E(N(t)) \tag{5.6}$$

$$Var(Y(t)) = Var(X_i)E(N(t)) + E(X_i)^2 Var(N(t)) \tag{5.7}$$

Let's define a sequence of independent renewal processes $N^k(t)$ on $[(k-1)T, kT)$ and the corresponding compound renewal processes

$$Y_k(t) = \sum_{j=1}^{N^k(t)} X_j^k \tag{5.8}$$

The cumulative loss reads

$$Y(t) = \sum_{j=1}^{k-1} Y_j(jT-) + Y_k(t) \tag{5.9}$$

where $(k-1)T \leq t < kT$.

Based on Lemma 3.1.1 and (5.6)- (5.9) we obtain the following

Theorem 5.2.1. *Set* $V_k^*(Y_k, \alpha_k, \beta_k) \equiv V_k^*$; *for* $k = 1, \cdot n$. *Then* $VaR_{\alpha_1}(R_T) = V_1^* - (u + cT)$ *and given then* n^{th} *period* $[(n-1)T, nT]$, *the the optimal solution has the following cumulative Value at Risk at time* $(n-1)T$

$$\begin{aligned} VaR_{\alpha_n}^*(R_nT) &= \sum_{k=1}^{n-1}(Y_k - I(Y_k) + P_I(Y_k)) + V_n^* - (u + cnT) \\ &= \sum_{k=1}^{n}(Y_k - I(Y_k) + P_I(Y_k)) + VaR_{\alpha_n}(Y_n) - (Y_n) - (u + cnT) \end{aligned}$$

Furthermore

$$E\big(VaR^*_{\alpha_n}(R_nT)\big) = \sum_{k=1}^{n-1}\big(E(N_k)E(X^{(k)}) - I(Y_k) + P_I(Y_k)\big) + V_n^* - (u + cnT)$$

and

$$P\Bigg[VaR^*_{\alpha_n}(R_nT) > \sum_{k=1}^{n-1} E(N_k)E(X^{(k)}) + \sum_{k=1}^{N_T}\big(P_I(Y_k) - I(Y_k)\big) + V_n^* - (u + cnT) + \sqrt{\sum_{k=1}^{n-1} Var(X^{(k)})E(N_k) + E^2(X^{(k)})Var(N_k)} \cdot z_{\alpha_n}\Bigg] \simeq \alpha_n$$

where $1 - \Phi(z_{\alpha_n}) = \alpha_n$ and $\Phi(\cdot)$ is the cdf of $N(0,1)$.

Corollary 5.2.2. *Let's consider n cycles of L periods with $(\alpha_i, \beta_i$ where $i = 1, \cdot, L$. As $n \longrightarrow \infty$, by the law of large numbers we have*

- *The expected Value at Risk per cycle*

$$\frac{E\big(VaR^*_{\alpha_n}(R_{nLT}\big)}{nL} \longrightarrow \sum_{i=1}^{L}\frac{(E(N_k)E(X^{(k)})}{L} - \sum_{i=1}^{L}\frac{I(Y_i)}{L} - \sum_{i=1}^{L}\frac{P_I(Y_i)}{L} - cT$$

- *The expected value of the portfolio per cycle*

$$\frac{E\big(VaR^*_{\alpha_n}(R_{nLT}\big)}{nL} \longrightarrow \sum_{i=1}^{L}\frac{(E(N_k)E(X^{(k)})}{L} - \sum_{i=1}^{L}\frac{I(Y_i)}{L} - \sum_{i=1}^{L}\frac{P_I(Y_i)}{L} - cT$$

$$\frac{E(X^*_{nLT})}{nL} \longrightarrow -\Big\{\sum_{i=1}^{L}\frac{(E(N_k)E(X^{(k)}))}{L} - \sum_{i=1}^{L}\frac{I(Y_i)}{L} - \sum_{i=1}^{L}\frac{P_I(Y_i)}{L} - cT\Big\}$$

$$= \sum_{i=1}^{L}\frac{I(Y_i)}{L} + cT - \sum_{i=1}^{L}\frac{(E(N_k)E(X^{(k)})}{L} - \sum_{i=1}^{L}\frac{P_I(Y_i)}{L}$$

where $\sum_{i=1}^{L}\frac{I(Y_i)}{L} + cT$ represents the Expected Income and $\sum_{i=1}^{L}\frac{(E(N_k)E(X^{(k)})}{L} + \sum_{i=1}^{L}\frac{P_I(Y_i)}{L}$ represents the Expected Loss.

Proposition 5.2.3. *These extensions could also be applied to the investment portion. Assume the premium collected is invested in different stocks at each time* iT , $i = 1, \cdots, n-1$. *and consider* (4.8) *such that* $\sum_{i=1}^{n-1} e^{(\mu_i - \frac{\sigma_i^2}{2})(n-i)T + \sigma_i (B_{nT}^{(i)} - B_{iT}^{(i)})}$ *is approximated by a single lognormal* $\hat{W} = e^{\mu+\sigma z}$ *using F.W.*

1. Then in the case of compound mixed Poisson process , using (5.1)-(5.4) and Theorem 5.1.1, we can find the insurer's value at risk with the following

$$P\Big[VaR^*_{\alpha_n}(R_nT)\big) > \sum_{k=1}^{n-1} E\theta\mu(t) \cdot E(X^k) + \sum_{k=1}^{N_T} \big(P_I(Y_k) - I(Y_k)\big)$$

$$+V_n^* - [u + (n-1)(1-w)cT + wcTE(\hat{W})]$$

$$+z_{\alpha_n}\sqrt{Var(X_i)E\theta\mu(t) + E(X_i)^2\big(E\theta\mu(t) + Var(\theta)(\mu(t))^2\big) + (wcT)^2 Var(\hat{W})}\Big] \simeq \alpha_n.$$

2. And in the case of compound renewal process, we can use (5.6)- (5.9) along with Theorem 5.2.1 to describe the reinsurer value at. risk as follow

$$P\Big[VaR^*_{\alpha_n}(R_nT)\big) > \sum_{k=1}^{n-1} E(N_k)E(X^{(k)}) + \sum_{k=1}^{N_T} \big(P_I(Y_k) - I(Y_k)\big) + V_n^*$$

$$-(u + (n-1)(1-w)cT + wcTE(\hat{W})$$

$$+z_{\alpha_n}\sqrt{\sum_{k=1}^{n-1} Var(X^{(k)})E(N_k) + E^2(X^{(k)})Var(N_k) + (wcT)^2 Var(\hat{W})}\Big] \simeq \alpha_n$$

where $1 - \Phi(z_{\alpha_n}) = \alpha_n$ *and* $\Phi(\cdot)$ *is the cdf of* $N(0,1)$

CHAPTER 6

FUTURE WORK

6.1 Conditional Tail Expectation (CTE)

In this dissertation, we studied periodic reinsurance with investment, where for each period a specific DRM, namely VaR, was considered. The main reason for this approach to risk assessment stems from the fact that for a general DRM, expressions involving the effective risk calculation have no closed form and become hardly tractable.

Our future research will be devoted to a hybrid method, whereby in the final stage, another concrete DRM with reasonable tractability, such as the Conditional Tail Expectation (CTE), can be used. The reasons for pursuing such direction are two-fold:

(a) CTE risk calculation can be readily achieved via Monte Carlo simulation

(b) CTE offers a remedy for under-estimating risks associated with VaR

Remark. Observe that for every level $0 < \alpha < 1$, $CTE_\alpha > VaR_\alpha$. Whence it follows that CTE offers a more informative risk estimate. Clearly, knowing the size of average loss given the loss exceeded L, provides much better prediction than just knowing that the loss is greater than L.

In what follows, we recall the definition of CTE and illustrate how this approach leads to a Monte Carlo simulation of CTE for our model with investment.

The CTE can be expressed in terms of distortion risk measure. Let's recall from chapter 2 the definition of DRM.

Definition 6.1.1. Let $g : [0, 1] \to [0, 1]$ be a non-decreasing function with $g(0) = 0$ and $g(1) = 1$. For a non-negative random variable X, the Distorted Risk Measure (DRM) is defined by

$$\rho_g(X) := \int_0^\infty g(S_X(x))dx$$

where $S_X(x) = P(X > x)$ is the survival function of X.

Given the parameter α, such that $0 < \alpha < 1$ if

$$g(t) = \begin{cases} 1 & \text{if } 1-\alpha < t \leq 1 \\ \frac{t}{1-\alpha} & \text{if } 0 < t < 1-\alpha \end{cases} \tag{6.1}$$

then

$$\rho_g(X) := \int_0^\infty g(S_X(x))dx = CTE_\alpha$$

As noted in [18], an equivalent way of defining CTE_α is given by the following

Definition 6.1.2. Let X be a random loss variable with density f. Then the CTE of the risk X is defined

$$CTE(X) = E[X|X \geq VaR_\alpha(X)] = \frac{1}{\alpha}\int_{VaR_\alpha(X)}^\infty xf(x)dx, \quad 0 \leq \alpha \leq 1$$

Assume that the pdf of the sum of two independent random variables has known closed form.

Example 6.1.1. Let $X, Y \sim Exponential(\lambda)$, then the sum $Z = X+Y \sim Gamma(2, \lambda)$ with pdf $f(z) = ze^{-z}$. At the level $\alpha = 0.05$ one calculates $VaR_\alpha = 4.743$, and the Conditional Tail Expectation of Z is

$$CTE(Z) = \frac{1}{0.05} \int_{4.743}^{\infty} z \cdot ze^{-z} dz = 5.9213$$

It turns out that if we simulate X and Y by Monte-Carlo using the following

$$\frac{1}{\alpha} \sum_{i=1}^{N} \frac{(X_i+Y_i) \cdot \mathbf{1}_{(X_i+Y_i > VaR_\alpha)}}{N}, \qquad P(X+Y > VaR_\alpha) = \alpha$$

we can then recover CTE_α with very good degree of accuracy. Namely, by running three simulations with $N = 10^7$ trials and taking their average we obtained 5.9214.

Based on this, we can apply Monte Carlo simulation to approximate CTE_α in the case the pdf of the sum is unknown.

Example 6.1.2. Let $X \sim Normal(13, 13)$ and $Y \sim Lognormal(0.163, 3.23)$, thus we do not have a closed pdf form for $Z = X - cwY$. This case corresponds to Lemma 4.0.1 regarding the risk model with investment.

At the level $\alpha = 0.05$ with $VaR_\alpha = 8.25$, $c = 1$ and $w = 0.4$, the Conditional Tail Expectation of Z calculated by analogous Monte Carlo simulation was found to be 9.59, thus greater than the value at risk VaR_α as expected.

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