EXPLORING DEVELOPMENT OF PROBLEM SOLVING STRATEGIES IN EMERGING MATHEMATICIANS

by

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Abstract

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To solve an unfamiliar mathematics problem, students of the subject must know more than the appropriate prerequisite content knowledge. They must also know how best to strategically apply their knowledge, how to monitor and gauge the effectiveness of their work, and how to respond (both cognitively and emotionally) to unanticipated results. Expanding current research on the types of experiences that foster these skills is the objective of this study.

In a sequence of two task-based interviews, eight graduate and four upper-division undergraduate mathematics students solved non-traditional mathematics problems, used their work on these problems as a basis to comment on their mathematical beliefs and problem solving strategies, and tied these aspects of their mathematical identity to the formative courses, instructors, and experiences that influenced them. I then analyzed these interviews using coding techniques appropriate for thematic analysis and created qualitative characterizations of the participants' problem solving strategy usage. By comparing these characterizations to the ways in which the participants reported that their problem solving had developed over time, I identified which types of formative experiences may have contributed to this development (and how they did so).

Findings suggest when participants had been encouraged to discover their own mathematical justifications in previous courses, they would be likely to display creative, exploratory behavior while solving interview tasks, even when frustrated. Participants who valued experiences in which they participated in meaningful mathematics as part of a group were more comfortable engaging in self-talk and considering multiple approaches to an interview task. Finally, many participants reported learning specific heuristic strategies, either explicitly from instructors or as a consequences of the nature of a course; findings indicate that the degree to which they could effectively leverage any of these heuristics depended on if they were comfortable and familiar with their application. These results point mathematics instructors towards ways that they can develop problem solving expertise in their students.

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Chapter 1

Introduction

Instructors of many mathematics courses expect their students to use content knowledge from that course to solve challenging or novel problems (Selden et al. 1999). But solving such a problem requires more than simply recalling and then correctly applying the relevant content knowledge: for example, one must first decide which content is in fact relevant, then whether the correct application they know is problem-specific or generalizable to other settings, and then finally, how to react, both cognitively, meta-cognitively, and emotionally, to unexpected or unfamiliar complications in their anticipated approach to the problem. Attempting to identify and rigorously study these additional dimensions of mathematical proficiency have provided a stable foundation for the realm of mathematical problem solving research for several decades. Over these years, and across countless studies, the intuitive notion of every mathematician that there is more to their subject than just memorization and reproduction has been validated (e.g. Schoenfeld, 1989; Lester et al., 1989; Selden et al., 1999; Selden et al., 1989, 1994; Campione et al., 1988; Álvarez, Rhoads, & Campbell, 2019).

With the implicit understanding that, like content knowledge, mathematical problem solving strategies must develop over time, researchers have studied the problem solving processes of different populations: ranging from undergraduate students (Selden et al., 1994, 1999; Maciejewski, 2019; Furinghetti & Morselli, 2009; Stylianou & Silver, 2004; Čadež & Kolar, 2015) to graduate students (Carlson & Bloom, 2005) to mathematics faculty/ Ph.D. recipients (Stylianou & Silver, 2004; Stylianou, 2002; Carlson & Bloom, 2005; DeFranco, 1996). These reports often include at least one bespoke analytical framework and an accompanying diagram for explicating the underlying structure of their subjects' decisions during the problem solving process. They then conclude with data-driven assertions about how problem solving might manifest in the context of challenging mathematics problems. These results might be leveraged, for example, to attempt to explain the failure of otherwise successful calculus students in solving particular kinds of calculus problems (as in Selden et al., 1999).

But most compelling (and least common) among these studies are those that adopt some permutation of an expert-novice paradigm; that is, instead of describing the isolated actions of one population, they attempt to compare problem solving strategies between two populations wherein one is comparatively more advanced than the other. This contrast could be between undergraduate students and Ph.D. mathematicians (Stylianou & Silver, 2004), graduate students and Ph.D. mathematicians (Carlson & Bloom, 2005), or even Ph.D. mathematicians of various levels of success in their field (DeFranco, 1996). The relative sparsity of these comparative studies (and in a related way, the limited research on the problem solving strategies of graduate students) motivates the current work. That is, a significant amount of research now exists which characterizes, either in isolation or through comparison, different populations' problem solving strategies. But where do these differences arise? How do the elegant and malleable approaches deployed by research mathematicians develop from the nascent strategies of undergraduate students?

An auxiliary benefit of any research in the field of mathematical problem solving is the ability to empower mathematics educators to be able to teach their students not only content knowledge but also more effective ways of leveraging this knowledge in unfamiliar problem solving situations. To this end, answering the above questions may appear unnecessary; after all, if we sufficiently characterize the problem solving behavior of experts, we could simply teach novices to do precisely what we have observed experts to do. But as noted by Silver (1985),

> This is specious reasoning and such advice may be counterproductive to learning. Surely the behavior of experts is highly efficient and often elegant, but the polished, mature problem-solving behaviors of experts fail to reveal the successive approximations that preceded these mature procedures (pg. 251).

When mathematicians read and interpret proofs from textbooks, they see the final product but not necessarily the careful thinking and refinement that enabled that elegant presentation. Similarly, the researchers cited above have depicted what the product of mathematical problem solving development might look like when they describe expert behavior; in my study, my goal is to capture the process involved in refining these behaviors.

To this end, I generated three specific research questions that I sought to answer in the present study.

1. In what ways do emerging mathematicians characterize their problem solving strategies? Do these characterizations manifest in practice?
2. What experiences contributed to the development of particular mathematical problem solving strategies in emerging mathematicians?
3. How do problem-solving strategies develop as a result of extended mathematical education?

Here, I define an *emerging* mathematician as a student of mathematics who understands foundational content knowledge but has had limited opportunities for refinement of their mathematical practice (Kercher & Álvarez, under review). Through a sequence of task-based, semi-structured interviews with upper-division undergraduate and first-semester graduate students, I attempt to describe their problem solving strategies and the experiences that they attribute to the development thereof. With the help of an accompanying questionnaire and interview protocol to facilitate the interview, I characterize different formative experiences that participants perceived as influential in their growth as mathematical problem solvers. By analyzing their work on the non-traditional tasks which formed the focal point of the interview, I then consider whether these anecdotal accounts of development are borne out in problem solving strategies actually employed.

Participants in this study valued mathematical experiences that allowed them to independently develop conceptual justifications, discover connections between topics, and revisit previous material. Findings suggest that the degree to which these explorations were actually independent appeared to influence the value each participant ascribed to creativity and how likely they were to work through temporary frustrations. Participants also valued mathematical activities in social settings, although the effect of these experiences appeared to hinge largely on the quality of the activity in question. Finally, many participants also reported learning specific problem solving heuristics from instructors or courses; they were more likely to leverage these heuristics in interview tasks if they were confident and familiar with their application.

Chapter 2

Literature Review

Before summarizing the relevant literature that I chose to review, it is worth describing some relevant literature that I chose *not* to review. First, to discuss problem solving in any great depth, one needs necessarily to know what precisely constitutes a *problem*. There exists good research elaborating on characteristics of or classifications for different kinds of problems (Jonassen, 1997, 2000; Selden et al., 1999). Instead of classifying and analyzing my interview tasks with one of these available frameworks, I chose instead to adopt Lester's (2013) broader definition of a problem as "a task for which an individual does not know (immediately) what to do to get an answer." By eschewing a close examination of the properties of the interview tasks themselves (that is, by treating them generally as problems and not one of the specific subtypes from a particular framework), my analysis aims to capture problem solving development independent of problem type or structure.

I identify and describe three prevailing themes in existing mathematical problem solving literature that have immediate relevance to my research questions. First, I attend to various problem solving frameworks that attempt to describe the key mechanical steps in the problem solving process; these steps capture the distinct cognitive processes of problem solving that, consciously or otherwise, mathematicians draw on when solving a difficult or nontraditional problem. More recent frameworks (Mayer, 1998; Lester et al., 1989; Schoenfeld, 2014) supplement the explicit cognitive steps by describing metacognitive and affective dimensions that may influence the order or effectiveness of said steps. I next describe precisely how researchers categorize these non-cognitive dimensions and explore findings from the literature which deal directly with the task of describing the effects they have on the problem solver. The literature review concludes with an exploration of classroom factors and societal influences which may contribute to the development of students' critical thinking skills or positive mathematical outlook.

2.1 Problem Solving Frameworks

Early literature on problem solving was often personal and anecdotal, in that the author (typically already a celebrated and successful mathematician) reflected on their own approach to difficult mathematics problems (e.g. Poincare, 1908; Hadamard, 1954; as cited in Furinghetti & Morselli, 2009). Attempts to describe the problem solving strategies of others in a single, comprehensive framework began in earnest when George Pólya published his book, *How to Solve It* (Pólya, 1945), in which the he delineates four phases of problem solving: understanding the problem, devising a plan, carrying out the plan, and looking back. The primary function of *How to Solve It*, however, is not to validate its framework with empirical evidence; it is to collect and present to the reader a series of heuristic strategies. These strategies (e.g. "Could you restate the problem?" or "Did you use all the data?"; pp. 75, 95) can then facilitate any of the four phases comprising the framework; in particular, many heuristics are geared towards the looking back or devising a plan phases, the latter of which Pólya warns may be "long and tortuous" (Pólya, pg. 23). Such a concession to the difficult nature of discovering a feasible approach to an unfamiliar problem foreshadows the fact that future frameworks would attend in greater detail to which particular cognitive and metacognitive strategies problem solvers undertake in order to transition from a basic understanding of the problem space to the generation of a workable plan for solving the problem.

Some researchers, instead of attempting to outline an approximately sequential process of solving a difficult problem, opt instead to describe necessary and sufficient qualities which contribute to successful problem solving: Mayer (1998) lists three categories in which a problem solver would need to excel in order to be consistently successful in their field: skills (i.e. domain knowledge), meta-skills (metacognitive knowledge), and will/ motivation; Lester et al. give five: resources, affects and attitudes, beliefs, control, and contextual factors (1989); Schoenfeld (2014) provides four categories, which he calls content knowledge, heuristic strategies, monitoring/ control, and beliefs/ affect. These frameworks are summarized in the table below.

Table 2.1 Different components of problem solving according to various frameworks

	Mayer (1998)	**Lester et al. (1989)**	**Schoenfeld (2014)**
Cognitive	Domain Knowledge	Resources	Content knowledge; Heuristic strategies
Metacognitive	Meta-Skills	Control	Monitoring/ control; Beliefs
Affective	Will/ motivation	Affects/ attitudes; Beliefs	Beliefs/ affect
Other		Contextual factors	

As illustrated in the table, each of the above frameworks is significant not only because of its componential rather than sequential structure but also because they explicitly acknowledge that problem solving may not require only cognitive proficiencies. First, the inclusion of a metacognitive dimension of problem solving makes explicit a combination of two thematically pervasive (but ultimately implicit) ideas in *How to Solve It*: that the problem solver must carefully justify the effectiveness and mathematical veracity of a particular choice of approach before committing to it completely; and that, once committed, they must regularly check their work to verify its success. Second, the inclusion in these problem solving frameworks of an affective (either in general, as in Lester et al. (1989), or a specific subset of affect such as motivation, as in Mayer (1998)) is also notable. That is, telling problem solvers that they should devise a plan certainly

20makes sense, but vastly oversimplifies the complex interactions between emotional and cognitive systems that manifest when the recipient of this advice acts upon it. The way in which problem solvers respond to the frustration engendered by their inability to create such a plan or modify it in the face of adversity often has just as great an effect on their success in problem solving as, for example, their prior content knowledge (e.g. Furinghetti & Morselli, 2009; McLeod et al., 1989; Carlson, 1998, 1999).

Other researchers expanded Pólya's framework to account for these non-cognitive aspects of problem solving while maintaining a sequence of cognitive steps. Garofalo and Lester initially cited Schoenfeld's (1983) preliminary observations of metacognition in problem solving and Silver's (1982, as cited in Garofalo & Lester, 1985) findings that exclusively cognitive frameworks may be neglecting an important metacognitive perspective when they published an expanded version of Pólya's framework that allows for metacognitive practices within the original four cognitive steps (1985). For example, Pólya's almost entirely cognitive execution step is supplemented by Garofalo & Lester's description of how successful mathematicians should also "[monitor] the progress of local and global plans" during this phase of problem solving (Garofalo & Lester, 1985, pg. 10).

Carlson and Bloom's (2005) problem solving framework, like that of Garofalo and Lester (1985), also comprises four sequential phases (largely isomorphic, on a surface level, to Pólya's) called orientation, planning, executing, and checking. However, they expand the planning stage of this newer framework into an entire planning subcycle that involves the following steps: making a conjecture, imagining its resolution, and evaluating its potential for success. This subcycle repeats until the problem solver establishes a satisfactory plan and proceeds to its execution, a process which necessarily involves elements of a metacognitive archetype (e.g. Schoenfeld, 1983; Silver 1982). A further cyclic layer envelopes the larger framework by allowing that, after checking their work, the problem solver may find their solution unsatisfactory and begin again at the orienting or planning stage.

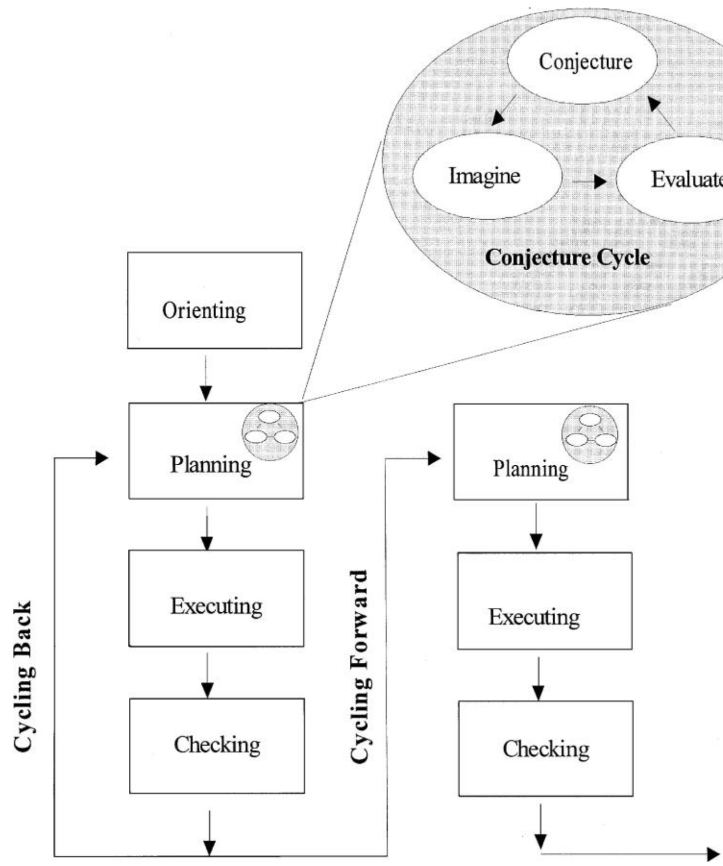


Figure 2.1 Carlson & Bloom’s framework (2005).

The cyclic models of problem solving above would not appear completely foreign to earlier problem solving researchers. As pointed out by Wilson et al. (1993), one does a disservice to Pólya’s initial framework by assuming that he intended his four phases to be treated as entirely linear steps. Much as researchers sought to bring the undercurrent of metacognition and affect evident in *How to Solve It* to the forefront by making them overt components of their problem solving framework, Carlson & Bloom only made explicit the already present subtext in *How to Solve It* that problem solving is an iterative and incremental process best visualized in a cycle or spiral. In fact, they were not the first authors to do so (see Wilson et al., 1993; Jonassen, 1997), and other authors since have presented frameworks that function in an essentially cyclic way (e.g. Maciejewski 2019, Stylianou 2002). For an example of one such earlier cyclic framework, Wilson et al. reimagine Pólya's four phases as arranged at the cardinal directions of a compass and note that students may travel from any phase to an adjacent phase either clockwise or counterclockwise (1993).

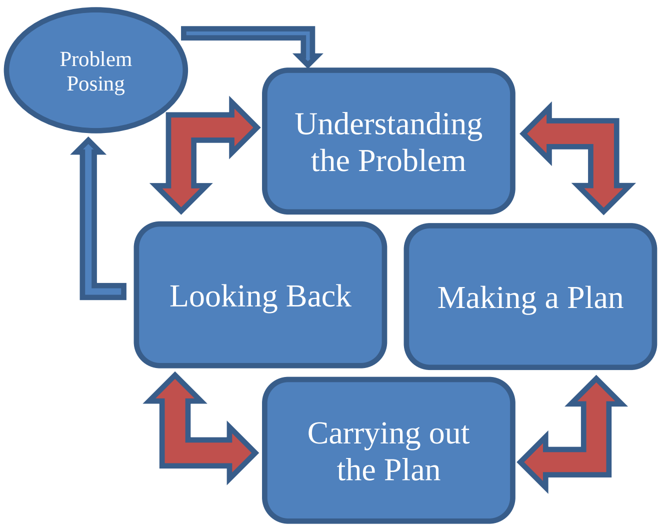


Figure 2.2 Wilson et al.'s framework (1993).

Some MPS frameworks parallel aspects of concept images (Tall & Vinner, 1981) and cognitive schema (Čadež & Kolar, 2015). When examining the work of differential equations students who completed a battery of non-routine calculus problems, Selden et al. (1999) posited the concept of a *problem situation image*. Such a construct would consist of the problem solver's various strategies, (non-) examples, related facts, beliefs about difficulty, etc. that they associate with a given type of problem situation. Importantly, problem situation images would include *tentative solution starts* (i.e. possible ways of approaching a given problem situation) collated from the problem solver's past experiences with similar problems; they would use these to jump-start their problem solving process. Tentative solution starts (e.g. trying to set the equation equal to 0 when given a problem about solutions to a quadratic function) set themselves apart from the more general heuristics espoused by Pólya and Schoenfeld in that they more closely associate with specific content areas. Problem situation images coincide in many ways with how Čadež and Kolar describe cognitive schema: as a method of organization which might allow an individual to associate similar experiences in a way that later allows them to handle new, but familiar, situations (2015). A diagram of one way in which schema enhance the descriptive capabilities of a cyclic problem solving model is provided by Jonassen (1997), wherein the planning phase is bypassed entirely when an appropriate solution start is recognized from the context of the problem.

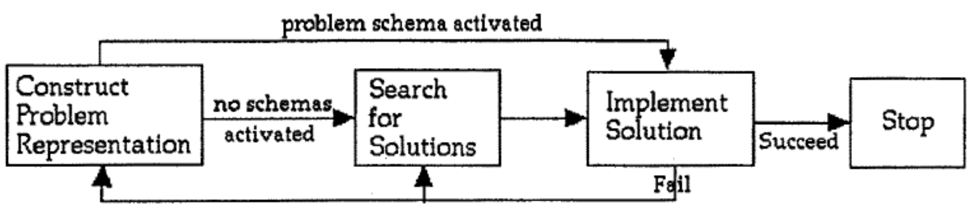


Figure 2.3 Jonassen's framework (1997).

Some researchers devoted their time to creating frameworks which might be used to analyze specific phases of problem solving rather than the entire process. Maciejewski's (2019) framework more deeply explores the planning stage of problem solving. He describes mathematical foresight as the ability to anticipate the form of a solution, which motivates the creation of a series of steps (in his words, a trajectory) required to reach a resolution. Constructing the trajectory uses episodic future thinking, a type of future-oriented conceptualization in which the problem solver envisions themselves experiencing the act of attempting a possible solution path in order to judge its viability; much like Carlson and Bloom's planning subcycle, it involves considering the likelihood of the success of each step in the problem solving process. Maciejewski viewed his framework as "a complementary characterization of problem solving processes" (2019, pg. 23) when compared to those of Pólya and Schoenfeld. He explains that in these heuristic-oriented frameworks, the planning phase largely consists of the problem solver choosing an appropriate strategy; in the mathematical foresight model, the agency is, in some sense, shifted from the problem solver to their visualization of the anticipated solution. That is, when correctly predicted, the form of the solution informs entirely how the problem solver should approach the problem in its current state (Maciejewski, 2019).

Stylianou hybridized the traditional orientation and planning phases of problem solving by chronicling how experts negotiated the interactions between visualization and analysis (2002). She described the process of constructing a visualization with some goal in mind, inferring additional aspects of the problem space from the visualization, elaborating on the mathematical implications of these new details, and then using this data to impose a new goal that allows for the refinement of the previous visualization or a more accurate second diagram; this process is repeated until a satisfactory plan emerges for finding a solution to the problem. Stylianou also notes the importance of self-talk for monitoring the feedback loop used to accurately construct a progressively more insightful visualization. Again, note the cyclic nature of this framework akin to previous researchers (such as Carlson & Bloom (2002), Wilson et al. (1993), and Jonassen (1997)).

3.2 Affect and Metacognition

The earliest research in mathematical problem solving tends to skew towards the "popular myth that mathematics is a purely intellectual endeavor in which emotion plays no essential role" (Goldin, 2002 pg. 59). This is in some way surprising, given that the earliest accounts of mathematical problem solving and other commentaries on mathematics in general (such as Poincare, 1910; Hadamard, 1945; Hardy, 1967; as cited in Zan et al., 2006) already contained allusions to the important role played by managing emotions and mathematical beliefs— two distinctly non-cognitive dimensions. On the other hand, perhaps the view of mathematics education research as purely a matter of cognition can be attributed to the strict emphasis on observable phenomena and quantitative methods inherent to behavioralist research that, for a time, dominated mathematics education. At the very least, it is non-trivial to define and measure what it means for someone to feel a particular way about mathematics, and this fuzziness was off-putting to researchers concerned with validity and consistency (DeBellis & Goldin, 2006). Eventually, though, enough researchers (such as Silver, 1985; Schoenfeld, 1989; Norman, 1980) challenged this strictly cognitive view of thinking that investigations into mathematical problem solving were finally forced to ask a difficult question: Why do students, who otherwise demonstrate sufficient cognitive resources (that is, they know and remember the mathematical content), still struggle to solve non-routine problems? At least some part of the answer to this question relies on the existence of a fundamental link between cognitive, affective, and metacognitive dimensions of thinking (Schoenfeld, 1989; Lester et al., 1989; Selden et al., 1994, 1999; Selden et al. 1994; Campione et al., 1988; Álvarez et al., 2019).

The term *affect* in mathematics education often functions as a catch-all indicating some subset of beliefs, attitudes, values, or emotions as related to the subject's perception of mathematics, their classroom environment, or themselves (Carlson & Bloom, 2005; Schoenfeld, 2014; Reyes, 1984; DeBellis & Goldin, 2006; Goldin, 2002; McLeod, 1989). Summarized (perhaps too concisely) by Liljedahl: "the affective domain is most simply described as *feelings*—the feelings students have about mathematics"

(2005). For a less heavy-handed simplification, one could consider affect as a dichotomy wherein *local affect* refers to the subset of affect which deals with the quickly changing emotional reactions to the current experience of being in a mathematics situation, and *global affect* entails more stable, long-term mathematical beliefs or attitudes founded over a series of experiences (DeBellis & Goldin, 2006; Goldin, 2002). Some researchers find even this approach too simplistic and have instead taken to describing the several affective constructs which comprise local and global affect, and further elaborating on how precisely they are tied to cognition.

Emotions are entirely local, quick to develop and quick to fade, and may be experienced consciously or subconsciously. Emotions are difficult to define and measure, but extant research still presents compelling frameworks describing the genesis of emotion from a constructivist or a cognitive psychologist perspective (Harré, 1986; Mandler, 1984; Case, Lewis, & Hurst, 1988; as cited in McLeod, 1992). Key to understanding the role of emotion in problem solving is the idea of *meta-affect*: essentially, how one manages and responds to affective states (DeBellis & Goldin, 2006). DeBellis and Goldin argue that the goal of mathematics instruction (especially when centered around challenging and non-routine problems) should not be to eliminate traditionally negative emotions such as frustration. Instead, students should be taught how to react to and manage frustration in a healthy and productive way. Riding a roller-coaster induces fear, yet people line up to ride them at amusement parks anyway. This is because, cognitively, they trust the safety of the ride and know that the exhilaration of surviving (so to speak) awaits them at the end. Analogously, problem solvers should learn that the frustration or anxiety associated with being confronted by an unfamiliar problem indicates an opportunity for creativity, learning, and a rush of elation and satisfaction that accompanies eventual success. This meta-affective response requires an underlying structure of beliefs about mathematics; that is, just as the theme park attendant believes in the safety of the roller coaster, so too must the mathematics student believe in the solvability of difficult mathematics problems with concerted and possibly prolonged effort (Goldin, 2006; DeBellis & Goldin, 2002). Corroborating the need for meta-affect as a construct, research shows both expert and novice problem solvers experience the same emotional reactions to solving non-traditional problems; however, experts more capably control the extent of these emotions (McLeod et al., 1989). Similarly, other researchers have demonstrated that persistence in the face of frustration serves as a remarkable indicator of problem solving success (Carlson, 1998, 1999) or that problem solvers who fail to manage their anxiety are both less proficient at orienting themselves in the problem space and re-orienting in the face of an unsuccessful attempt (Furinghetti & Morselli, 2009).

Beliefs involve some attribution of a truth value to a structure built from cognitive/affective pathways. Beliefs require some warrant, or justification, on the part of the belief holder but do not have to represent an incontrovertible truth (in the Platonic ideal sense of the word). That is, a problem solver may believe that they can solve any mathematics problem within 10 minutes based on the warrant that, in their mathematics class, they can solve any problem given to them within that timeframe. Beliefs demonstrate remarkable stability, a virtue owed to their construction and reinforcement at the hands of both cognitive and affective experiences. Some researchers (Schoenfeld, 1987) categorize beliefs as an aspect of metacognition rather than affect, since the requirement that beliefs have a truth value and associated warrant requires some amount of cognitive introspection on the part of the believer. Other researchers (Silver, 1985) distinguish beliefs from both affect and metacognition, choosing instead to attribute the development of beliefs to enculturation. Regardless of their classification, beliefs have an immediate and often profound impact on problem solving. Belief in one's own ability to do a particular mathematics problem (that is, one's self-efficacy) was found to have a greater effect on success on that problem than several other variables, including prior mathematical experience (Pajares & Miller, 1994). Furinghetti and Morselli found that students who believed mathematics boils down to finding the right formula and following the associated steps became frustrated more easily in the face of difficulty; when the next step of the problem was not obvious, even if all of their prior work was correct, these students attributed their standstill to personal shortcomings (for example, one participant lamented that "in algebra, indeed, we have worked a lot on these things! […] But does this mean I have not understood what I studied???"; 2009, p. 82). Schoenfeld found that students who believed problems in mathematics courses should take no longer to complete than a few minutes were less likely to persist through adversity (1989). Finally, competent mathematics students who believe that a given problem will be fun or interesting to solve may be more likely to think carefully and creatively when formulating an approach to that problem; if they perceive a problem as either rote or challenging in a frustrating way, they may only go through the motions of problem solving (Callejo & Vila, 2009).

Importantly, one should understand that individual beliefs taken in isolation may not always be entirely accurate indicators of problem solving actions. Schoenfeld (1989) found that students who believed proof and construction in geometry were closely related mathematical activities still struggled to leverage the structure of a familiar proof to allow them to make a corresponding construction. Furthermore, these same students expressed that mathematics is ripe with opportunities for the application of logic and creativity, neither of which they were able to successfully utilize in the construction task. Schoenfeld hypothesizes that this discrepancy derives from students absorbing "the r hetoric— but not the substance" of well-meaning instructors who glorify mathematical thinking but only prescribe simple exercises that test memorization and algorithmic proficiency (pg. 349). Other researchers (Callejo & Vila, 2009; Ruthven & Coe, 1994) have pointed out similar inconsistencies between beliefs and action in problem solving contexts. They attribute these perceived contradictions to the fact that students often build entire belief systems in which multiple, separate beliefs can contradict each other and result in problem solving activity that does not always align with individual student beliefs. In line with this observation, Schoenfeld (1989) also noted that despite his students' claim that mathematics encouraged creativity, they also felt that memorization was the most effective way to learn it.

Values and *attitudes*, the final affective constructs worth discussion, share many similarities with beliefs. Whereas beliefs build on cognitive/ affective pathways, it could be said that values build on affective/ cognitive pathways. That is, the problem solver may justify a statement of value by employing a warrant founded mostly on emotions, making it even less amenable to logical discourse or other cognitive intervention than beliefs (Callejo & Vila, 2009; Goldin, 2002; DeBellis & Goldin, 2006). Attitudes are one's predisposition towards a (set of) emotion(s) when exposed to a particular situation (Goldin, 2006). Attitudes may develop as a result of repeatedly experiencing the same emotional responses to similar situations, such as a student who often finds themselves frustrated by geometric proofs. To Liljedahl, this can be interpreted by treating attitudes as the physical manifestation of beliefs. For example, "a belief that 'math is all about formulas' may manifest itself as an attitude of disregard for explanations in anticipation of the eventual presentation of a formula" (2005, p. 221). Attitudes can also accommodate new situations, similarly to cognitive schema, such as when a student who has been frustrated by proof geometry reacts negatively to a proof in an algebraic setting, despite never having seen a proof in that context before (McLeod, 1992). Carlson reported that successful problem solvers believe that challenging problems situated in a meaningful context provide the best environment for learning mathematics; consequently, their attitude towards memorization and rote exercise was decidedly negative (1998).

The problem solving research community originally perceived metacognition, like affect, as a dimension of problem solving auxiliary to cognition despite the fact that elements of metacognition, variously referred to as control processes, executive function, or reflective intelligence, existed in cognitive models based on information-processing theory concurrently to problem solving research (Atkinson and Shiffrin, 1968; Butterfield and Belmont, 1975; Miller et al., 1960; as cited in Silver, 1985). Also like affect, metacognition suffered somewhat from competing definitions that vary widely in scope. On the one hand, some audiences may find it sufficient to call metacognition “thinking about your own thinking” (Schoenfeld, 1987; p. 189); on the other, this pithy definition oversimplifies a complex phenomenon that encompasses multiple distinct aspects of thought. I next consider various models of metacognition and attempt to describe how they relate both to each other and to student success in problem solving.

Schoenfeld (1987) describes three subsets of metacognition, two of which also directly correspond to components of his framework for describing successful problem solving (Schoenfeld, 2014; see section 1). These are *control* and *beliefs*. A problem solver exhibits control throughout the stages of a solution attempt, whether in the form of assessing their initial understanding of the problem space, planning an approach, or monitoring its execution. Control is the embodiment of the phrase “It’s not only what you know, but how you use it (if at all) that matters” (Schoenfeld, 1987; pg. 192). This aspect of metacognition parallels Goldin’s concept of meta-affect, seen earlier (2006), but emotions do not necessarily need to trigger it; in fact, Schoenfeld stresses that effective problem solvers often stop to monitor and assess the progress of their solution attempt in the absence of any specific prompts (Schoenfeld, 1987). Flavell (1976) notes that conscious responses to both affective and cognitive triggers (that is, meta-affect and metacognition) can both engender “a kind of quality control” with respect to effective thinking (pg. 908). Unfortunately, control of any kind seems to be a difficult skill to acquire; both talented undergraduates and graduate students display an inability to access and regulate their use of content knowledge when solving challenging or non-traditional problems (Carlson, 1999; Schoenfeld, 1989).

Beliefs, Schoenfeld’s second subcategory of metacognition (1987), have already been described as part of mathematical affect. Discussing them briefly again, however, in tandem with Schoenfeld’s third subcategory (and through the lens of metacognition) may clear up some ambiguity. This third subcategory, *knowledge about your own thought processes*, is in some ways hard to distinguish from beliefs; evidence of this ambiguity is given by the definition, in Lester et al., of beliefs as “subjective knowledge” (1989, p. 77). In fact, the distinction between beliefs and knowledge has instigated disagreement in the research literature (see Goldin, 2002). To avoid this confusion, we might choose to adopt Flavell’s (1976) use of the term *metacognitive knowledge.* Flavell uses this construct, despite its name, to refer to both knowledge and beliefs, so long as they pertain to the cognition either of oneself or another person. An example Flavell provides of metacognitive knowledge is one’s awareness that they “can learn most things better by listening than by reading” (1979, pg. 907). Regardless of whether one considers this statement knowledge or belief, it certainly implies a recognition of one’s own cognitive capabilities and represents a decidedly metacognitive mindset.

3.3 Classroom Practice

All of the preceding definitions and frameworks, when used to better model problem solving success, can also be leveraged to create an actionable prescription for teaching problem solving in mathematics classrooms. The necessity of such a prescription is predicated on clear evidence that students with sufficient content knowledge may still fail to transfer this knowledge to unfamiliar or non-traditional problems; that is, something other than their mathematical resources has a hand in their success as problem solvers (Schoenfeld, 1989; Lester et al., 1989; Selden et al., 1999; Selden et al., 1989, 1994; Campione et al., 1988, Álvarez et al., 2019). Unfortunately, traditional school instruction almost entirely focuses on the conveyance of mathematical knowledge. Campione et al. elaborate on exactly what aspects of traditional instruction they feel contribute to the so-called *transfer problem* described above: lack of meaningful discussions involving students, lack of formative assessment, emphasis on procedural rather than conceptual knowledge (especially for poorly performing students), and little instruction on metacognitive and heuristic strategies (1988). While various organizations have, in the intervening years, made suggestions for classroom reform that emphasize problem solving to some degree (e.g. NCTM, 2000), value still exists in considering some of these perceived shortcomings of traditional instruction in greater depth.

The first of these shortcoming is the emphasis on procedural rather than conceptual knowledge, which leads to the belief that mathematics, as a subject, essentially amounts to the reproduction of facts and algorithms. Such belief leads to mathematical behaviors that do not transfer to more general problem settings; such knowledge is called *inert* (Whitehead, 1929; as cited in Renkl et al., 1996). In the vocabulary of Selden et al. (1999), the problem situation image of the problem solver becomes overly localized. The associated tentative solution starts do not apply to a wide variety of problems and cannot be leveraged in environments that do not overwhelmingly resemble the original problem situation. This aligns with existing research on differences between expert and novice problem solvers, which observes that experts' initial attempts to solve unfamiliar problems feature qualitative assessments that rely on the conditions of the problem and relationships between its elements; novices tend to apply quantitative and computational approaches (Silver, 1985).

If another issue with standard classroom pedagogical practices involves the dearth of instruction on metacognitive and heuristic strategies, one might assume that a reasonable correction would be to simply teach more metacognitive and heuristic strategies. Unfortunately, the situation may not be so simple. Research indicates that it is not sufficient for students to know that they *should* employ metacognitive and heuristic strategies; they must also be aware of *how* and *when* these strategies effectively and appropriately steer their thinking (Lester et al., 1989; Renkl et al., 1996). Furthermore, some authors argue, teaching problem solving strategies outside of a specific mathematical context is especially ineffective: "Teachers who expect their students to be metacognitive must insure that their students have something to be metacognitive about" (Lester et al., 1989; pg. 86). Schoenfeld (1985; as cited in Arcavi et al., 1998) refers to this observation as the *specificity problem* of teaching heuristics and notes that it has a converse: teaching heuristics that are perhaps too closely tied to the specific content being delivered renders them less useful in general settings, and the sheer number of heuristics students feel compelled to know skyrockets. Schoenfeld also notes other problems inherent to teaching heuristic strategies, one of which he identifies as the *resource problem*: to correctly gauge the suitability of a heuristic strategy often requires a particular mastery of the underlying content that makes delivery of heuristics in isolation unreasonable. If, as established above, standard educational practices over-emphasize algorithmic understanding at the expense of well-connected conceptual understanding, students may not have the robust resource base required to adequately appreciate the generality of heuristics even if taught in tandem with content.

How, then, can we overcome the above shortcomings of traditional classroom instruction? One possible approach involves adopting a situated learning model of general instruction, a type of content delivery that emphasizes the interdependence of context and knowledge application (Renkl et al., 1996), which is epitomized in a purely mathematical setting by Schoenfeld's problem solving course (described in detail in Arcavi et al., 1998). While there exist different interpretations of situated learning (cf. *anchored instruction* (Bransford et al., 2012); *cognitive flexibility theory* (Spiro et al., 1991); *reciprocal teaching* (Campione et al., 1988)), *cognitive apprenticeship* (Collins et al., 1988) most effectively captures the pedagogical principles Schoenfeld espouses in his research and exemplifies in his classroom. Collins et al. argue that three aspects of cognitive apprenticeship help students form transferrable rather than inert knowledge: first, the instructor should identify the *process* rather than the *outcome* of the task to be taught and make it visible, through modelling, to students; second, they should situate the task as much as possible in an authentic context rather than presenting it abstractly; and finally, they should draw parallels between the processes of multiple tasks to encourage transfer between them (1988). One can see clearly how Schoenfeld (as in Arcavi et al., 1998; Schoenfeld, 1987) epitomizes the first of these pedagogical tenets when he models his own problem solving behavior in front of his class, taking care to illustrate how he handles incorrect approaches by leveraging his expert knowledge of heuristics and control strategies. After introducing them, Schoenfeld further references these heuristic and control strategies repeatedly throughout the course both in situations where they are and where they are not appropriate, fulfilling the third criteria. Finally, with respect to the second criteria, Schoenfeld takes care to point out that working problems mathematically abstracted from a physical setting does not mean they are devoid of an authentic context; rather, that doing mathematics in “a microcosm of mathematical culture” (Schoenfeld, 1987; pg. 213) provides the authentic context. That is, Schoenfeld’s approach to problems in his course represents how mathematicians often actually do mathematics: out loud, collaboratively, and with no expectation that they can solve the problem quickly or easily.

Other researchers advocate for the of doing mathematics collaboratively and in the open. The effectiveness of solving difficult problems in a group setting, especially one in which an expert instructor oversees progress, has a sound theoretical basis. Vygotsky (1980) hypothesized that relationships between individuals, once internalized, become higher cognitive functions. For example, the ability of a problem solver to choose from among many different tentative solution starts (and later, to monitor the effectiveness of their eventual choice) is predicated on their ability to first select from and then monitor the many different approaches generated by members of their cohort in a collaborative effort to solve a difficult problem (Schoenfeld, 1987). Problem solving research positions group work as a key component of problem solving for other reasons as well. Silver notes that the necessity of communication in groups makes the metacognitive and control processes external and thus both easier to study (from a research perspective) and learn from (in the sense that others in the group have a model for traditionally invisible practices) (1985). Schoenfeld argues that students working in groups operate within their zone of proximal development (Vygotsky, 1980) and thus solve more challenging problems than they would alone (1987).

Some researchers have used empirical data to collate the above ideas into a framework for describing what kinds of classroom environments best promote the kinds of powerful thinking that improve problem solving performance. Liljedahl provides nine components of what he calls *thinking classrooms*, among which are recommendations for group work and an emphasis on process over product (especially in assessment settings, a recommendation shared by Silver (1985) and Collins et al., (1988)). Liljedahl also advocates for the implementation of vertical, non-permanent surfaces (that is, groups should work at whiteboards on the wall; see also a similar recommendation in Campione et al., 1988). He argues that the ability to easily erase their work encourages groups not to feel self-conscious of incorrect initial approaches and that vertical writing surfaces lead to collaboration among groups and a greater sense of mathematical community (2016). Besides recording the structure of his own classroom at great length, Schoenfeld also provided a framework for assessing the classrooms of others. He gives five components of what he calls *powerful classrooms*. The combination of the first two components (that the mathematics presented is deeply connected; that the cognitive demand on students is appropriate for their level) expand on Liljedahl's broad recommendation that students should only work on "good problem solving tasks" (2016; pg. 381). Most importantly among the remaining components is the stipulation that the students, rather than the teacher, are expected to conjecture and explain mathematical arguments; that is, the students have agency over the content, and this encourages them to identify with the mathematical community as doers of mathematics.

3.4 Summary

As described above, there are many extant MPS frameworks that can be used to describe students' problem solving strategies. Earlier frameworks dealt with linear, cognitive steps and associated heuristics (Pólya, 1945); soon after, frameworks began to integrate non-cognitive dimensions (Lester et al., 1989; Schoenfeld, 2014) and made explicit the fact that problem solving can (and often does) transpire through a series of nonlinear phases (Carlson & Bloom, 2005; Wilson et al., 1993). Further research generated frameworks for these phases specifically, such as Maciejewski's examination of the planning cycle (2019) or Stylianou's description of how visualizing affects both the orientation and planning phases (2002).

Because most modern frameworks contain non-cognitive (e.g., affective and metacognitive) components, the nature of such constructs benefit from precise characterization; this is doubly true in the face of historical difficulties surrounding their definition and measurement (DeBellis & Goldin, 2006). In the affective domain, emotions, attitudes, beliefs, and values have a variety of both positive and negative effects on problem solvers (cf. McLeod et al., 1989; Furinghetti & Morselli, 2009; Schoenfeld, 1989, 1987; Carlson, 1998). Not all of these effects are predictable, though, in particular because it is possible to hold many different (sometimes contradictory) beliefs (Callejo & Vila, 2009; Ruthven & Coe, 1994; Schoenfeld, 1989). Researchers often delineate metacognition into (roughly) two different types of thought: control systems and metacognitive knowledge (Schoenfeld, 1987; Flavell, 1979). Most novice problem solvers demonstrate little metacognitive control (Carlson, 1999), an important aspect of effective application of mathematical content and heuristics.

To teach the types of successful problem solving strategies observed in expert mathematicians, curriculum designers must first understand the shortcomings of traditional instruction. These include an emphasis on process rather than concept, no instruction on metacognitive or heuristic strategies, and lack of formative assessment and student-centered discussion (Campione et al., 1988). Aspects of cognitive apprenticeship (Collins et al., 1988) seem to do well in resolving some of these perceived issues (as seen in Arcavi et al., 1998). Instead of directly looking to address the problems of traditional instruction, some researchers have instead sought to categorize the positive qualities that they observe in successful classrooms (Liljedahl, 2016; Schoenfeld, 2014). Such qualities might include the integration of vertical non-permanent writing surfaces, observably randomized groups (Liljedahl, 2016), or discussions which place the impetus on students to generate and discover mathematics (Schoenfeld, 2014).

Chapter 3

Methodology

3.1 Setting and Participants

This study took place at a large, urban, public university (approximately 42,000 total enrollment) primarily during the Fall 2019 semester. The mathematics department in this university has a Ph.D. program and has granted an average of 11 doctoral degrees each year since 2016. In Fall 2019, there were 113 graduate students in the department (42% female, 58% male; 74% domestic, 26% international) and 264 declared undergraduate mathematics majors (44% female, 56% male; 89% domestic, 11% international). I sourced participants from both an upper-division undergraduate real analysis and a first-semester graduate linear algebra course. In choosing these courses, I aimed to capture data that presented a pseudo-longitudinal view of mathematics students who could personify the transition from a novice to an expert approach to mathematical problem solving. That is, their base of mathematical content knowledge was sufficiently comprehensive but, as detailed below, there still existed avenues for each population to develop as doers of mathematics.

I recruited undergraduate participants from an undergraduate real analysis course at the university. This is a proof-based course that is not used to introduce students to proofs, but does expect them to have prior experience reading and writing proofs; as such, I reasoned that a population of undergraduate students drawn from this class could be considered emerging in the sense that they would have tools with which to think critically about mathematical justification but may not yet have refined their critical thinking and heuristic strategies. Similarly, the population of graduate students taken from first-semester linear algebra have a more comprehensive baseline of mathematical experience. However, they may not yet have encountered the increased demands of graduate course work or had time to become encultured in the more academic socio-mathematical environment of graduate school (i.e., they would not yet have experience in research mathematics).

Eleven undergraduate and twenty graduate students completed the initial demographic survey and questionnaire, although one graduate student's responses to the questionnaire were disregarded because it was not clear that they correctly understood the instructions. Of the undergraduate students, all were enrolled in at least one other upper-division mathematics course at the time. Four indicated that they are planning on going to graduate school (four were not, and the remaining three were undecided). Undergraduate participants were almost entirely in the 18-22 age bracket; only two were aged 23-28, and one was older than 33. Of the graduate students, three were unique in that they were taking both graduate and undergraduate courses simultaneously as a means of leveling. Of the remaining graduate students enrolled only in graduate level courses, ten were taking both graduate real analysis and numerical analysis courses. This is a traditional first-semester course load for graduate students at this university. Another four graduate students were enrolled in at least one of these two courses, a fact which was largely predicated on the degree track they were seeking or if they were in leveling courses. Eight of the graduate participants held a master's degree. The graduate students also displayed a much more diverse range of ages: five were 18-22, five more were 23-28, six were 29-33, and the remaining three were older than 33.

Twelve graduate and seven undergraduate students indicated their interest in the interview sequence. Every undergraduate was invited, as were ten graduate students. Ultimately, eight graduate and five undergraduate students participated in the first interview, and all but one undergraduate student returned for the second interview. The majority of participants in the combined graduate and undergraduate population identified as female. One participant identified as nonbinary; in order to protect the anonymity of this participant, however, I refer to this participant using a gendered pseudonym and pronouns. Relevant demographic data for those students who took part in both interviews is presented in the tables below.

Table 3.1 Interviewed undergraduate demographic data.

Participant	Age Bracket	Graduate School?
Will	18-22	Yes
Riley	18-22	Yes
Brady	18-22	Undecided
Verna	33+	No

Table 3.2 Interviewed graduate demographic data.

Participant	Age Bracket	Degree(s) Held
Anne	18-22	BS (Mathematics)
Greg	23-28	BA (Mathematics)
Taylor	28-33	BS (Mathematics) AS (Computer Science)
Mia	18-22	BA (Mathematics) AS
Sara	18-22	BS (Mathematics)
Frank	33+	BS (Mathematics) BA (History)
Julie	18-22	BS (Mathematics)
Carol	23-28	MS

3.2 Procedures

I first administered the questionnaire and demographic survey (see Appendix A and Appendix B, respectively) to the eleven undergraduate students and twenty graduate students during the first ten minutes of their respective class periods within the first week of the Fall 2019 school year. The questionnaire contained nineteen Likert-scale (from 1, strongly disagree, to 6, strongly agree) items divided amongst three parts. I designed part one, comprised of eight items, to allow students to rate how highly they valued different potential aspects of mathematical problem solving (see section 3.3 for a detailed explanation of questionnaire design). Part two, a total of five items, gave participants the opportunity to indicate how much they felt certain personality traits (e.g. natural talent, persistence) contributed to success in mathematics. Part three, containing the final six items, prompted participants to choose how much they agreed with certain broad mathematical beliefs. See sample items below:

	In mathematics...						
1.	An answer is either right or it is wrong	1	2	3	4	5	6
2.	There is often only one correct approach to a particular problem	1	2	3	4	5	6

Figure 3.1 Sample questionnaire items from part three.

The intent of the questionnaire was two-fold: on one hand, I intended to use participant responses to help select interview participants that I anticipated might provide interesting or unique perspectives on the value of certain aspects of mathematical problem solving; on the other hand, I expected that participant responses to the questionnaire would allow me to better interpret their work on the interview tasks through the lenses provided by relevant research (for example, on the relationship between professed student beliefs and the manifestation (or lack thereof) of these beliefs in their work, as seen in Schoenfeld (1986)).

The demographic survey allowed participants to indicate whether or not they were interested in participating in a sequence of follow-up, video-recorded interviews that would allow me to further probe their responses to the questionnaire and observe their problem solving practices in-person. Of the eleven undergraduate students, eight indicated that they were open to participating in the interview process. Of the twenty graduate students, thirteen indicated the same. Interview invitations were sent to all eight undergraduates and ten of the thirteen graduate students; of these, five undergraduates and eight graduate students accepted the invitation and took part in the first interview. This invitation also made invitees aware that if they completed the entire two-interview sequence, they would be entered into a raffle to receive a $50 gift card.

Because of the nature of the courses from which the participants were sampled, I assumed that their problem solving strategies or beliefs about mathematics might change due to events over the course of the semester. For this reason, the first of the two anticipated interviews were scheduled as close to the beginning of the semester as possible, which for no participant was any later than the beginning of the third week of class. Both interviews were semi-structured and task-based; I oversaw participants as they completed a sequence of tasks in which they solved three non-standard problems (and a possible fourth problem, time and circumstances permitting. See section 3.3 for a detailed explanation of how tasks were chosen; see Appendix C for all interview tasks). Participants worked on each problem for as much time as they liked, except in circumstances wherein continued efforts to complete their current task would prevent them from seeing one of the three required tasks. Before the interview, I encouraged participants to think aloud as they worked, such as by explaining why they chose to make (or, in fact, not make) certain strategic decisions in their attempt to solve the problem. During the course of the interview, I prompted students to explain what they were currently thinking about (if they had been silent for a moderate stretch of time) and to elaborate on why or how they decided to make a particular strategic decision (if they did not volunteer a reason independently). During Summer 2019, I conducted a pilot study to explore the effectiveness of other prompts in eliciting meaningful responses from interview participants. After revising and expanding on fruitful lines of questioning, examples of interview questions used during this study include:

- What aspect of this problem do you find most interesting/ challenging? Why?
- What strategies, if any, did you consider using but ultimately rejected? Why?
- Give me a summary of your approach to this problem, explaining the rationale behind your decisions when appropriate.
- In what way have you chosen to solve this problem (like/ unlike) other mathematics problems you have worked in the past?

The first interview tasks are pictured in Figure 3.2 below. Tasks in both interviews (see also Figure 3.3 for second interview tasks) are identified in their respective Figures and referred to in this report using a label (e.g. LV-Pre, MV-Post) that was not visible to participants during interviews. The suffix of this label indicates whether the task was used in the first or second interview; the prefix refers to the amount of anticipated visualization (LV, MV, and HV stand for low, medium, and high visualization respectively; see section 3.3 for more details). LV-Pre is adapted from Vaderlind (2002, pg. 6); MV-Pre is adapted from Schoenfeld (1998, pg. 98); HV-Pre is adapted from Levasseur & Cuoco (2003, pg. 16); BP is adapted from Kahan & Wyberg (2003, pg. 27).

Problem 1.1 (LV-Pre). *The entire population of a small town has an 8:7 ratio of men to women. After 90 men and 80 women move to the town, the ratio of men to women is now 17:15. Assuming no other changes in population have occurred, what is the new population of the town?*

Problem 1.2 (MV-Pre). *What value(s) of α maximize the number of solutions to the following system of equations?*

$$x^2 - y^2 = 0$$
$$(x - \alpha)^2 + y^2 = 1$$

Problem 1.3 (HV-Pre). *Is it always possible to cut an arbitrary right triangle into two isosceles triangles? Explain why or why not.*

Problem 3.1 (BP). *Is it possible to use precisely 100 U.S. coins, none of which are nickels, in order to make precisely \$5? If so, how?*

Figure 3.2 First interview tasks.

After completing the first three tasks, some participants briefly attempted the fourthtask. Most, however, transitioned to the second part of the interview during which they elaborated on their responses to the questionnaire and answered additional questions about the development of their mathematical problem solving strategies. In order to structure this portion of the interview, I calculated the mean and standard deviation of each Likert item on the questionnaire and tabulated which of the participants' responses fell outside one standard deviation from the mean response. For this analysis, graduate and undergraduate responses were treated separately. During the interview, participants were then prompted to further elaborate on their answers to these items in particular; they were also given the opportunity to justify their response for any other item that they personally felt required explanation. Finally, participants answered questions about their mathematical development and trajectory, especially with regards to problem solving. These questions included:

- How has your approach to problem solving changed over time?

- Has any singular (class/ person/ event) significantly affected your approach to problem solving? How?

Based on previous interviews, foreknowledge of the participant, and time constraints, I sometimes also asked participants about their involvement in teaching or recreational mathematics and how those activities in particular may have affected their mathematical problem solving strategy development.

I contacted every participant from the first set of interviews near the end of the Fall 2019 semester to set up a time for their second interview. One undergraduate did not respond; the remaining four undergraduates and six of the eight graduates partook in the second interview during the last week of the semester or the week immediately after the semester concluded. The final two graduate students finished the interview sequence within the first week of the Spring 2020 semester instead due to scheduling difficulties. The format of the second interview was largely the same as the first, in that it was centered around the completion of three (sometimes four) non-traditional mathematics problems. Again, I asked participants to verbalize their thoughts as much as possible; when they did not, I used the same questions as during the first interview to elicit explication. The four tasks from this interview are pictured below. LV-Post is adapted from Vaderlind (2002, pg. 2); HV-Post is adapted from Schoenfeld (1998, pg. 98); BP is adapted from Kahan & Wyberg (2003, pg. 27).

Problem 2.1 (LV-Post). *At a particular boarding school, every student plays at least one of two sports: tennis or golf. A survey finds that $\frac{1}{7}$ of the tennis-players also play golf, while $\frac{1}{9}$ of the golf-players also play tennis. What fraction of the student body plays tennis?*

Problem 2.2 (MV-Post). *For which positive integer(s) α does the number of solutions to the following system of equations equal α?*

$$y = x$$
$$y = sin(\alpha x)$$

Problem 2.3 (HV-Post). *Is it always possible to inscribe a square in an arbitrary triangle? Explain why or why not.*

Problem 3.1 (BP). *Is it possible to use precisely 100 U.S. coins, none of which are nickels, in order to make precisely \$5? If so, how?*

Figure 3.3 Second interview tasks.

Note that BP, the optional fourth task, is the same as the optional task used in the first interview. This led to three different scenarios, the implementation of which depended on the participants' progress during the first interview. If participants completed the other three tasks during the second interview quickly and had time to attempt BP:

1. If they had not already seen BP during the first interview, they worked the task without further comment.
2. If they had seen but not completed BP during the first interview, they first reviewed their previous work and then continued/ revised this work to attempt to find any solution.
3. If they had both seen and also completed BP during the first interview, they first reviewed their previous work and then were challenged to expand/ adapt their approach to find a different solution.

The interview protocol questions differed slightly from those of the first interview to emphasize changes that might have occurred specifically during the intervening time since the first interview (bold text indicates wording changes):

- How has your approach to problem solving changed **in the past semester**?
- Has any singular (class/ person/ event) **in the past semester** significantly affected your approach to problem solving? How?

Another difference between the first and second interviews involved the generation of follow-up questions based on the questionnaire. After completing the three tasks, participants completed the same questionnaire that they took prior to the initial interview invitation. Instead of elaborating on responses that differed significantly from the mean, participants instead explained any responses that differed significantly from their original questionnaire responses (paying special attention to why this shift might have occurred). Answers that differed significantly included responses that changed by more than two points (e.g., from a 6 to a 4 or below) or those for which the participant's answer indicated a shift from agreeing to disagreeing with a questionnaire item (e.g., from a 3 to a 4 or vice-versa). I intended this approach to questioning to uncover any potentially influential events or experiences from the participants' most recent semester that affected their mathematical beliefs or self-reported problem solving strategies.

Near the end of the second interview, I also made a point to ask the following two questions:

- How would you describe your usual relationship with your mathematics teachers/ professors?
- Describe the particular qualities of your favorite math course that made it stand out.

Based on participant responses to questions posed in the first interview, I suspected that it might be difficult for participants to retroactively pinpoint specific experiences during their mathematics education that improved their mathematical beliefs or problem solving strategies. As such, these questions were intended to be easy for participants to answer in a meaningful way and still give me some insight into the mathematical environments that could have had a profound effect on their mathematical mindset and activities.

All interviews were video-recorded and selectively transcribed by the researcher (self-talk during problem solving was typically not transcribed, but direct answers to questions about problem solving development were transcribed entirely and verbatim). Quotes presented in later sections of this report have been lightly edited for clarity by removing speech disfluencies (e.g. “like”, “um”), except when doing so would alter the tone or meaning. Identifying information was removed from the survey and questionnaire, and all consenting students (including those who did not participate in the interviews) were assigned a pseudonym with which I associated their physical documents and interview recordings. The demographic survey, both questionnaires, as well as any physical artifacts (i.e., student work) produced during the interviews were scanned and uploaded to a secure data-hosting website. Physical copies of all items were stored in a locked office.

3.3 Interview and Questionnaire Design

When selecting items for part one of the questionnaire, I first identified a collection of aspects, such as drawing a diagram or checking one’s work, that represented a set of actions whose union I perceived as representative of the various cognitive processes described in published mathematical problem solving frameworks. In Table 3.3 below, I correspond each of these questionnaire items to at least one problem solving study in which the listed behavior is measured, observed, or otherwise described as having an effect on the subjects’ success in solving a non-traditional problem. For the rest of this report, when referring to a specific questionnaire item from part one, I will do so using the italicized label given in parentheses at the end of the item's text (see the first column of Table 3.3). These labels did not appear on the physical questionnaires given to participants.

Table 3.3 Justifying the inclusion of items from part one of the questionnaire.

When solving a difficult math problem, it is helpful to…	**Relevant Literature**
Read the problem carefully. (*Read Carefully)*	Schoenfeld (1987) Furinghetti & Morselli (2009) Pólya (1945)
Try and imagine what a solution might look like. (*Imagine Solution*)	Carlson & Bloom (2005) Maciejewski (2019)
Consider multiple different ways in which the problem might be solved. (*Consider Approaches*)	Schoenfeld (1987)
Plan the steps I will take in advance. (*Plan Steps*)	Wilson et al. (1993) Carlson & Bloom (2005) Pólya (1945)
Draw a diagram or sketch. (*Draw Diagram*)	Stylianou (2002) Álvarez et al. (2019) Pólya (1945)
Check my work at every step. (*Check Work*)	Lester et al. (1989) Garofalo & Lester (1985) Schoenfeld (2014) Pólya (1945)
Justify each step of my approach. (*Justify Work*)	Álvarez et al. (2019) Schoenfeld (2014) Pólya (1945)
Verify my answer at the end. (*Verify Solution*)	Wilson et al. (1993) Arcavi et al. (1998) Pólya (1945)

In part two of the questionnaire (see Table 3.4 below), I did not always choose to include a quality because there existed research indicating that it might influence mathematical problem solving ability. Rather, these were qualities that existing questionnaires (e.g. Schoenfeld, 1989; Perrenet & Taconis, 2009) had indicated problem solvers may believe are important for problem solving. For example, Schoenfeld (1989) does not claim that natural talent is a requirement for successfully solving non-traditional mathematics problems; but 230 high school participants who completed a mathematical beliefs questionnaire indicated that it was “sort of true” that their mathematical success was because they have “always [been] good at math” and that some people “just aren’t” (pgs. 351, 352). Similarly, novice undergraduates were more likely to agree with the statement that “Mathematics is 90% insight and 10% work” than undergraduates with more experience (Perrenet & Taconis, 2009; p. 186). Some of the qualities, though, were substantiated in other research as qualities that may contribute to problem solving success. For example, Carlson (1998, 1999) demonstrated that persistence in the face of difficulty is a strong indicator of success in solving difficult problems. For the rest of this report, I refer to items from part two using only the quality in question.

Table 3.4 Justifying the inclusion of items from part two of the questionnaire.

In general, success in mathematics requires…	**Relevant Literature**
Natural Talent	Schoenfeld (1989) Perrenet & Taconis (2009)
Hard Work	Schoenfeld (1989) Perrenet & Taconis (2009)
Creativity	Schoenfeld (1989) Perrenet & Taconis (2009) Callejo & Vila (2009)
Persistence	Perrenet & Taconis (2009) Carlson (1998, 1999)
Memorization	Schoenfeld (1989) Perrenet & Taconis (2009) Furinghetti & Morselli (2009)

Finally, in part three of the questionnaire, students were asked to indicate how much they agreed with statements of broad mathematical belief (see Table 3.5 below).

Again, most of these items were adapted from existing questionnaires (Schoenfeld, 1989; Perrenet & Taconis, 2009) and corroborated when possible with relevant results from problem solving literature. For example, Schoenfeld found that students largely disagreed with the statement that “Math problems can be done correctly in only one way” (1989; pg. 352); similarly, Perrenet and Taconis found that both novice and experienced undergraduates did not agree with the claim that “Most mathematical assignments can be solved in one way only” (2009; pg. 186). Just as in part one of the questionnaire, I refer to items from part three using an abbreviated label. These are found in parentheses after the text of the item in the first column of Table 3.5.

Table 3.5 Justifying the inclusion of items from part three of the questionnaire.

In mathematics…	**Relevant Literature**
An answer is either right or it is wrong. (*Right or Wrong*)	Schoenfeld (1989)
There is often only one correct approach to a particular problem. (*One Correct Approach*)	Schoenfeld (1989) Perrenet & Taconis (2009) Arcavi et al. (1998)
It is possible to discover how to do a problem you have never seen before. (*Discover Solution*)	Schoenfeld (1989) Perrenet & Taconis (2009)
I can solve any homework problem assigned to me if I do not give up. (*HW Confidence*)	Liljedahl (2005) Carlson (1998, 1999)
Getting an answer correct is often a matter of knowing the correct formula. (*Knowing Formula*)	Schoenfeld (1989) Perrenet & Taconis (2009) Furinghetti & Morselli (2009)
I often consider whether my solution was the best way to solve a given problem. (*Best Approach*)	Perrenet & Taconis (2009) Wilson et al. (1993)

Because the questionnaire items from both the second and third section were adapted from items featured in the previously validated mathematical beliefs questionnaires in Schoenfeld (1989) and Perrenet & Taconis (2009), it is worth addressing why I did not use these questionnaires in their entirety. The Schoenfeld (1989) questionnaire sought to identify differences between mathematical beliefs and beliefs of other subjects; thus, for every mathematics belief question there were corresponding beliefs questions for other subjects (e.g. respondents were prompted to rate how much they agree with "In mathematics something is either right or it's wrong" as well as "In English something is either right or it's wrong", (Schoenfeld, 1989; pg. 352)). It also focused more specifically on beliefs about geometric proofs and constructions (e.g. "Geometry constructions are fun to do" (Schoenfeld, 1989; pg. 353). Both these perspectives are beyond the scope of my study and, along with the inclusion of several short answer questions, would have made it difficult for participants to give this questionnaire due consideration in only the first ten minutes their classes. Perrent & Taconis' (2009) questionnaire was of appropriate length but geared towards applied mathematics. As such, it included questions that did not apply readily to mathematics in general (e.g. "Applied mathematics has available ready-made methods to solve mathematical problems from other technical disciplines", (Perrenet & Taconis, 2009; p. 186)).

I chose interview questions according to criteria described in other studies for the selection of non-traditional or non-routine problems (e.g. Carlson & Bloom, 2005; Calejo & Vila, 2009; DeFranco, 1996; Stylianou & Silver, 2004): first, I ensured that none of the questions required overly specific or advanced mathematical content knowledge; second, I avoided the inclusion of interview questions that had only a single reasonable method of solution; and third, I avoided selecting problems for which there existed a well-known or canonical method of approach. The goal of this selection process was to guarantee that the problems chosen were indeed non-traditional to as many of the interview participants as possible, in that they afforded the opportunity for relatively creative thought, multiple representations, and correct but different approaches. In order to better gauge the propensity of the participants to solve problems visually versus algebraically, I also chose each question to suggest variable levels of natural visualization. That is, it is difficult (but not impossible) to attempt to represent LV-Pre and LV-Post with a diagram or visual aide, but they are readily translated into algebraic expressions; here, LV stands for Low Visualization. On the other hand, HV-Pre and HV-Post are difficult (but not impossible) to do without drawing a picture of the situation before considering the assignment of variables. These are High Visualization problems. Splitting the difference, I perceived MV-Pre and MV-Post as amenable to both types of approaches, or as Medium Visualization tasks. Finally, because I foresaw participants being nervous about (or at the very least, unfamiliar with) the work-aloud protocol and the formal interview environment, I chose the first task of each interview (LV-Pre and LV-Post, respectively) to be less inherently challenging than the latter two. This was accomplished, in part, by selecting problems anchored to a real-world context; this allowed participants to more easily interpret the results of their work. I also chose to lead with these tasks because they were most susceptible to traditional algorithms with which participants might already be familiar.

As indicated above, the tasks from the second interview intentionally mirror certain aspects of the tasks from the first interview. In order to most accurately assess changes in problem solving strategies, I decided that it was important that, for example, both LV-Pre and LV-Post could be solved with proportional reasoning and the creation of a basic system of equations. MV-Pre and MV-Post provide an explicit system of equations to the problem solver, but with an additional parameter and whose solution does not represent the solution to the overall task. This makes MV-Pre and MV-Post less amenable to approaches that rely on the traditional algorithms for solving systems of equations. Finally, both HV-Pre and HV-Post involve geometric proofs featuring triangles in some important capacity. If a participant solved a pair of corresponding tasks from the two interviews in dramatically different ways, that presented an opportunity during the interview to discuss developments in their problem solving mindset or strategies.

3.4 Data Analysis

For the first part of the interviews, in which participants attempted to complete the interview tasks and provided simultaneous running commentary on their process, participants' problem solving behavior often manifested in their actions rather than their speech. This was especially true of those participants who were uncomfortable speaking aloud while they worked during the interview. In light of this difficulty, I created qualitative outlines of each participant's specific problem solving behavior during the tasks as opposed to explicit transcripts. Table 3.6 below captures the types of behaviors (actions or verbal admissions) that I observed and recorded during this phase of data analysis as well as the problem solving strategy they embodied.

Transcriptions of the later question-and-answer portions of the interviews, in which participants discussed their questionnaire responses and problem solving development, were coded using open and axial coding techniques (Strauss & Corbin, 1998) that aligned with techniques from thematic analysis (e.g. Braun & Clarke, 2006; Nowell et al., 2017). The first coding pass relied on preliminary codes generated by the questionnaire items and interview protocol. New codes were devised to capture emergent themes and to delineate subcategories of new and existing themes. For example, the *Conceptual v. Procedural* code arose when it became clear that many participants closely associated their development as problem solvers with the development of their mathematical content knowledge rather than due to any specific problem solving strategy or heuristic. In response, this code captures the belief, held by many participants, that their mathematical content knowledge has grown to reflect the increasing emphasis on justification and understanding that they have observed throughout their continued mathematics education. In light of new codes, previously coded transcripts were revisited to ensure consistency and completeness of thematic saturation. When necessary, codes were subsumed, split, or otherwise adjusted to account for shifts in perception and focus. Once I had independently generated a complete codebook (see Appendix E), my dissertation advisor coded an interview transcript for comparative purposes; that is, discrepancies between the two sets of codes were discussed and resolved until we had reached an agreement. Subtleties brought to light by this process were used to modify codes for all other interviews.

Note the inclusion in Table 3.6 of *Explore Setting*, an emergent type of problem solving strategy not originally present in part one of the questionnaire. During preliminary analysis, it became clear that the eight categories of problem solving strategy were unable to account for certain behaviors prevalent in the interviews. These behaviors typically involved participants' preliminary efforts to make sense of the setting (through hypothetical examinations, considerations of analogous problems, etc.) that preceded or supplemented their execution of a single approach. That is, *Explore Setting* encapsulates participants' efforts to understand the problem before attempting to solve it.

Table 3.6 Behaviors attributed to each problem solving strategy

Strategy	Observed Behaviors
Read Carefully	• Identifying important numerical data/ key words/ goal of problem • Underlining/ circling text • Organizing/ rewriting information from text • Defining important vocabulary and clarifying concepts
Explore Setting	• Test sample values of parameter/ variables • Delineate/ examine separate cases • Interpret the relevance of relationships and quantities described in task description • Manipulate/ recombine equations, variables, or numerical data
Imagine Solution	• Clarify what type of mathematical object the solution will be • Use the form of the solution to choose or evaluate the appropriateness of mathematical manipulations • Check work/ solution by comparing against properties of the anticipated solution
Consider Approaches	• Employing/ considering more than one way to approach the problem at hand • Comparing/ contrasting/ relating one approach to another (retroactively)
Plan Ahead	• Considering the effectiveness of a potential approach • Imagining the difficulties a particular approach might entail • Comparing/ contrasting/ relating one approach to another (proactively)
Draw Diagram	• Drawing a visualization • Describing a mental image • Using dynamic language or hand gestures to describe objects (or relationships between them)
Check Work	• Periodically reviewing work (unprompted) • Checking work because of some stimulus (incongruent conclusions, switching to new approach, etc.) • Revisiting task description to verify information
Justify Work	• Explaining the mathematical principals underlying their work • Explaining why their choice of method is appropriate • Explaining why they have decided NOT do something
Verify Answer	• Checking their work comprehensively after finding a solution • Corroborating their answer by examining how another approach would lead to the same solution

Chapter 4

Findings

This chapter is divided into graduate and undergraduate subsections, in which I provide a brief overview of the responses given by each total population (i.e., both interviewed and non-interviewed participants); the graduate and undergraduate subsections are then divided into further subsections devoted to each participant. I do not provide overviews for the second questionnaires given that the sample size of responses (only the interviewed participants) was necessarily much smaller. Instead, with respect to reporting on the second questionnaire responses, detailed discussion of significant changes in individual participant's opinions on questionnaire items are typically discussed in the corresponding subsection of the participant in question. Also in each participant's section, I describe the formative experiences they recounted in their interviews, their usage of different problem solving strategies while solving the interview tasks, and their mathematical affect (as evidenced through questionnaire responses and behavior during interviews). I conclude each participant's section in summary with a characterization of their problem solving identity.

Each subsection devoted to a specific participant contains a synthesized collation of several relevant codes from the respective participant's interviews. For example, in the subsection *Formative Experiences*, I first describe the experiences that may have contributed to the development of that participant as a problem solver and mathematical thinker. These introductory paragraphs are constructed primarily from the *Formative Experience*, *Favorite Class*, and *Last Semester* codes and their subcodes; when appropriate, interview excerpts from other codes are used to provide context or explanation.

Next, in *Problem Solving Strategies*, I draw attention to specific problem solving behaviors (or the lack thereof) from each participant's work during the task-focused portion of the interviews. This subsection is organized by addressing, in order, the strategies found in part one of the questionnaire (and including also the emergent strategy, *Explore Setting*, between *Read Carefully* and *Imagine Solution*). This section also includes that participant's questionnaire responses with respect to each problem solving strategy and any accompanying context they used to explain their responses (or their change in responses between questionnaires). Immediately after examining their questionnaire responses, I describe each participant's problem solving behavior with respect to the strategy in question. These vignettes should not be interpreted as a comprehensive recounting of every problem solving behavior, but rather as a representative sample of behaviors that characterized the participants' overall application of that strategy.

The next subsection, titled *Mathematical Affect*, describes the emotions, attitudes, and beliefs that each participant expressed during the interviews. When relevant, specific questionnaire responses (from part two or part three) supplement the qualitative reporting on participants' statements and behaviors from the interviews; not every questionnaire response from parts two and three is addressed in this subsection. Finally, in *Problem Solving Characterization*, I synthesize the observations from each subsection in order to generate a broad-strokes overview of how each participant appears to handle problem solving situations.

4.1 Graduate Students

Figure 4.1 below includes eight diagrams, each one representing the distribution of responses to the eight questionnaire items from part one. The horizontal axis contains the possible Likert scale responses and the vertical axis measures frequency. Each bar is color coded according to the following key:

- Orange bars are more than one standard deviation below the mean response.
- Green bars are more than one standard deviation above the mean response.
- Blue bars are within one standard deviation of the mean response.
- The darker portion of each bar represents responses from interviewed participants.
- The lighter portion of each bar represents responses from non-interviewed participants.

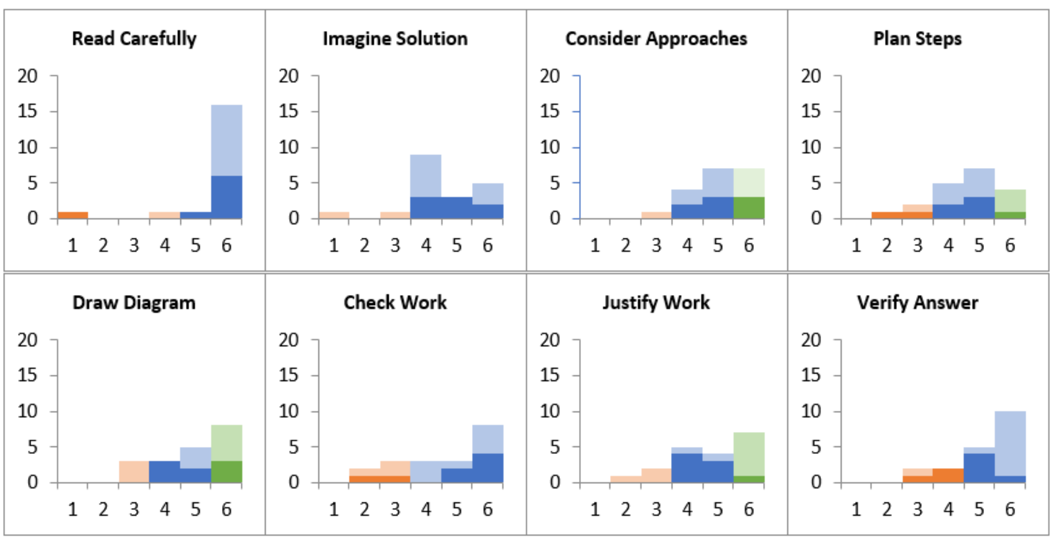


Figure 4.1 Graduate responses to part one of the first questionnaire

Among the items on this first portion of the questionnaire, *Check Work* had the highest standard deviation in responses and *Consider Approaches* had the lowest (although, it should be noted that *Read Carefully* would have a much lower standard deviation if Greg's strong disagreement was not such an extreme outlier). The only individual item with a noticeable increasing or decreasing trend between the first and second questionnaire was *Verify Answer*, for which every graduate student's opinion increased.

Figure 4.2 and Figure 4.3 below follow the same color-coding scheme as Figure 4.1 (pg. 61) and present the total graduate responses to parts two and three of the first questionnaire.

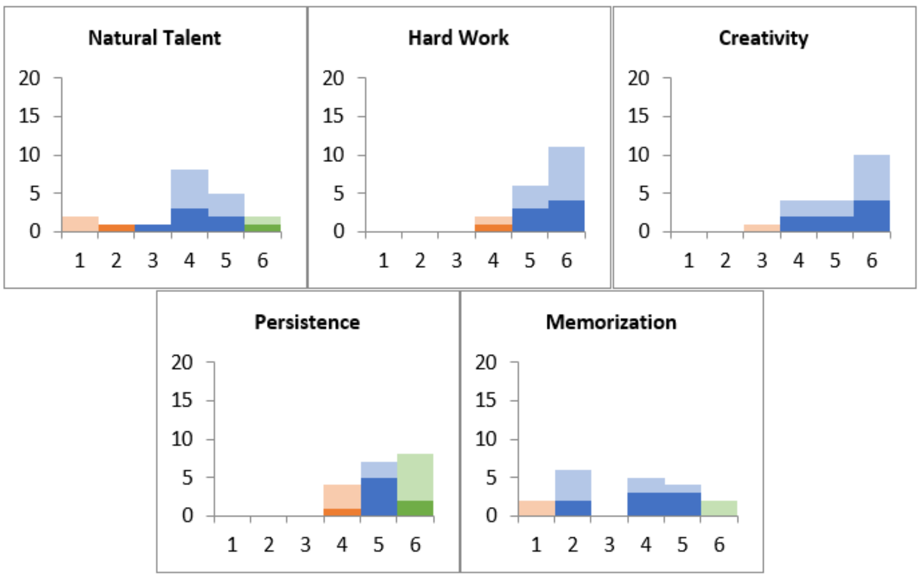


Figure 4.2 Graduate responses to part two of the first questionnaire

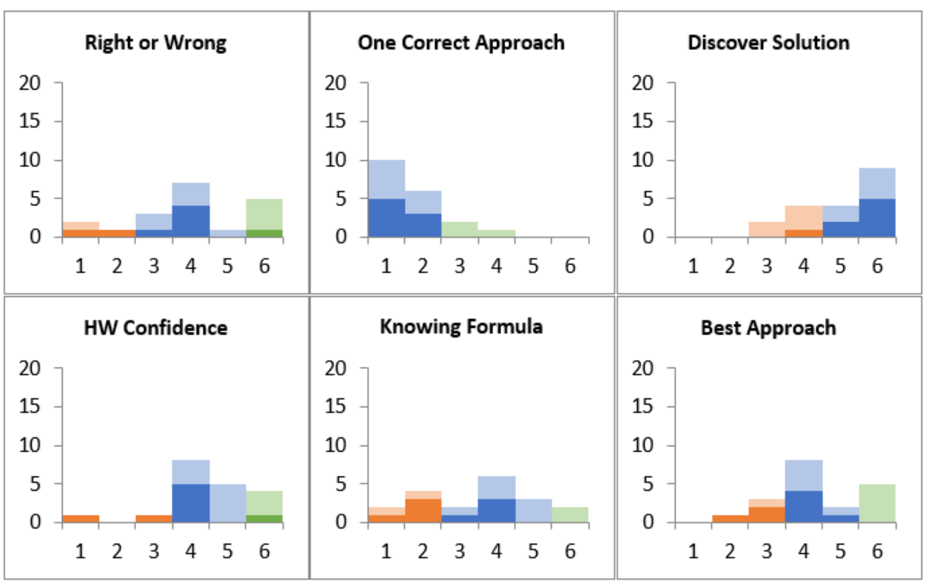


Figure 4.3 Graduate responses to part three of the first questionnaire

The item with the largest standard deviation from part two was *Memorization*; the item with the smallest standard deviation was *Hard Work*. Similarly, from part three, the items with the largest and smallest standard deviations were *Right or Wrong* and *One Correct Approach*, respectively.

4.1.1 Anne

4.1.1.1 Formative Experiences

During the first interview, Anne began by describing the way in which her work as a tutor affected her own problem solving strategies. In particular, she found that teaching students “things to help them figure out what to do on a problem when they see it, like, help me to do that.” She qualified this statement by admitting that, now that she is primarily taking proof-based courses, the strategies she teaches her students are less likely to apply to her own work. The primary advice Anne gives her students that she also felt helped her in proof writing had to do with maintaining good study habits; such habits include utilizing flash cards and redoing homework problems. Anne herself developed these study habits in response to advice from a calculus professor in whose class she had struggled. Anne also recalled the effect of her undergraduate linear algebra course on her problem solving. This course marked her first experience with proof-writing, which showed her that she needed to pay closer attention to *why* a certain approach to solving a problem worked, and not just that it did. Teaching only reinforced this outlook on understanding mathematics; generating meaningful problems for her students “would make me have to know the skill well enough to be able to give them a problem.” Anne also appreciated the chance, through teaching mathematics, to become reacquainted with areas of the subject that she had only learned at a cursory level up until that point.

During the second interview, Anne reflected on her recent graduate linear algebra class; she felt this course had affected the way she approached problems. In particular, she found that she often ended up starting problems on the wrong foot. She began to, “whenever I like get a problem, write down—like, actually like write down what I'm supposed to be doing,” hopefully to avoid spending time on fruitless approaches. The instructor of this course reassured Anne by explaining that, often, “the hardest part of the question is approaching it.” They also helped Anne improve her study habits, something that she explained she valued highly in the first interview.

Finally, Anne also described her favorite mathematics course: her secondary school calculus course, which was a flipped classroom. Anne valued that, in such a classroom, the instructor was immediately on-hand in the classroom to answer questions she had about the problems on which she was working (compared to in a traditional classroom, wherein homework was traditionally independently completed at home). Anne compared this to the same linear algebra course, in which she felt that she

> struggled because I don't know how to do the questions. We didn't do a lot of like *actual* examples. Like, we do a lot of examples, but it's not—I like seeing exactly how to do a question. And then I can see how we did that and then apply it to other questions. And so, a flipped classroom, you spend like a whole class doing questions.

Besides having the instructor nearby, Anne also appreciated that her calculus course forced her to work with partners—even on exams. This created an environment of both friendly competition and responsibility to one’s peers that, in combination, promoted mathematical thinking.

4.1.1.2 Problem Solving Strategies

Anne reported that she strongly agreed with the statement that reading carefully is helpful for solving difficult mathematics problems in both questionnaires. In LV-Pre, Anne rewrote the numerical data from the task description in the space provided for working the task. In the corresponding task from the second interview, LV-Post, she repeated this procedure but took additional time to label and assign appropriate variable names to the data. At one point while solving LV-Post, she revisited the task description to clarify her image of what form the solution should take (and discovers that it should be a fraction, not an absolute magnitude). Anne's first written transcriptions on every other problem from either interview were the first steps of the procedure she had chosen in order to solve the problem. The above example from LV-Post was also the only instance in which she clearly revisited the task description.

When asked by the interviewer what she looks for when reading a problem for the first time, Anne said that she looks for information that she identifies as "important." As an example, she indicated that the ratios given in LV-Pre are important. She does not, however, elaborate on what constitutes "important" information in general. Overall, Anne appeared to equate "reading the problem carefully" with identifying and isolating numerical data points explicitly given in the task description. When no such data was provided (e.g. in MV-Pre, HV-Pre, MV-Post, and HV-Post), it is not clear how closely Anne attended to the task description. In at least one task (MV-Post), she misinterpreted a fundamental aspect of the task (which values of α are solutions) that could have been clarified by closer attention to the text.

As noted above, Anne did not typically make any preliminary explorations of the problem space on paper before setting out on what she hoped would be the correct approach. When possible, she appeared to test certain sample values in her head or on a calculator to get a feel for the task. For example, when solving BP, Anne commented that she was mentally checking different amounts of coins before committing any of her work to the page. She demonstrated a similar reluctance in LV-Pre, wherein she spent a significant amount of her initial time with the task making computations in her head or on a calculator. In contrast, on LV-Post, Anne's inability to assign variable names to or organize the given information also prevented her from making exploratory computations.

Anne did not explore the setting of tasks in which she could remember the exact method, since there was no need. This behavior could be observed in her solution to HV-Pre, which she produced immediately upon reading the task description (Figure 4.4, below).

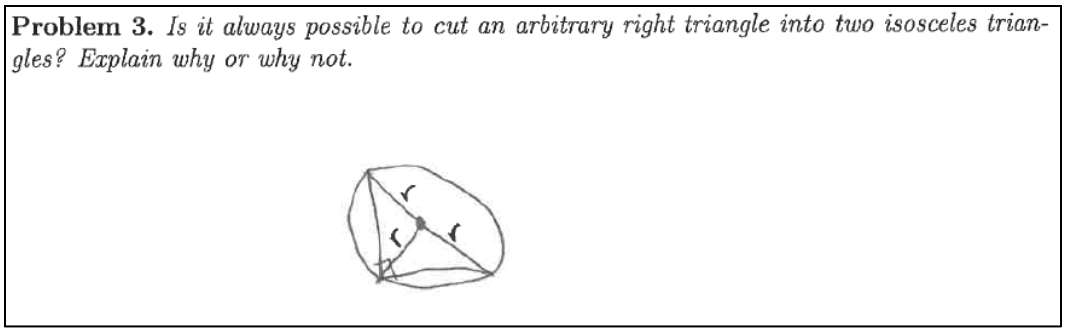

Problem 3. *Is it always possible to cut an arbitrary right triangle into two isosceles triangles? Explain why or why not.*

Figure 4.4 The entirety of Anne's work for HV-Pre.

Compare this to her work on HV-Post, for which Anne did not immediately remember the outline of a canonical proof. There, she began by drawing an inscribed square both to verify with the interviewer that her understanding of the task description was correct and to find at least one case in which the proposition was true. She then explored different cases by continuing to draw follow-up examples of different triangles. This allowed her to eventually draw an empirical conclusion about whether the statement in question was true or not.

Anne strongly agreed with the statement that imagining what a possible solution might look like is helpful when solving a difficult mathematics problem on both questionnaires. At the end of the first interview, Anne explained why she valued this aspect of problem solving more than other participants. To do so, she first clarified that certain "critical thinking skills" are the most important aspect of successful problem solving; she then listed "knowing what your end goal is and trying to figure out how to get there" as a critical thinking skill. She contrasted this against the less successful practice of trying to pick the correct approach without first considering what the goal of the problem may be.

The interviewer asked Anne what a solution to LV-Pre might look like while she worked on that task; in response, Anne observed that the final population should be relatively greater than the initial population. While correct, this description did not later have any effect on how Anne approached the task. She did, however, use the fact that the population should be a whole number later in while using the *Check Work* strategy. In LV-Post, Anne initially expected her answer to be the explicit population of tennis players. As previously mentioned, she revisited the task description to correct this interpretation and conclude that her solution should actually be a fraction. Despite acknowledging verbally that her solution should be a fraction, Anne repeated several times while working the task that she needed (or at least, expected) to be able to find the explicit population of students who play golf.

In MV-Post, Anne made an incorrect statement about the form of the solution that she did not later correct: she indicated that the value of α could be anything on the positive real line and supplied π and 2π as example values. The task description (where α is declared to be a positive integer) and the circumstances of the task (α is also the number of solutions to a system) would have both refuted Anne's interpretation of α's domain. Here, Anne appeared motivated by the fact that answers to previous trigonometry problems with which she was familiar typically had solutions that were scalar multiples of π. While working MV-Pre, Anne more explicitly referred to a previous problem that she taught in her calculus lab. This problem also featured the key word "maximize," and ultimately resulted a pair of solutions; Anne used this memory to claim that there would be at most two solutions to MV-Pre. She amended her image, noting that the task at hand might have up to four solutions because both x and y are squared; in the problem she remembered teaching, there was only one quadratic variable.

At first, Anne only slightly agreed with the statement that considering multiple approaches is helpful when solving difficult problems. She commented on this opinion, which was lower than other participant responses, when prompted by the interviewer at the end of the first interview. Anne explained that she did not feel compelled to consider multiple approaches when she encounters a problem that she recognizes can be solved with a familiar method; on the other hand, she might consider multiple approaches only if she cannot remember the correct approach that she had already learned. On the second questionnaire, Anne came to strongly agree that considering multiple approaches can be helpful. She explained this change in opinion by recalling an exam from her graduate linear algebra class from the previous semester during which, after an initial approach proved unhelpful for solving a hard problem, Anne was able to remember an alternative approach that led to a satisfying solution. Furthermore, she felt that being open to considering multiple approaches had also benefitted her in an undergraduate abstract algebra class that she was enrolled in that was simultaneously taught by the same instructor.

During the interviews, Anne occasionally mentioned alternative methods for solving certain tasks besides the one that she would eventually choose to employ. For example, in MV-Pre she noted that one of the provided equations could be represented visually by a circle. Despite this observation, she made no effort to utilize such a visualization; instead, she attempted to solve the task algebraically. Similarly, in LV-Pre, Anne briefly considered a Venn diagram but elected instead to try and write down a system of equations. Finally, in MV-Post, Anne drew a basic sinusoidal curve first before stopping to consider how she might solve the system of equations algebraically. Here, like in previous tasks, she did not actually implement this secondary consideration until prompted by the interviewer. In all other tasks, Anne did not indicate that she had more than one approach in mind.

Anne's work in the interviews is evidence that she sometimes considered multiple approaches. However, it also indicates that she preferred not to shift to a different approach unless she was confident that doing so would lead to meaningful results. That is, she did not use alternative approaches as a way to explore the setting of a task even when her initial strategy had proven ineffective. Anne's ability to seriously consider multiple approaches was also limited by her reliance on comfortable rather than mathematically appropriate strategies. When Anne foresaw that her first, familiar approach would not pan out in a meaningful way, she was sometimes left at a standstill.

Anne agreed with the statement that planning one's steps in advance is helpful for solving problems on the first questionnaire, and only slightly adjusted this opinion (to strongly agree) on the second questionnaire. There was some evidence in the interviews that Anne planned ahead when she rejected certain approaches. For example, in MV-Pre she initially decided not to solve the system by traditional algebraic means when she noted that "it's gonna be difficult to get alpha by itself." Similarly, in MV-Post, she again rejected her plan to solve for α algebraically when it did not appear to be helpful. In this same task, though, there was also evidence that Anne sometimes did not plan ahead: before rejecting her algebraic approach, Anne had originally drawn the first period of the sine function. When the interviewer asked how she planned to use her diagram, however, she did not answer at all.

Anne only slightly agreed, on both questionnaires, that drawing a diagram or sketch of the situation might help her to solve a difficult mathematics problem. Of this strategy, she explained that "I feel like I don't take the time to do it unless I really think it will help me." She clarified that this was mostly for geometry problems and when visualizing trigonometric identities on right triangles. Near the end of the first interview, Anne also added that she sometimes does not choose to produce a diagram because she is "afraid that my graph is going to be wrong. And then I'm going to get confused."

In line with her comments, Anne did use visualizations on those interview tasks with an overt geometric emphasis (e.g., HV-Pre and HV-Post) or when the task involved a trigonometric function (MV-Post). In HV-Pre, because Anne recalled exactly the steps required to answer the question from a previous geometry class, the one diagram she produced simultaneously represented both her entire approach and her solution. She added, however, that she "would probably have to use, like, the properties of a circle and triangles to actually prove it." She was convinced by her diagram because she reproduced it from memory, but she did not find it mathematically sufficient to prove the proposition. In HV-Post, where she did not immediately know the correct diagram to draw, Anne produced multiple diagrams that were much more exploratory. Finally, in MV-Post, Anne drew a perfunctory sinusoidal curve but was unable to use it to move forward in answering the question at hand. Near the end of the task, after being prompted by the interviewer, Anne added the line $y = x$ to the same plane as her sinusoidal curve; this move, while potentially helpful, still did not provide her with an actionable strategy. In two other tasks, MV-Pre and LV-Post, Anne mentioned a possible visual approach (graphing the equations and drawing a Venn diagram, respectively) but did not act on them.

In both interviews, Anne strongly agreed with the statement that checking your work at every step is helpful when solving a difficult mathematics problem. When solving the interview questions, though, Anne did not often generate enough work that warranted checking. The exception to this claim was LV-Pre, in which she did review the reasonability of certain interstitial numerical values in her approach. These brief monitoring episodes were tied to Anne's use of *Imagine Solution*, in that she checked her computations for whether or not they made sense in the real-world context of the problem setting; for example, at one point Anne computed the quotient of 90 and 17. She then remarked on her discomfort with the fact that this was not an integer. Throughout her work on this task, she decided to round this quotient and other various non-integers up or down to some whole number according to what "makes more sense" to her. Anne's recognition that non-integer values in this task were an issue did not prompt her to reconsider her overall approach.

Anne initially somewhat agreed with the claim that justifying each step of an approach is helpful when solving a difficult mathematics problem; she would later strongly agree in the second questionnaire. By way of explanation, Anne recounted that she has, in the last semester, started rewriting homework problems to study for a test because it helps her to be able to remember what to do on the actual exam. While she was not clear whether this included writing down the justification for each step of her homework, her explanation of this change implies that she only thinks about the justification for an approach after it is already known to lead to a correct answer and not as a way to navigate the problem space for the first time.

Anne's mathematical justification while working interview tasks was often vague or imprecise. For example, while solving HV-Post, she suggested that the height of any square inscribed in a right triangle will be half the height of the triangle itself. She claimed that she had learned some theorem to this effect in the past but could not, at that moment, remember the specifics. Throughout the rest of her work in HV-Post, Anne continued to make reference to this theorem despite numerical evidence that several of her alleged squares were in fact rectangles. In other tasks, Anne similarly relied on her memory of particular processes or theorems to justify her approaches. Even when her reliance on memory gave her an ostensibly correct approach (as in HV-Pre, see Figure 4.4 above), any additional mathematical justification beyond her memory of the method being correct was vague or nonexistent.

In the first questionnaire, Anne somewhat agreed with the statement that verifying one's final answer is helpful when solving a difficult mathematics problem. In the second interview, she changed her answer to strong agreement. She explained this change by describing the need to teach her calculus students ways to check their work. In her own classes, Anne also recalled at least one exam question wherein she was able to avoid an incorrect answer by carefully rereading the task description after noticing her solution did not meet her expectations.

Anne did not produce enough complete solutions to accurately ascertain how often she might verify them; she only reached a verifiable solution on LV-Pre. In this task, after arriving at a value for the final population, she subtracted the number of people who moved to the town to find an initial population. This initial population value was close to what she thought the initial population might have been based on some earlier computations, which satisfied her. Notably, she does not at any point check to see if the male and female populations that she calculated satisfy the ratios given in the task description; they did not.

4.1.1.3 Mathematical Affect

In the first interview, Anne claimed that she has always known that she was good at math. In the second interview, however, she admitted that her secondary school calculus class was her favorite math course in part because it "made me realize I was actually good at math." She resolved this discrepancy by explaining that, while she had always gotten good grades in mathematics throughout primary school, the group work activities in her secondary school calculus class showed her that she was not just a competent mathematician in isolation; compared to her peers, whom she perceived as especially intelligent, Anne held her own and contributed on assignments.

Anne felt, up until her experiences in an undergraduate linear algebra class, that most of mathematics work could be accomplished through the application of known formulas. In linear algebra, however, her belief shifted: she then realized proof was an integral part of mathematics wherein one must provide mathematical justification for procedures or claims. Anne noted that this perspective bled into other mathematics classes and her work as a mathematics tutor, where she explained that she often chooses to emphasize why things work the way they do. During the first interview, Anne expressed another interesting belief about meta-affective control: namely, that she considered the ability to think about solving problems without "getting flustered" as something that she considers a problem solving skill.

On the questionnaire items from part two and three, Anne's responses were almost entirely in-line with other graduate students; furthermore, her opinion did not noticeably shift on any item between the first and second questionnaire. Anne's only meaningfully different opinion on these items was her claim that she somewhat disagreed with the statement that homework problems are always possible if she does not give up. She explained that, when stuck on a difficult problem, she is not opposed to consulting with an outside source to help her overcome her current roadblock. Anne added that "I don't necessarily say, like—think of it as I'm giving up, but I'm giving up on doing it on my own."

4.1.1.4 Problem Solving Characterization

Overall, Anne relied on her memory of similar problems to plan, justify, and execute approaches to non-traditional problems. This is not surprising, given that during the interviews, she made several references to the fact that memorizing homework problems through repetition is how she studies for exams in especially difficult mathematical courses. This reliance on memorization culminated in her apparent belief that certain tasks have specific tricks or methods that one must remember in order to solve them: “I feel like this problem... there's probably like a certain method that's the best way to approach this, and if you don't know it—or if you don't remember it—you’re kind of out of luck.”

Anne’s justifications which were not based on memorized properties or procedures were mathematically insubstantial or built shallowly on surface features of the task. She correctly recognized that many of these tenuous justifications were not appropriate; however, this often left her with no actionable strategy for approaching tasks that she did not remember. Anne recognized that it was important not to be flustered in these situations, but her usually strategy for avoiding negative emotions appeared to be consulting a mathematical authority (such as correct homework problems, an instructor, or the internet). With no such resources available to her during interviews, she made little headway once any initial idea proved insufficient. This issue was compounded by her reluctance to explore the problem space on paper and her limited, perfunctory use of diagrams.

4.1.2 Greg

4.1.2.1 Formative Experiences

Greg stressed, in both interviews, that he did not think any single mathematical experience he had had in his formal education (or anywhere else) was especially influential on his problem solving technique. He also asserted that he did not perceive any meaningful differences between his undergraduate and graduate experiences in mathematics. Instead, he attributed his improvement in problem solving over time to having “more tools”; that is, “learning new techniques of proof or approaches that change how you look at problems in the future." Correspondingly, this was the aspect of Greg's favorite math class that stood out most to him; this course, called "special topics in mathematics," was organized in Greg's secondary school in order to create a meaningful mathematical experience for those students who had finished the traditional calculus sequence especially early (Greg enrolled in the course as a junior and again his senior year). He explained that

> the professor, or the teacher, for that did a fairly good job at explaining different methods of proof and logic. And then how we apply them. I think that that gave me a leg up going into college, because I understood how things should look going forward.

In particular, the first year of the course was dedicated to introductory number theory and helped Greg to appreciate a conceptual side of mathematics not traditionally included in secondary school curricula.

4.1.2.2 Problem Solving Strategies

In the first questionnaire, Greg strongly disagreed with the statement that reading a difficult problem carefully would be helpful when solving it. In filling out the second questionnaire, he indicated that he now agreed with this statement. When the interviewer asked him to address this change, Greg observed that "nothing particularly traumatizing" had "caused a total shift in my way of doing things." He conceded that he simply might not have given careful thought to the question when he answered it on the initial questionnaire; ultimately, he modified his second questionnaire response to indicate that he probably only slightly agreed with the statement. Furthermore, he felt that this was probably what she *should* have answered on the first questionnaire as well.

Greg's work on any given task from either interview began very quickly after his initial reading of the task description. Only in HV-Pre and HV-Post did he stop to clarify some aspect of the text when he asked the interviewer to remind him of the formal definitions of an isosceles triangle and what it means to inscribe a shape, respectively. In every other task, he transitioned from reading the text to working almost immediately. Neither did Greg appear to revisit the task description after his initial reading, even when he was changing approaches or resolving a perceived error in his work. As the exception that proves the rule, he briefly verified that he had pulled the correct numerical data from LV-Pre and LV-Post. After completing LV-Pre, Greg explained to the interviewer that he does not think he looks for anything specific when reading “a problem like this” and that he is pretty confident in whatever understanding of the problem space he develops from his first reading.

In the course of working those tasks for which he did not create a visualization, Greg very seldomly explored the setting. That is, for LV-Pre, LV-Post, and MV-Pre, he very quickly perceived and executed on what he considered a “standard” approach to the task (either constructing an equivalent ratio or a system of equations). Even in LV-Pre, for which arithmetic errors initially stymied Greg’s first approach, he did not appear to second-guess his instinct that the task could be solved using the careful application of a familiar algebraic methodology.

When starting HV-Pre, on the other hand, Greg noted that his “intuition […] is occasionally just… draw some picture.” He tried to cut an isosceles right triangle into two smaller isosceles triangles, saw that this was possible, and concluded that the statement of the task was probably true within the first few seconds of work. He would add, after completing the task, that he suspected the statement was true even while reading the text for the first time, but that he could not immediately provide sufficient justification for this claim. The rest of Greg’s work in this task was generated under the assumption that the statement he was meant to prove was indeed true; that is, while he continued to draw and manipulate various triangles in an exploratory manner, this activity was largely in service of uncovering reasonable justification for this claim. Greg’s exploration in HV-Pre, and the sister task HV-Post, revolved around establishing *how* he could do something, not *whether* he could do something.

This description also characterizes Greg's work on MV-Post. There, he drew a preliminary sketch and extrapolated the entire, correct solution within four minutes of beginning the task. The remaining twenty minutes of his work might be considered exploratory justification rather than exploratory sense-making. Greg appeared to be searching for a method of sufficiently supporting a conclusion that he felt confident was already correct but that he had not yet formally substantiated with mathematical logic.

Greg first agreed with the statement that imagining what a solution might look like could be a helpful strategy when solving difficult mathematics problems. In the second interview, he only slightly agreed. In almost none of Greg's work did he appear to lean heavily on his perception of what he imagined the solution might look like. In a discussion with the interviewer following LV-Pre, Greg even acknowledged that he does not normally try to imagine the answer before beginning a problem. Ironically, it was also in LV-Pre where the only instantiation of the *Imagine Solution* strategy manifested in Greg's work: during his initial approach, he realized that he was going to get a negative population that would not make sense in the real-world context of the task's setting. This caused him to re-evaluate his work and, when he could not see his error, to switch methods. In every other task, it does not appear as though Greg's choice of approach or understanding of the problem space is influenced by his perception of what a solution might look like.

Whereas Greg strongly agreed with the statement on the first questionnaire that considering multiple approaches might be helpful when solving a difficult mathematics problem, on the second questionnaire, he indicated that he now only slightly agreed. Greg expanded on his initially high opinion of this strategy during the first interview: he explained that, when working a problem, he often considers several possible first steps which he might take to begin a problem. After choosing one, he then considers an array of next steps that might be most profitable, and so on until he finds a reasonable solution. That is, “instead of using a depth-first search I'm using a breadth-first search.” See his accompanying diagram in Figure 4.5 below. Furthermore, because he does “tend to get pretty impatient,” Greg added that “I've got in the back of my mind these other approaches, so it's not too hard to shift gears and try one of those instead” when his initial choice of approach doesn’t pan out. During the second interview, Greg attributed his change in opinion (with respect to this strategy) only to his own perception that there is not a significant difference between strong and slight agreement with a statement.

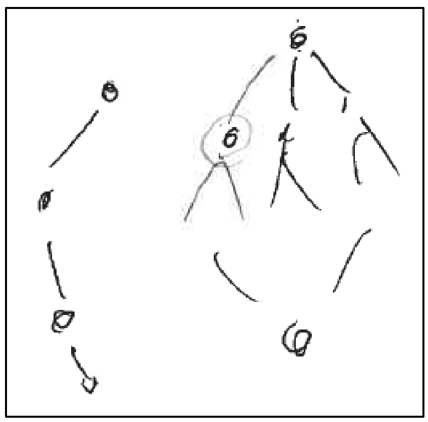


Figure 4.5 Greg's tree diagrams, which he used to compare different ways in which one might *Consider Approaches*.

As demonstrated in preceding paragraphs, Greg’s work is characterized by decisive action; when a task could be solved using what he deemed a traditional algebraic approach, he did not consider other methods but instead applied this approach. At best, Greg navigated between two traditional approaches when his initial attempt was thwarted in some way; this happened in LV-Pre, when Greg switched from using equivalent ratios with an unknown constant of proportionality to instead considering a system of equations after failing to locate an arithmetic error. In other tasks, such as HV-

Post and MV-Post, Greg also attempted a number of different approaches. In these tasks, he switched between approaches when he was not able to generate sufficiently precise mathematical justification for his current method. For example, in MV-Post, he first attempted to translate his graphical intuition of the task into a meaningful algebraic inequality that might bound the number of viable α values. When this inequality did not lead to a justification that could be easily generalized to handle any value of α, he attempted an analytic approach to the system of equations using Taylor series expansions. In no task from either interview did Greg return to an approach that he had already tried, either for a second attempt or to rationalize why there were similarities and differences between approaches (unless, in the latter case, the interviewer requested this of him).

In both questionnaires, Greg slightly disagreed with the statement that planning one's steps in advance could be helpful when solving a difficult mathematics problem. This was a lower opinion than the majority of the other participants held of this strategy, so during the first interview, Greg elaborated. He said that

> I find that my math tends to be a bit more free-flowy than that. I might have an overall picture of what I'm going to do. What the end result is going to look like. But along the way, I don't have a particular path mapped out, really ever. I think that I'll start with like, general rules that I know and move from there.

This perspective aligns with the tree diagram Greg would later draw with respect to *Consider Approaches*, in which he explained that he seldom plans an approach all the way through to its logical conclusion; instead, he chooses each step of his approach as it becomes necessary to do so in the problem solving process.

During interviews, Greg only admitted to using the *Plan Steps* strategy in retrospect. After ending his work on MV-Post, he mentioned that he had originally considered solving the task by manipulating the given system of equations. This was the strategy he had previously used for all of LV-Pre, LV-Post, and MV-Pre. However, in the circumstances of MV-Post, he did not think it would pan out; instead, he only resorted to this approach after he was unable to sufficiently justify his conclusion only by examining the graphs of the functions in question. In other tasks, even those for which Greg eventually attempted more than one approach, he did not appear to spend any significant amount of time selecting from more than one possible strategy either at the outset of the task or when he switched between two methods.

Greg initially expressed slight agreement with the statement that drawing diagrams could be helpful when solving a difficult mathematics problem; during the second questionnaire, however, he indicated that he now slightly disagreed with the same statement. While he did not provide justification for this shift, he did elaborate on exactly when and how frequently he tends to draw a diagram: after solving HV-Pre, Greg told the interviewer that he tends only to draw a diagram for certain “sketchable” geometry problems, and then only to get an initial intuition of what the situation might look like; later, after solving MV-Pre, he added that he only “very very rarely” considers graphing equations of functions.

These characterizations of Greg’s visualizing propensities were certainly true for the first interview, in which he only drew a figure of any kind in HV-Pre. However, in the second interview, he drew at least one diagram for each of the three tasks. Greg could not explain this change, which was seemingly antithetical to the small change in the opposite direction that he himself self-reported in his second questionnaire response. For example, when the interviewer asked him to explain why he had chosen to start LV-Post by drawing a Venn diagram, Greg said that it was “just because.” He admitted that the visualization ultimately helped him to see how to write an equation for the total student body without overcounting any subgroup of the population; he added, though, that he probably could have completed the task anyway without the Venn diagram.

After finishing MV-Post, the last task Greg worked in the second interview, the interviewer asked him to explain how he had used diagrams to help him solve that interview's three tasks. First, Greg implied that the benefits of drawing a diagram for a plane geometry problem such as HV-Post (a so-called "sketchable" geometry problem) were self-evident. His diagrams for MV-Post, on the other hand, were brought about out of necessity; his initial analysis of the task led him to suspect that taking an analytical approach would not be fruitful (or at least not "a really fast way of finding all the solutions there."). Instead, he graphed the functions of interest to "get a sense of what was going on." This was certainly successful, as after graphing the equations of MV-Post, Greg immediately used his diagram to make a number of important observations that he would leverage over the course of his work on that task. This included, eventually, the correct solution set.

During both interviews, Greg several times alluded to the fact that his diagrams might be convincing but that they also did not really provide a satisfyingly rigorous proof of whatever conclusion or solution he eventually arrived at. This was perhaps most evident in the way he approached HV-Pre and HV-Post. After a brief exploratory phase, in which Greg used diagrams to generate a solution and identify how it might be justified, he always attempted to provide this justification symbolically by coordinatizing a triangle and assigning variables (see Figure 4.6 below).

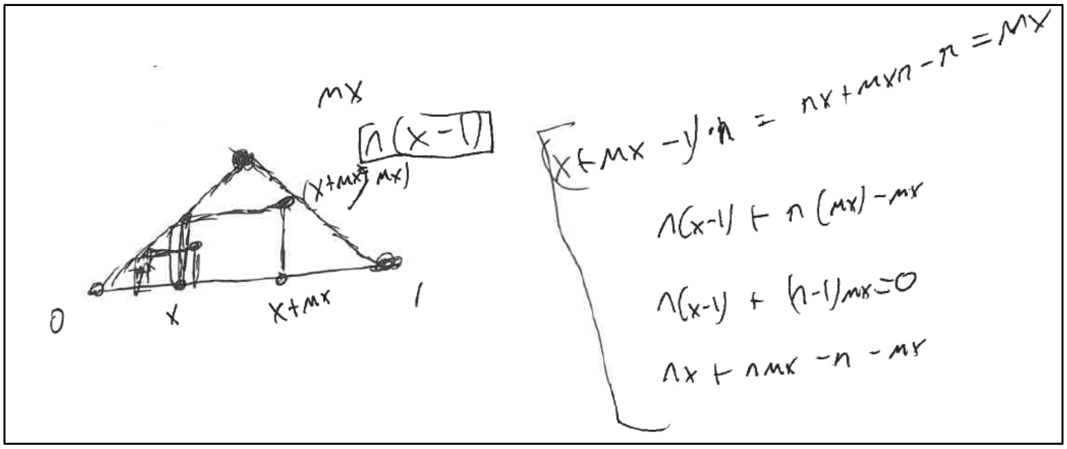


Figure 4.6 Greg's attempt to justify symbolically that a point exists for which the inscribed rectangle is also a square (from HV-Post).

In neither questionnaire did Greg appear to value checking his work at each step as a helpful strategy when solving problems. In the first questionnaire he disagreed (and in the second questionnaire, he strongly disagreed) with this statement. Despite having a lower opinion of this strategy than other participants, Greg never explained why he does not value checking his work at every step besides to point out, simply, that "I don't do that." While working on interview tasks, Greg lived up to this claim except when presented with some kind of contradiction or obvious inconsistency that alerted him to the fact that something had (uncharacteristically) gone wrong.

For example, on MV-Post, Greg made (and later, correct) a conceptual error involving the relationship between the number of solutions to the system and the number of periods of sine that he would need to fit in the interval $[0,1]$ on the x-axis. In particular, he initially failed to notice that the first period would create only one solution in the first quadrant, while each additional period would create two. This error was made at a point when Greg was trying to justify his initial, visual intuition with a concrete inequality; it was the fact that the inequality he had developed refuted his intuition that led him to reevaluate his work. Greg also checked his work on LV-Pre when he realized that he was heading towards a negative population, which would not make sense in the context of the task. Interestingly, when Greg was unable to quickly find the basic arithmetic error that led to this contradiction, he stopped looking and instead began working on the task using an entirely different approach. During this secondary approach, Greg made two more arithmetic errors (only one of which he found) but still arrived a correct answer.

Greg slightly agreed with the statement that justifying his work at every step is a helpful problem solving strategy on the first questionnaire, and in the second questionnaire, he agreed with same statement. Other participants rated this strategy higher; to explain why he did not, Greg isolated the word “helpful” in the questionnaire prompt. He argued that justifying one’s work, while important, might not actually help to find an answer during the process of solving the problem. Instead, after finding a solution, he would “go back and double-check my justification for steps. But as I'm going through it I don't think it's necessarily helpful to have a justificaiton for each thing.”

Greg was sometimes dismissive about the need to provide certain justifications for his methods, even after finding a solution. For example, he did not feel compelled to explain why those of his approaches that he deemed “standard” (such as for LV-Pre, LV-Post, or MV-Pre) were appropriate. He argued that “I don't need to be justifying everything [in that type of problem]. That's just techniques from algebra.” On the other hand, Greg did justify his approach to other less-routine tasks when prompted by the interviewer. This occurred, for example, after HV-Pre; there, he reviewed the steps he took to solve the task and what mathematical reasoning led him from one step to the next.

When not applying what he considered standard methods, Greg still sometimes made mathematical assertations that he did not justify. This was usually when the assertation itself was being used as justification. For example, after some exploration of the problem setting of HV-Pre, he quickly concluded that the statement in the task description was indeed possible. To justify this overarching claim, he made several sub-claims (about perceived congruencies, which vertex the cut should pass through, and which two sides of the isosceles triangle are the congruent pair) that he did not in turn substantiate with mathematical evidence. That is, Greg followed through with justifying solutions he ultimately gave to tasks when they relied on bespoke mathematical reasoning and not rote application of commonly accepted methods (and in fact, discovering sufficient justification in these situations constituted the bulk of his work in the interviews); however, he appeared to feel free to make smaller claims towards this end without justifying those in turn.

When the interviewer asked Greg to explain to him why certain statements he made were acceptable, he usually provided sufficient justification. For example, when drawing a coordinatized triangle in HV-Pre (see Figure 4.7 below), he used appropriate logic to justify his assignment of certain labelled coordinates at the behest of the interviewer.

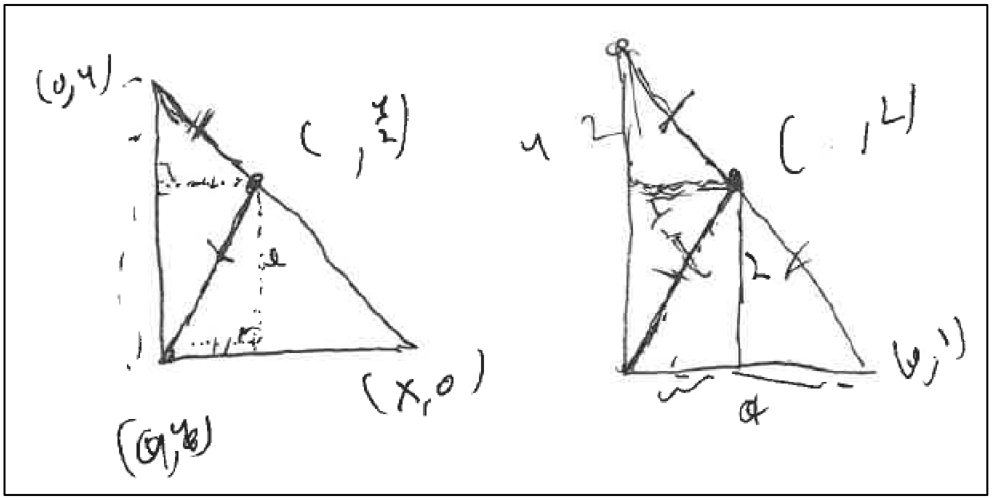


Figure 4.7 Two of Greg’s diagrams from HV-Pre, with variables (on the right) and with sample values (on the left).

Similarly, after generating an inequality that he planned to use in MV-Post, he demonstrated to the interviewer how the inequality aligned with the diagrams he had drawn up until that point. Not all of Greg’s justifications were correct in these circumstances, though; in MV-Pre, after solving the tak quickly with an algebraic method, the interviewer asked Greg to analyze a case he had not considered: whether the system could ever have three solutions. Greg quickly dismissed this case using faulty logic caused by a poor choice of notation.

On the first questionnaire, Greg slightly disagreed with the statement that verifying his solution at the end of a problem is a helpful problem solving strategy. On the second questionnaire, however, he initially indicated that he now strongly agreed with this claim. He later softened his second response to only slight agreement. Like for *Read Carefully*, Greg did not feel that his change in perspective was caused by some aspect of his first semester of graduate school. Instead, he explained, the change might be attributable to a difference in how he interpreted the questionnaire prompt. In filling out the first questionnaire, Greg thought he might have given more weight to the word "helpful"; that is, while verifying an answer might be "important," it is obviously not "helpful" in the sense of helping you get an answer; you have, after all, already gotten an answer if you are attempting to verify it. He elaborated that "if I did the problem right, I know that I have the right answer, so I don't feel a need to verify it." This explanation did, in fact, very closely mirror Greg's initial reasoning (given during the first interview) as to why he rated this strategy lower than other participants. At that time, he had argued that "most of the time, I feel like if I've done everything properly up until that point, then I don't need to verify my answer to know it's right."

Greg did verify his answer in LV-Pre, but only when his answer was called into question by the interviewer. Specifically, during Greg's work, he simplified an equation to solve for a variable and found that $W = 10$; the interviewer clarified with Greg that he chose the variable W to represent the number of women before pointing out that, later in his work, Greg established that there were never less than 70 women in the town. This observation prompted Greg to verify that the final population he claimed to be the solution did in fact satisfy the criteria of the problem space. When it did, he was satisfied that his work was correct. As a result, he did not address the fact that (despite inadvertently arriving at the correct answer) he had committed a sequence of errors that led to both $W = 10$ and $W = 70$ at different points in his approach. In some sense, verifying that his answer was correct dissuaded Greg from pursuing and resolving a noted conceptual contradiction in his work leading up to his answer.

4.1.2.3 Mathematical Affect

Greg's responses to parts two and three of the questionnaire were, by far, the most unique as compared to his peers in the graduate population; in fact, of the eleven items in these two sections, Greg's responses to seven of them were atypical. Many of these differences can be explained, however, by recognizing that Greg chose to interpret success in mathematics as proving meaningful results on the cutting edge of mathematics research; he jokingly explained that one is only successful in mathematics after winning a Fields medal. While this was clearly meant to be an exaggeration, the fact that Greg did not consider succeeding in an educational mathematics setting to really constitute success in mathematics does give perspective on his beliefs about the subject in general. For example, on the first questionnaire Greg agreed that persistence is an important quality for success in mathematics; on the second questionnaire, he strongly disagreed. He characterized this difference by explaining that, on the second questionnaire, he was considering the item from an academic perspective. That is, while learning mathematics, “it's better to know all the techniques before you then try to throw yourself at a wall over and over again.” This is because it’s unlikely for a mathematics learner to “discover the most optimal way to do it” on their own, so they should focus their efforts on absorbing content that has already been proven instead of persistently trying to reinvent known procedures. On the other hand, “once you have everything else down, persistence is important”; but this is only true for very cutting-edge mathematics work done by very talented thinkers and not average students.

In part two of the first questionnaire, Greg strongly agreed that natural talent is important for mathematical success; however, he only somewhat agreed with the importance of hard work and creativity. All of these responses were beyond one standard deviation from the mean response given by other graduate students. Again, Greg’s opinions here must be contextualized by his interpretation of “success in mathematics” as making a revolutionary mathematical discovery. He said,

> I think that generally, what we've seen historically is that good mathematicians are good from very very very young ages. I think that there's a specific—that like, there's some people, no matter how much

> hard work they put into math, they just cannot quite get the insight that's necessary to be on the cutting edge.

Thus explaining his high rating for *Natural Talent* and low rating for *Hard Work*. He added that *Creativity*, at the absolute highest level of mathematical achievement, was very important—except in certain areas of mathematics, such as applied mathematics, which Greg argued are less inherently demanding of creativity. When asked by the interviewer to reconsider his responses to these items if "success in mathematics" was calibrated to only earning a Ph.D., Greg noted that his response to *Creativity* would go way down ("I think that there are some people in some Ph.D. programs who could get away without really having to do anything creative at all.") but that his response to *Natural Talent* would likely be very similar.

In part three, Greg's lower responses to *Discover Solution* and *HW Confidence* (somewhat agree and strongly disagree, respectively) were both predicated on his experience with a textbook that included the Riemann hypothesis as a student exercise ("Each of the questions was weighted on difficulty of 1 through 5 stars. That one was only a 4 star."). Disregarding this extreme case, however, Greg said that he felt confident that he could answer any question given to him on his usual homework assignments. Finally, Greg commented briefly on the *Best Approach* item: while his response was not significant compared to his peers, he explained that he enjoys looking for ways to "nuke mosquitoes" after solving a problem. That is, instead of reconsidering his work to decide if his initial solution was efficient, he instead tries to find extremely *inefficient* methods that leverage advanced mathematics in unexpected ways. Greg added that he does not feel compelled to consider whether his approach is the best way to solve something because he is usually very confident in his work.

4.1.2.4 Problem Solving Characterization

Greg privileged non-visual problem solving methods over visual representations, even when he used visual methods to get started. When he drew a diagram to facilitate his work on any task, he was then quick to either translate his visual intuition into symbolic logic or otherwise downplay the diagram's importance in contributing to his understanding. This proclivity towards analytic or algebraic approaches appeared to stem from Greg's perception of his diagrams as convincing but ultimately informal and insufficient as proof. It was his efforts to provide what he considered sufficient proof that constituted the bulk of Greg's time during the interviews. He sometimes very quickly solved a task using what, to him, was a commonly accepted method that required no additional explanation; on the other hand, if he could not use such a method, he very quickly arrived at a correct solution anyway using observational intuition but then felt compelled to spend a significant amount of time trying to provide a rigorous justification.

This rigorous justification often required Greg to make other, smaller claims that he only explained when prompted by the interviewer. That is, just as he assumed certain overarching algebraic methods were standard practice and thus beyond justification, he also appeared to believe that certain mathematical micro-claims should be accepted without explanation as well. Greg's disposition towards treating some aspects of his work as obvious, in this way, was reflective of his belief that a significant portion of problem solving success is dependent on the prerequisite mathematical knowledge held by the problem solver. That is, an individual's inherent problem solving capability is relatively fixed; thus, one's ability to solve difficult problems hinges largely on how many known mathematical facts and procedures they have acquired. Just as he chose to forgo some justification, Greg did not routinely check his work or verify his solution unless prompted by the interviewer or an obvious contradiction alerted him to a possible underlying error.

Despite this perceived lack of monitoring both with respect to mathematical logic and algebraic manipulations, Greg's work was usually both fast and precise.

In fact, every aspect of Greg's work could be characterized as fast (or, alternatively, immediate). He did not appear to spend much time reading the task description or planning how to proceed once he had finished reading. His first choice of approach was usually the correct one despite little apparent time spent deciding on which approach to take. If this initial approach did not pan out, Greg transitioned to another approach and seldom looked back. This transition was often triggered by a perceived need for more mathematical rigor rather than because of a conceptual roadblock. Such a shift, then, did *not* occur when the initial approach was one that Greg appeared to find inherently rigorous by some unspoken agreement in the mathematical community. Overall, Greg's confidence in his work was well-founded but engendered certain assumptions of self-evidence that sometimes led to errors and almost always obfuscated his underlying problem solving strategy.

4.1.3 Taylor

4.1.3.1 Formative Experiences

Taylor was one of the only participants to reference experiences outside of mathematics courses that helped him improve as a problem solver. Specifically, he learned certain heuristics through both engineering and computer science; these were, respectively, that he should pay careful attention to computational accuracy (especially with regard to rounding errors) and that he should choose variable names to be representative of the quantity that those variables represent (so that "there's no need to waste time processing what I wrote down."). In terms of mathematics courses, Taylor identified an introductory proofs course as both his favorite course and one of the most impactful. He described proofs as "the first time that I felt like I took a math class," and included it with abstract algebra and analysis aexamples of courses that paint for undergraduates a more accurate picture of what “math entails.” In terms of problem solving, Taylor recalled a specific problem from his proofs course which he solved “in a very roundabout way”; finding such an unexpected solution taught him not to

> be afraid to think outside the box. That problem, we worked on it for a long time. It was a problem that people didn't quite have satisfactory proofs for for weeks. Like 2 weeks, easily. And so: just don't give up. Think about things, approach them in different ways. Maybe see what connections you can make.

Taylor added that he has enjoyed other mathematics classes as well, though, and that the instructors of these courses tend to have certain commonalities. Good instructors, according to Taylor, enjoyed both their subject and the act of teaching; furthermore, they were very knowledgeable and “could help us [the students] guide our questions and think about things.” In fact, Taylor admitted that

> I don't know how effective I would have been at solving problems if I wasn't guided at certain key points. When I was taking proofs, or when I was doing these more like—where it required more thinking. Being guided to more effectively approaching problems or just getting guided on pitfalls and common things and getting feedback on how to improve.

He compared this process to having a creative writing coach who could point to specific parts of written proofs and explain whether they were or were not effective. While this sort of guidance is unique, in a way, to proof-based courses, Taylor also argued that “advice on how to think about problems, I think, could be useful no matter what kind of problems you're solving.” Such advice might be to read the text of problems carefully, to look for patterns and similarities between problems, and not to be afraid to return to definitions and think about foundational concepts. When asked to identify a piece of problem solving advice that he had received and found particularly helpful, however, Taylor could not pinpoint anything specific that he had taken to heart.

Finally, Taylor noted that his first semester of graduate school coursework did not feel significantly different from his educational experiences as an undergraduate. Instead, he identified his recent experiences as a graduate teaching assistant as formative with respect to problem solving (see, for example, their discussion about *Draw Diagram* in the following section).

4.1.3.2 Problem Solving Strategies

In both questionnaires, Taylor strongly agreed with the statement that reading a problem carefully helps when solving difficult mathematics problems. In tasks with numerical data (namely, LV-pre and LV-Post), he lived up to this claim by transcribing the data from the task description, assigning variable names, and using equations to connect variables according to the setting. Taylor placed an emphasis on reading in order to understand relationships between different aspects of a task; for example, when he failed to find an actionable plan for solving LV-Pre, he revisited the task description to examine whether or not he had captured all the given information before explaining that there seemed to be a “missing connection” that he was not seeing. Taylor had a similar realization in LV-Post, when he remarked that “it almost feels like there's not enough information to make any conclusions.” However, he stressed that this was actually a “very cool” aspect of the task that he was interested in resolving. To do so, Taylor took time to carefully sort out “what all the pieces mean.” In every interview task, Taylor occasionally underlined key words or phrases (such as “arbitrary” in HV-Pre and HV-Post, for example) when explaining to the interviewer how he was thinking about that task.

When prompted by the interviewer, Taylor explained that he first reads a problem to find out what he is expected to solve for. He then uses this information to review any given data so that he can effectively organize it in a way that will lead to a solution. He added that this includes carefully choosing variable names to be reflective of the quantities that those variables represent. In tasks without numerical data, Taylor appeared to immediately begin exploring the problem space without these preliminary steps.

Taylor spent significant time exploring the setting of various interview tasks, especially HV-Pre and HV-Post. In both, Taylor initially tested various example triangles to try and find a counterexample to the tasks' respective claims. When he found a triangle that was not a counterexample, in either task, he attempted to produce an explicit procedure for the method he used and to apply this method to another possible counterexample. This exploratory, comparative, visualizing phase constituted the majority of Taylor's work in these tasks. In MV-Pre and MV-Post, Taylor also constructed a sequence of different visualizations; this time, however, he focused on drawing graphs that represented different sample values of the parameter α. This helped him, in each task, to assess how different magnitudes of α could affect the number of solutions to the given system. In particular, he often chose extreme values; for example, in both tasks, he discussed what would happen if $\alpha = 100$ or if $\alpha = 1$. In both HV-Pre and MV-Post, he also remarked that he would typically use some kind of visualization software to more quickly and accurately explore the different examples they were forced to construct by hand in the interview.

Taylor also explored the settings of LV-Post, and to a lesser extent, LV-Pre, although he did so symbolically rather than visually. That is, he assigned variable names and wrote equations that he hoped would capture the relationships described in the task description. In LV-Post, Taylor tested several iterations of variable names to explore the most effective way of organizing the data (see Figure 4.8 below).

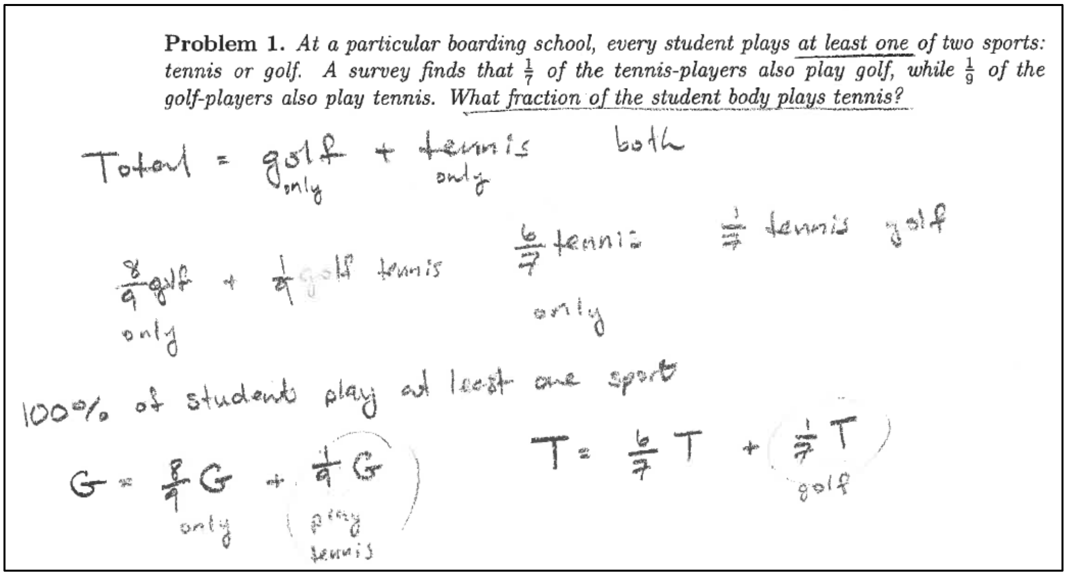


Figure 4.8 Taylor's refinements of his variable scheme in LV-Post

In the first questionnaire, Taylor agreed with the claim that imagining what a solution might look like is helpful for solving unfamiliar mathematics problems; in the second questionnaire, he only slightly agreed. Taylor explained during the first interview that when he reads a problem, he looks to first understand "What is the problem asking of me. And then I need to figure out what I know and how can I use that to get to what I need." This description was reflected in Taylor's work during many of the interview tasks.

When solving LV-Pre, Taylor stated that "we're trying to find the total new population, so we need to know the old population." Here, the form of the expected solution influenced his approach to the task by leading them to solve for the initial population. He added, after completing his work on this task, that he initially "had no insight as to what the scale of this would be." In the corresponding task from the second interview, LV-Post, Taylor initially began to name variables and organize numerical data before stopping to revisit the text and underline the final sentence, remarking that "it might be good to identify the actual question!" Because he understood that he did not actually need the absolute magnitude of any population, he reacted positively when he realized that manipulating his system of equations would result in a proportional relationship between tennis players and the school population.

After reading the text of MV-Pre, Taylor remarked that maximizing the number of solutions is an unconventional and interesting problem. This is because, he explained, it is normally known what the hypothetical maximum number of solutions for any system might be and the problem solver's job is to find them. When the form of a solution was in some way novel, Taylor's positive emotional reaction to an unfamiliar situation appeared to motivate a closer examination of the problem space in order to reframe familiar concepts. In MV-Post, another task whose circumstances Taylor found interesting, he correctly described his interpretation of the situation (and what might constitute a solution) in order to verify with the interviewer that his understanding was correct. He then used this firm conceptual grasp later in his approach to list a table of possible solutions that he used to facilitate his work.

Taylor slightly agreed with the statement, in both the first and second questionnaire, that considering different ways of solving a problem is helpful. Despite the fact that his opinion of *Consider Approaches* was lower than other participants, Taylor stressed that he "didn't say it was *not* helpful." Instead, he explained, considering multiple approaches is a strategy that is most useful "once you've reached stopping points. Road blocks. But it's not something, I think, when you're starting a problem, you should immediately see all of that." Taylor argued that considering too many approaches too soon can lead to getting "paralyzed by choice."

Taylor explained that his first approach when solving a problem usually mirrors any intuitive understanding that he has of the mathematical concepts involved; ideas he understands visually prompt him to use diagrams, for example. In MV-Pre, Taylor first used algebraic manipulations to write the system of equations as a single quadratic equation in terms of x. He also remarked that the word “maximize” in the task description naturally prompted him to consider using calculus techniques to analyze this new equation. He rejected this approach and instead decided to interpret what the original equations are “actually saying,” which he did using diagrams after he recognized the second equation as that of a circle being translated along the x-axis.

On the other hand, in MV-Post, Taylor began the task visually by placing a sinusoidal curve and the line $y = x$ on the same coordinate plane. He then stopped to consider whether the question might be better answered by simultaneously visualizing a unit circle. When, after significant work, Taylor found himself still unsatisfied by the limitations of hand-drawn trigonometric functions, he finally turned to an algebraic approach. Taylor also considered both visual and algebraic options in LV-Pre, where he first noted that this task felt like one that he had seen before and involved a Venn diagram. Despite this observation, he began the task with using variables and equations until completely stymied, at which point he drew the Venn diagram.

While Taylor initially slightly agreed with the claim that planning out steps in advanced is helpful for solving difficult problems, he came to strongly disagree with this statement on the second questionnaire. He explained this change by noting that he was probably influenced, in the second questionnaire, by the interview tasks he had just worked. In these tasks, he admitted that he did not really plan anything too far in advance; this was because the tasks were “less constructed like a homework assignment.” Here, he elaborated further, he meant that it was less clear what theorems or definitions he could chain together to reach his goal. He added that he might plan his steps in advance on a problem if he could identify certain subgoals or break the problem into smaller pieces that might need to be completed in order.

True to his own reflections, Taylor did not engage in much noticeable planning when he worked interview tasks. Instead, he appeared to choose an initial approach that allowed him to best leverage his personal, internal representation of any underlying mathematical concepts. Then, he manipulated that representation or tested various cases until a pattern or underlying structure revealed itself. For example, in the midst of working LV-Pre, Taylor admitted that he did not “really know exactly how I'm gonna get to a final solution”; instead, he resorted to manipulating one of their equations until he saw that he could isolate a variable. At this point, he proclaimed that “we might be getting somewhere here.” Similarly, when using the quadratic formula to attempt to solve MV-Pre, Taylor conceded that “I haven't even thought about what this—what interpretting this means. At the moment, it's just I have a quadratic equation in one variable and a parameter. And seeing: what does that lead me?”

Taylor shifted from slightly to strongly agreeing with the statement that drawing a diagram or sketch is helpful when solving a difficult mathematics problem. Taylor acknowledged that this change might have been affected by some of his own graduate coursework but that the bigger influence was probably his work as a tutor in an undergraduate abstract algebra class. He found that explanations for students came across much clearer with some kind of diagram, even if it was “just getting relationships on the board and drawing a line between them.” Taylor also noted that

> “I find it much easier to guide other people with this kind of stuff [diagrams and sketches]. And I think, part of it for me, is it helps illustrate what I can imagine—or *my* intuition—more directly to the student. Because I can't just explain: I see it this way.”

He added that because his intuition was usually visual, “a diagram’s a way to get that information across.” With regards to his own work, Taylor noted that “If you can do so, I like to do so… get some visual intuition.” Furthermore, when prompted by the interviewer, Taylor explained that when drawing a representation of a problem, he will often “visualize these things as being active," especially in analysis courses. He gave an example of how, to him, checking an epsilon neighborhood around a point is like "wiggling" the point around. He cautioned, however, that visualization "might waste time. Because what we kind of saw was that it's easy to make things that are not accurate or could be misleading."

Taylor used some kind of diagram or sketch in five of the six tasks that he worked in the two interviews. Even in the outlying task, LV-Pre, he circled certain lines of symbolic manipulations and drew arrows between steps to illustrate a flow of ideas when justifying his methodology. He would later explain that he still tries to visualize in problems that he first solves, like LV-Pre, strictly arithmetically; however, the arithmetic in LV-Pre did not have to do with "areas or volumes where you might be able to make a picture that's more helpful."

In the other tasks, Taylor's visualizations were much more explicit: in HV-Pre and HV-Post, almost his entire work was done by drawing and examining pictures of different triangles; in LV-Post, a Venn diagram led to an important insight about the relationship between two quantities and, ultimately, a solution; and in MV-Pre and MV-Post, he used graphs (of circles and sinusoidal curves, respectively) for the majority of their work. Taylor also demonstrated that he supplements his diagrams with pertinent symbolic work when his visual intuition is not obviously generalizable or is too reliant on whether or not some diagram has been drawn correctly to scale (see Figure 4.9 below).

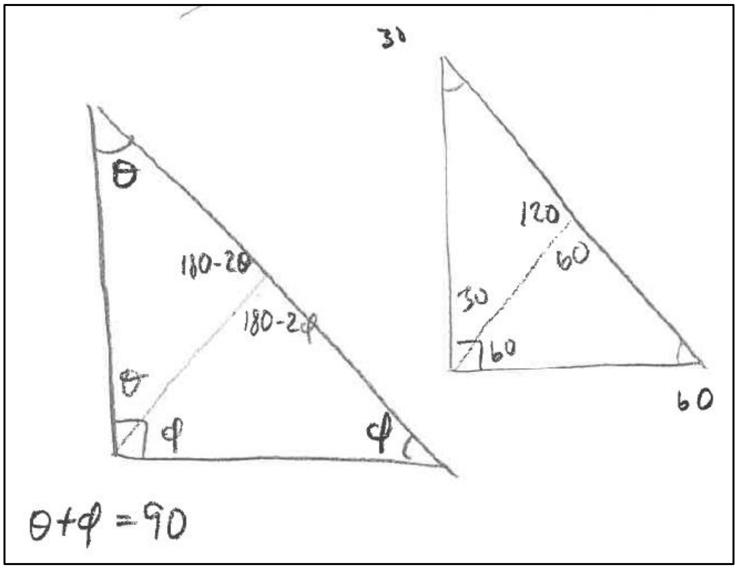


Figure 4.9 Taylor's general diagram (left) and diagram with sample values (right), both for HV-Pre.

Taylor first slightly disagreed with the statement that checking one's work at each step is helpful when solving a difficult mathematics problem. In the second questionnaire, though, he indicated that he now slightly agreed. In the first interview, Taylor explained that checking one's work is most helpful in "problems with a lot of algebra […]; explicit calculation, I mean. Like solving a differential equation. Whereas making a logical argument, I tend to be slower about how I process that anyway and so I'm not really needing to check my work at every step." During the second interview, Taylor did not comment on his slightly improved perception of this strategy; however, he did add another example of a situation in which checking one's work might be easy to do, and thus, more helpful. He explained that when solving an "applied problem," the answer should make sense in some given, real-world context in which the mathematics had been couched. This sentiment towards problems with real-world context was most clearly demonstrated when Taylor solved LV-Pre. He explained that, because he was tasked with finding a population (and thus, a positive integer), it was possible to gauge whether or not he was on the right track by periodically checking his work for negative numbers and non-integer values.

Taylor also made some effort to check his work in both MV-Pre and MV-Post. In MV-Pre, he applied the quadratic formula in order to attempt to verify where the solutions to the system were located in the $\alpha = 0$ case that he had already drawn on a diagram. In doing so, he made a computation error that led to different, incorrect solutions. When Taylor realized this difference, he attempted to reconcile conceptually why his visual and algebraic approaches might have produced different answers; he did not look back over his work to try to see if the difference might have been due to an execution error. In MV-Post, Taylor incorrectly graphed the function $\sin(7x)$ and did a significant amount of work based on this misrepresentation until realizing his error. It is possible that Taylor's delay in recognizing this error was because, unlike the error in MV-Pre, it did not immediately trigger a moment of cognitive inconsistency. Other errors, in LV-Pre and elsewhere in MV-Pre, also went unnoticed until the interviewer explicitly drew Taylor's attention to them.

In both questionnaires, Taylor indicated that he slightly agreed with the statement that justifying one's work at each step is helpful for solving a difficult mathematics problem. In general, Taylor justified both his use of mathematical principles and why he was approaching a task in a certain why. For example, in working LV- Pre, he pointed out that he knew why a certain fraction was actually an integer by citing (and explaining why he could apply) the divisibility rule for six. In the same task, he justified his decision not to reduce fractions until the final step of his computations: his background in engineering had made him especially cognizant of round-off errors, and he saw in his work at least one fraction with an unappealing decimal representation. Finally, he justified his overall approach post hoc by pinpointing the line of his work at which he was able to isolate one variable, which indicated to him that he would be sure to get some kind of answer to the system of equations he had created.

These three types of justification (explaining mathematical properties that he used, justifying certain micro-decisions, and reflecting on a more comprehensive justification after reaching a solution) apply to the majority of Taylor's work. In MV- Pre, for example, he explained that he could characterize isosceles triangles by the number of congruent angles as well as sides. He justified an approach centered around angle measure instead of side length by pointing out that he also knew that the sum of the interior angles of a triangle is 180 degrees, which he expected to leverage during his solution attempt. Finally, he summarized his approach to the task and argued why it worked: by carefully considering a hypothetical counterexample, he actually produced a generalizable procedure for demonstrating that no such counterexample could exist. Had he assumed from the start that such a procedure could be found, it would not have been obvious how to describe it.

In the first questionnaire, Taylor agreed with the claim that verifying one's eventual solution is helpful when solving a difficult mathematics problem. In the second questionnaire, he indicated that he now strongly agreed with this claim. Taylor did not make outright attempts to verify their solutions to either LV- Pre or LV-Post at the end of the task, despite these questions being the most amenable to such practice; the solutions he produced seemed reasonable enough, so he did not appear compelled to check them. Only in HV-Pre did Taylor engage in a final verification process: after describing a generalizable process for cutting a right triangle into two isosceles triangles, Taylor used a specific, extremely acute triangle (see Figure 4.10, below) to confirm that his procedure would not lead to any geometric contradictions in such circumstances.

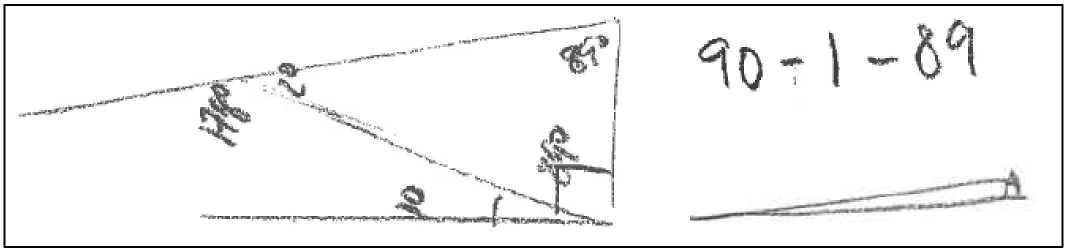


Figure 4.10 Taylor verified their approach using extreme sample values.

4.1.3.3 Mathematical Affect

There were two interesting aspects of Taylor's mathematical affect that arose during interviews. First, he responded to many interview tasks with explicitly verbalized intellectual curiosity; on each of MV-Pre, HV-Pre, and LV-Post, Taylor commented that the task was interesting or in some way unexpectedly novel. Notably, one of these remarks came after Taylor discovered, on HV-Pre, that attempting to justify a counterexample had inadvertently led him to the conclusion that the proposition was, in fact, true. This was notable because it illustrated Taylor's positive emotional reaction to incorrect work; rather than responding with frustration or embarrassment, he described their revelation as sparking curiosity. Taylor elaborated specifically on this mindset later in the interview when he described why he strongly agreed that persistence is important for success in mathematics:

> Because it's just not giving up; don't get frustrated; don't get discouraged. Just keep at it. Try different things. Maybe come up with a different approach. Maybe try something else. Look at a different problem. But just keep pushing forward is, I think, really what can make this fulfilling. Because when you get that answer that you've been working on for so long, it's awesome!

That is not to say that Taylor was never frustrated or embarrassed. More than any other participant, Taylor commented on his discomfort with the interview process and how he was constantly "trying not to look silly," which made him "more self-conscious about what I'm trying." He also repeatedly qualified his work by admitting that he was not especially competent at mental arithmetic or number sense. Still, Taylor argued that "math is about making mistakes enough to where you learn how to overcome them. Because a big part of intuition is it's applied experience." As a result, despite his apprehension, Taylor was clearly willing (and often, eager) to explore unfamiliar settings.

The second interesting aspect of Taylor's affect was his disregard for natural talent as a prerequisite for mathematical success. In fact, he was the only interviewed graduate student who disagreed, to any extent, with the *Natural Talent* item. In order to couch this response, he first explained that success in mathematics was dependent on context. But, ultimately, being

> satisfied is a big part of all of it. You're satisfied in what you're using the mathematics for. Like, there's a personal satisfaction there. If you're doing research, you're uncovering new results. Or just proving things or whatever. You're making progress. You're developing math— the body of mathematics. If you're teaching, you are not only effectively teaching people how to do mathematics, but you're inspiring, you're giving off some of your passion. You're igniting a passion in other people. I think that's success in that regard.

That being said, Taylor did not "think people need to be naturally gifted at it [mathematics] to succeed at it. To find it fulfilling. To find it engaging. I think it's something that can be cultivated." He commented again on *Natural Talent* during the second interview, unprompted by the interviewer, to remark that "natural talent is really overrated" and that his own skill in mathematics was the result of key guidance provided by instructors, as described in *Formative Experiences*. That is, Taylor appeared to consider himself first-hand proof that natural talent is not required for finding satisfaction (and thus, success) in the study of mathematics.

4.1.3.4 Problem Solving Characterization

The most striking aspect of Taylor's problem solving strategy during interviews was his proclivity for substantial exploration of the problem space. In particular, Taylor emphasized that he often sought to understand any explicit or implicit relationships or connections in the setting. Taylor also captured connections between concepts in the problem space and his own content knowledge when he had preexisting experience with relevant properties or theorems. As a result of this sense-making, he did not appear to rely on his recollection of entire algorithms or procedures. Taylor's exploration was largely visual, which he admitted is because most of his intuitive understanding of mathematical concepts is visual. That is, he gravitated toward approaches that mirror his own representations of the underlying mathematical concepts and not necessarily those that he had learned to be most normative in the mathematical community.

Taylor also spent significant amounts of time exploring connections within a task. This appeared to be his default problem solving mode when he did not perceive a clear method for reaching a solution. He did not spend long, though, looking for such a method; Taylor was quick to try something, so long as there was a chance it might spark a moment of insight. This could be, at least in part, due to his apparently genuine interest in understanding novel mathematical situations (seen through his positive emotional reactions to certain tasks). Ultimately, Taylor's lack of planning was overcome by his willingness to change course and try a different approach to a task, a strategy that exemplified Taylor's belief in the value of persistence. Furthermore, after changing course in this way, Taylor was not frustrated and did not treat each approach as a separate, disconnected entity. He used the ideas from one approach to verify and justify the work in another approach until he arrived at a conclusion that satisfactorily coincided with his understanding of the problem space.

4.1.4 Mia

4.1.4.1 Formative Experiences

In the first interview, Mia reflected on undergraduate discrete mathematics course. This course functioned as her introduction to proofs, and was influential for her problem solving

> because before, you were taught how to *do* a problem, not necessarily how to problem solve. They would give you a problem, and they would be like, okay, now use this method to do it, right? Or they would actually list it in the problem. Be like, use such-and-such to do this. Or even if you're talking about proofs: use proof by contradiction. But, ever since that class, it became more open-ended. So it kind of—we started more question like this: *is* it true? Is it *not* true? And if so, find a counterexample or prove it. So I guess that was my first real taste of these kinds of open-ended questions.

Mia added that the instructor of the course helped orchestrate this shift in perspective: “he challenged you to think by yourself and do a lot by yourself, so you really did have to think it through. He didn't *tell* you the answer.”

During the second interview, Mia focused more on her experiences from her recent first semester of graduate school. She found that, broadly speaking, she was expected to learn more content more quickly than she had been required to learn as an undergraduate. This led Mia to adopt a more open mindset towards trying a variety of approaches when solving difficult problems, partly because so many more problems were more difficult for her now. She had a greater appreciation for the fact that “everything is so connected.” When considering which mathematical tool to use to solve a problem, “Not every single formula is going to have a place in the problem that you're working at, but... it doesn't hurt to try.” Linear algebra, the graduate course that she found most difficult, was also her favorite. This was because the content, while significant in volume, also “fits together nicely. Like, you're building something. I feel like you start from the ground up.” As a graduate student, Mia also found that her teaching duties had an effect on her mathematical outlook. Having to generate meaningful explanations for properties or methods she taught to her students encouraged her search for similarly deep understanding in her own courses. Mia argued that this can actually be detrimental when she gets “caught up on the things that I don't understand as well.”

4.1.4.2 Problem Solving Strategies

On both the first and second questionnaire, Mia strongly agreed that reading a difficult problem carefully is helpful for eventually solving that problem. After solving LV-Pre, Mia explained (at the behest of the interviewer) how she approaches the text of a difficult problem when she first reads it: initially, she takes note of numerical data. Then, she identifies what the question is asking her to find. Finally, and most importantly, Mia claimed that she uses the prompt to reassess the numerical data, identifying what resources were given to her that might lead to the desired solution. In LV-Pre, for example, Mia pointed out that she recognized the provided ratios as a tool that she could use to set up an equation that would represent the population of the town.

Throughout the two interviews, Mia asked the interviewer to clarify the text of almost every task in some way. For example, in HV-Pre and HV-Post she double-checked her understanding of the geometric vocabulary (i.e. “isosceles”, “inscribe”) used to establish the problem situations. When she was initially surprised by the unfamiliar structure of MV-Post, Mia questioned whether the parameter α in the text was actually intended to serve two different functions (both as a parameter and to represent the number of solutions to the system) or if that was a typo. After receiving clarification from the interviewer, she restated the task in her own words to verify with the interviewer that she now understood the premise. When given BP, Mia asked the interviewer what kinds of coins she may or may not use in solving the task.

Mia returned to the task description on at least a few occasions (when solving LV-Pre and LV-Post) in order to correct small transcription errors when she rewrote the given data. In LV-Post, she also revisited the task description in order to remind herself of what the question was asking her to find; she had reached an impasse in her algebraic manipulations that could only be resolved by a better understanding of the type of solution she should provide. As perhaps indicated by these relatively isolated instances, Mia did not appear to routinely revisit the task description unless prompted by an error or an immediate difficulty.

As with most participants, Mia undertook significant exploratory diagramming as her first step on both HV-Pre and HV-Post. Her exploration of these problem spaces did not appear especially structured; that is, she did not comment on her choice to draw specific types of triangles (or *not* to explore other types) and also did not appear to systematically explore the propositions by exhaustively considering cases. Each of her exploratory triangles was roughly identical in size and shape, although Mia often explored them in different ways. The transition from one triangle to another appeared to be relatively arbitrary rather than because Mia was responding to a particular discovery.

Mia also used sample values for the parameter α in both MV-Pre and MV-Post to further her understanding of the effect α had on these systems of equations. This practice was most apparent in MV-Pre, where Mia first made a substitution of one equation into the other and expanded a squared binomial (she commented: “I don't know... what exactly I'm going to do with this, but I'm just moving stuff around to see if I can figure something out”). This itself was an act of exploring; ultimately, Mia hoped to find “an idea of what a possible solution might look like.” When none was apparent, she decided to test sample values of α to see how it transformed the quadratic equation she had produced. After testing $\alpha = -1$, $\alpha = 0$, and $\alpha = 1$, Mia compared the resultant parabolas to interpret

α's effect. From her observations, she hypothesized that her eventual solution would be an interval of α values symmetric about 0. Similar exploration of the parameter α in MV-Post was stymied by Mia's admittance that trigonometry was one of her mathematical weaknesses. She cursorily tested two values of α ($\alpha = 0$ and $\alpha = 1$) before halting this exploratory phase in favor of another approach. In MV-Pre, each test value of α was accompanied by a corresponding diagram; in MV-Post, Mia's diagramming was more limited.

Between the first and second questionnaires, Mia shifted from slightly agreeing to slightly disagreeing with the statement that imagining what a solution might look like is helpful when solving difficult mathematics problems. As discussed above, Mia claimed on the first interview to pay special attention to identifying exactly what type of solution she would be required to produce and how the information given in the problem might facilitate that work. It was not clear, in the second interview, why Mia's opinion of this strategy changed.

While working interview tasks, Mia sometimes lost sight of the form of the solution even after first identifying it in the task description. For example, while solving LV-Post, Mia manipulated the data in the task to set up a system of equations. She was able to simplify this system to eventually produce one equation that gave an explicit proportional relationship between the student body and its tennis population. This is exactly what the solution to the task requires, but Mia did not immediately recognize it as such. Even after revisiting the task description and clarifying that her answer should be a fraction, she still spent several more minutes making additional algebraic manipulations until she arrived at the same fraction from another, slightly more roundabout approach. Ultimately, while she did use the question's prompt in order to parse numerical data and decide on an initial approach, it is not clear that she kept the actual form of the solution in mind while she worked.

Interestingly, Mia also alluded to the form of the solution of HV-Pre, even though that task does not have an explicit numerical answer: she lamented that the task would be much easier if she knew from the outset whether it was true or false and only had to find justification for either. This admission provides further evidence that Mia does use the form of the solution to decide on an initial approach.

Mia agreed, in the first questionnaire, with the statement that considering multiple approaches is helpful when solving difficult mathematics problems. In the second questionnaire, her sentiment shifted towards slight disagreement. Mia seemed surprised that she had agreed with the statement at any point in time. She admitted that “I still do it [consider approaches] in the back of my mind, but I guess I just don't think it's as conscious as I used to do it, now that I think about. I think now it's just, like: go a little bit more with my gut feeling.” She summarized her philosophy towards considering multiple approaches by asking “Why think about it so much? Just try. See what happens.” At the end of her first interview, Mia clarified that she is not unilaterally against multiple approaches and that “if that [her initial approach] starts getting kind of complicated, *then* I switch.”

In both LV-Pre and LV-Post, Mia appeared to immediately gravitate toward familiar approaches; this aligns with her stated strategy for problem solving in general, which was: “Work with what I know first.” In LV-Pre, she observed that “I remember doing these fractions in algebra,” which prompted her to set up two ratios that she could solve as a system of equations. Similarly, in LV-Post, she explained to the interviewer that her first instinct was also to set up a system of equations. However, the context of the task (intersecting populations) reminded her of a statistics course; in this course she had used diagrams to compartmentalize populations, so she attempted to utilize two such diagrams in this task. See Figure 4.11 below.

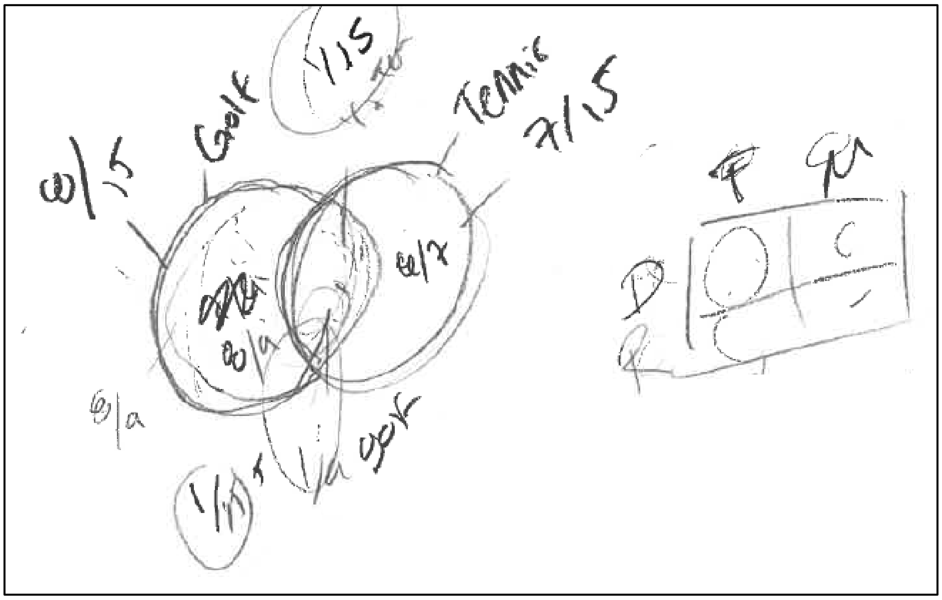


Figure 4.11 Mia's two different diagrams for LV-Post.

Finally, it should be mentioned that on at least one occasion Mia asked the interviewer whether a possible approach she was considering would be appropriate: when solving BP during the second interview, she asked whether she could use a known solution as a starting point and switch out groups of coins to make another solution.

Mia associated HV-Post and HV-Pre with a foundational geometry course she had taken as an undergraduate. However, she did not remember clearly enough what approach was best to take for either task. Instead, she appeared to consider different approaches while she explored the setting by drawing different triangles. For example, in HV-Pre, she tried labelling the isosceles triangles first by their congruent sides and then by their congruent angles. In HV-Post, she hypothesized that she might be able to inscribe a square by constructing two isosceles right triangles in a particular fashion. In both tasks, she at times considered using either the Pythagorean Theorem or the triangle inequality as a way to identify a counterexample. Again, this aligned with her philosophy of using what you know: she explained that, despite the fact that she did not expect the triangle inequality to be helpful, it was a tool that she knew she understood and so she considered an approach that would leverage it.

Mia’s response on the first questionnaire indicated that she disagreed with the claim that planning one’s steps in advance is helpful when solving a difficult mathematics problem. On the second questionnaire, she now strongly disagreed with this statement. Both responses align with Mia’s stated philosophy of immediately beginning a problem and getting to work without spending a lot of time thinking beforehand. During the first interview, she even argued that “if you plan out your steps in advance, you're limiting yourself.” She backpedaled slightly and added some clarification when she acknowledged that, “obviously, knowing what's the next step is cool. But if you are talking about 2, 3 steps in advance—it’s too much. No. Just one step at a time.”

As seen in previous examples, Mia sometimes committed to an approach without first gauging its effectiveness because she intended to use that approach as a means to better understand the problem space. In MV-Pre and LV-Post, Mia made some comment similar to one that she would later make while solving BP: “I’m just gonna try something and see what happens.” In each of these three tasks, despite no explicit planning, she forged ahead with some approach while acknowledging that it might not pan out. In LV-Post, Mia demonstrated another, different way to avoid planning out how an approach might unfold: when asked about a rectangular diagram that she had drawn and almost immediately disregarded, Mia admitted to the interviewer that she simply had not thought deeply about the efficacy of that diagram because she was already invested in another, more familiar approach.

In at least one task, Mia implied that she might sometimes plan at least a few steps in advance when considering different approaches. While solving HV-Pre, she conceded that any attempt to use the Pythagorean Theorem would ultimately become too unwieldy. In the same task, she adjusted the way she was labelling her isosceles triangles because she “didn’t like” her current method of doing so (identifying the pair of congruent sides; she began identifying the pair of congruent angles instead). While this activity might constitute a form of planning, it is clear that Mia’s appraisal of her tentative plans was largely based on her intuition. When she rejected a considered approach, her reasoning was often only that it did not seem like a good idea without any supplementary detail that might have explained why.

On both the first and second questionnaire, Mia agreed with the statement that drawing a diagram is helpful when solving a difficult mathematics problem. Still, after solving MV-Pre, Mia explained to the interviewer that her preference for solving difficult math problems is “definitely algebraically first, then visual.”

Mia drew the most diagrams in HV-Pre and HV-Post. In these tasks, she did not use her diagrams to enhance a rigorous mathematical justification of her claims; as such, she tended to rely heavily on her visual intuition (i.e., conclusions implied by the relative accuracy of her drawings and not by any mathematical underpinnings). For example, when solving HV-Pre, she initially attempted to cut an arbitrary right triangle from its right angle before concluding that that did not look right (with no further explanation). She then attempted to draw a “prettier picture” so that she might more accurately decide whether she had or had not actually produced two isosceles triangles. She eventually concluded that she had not because “they just look different sizes.” In HV-Post, Mia claimed that it was clear you could always inscribe a square in an isosceles triangle, but only came to this conclusion through visual appraisal of her exploratory diagrams.

In both MV-Pre and MV-Post, Mia often turned to the interviewer to reassure herself in situations that involved her visualizations. For example, In MV-Pre, she petitioned the interviewer to check the accuracy of her diagram of a specific parabola. She had redrawn this parabola at least once already after realizing that her initial diagram was incorrectly oriented. In MV-Post, Mia asked the interviewer to remind her how the parameter α affects the graph of the sinusoidal curve $\sin(\alpha x)$. In these tasks, also Mia often relied on informal visual intuition. Just as in HV-Pre and HV-Post, she again remarked that she would prefer something “prettier” and wished that she had access to computerized graphing software.

Mia’s best use of a diagram came in LV-Post. In this task, her first act after reading the task description was to draw a Venn diagram (see Figure 4.11 in discussion of the *Consider Approaches* strategy later in this section) and attempt to label it with data from the task description. She referred back to this diagram often while organizing the data into an appropriate system of equations, added additional labels as she developed a deeper understanding of the relationships in the task, and relied on the diagram to justify to the interviewer why she felt confident that her ultimate answer was correct.

In the first questionnaire, Mia strongly agreed with the statement that checking one’s work at every step is helpful for solving difficult mathematics problems. However, in the second questionnaire, she now disagreed with this same statement. Mia suggested that, in the first interview, she had been interpreting the question as it pertained to the act of proving. In proofs, “one mistake is going to make the entire thing wrong.” In contrast, while solving for a specific answer in a difficult problem, “one mistake here could lead you to a different idea.”

On tasks in which she produced a significant amount of work that could be checked (such as LV-Pre, LV-Post, and MV-Pre), Mia did monitor her progress. She reviewed her work both when confronted by cognitive discrepancies and when rereading the task description. For example, in LV-Pre, she noted that a particular product that she had written would be negative; if the final population of the town also came out to be negative, she recognized that something must have gone wrong. Mia argued, however, that this would not happen and pointed out where she would expect to later multiply by another negative value. Mia worked much of this task on her calculator, and at one point, she also stopped to check an intermediary output that was given in an unexpected and "really weird scientific notation" that she correctly interpreted as the result of an input error. Similarly, in MV-Pre, Mia was triggered to review her graph of a parabola when she found that two x-values unexpectedly produced the same y-value. During this task, it should also be noted that Mia at times petitioned the interviewer to check her work for her. In LV-Post, Mia checked all her arithmetic operations after reaching the end of a computation and realizing that she did not know if the equation she had produced was actually what the task description asked her to solve for.

Mia strongly agreed with the statement that justifying one's work at every step is helpful for solving difficult mathematics problems when she completed the first questionnaire; when she completed the second questionnaire, however, she indicated that she now slightly disagreed with this statement. Her explanation for this change mirrored her explanation for the similar change in her opinion of *Check Work*. Namely, rigorous justification is important for proofs but not for solving problems with a predetermined answer. In the latter case, making mistakes can actually lead to insight.

While working on tasks in the interview, Mia often verbalized specific justifications for why she chose certain approaches; however, many of these justifications were superficial ("I'm going to set up a ratio. I remember doing these fractions in algebra."; "I'm going to start working with the angles because I don't like working with the sides.”). Mia also provided relevant justification when she made specific mathematical claims; however, these were sometimes (but not always) spurious. For example, in LV-Post, she gave an excellent justification of how she knew that her answer was correct and how she could use the Venn diagram to explain her thought process. On the other hand, Mia’s mathematical justification for how she set up a system of equations in HV-Pre was incorrect and let to an inconsistent system. Sometimes, even when Mia’s justification was appropriate, she asked the interviewer whether or not she had stated something correctly. In MV-Pre, this happened when Mia claimed that she should look for values of which gave two solutions to a particular quadratic equation; she then asked the interviewer whether this decision was mathematically sound.

Overall, Mia’s justification is notable for its volume. She gave some form of justification after most decisions, despite the fact that it was at times insubstantial or imprecise. It was during the tasks in which she had trouble giving justification (such as HV-Pre and HV-Post) that Mia appeared most unsure of herself and how to proceed in with her work.

Mia agreed with the claim that it is helpful to verify the answer to a difficult mathematics problem, both on the first and the second questionnaire. In tasks for which she produced a complete, verifiable numerical solution (LV-Pre, LV-Post, and BP), Mia always verified that solution. In some cases, this was routine; Mia appeared confident in the answers she gave for LV-Post and BP but checked them against the stipulations of the task anyway. In LV-Pre, Mia was initially suspicious of her solution. She observed that the initial population of 80 men and 70 women was uncomfortably similar to the provided ratio of 8:7 and worried that she had made an unwarranted assumption somewhere. After checking that the final population also satisfied the given ratio, she seemed confident that she had done everything correctly. Mia was much less likely to verify her answer in the other tasks; this was partly due to the fact that her answers to these tasks were incomplete, and thus, not easily verifiable.

4.1.4.3 Mathematical Affect

Mia made several comments throughout both interviews about how she found particular tasks (every task on the first interview, as well as LV-Post and BP) interesting or even entertaining to solve. However, Mia was never initially interested in approaching a task and had to be encouraged to make a thorough attempt by the interviewer. For example, after briefly sketching a verbal outline how she might approach BP, the following exchange occurred:

> Mia: Or kind of do something like that, but... Do you want me to try to find one [an answer]?
>
> Interviewer: It doesn't appeal to you, does it?
>
> Mia: Noooooo…

However, after finding a solution, she conceded that “it was pretty fun, actually.”

During the first interview, Mia also commented multiple times on the fact that mathematics problems typically have more than one solution. However, when asked to elaborate on why she disagreed with the statement that solving a mathematics problem often requires one to know the correct formula, she clarified that “you have to know the method for some things. Let's be honest. You do.” By way of example, she pointed out that it would be especially difficult to solve for a missing side of a right triangle without knowing and applying the Pythagorean theorem. During the second interview, Mia spoke at length about her change in opinion (from somewhat agree to somewhat disagree) on the *HW Confidence* item from part three of the questionnaire. This change was brought about primarily because of the advanced level of graduate coursework Mia encountered during the previous semester; in particular, she found it difficult to learn more material at a faster rate. Despite the increase in difficulty, Mia remained cautiously optimistic in the face of graduate coursework: “I'm doubting it [her mathematical ability], but I know I can.”

4.1.4.4 Problem Solving Characterization

Mia often defaulted to familiar algebraic approaches. Her use of non-algebraic approaches was most effective when these visual methods were supplementary to a (successful) algebraic approach. When she was forced to lean too heavily on diagrams to provide a solution (either because her algebra was unsuccessful or because the tasks themselves were inherently visual), Mia was unconfident and imprecise. This was especially true of her justification, which relied often on visual intuition and half-remembered theorems. In these situations, Mia appealed to the interviewer as an authority figure.

In fact, Mia’s reliance on authority was a particularly striking aspect of every task in the interviews. Besides asking the interviewer to check her visualizations for accuracy, she also spent longer than other participants asking the interviewer about the problem settings after initially reading the text. She also, at least twice, asked the interviewer whether or not a stated justification is mathematically sound; in another task, she asked the interviewer if a possible approach was worth pursuing. Mia also spent more time than most other participants working with the calculator, and in both interviews remarked that she wished she had more advanced technology with which to verify her work. When she worked independently, without seeking guidance from an authority, Mia often chose to avoid formally planning how to approach a problem; instead, she valued improvised exploration that she argued could lead to unexpected moments of insight.

Mia’s habit of deferring to an immediate authority is perhaps surprising, given two facts. First, she did a good overall job of independently considering multiple approaches, drawing diagrams, verifying her work, and interpreting the task description correctly—

even in situations that she found unfamiliar. Second, she seemed to enjoy the work; she made several comments that certain tasks were interesting or fun and admitted to thinking about one of the first interview tasks in her spare time over the course of the intervening semester.

4.1.5 Sara

4.1.5.1 Formative Experiences

In both interviews, Sara had a lot to say about her teaching experiences (both as a tutor both in her university's mathematics clinic and at a local community college). As a teacher, she found that she was often motivated to find multiple, more effective methods for solving problems: "When students come up and they come with questions, at that point sometimes we just help them out with the question. Student leaves, when we're still stuck on the problem trying to make it easier so that some student comes, we can help again with much easier method." Simultaneously, teaching required Sara to know more than just formulas. She had to understand why a formula worked the way it did and how to explain this to someone else. These insights carried over into Sara's own work; as an undergraduate, she often "crammed up" in order to take exams and adopted the perspective that "I did [the] practice problem like this, and I'm just gonna do it again like that [on the test]." Now, as a graduate student, she often had to rely on her intuition and critical thinking skills in order to generate new or different ways of solving problems during exams.

Broadly speaking, Sara found that the difference between undergraduate and graduate mathematics was that "grad school professors really want you to *know* what you're doing, because yeah, you are studying for a Ph.D. or masters or whatever. Undergraduate, […] they give a lot of stuff right out of book." Sara's favorite mathematics course was undergraduate analysis, but not because the work was prescriptive in this way; this instructor emphasized connections between concepts and tried to ensure that "they have you part of the conversation. So you know what you're talking about." One way that they accomplished this was by prompting students to help create definitions, something that Sara "never knew that you're allowed to do!" They also imparted, explicitly or implicitly, a number of problem solving strategies. These included the power of visualization and the importance of clear and connected justifications, as well as specific heuristics like to "start with counterexamples. If you can't find any, then you try to prove it."

4.1.5.2 Problem Solving Strategies

Sara at first agreed with the statement that reading a problem carefully is helpful when solving that problem. On the second questionnaire, she indicated that she now strongly agreed with the same statement. While the interviewer did not ask her to elaborate on this change, Sara touched on her perception of *Read Carefully* while discussing *Plan Steps*: namely, she observed that during her previous semester of graduate school, she missed several questions in her analysis class because she read part of the problem, saw something that inspired a particular approach, and immediately began that approach before guaranteeing that she had a thorough understanding of the entire problem setting. This led to situations in which she proved related theorems similar (but not identical) to the theorem she was supposed to be considering. During the first interview, Sara also acknowledged her propensity for reckless problem solving; she said that "when I am way too prepared for anything, I rush into it and I don't read the questions."

Sara's impatience sometimes surfaced during the interview. In MV-Pre, she identified that she could use substitution in order to rewrite the given system as a single quadratic equation in x (that also involved the parameter α). She solved this new equation using the quadratic formula and identified solutions to the system in terms of α. However, she appeared to believe that this was all that was required of her and made no attempt to find values of α maximizing the number of viable solutions to the system until prompted by the interviewer. In MV-Post, even after clarifying the task description in a discussion with the interviewer, Sara approached the task by trying to solve the equation $\sin(\alpha x) = x$ for α after fixing a value of x. This led to conclusions which, while mathematically valid, did not help to answer the task at hand.

Sara had similar difficulties in HV-Pre and HV-Post. In each task, she concluded her work after showing the proposition was true for *some* triangle (but not necessarily *all* triangles). Furthermore, in HV-Post, she first attempted to inscribe a rectangle instead of a square; then, she attempted to circumscribe a triangle about a square instead of inscribing a square inside a triangle. Each of these related hypotheses could have been used as the foundation for a valid approach to the actual task, but it was clear during the interview that Sara had not chosen to consider related problems as a heuristic strategy; she simply did not attend carefully to the task description.

The only tasks in which Sara did not solve a slightly different problem were LV-Pre, LV-Post, and BP. In fact, for both LV-Pre and LV-Post, Sara carefully read the task description several times and revisited it for clarification while solving the task. In each of these tasks, Sara quickly and effectively parsed the task description for important values, labeled them with appropriate variable names, and generated an effective system of equations with little to no confusion about what the task required of her.

Sara engaged with the most exploratory practices when solving MV-Post. In this task, she substituted different values of x into the equation $\sin(\alpha x) = x$ in order to understand what values of α could then be used to satisfy the equation. Despite choosing a variety of different values ($x = 0$, $x = 1$, $x = 2$, and $x = \frac{\pi}{2}$) and drawing a number of appropriate mathematical observations in each individual case, Sara was unable to unify these observations into a single conclusion that might have helped her identify values of α that were a part of the required solution set. In the corresponding task from the first interview, MV-Pre, Sara only tested the effect of different values of α at the very end of her time with the task; at that point, the interviewer had pointed out to her that her previous efforts did not address the prompt given in the task description. In response, she examined the system under the assumption that $\alpha = 0$. When this led to values that contradicted her previous results, she did not test any other values.

In HV-Pre and HV-Post, Sara only drew at most one diagram that could be considered exploratory. She typically spent very little time with this diagram, moving on quickly to a specific approach once she identified something familiar during her exploration. For example, in HV-Pre, Sara drew a right triangle, asked herself (out loud) if bisecting the right angle would lead to a solution, and then immediately drew and labelled the sides of a more precise triangle with this method in mind. Reading the task description and exploring her first rough diagram took Sara roughly 20 seconds.

On both the first and second questionnaire, Sara slightly agreed with the statement that it is helpful to first try and imagine the form of the solution to a problem in order to solve it. This ambivalence towards visualizing a solution manifested in several ways throughout Sara’s interviews. As described at the beginning of this subsection, she sometimes solved related problems that did not align with the provided text. It is not likely that this would have happened if Sara had been especially mindful of the form of the solution.

On some tasks, Sara concluded her work and seemed satisfied with a solution that only partially (or did not at all) resemble what she was supposed to find. In MV-Pre, Sara did not reflect on whether her answer, given as (x, y) coordinate pairs, was what the task had asked her to find. On the other hand, Sara paid more careful attention to the form of her solution in those tasks that involved real-world contextual clues; for example, she seemed aware that BP required her to provide positive integer solutions and, while working the task, rounded off decimals in order to massage her answer into a form that agreed with the its constraints. On the other hand, after completing LV-Pre, she remarked that she had not considered the relative size of the final population before reaching her solution. While working the task, she also appeared not to assess the validity of her computations until they had entirely resolved. That is, while this task also required positive integer solutions, Sara did not seem wary of fractions or negative numbers that appeared during intermediate steps of her computation.

On the first questionnaire, Sara strongly agreed with the statement that considering multiple approaches to a problem is helpful when solving that problem. Despite valuing this problem solving strategy very highly, Sara explained that she was only likely to consider multiple approaches after attempting to solve the problem in a way that seemed most natural: “I usually go with my instincts. The first approach which comes to my mind when I read the problem. That's the one when I do that. And yeah, that's the first approach I try. Then if it doesn't work, I go back and think of other approaches.” Later, on the second questionnaire, Sara only agreed that considering multiple approaches can help to solve difficult mathematics problems.

Despite the above explanation, on some of the tasks, Sara entertained multiple approaches before deciding on one that she preferred. The different ways in which Sara used *Consider Approaches* are highlighted by the four tasks which she solved using a system of equations: On MV-Pre, she first considered representing the system using a matrix; she decided against this approach in when she realized that the equations were nonlinear. Instead, she tried to solve the system algebraically. When this failed, she recognized that one of the given equations described a circle with center $(\alpha, 0)$ but was unable to leverage this observation in order to begin another, distinct approach. On the other hand, In BP, Sara followed through with her initial decision to represent the system as a matrix. Only when it did not give her integer solutions did she try an algebraic approach, and only when this algebraic approach also did not give her integer solutions did she try a second (slightly different) algebraic approach.

In LV-Post, Sara did not attempt to use a matrix the solve the linear system she created. However, she did associate the task with others that she had worked in the past that involved overlapping populations; because of this perceived similarity, she wrote an equation (that she remembered using before) involving the probabilities that a given student played each sport. While she did not follow through with this approach, Sara demonstrated that she actively considered how the interview tasks might be similar to problems she had seen before. Finally, LV-Pre also featured a system of equations. However, Sara explained that she knew she could write and solve a simple linear system for LV-Pre after her first time reading the task description and felt that she did not need to consider other approaches.

Sara slightly agreed that planning one's steps in advance can help to solve a difficult mathematics problem when she completed the first questionnaire. However, on the second questionnaire, she slightly disagreed with this sentiment. Sara explained that, when she tries to plan too far in advance, it only causes her to make mistakes on whatever the current step is. Sara noted that this happens in particular when she is too eager to start a problem without fully understanding the problem situation first.

In both interviews, Sara's admission that she engages with problems after very little mathematical forethought held true. The most notable exception was in LV-Post, where Sara decided not to use an equation that tied together the probabilities of selecting a student from each population. While working the task, she initially rejected this approach because she anticipated that it would ultimately prove incompatible with the variables she had chosen to define (see Figure 4.12 below for both the equation and Sara's variables).

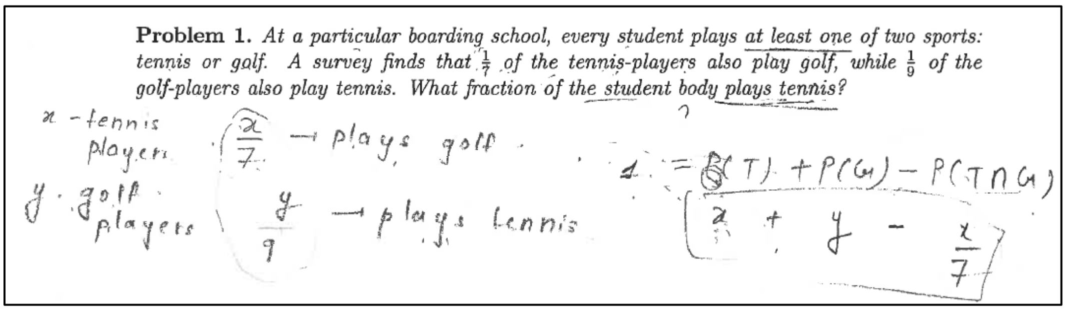


Figure 4.12 Sara's delineation of the task's constraints using variables and probabilistic equations.

At the end of solving LV-Post, when reflecting on the probability equation, Sara reconsidered her earlier judgement and hypothesized that it would be possible to solve the task with this equation and her existing variable structure. However, she now argued that such an approach would be needlessly complicated and inferior to the solution she ultimately chose.

On the first questionnaire, Sara strongly agreed with the statement that drawing a diagram of the situation can be helpful when solving a difficult mathematics problem. She only agreed with this same statement on the second questionnaire.

Despite her agreement with this claim, Sara drew relatively few diagrams compared to other participants. For example, she did not graph either equation in MV-Pre even after recognizing that one of them represented a circle being translated along the x-axis by the parameter α. She also did not draw any sinusoidal curves in MV-Post. Sara did use a Venn diagram in LV-Post, but only because she was prompted by the interviewer to explain why she chose to abandon her initial probabilistic approach and she found it “easier to communicate through a diagram.” It is not clear from the interview whether this sentiment about communicating with diagrams was made with respect to ideas in general or the specific idea Sara was explaining to the interviewer at the time.

On the other hand, Sara did engage with significant visualization while solving HV-Pre and HV-Post; for both of these tasks she drew many triangles. Notably, only one triangle from each task was a quick, unmarked sketch without labeled variable names or angle measures. These were the exploratory triangles that Sara used only to decide quickly on an approach to investigate with more depth and precision. Her other diagrams integrated a significant amount of hypothetical numerical data and variable names that Sara used in computations (see Figure 4.13 below).

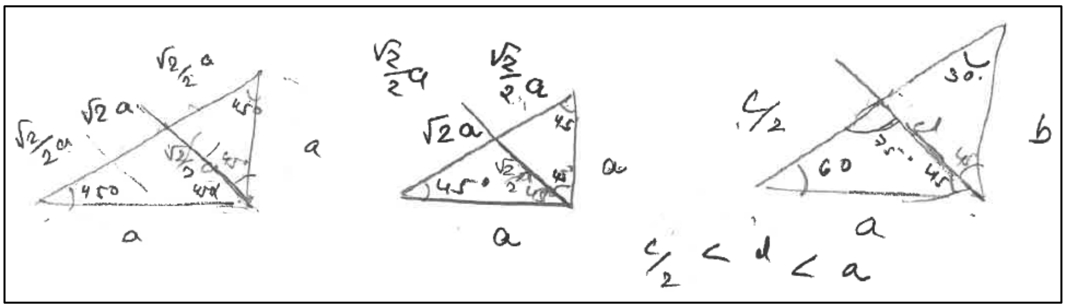


Figure 4.13 Sara’s heavily annotated diagrams from HV-Pre.

Sara strongly agreed that checking one’s work at each step is helpful, both when she filled out the first and the second questionnaire. She did check her work at different times throughout the interviews, although it was not clear at times what prompted her to do so. In MV-Pre, for example, she incorrectly applied the quadratic formula and found imaginary solutions to the underlying system of equations, which did not trigger her to review her work; on the other hand, in solving HV-Pre, Sara labeled the hypotenuse of an isosceles right triangle as the length of a leg multiplied by $\sqrt{2}$. She stopped to check this value, even though it had no bearing on her further work in the task and was not initially incorrect. Sara also used the provided calculator for a number of important calculations, sometimes immediately recalculating a value she had just computed.

Sara agreed that justifying one's work at every step is helpful when solving a difficult problem when she first responded to the questionnaire. When she filled out the questionnaire a second time, she only slightly agreed with the same statement. Sara engaged with justifying several claims throughout her work during the interviews, with inconsistent results. For example, after solving LV-Pre, she was able to appropriately justify her decision to create a system of equations by appealing to the data given in the task description. In LV-Post, she also provided good justification for several claims about the relationships between the variables she chose to define. In HV-Pre, she was able to eventually explain a process for cutting an arbitrary right triangle into two isosceles triangles and why other processes would not be as effective.

On the other hand, Sara made several mathematically poor justifications. When the ordered pairs that she thought would solve the system in MV-Pre turned out to be imaginary, the mathematical explanation she provided to justify this fact was incorrect. In HV-Post, one of her first approaches revolved around cutting the base of the triangle into three equal parts; no justification for this ratio was ever explained, and it is ultimately unrelated to the task at hand of inscribing a square in a triangle. While solving both HV-Post and HV-Pre, Sara often integrated the midpoint of one or more sides of a triangle into the approach with which she was currently engaged. She justified this practice by explaining that the midpoint is often associated with important geometric properties and allowed her to divide a side evenly in half; despite the fact that both these claims are true, in her efforts to incorporate the midpoint, Sara often relied on spurious mathematical claims that she did not entirely remember, and which generated equally incorrect justifications.

On the first questionnaire, Sara agreed with the statement that verifying one's answer at the end of the problem is helpful for solving that problem; on the second questionnaire, she strongly agreed that verifying one's answer is helpful.

Sara only reached a verifiable numerical answer in LV-Pre and LV-Post. She failed to notice a small arithmetic error in LV-Pre that could have easily been corrected by comparing her final answer against the provided ratios in the task description, which she did not do. On the other hand, after mentioning that her final solution "sounds right" in LV-Post, the interview prompted Sara to be more explicit. She then gave a thorough explanation of why this answer was reasonable. Furthermore, the original assertion that the fraction "sounds right" was in response to the correction of another arithmetic error also pointed out by the interviewer. Given the similarity of these two situations, it is unclear whether Sara was ever independently motivated to verify her own solutions.

4.1.5.3 Mathematical Affect

None of Sara's responses to parts two or three of the first questionnaire stood out significantly when compared to other graduate students. However, several of her opinions changed notably between the two questionnaires. For example, on the first questionnaire, Sara somewhat agreed that she often considers whether her solution was the best way to solve a problem. On the second interview she now strongly disagreed. Sara hypothesized that, on the second questionnaire, she was likely considering the item as it applied to an exam rather than homework problem. She explained that on exams, "when I'm usually time crunched, I try to go by the method I am on. So that the point is getting the answer and knowing what you did and being confident about what you wrote." On the other hand, after homework assignments, she preferred to look back over her work in order to make sure she has a thorough conceptual grasp of the material. Sara also lowered her opinion of *Memorization* as a contributing factor for problem solving success. She attributed this change in belief directly to a professor from her first semester of graduate school, who showed her that multiple, equally functional definitions can describe the same phenomenon. As a result, “maybe you can write it in your own words but explain what you mean. So priorly [sic] I never knew that you're allowed to do that with definitions!” Sara added that, if you were determined to write the exact definition or formula, “at a point, everything is derived from something, so you can derive what you're talking about.”

Sara also reflected on her love for computations at various points throughout the interviews. She described doing her friends’ homework assignments for fun when they were in a college algebra class; as a young child, she also remembered learning how to factor quadratic expressions and practicing pages of such problems independently in her spare time. Ultimately, this love of computations culminated in a calculus course; Sara mentioned several times that she enjoyed taking derivatives or calculating integrals. On the other hand, she also repeated that she did not like any of geometry, linear algebra, or abstract algebra as subjects.

4.1.5.4 Problem Solving Characterization

In many interview tasks, Sara’s impatience to begin working on the first viable approach that she could conceive of led her astray. Often, this was because the approach she chose only attended to the easiest or most familiar part of the task; Sara sometimes found herself with a related or partial solution that did not answer the actual question at hand, but with which she was nonetheless satisfied. During this phase of her problem solving, Sara usually provided reasonable justification (at least in the scope of the sub-problem that she had chosen to solve). If the interviewer pointed out that she had not solved the entire task as stated, Sara’s quality of justification and confidence in her work often declined as she was forced to use properties and approaches with which she was less familiar. This trouble was compounded when she arrived at a new conclusion that in some way contradicted her prior work. Tasks with numerical data in the associated text were the exception to this pattern. Sara read these tasks carefully, revisited the text often, correctly defined variables, and implemented methods with complete and mathematically sound justification that answered the task as written. This is reflective of Sara's admitted preference for computational, symbolic problems.

Sara used relatively few visualizations when solving the interview tasks, and when she did, these visualizations often served as organizational tools for collating variables and data rather than as dynamic representations of important aspects of the task. Sara occasionally used diagrams to facilitate her ability to explain her thought process and mathematical conclusions, but only when she felt compelled to do so for the benefit of the interview process. When she explained her work, she was also more likely to retroactively catch arithmetic errors that affected her solution. Otherwise, she checked her work seemingly arbitrarily.

4.1.6 Frank

4.1.6.1 Formative Experiences

In the first interview, Frank explained that he didn't feel as though any singular experience had had a dramatic effect on his problem solving strategies. Instead, he argued that his approach to solving difficult mathematics problems has remained "generally the same, and maybe has become refined through experience." He added that "I often find myself, doing math now, being reminded that it's the same way I approached it when I was in high school." The one exception, according to Frank, was that he began to have more trouble as he got older storing and retrieving important facts from his long-term memory.

In the second interview, reiterated that most of his problem solving development was owed to the "cumulative effect" of simply doing more mathematics. He further described that there was a "benefit of seeing some things for the second time. Having a better understanding of material and a better understanding of my own strengths and weaknesses." However, he did also admit that a recent sequence of exams may have changed his problem solving strategy in a specific way: he found himself "more willing to just start writing and playing with what I know" because he hoped to get as much partial credit as possible on those exams. Frank also mentioned that, by virtue of trying to find ways to give his own students partial credit while grading assignments as a GTA, he had become more aware of the need to provide thorough justification and show a lot of work in his own problem solving.

Frank described two courses in which he had a notably positive mathematical experience: a secondary school physics class and a numerical methods course. In the former, Frank found the teacher engaging—but more importantly, he appreciated that physics provided a practical application that contextualized the abstract mathematical content he had so far learned. He noted, "It kind of added a 'so what?', right, to the years of studying math." Furthermore, these application problems were typically given as word problems that required him to think carefully about and make sense of the underlying problem setting. On the other hand, the content of Frank's numerical methods course was not what he found compelling; instead, he valued his professor's office hours. There, he had the opportunity to explain his thought processes to classmates who also attended those office hours (often, with minimal input from the instructor). Frank reflected on this experience that "you learn different when you are teaching others than when you teach yourself."

4.1.6.2 Problem Solving Strategies

On both the first and second questionnaire, Frank indicated that he strongly agreed with the statement that reading the text of a problem carefully is helpful for solving that problem. In fact, during the second interview, Frank claimed that "reading a problem carefully is one of my strengths." During the interviews, Frank did read each task one or more times before beginning and often revisited the text when he was stuck or when checking his comprehension of the problem situation. For example, after completing LV-Pre relatively quickly, he double checked that there was no "wording that could trip you up" in the task description; that is, he wanted to make "sure I fully understand exactly what it's asking" before committing to the solution he had found. Frank later explained that he often rereads problems in this way, even before starting the problem itself, because he especially wants to avoid doing a lot of work in service of an approach that may have been founded on a fundamental misunderstanding.

In a number of specific instances, Frank demonstrated that his careful and continuous readings of the task descriptions facilitated his ability to solve the task at hand. In LV-Pre, for example, his approach generated both the current and former populations of the town; he committed to choosing one as a final answer only after verifying that the question asked for the newer population. In both LV-Pre and LV-Post, Frank also defined a reasonable number of variables and took care to use them appropriately for capturing relationships between the data in the task. In HV-Pre, the first thing that Frank did after reading the task was remind himself of the definitions for right and isosceles triangles. After doing so, he drew a representative example of each and revisited the task description to make sure there were no other explicit references to fundamental geometry concepts that he would have to know in order to understand the task. In several problems from both interviews, Frank underlined or circled parts of the task description as they became relevant to his current approach.

Frank took steps to explore the problem space if doing so could help him confirm or deny his initial intuitions about a task. For example, after reading the text of LV-Pre, he immediately constructed two equations with the variables x and y that represented the (old and new, respectively) common factor between the male and female populations of the town. When asked to explain to the interviewer exactly what these variables represented, however, Frank had trouble justifying his intuitive notion that this approach was correct; to compensate, he tested different values of x and y to imagine what that would mean in the situational context of the task.

In other circumstances, when Frank explained that no approach immediately jumped out at him, he opted to "play with it [the components of the task] and see what I can do." This was a strategy which he referred to as both "trial and error" and "thinking on paper" and which he applied to both MV-Pre and MV-Post. In MV-Pre, exploratory algebraic manipulations led to an important structural observation: the system of equations could be reduced to single a quadratic equation. In MV-Post, a similar algebraic exploration proved fruitless; here, Frank (reluctantly) turned to visual methods in order to compare and contrast $\sin(x)$ and $\sin(2x)$. In both tasks, Frank eventually decided to conduct numerical tests by substituting in different values of α and attempting to reason about resultant changes to the system. As explained in previous paragraphs, however, Frank was reluctant to perform significant computational work if he was not convinced it would be worthwhile; thus, his exploration of these tasks by testing sample values was half-hearted and did not lead to significant insight. For example, in MV-Pre, Frank identified the second equation in the system as some kind of circle. However, other than $\alpha = 0$, he balked at the prospect of testing sample values of α and, as a result, could not make sense of how precisely α transformed the circle.

Frank's resistance towards engaging with numerical exploration stands in stark contrast with his willingness to visually explore both HV-Pre and HV-Post. In the latter task, he also remarked that "I'm going to do some trial and error, what I often do when I don't know the right way to do things." Presumably because a quick geometric sketch felt less committal to Frank than a numerical computation, he spent most of his time with HV-Pre and HV-Post exploring a variety of different test triangles. In particular, he commented that "the insight will come when I look at the extreme versions of triangles" and followed through by imagining a diverse array of different cases. What exactly differentiated one case from another in each task was often the focus of Frank's exploration; he spent time identifying similarities between different triangles and attempting to understand how, for example, his strategy for inscribing a square in a right triangle differed from or was similar to his strategy for inscribing a square in an isosceles triangle. The fact that these two types of triangles were not necessarily mutually exclusive led to an abundance of exploratory diagrams as Frank sought to organize his intuition for different cases into a mathematical framework.

Frank first agreed with the statement that imaging what a solution might look like is helpful when solving a difficult problem. In the second questionnaire, he strongly agreed with the same statement. Frank was able to follow through on this assertion when solving the two interview tasks with real-world context, LV-Pre and LV-Post. In LV-Pre, after writing down a system of equations that described the population of the towns, Frank spent some time considering whether the relative size of the town (either before or after the increase in population) could be inferred by how much the gender ratio changed as the population increased. In LV-Post, one of the first comments Frank made about the task description was that the size of student body was irrelevant because the task only required the fraction of the student body that played tennis. Returning to this fact later in his work, Frank repurposed a pair of variables he had already defined into one expression that represented the fraction of tennis players in the student body. He centered the rest of his approach around understanding this fraction in terms of the data given in the task description. When he eventually got an answer to LV-Post and checked this solution with an arbitrary student population, Frank found that the number of tennis players turned out to be a non-integer. He clearly understood that this hypothetical solution did not make sense in the context of the task.

In comparison, Frank exhibited some difficulty correctly recognizing that the non-integer form of his solution to MV-Post was also inappropriate. After starting that task by considering the case of $\alpha = 2$, he eventually entertained the use of non-integer values of α despite the fact that α was not only stated explicitly to be a positive integer but also implied as such by the fact that it must simultaneously represent the number of solutions to the system. His eventual solution, that $\alpha = \frac{\pi}{2} + 2\pi k$ for k in the natural numbers, reflects the fact that Frank lost sight of the form of the solution.

On both the first and second questionnaire, Frank agreed that considering possible approaches is helpful for solving a difficult problem. Indeed, in almost every interview task, Frank entertained a number of possible approaches and enacted most of them. The exception to this rule was perhaps LV-Pre, which he answered quickly (and accurately) via his first and only approach.

To illustrate Frank's propensity for multiple approaches, consider his work on MV-Pre: first, he a standard elimination algorithm to solve the system. When this approach produced a quadratic equation equal to zero, he attempted to factor it; difficulty here led him to consider the quadratic formula, which he deemed too computationally intensive. Instead, he elected to try testing values of α. He used these test values to facilitate further factoring attempts but was ultimately unsuccessful. Next, he returned to the initial system and considered a substitution algorithm. In little time, however, he realized that this would generate the same quadratic equation. Finally, he attempted to visualize the two individual equations graphically despite the fact that he anticipated that this approach would be difficult and time-consuming. At this point, time constraints forced Frank to move on to a different task. He asked to take a copy of the task with him so that he could continue to work on it after the interview.

This vignette of only one interview task is representative of Frank’s work in the rest of the tasks. He considered many approaches and tended to avoid those that would be especially time consuming. For another example, in HV-Pre, Frank briefly considered an approach that utilized the Pythagorean theorem; he decided against it when he was uncertain how this approach would handle his current understanding of the multiple possible cases involved in the task without a lot of algebraic tedium. Frank also spent more time working, in general, than other participants: he was one of only two who had to be stopped from continuing their work due to time constraints, and the only one who faced this issue in both interviews. Finally, he appeared to cross-reference his multiple approaches in an effort to understand how they were related. Using HV-Pre again as an example, at one point, Frank attempted to “build” a right triangle out of two isosceles triangles, inverting the actual requirements of the task. Later, when he was eventually successful in cutting a right triangle into two isosceles triangles, he described how he might have “built” this right triangle under his previous approach.

Frank indicated on the first questionnaire that he agreed with the statement that planning one’s steps in advance is helpful when solving a difficult mathematics problem. On the second questionnaire, he appeared to have changed his mind; he indicated that he now slightly disagreed with this statement. By way of explanation, Frank pointed out that his answers on the second questionnaire could have been influenced by the recent interview tasks; in them, he often made some amount of headway by recombining known elements in different ways. This, he argued, did not involve any explicit plan. Instead, it relied on his willingness to experiment and ability to recognize familiar structures that may be uncovered during his exploration. Frank expressed the opinion that most of mathematics, generally speaking, can be done the same way.

Certainly, as discussed in prior sections, Frank was more than willing to consider a multitude of viable (but largely exploratory) approaches in hopes of discovering a breakthrough into some familiar, rigorous methodology. However, Frank engaged with more planning than he gave himself credit for. When considering a possible approach, he often rejected those that he did not anticipate leading to useful conclusions. As previously noted, sometimes Frank saw a possible alternative approach would not resolve his current difficulties (elimination and substitution algorithms in MV-Pre would both lead to the same intractable quadratic equation). Other times, Frank recognized approaches and gauged how effective they might be before employing them; in HV-Post, he briefly considered circumscribing an arbitrary triangle about a square before declaring that that approach would likely be a “mess.” Finally, on some occasions Frank weighed the comparative benefits of two different approaches before deciding on one that was easier. This typically occurred when he was deciding on one of the two variables to solve for in a system of equations and what method to use to do so.

Frank agreed that drawing a diagram can help one solve a difficult mathematics problem, which he indicated on the first questionnaire. On the second questionnaire, however, he now strongly agreed with this sentiment. Despite his appreciation for visualizing, while solving MV-Pre, Frank noted that he was not fond of using graphing techniques when a problem can be solved symbolically. When he eventually conceded to attempt the task by graphing the two equations, he lamented that “graphing relies on having memorized a lot of rules,” and that he has trouble remembering them. To prove this point, he noted that both equations were almost certainly conic sections but that he was not familiar enough with those types of equations to graph them quickly and insightfully. While comparing his symbolic and visual approaches on MV-Pre, Frank also observed that graphing functions can be beneficial, despite initial difficulties, because purely symbolic approaches make certain properties harder to spot; later in this task, Frank’s sketch of a circle and a hyperbola would reveal to him a symmetry in the system of equations that, up until that point, his algebraic manipulations had not revealed.

Frank also drew a Venn diagram for LV-Pro, which he described as a device to help him conceptualize the problem space: “the algebra became more obvious when I looked at it as […] set intersections.” In comparison, he remarked that HV-Pre and HV-Post might have actually been impossible to solve without drawing the prerequisite shapes. That is, the visuals he employed in those tasks were so closely tied to the tasks themselves that he found them unavoidable necessities, whereas the Venn diagram was only a convenient tool. In line with his earlier claim that graphing relies on some amount of memorization, Frank noted during his work on HV-Pro that there were probably some “10th grade geometry rules” that he should remember that would add rigor and merit to his drawings. In HV-Pre, he also alluded to a “high school geometry way of doing it” that he must not be aware of and which would facilitate a straightforward approach.

On both the first and second questionnaire, Frank strongly agreed with the statement that checking one’s work at every step is helpful when solving a difficult mathematics problem. After solving LV-Pre, Frank pointed out that he does a lot of double and triple checking of his work because he feels prone to making silly mistakes.

This sentiment reappeared in the second interview: while graphing the sine function in MV-Post, Frank said "I still don't trust myself" after labeling his graph with important intercepts. He then spent time drawing a unit circle so that he might recheck his graph. Later in the interview, when he still found himself unable to trust his memory of trigonometric values on the unit circle, he appealed to a calculator in order to fill in certain points on his graph of a sinusoidal curve.

During the interviews, Frank checked his work almost continuously. In this semi-perpetual state of review, certain notable instances stood out because they were caused by particular stimuli and not because they were an unconscious, default action. First, Frank checked his work on MV-Pre before switching from one approach to another: he did not anticipate the new approach (applying the quadratic formula) to be especially easy, so he wanted to make sure that the difficulties he was experiencing with his current approach (factoring) were not due to an avoidable arithmetic error. Second, in LV-Pre, Frank checked his work when the two variables defined for the task, x and y, turned out to have the same value. Because he had previously spent some time justifying that these variables should be differently named because they could have different values, this disconnect appeared to trigger his monitoring instinct.

At most other times, Frank appeared to check his work constantly, especially whenever he found himself stuck or unsure of his direction in the greater landscape of a task. This pattern of behavior only did not apply to HV-Pre and HV-Post, wherein Frank never really produced any formal, substantive mathematics that might be considered checkable. In these tasks, whenever he found himself stuck, Frank simply drew a different diagram instead of reviewing his current one.

On the first questionnaire, Frank agreed that justifying each step one takes in solving a difficult mathematics problem is helpful when solving that problem. His opinion was unchanged on the second questionnaire. When working on the interview tasks, Frank's justification for *why* he was choosing a particular approach was usually reasonable. For example, he identified in both HV-Pre and HV-Post that he only needed to find a single counterexample in order to know the proposition was false; he spent a lot of time, then, examining a variety of differently-shaped triangles to try and find one. In other tasks, he explained why he had chosen to solve for a certain variable a certain way in a given system of equations.

On the other hand, Frank's justification of certain mathematical claims which he made while solving a task were often weak. While solving MV-Post, for example, he attempted to appeal to the intermediate value theorem; his reasoning was imprecise and led to an incorrect conclusion about the value of $\sin(x)$ over a given interval. In HV-Post, he suspected that he would need one or more of the triangle congruency theorems but did not specify which or actually attempt to use any of them. These instances were few, however, because Frank did not often appeal directly to specific mathematical principles. He explained that he would like to but was limited by his failure to remember the specific properties that he would need. For example, while solving MV-Pre, he lamented that "there's some geometry theorems that would make this provable," but that he did not know them. Instead, he joked that his conclusions were necessarily justified "by inspection."

Frank agreed with the statement that verifying one's answer to a difficult problem is a helpful practice when he completed both the first and second questionnaire. After solving both LV-Pre and LV-Post, Frank took steps to make sure that his answer aligned with the numerical data given in the task description. In MV-Post, when he also reached a numerical solution, he did not have time to check his answer; still, he commented that he would like to continue checking different values of α that might confirm or deny the intuition he had used to reach his solution.

Interestingly, Frank also appeared to verify his answer for HV-Pre. Despite the fact that this task was more proof-like than those that preceded it (and thus was less susceptible to traditional methods of verification), Frank continued working on the task after reaching an initial conclusion after 7.5 minutes. In fact, he changed his conclusion three times over the course of the 23.5 minutes that he spent on the task. In each case, once Frank had declared that the proposition was true or false, he tried to justify this reasoning; invariably, he would come to a new realization during this process that he felt actually refuted it. Frank was noticeably unhappy relying on any mathematical claim of which he was personally unconvinced, and his continued work on this task after each failed attempt to verify his current conclusion was proof of this trait.

4.1.6.3 Mathematical Affect

Like several other participants, Frank first contextualized his answers to part two of the questionnaire by explaining that “the bar is kind of low” when it comes to “success in mathematics.” That is, “there's plenty of mediocre mathematicians all over the world making a living, doing good stuff,” and one does not need to be the next Newton or Gauss to find success in mathematics. He added that he disagreed fundamentally with the idea that any one of the listed characteristics is *required* for success, and that shortcomings in any one area could usually be compensated for with strength in another. This is why he only somewhat agreed that creativity is important for mathematical success, an opinion lower than average for graduate students. Frank argued that

> if you're somewhat lacking in creativity, you can often get by just memorizing or, you know—there might be limits to what kind of math you can become good at. But it's not—I would say creativity allows you to excel, but it's possible to be successful to a certain level without being terribly creative.

Frank found that success in mathematics classrooms specifically was entirely possible for students who supplant other positive characteristics with memorization:

> there can be, and are, students who are just good at memorizing what the professor demonstrates. They can recreate it. Don't have a creative bone in their body and just can kind of brute force their way through.

Frank’s inability to memorize important identities and concepts was an issue he repeatedly lamented as he worked on interview tasks, particularly where trigonometry was concerned. His attempt to surmount this shortcoming relied on visualizations and intuition (which he suspected was inefficient); however, Frank admitted that he did not feel as though he had the natural, powerful intuition that he observed in younger mathematicians.

After filling out the second questionnaire, Frank only demonstrated a few small differences in opinion: on both *HW Confidence* and *Best Approach*, Frank shifted from somewhat agreement to somewhat disagreement. He explained that this slight change was probably due to the fact that he often feels “more humbled” at the end of a semester (or conversely, that he feels more optimistic near the beginning of a semester). In his first semester of graduate school, for example, he routinely encountered proofs wherein “you kind of just have to know what the trick is and regurgitate it,” but that that trick was often “a really clever idea that I wouldn't come up with in a million years.” Frank did not explain why he originally agreed with *Best Approach* but noted that now, “when I come up with a solution I'm just happy I have a solution. I don't try to think whether it was the best.”

4.1.6.4 Problem Solving Characterization

Frank operated under the assumption that, often, problem solving success was reliant on knowing more mathematical facts. Reflective of this mindset, his self-described approach to problem solving was to use his mathematical intuition as much as possible; he felt that this compensated for his inability to memorize and apply the specific facts and/ or techniques that he had been formally trained to use but could not remember. He elaborated that this application of intuition often manifested as an attempt to “muddle through it” and that he sometimes spent a long time thinking about problems without reaching a solution. However, because “I enjoy the process, even if I’m going in the totally wrong direction,” Frank did not particularly mind following through on multiple approaches. This was clearly observed in his interviews; more than many other participants, Frank was unwilling to stop working on tasks that he had not yet answered satisfactorily and often went through several alternative approaches instead of identifying why his current approach may not have worked. Despite his willingness to attempt an interview task multiple times, he often balked at certain approaches that he thought would be especially messy or computationally intensive. Frank filled any downtime during this process, which he broadly referred to as “trial and error,” with reviews of both his arithmetic and the task description.

When stuck on a task, even as he continued working, Frank sometimes expressed his belief that there must be some necessary technique that he simply did not know or could not remember that would make the task trivial. This was especially true when he produced visualizations: Frank lamented his failure to memorize both important geometric principles and the graphs of certain equations. Ultimately, Frank seemed to feel that his lack of specific content knowledge caused his approaches to be inelegant and particularly arithmetic-heavy. For example, after working on one task (that he had failed to solve after significant effort), he remarked that someone who “knows their stuff” could have probably finished it in 30 seconds.

4.1.7 Julie

4.1.7.1 Formative Experiences

Julie described her favorite mathematics experience, a secondary school precalculus class, in significant detail. She enjoyed this class primarily because of the efforts her teacher took to make exams less stressful (e.g., allowing students to create and use memory aids). However, Julie also commended the fact that projects in this course required her to link mathematics concepts to practical applications in interesting ways; this perspective was augmented by the instructor's frequent use of videos that related new topics to "real-life concepts." Finally, Julie appreciated that instruction included many worked examples and that she had the opportunity to engage in mathematical discussion with her classmates while completing group worksheets.

From her first semester of graduate school, partner discussions in her analysis course helped Julie to realize that problems could have multiple solutions because "I know what I did was right, and then I listen to my partner and I know he's also right." This course also impressed upon Julie the need for mathematical formality when proof-writing, which she finds particularly difficult. However, she noted that ""even though I don't formally write the correct answer, if it sounds like I know what I'm talking about, then she [the instructor] would give me credit, and that makes me feel way more—way less pressure." This, in turn, has given her more confidence in her own mathematical abilities. Finally, the instructor of this course drew many visual representations of mathematical concepts, which Julie found helpful. She argued that other courses are difficult not because the content is difficult, but because they "write a bunch of math symbols"; instead, Julie would prefer if they "took a minute to draw it out."

Julie mentioned one other formative mathematical experience: an undergraduate introduction to proofs course in which the instructor, somewhat ironically, "didn't really explain how to approach proofs or anything. But he just like wanted us to go up on the board and like stumble." Julie admitted to preferring the more-structured approach of other courses but appreciated seeing a less rigorous treatment of mathematics content. Her biggest take-away from this course, however, came after hard work to pull up a low initial grade: "it's okay to fuck up […] and then like, come back because you understand things."

4.1.7.2 Problem Solving Strategies

Julie strongly agreed with the statement that reading a problem carefully is helpful when solving that problem, and she did so on both the first and second questionnaire. She explained to the interviewer that the most important parts of a problem might include any numerical data as well as "what they're asking, and then I read back the numbers and see if they're relevant." She added that she has been told since kindergarten to always annotate the text of word problems and that, apparently, this lesson had stuck.

When she first encountered LV-Pre, Julie circled and underlined what she considered the important parts. Later, after her initial approach had failed to provide an actionable strategy, Julie returned to the task description and tentatively explored a secondary approach in the margins surrounding the text, drawing arrows to connect her work to the numerical data provided. Julie made a similar number of markings on the text of LV-Post, which also contained significant numerical data.

Julie repeatedly revisited the text of some tasks when exploring the setting or justifying her work. For example, after creating some cursory diagrams of triangles in HV-Pre, Julie reread the stipulations of the task to assess how she might now cut the triangles. After Julie had stopped working LV-Post, the interviewer asked her to explain why her first step when approaching the task was to draw a Venn diagram. Julie responded by pointing to the part of the task description that implied there was an overlap between the populations: “the information they gave us was saying some people did both sports, so I thought it would be helpful if I wrote out what we knew.”

Julie’s work for both LV-Pre and LV-Post was entirely characterized by numerical exploration. That is, she spent her entire time on these two tasks manipulating the given numerical data. She assigned no variables and wrote no equations, but instead searched for common denominators, added or subtracted different values, and looked for patterns. When asked by the interviewer how she planned to use one of the several sums that she produced, she said that she had not yet finalized any further strategy.

Julie appeared apprehensive about using geometric sketches to explore HV-Pre and HV-Post. In HV-Pre, she spent four minutes thinking about the tasks without drawing any triangle; after drawing her first triangle, however, she concluded the statement in the task description was false after only one additional minute of consideration. Similarly, Julie concluded that HV-Post was also false within one minute of drawing any triangle on the page. She was less conservative about visualizing in MV-Post, the other only other task for which she spent some time exploring the setting. Here, she used her graphs of the two given equations to reinforce her intuition about how the parameter α affected the shape of the sinusoidal curve and to try and characterize which values of α might be solutions. After remarking that she was not sure what would happen if α gets “really big,” for example, Julie attempted to draw a sinusoidal curve with an arbitrarily short period to observe what the number of intersections might be.

On the first questionnaire, Julie indicated that she somewhat agreed with the statement that imagining a solution would help to solve a difficult mathematics problem; on the second questionnaire, she agreed. During interviews, Julie exhibited some awareness of how the form of the solution might affect her approach. In LV-Pre, for example, Julie spent a significant amount of time manipulating and reducing the ratios given in the task description. Her focus on ratios also caused her to avoid manipulating certain numerical non-ratio data, such as the 90 men and 80 women who moved to the town. Despite her focus on ratios in fraction form, however, Julie remarked that she would ultimately need to provide the final population as a positive integer value; it was this necessary shift that, in part, prevented her from making further progress towards a solution.

In other tasks, however, the way that Julie imagined the solution might look did not appear to affect her choice of approach. In MV-Post, Julie claimed that no value of α that was an odd number would be part of the solution set. Despite the fact that her claim was incorrect (α, in fact, *must* be odd), she did not leverage this observation about the form of α to refine the way in which she explored the problem space or decide on approaches. In LV-Post, Julie appeared aware that her answer should ultimately be a proper fraction. This fact only affected her work when she guessed that one of the many proper fractions she had produced through numerical manipulations was the answer to the task, after which she immediately refuted herself and returned to her work.

Julie agreed that considering multiple approaches could be helpful when solving a difficult mathematics problem. She strongly agreed with the same statement on the second questionnaire. She added that she only tends to consider more than one approach, however, if her initial idea for how to solve a problem fails to pan out.

During interviews, Julie only appeared to consider multiple approaches for LV-Pre and LV-Post; and, as promised, only after failing to make progress with an initial approach. In LV-Pre, this manifested as multiple ways that she might combine and compare the several ratios given in the task description. In LV-Post, Julie drew two different styles of Venn diagram in an attempt to quantify the intersection of the two populations in question. She explained that the first Venn diagram was modelled after the methods that she remembered from past mathematics courses; these methods involved drawing a Venn diagram and subtracting the intersection from the whole. Notably, Julie entertained any secondary approaches for much less time than her first approach unless indirectly encouraged by the questioning of the interviewer; when her secondary approach was actually more effective, this proved problematic.

On both the first and second questionnaire, Julie strongly agreed with the statement that planning one's steps in advance would be helpful when solving a difficult mathematics problem. She explained that, before committing to any one approach, she would "want to think about it for a second. Like, will this lead me to my goal?" Julie described a situation in which such forethought would be beneficial: some problems in mathematics have only one method of solution (such as, she claims, the proof that the square root of two is irrational; this must be done by contradiction). If one were to start such a problem without planning ahead, one might accidentally choose to approach the problem via a method that would never reach a solution.

Because Julie seldomly considered multiple approaches during the interview tasks, she correspondingly did not often have many opportunities to plan ahead. Most strikingly, in fact, was that the limited planning with which she did engage appeared to backfire more often than not: for example, in LV-Post, she rejected a strategy involving a rectangular Venn diagram because she did not anticipate that it would be significantly different from her first approach using a traditionally circular Venn diagram (see Figure 4.14 below). This alternative diagram, and the way she described using it (see the discussion of *Draw Diagram*, below, for details), was actually an efficient way to solve the task.

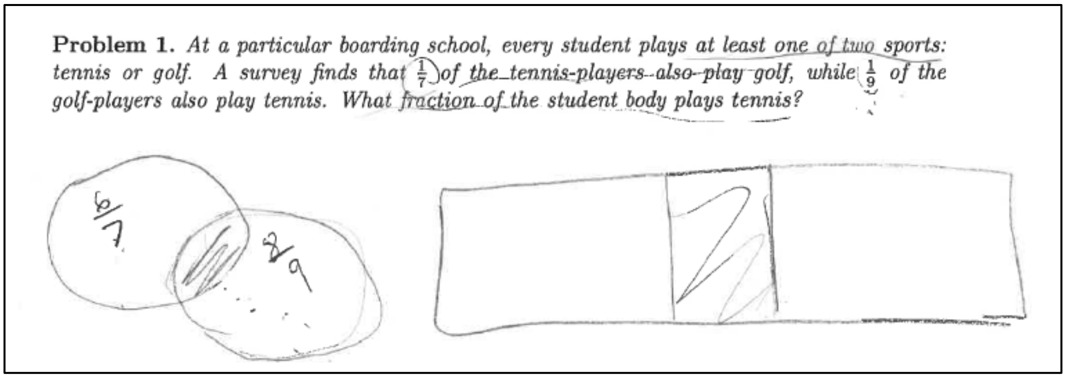


Figure 4.14 Julie's two Venn diagrams from LV-Post.

Initially, Julie strongly agreed that drawing a diagram would help to solve a difficult mathematics problem. She explained that she valued drawing a diagram so highly because it often helped her to interpret the dense "symbols and notation and short-hand" of difficult problems. However, on the second questionnaire, she indicated that she now only slightly agreed with the same statement. She explained this change by noting that she now thought that not all problems require a visualization to solve. She also tentatively speculated that maybe, in classes from her most recent academic semester, she did not use as many diagrams as she had seen in previous mathematics courses.

During the first interview, Julie only drew a few small diagrams to facilitate her work for HV-Pre but did not produce any visualizations for the other tasks; she did, however, draw a similar number of small diagrams for HV-Post on the second interview. In neither of these tasks did Julie appear to be comfortable handling the stipulation that the triangles be, to some degree, arbitrary. In HV-Pre, her first (and most referenced) diagram was of only a right angle rather than an entire right triangle. In HV-Post, her first diagram was of a square about which she eventually attempted to circumscribe the requisite triangle. Julie rationalized her decision to draw the square first because the square is a more predictable shape than an arbitrary triangle, which "could be anything."

Presumably, this is why she avoided initially drawing any arbitrary triangle for both tasks despite their centrality to the questions.

Julie also produced a graph of the two equations in the system given for MV-Post, which she used to interpret how changes in α affected the period of $\sin(\alpha x)$. With this insight, Julie roughly sketched several copies of $\sin(\alpha x)$ for different values of α. She also reproduced the first quadrant of the unit circle for reference. Although Julie utilized several different diagrams when solving this task, small inaccuracies in the scale of her functions ultimately led her to draw the wrong conclusion about the number of intersections and, thus, the number of solutions.

The only other task for which Julie used diagrams was LV-Post; here, she first drew a Venn diagram after recognizing the importance of being able to name and measure the intersection of the two student athlete populations. Of more interest, though, was a second, rectangular Venn diagram that she drew at the end of her time with the task (see Figure 4.14 above). This diagram was originally conceived when Julie hypothesized that her original Venn diagram did a poor job of representing the populations proportionally. That is, she had drawn the two circles in her Venn diagram the same size and she suspected that this was inaccurate. In the rectangle, she hoped to be able to capture the fact, which she indicated in the task description, that the intersection was one ninth of one population and one seventh of the other. This conceptualization, utilized in the way Julie described, had potential as a relatively straightforward method for finding the solution. Julie gave up on this diagram, however, and lamented that her rectangle was ultimately too similar the Venn diagram to be of use.

On the first questionnaire, Julie agreed that checking one's work at every step is beneficial when solving a challenging mathematics problem. On the second questionnaire, she softened this response to only somewhat agreement. Julie produced a number of computations while solving LV-Pre and LV-Post that she did not appear to review during her time with those tasks. This might have been because she never attempted to connect any of her computed values together in a way that would have produced algebraic statements worth checking. In other words, it is unfair to draw attention to the fact that Julie did not check her computations when they were largely isolated multiplication or addition facts done on a calculator. On the other hand, Julie also did not check the accuracy of any of the diagrams she drew for MV-Post and HV-Post; these did contain important (non-computational) errors.

The only task in which Julie did review her work while she was working was MV-Pre. Here, she simplified the equation $x^2 = y^2$ to $x = y$. After noting that “that seems weird” and that something might go wrong for negative numbers, Julie substituted $x = -1$ and $y = 1$ into $x^2 = y^2$ and seemed satisfied that she had not made an error when this process did not produce a contradiction. Despite the fact that her mathematical instincts alerted her to the possibility of an error, her inability to correctly review her work prevented her from isolating it.

On both the first and second questionnaire, Julie only somewhat agreed with the statement that justifying one’s work at every step is helpful when solving a challenging mathematics problem. Julie typically only gave justification when prompted to do so by the interviewer. When provided, it was usually vague, unclear, or incorrect. For example, Julie justified her decision to find common denominators in LV-Pre by pointing out that “you can’t really compare ratios properly if you don’t have a common denominator.” While a reasonable statement when taken in isolation, Julie did not elaborate on how comparing the provided ratios would facilitate her attempts to find a solution to this task. In a similar vein, Julie explained to the interviewer that, while working MV-Pre, a number of algebraic manipulations she made were in service of trying to “single out alpha,” though she could not verbalize why she thought it important to isolate this parameter. Perhaps Julie's best justification for an approach occurred in LV-Post, where she recognized that a Venn diagram might be able to help her quantify the intersection in a way that was more accurately proportional to the two populations. Again, though, she was decidedly unclear about how she could actually use the intersection once she found it. Julie's justification was especially superficial in HV-Pre and HV-Post; she made a number of poor observations in the latter task when trying to justify that a square could only be inscribed in an equilateral triangle. In the former task, Julie argued that she might be able to use the Pythagorean theorem or triangle inequality to support an incorrect justification but could not elaborate on how those theorems might be utilized.

All of Julie's justification was grounded in familiarity; she often remarked that her diagrams or the tasks themselves looked familiar and that she felt like there was some mathematical content that she should remember in order to better justify her methods and claims. For example, her choice of approach for MV-Pre was based off of her experiences solving systems of equations in high school, and the diagram she leaned on most heavily to justify her claims in HV-Post was something that Julie was sure she recognized from high school geometry.

Julie, when answering the first questionnaire, somewhat agreed with the statement that verifying one's final answer would be helpful when solving a mathematics problem. However, on the second questionnaire, she indicated that she now strongly agreed with the same statement. To explain this change, Julie described a process for proof-writing that she had developed over the past semester: first, she completes the proof to the best of her ability. Then, she reviews what she has written to "be sure every step you did was just to get to your goal. Nothing extra." This strategy resulted from feedback from a professor who valued clarity and succinctness in written proofs.

In HV-Pre and HV-Post, Julie answered that both task statements were probably false. In HV-Pre, she could not justify this intuition in any way (although, as discussed above, she suspected that she would need the Pythagorean theorem or triangle inequality). In HV-Post, she provided justification by marking one of her existing sketches with certain indicators of congruency while providing an accompanying verbal explanation. Whereas these two tasks were most proof-like and the two likeliest places for Julie to apply her stated procedure for checking proofs, she approached both tasks informally and it would have been very difficult for her to verify her claim by reviewing her work (which was mostly verbal).

In all other interview tasks, Julie did not arrive at a substantive enough answer to which she could apply a verification procedure. However, she did acknowledge that it is sometimes possible to check one's work for problems with a numerical solution by thinking about it in a "logical way." She described a recent experience in which she was tasked with using numeric methods in order to integrate one period of the sine function, an integral which she could reasonably conclude should be equal zero based on an analysis of the graph of the function. But if she had gotten an unreasonable answer "like -100, like okay, maybe this isn't right. Like your error is really big."

4.1.7.3 Mathematical Affect

During the first interview, Julie explained why she disagreed with the statement that solving a problem is often a matter of knowing the right formula. In her opinion, "if you just know a bunch of math terms, like axioms and definitions, I think you can, like, stumble your way across an answer instead of just knowing a formula." She argued, based on experience in a recent analysis course, that she could even write a serviceable proof without knowing the exact definitions required "if it sounds like I know what I'm talking about." Julie felt that this newfound belief in the subjectivity of mathematics had given her more confidence. This perspective may have bled into her response to the *HW Confidence* item, which she strongly agreed with. She described “my personal values of, like, I feel like I can do anything I can put my mind to,” and further explained that “If I'm sitting here, I can do anything.”

This outlook had notably shifted by the time of the second interview. Julie now somewhat disagreed that she could do any homework problem given to her, citing the fact that “my classes are proof-based and I'm not the best at proofs. I found it really difficult to like, get help on proofs and stuff.” Furthermore, “this semester they're making us more formal with our proofs.” While filling out the second questionnaire, she made a self-deprecating joke about how she had previously responded to the *Homework Confidence* item in particular: “I was like, yeah, I'm so smart, I put a circle on this one [strongly agree]. I'm like, pfft, I dumb. I had to drop a class this semester, I've never dropped a class before. I feel dumb. I'm not dumb, but...” This profoundly negative affect was indicative of Julie’s disposition while completing the interview tasks, even during the first interview; for example, she remarked while attempting LV-Pre that “maybe I’m just dumb” when she had trouble integrating numerical data into her approach. At the end of the first interview, she exclaimed that her work made her “so embarrassed! Like, why am I a math major?” These were not isolated incidents, and despite her self-proclaimed belief in her ability to accomplish whatever she sets her mind to, Julie reacted with immediate and profoundly negative emotions when dissatisfied with her work.

4.1.7.4 Problem Solving Characterization

Julie’s problem solving strategy was best characterized by low confidence and overall hesitancy, which limited her ability to problem solve. She often seemed unsure of herself and her abilities, especially when faced with an unfamiliar task or when an approach did not pan out in the way she had hoped it might. This negative affective response to failure appeared to lead Julie, in some cases, to avoid committing to paper certain approaches in which she was not confident; this was especially true if a new hypothetical approach seemed too similar to a previously failed attempt.

Julie was studious about circling and underlining important parts of the task description and revisiting those markings during her work. However, her awareness of important parts of the task did not often translate into actionable work: she seldomly introduced variables and never wrote equations that captured relationships between parts of the tasks. She did draw some diagrams in order to interpret relationships and equations from the task description, but only when she could readily link that part of the task to a known diagram from a previous, related experience. Her limited perception of important connections was sometimes remedied by an exploratory sense-making phase, but her exploratory sense-making phases were often themselves limited by anxiety and fear of failure. Similarly, Julie was hesitant to try new approaches that did not seem familiar to her (by being related to similar problems or known procedures from her educational history).

4.1.8 Carol

4.1.8.1 Formative Experiences

In the first interview, Carol pointed out that it was not a specific experience but in fact a *repeated* experience that affected her problem solving: many of her past instructors had emphasized to her that you should always start by considering new problems geometrically (i.e., using visualizations). She also elaborated on the ways in which teaching mathematics had changed the way she thought about her own work as a mathematician. That is, teaching had improved the depth and breadth of her mathematical knowledge because it required her to explain concepts in great detail and also be able answer questions such as “What are the other options to solve this? What can be the different methods?”

In the second interview, Carol touched on notable difference between being a graduate versus undergraduate student. First, as an undergraduate, she “just started the book and solving the problems. And again, the thing is like the problems are in the sequence, so we don't have to think that much.” This did not require much planning or creative thought; in fact, Carol reported being able to memorize almost everything she needed to know (such as theorems and properties) in order to solve problems. She now finds this strategy counterproductive, and that “if you're just looking [at] a very new and different problem, and if you want to see the creativity, you have to forget other things.” Without relying on a memorized procedure, Carol has found that she is more likely to plan ahead and consider many different aspects of the problem. This change in approach (and the in-depth thinking it promoted) had the added benefit of making mathematics more enjoyable for Carol.

Perhaps for this reason, Carol’s listed her recent graduate analysis course as her favorite math experience. This course used problems that Carol had always considered as having a “traditional solution,” such as the evaluation of the limit of sequences; however, Carol now saw that many of these problems could have surprising results. This led to a direct change in problem solving strategy: in order not to make careless assumptions, she found that she needed to “focus on the problem, read it many times. And then I go for the solution.” Homework problems, in particular, necessitated this change. However, Carol also observed that exam questions required similarly careful thinking. Traditionally, on an exam, “you have some questions and you started writing the solutions.” But now, “you have to think in the exam.” The instructor of this course also took time to help students visualize familiar proofs using intuitive geometry rather than symbolic manipulations (and encouraged them to do the same when solving problems), which Carol found enlightening. Finally, Carol noted that, even though she found the content engaging, it helped that “the professor was more interesting!”

4.1.8.2 Problem Solving Strategies

Carol strongly agreed, on both the first and second questionnaires, that reading a problem carefully is helpful for solving that problem. Despite her inability to give it a better score on the second questionnaire, Carol chose this strategy as one in particular that she valued more highly after her experiences during the semester. She said that “earlier, I used to just read the lines, read the problem, and start it. I started doing that. But now, I try to more focus on the problem, read it many times. And then I go for the solution.” Paying closer attention to problems was a beneficial practice in her analysis course, wherein a number of surprising proofs hinged on small details in the text of their hypotheses.

After initially approaching MV-Post by trying to characterize the ordered pair(s) that would solve the system of equations for a given value of α, Carol revisited the task description and saw that this was not necessarily a complete answer. She used this opportunity to reiterate her belief that “It’s very necessary to read the question many times.” After finishing this task, she again emphasized that you have to read the question carefully to understand exactly what is required of you. During MV-Pre, however, Carol was not as cautious. In that task, the interviewer had to eventually point out that she was looking for *all* values of α that would maximize the number of solutions to the system and not just *any* α value. An improvement in reading comprehension between the first and second interview aligns with Carol’s greater appreciation for *Read Carefully*.

Carol often circled or underlined important phrases or data points in longer questions such as LV-Pre and LV-Post. She also drew attention to important key words, such as “maximize” in MV-Pre. When she found herself stuck on a task, Carol returned to the task description to make sure that she was interpreting its constraints correctly. For example, she had an unexpected amount of trouble with LV-Pre despite immediately characterizing it as a standard “ratio problem.” When both her initial and second attempt to solve the task failed, she carefully aligned her work with the text of the task so that she could physically point to it while she reconsidered the constraints of her linear systems.

Despite the diligent way in which she read and returned to task description, Carol still sometimes had difficulty interpreting it correctly. On LV-Pre, for example, she never transcribed the text into a system that correctly captured the relationships therein. Carol had similar difficulties on LV-Post, although in that task they were not fatal. As noted above, in MV-Pre, she also misinterpreted how she was intended to answer the task. However, even when she experienced these difficulties, Carol recognized the importance of the task description. For example, while working LV-Post, she admitted at one point that her current approach did not appear to use some of the given numerical data and that she had little hope that it would lead to a useable solution.

Carol did quite a bit of exploratory sense-making in both interviews. As with most participants, she drew a number of triangles in both HV-Pre and HV-Post to better understand the circumstances under which the propositions in question might be true. She explained that such exploration could also lead to the discovery of a direct counterexample, significantly simplifying her responsibilities for the task.

Carol explored several other tasks as well, including LV-Post, MV-Pre, and MV-Post. She noted in the first interview that her exploratory tendencies tended to manifest when she was unfamiliar with the type of problem she was solving, which would explain why she did not leave much time for such sense-making in LV-Pre; in this task, she immediately recognized that there was a familiar schema she could use to reach an answer. Accordingly, she produced a system of equations and attempted to solve it through traditional algebraic steps. On the other hand, in LV-Post, this schema was not triggered (despite the fact that a system of linear equations could also be used for the task at hand). Instead, Carol took some time to test numerical values that might hypothetically represent the populations of students in order to see if they might satisfy the constraints of the problem situation. She applied a similar technique for MV-Pre and MV-Post, wherein she chose different numerical values of α to get a feel for the parameter's effect on each systems of equations. In these tasks, exploring different values of α was done visually rather than algebraically.

On the first questionnaire, Carol strongly agreed that trying to visualize the solution to a difficult mathematics problem would be helpful for solving it. She explained that her general problem solving strategy primarily concerned taking note of "what is given to me and what I need to do. I should start from there." However, she also acknowledged that her valuation of his strategy was probably too high; sometimes, "I have to do something [first]—then maybe I check the goal. So just starting from seeing the solution and then doing something might not be helpful always." This opinion was more pronounced on the second questionnaire, where she only slightly agreed that imagining the solution is helpful. To further explain this change, Carol referred to her experiences in her analysis class from the past semester. In this course, the instructor gave the class homework assignments that asked them to prove or disprove a collection of mathematical claims. Carol found that when she went into a problem with a preconceived notion of whether or not the statement was true, it often led to dead ends and false assumptions. The only tasks from the interviews that might have presented a similar danger were MV-Pre and MV-Post; in both of these tasks, Carol did not form an immediate opinion about whether the proposition was true or false only from reading the task description.

LV-Pre was the task for which Carol engaged most heavily in imagining the form of the solution. Because of the real-world context of the task, she was immediately able to recognize that the solution should be comprised of positive integers. This observation informed her ability to gauge whether or not each of her approaches had been successful after they had reached a conclusion; it is not clear from the interviews whether or not it also affected her choice of approach or the way she monitored each approach's progress. On the other hand, Carol did use the form of the solution to determine a method of approach in MV-Post. In this task, α also had to be a positive integer. With this in mind, Carol (gratefully) recognized that she would not need to draw a diagram for the case when α was a proper fraction, which she had been struggling to conceptualize.

On both the first and second questionnaires, Carol strongly agreed that considering a number of possible approaches to a difficult problem might help one to solve that problem. This attitude held true throughout most of the interview tasks. In LV-Post, for example, Carol considered: using a formula she remembered for the union of two sets; using a system of equations; and generalizing a solution from conclusions she observed from working with a specific, arbitrary population. In MV-Pre, she tried to draw reasonable conclusions from her diagrams, used two different applications of calculus (maximization techniques and equations of tangent lines), and isolated a quadratic equation by solving the system of equations with substitution. In HV-Pre, she considered using both the Pythagorean theorem and the law of cosines.

The only task in which Carol distinctly did not appear to consider multiple approaches was LV-Pre; at least at first, her approach to this task was entirely dictated by the fact that she recognized it as a familiar system of equations. It was not until her system of equations had failed several times that she considered a different approach. And still, this different approach was only to try and solve the system of equations with a matrix instead of an algebraic algorithm. While this relative single-mindedness goes against Carol's otherwise studious consideration of multiple approaches, it was understandable in light of the fact that the approach she fixated on (solving a system) was certainly an appropriate choice.

Carol indicated, on the first questionnaire, that she agreed with the statement that planning one's steps in advance can help one to solve a difficult mathematics problem. On the second questionnaire, she instead indicated that she now slightly disagreed with that idea. Like with *Imagine Solution*, Carol again drew on her recent experiences in her first semester of graduate school. She found that, unlike as an undergraduate, homework problems did not often mirror the exact sequence of steps that the instructor had demonstrated as an example in class. Instead, she identified a greater need for flexibility and openness when solving difficult mathematics problems; assuming that homework would follow a prescribed sequence of steps often led to dead ends.

In MV-Pre, Carol did assume that the task would follow a prescribed sequence of steps when she noticed that its text asked her to maximize; this key word triggered a calculus schema, so she calculated a derivative and found critical points. Carol undertook this approach without first considering whether the function she wrote was continuous with respect to the variable of differentiation; correspondingly, she was unable to interpret her eventual answer. At other times in this task, Carol did use brief planning phases to assess the validity of potential or ongoing approaches. For example, she chose not to solve the system of equations through traditional means because it was nonlinear and she anticipated that the arithmetic would be too cumbersome; similarly, she abandoned an approach to find a value of α that would give only two solutions after concluding that doing so would be irrelevant in light of the fact that she had already found values of α that would give more solutions. In HV-Pre, Carol avoided the Pythagorean theorem and law of cosines after considering whether she had enough information to use them effectively in the task. Similarly, Carol did not use her formula for the union of two sets in MV-Post after recognizing that she was not given the right sort of numerical data in the task description to use this formula as she had hoped.

Carol strongly agreed, on both questionnaires, with the statement that drawing a diagram of a difficult problem can be helpful when solving it. She lived up to this expectation; in every task except for LV-Pre, Carol engaged with diagramming or visualization at some point in her problem solving process. This was least evident in LV-Post, where she only drew a single Venn diagram early in the process in order to help her interpret the task description. Carol admitted that she did not find it very helpful, though, and that her subsequent symbolic manipulations were not facilitated by the Venn diagram.

Carol was especially keen to draw diagrams for both MV-Pre and MV-Post. In fact, she attributed her misinterpretation of the task description of MV-Pre to the fact that she almost immediately recognized that she could easily draw the system of equations on a coordinate grid; because she was so eager to do so, she did not read the task as carefully as she might have otherwise. Initially, her diagrams in both tasks were used for exploratory sense-making. That is, she chose a value of α and examined the system of equations for important mathematical conclusions that she might be able to expand upon or corroborate in other cases. This was especially evident in MV-Post, where drawing a simple diagram of the system immediately revealed to Carol that α had to be odd because of the persistent solution at the origin and the symmetry of $\sin(x)$. In MV-Pre, these exploratory diagrams also played a role in helping her identify important elements of her eventual symbolic approach. For example, after imaging the circle sliding along the x axis via changes in the parameter α, Carol identified that there was a point at which the circle would be tangent to the degenerate hyperbola representing the other equation (see Figure 4.15 below). She then spent some time identifying the α value for which this circumstance was true. Carol heavily relied on visualizations for these tasks, which caused her to lament that she did not have enough time to produce less "subjective" work.

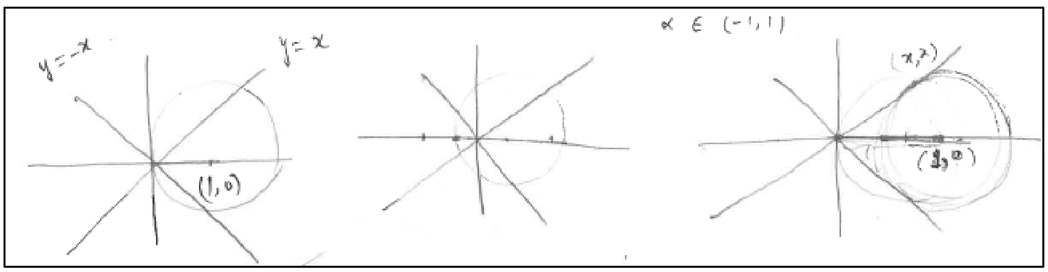


Figure 4.15 Carol's sequence of diagrams handling different cases of the parameter α.

As indicated earlier in this subsection, Carol's diagrams in HV-Pre and HV-Post were largely for sense-making purposes. Importantly, however, it should be noted that these diagrams also often included relevant symbolic labels and sometimes arbitrary numerical side lengths. For example, in HV-Pre, she at one point sought to show that one line segment in her diagram was half as long as another; Carol admitted, however, that "I've forgotten most of the geometry; I'm not sure how I'm going to do that." By supplementing her visualizations with symbolic placeholders for side and angle length, Carol was able to use what she did remember of geometric principles to draw a reasonable conclusion. After solving HV-Post, Carol observed that the hardest part of the task was trying to justify her visual intuition with rigorous mathematics.

On the first questionnaire, Carol agreed that checking one's work while solving a difficult mathematics problem can be helpful when solving that problem. On the second questionnaire, she indicated that she now strongly agreed with this statement. During the interviews, Carol did not routinely check her work while she was working; however, she also made no obvious arithmetic errors that would have immediately triggered a monitoring response. Most of her errors were conceptual in nature and did not lead to obvious, immediate inconsistencies in her work..

For example, consider Carol's work in LV-Pre. In this task, one of the two equations that comprised the system with which Carol was working had been constructed incorrectly: it included both sign errors and incorrectly placed coefficients. These mistakes derived from her understanding of the task description, not computational errors, and did not cause Carol problems until her work produced a bogus answer that did not make sense in the real-world context. Similarly, in MV-Post, Carol incorrectly drew the function $\sin(x)$ as having multiple intersections with the identity line $y = x$. While incorrect, this diagram did not lead to any conceptual inconsistencies that would have triggered Carol's monitoring instincts.

Carol initially agreed, on the first questionnaire, that justifying one's work at every step is helpful for solving difficult mathematics problems. When she completed the second questionnaire, she strongly agreed with this statement. Despite the fact that Carol found justification very important in the problem solving process, her justifications were often absent or vague allusions to properties half-remembered. This was especially true for HV-Pre and HV-Post. These tasks, Carol explained, required a different kind of thinking and a different subset of mathematics knowledge that she could not always manifest. In HV-Pre, she was able to eventually just remember an exact proof; even then, however, she was not able to justify this proof to her satisfaction and never felt convinced that she had produced a complete and formal explanation. In HV-Post, after a sequence of failed attempts, Carol remarked that her cursory visual explorations were "not the proper reasoning for solving a maths problem! It's good for the intuition, but… I need to use some more properties of the triangle which I'm not able to recall."

Carol also mentioned during the second interview that her difficulty with HV-Post was due to the deterministic style of teaching in her only geometry course. That is, because the course lectures were largely isomorphic to the textbook chapters, it was always clear what theorems you should apply to a given problem. HV-Post, disconnected from a structured course, could not be resolved so trivially.

Carol agreed with the statement that verifying the answer at the end of solving a difficult mathematics problem is a helpful practice. She strongly agreed with the same idea on the second questionnaire. When Carol produced an answer worth verifying during an interview, she had mixed results when verifying that answer. For example, in LV-Pre, Carol revisited the task description and her accompanying work when she found first a non-integer solution and then a negative solution to the task (she anticipated a positive integer). She also checked her solutions, however incorrect she suspected them to be, against the constraints of the task as outlined in its text. This helped her, at least in one case, to identify the source of her error. In LV-Post, Carol reached another incorrect answer that she had difficulty verifying. Her answer, $\frac{61}{101}$, triggered her monitoring instinct (it seemed "quite complicated", whereas she was expecting that she would be able to "find a simple fraction.") However, she could not generate a simple, alternative method for checking her solution and never compared it against the constraints of the task.

In both MV-Pre and MV-Post, Carol's solutions were correct but incomplete. Because her monitoring instinct was not triggered, she did not verify them. Ultimately, such a verification might have proved fruitless; her answers were incomplete because of unconsidered cases that a review of her existing work probably would not have prompted her to consider.

4.1.8.3 Mathematical Affect

Carol's beliefs about mathematics blurred the line between the *Hard Work*, *Natural Talent*, and *Persistence* items on part two of the questionnaire. In the first interview, Carol explained why she strongly agreed that persistence was a key element of mathematical success:

> So, most of the time when we say in the class, some teacher was teaching and I thought like it's very easy, and I can do that do that and I'll leave it like this. When it came in exam, I thought like, it is so easy but why am I not able to do that? Maybe I didn't solve the exercises. And I was not persistent in that.

That is, mathematics requires constant reinforcement and practice. Carol provided additional insight into this perspective when, during the second interview, she explained why her responses to the *Natural Talent* item fell to somewhat disagreement from somewhat agreement. Namely, "hard work and all these things work a lot instead of the talent. Someone is more talented [but] he or she is not doing anything? […] It doesn't matter." Even though Carol found that she needed to consistently apply herself to mathematical thinking to stay sharp, she admitted that applying herself to one difficult problem continuously sometimes felt counterproductive. That is, she believed that it was important to take breaks from challenging mathematical tasks in order to return to it later with a fresh perspective. Carol explained that completing such a difficult problem was immensely rewarding, however, as was reviewing her work to see if she could find a simpler or more interesting approach. This is a perspective Carol only developed as a graduate student; as an undergraduate, she reflected that she was "careless" and only wanted the correct answer. Now, however, she observed that "as I go into the depth, I start enjoying that thing."

During the second interview, Carol also touched on the fact that she no longer agreed that memorization is important for success in mathematics; now, she somewhat disagreed with that statement. She found that, during her first semester of graduate school, she was often required to solve problems in a mathematically creative way. This was more difficult if she was focused on using specific theorems that she had memorized. As a result, Carol came to the conclusion that “if you want to see the creativity, you have to forget other things.”

4.1.8.4 Problem Solving Characterization

Carol’s problem solving strategy was characterized by her belief in known methods and perseverance in applying them, even when she recognized that they would be ineffective (or had already produced clearly incorrect work). When she could not see a familiar schema that she might apply to a given task, Carol relied on exploratory sense-making with diagrams and sample values. Sometimes these explorations were misleading; a number of Carol’s mistakes during the interview tasks could be attributed to inaccurate diagrams or improper generalizations therefrom. More often, however, these visualizations gave Carol a sense of direction and a physical framework onto which she could attach variables and numerical data.

Carol’s mathematical intuition was sharp enough that she was often cognizant of the shortcomings of her mathematical justifications. She was unsatisfied with her work when it did not meet her high standards for mathematical rigor, even when she had correctly outlined the broad strokes of an appropriate overall approach. This attitude was fairly unique among interview participants, many of whom were satisfied when something felt intuitively clear to them without formal logical underpinnings. Despite Carol’s eye for thoroughness, however, her tendency not to check conceptual assumptions led, in some tasks, to unresolved errors that produced incorrect or incomplete solutions.

4.2 Undergraduate Students

The total undergraduate responses to the first questionnaire are visualized, as charts, in Figure 4.16, Figure 4.17, and Figure 4.18 below; they follow the same color-coding system as the graduate student charts in Figure 4.1, Figure 4.2, and Figure 4.3.

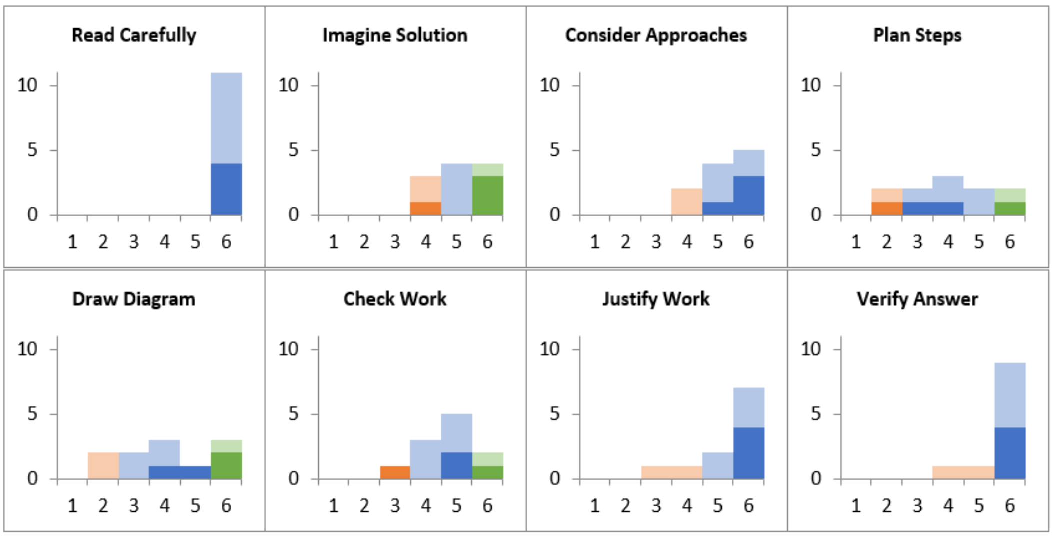


Figure 4.16: Undergraduate responses to part one of the first questionnaire

From part one, *Read Carefully* clearly had the lowest standard deviation; *Draw Diagram* had the largest. From part two, the item with the largest standard deviation was *Natural Talent*. From part three, two items actually tied for the largest standard deviation: *Right or Wrong* and *Knowing Formula*. The item with the smallest standard deviation from part two was *Persistence*, whereas for part three it was *Discover Solution*.

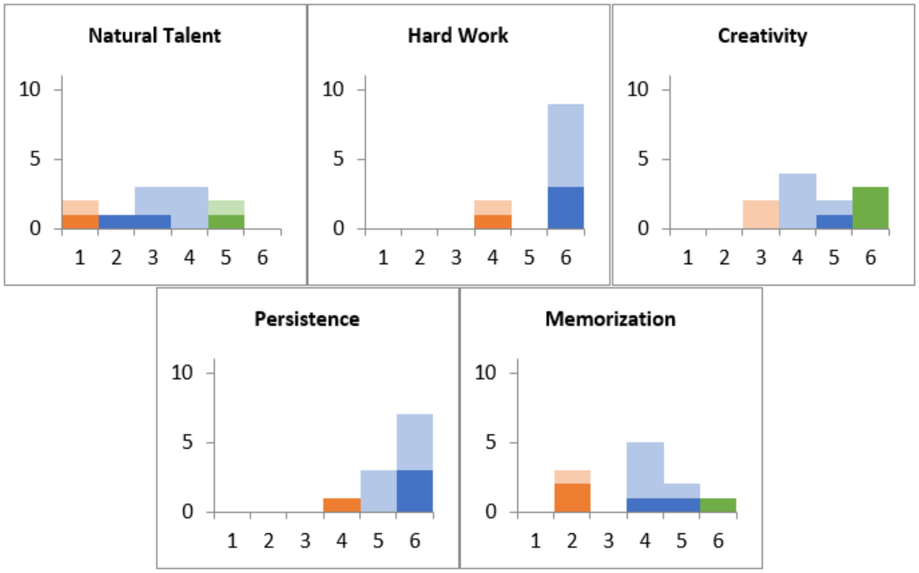


Figure 4.17: Undergraduate responses to part two of the first questionnaire

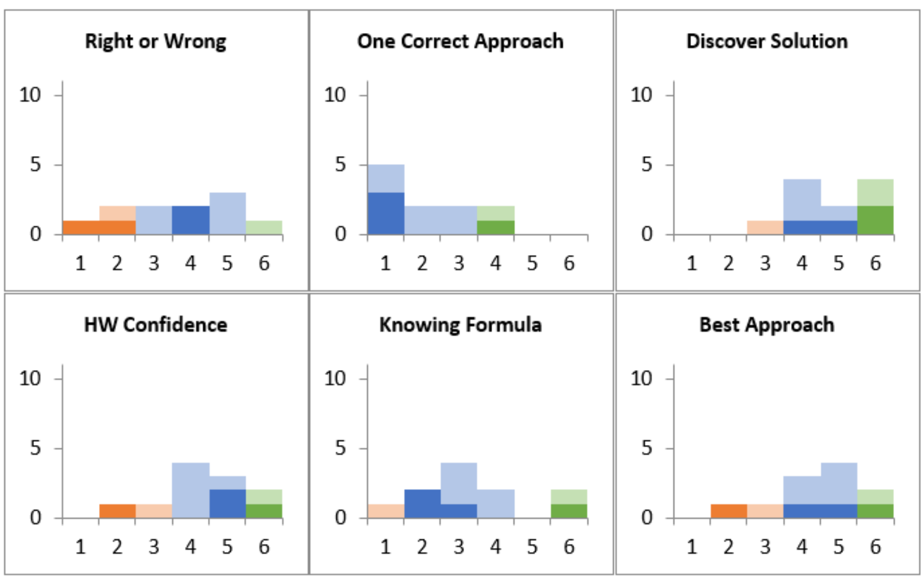


Figure 4.18: Undergraduate responses to part three of the first questionnaire

4.2.1 Will

4.2.1.1 Formative Experiences

In both interviews, Will spoke at length about the ways in which his first discrete mathematics course (which he described as his “first real math class”) affected his problem solving and disposition towards mathematics. Primarily, he appreciated that none of the exam problems were exactly like prior homework problems. This forced Will and his classmates to “learn the very *idea* of what you are doing and have a very deep understanding of what you're doing instead of just learning the steps.” However, the homework problems themselves were also difficult and required Will and his classmates to meet outside of the class itself to work on problems together at a whiteboard in the library. He appreciated that

> such difficult problems that required such a long math thinking. And being in a group with others made it so you could hear other people's approaches. It wasn't just you sitting silently by yourself and just being stuck in your own little bubble of not being able to consider these ideas. But being able to hear other people's approaches that then add to your bubble, basically.

Will felt that spending so long thinking about one problem at a time improved his ability to think about difficult problems in general. Completing difficult homework problems in this course led Will to carefully develop his mathematical intuition; specifically, this was because

> it wasn't just like I could get a straightforward answer from any of these problems. I have to now consider different possibilities and think about this for a lot longer. And so, then when it came to other problems, I became a little bit better at considering what other options I had or what I *could* do to approach this problem.

Will also appreciated that the instructor of this course was approachable and interested in teaching, which made learning difficult material less strenuous. He contrasted this later in the interview with “a teacher in my geometry class who made fun of the fact that I would ask so many questions,” which he found annoying and disrespectful.

Finally, Will was one of the only participants to cite the first interview as a formative experience during the second interview. Specifically, he recalled continuing to think about HV-Pre for a week after the interview. Eventually, he realized that his initial conclusion, that the proposition was false, wasn't actually correct. Will claimed that this experience suggested that he should be less rash when creating proofs, which are "not something where it's like, boom: here's my answer. You know? So I guess that's kind of led to me being more slower in my approach and considering more options."

4.2.1.2 Problem Solving Strategies

On both questionnaires, Will strongly agreed with the statement that reading a problem carefully is helpful when trying to solve that problem. Before beginning HV-Pre, he explained to the interviewer that when reading a problem, he often tries to connect the task description to his existing body of content knowledge in order to gauge whether or not he knows of a tool that will be especially helpful for solving the problem at hand.

Will was largely successful in analyzing and interpreting the text of the interview tasks. He did not annotate the task descriptions themselves directly but was still usually able to come to useful conclusions from an initial reading. For example, in both LV-Pre and LV-Post, Will anticipated that both tasks could likely be solved with a system of linear equations; he came to this conclusion after pointing out that both tasks involved two different populations that were related in some way, which was a setting that lent itself naturally to some kind of system. While this observation was correct, in both tasks Will still had difficulty confidently defining the variables involved in such a system so as to facilitate a method of solution. This was exemplified by his work in LV-Post, in which he ended up with a system of three ambiguously connected equations that had four tentative variables (Figure 4.19 below).

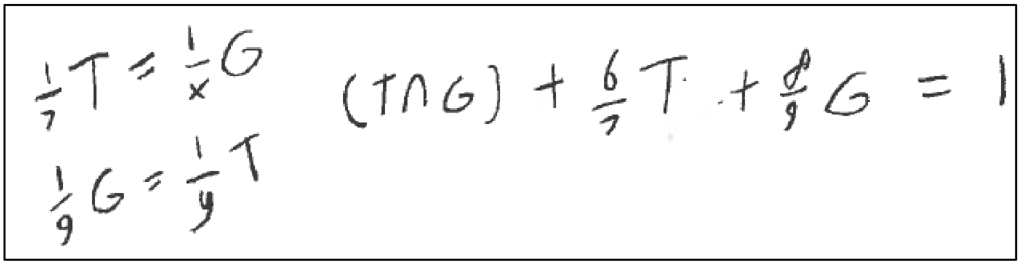


Figure 4.19 Will's tentative system of equations (LV-Post).

When necessary, Will revisited task descriptions to clarify both constraints and the eventual form of the solution. In LV-Pre, after finding an initial male and female population that satisfied the requisite ratios, he remarked that he was satisfied with his work but still needed to identify what the task actually wanted him to solve for and returned to the text to do so. In HV-Pre, after finding a right triangle that he could successfully cut into two isosceles triangles, Will revisited the task description to circle the word "arbitrary" and remind himself that he needed to consider more than just a single case.

Will mentioned during the first interview that he often likes to choose sample values to help him make sense of a task with unknown variables, a strategy he first found useful during his discrete mathematics course. For example, in LV-Pre, he chose an arbitrary common factor of 10 to apply to the ratio of 8 men to 7 women in the initial population; that is, he assumed there were 80 men and 70 women. Will found that this choice in fact solved his system and allowed him to retroactively explain why some of the tentative decisions he had considered while writing down his equations were, in fact, correct. In BP, his first plan of attack for exploring the problem space was to try and find any combination of coins that got close to $5.00 so that he could better make sense of what precisely had gone wrong. Finally, while solving HV-Pre, Will chose to assign one of his right triangles specific side lengths to test whether a particular working hypothesis (he needed to cut from the right angle to the midpoint of the opposite side) might hold.

Will also used algebraic exploration as a strategy to find inspiration when he was not sure how to proceed in a task. When he had difficulty constructing a system of linear equations in LV-Post, Will chose to work with a preliminary, unfinished pair of equations anyway and said, “I know this formula isn’t going to exactly work, but it’s just going to help me visualize it.” Later on in this task, after again finding himself stuck, he revisited the task description to “think of what options do I have of messing with things.” In MV-Pre, Will made a number of algebraic manipulations to the pair of given equations just to see what would happen; he hoped that moving around parts of the task would give him a useful realization. This algebraic exploration was not limited to the beginnings of tasks, either. Part way through his work on MV-Pre, Will briefly stopped working and said he was now “just looking at this and seeing: what can I do to this problem to just break it up even more and see what happens?”

On both the first and second questionnaire, Will strongly agreed that imagining the form of the solution is helpful when solving a difficult mathematics problem. To explain why he valued imaging a solution as a problem solving strategy, he pointed to HV-Pre and said that “drawing a solution or drawing an idea or imagining just some examples is always, for me, the best thing that helps with solving—just, trying to get an idea of the steps that are actually involved in the process of the example. That always leads to some sort of idea at some point.” At least in this example, Will appeared to associate *Imagine Solution* closely with *Draw Diagram*.

During interviews, Will was especially capable of understanding precisely what a given task was asking him to solve for. Despite this, on most problems, Will did not usually use this information to guide his problem solving approach. For example, as previously noted, Will completed almost all of LV-Pre before returning to the task description to remind himself of what the actual answer should be. On the other hand, for at least one task (MV-Pre), the form of Will's anticipated solution greatly affected how he approached his work. On that task, previous experience with systems of equations led Will to suspect that, for a particular α, the number of solutions to the system might be infinite. He spent a significant amount of time explaining precisely what conditions would lead to an infinite number of solutions; he also referenced previous problems with structural similarities and applied similar techniques from those problems in order to attempt to manipulate the system of MV-Pre into having an infinite number of solutions. This is impossible, and so Will was ultimately unsuccessful, but this initial idea led to three distinctly different approaches that had the effect of refining his understanding of the problem space.

Will strongly agreed, on the first questionnaire, with the statement that considering multiple approaches to a problem is helpful for solving that problem. He also strongly agreed with this statement on the second questionnaire. During the interviews, Will often applied alternative approaches with the goal of refining or connecting to previous approaches (compared to, instead, finding an entirely different approach altogether). This distinction is best illustrated by his work on MV-Pre. There, Will initially approached the task under the assumption that he might be able to demonstrate that the system had infinitely many solutions for some α. With this in mind, he first let $\alpha = 0$; he recalled a similar problem featuring a quadratic function (also with a parameter) that was greatly simplified by letting that parameter equal zero. He hoped a similar simplification would occur here, trivializing the system in such a way that it was clear whether or not there were infinitely many solutions. When this did not happen, he saw instead that letting

$\alpha = 1$ (after combining the two equations via substitution) would allow him to eliminate a term from the quadratic equation (see Figure 4.20 below).

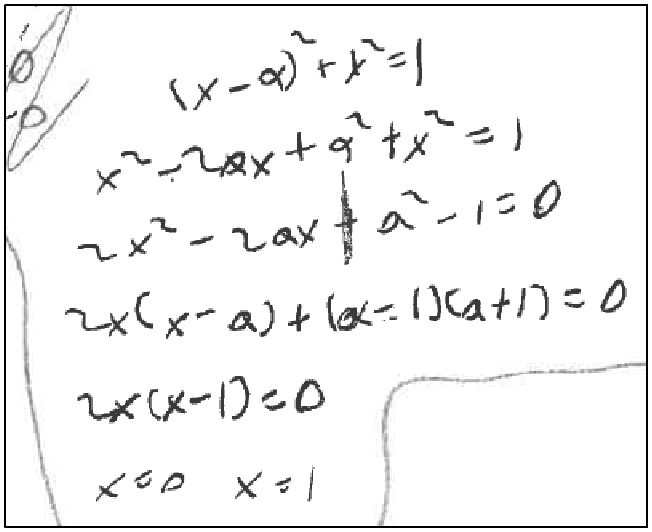


Figure 4.20 Will's attempt to find a value of α that would confirm his intuition about the system of equations in MV-Pre.

When this value of α still did not have infinitely many solutions, Will next sought to compare the effect of substituting $x = k$ versus $x = -k$ in order to better understand whether there was underlying symmetry he could exploit; while making the $x = k$ substitution, he pointed out that he was deriving the same quadratic structure that led him to try $\alpha = 1$ in the previous approach.

In the other interview tasks, Will considered multiple approaches but never actually attempted them. In LV-Pre, he briefly considered using complex numbers to factor a sum of squares before coming to his senses. In HV-Pre, he considered using the Pythagorean theorem but demonstrated with a numerical example that it would not serve his purpose effectively. In LV-Post, Will chose not to employ his usual strategy of picking sample values when he needed to find two numbers whose product was 63. He explained that there were infinitely many such pairs and that because he had no mechanism for evaluating whether or not his choice had been reasonable, it would be ineffective to proceed with approach.

On both questionnaires, Will indicated that he disagreed with the statement that planning one's steps in advance is helpful when solving a difficult mathematics problem. He explained his particularly low regard for this strategy:

> "It's good to [plan steps in advance], but I think in actuality, that's not what happens. Like you can even see there [indicating an interview problem], I was kind of just like, let's just see what happens when I do something. […] It's good to [plan ahead], but in actuality you're never—you're not really gonna do that. Or at least, I'm not."

Despite these misgivings, Will did reject some alternative ideas (such as an application of the Pythagorean theorem or the introduction of imaginary numbers) during the interviews when he did not foresee that they would improve his current situation. On the other hand, when choosing an initial strategy, Will behaved more in line with his questionnaire responses; that is, he often began tasks without a clear understanding of whether his approach would be effective or not. Instead, Will used tentative algebraic manipulations and specific test values to get an intuitive feel for the task's setting.

On the first questionnaire, Will only somewhat agreed with the claim that drawing a diagram is helpful when solving a difficult mathematics problem. On the second questionnaire, though, he now agreed with the same statement. Will did not draw noticeably many more diagrams than other participants, but he did use more dynamic language when referring to what he had drawn. For example, on HV-Pre, he explained how he had been visualizing different triangles: "If I had a string to basically pull this around, I can change what these two angles are." On HV-Post, he described how one might turn any inscribed rectangle into a square by "sliding" its sides along the base; on the same task, he explained that he first thought of inscribing a square in a right triangle and then "pulling" one of the triangle's sides away to make it an obtuse triangle. On MV-

Post, he referred to α as “squishing” the graph of $\sin(\alpha x)$. Each of these descriptions was accompanied by gestures or approximations of physical manipulations. That is, Will used his hands to simulate stretching or sliding the diagrams instead of drawing new, static images at a different position. Finally, also on MV-Post, Will explained how seeing a video of the arm in a unit circle “spinning” to produce a sinusoidal curve helped him to graph $\sin(x)$ correctly:

> I feel like it's really helpful to just visualize, okay, sine is our y value throughout the unit circle and that helps me visualize, okay, well, what am I actually doing with the sinewave? […] So, those [dynamic visualizations] are always more helpful. Visualizing something like that with this helps.

Besides a slew of triangles (on HV-Pre and HV-Post) and a Venn diagram (on LV-Post), Will also drew a scale balance when explaining how he planned to approach BP (see Figure 4.21 below). He did not, however, draw any kind of visualization for MV-Pre even after identifying that there might be some underlying symmetry or after isolating a quadratic function.

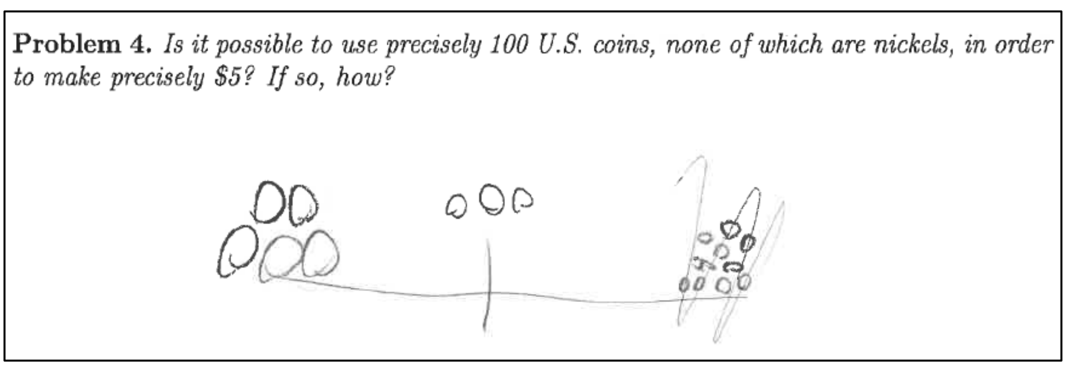


Figure 4.21 Will’s diagram for BP, which he used to explain how he planned to systematically employ a guess-and-check approach.

Initially, Will agreed with the statement that checking one’s work over the course of solving a difficult mathematics problem is helpful; later, on the second questionnaire, he indicated that he now strongly agreed. None of Will’s shortcomings while working on interview tasks was due to an unperceived arithmetic or algebraic error. When he produced such an inaccuracy (for example, he made a few sign errors when carrying out the substitution $x = -k$ on MV-Pre; on MV-Post, he incorrectly labelled a point on one of his diagrams), Will normally corrected himself before any major damage was done.

Notably, that meant that the above instances of review and correction did not appear to be motivated by a moment of cognitive dissonance; that is, rather than reaching an unexpected or troubling conclusion and checking his work accordingly, Will seemed to notice errors with unprompted spontaneity. This should not be taken to mean that Will did not actively monitor his work, however: for example, while working on MV-Pre, he stopped several times to assess his work and gauge which of certain algebraic manipulations would be more profitable (in his words, less “annoying”).

On both questionnaires, Will strongly agreed that justifying one’s work at every step is helpful for solving a difficult mathematics problem. Will followed through on this claim, making a noticeable effort to explain why he chose certain approaches and why they were mathematically sound. In HV-Pre, he argued that the cut in the right triangle should be made from the right angle, because that was the only specific trait given by the task description to the arbitrary triangle in question. This observation, even though it was not actually founded in mathematical principle, informed the rest of his work on the task. At one point, while solving LV-Post, Will made several salient justifications to explain how he generated an equation that captured the relationships between the populations in question without overcounting. Later in the task, he made another justification for why he knew the product of two variables must be 63: if it was not, his work implied that there were no tennis players at the school, which could not be true. On MV-Post, Will honed in on the fact that sine is an odd function to justify why $\alpha = 2$ is not part of the solution set.

Not all of Will's justifications were appropriate, but when they were not, they sometimes led to moments of insight that allowed him to refine his original claim. For example, in HV-Post, Will thought he had identified a counterexample to the proposition. In order to rigorously justify why a square could not be inscribed in the obtuse triangle he had drawn, Will began to explain that only one side of the triangle could contain two of the square's vertices. In doing so, he realized that he had to account for all three sides and not just the base of the triangle as he had drawn it; he quickly found a way to inscribe a square in his counterexample and had to reorient himself accordingly in the problem space. Other times, especially in HV-Pre and HV-Post, Will might make a mathematical justification that was completely unfounded and that did not lead to an associated moment of insight. This occurred at least once in HV-Pre when he claimed that a line drawn from the right angle of a triangle to the midpoint of the hypotenuse would necessarily bisect the angle.

Will strongly agreed, on both the first and second questionnaire, that verifying the purported answer to a difficult mathematics problem is a helpful practice. During the interviews, he only produced an answer that could be reasonably verified on LV-Pre and MV-Post. On both tasks, he did verify his work. On LV-Pre, while checking his answer, he added that he was "just making sure I'm not being stupid. Because I do math tutoring for a living, and I know there are several times where stupid moments get you in the end!" On MV-Post, he used a calculator to check that his hand-drawn graph of $y = \sin(3x)$ was accurate when it depicted three intersections with the line $y = x$.

4.2.1.3 Mathematical Affect

During the second interview, Will was surprised to learn that he had even somewhat agreed that memorization is important for mathematical success, given that he now disagreed. He guessed that the change might have been brought about by several experiences from the recent semester in which he had been able to circumvent a requirement for memorization by re-deriving a formula or process. As such, he now found that knowing formulas by heart could be used (but was certainly not required) to succeed in mathematics. That is, “as long as you understand it intuitively, enough to be able to be where you can replicate it in another situation, then you don't need to put in all that effort of trying to memorize these steps.”

Will also commented on his especially high responses to the *Creativity* and *Discover Solution* items, observing that “I mean, that's how people invent math.” He later lamented that he himself is not creative enough to invent recreational math problems to consider on his own time, which he normally enjoys doing. On the other hand, Will rated the *Hard Work* and *Persistence* items lower than his undergraduate peers. Throughout the interviews, however, it appeared that Will really did value these qualities. For example, in explaining why he can do any homework assignment given to him, he described an especially long numerical analysis assignment that he had to work hard to finish over a number of hours. At another point, he described his method of preparing for a geometry exam that involved summarizing and organizing his notes into a study guide. This process took a few hours of dedicated work, but ultimately allowed him to easily review important material at a glance. He stated that a person unprepared for that level of work would not “be successful with the amount of effort you do in math.”

Finally, Will made a number of offhand statements of mathematical belief that were not discussed at length during the interview but are worth noting as elements of his mathematical affect. First, he noted that, in mathematics, whether an answer is right or wrong is entirely dependent on the definitions and axioms chosen as part of the system in which it is derived. He also observed that “some difficult math problems involve very easy math” and that “it's important to me to actually understand what I'm doing.” In service of this latter point, Will tries to contribute to lectures by asking questions because he would "rather look like an idiot now than *be* an idiot when I turn in the homework!"

4.2.1.4 Problem Solving Characterization

Will's problem solving strategy was robust and characterized both by adaptation and the value he placed on his ability to use what he knows to recreate important mathematics that he might not remember explicitly. He seemed equally confident using algebraic techniques as he did diagrams or visualizations. In either case, he typically relied on the same suite of all-purpose heuristics: trying sample values, applying his experience from similar problems, and attempting to handle simpler cases first. After he had done enough work to warrant a justification, verbalizing such mathematical reasoning often gave Will insight to ways that he might improve or adapt his current approach. Sometimes, however, an inappropriate justification that went unnoticed led Will astray.

When his current approach proved insufficient in some way, Will did not typically jump to an entirely different approach. Instead, he sought to better understand the problem space and make appropriate adjustments to his current approach in order to overcome its shortcomings. This proclivity towards transformation rather than replacement also characterized Will's treatment of diagrams. Instead of drawing many separate diagrams for each case, Will tended to treat his initial visualization as a dynamic object that could be stretched or pulled to handle each case. All in all, Will ended up with diagrams and approaches that were all related to each other in some way, allowing him to trace his growing understanding of the task at hand.

4.2.2 Riley

4.2.2.1 Formative Experiences

One experience that affected Riley's problem solving strategies was her enrollment in a second-semester calculus course. The instructor of that course first introduced Riley to the conceptual (as opposed to computational) side of mathematics as a subject. In calculus, Riley also learned that not knowing a prescriptive method by which she could obtain an answer to a given problem does not make that problem unsolvable; that is, “you might not know what you're doing, but you can definitely stumble upon something.” This lesson deemphasized the need for an initial justification when solving a problem. Riley felt more comfortable going down “the wrong path. Because you could figure out your way back to the right path. It might not be like the easiest route, but you could stumble upon something and fix that.” Ultimately, Riley’s introduction to conceptual rather than computational mathematics in calculus caused her to change from an engineering to a mathematics major.

Riley took a similar lesson away from her discrete mathematics course, which she characterized as her favorite mathematics experience. The instructor of the course influenced Riley’s opinion significantly by choosing engaging problems that were sometimes simply “logic puzzles”; these were similar to the kinds of “math riddles” that Riley sometimes worked recreationally and that she found improved her mathematical creativity. Instead of immediately explaining how to solve the problems in her discrete course, Riley’s professor gave the class a collection of applicable strategies that would allow them to independently work towards a solution. This “made us think harder and actually use our brains”; Riley explained that

> it's one thing for a professor to give you a function and give you a problem, give you an example, and show you how it works. And that's—you could mimic that, a lot of the time. But to have a professor that gives you a problem that you haven't seen before and give you certain tools that will help you solve the problem, and so you know that those tools are there, you'll be able to solve the problem a lot easier than someone that's mimicking.

Riley gave an example of one such tool that mirrored her understanding about the value of going down the “wrong path”: when she was initially overcome by ways to start a problem, her discrete instructor encouraged her to, “even if you're not sure about going about a problem, like, draw something. Put *something* down. Because it's probably worth it in the end.” Despite teaching in a lecture format, this instructor also “made sure to include everybody” and engaged the class by trying to “think of examples that would apply to us.”

Riley also expounded on ways in which working as a personal tutor had affected the way she approached difficult problems. First, and most generally, it was a way for her to reacquaint herself with familiar mathematics concepts that she had not otherwise had a reason to revisit. However, she has also learned specific strategies from tutoring. One of her students, for example, learns well when they are given a solution and asked to reverse engineer a logical sequence of steps that would lead to that answer. This is a strategy that Riley herself found effective, and which mirrored a lesson she simultaneously learned in her recent analysis course: “most of the time, if you don't know how to do something, you can figure out that it works and then come to the justification later.” Riley also argued that tutoring improved her interpersonal skills, which was valuable for explaining ideas in mathematical settings that involved group work.

4.2.2.2 Problem Solving Strategies

Riley strongly agreed, when answering the first questionnaire, that reading a problem carefully is helpful when solving that problem. On the second questionnaire, she only agreed with this statement. After reading LV-Pre, Riley underlined the question at the end of the text and explained that this was to help her avoid focusing on unimportant details. In this same task, when briefly stuck, she declared that she was “going to reread the problem to see if I catch anything useful.” While working on LV-Post, she added that she likes to look for key words the question, but that this might backfire. For example, because she had only internalized certain key *words* in the current task, she had missed at least one important detail that appeared as a *phrase*. That phrase was “of the entire student body,” which Riley eventually used to justify the inclusion of another important variable, S, in her system of equations which represented the school’s total population.

Riley not only underlined key words but also referred back to them when considering how to approach a task or whether to draw a diagram. For example, On MV-Pre, she noted that “there are some key words in here that allude to some things I could do,” and specifically pointed to “maximize” and “system of equations” as calling to mind previous experiences in mathematics classes. In LV-Post, the key word “also” reminded Riley of statistics courses and overlapping populations, which prompted her to draw a Venn diagram. In HV-Pre, Riley began the task by drawing a right isosceles triangle because it used both of the important geometric key words that she identified in the text.

Riley did the most exploratory sense-making on LV-Post. In this task, she began by first assigning variable names to the several populations of interest and then by trying to use these variables to write equations that captured the relationships described in the task description. To that end, she drew and labelled a Venn diagram and created four different equations using these variables; at that point, she recognized that two of these equations could be used to construct a system of equations and proceeded with a more methodical approach.

In other tasks, Riley engaged with less clear-cut exploratory behavior. In LV-Post, for example, she claimed that she was testing sample populations in her head, but never wrote anything down or mentioned any conclusions that she had drawn from her testing. On the other hand, in neither MV-Pre nor MV-Post did she choose any sample values of α to help her understand particular cases; she did spend some time “just moving things around” to see if she could isolate or eliminate a variable while solving MV-Pre, however.

On the first questionnaire, Riley somewhat agreed that imagining the form of the solution to a problem is helpful when solving that problem; she would later agree with this statement when completing the second questionnaire. Riley elaborated on this strategy by pointing out that “you have to see some of where you're going. You can't just go in blind. Because even if you do go in blind, you might waste your time on something—like, going one route. You have to have something that you can see.” She then immediately qualified that “you can still solve a problem without having an ending,” but that she tends to have the most problem solving success when she can visualize the “big picture” from the outset.

During both interviews, Riley appeared to value identifying the form of the solution more than her questionnaire responses would indicate. For example, on LV-Post, she explicitly wrote down the type of equation she hoped to arrive at once she had solved her system of equations (see the right-hand side of Figure 4.22 below); importantly, she recognized that her answer would not appear as a single number or ordered pair but as some proper fraction.

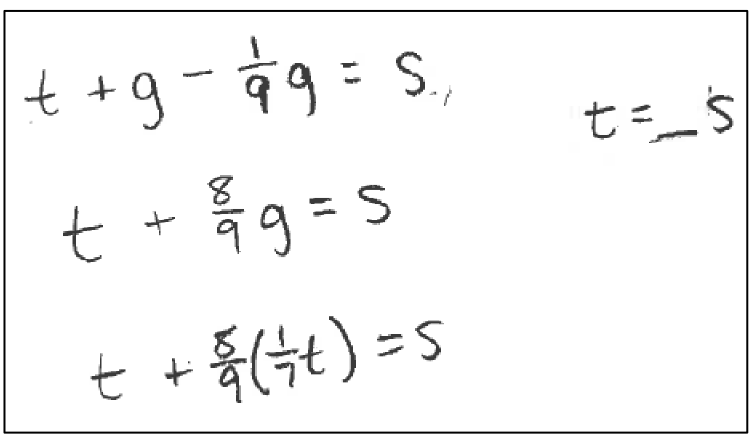


Figure 4.22 Riley’s image of the anticipated solution to LV-Post was of the form $t = \_\, s$.

Conversely, Riley identified that her biggest roadblock when solving MV-Pre was that she was “not completely certain what the answer’s supposed to look like.” She compared this to the previous task she had solved, LV-Pre, and pointed out that she had easily understood the form of that solution. In fact, after working LV-Pre, Riley explained that she even had a good idea that the final population (i.e., the answer she needed to provide) would have roughly equal amounts of men and women because the ratio was close to being 1:1.

After solving HV-Post, Riley discussed different strategies that she uses on proof-based problems as compared to those she uses on computationally-oriented problems. One of her strategies for proof-based problems required an application of the *Imagine Solution* strategy: Riley mentioned that she has found a useful approach to those problems might be to assume the proposition is true, draw (or write algebraically) the proposition under that assumption, and then to work backwards. She had actually just applied a version of this strategy on HV-Post, where she considered how the task might be different if she started with a square and circumscribed a triangle around it.

Riley's agreement, on the first questionnaire, with the statement that considering multiple approaches is helpful when solving difficult mathematics problems later increased to strong agreement on the second questionnaire. While working on LV-Post, Riley at one point considered whether she needed to create an equation with similar ratios, which was a technique she recalled using in the past. She further explained that, at that point in her work, she was "going through what I could potentially use to solve this problem." When the interviewer prompted her to elaborate, she added that "math gives you a toolbelt" for solving problems, and each mathematics problem normally allows for multiple viable methods or approaches; it is only a matter of finding one that works. Later, during HV-Post, Riley continued with this analogy by adding that her "toolbelt" for proof-based problems was fundamentally different than the one she uses for "normal math problems," and furthermore, that her proof-based tool belt was currently less developed.

At various times throughout the interview tasks, Riley considered different hypothetical tools from her belt. For example, in both LV-Pre and MV-Pre she considered using matrices to solve the system of equations, but in both circumstances opted instead for traditional substitution methods. Furthermore, in MV-Pre, she used two distinct substitution strategies (solving first for x and then for y) over the course of two approaches. During her work on HV-Pre, Riley considered looking for similar triangles in her diagram in order to use familiar geometric properties (and, in fact, ultimately based her incorrect answer on similar triangles). Interestingly, in LV-Post, Riley mentioned that she sometimes feels that considering too many approaches is “overthinking” and can ultimately distract her from working productively towards a solution. Later in that interview, she mentioned that she was, “like usual,” overthinking her work on MV-Post and committed herself to finding one working example of a square inscribed in a triangle instead of trying to consider a general strategy that would handle multiple cases at once.

On both the first and second questionnaires, Riley somewhat disagreed that planning one’s steps is helpful when solving a difficult problem. At certain times during the interviews, however, Riley did plan out a few steps in advance while considering an approach; for example, in MV-Pre, she ultimately rejected a matrix-based approach because she was not sure how effectively it would handle the additional parameter α. In LV-Post, she anticipated that using a system of linear equations would be effective despite the fact that she would not be able to explicitly solve for any variable. She came to this conclusion after returning to the task description and recognizing that she only needs to simplify the system up until the point when a proportional relationship is established (see previous discussion of the *Imagine Solution* strategy).

At other times, though, Riley lived up to her somewhat disagreement of this strategy. She remarked, in multiple circumstances, that she was happy to proceed with an approach that she was unsure would pan out effectively. In MV-Pre, she paused and observed that, from her work's current position, "I could expand it [a binomial squared] out. But I feel like that would put too many variables in it… Who knows? I'm going to do it anyway." She described this process as "taking a hunch and going with it until I got somewhere. Trying to look for something that looks familiar that I could use to find the answer."

Riley agreed, on both the first and second questionnaires, with the statement that drawing diagrams is helpful when solving a difficult mathematics problem. After solving MV-Post, Riley explained that she tends to draw diagrams when the mental picture she has of the problem space is very detailed; in that case, she reproduces her mental picture as a diagram and gives it the necessary detail, freeing up space in the forefront of her mind for her to use for big picture, conceptual ideas. Not all of the mental detail is added to a diagram immediately, however; in HV-Pre, Riley noted that the triangles that she had drawn on the page did not include angle measure, at first, while her mental image did. It was only when she realized that angle measure would be pivotal for providing a rigorous justification that she added it to the physical diagram. On the other hand, some mental images Riley generated were deemed unimportant for actually solving the task and did not make the transition to physical form at all. For example, she described her mental image of the populations in LV-Pre as two circles with a bunch of people, represented by dots, inside of them. She did not anticipate this image serving any practical purpose, and so did not draw it.

Riley's most effective visualization was the Venn diagram she drew for LV-Post. From it, she was able to distill and interpret important relationships about the populations of interest in the task. She also drew a number of diagrams for MV-Post; in particular, she began with a sinusoidal curve and supplemented it with a unit circle. Riley had trouble, however, coordinating between these two diagrams in order to make effective headway in the task. She treated the ordered pairs of the unit circle as corresponding to ordered pairs on the function $y = \sin(\alpha x)$, and because she had only drawn $y = \sin(\alpha x)$ for positive x-values, only considered points on the unit circle in the first quadrant. This ultimately led to an incorrect answer based on mathematically unsound justification. Finally, Riley drew relatively few triangles for both HV-Pre and HV-Post; in the former task, she had to be encouraged by the interviewer to consider more than just one case. In the latter, she had to be similarly prodded into attempting any kind of justification with her diagrams.

On both the first and second questionnaires, Riley somewhat disagreed with the statement that checking one's work at every step is a helpful practice when solving a difficult mathematics problem. While working LV-Pre, though, Riley admitted that "I always have to check my math because I don't trust myself." During this task, there were several instances wherein Riley reviewed her work; she also opted to use a calculator for all her computations to avoid making an arithmetic error. Finally, she also demonstrated some monitoring tendencies as she solved LV-Pre: at one point, Riley explained that she would not reduce an improper fraction to a decimal, which she feared would lead to more difficult computations in the future.

Riley also checked her work for MV-Pre when she was alerted to the possible presence of an error by a mathematical inconsistency. In that task, she made a small arithmetic error (that initially went uncorrected) while applying a substitution method to solve for x in the system of equations. Later, when she chose to solve for y instead, she realized that what should have been a largely isomorphic procedure had culminated in a different equation; this prompted her to revisit her first approach and identify her error.

On the first questionnaire, Riley strongly agreed with the statement that justifying one's work at every step is helpful when solving a difficult mathematics problem. However, on the second questionnaire, she indicated that she now somewhat disagreed with this statement. Riley explained that she valued justification less because, in her recent analysis class, she sometimes found it useful to explore an option for solving a problem even if you could not justify its use. This technique is useful "because you could stumble on something that might give you the actual solution. Or a better way to do it." She added that, while working, you might need to focus more on making something work rather than explaining *why* it works, then retroactively consider the justification. This is an outlook Riley saw in practice in analysis, but she added that previous professors (from discrete mathematics and calculus courses) encouraged this type of mathematical activity as well.

Riley used thorough and sufficient justification for both LV-Pre and LV-Post. In those tasks, she adequately described how the equations she had generated were good representations of the relationships described in the task description. She also justified her use of a system of equations to solve both tasks, and in the case of LV-Post, accurately labelled her Venn diagram based on the constraints of the task description. Riley's justifications on MV-Pre more closely aligned with the outlook she espoused on the second questionnaire. That is, her work on that task was characterized by significant algebraic exploration and the application of procedures that she could not (and did not attempt to) justify using.

On the other hand, Riley's explanations of the reasoning she applied on HV-Pre and HV-Post was largely inaccurate or misleading. For example, she argued that there were only three types of triangle (right, equilateral, and isosceles) so she needed to only consider these three cases in HV-Post. Similarly, in HV-Pre, she argued that all right triangles fall into two categories (45-45-90 and 30-60-90), and implied that any right triangle that was not isosceles was of the latter variety. Not all of her justification was erroneous, however; Riley made a cogent and precise justification during HV-Pre for why drawing the altitude from the right angle of a right triangle always produced two smaller, similar triangles. She then argued that, if the original right triangle was not isosceles, neither of the two smaller triangles would be. However, she then applied this reasoning (in conjunction with her previous claim that right triangles were either isosceles or 30-60-90) to inaccurately conclude that she could provide a counterexample to the proposition.

Riley strongly agreed, when she completed the first questionnaire, that verifying the answer at the end of a solving a difficult problem is a helpful practice. She only agreed with this statement on the second questionnaire. Riley produced an answer that she could have verified on LV-Pre, LV-Post, and MV-Post. She did not appear to verify any of them, either by going back over her work or by using the constraints of the task description to check that solution was valid. For example, in LV-Post, Riley found that $\frac{63}{71}$ of the student body played tennis after accidentally using the wrong equation as part of her system. Had she used the information provided in the task description to corroborate her answer, she might have found that this fraction is much too high if indeed $\frac{1}{7}$ of the tennis players and $\frac{1}{9}$ of the golf players played both sports. Similarly, in MV-Post, Riley focused too closely on finding ordered pairs for which the x and y coordinates were the same; this caused her to hypothesize that $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ was a solution to the system of equations. She only realized something was wrong when prompted by the interviewer. However, she then claimed that $(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$ was a better solution (given $\alpha = \frac{\pi}{2\sqrt{2}}$), an answer that is possibly more incorrect. Here, Riley was focused more on the mathematical precision of her calculations than on whether the computations she was undertaking served the purpose of the task.

4.2.2.3 Mathematical Affect

When asked to explain why she strongly agreed that creativity is important for mathematical success, Riley took the opportunity to compare mathematics to other STEM fields:

> I fully believe in this theory that if you give a mathematician a problem, they can figure out a way to solve it; but if you give, like, an engineer a problem, a math problem, they won't be able to solve it as easily mainly because you need to think outside of the box sometimes if you have no idea what you're doing with this question. You have to think of other things that *could* lead you to a solution or could get you closer to the right path to do it. And so sometimes those aren't always by the book.

Still, even though she disagreed that memorization was important for mathematical success, Riley believed that memorizing could be helpful in that it gave one more tools with which to ultimately be creative. Despite her response to the *Persistence* item being relatively close to the mean, Riley also described the importance of “powering through something,” especially in areas where one is uncomfortable. That is, even if she feels frustrated or has come to a dead end on a difficult problem, eventually finding the solution “helps expand your knowledge.” She felt that she has “always been an optimist, in that sense. So, just pushing through. Because you won't get anywhere if you stop moving.” This attitude is supplemented by the fact that Riley also felt that she has always been a competent mathematician, so she is not deterred by temporary setbacks.

None of Riley’s responses to an item in part two or three of the questionnaire shifted dramatically between the first and second questionnaire. Instead of commenting on questionnaire items, Riley instead had the opportunity to talk about her perception of mathematics classrooms and how she learned most effectively. Instead of engaging directly with discussions or lectures, Riley explained that she preferred to listen to and compare the different perspectives given by those who do contribute. Despite not characterizing herself as an "outgoing person," Riley did feel that she was an effective speaker and good at explaining her mathematical ideas. Not being able to communicate and interact with others in a mathematical setting, she argued, is "one of the things that kind of like, holds people back from forming groups and having those different ideas." She especially valued tutoring as an opportunity to practice this skill. Tutoring also reinforced to Riley that "no one person is the same. And so everybody learns differently." She explained that she herself dislikes lecture formats and would prefer to read, rather than listen to, content being delivered.

4.2.2.4 Problem Solving Characterization

When Riley could solve an interview task with strictly computational algebra techniques, she was (barring a single careless error) overwhelmingly successful. In these tasks, she adeptly distilled important information from the task description, drew relevant diagrams when appropriate, and used her image of the form of the solution to motivate and guide her choice of approach. However, when a computational task had a novel conceptual twist, Riley had trouble identifying precisely how her familiar repertoire of approaches was practically affected. To embrace her toolbelt analogy, Riley appeared very comfortable hammering nails and tightening screws; when presented with a nail and told not to use a hammer, though, she became lost. As she identified herself, she had more difficulty telling what was important and what form the solution would take in problems that fit less snuggly into her area of expertise.

At the other end of the spectrum, the geometric tasks that required much more mathematical justification than accurate computation were especially difficult for Riley. In line with her belief that she should just "power through" such situations, she did not completely avoid making a meaningful attempt to solve these tasks; still, she had trouble (or was hesitant to try) drawing diagrams for more than one case. The associated justifications she provided to explain her diagrams were also sometimes inaccurate, leading to illogical claims that undermined any success that she otherwise found. Interestingly, unlike some other participants, Riley did not attempt to force computational elements into these types of tasks by assigning hypothetical numerical values that she could manipulate. Instead, she seemed to accept that she simply was not prepared to handle these tasks because she did not have the associated content knowledge.

4.2.3 Brady

4.2.3.1 Formative Experiences

Brady found several previous mathematics courses to have influenced his problem solving strategies. For example, of analysis and probability, he found that "it feels like we're taking all these logical steps, but then sometimes it leads to some really really surprising results." As a result,

> I try to keep in mind whenever I'm solving problems—is that, it may not be an intuitive next step for each in the sequence of solving it. There may be a point where you need to maybe just, I don't know, see it from an entirely different perspective or use some non-intuitive trick to solve it.

Brady found this to be interesting rather than discouraging, however. For example, seeing a proof by contradiction for the first time "got me interested in this stuff and made me want to, I guess, learn new ways to solve problems." These courses also taught him to the value of "thinking exhaustively"; that is, considering different interpretations of a task and accounting for multiple cases that he might otherwise overlook. Finally, analysis in particular helped Brady train his ability to evaluate logical arguments.

Brady's appreciation for rigorous, logical justification began earlier in introduction to proofs, which he identified as his favorite mathematics course. Introduction to proofs showed him the importance of justification because, in that class, writing a proof

> felt like it was pretty much line by line, you know? Every single—like, even commutativity and associativity, basic stuff. But what that ultimately leads to is that you never make a false step. You're working with every single line. And I felt like, in the beginning, that that was very important. Because now, I don't just, you know, skip a few lines of work without really thinking about it first.

The instructor encouraged their students to take part in the mathematical activity of lectures by volunteering them to answer questions or present proofs in front of the class. Brady noticed that this “added a community feeling.”

Finally, Brady briefly commented on his informal experiences teaching friends and classmates in study groups. He felt that this experience forced him to “over-learn” material, in that he had to evaluate and understand multiple different approaches to a problem. However, he also felt that attempting to teach unsuccessful classmates convinced him, unlike he previously thought, some people who are not sufficiently naturally talented in mathematics might not be able to succeed in the subject.

4.2.3.2 Problem Solving Strategies

On both the first and second questionnaire, Brady strongly agreed with the statement that reading a problem carefully is helpful when attempting to solve that problem. When asked by the interviewer at the beginning of HV-Pre what he looks for while reading a problem, Brady explained that he searches primarily for key words. To illustrate, Brady noted “always” and arbitrary” in the text of that task tell him he will have to make an argument that holds generally for all triangles. Additionally, the word “isosceles” constrains the way in which one cuts the arbitrary right triangle. Brady commented again on the key word “arbitrary” in the text of HV-Post, where it prompted him to “immediately begin thinking of extreme cases.”

While working on the interview tasks, Brady often revisited the task description for inspiration when stuck or to verify that he was on the right path when about to apply a new approach. For example, after applying techniques involving conditional probability to

LV-Post with little success, Brady reread the task description and made two important observations: first, that he had so far only been incorporating half of the provided data points; and second, that the wording of the question suggested that there might be some use in identifying which of the populations in question were partitions of others. On LV-Pre, Brady originally set up a system of equations that would allow him to solve for the initial population. After revisiting the task description, however, and realizing that he actually needed to find the final population, he introduced another pair of variables to represent the number of men and the number of women in the final population of the town. This change came despite the fact that, earlier in the task, he remarked that he could easily find the final population if only he knew the initial population. The decision to introduce additional variables to his system of equations based on the wording of the task description ultimately inhibited Brady from solving for either population.

In many of the interview tasks, Brady made sense of the relationships described in the task description by defining a multitude of variables or equations, not all of which ended up being necessary. For example, in HV-Pre, he drew and cut a right triangle into two smaller triangles, then used seven different variables to label a combination of edges. By considering which of the variables might actually be the same (or might have some more complicated underlying relationship which could allow him to represent one variable in terms of another), Brady sought to increase his understanding of the situation and identify the most profitable approach to the task. In LV-Pre, he created a system of two equations with four variables. Finally, in LV-Post, Brady listed as many different equations as possible that might describe the ways in which the populations in that task were related.

To solve both MV-Pre and MV-Post, wherein the tasks themselves already provided him with variables and equations, Brady had to rely on two different approaches in order to explore the space. First, he sought to manipulate and recombine these equations algebraically as a means of sense-making. The statement Brady made about MV-Pre, that he could "just play with it [the system of equations] and see what comes of it," embodies this approach. After making the above observation, Brady followed through by substituting $x = y$ into the second equation and attempting to simplify the result.

Second, in both MV-Pre and MV-Post, Brady also made exploratory substitutions for the parameter α to make sense of simpler cases. For example, as a preliminary step towards solving MV-Post, he said that was "going to imagine throwing in some values. What if I put in $\alpha = 0$?" Brady briefly considered this case before observing that α was required to be a positive integer; afterwards, he sought instead to understand the case when $\alpha = 1$. Finally, after observing that $\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)$, Brady hypothesized that $\frac{\pi}{4}$ could be an important representative value in some way (perhaps based on structural similarities between the preceding equation and the equation $x = y$), which he resolved to "feel out for a little bit."

Brady strongly agreed, on both the first and second questionnaire, that imagining the form of a solution is a helpful practice for solving difficult mathematics problems. At the end of the first interview, he explained this opinion by pointing out that "if you don't already know what a question is asking for, it may be kind of difficult to realize what concept it's testing." Living up to this claim, Brady attributed his difficulties with LV-Pre to his inability to efficiently visualize the changing population of the town when the primary source of numerical data to work with was given in ratio form. He added that he was "roadblocked by my own imagination"; later, when he tried to manipulate the ratios into a more familiar shape (in this case, by treating them as fractions and finding a common denominator), Brady was only further confused by the contextual meaning of the new fractions he had created.

On other interview tasks, Brady was more successful in imagining some form of the solution; however, his mental image was not always correct. When it was, it was not always helpful. For example, in MV-Pre, Brady hypothesized that the set of solutions to the system of equations was likely to be infinitely large. Operating under this incorrect conclusion led him to lose sight of exactly what quantity in the task he was meant to maximize. That is, after choosing a value of α, Brady calculated the number of points that would satisfy only the equations in the system which contained α, rather than the system as a whole. From this perspective, it appeared to Brady that any choice of α would lead to an infinite number of solutions to the task (since $(x+\alpha)^2+y^2=1$ contains infinitely many points no matter the choice of α). On, HV-Pre, Brady briefly entertained an approach to the task in which he began with the solution (two isosceles triangles) and tried to connect them to form a right triangle. This expedition was short lived, however, and ultimately did not appear to offer Brady any substantial insight or otherwise affect his eventual solution.

On the first questionnaire, Brady strongly agreed with the statement that considering multiple approaches is helpful when solving a difficult mathematics problem. On the second questionnaire, this opinion was unchanged. Early in the first interview, before writing anything down for LV-Pre, Brady said that he was “focused more on how to do it without having to erase a bunch. Just, approaching it from the correct direction the first time.” This may have been an artifact of Brady’s discomfort with the interview environment; eventually, he decided to just “brainstorm it” instead, which he described as attacking the task from multiple angles. In that task, he went on to apply several different approaches. These included: relating the male and female populations using an equation and the given ratios; “parameterizing” the task, which Brady did not explain further or seriously pursue; leveraging techniques from related rates problems; and, eventually, solving a system of equations. In this and other tasks, Brady was quick to list different possible approaches, some of which he appeared to remember from previous courses. He very seldomly followed through with these considered approaches. For example, in LV-Post, Brady first thought that he might need what he called “inclusion and exclusion techniques” from set theory, but like when he considering “parameterizing,” he did not clarify what techniques he was referring to and did not appear to actually employ them. Instead, he went on to write a number of probabilistic equations that he derived from Bayes Theorem, but eventually concluded that he was “just forgetting one formula to help me out here.” In HV-Pre, Brady wrote the Pythagorean theorem down (but did not use it) and wondered aloud whether comparing the sides of the triangle with a ratio might allow him to see if “maybe anything trip pops up”; despite making this consideration, he never wrote down a ratio or applied any trigonometry. In MV-Post, a task that already included explicit trigonometric functions, Brady wondered aloud if he might need to employ the double-angle formula but never used it in pursuit of a solution.

Brady somewhat agreed, on both questionnaires, that planning one’s steps in advance can help one to solve a difficult mathematics problem. He explained that “I want to plan it, but I don't want to plan it really thoroughly. Just kind of get a feel for it,” in order to avoid “over-thinking.” As seen in *Consider Approaches*, Brady appeared to list off several different potential solution methods with which he did not ultimately follow through. Whether this was because he had planned ahead and had deemed these approaches inadequate, though, was unclear. Indeed, early in the first interview, Brady admitted that his lack of focus while solving difficult mathematics problems often led him to jump around haphazardly between approaches.

While it was not always clear whether or not Brady planned ahead when he changed approaches, he did sometimes acknowledge that he was doing the opposite, i.e.

proceeding with some initial strategy without any foreknowledge of whether or not it would be successful. For example, on LV-Post, he remarked that "I'm not sure that this is going to lead anywhere, but I'm just going to try it," before applying a substitution technique to attempt to solve for the probability of a student playing tennis. In LV-Pre, in order to explain to the interviewer why he had opted into factoring a particular difference of squares, Brady said that "If I didn't feel like I know which direction to go, I figured I should take one direction and see what happens. So there wasn't necessarily a reason behind that."

On both the first and second questionnaire, Brady strongly agreed with the statement that drawing a diagram is helpful for solving a difficult problem. He ascribed his propensity for drawing diagrams to the fact that he is so "scatter-brained"; that is, he "can picture it [a problem setting] in my head, but if I don't draw it, I may not remember what I was thinking about." He used multiple integration as a specific, recent example of content that he found especially hard to interpret and keep track of symbolically: it was "a lot more helpful for me, instead of having to contain all that information in my head, actually just drawing it."

During the interviews, he did follow through with this claim by drawing a number of diagrams; although, when he felt uncomfortable visualizing a particular type of diagram, he often chose not to draw it at all. For example, near the end of his work on MV-Pre, Brady admitted that the equation $(x + \alpha)^2 + y^2 = 1$ had some kind of geometric interpretation that he could not remember. He suspected that it was related to the unit circle, but that the parameter might make it an ellipse. Brady also rejected his initial plan to graph the equation $x^2 = y^2$ in favor of algebraic manipulation. At several different times in the interviews, Brady mentioned that he did not think he was especially strong at trigonometry; he was also one of only two participants not to draw a sinusoidal curve on

MV-Post (although he did draw several circles instead). On the other hand, Brady's relative confidence with probability led him to produce three different Venn diagrams that he used to orient himself on each of several different approaches to LV-Post.

Brady drew his most interesting (but perhaps not most effective) visualization for HV-Post. In order to represent all possible types of triangle and thus consider all possible cases, Brady drew a circle. Inside, he created a number of triangles such that two of the triangle's sides were radii of the circle (see Figure 4.23 below).

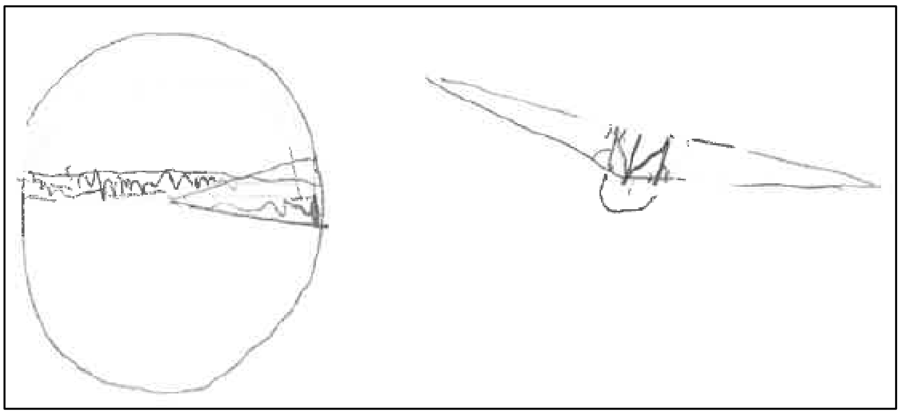


Figure 4.23 Brady used this diagram in an attempt to ensure he handled every possible case of triangle in HV-Post.

Brady claimed that, by "spinning" one leg of the triangle around the interior of the circle, he could create any possible triangle. This dynamic visualization appeared to help him coordinate a systematic effort to inscribe a square in a large number of different kinds of triangles, although he did not appear to notice that all his triangles were necessarily isosceles as a result. Brady also anticipated that his diagram would give him the ability to apply useful "tricks" invoking the geometric properties of circles, although this opportunity never resolved. In this same task, Brady used additional dynamic visual language when he referred to his ability to "shrink" any rectangle into a square.

Finally, in both HV-Pre and HV-Post, Brady relied more on his visual intuition than he was sometimes comfortable with. For example, when working on HV-Pre, he failed to justify why his (correct) method for cutting a right triangle into two isosceles triangles would always work, primarily because he relied on the accuracy of his drawings and not geometric properties. Later, during his work on HV-Post, he also admitted that he was “starting to wonder if I’ve been fooled by my own drawings.”

Brady’s opinion on checking his work at every step of the problem solving process remained unchanged between the first and second questionnaire; in both instances, he agreed with that statement. Brady did check his work often; for example, while solving LV-Pre, he corrected a number of small algebraic errors, checked some numerical computations that he suspected might be wrong (and discovered they were not), and revisited the task description to double check that he was solving for the correct population. He also assessed whether certain algebraic manipulations would efficiently allow him to reach his goals.

On the other hand, on some tasks, Brady’s work was crippled by unperceived errors. This was most evident on MV-Pre; while solving that task, Brady made a small sign error and incorrectly expanded a binomial. In combination, these mistakes allowed him to isolate α in terms of x. This led him down a completely unproductive path of inquiry in which he focused on interpreting this (incorrect) result. Later in this task, he committed another error when he justified a mathematical claim using the fact that he believed $(\frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2})$ to be a point on the unit circle.

Brady strongly agreed, on both the first and second questionnaire, that justifying one’s work at every step is helpful when attempting to solve a difficult mathematics problem. Over the course of both interviews, he lived up to this claim by attempting to justify almost every step of a given task. It is perhaps this need to justify everything that led Brady to occasionally give incorrect (but detailed) mathematical explanations. For example, in HV-Post, he based a particularly long justification on the fact that any angle bisector drawn from the vertex of a triangle would coincide with the altitude drawn from that vertex. Even when he later realized the inaccuracy of this claim, Brady continued to use it as justification for a number of other incorrect conclusions. Other times, Brady's mathematical justifications were interrupted by realizations or shifts in thinking that left his thoughts uncompleted.

On the other hand, Brady's justifications for why he took a particular approach were often reasonable and productive. In HV-Post, he noted before beginning the task that "arbitrary" in the task description prompted him to consider extreme cases in order to possibly identify a counterexample. In HV-Pre, he anticipated that each right triangle would have a unique solution that cut it into two isosceles triangles; he hoped to find a few working examples from different triangles and look for characteristics that all the cuts had in common in order to formulate a generalizable procedure..

On the first questionnaire, Brady strongly agreed that verifying one's answer at the end of the problem solving process is helpful when working a difficult mathematics problem. On the second questionnaire, he only agreed with the same statement. Brady produced verifiable solutions on MV-Pre and MV-Post. In both cases, he first stated what would eventually be his solution as a tentative mathematical hypothesis; then, he spent the bulk of his time on the task verifying these claims. For example, in MV-Pre, Brady hypothesized that, when $\alpha = 0$, the equation $(x + \alpha)^2 + y^2 = 1$ would have infinitely many solutions; thus, $\alpha = 0$ was part of the solution set. He verified that $x^2 + y^2 = 1$ did contain infinitely many points by recognizing it as the equation of the unit circle. Prompted by the instructor, he then considered the case when $\alpha = 1$ and identified, again, that the new equation contained infinitely many solutions. At this point, he seemed confident in claiming that any choice of α would give infinitely many solutions. Brady employed a similar process in HV-Post; there, he hypothesized which values of α might be a solution and used mathematical reasoning to either prove or disprove each hypothesis. Afterwards, he noted that the particularly specific nature of the given constraints on α meant that there would not likely be very many solutions. This reasoning, in part, led him to accept his eventual answer of $\alpha = 1$.

As seen in the above processes, Brady appeared to form a solution relatively early in each task and spend the bulk of his time verifying whether his initial guess was correct. Part of Brady's difficultly providing any answer to LV-Pre or LV-Post, then, might be attributed to his inability to guess at what a reasonable answer might be; unlike in MV-Pre and MV-Post, those questions are not as susceptible to the application of test values.

4.2.3.3 Mathematical Affect

Throughout both interviews, Brady repeatedly observed that difficulties he had on both interview tasks and in other mathematical settings could be attributed to the fact that he considered himself "scatter-brained." For example, after working on MV-Pre for an extended amount of time, Brady lamented that the task "looks like it should be simple, but for whatever reason I'm not able to piece it together because my thoughts are scattered or whatever the reason." He also described certain generalized problem solving behaviors that he developed in response to his perceived lack of focus; for example, he "may not remember what I was thinking about" if he doesn't draw a diagram of his thoughts, given that he "can't store that much in my head at once." In the second interview, Brady also explained that his higher valuation of the importance of memorization for mathematical success was due to the fact that he hated not being able to remember the exact formulation of certain important definitions (e.g. uniform continuity) in his recent analysis course during an exam, which he felt ultimately lost him points. Still, though, he did not strongly agree with the *Memory* item because “knowing the correct formula can definitely help, but given that there're often so many different avenues to solving a problem, I could see a lot of problems with enough time you could find a workaround for it.”

Besides these beliefs in his own ability to utilize short-term memory, the next most significant facet of Brady’s mathematical affect centered around his belief in the importance of natural talent. On the first questionnaire, Brady strongly disagreed with this characteristic’s relevance for achieving mathematical success. This was a notably lower opinion than other undergraduates. However, on the second questionnaire, Brady only somewhat disagreed; this was now one of the two highest ratings given on the *Natural Talent* item. At first, Brady was “pretty passionate” when he explained that

> I don't believe you have to be naturally good at math. Because I believe natural talent, in itself, is just—okay. So something may come to you easily, like Cal 2 and Cal 3 came to me pretty easily. But suddenly probability didn't. Or linear algebra *really* didn't, for whatever reason. And I was able to, at least, try to do better in those classes by working hard and just going out of my way to figure out what about it I wasn't understanding. […] Maybe the only thing that you'd need natural talent for is a natural talent for figuring out how to do things you can't do. […] I guess, being naturally curious is probably more important than natural talent.

This opinion was ultimately supported by Brady’s claim that, despite the fact that he found MV-Pre frustrating, ultimately this made him curious to know more about the task and how it could be intuitively understood. Brady’s outlook on natural talent shifted during the semester when he and some classmates hosted a reviews session for their probability course. During those review sessions, some of Brady’s classmates “were working really hard and some things just don't click.” Despite his best efforts, none of his explanations of abstract concepts appeared to make sense to them. In his opinion, this experience lent credence to the claim that “some people's brains are wired, you know, in a different way than other people's are.” He qualified this conclusion by pointing out that he had previously had success tutoring in calculus: “maybe it's just that the more abstract something becomes, the more natural talent it requires.”

4.2.3.4 Problem Solving Characterization

Brady’s problem solving was based largely on volume rather than accuracy. After overcoming his initial desire to find exactly the right approach and avoid making any erasures, he ended up producing a substantial amount of written work that he supported (to the best of his ability) with significant verbal explanations. Much of Brady’s work could be attributed to the exploratory algebraic manipulation of variables and equations until he was struck by some insight that might prompt him to generate either a hypothesis about the form of the solution (that he could then try to verify) or to a more rigorously defined method of approaching the task. However, even transitioning out of his initial exploratory phase, Brady was prone to subsequent new ideas that sometimes pulled him away from testing hypotheses or approaches that he had not yet completely borne out.

Brady usually justified why each change of approach was appealing, but when he attempted to then justify the mathematical properties it relied upon, the results were uneven. For example, in the process of justifying a mathematical claim, he might recognize that it was actually untrue; however, when this realization came after he had already switched between several approaches, Brady had trouble disentangling his incorrect assumptions from what parts of his work were still correct. This difficulty was especially pronounced in tasks with high visualization requirements. While Brady appeared to appreciate the value of well-constructed diagrams, he sometimes doubted his ability to produce them and opted for symbolic manipulation instead. When he did produce a diagram, Brady did not always recognize when he was relying on his visual intuition to support a claim.

4.2.4 Verna

4.2.4.1 Formative Experiences

Verna identified an introduction to proofs course as her favorite mathematics class. She appreciated that the instructor of the course was very organized and gave regular, pre-class quizzes so that students could later use as reference when studying for exams; she also preferred that the instructor lectured using a document camera rather than by writing on the whiteboard, which Verna sometimes found harder to follow. Most importantly, however, was the fact that the instructor also emphasized the need for each step of a proof to have detailed justification: “she told us to go step by step and don't, like, don't ignore this. This is have some meaning. *This* is have some meaning. You must understand what you [are] talking about.” She also claimed to like other mathematics courses as well, but only for the type of content they delivered.

Verna mentioned, in both interviews, that she learned about the importance of visualizations from a series of instructors who all recommended that she draw more diagrams when solving problems. She listed other heuristics she had learned as well as their origins; for example, her proofs instructor taught her the importance of circling important parts of a question’s text. This lesson was reinforced during Verna’s time as a tutor in the math clinic. In her capacity as an educator, she found it useful to

> ask them [her students]: look what they ask here. They figure out, oh, this is what they ask? Like this. So I make them to understand the question and what they ask about. And they have the knowledge, but they didn't know. That's why—that’s helped me

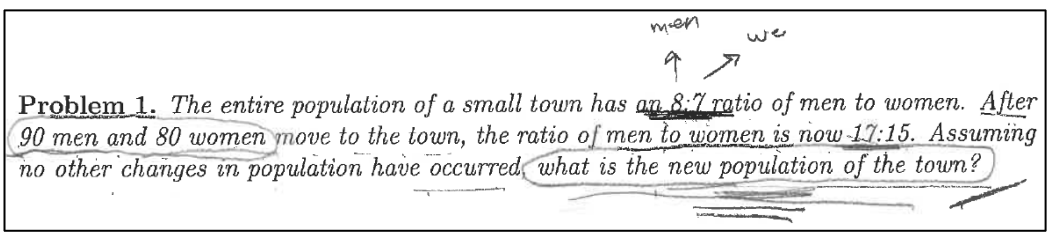
Finally, Verna’s recent analysis class helped her see the value in starting from a known solution and working out appropriate justification retroactively.

4.2.4.2 Problem Solving Strategies

On both questionnaires, Verna strongly agreed with the statement that reading a problem carefully is helpful when solving that problem. Verna explained during the first interview that

> My problem is *not* read it carefully! Like when I see the problem: What?! Okay, calm down. Read it carefully. He asked about this, not this. Because sometimes, I lose grades because I didn't read it carefully. Or there is something, and they ask about something else. I do that, I forget the other one.

Perhaps because of this awareness, when reading the interview tasks, Verna made a habit of judiciously underlining and circling key parts of the text (e.g. words like “always”, “integer(s)”, “one or more”; any numerical data) that would affect the way she solves the task (see Figure 4.24 below).



Problem 1. *The entire population of a small town has an 8:7 ratio of men to women. After 90 men and 80 women move to the town, the ratio of men to women is now 17:15. Assuming no other changes in population have occurred, what is the new population of the town?*

Figure 4.24 The text of LV-Pre, to which Verna appeared to carefully attend while reading and working on the task.

Sometimes, though, despite her conscientiousness while reading the task, her work would fail to account for the subtleties that she drew attention to in the text. For example, in HV-Post, Verna returned repeatedly to the fact that she had to “always” be able to inscribe a square in an arbitrary triangle; this consideration influenced the amount and direction of her work. However, in MV-Post, Verna repeatedly lost track of which variable in the system of equations must be an integer value. At other times, Verna appeared to ignore or misunderstand key parts of the task description entirely. For example, in MV-Pre, she did not mention the fact that she was being asked to find a value of α that maximized the number of solutions to the system; in fact, she did not attend to α at all in her approach to the task.

In LV-Pre and LV-Post, Verna's first action after reading the text was to transcribe and reorganize the numerical data with additional labels and arrows to capture relationships. However, her difficulty interpreting what a given question actually asked her to find sometimes inhibited her from using the numerical data effectively no matter how she had organized it. In LV-Pre, for example, she fixated on finding the percent of population growth and not the actual magnitude of the population. As a result, her treatment of that task's numerical data was incorrect.

Verna engaged with almost no exploratory sense-making, especially during the first interview. She did not choose sample values for the parameter α in either MV-Pre or MV-Post, and she drew only one and two triangles in HV-Pre and HV-Post, respectively, before making a claim about the truth of those statements. While working HV-Pre, Verna did at one point clarify that the arbitrary right triangle she was currently considering was also isosceles. However, this was not treated as a tentative exploration of a simpler case and she did not further refer to the congruency of sides or angles as she manipulated that triangle towards a solution. In those tasks (such as LV-Pre and LV-Post) which might have been facilitated by an algebraic exploration of variables and equations, Verna instead relied on seemingly arbitrary arithmetic computations. That is, she recombined numerical data from the task description without seeking first to make sense of the task with symbolic representations.

Verna strongly agreed (on both the first and second questionnaire) that imaging a solution is helpful when solving a difficult mathematics problem. In most tasks, Verna had trouble interpreting what form the solution should take. As a consequence, she could not then use the form of the solution to motivate her choice of approach or to check her work.

For example, in MV-Pre, Verna did not acknowledge that she was finding a set of α values that maximized the number of solutions. Instead, she thought there must be a single value of α that would arise from solving the system of equations with traditional algebraic techniques. She had similar difficulty interpreting the form of the solution for MV-Post, LV-Pre, and HV-Pre. The only task in which Verna appeared to recognize correctly what was being asked of her was LV-Post.

Verna's shortcomings in this area hinged largely on the fact that she was not a native English speaker; even with clarification and reminders from the interviewer, however, she still had trouble keeping the goals of the tasks in mind when she became enmeshed in a particular solution approach.

On the first questionnaire, Verna strongly agreed that considering multiple approaches is helpful for solving difficult mathematics problems. On the second questionnaire, she indicated that her opinion had not changed. Verna only considered taking an alternative approach a small number of times during both interviews, and usually only in a very superficial manner. For example, in MV-Post, after working for a while with visualizations, she remarked that a guess-and-check approach for finding values that satisfied the equation $\sin(x) = x$ would be impractical. In MV-Pre, Verna first took an elimination approach to solving the system of equations; she added, however, that substitution would also be possible. Notably, Verna did not do any perceptible shifting from one approach to another, at least not without explicit prompting from the interviewer.

While solving LV-Post, Verna shed some light on why she did not consider many different approaches. She explained that, for proofs, she finds it helpful to try as many different approaches as possible; otherwise, however, she prefers to stick with what she thinks is the most correct method for as long as possible. This strategy, she argued, is more likely to get her the maximum amount of partial credit on exams. At the end of the second interview, Verna further explained her typical strategy for starting problems: foremost, she tries to pick a known formula that applies to the current context. If she cannot think of a formula, she tries to solve the problem organically. However, formulas give her “the perfect answer in shorter time” and helps her to make “sure she’s correct.”

Verna strongly agreed with the statement on both questionnaires that planning one’s steps in advance is helpful when solving a difficult mathematics problem. Despite this praise, during the second interview Verna identified *Plan Steps* was the problem solving strategy that she found least helpful among those on the questionnaire. She explained that “sometimes I see the question, I—like, all the information is coming, coming. So I start!” This low opinion of planning ahead coincided with her self-assessed weakness for reading a problem slowly and carefully. Because Verna engaged with very little consideration of alternate approaches, it follows that she did even less planning. The extent of her perceptible planning was mentioned in earlier discussion of her us of the *Consider Approaches* strategy: during MV-Post, she rejected a trial and error method that would have been, practically speaking, impossible to use to find a solution.

On both questionnaires, Verna strongly agreed with the statement that drawing diagrams can be helpful when solving a difficult mathematics problem. In the first interview, she clarified that this strategy, while very important, is not one that she herself tends to engage with. In that same interview, Verna only drew a visualization or diagram for HV-Pre. After ending her work on this task, when asked about her visualizing habits, she explained that she really likes drawing pictures to help her understand difficult problems. However, she said that LV-Pre had no geometric figures, functions, or points that might have prompted her to produce a diagram. In the case of MV-Pre, she felt that the task was so straightforward that drawing some kind of visualization was unnecessary.

While working on LV-Post in the second interview, Verna explained that her experiences in her most recent semester had cemented her belief in the value of visualizations for seeing the “big picture.” Correspondingly, she did draw more diagrams in that interview when solving the non-geometric tasks. In LV-Post, she produced a Venn diagram to help her keep track of the different overlapping populations; in MV-Post, her first instinct was to draw both equations in the given system. These graphs were imprecise and did not appear to meaningfully affect her ability to draw mathematical conclusions, however. In HV-Post, Verna drew about the same number of triangles as she did in HV-Pre.

Verna initially strongly agreed that checking one’s work over the course of solving a difficult mathematics problem is helpful; she maintained her strong agreement on the second questionnaire. She even stressed, in both interviews, that “when I solve something, I didn't go to other steps until I very agree 100% about the steps.” She had little opportunity to exhibit this practice while working the interview tasks, however; Verna only produced a nontrivial amount of written, algebraic work in MV-Pre, and when she did, she did not correct several unnoticed errors. In other tasks (LV-Pre, LV-Post, and HV-Post), Verna only engaged in basic arithmetic that was either correct or easily corrected after an initial error. Finally, as mentioned in a previous paragraph, at least one of Verna’s sinusoidal curves in MV-Post was not actually a sine function of the form $\sin(\alpha x)$. She did not catch this error either; however, this diagram appeared to be an especially quick sketch and was immediately set aside for a more precise graphical representation.

Verna strongly agreed on both questionnaires with the statement that justifying one’s work at every step is helpful when solving a difficult mathematics problem. Verna’s justification in LV-Pre and LV-Post was nonexistent, which she blamed on her inability to remember discrete mathematics, statistics, and probability concepts. In LV-Post, at least, Verna did correctly justify *why* she was having difficulty with the task beyond her inability to remember the correct method. That is, she correctly identified that quantifying the intersection of the tennis and golf populations was the most important step, but that she was unsure how to do so given the data in the task.

In HV-Pre, HV-Post, and MV-Post, Verna often attempted to justify her claims with unfounded mathematics. In HV-Pre, she explained that "I would use the area of triangles" to prove that she had produced two isosceles triangles with her cut. This would involve the formula $r^2 = x^2 + y^2$ in some way, although she never explained how. In HV-Post, when asked how she would justify that the rectangle she inscribed in the triangle was also a square, she claimed that "there's a formula for that," but that she had forgotten it. Instead, she improvised a method using the altitude of the circumscribed triangle: if that altitude cuts the rectangle into two pieces of equal area, it must be a square. Finally, in MV-Post, Verna argued that because the range of $\sin(\alpha x)$ is $[-1,1]$, the solution set of α values should be $\{-1,0,1\}$.

In MV-Pre, Verna justified her decision to attempt to solve the system of equations using elimination methods by describing exactly how such a solution would play out. This was an excellent explanation of how to solve a *linear* system of two variables. It is likely that this was the algorithm that the instinctively applied to any two-equation system. However, her explanation did not take into account the additional parameter or the fact that both variables were quadratic. Only when trying to follow through on her outlined approach did she realize that her methodology was not applicable to the current setting.

On both the first and second questionnaire, Verna strongly agreed that verifying the answer at the end of a difficult problem is a helpful practice. On both LV-Pre and LV-

Post, Verna produced verifiable answers that she could have checked against the constraints of the task. In LV-Pre, she described her answer of 32% as a “huge number” and suspected that she might have needed to divide it by something. In LV-Post, Verna also admitted that her answer of $\frac{16}{63}$ was probably not correct and referred to the task description to justify this hunch. In both tasks, despite her misgivings about her solutions, Verna did not rework the task in any way or use the constraints given in the task description to verify that they were definitely incorrect.

The only other task for which Verna produced a solution that could have been verified was MV-Post. After claiming that the solution set was $\{-1,0,1\}$, however, she did not take any steps to validate this assertion.

4.2.4.3 Mathematical Affect

Verna’s mathematical affect was defined by her classification of which specific subject areas she did or did not like. At various times during the first interview, for example, Verna claimed to enjoy linear algebra, any course in the calculus sequence, and college algebra. On the other hand, she disliked discrete mathematics, probability, and statistics. Verna appeared to enjoy classes that she found easy, with the exception of statistics; in that case, “if you read the question correctly and if you know what the x, what the y, what the variance, what the—you will understand it, and only the formula? Put the formula, put the numbers. Nothing to do it. It's not use your brain to do it.” On the other hand, she disliked discrete mathematics because it was her first proofs course, and she felt the instructor expected too much of her prerequisite mathematical knowledge.

Verna’s other mathematical beliefs were not necessarily tied to a specific course. For example, she agreed that natural talent was important for mathematical success; this was a higher opinion of that characteristic than any other undergraduate. When asked to explain her rationale, she instead described why she had not strongly agreed: “So it's not always I succeed because I'm only [talented]. I cannot count in this. I should practice more and more and improve myself.” She used the fable of the tortoise and the hare as an analogy; whereas the hare should have won due to its natural talent, it did not simultaneously work hard enough to beat the tortoise in the race. Unlike in her home country, Verna was surprised to find that professors in the United States often allowed formula sheets for exams. This was a pleasant surprise, however, because Verna admitted that she did not often feel as though she excelled at committing important formulas to memory. Finally, Verna enjoyed proof-based courses for the satisfaction she felt after completing a difficult problem. Even after completing a proof, she sometimes “will keep looking. Because if there's [another approach], and short, and it explains more, I will take it. I will exercise or practice.” This corresponds to her especially high response to the *Best Approach* item of part three from the questionnaire.

4.2.4.4 Problem Solving Characterization

.Verna concluded her work on the interview tasks abruptly. If she thought she could remember what to do, the task was “very easy,” and she did not feel compelled to actually follow through with any calculations. On the other hand, if she anticipated that she would need a specific formula and could not remember one, she did especially little original, independent work to try and compensate for this lack of foreknowledge. This meant that, despite her claims about how important each of the problem solving items was on the questionnaire, Verna did very little exploratory sense-making, seldom checked her algebraic manipulations or solutions, and only gave justification when it was something she was extremely familiar with or when prompted explicitly by the interviewer. In either case, these justifications were still incomplete, specious, or both.

On the other hand, Verna did take great care to read the task description, draw attention to the most important parts, and ask for clarification from the interviewer when it was needed. Despite these efforts, however, her ability to translate a careful reading of the task description into actionable strategies or salient mathematical insight was hit-or-miss. In particular, Verna sometimes had difficulty working towards the stated goal of the task, even after underlining it on an initial reading. Similarly, while she did produce some visualizations on the second interview, Verna was not usually able to leverage them to improve her justifications or to inform the direction of her solution attempt.

Chapter 5

Discussion and Conclusions

In this study, a number of descriptive themes arose with respect to the types of formative experiences recognized by participants as contributing to their problem solving development. First, many participants recognized that their mathematical content knowledge (or their perception of how such knowledge could/ should be constructed) had expanded. This development was often a consequence of a specific instructor's pedagogical style, a particular course's content, or the opportunity to engage in extended study. Second, participants found that the social context in which their mathematics education took place sometimes left an indelible mark on their mathematical beliefs and problem solving strategies. Finally, participants also acknowledged that they had picked up various explicit mathematical heuristics that necessarily allowed them to develop more robust or fine-tuned problem solving strategies. These themes are discussed below.

5.1 Mathematical Content Knowledge

Some participants noted that there had not been any significant event that they felt affected their problem solving strategies in a specific way. Instead, they conveyed that their approach to solving difficult problems had incrementally improved over time as they accumulated more mathematical tools and relevant experience in using them. This perspective was not unique to the participants who did not have an identifiable formative experience; even those participants who recalled specific influential courses or instructors still noted that their problem solving tended to improve naturally over time as a consequence of learning more mathematical content. This result is not unreasonable, given that research has shown that increases in content knowledge may have a perceptible effect on problem solving even for Ph.D.-holding mathematicians (DeFranco, 1996). Furthermore, content knowledge is one of the central pillars of several problem solving frameworks (Mayer, 1998; Lester et al., 1989; Schoenfeld, 2014). Content knowledge is not only necessary for problem solving in the obvious, prerequisite sense (e.g., to identify whether a particular sequence is convergent requires one to know the definition of convergence of a sequence). Schoenfeld, as described in Arcavi et al. (1998), also recognized that poor content knowledge can inhibit problem solvers from successfully applying heuristic strategies that are, on the surface, independent of specific content knowledge.

In this study, several participants noted that the way in which they learned or learned to perceive mathematical content knowledge was a more important factor in their development as problem solvers than the information that they acquired. That is, whereas increasing one's content knowledge is certainly a boon for problem solving, the situational context in which that content is learned and the way that the learner assimilates new information into their existing body of knowledge can affect problem solving strategies as well. In the following sections, I delineate various subthemes dealing with the relationship between developing content knowledge and developing problem solving strategy.

5.1.1 Connectedness

Multiple pedagogical frameworks for promoting powerful mathematical thinking strategies (e.g. Schoenfeld, 2014; Collins et al., 1988) explicitly include that new mathematical content should connect theory, procedure, and context in a meaningful way. Other frameworks, especially those in situated cognition models (e.g. Bransford et al., 2012; Spiro et al., 1991; Campione et al., 1988), particularly emphasize the importance of decoupling mathematical content from an abstract pedagogical environment by introducing material using a real-world context. Even studies that do not propose a resultant framework report that students "prefer mathematics to be taught in context” (Carlson, 1998; p. 143). Such connections can also simply be engaging, and Liljedahl (2016) identified a high propensity for engendering engagement as a key facet of tasks that promote problem solving. Participants in this study recalled instances in which the connections made with (and within) mathematical content had been particularly influential.

Julie, Frank, Taylor, and Sara all identified prior experiences in which the primary benefit was the introduction or study of mathematical connections to situational contexts. Julie appreciated a precalculus instructor who assigned projects in which she had to “apply something towards a real life concept,” and noted that the instructor’s explanations of new material were augmented by relevant videos of real-life concepts. Frank found that a secondary school physics class presented him with contexts in which he could apply his (previously entirely abstract) mathematical content knowledge. Conversely, Taylor derived problem solving heuristics from prior experience in STEM fields that he was able to apply to abstract mathematics problems. In her graduate analysis class, Sara appreciated that her professor made vertical connections between content by building new concepts from familiar foundational ideas and explicitly pointing out (or asking students to hypothesize) when different concepts could be connected. Similarly, Sara found that the explanations that she herself provided as an instructor while tutoring could be enhanced by couching them in relevant real-world contexts. Will did not pinpoint any specific experience that helped him to make connections to or within mathematics content knowledge; nonetheless, he explained that

> I like to think of math as like kind of a big web of things. And there're different parts of the web that you wouldn't think connect, but you can connect them in certain ways and stuff like that. So that's what I mean with that problem, where it's like it may be something you've never seen before, but you can still draw a connection form something you have seen before.

One might interpret the promotion of a mathematical perspective based on the connectedness of topics as a direct attempt to avoid the transfer problem (that is, successful students failing to solve novel problems using mathematical content of which they have demonstrated mastery (Campione et al., 1988)). Students who are aware of relevant connections between problems may be more likely to have robust problem solving schema (in the sense of Selden et al., 1999) that encompass a variety of tentative solution starts. A reasonable hypothesis, then, is that those participants who were particularly influenced by the connectedness of mathematics in some prior setting or who valued the observance of such connections might demonstrate a higher propensity for the *Consider Approaches* strategy. However, only one of the above participants (Sara) rated *Consider Approaches* higher than average, and only on the first questionnaire; in fact, Taylor offset this data point by rating *Consider Approaches* lower than average on the first questionnaire. Ironically, both Sara and Taylor explained their disposition towards *Consider Approaches* in the same way: each felt that it was a useful strategy to consider after a first, most obvious approach had been exhausted and one was at a loss for how to proceed.

While attempting to solve the interview tasks themselves, the above participants also engaged (to varying degrees) with the *Consider Approaches* strategy. All of them applied Sara and Taylor's style of *Consider Approaches*, which was to only seriously consider an alternative approach when an intuitive initial strategy failed. Frank, in particular, was quick to completely switch approaches to a problem when he perceived that his current approach would not pan out. Julie, on the other hand, applied this strategy comparatively less than her peers. The difference between Julie's and Frank's propensity for *Consider Approaches*, despite their similarly formative experiences with the connectedness of mathematics, could be attributed to their distinctly different affective responses to failure. That is, Frank admitted that “I enjoy the process, even if I’m going in the totally wrong direction”; on the other hand, Julie reacted to difficulty with self-deprecation and repeatedly questioned her competency as a mathematician. As noted in the literature, negative affect can lead to difficulties when problem solving (Lester et al., 1989; Furinghetti & Morselli, 2009). In particular, Julie’s rejection of the “rectangular Venn diagram” approach to LV-Post is evidence of Furinghetti & Morselli’s conclusion that students with poor mathematical affect sometimes reject valid approaches that have not explicitly been taught to them. This behavior is warranted by the faulty belief that mathematics should always look familiar (2009).

5.1.2 Understanding Processes

Students are often unable to solve novel problems using content knowledge of which they have demonstrated prior mastery (c.f. Schoenfeld, 1989; Selden et al., 1999; Lester et al., 1989). Sometimes, this is due to poor or nonexistent metacognitive strategies (Schoenfeld, 1989); other times, it is because of a compromised affective state (Lester et al., 1989). But as Campione et al. (1988) suggest, some instances of this phenomenon are due to mathematics courses that emphasize correct answers at the cost of thorough understanding; furthermore, such courses usually also fail to engage students in meaningful mathematical discussion. This combination of shortcomings directly contradicts an important recommendation of existing pedagogical frameworks: namely, that students have a direct hand in explaining why mathematical processes produce the results that they do (Schoenfeld, 2014; Liljedahl, 2016; Collins et al., 1988). When students do not have time for meaningful exploration and reflection in this way, they often fall back on memorization of specific techniques (Carlson, 1998). As a corollary, these same frameworks assert that assessments should focus on allowing students to apply their conceptual knowledge rather than simply reproduce known algorithms. Several participants in this study reflected on specific formative experiences that required them (independently or through guided instruction) to justify, rather than simply perform, mathematical procedures.

One class that many participants perceived as especially influential in this regard was whatever course they took that first formally introduced them to the idea of proof-writing. For both Brady and Taylor, this class was introduction to proofs. For Mia, Riley, and Will, this class was discrete mathematics. For each participant, regardless of the course itself, they found that the instructor encouraged them to question why procedures worked the way that they did and to justify their use of those procedures. As Taylor explained, "it was not computation—it was actually thinking and trying to come up with solutions and seeing all the pieces connect." Will noted that exams in his discrete mathematics course reflected this principle: he found it absolutely necessary to understand why underlying concepts were useful for solving homework problems because exam questions were dissimilar enough from homework problems that a one-to-one transfer of procedure between homework and exam questions was not feasible. Riley and Brady recognized that the instructors of their courses, despite teaching in largely lecture format, managed to create a sense of agency and inclusion by encouraging participation (either by asking students to contribute to discussions or to present problems at the board for consideration). By way of contrast, Will remembered a particularly disliked secondary school instructor "who made fun of the fact that I would ask so many questions."

Participants also found that teaching mathematics forced them to be able to understand and justify material at a deeper level than they had before. Anne explained that

> That's like the reason why people need tutoring, I feel: they don't understand why you're doing what you're doing. And so, I can't just be

> like, well, use this formula. I don't know why it works, but it does. […] It just confuses them, I feel like, more. So it's made me have to figure out like, why I was doing things in previous courses as well.

Mia had a similarly reflective experience, in that teaching forced her to examine the extent to which she truly understood mathematics. Previously, “when I used to do math but I didn't understand something, I feel like I would kind of—like, just skip it. And then move on, take it as: okay, this is fact.” After answering questions and making justification for her students, Mia no longer felt comfortable relying on the results of a procedure to justify its use; she found that “I'm not able to let it go how I used to be able to let it go.” Sara also noticed that students often requested in-depth justifications for results that she was sometimes unable to provide. This, in turn, motivated her to discover independently how certain procedures were derived. Finally, Carol and Brady both worried that they would disappoint inquisitive students. Brady pointed out that “you can memorize a formula, or you can actually understand what the formula's doing.” He argued that teaching for the latter type of comprehension was much more beneficial for problem solving in general.

Formative experiences centered around mathematical agency and depth of understanding might reasonably be construed to induce a higher rate of the *Justify Work* strategy. That is, participants with a proclivity towards understanding why certain procedures work might be more likely to justify the application of such procedures or construct a sound mathematical argument for their approach to a problem. The above participants, however, seemed to have different ideas of when and how justification could be helpful. For example, both Mia (the only participant to rate *Justify Work* highly on the first questionnaire) and Riley scored that strategy significantly lower on the second questionnaire than they had on the first; they both shifted from strong agreement to somewhat disagreement. Both participants explained their shift in a similar way: they discovered that it is often helpful to initially forgo a rigorous justification of their mathematical decisions and instead explore the space in order to, as Riley put it, "stumble on something that might give you the actual solution." In contrast, Anne came to value *Justify Work* more strongly over her first semester of graduate school. She explained that reworking homework problems before an exam in her graduate linear algebra course helped her to more quickly remember appropriate justification for procedures she would need to use during the test. While attempting to solve the interview tasks, Anne's justifications (when provided) reflected this mindset accordingly. That is, she warranted her mathematical behavior by referencing memorized theorems (or theorems that she did not have memorized but thought probably existed) even in the face of explicit evidence that the theorem she sought to use was inappropriate or incorrectly stated. Differences in the above explanations illustrate that, of formative experiences in this theme, it may not be the emphasis on conceptual justifications themselves that has an effect on a participants' problem solving strategies. This is due to the fact that participants who memorize justifications just as they would memorize procedures render those justification inert, in the sense that they have difficulty transferring them to relevantly similar problem situations. On the other hand, participants sometimes eschewed formal justifications in favor of unstructured sense-making.

With this in mind, we might consider that participants who have been encouraged to independently seek conceptual bases for mathematical procedures could value *Creativity* over *Memorization*. In fact, of all surveyed undergraduates, Will, Riley, and Brady (who all recalled formative experiences in this theme) were the only three who rated *Creativity* one standard deviation above the mean response: all three strongly agreed that it is an important component of mathematical success. Similarly, Riley and Brady were also two of only three surveyed undergraduates who rated *Memorization* one standard deviation below the mean response. Riley directly compared her responses to these two items, explaining that she memorizes certain rules in order to use them creatively. For example, when taking the derivative of a quotient, there is no room for creativity; she must know the quotient rule. However, when solving a novel problem for which multiple approaches might exist, having memorized the quotient rule might allow her to creatively leverage her calculus content knowledge to solve that problem. Mia made a similar comparison during her first interview, pointing out that it would be difficult to solve for the missing side of a right triangle without knowing and applying the Pythagorean theorem. Here, though, both Riley and Mia draw a distinction between routine and non-routine problems often seen in literature (such as in the two different types of exam given to calculus students who participated in the study described in Selden et al. (1994); notably, in that report, the authors observed that almost none of the participants *could* leverage their content knowledge creatively, in the way Riley described). Will was surprised, during the second interview, to learn that he had slightly agreed with the importance of memorization on his first questionnaire. He explained that, even in previous semesters, classmates who used memorization of procedure as a problem solving strategy were at a noticeable disadvantage:

> In linear algebra there's a lot of situations where you don't do that [memorize procedures]. You're just like—you have to logically consider what you're doing. And so what happened—what I noticed was that a lot of students in that class were trying to take the math—or would try to memorize the steps but would run into this issue of, well, there's so many situations that go against the steps! And then I have to memorize *these* steps! And they make the situation much harder on themselves.

From the above observations, I argue that the key aspect of formative experiences involving the deepening of mathematical knowledge was not the expectation that participants come to know any underlying conceptual justification for a procedure or theorem; instead, it was the agency given to participants for independently creating such a justification. By promoting intellectual curiosity and autonomy as mathematical thinkers, these particular experiences appeared to give participants greater confidence when *not* applying *Justify Work*. That is, their familiarity with independent, exploratory sense-making in service of discovering a reasonable conceptual justification promoted similar activity while problem solving. While working interview tasks, almost all of the above participants demonstrated a robust and extensive use of exploratory heuristics (e.g. testing sample values, attending to similar/ simpler problems, drawing diagrams), behavior that could be attributed to pedagogical settings that encouraged the independent discovery of mathematical justification for known procedures. Furthermore, every participant mentioned in this subsection agreed or strongly agreed with the *Discover Solution* item from part three of the questionnaire. This was especially notable for Brady and Will, whose strong agreement with this item was more than one standard deviation above the undergraduate population's mean response.

5.1.3 Benefit of Experience

As noted in the previous section, Selden et al. observed that calculus students of all levels have difficulty applying their content knowledge to solve novel calculus problems (1989, 1994); later, they found that even differential equations students have similar difficulties (1999). While none of these groups had developed problem solving skills with respect to calculus techniques, the authors do note that the differential equations students' "algebra skills seem to be relatively sophisticated and readily accessible" (p. 14). In light of this observation, the authors suggest that incremental improvements in problem solving may accumulate more slowly over time than expected. Carlson (1998) also observed that even "high performing students do not appear to access recently acquired function information to solve unfamiliar problems" (p. 142), but instead need years of additional study for reflection and exploration with the concepts in question. Several participants in my study identified a formative problem solving experience that was noteworthy for the opportunity it gave them to revisit or extend their study of some already-familiar mathematical content.

Frank and Sara both recognized that they were able to approach an analysis course much differently after they already had relevant experience in another analysis course. Frank struggled to understand core concepts the first time he took analysis but had a “better understanding of material and a better understanding of my own strengths and weaknesses” when he took the course again. Unfortunately, he did not think that this translated into meaningful improvement in problem solving ability; in fact, he was disheartened that “this time, you know, even when I *did* get it, I couldn't come up with some of the proof techniques and some of the counterexamples.” Sara had to rely mainly on memorizing definitions when she first attempted analysis as an undergraduate and, as a result, fared poorly. When she retried the course later, still as an undergraduate, she found that constructing visualizations and her improved confidence in the material (“I know what I was talking about.”) helped her succeed; confidence in one’s ability is a notable indicator of problem solving success (Furinghetti & Morselli, 2009; Pajares, 1994; Lester et al., 1989; Schoenfeld, 1989). Taking the course for a third time as a graduate student, she now felt that it was one of her favorite courses largely because of the way visualizations could lend context to abstract mathematical vocabulary By way of example, she explained that a diagram could illuminate exactly what one means when they use the phrase “it’s approaching” in reference to a limit. Sara also found that retaking calculus was a significantly different experience after she had already succeeded in the class at a previous time. She added that “once you get used to this stuff, you always see it in a complex way, you know?”

Other participants found that teaching material they had already learned as a student gave them a different conceptual outlook on (and in some cases, altered an existing attitude they held towards) that material. Julie experienced the most dramatic example of this phenomenon. She said:

> I see a lot of connections now, like it's kind of weird. […] When I was taking calculus, I feel like I was like "I don't know any of this!" and now teaching it, I'm like, oh my god this is so fun! Like, this is cool! Okay, I get so excited about this.

Anne noted that tutoring gave her the opportunity to practice applying certain techniques multiple times and that, eventually, she would master that content; this allowed her to more quickly and confidently apply these techniques to her own problem solving situations. Both Verna and Brady also appreciated that tutoring allowed them to revisit previously-learned content, although they responded to this old material in different ways. Verna used her role as a tutor to "memorize everything," especially those topics that she felt weak in. On the other hand, Brady found that preparing to teach material "changed my thinking on looking at problems that I already knew how to do," because he found himself searching for deeper meaning in material that he had once been content to memorize.

Notably, the majority of participants who valued experiences that allowed them to revisit concepts were also those participants who demonstrated an over-reliance on memorized techniques during interview tasks. Anne, Frank, Verna, and Julie all repeatedly attributed their inability to solve one or more tasks to their poor memory of important theorems or techniques that they would need to solve the task. Anne, Frank, and Julie were also the three graduate students simultaneously enrolled in undergraduate leveling courses. These participants, with the exception of Frank, described other formative experiences that were notable for the way that they emphasized or reinforced the importance of memorization or known formulas. For example, Anne felt that her recent graduate course in linear algebra was difficult primarily because the instructor did not provide enough worked examples that corresponded to exam questions; instead, she had to revisit homework problems multiple times in order to more readily recall certain procedures. Julie explained that one aspect of her favorite mathematics course was one in which the instructor lessened the burden of memorization by allowing the extensive use of formula sheets during exams. Finally, Verna, Anne, Julie, Frank, and Brady all agreed (or strongly agreed) that *Memorization* is an important quality for success in mathematics on the second questionnaire; none of the other participants who took the second questionnaire rated this item as highly. Correspondingly, one might assume that these participants valued the opportunity to revisit material in order to, as Verna admitted, memorize forgotten information rather than to explore concepts deeply for meaningful connections or justification.

5.2 Social Learning

Certain problem solving frameworks include an explicit dimension for social interactions, acknowledging the fact that sociocultural factors can affect how students eventually handle novel problem solving experiences (Lester et al., 1989). Other frameworks implicitly suggest the importance of collaborative learning environments; Schoenfeld (2014), for example, recommends that students have “opportunities to […] build on one another’s ideas” and that activities should “support the active engagement of all the students” in order to grow robust mathematical strategies.” Each of these suggestions is clearly facilitated by pedagogical approaches such as small group discussions or peer-to-peer explanations. Liljedahl (2016) recommends that both classwork and exams be given in groups. Empirical evidence (c.f. Liljedahl, 2016; Silver, 1985) also suggests that group activities and other types of interpersonal interactions in mathematics classrooms may promote powerful thinking strategies appropriate for problem solving. In particular, teaching by learning is an effective means of understanding new mathematical content (Fiorella & Mayer, 2016), developing problem solving skills (Carlson, 1998), and necessarily relies on the interactions between students in group settings. During interviews, some participants reflected on experiences centered around particular kinds of social exchanges that affected the way they learned content, and as a result, may have modified their problem solving strategies. These social exchanges were sometimes orchestrated by an instructor as part of a classroom in which the participant was a student. Other times, they were informal out-of-class group settings organized or attended by the participant. Finally, participants also reported on the effect of social interactions that they necessarily experienced through their work as instructors, tutors, or teaching assistants.

Both Anne and Julie recalled formative mathematical classes in which a key component was formal, regular group work. For Anne, this class also included group exams; the effect of both types of group work was largely affective, in that they reinforced her self-confidence and convinced her that she was a competent mathematician. They also created a competitive atmosphere that, at least for Anne, encouraged continuous effort so that she did not disappoint her groupmates. Julie was unable to pinpoint the precise aspect of group work that she found most influential. Other participants explained that group learning experiences outside the classroom influenced their problem solving by providing them with a different mathematical perspective. Will, for example, valued a study group formed in his discrete mathematics course because “it wasn't just you sitting silently by yourself and just being stuck in your own little bubble of not being able to consider these ideas. But being able to hear other people's approaches that then add to your bubble, basically.” Frank found that working with classmates to solve problems in one professor’s office hours, even with minimal input from that professor, was particularly enriching because it allowed him to test his ability to provide thorough and convincing explanations.

Participants who are exposed to the thinking of groupmates, like those whose experiences instilled in them an appreciation for mathematical connections, might be expected to spend more time with the *Consider Approaches* strategy. That is, successfully organized group work might expose the problem solver to tentative solution starts that had been generated by others in their group. Alternatively, as posited in Silver, (1985), group work might promote the externalization of traditionally internal problem solving practices such as monitoring or justification. Of the above participants, however, Julie and Anne were two of the least likely to verbalize self-talk of any kind. As noted in previous analysis, they were also very unlikely to consider multiple approaches that were unfamiliar or anticipated to stray from known procedures. On the other hand, Will and Frank were especially forthcoming in the interview process with mathematical rationale, which they applied to many different approaches.

One notable difference between the two sets of participants in this theme was that Frank and Will both experienced social learning episodes in courses that were largely proof-based. Julie and Anne, however, collaborated in groups to solve exercises rather than problems; that is, the mathematical activity that contextualized their social learning consisted of replicating a procedure seen in worked examples provided earlier by the instructor. For example, Julie recalled that “she [the instructor] would just go thoroughly through a problem and then we had a worksheet of them” to complete with a partner. Pedagogical frameworks that emphasize collaboration often stress that collaborative tasks should “drive students to want to talk with each other as they try to solve them” (Liljedahl, 2016; p.381) or that they feature “productive intellectual challenge” (Schoenfeld, 2014; p. 407). While proof-based group work does not automatically meet these requirements, it is likely that the completion of familiar exercises with a partner do *not*. Finally, both Frank and Will also described collaborative environments that featured vertical, non-permanent surfaces rather than individual worksheets; this is a direct recommendation of Liljedahl for group activities (2016).

5.3 Specific Heuristics

Beginning with Pólya's *How to Solve It* (1945), heuristic strategies have been a key aspect of problem solving research. Heuristics are often explicitly named as a central pillar of a problem solving framework (c.f. Schoenfeld, 2016; Carlson & Bloom, 2005; DeBellis & Goldin, 2006), although they are variously sorted as either a type of content knowledge (Schoenfeld, 2016) or metacognition (DeBellis & Goldin, 2006). One of the central goals of Schoenfeld's problem solving class (described in Arcavi et al., 1998) was to teach a variety of heuristics and model the successful (and sometimes, unsuccessful) application thereof. Combined with evidence that successful problem solvers often identify that their mathematical mindset was shaped by an influential mentor (Carlson, 1998), this is reasonable evidence to suggest that an instructor might convey when and how to use correct heuristic strategies. Participants in this study reported that they were in fact influenced by their instructors to focus on certain areas of problem solving. However, they also alluded to situations, rather than specific people, which instilled in them an appreciation for one or more applications of specific heuristic strategies.

Of the strategies encapsulated on part one of the questionnaire, participants most commonly recalled instructors who explicitly taught them to use the *Draw Diagram* strategy. Carol, Julie, and Sara were all positively influenced by the same analysis instructor who encouraged the use of visualization techniques for making sense of important theorems. Carol appeared to take this to heart; she drew more diagrams on her interviews than any other participant. On the other hand, Julie and Sara drew some of the least; for example, unlike almost every other participant, Sara did not produce any visualizations for either MV-Pre or MV-Post. She admitted that, despite this instructor's demonstrations, "sometimes I still have a hard time with visualizing." Julie only drew four diagrams for HV-Pre and two diagrams for HV-Post, in both cases less than almost every other participant. During the second interview, Verna claimed that several different instructors across multiple courses from her most recent semester had also demonstrated the value of visualization. She drew roughly the same number of diagrams for tasks in the first interview as she did for tasks in the second, however, with the exception that Verna's work on HV-Post involved much more visualization than her work on HV-Pre. The value these four participants place on *Draw Diagram* is reflected in their questionnaire responses; Carol, Julie, and Sara were the only three interviewed undergraduates who strongly agreed that that strategy was helpful on the first questionnaire. Additionally, this was more than one standard deviation above the mean opinion held by the total graduate population. Verna also strongly agreed with the utility of the *Draw Diagram* strategy on the first questionnaire, but this was not a significantly different response in context of the population of surveyed undergraduate students.

It should still be noted that the quantity of visualizations produced by a participant, as discussed above, can be a misleading metric for determining a participant's engagement with that strategy; one must also consider the extent to which each participant engages in meaningful mathematics with their diagrams. For example, despite the fact that Carol and Sara relied on a different number of visualizations across interview tasks, when they *did* employ a visualization, the way in which they leveraged those diagrams or sketches were similar and often aligned with observed behavior of successful problem solvers, as per Stylianou (2002, 2004): they both used diagrams first to make sense of qualitative aspects of the task in order to infer algebraic consequences.

This allowed them to then use their visualization as a staging ground for quantitative, symbolic work that motivated additional ideas for possible directions in which to focus their problem solving efforts. In contrast, Verna and Julie relied almost exclusively on visual intuition rather than underlying mathematical relationships to make claims about their diagrams. They also employed visualizations with much less mathematical foresight. As noted by Schoenfeld, students of mathematics sometimes “pick up the rhetoric—but not the substance” (1989, p. 349) of beliefs or practices espoused by their instructors. Julie and Verna’s limited use of *Draw Diagram* (in terms of both number and quality of visualizations) could be explained if they had not had personal experiences solving problems using effective visualization techniques in the classroom and had only been encouraged to use them by their instructor. That is, even though they “were able to anticipate the drawing of a diagram, they knew very little about how to make diagrams a helpful tool. […] They lack the necessary procedural knowledge that would allow them to use visual representations functionally and efficiently” (Stylianou, 2002; p. 380). The failure of students to employ the visual register for problem solving is a phenomenon also reported by Dawkins & Epperson (2014), who observed that successful calculus students were overly eager to apply algebraic approaches to non-routine problems even when doing so was cumbersome or inappropriate.

Next, many participants recalled learning the value of *Read Carefully* from an influential individual. Anne credited her recent graduate linear algebra professor with helping her focus on reading and interpreting the important parts of a novel problem; however, it was not clear that she was able to apply these techniques to the second interview tasks. Sara attributed her habit of rewriting important information from a problem’s text to her mom, who encouraged this behavior as a memory aid. In both LV-Pre and LV-Pro, Sara did rewrite the important numerical data with categorical labels to organize and draw her attention to this information. Anne and Sara indicated, on the first questionnaire, that they agreed or strongly agreed (respectively) that this strategy is helpful when solving difficult problems. Neither opinion was significantly different from the mean opinion of *Read Carefully* held by the graduate student population.

The difference in these participants' application of *Read Carefully* could be attributed to either of two different explanations. On one hand, Sara learned the value of this heuristic from her mother early in her mathematical career, whereas Anne only recently became acutely aware of the value of *Read Carefully* in her most recent proof-based graduate course. This would have given Sara much more time to learn to effectively apply this heuristic in a variety of different problem settings. If one considers heuristics to be a type of mathematical content knowledge (as in Schoenfeld, 2016), then this aligns with results in literature that students may need several years to learn to leverage their mathematical content knowledge effectively in novel situations (Selden et al., 1999; Carlson, 1998). On the other hand, the difference between Anne and Sara's use of this heuristic could be affective. As observed in Furinghetti and Morselli, initial orientation in a problem space (i.e., the degree to which a participant applies *Read Carefully* and is able to *Explore Setting*) can be drastically altered by low self-confidence (2009). In interviews, Anne seldomly appeared comfortable relying on her own mathematical intuition and preferred to remember known procedures, a hallmark of low mathematical confidence. Thus, despite her admission that she *should Read Carefully*, she was prevented in doing so by an underlying attitude towards unfamiliar problems.

The last questionnaire heuristic that participants learned from instructors was *Justify Work*. Brady and Verna both had the same introduction to proofs professor, who stressed to both of them the value of using careful reasoning to warrant each step of an approach. Brady was much more successful in applying this advice to interview tasks; in fact, he often appeared overwhelmed by the amount of justification he felt required to give. Verna, on the other hand, justified very little in either interview. Despite their differences, Brady and Verna both strongly agreed that *Justify Work* was a helpful strategy when solving difficult math problems on both questionnaires.

Other participants learned various heuristic strategies not from instructors but instead from the nature of certain courses or the content therein. Both Will and Taylor recalled difficult problems from their discrete mathematics course that took them an extended period of time to solve; they both felt that such problems encouraged them to *Consider Approaches*, which Taylor likened to not being "afraid to think outside the box." Carol, on the other hand, was more influenced by a previous analysis course. Her biggest takeaway from this class was that she should avoid coming into a problem with a preconceived idea of whether the problem was true or not. That is, she discovered that *Imagine Solution* can sometimes be actively detrimental to her problem solving efforts. This discovery was also, in part, facilitated by Carol's realization that her failure to engage sufficiently with the *Read Carefully* and *Plan Steps* strategies before beginning a problem sometimes led to her premature, incorrect images of the solution. When a problem solver cannot rely on their established problem schema to automatically resolve a problem (via bypassing the planning phase of problem solving, as described in Čadež and Kolar, 2015; Jonassen, 1997), Maciejewski (2016) argues that they instead fall back on mathematical foresight (if they direct their work using an image of the problem's anticipated solution) or mathematical exploration (if they do not). Through this lens, Carol's experiences in analysis can be interpreted as an example of mathematical foresight in which her incorrect image of the solution hampered her approach; Will and Taylor's admission that they had to "think outside the box" can similarly be interpreted as the requirement that they work without immediately familiar problem schemas, although they did not give enough detail to determine if they were engaged with mathematical foresight or exploration.

5.4 Summary

The most common formative experiences described by participants were those that influenced how they acquired, understood, and contextualized content knowledge. This content knowledge could be either entirely new or already familiar; furthermore, participants interacted with mathematical content both as students and as teachers of the subject. When participants described formative experiences in which they viewed mathematical content as an interconnected web of related ideas, this perspective did not always clearly manifest in observable problem solving activity. These participants were sometimes more likely to apply multiple approaches to an interview task, but this behavior appeared to be more closely tied to the participant's underlying mathematical affect (in particular, their confidence in their ability to handle unfamiliar mathematics). On the other hand, participants whose formative experiences centered around the discovery of conceptual justifications for mathematical claims did appear to display markedly different problem solving strategies than their peers: they were more willing to use mathematical creativity in order to explore the problem setting and make sense of the underlying relationships. From this process, they sought to eventually construct a reasonably justified solution. However, if the participant's experience of this type emphasized justification that was presented by an authority rather than justification that was created by the learner, the resultant problem solving activity was much less organic. That is, even if the end goal was still to produce a reasonably justified solution, this latter group of participants typically relied on memorized theorems or procedures to warrant any justification. Finally, some participants mentioned that revisiting familiar content facilitated an improvement in their problem solving methods. With limited exceptions, however, participants who appreciated the opportunity to revisit content felt this way because it helped them better remember that content in the future, not because it gave them the opportunity to understand that content more robustly or in a different, more effective way.

Participants in the study also observed that certain mathematical experiences centered around social interactions could have affected their problem solving strategies. This observation was warranted when the social interactions in question were founded on meaningful mathematical inquiry. When, instead, the experience was only partnered work on exercise worksheets, the participant displayed much less of the resultant problem solving behavior (e.g. self-talk, consideration of multiple approaches).

Finally, many participants reported that they had, at some point, learned a specific heuristic in response to a particular instructor's direction or through an experience whose characteristics promoted the application of that heuristic. These heuristics were the viability of drawing diagrams to understand concepts, the necessity of a careful reading and interpretation of the task, the requirement that mathematical manipulations be founded on sufficiently justified principles, and the importance of maintaining a willingness to consider many unconventional approaches to a problem. No matter the heuristic, however, some participants only absorbed the attitude of their instructor towards the strategy rather than a meaningful understanding of how to apply it (for example, drawing a diagram without utilizing it effectively; recognizing the need for justification without knowing how to provide it). More experience personally using a heuristic in a variety of relevant applications would likely rectify this shortcoming; those participants who had learned a particular heuristic longer ago had more success in leveraging it.

Conclusions

Several important results span the above thematic areas. First, a key aspect of these findings centered around participants (e.g. Anne, Sara, Carol, Brady, and Mia) who identified teaching opportunities as a type of formative problem solving experience that afforded them with opportunities to improve the depth of their content knowledge. This was sometimes because, while teaching, they sought out underlying concepts that warranted the use of certain processes in order to improve their instructions; they identified that inquisitive students might naturally ask *why* certain techniques worked the way they did, and that simultaneously, struggling students might benefit from in-depth explanations of relevant procedures. Carol and Sara also noted that teaching mathematics encouraged them to explore alternative approaches to familiar problems so that they might better address situations in which their students do not immediately understand a standard technique. Other times, participants found that revisiting material as instructors rather than students simply allowed them to explore content with more intellectual freedom, granted to them by their pre-existing familiarity with the foundational concepts of the subject in question. Finally, participants such as Anne, Verna, and Taylor all noted that they were more likely to discover and use certain heuristic strategies if they first used these strategies to facilitate their teaching. Learning by teaching has already been documented as an effective means of understanding new mathematical content (Fiorella & Mayer, 2016) and developing problem solving skills (Carlson, 1998). However, precisely *how* teaching experience affects problem solving is a novel result; furthermore, the variety of types of teaching experiences (and the corresponding variety of changes in problem solving strategy that participants perceived they had undergone as a result of these experiences) is also noteworthy.

Next, the findings indicate that some participants' (e.g. Will, Taylor, Mia, and Brady) first proof-based mathematics class had a significant effect on their mathematical disposition and problem solving strategies. With respect to mathematical affect, many participants felt that their initial introduction to proofs was their "first real math class," in that it showed them mathematics could be creative and engaging; by way of contrast, these participants often felt that their mathematical work as undergraduates up until that point had been prescriptive and focused largely on reaching numerical solutions through algorithmic means. They also found that being temporarily frustrated by difficult homework problems in these courses, along with the value their instructors placed on independent thought, encouraged them to persevere through difficulties. Cognitively, proofs, by nature of being relatively more open-ended and reliant on critical thinking (as opposed to memorization), also act as a meaningful foundation on which instructors might model successful problem solving strategies for their students. Participants reported as such, while simultaneously noting that they were given the opportunity to apply these problem solving techniques themselves on challenging and meaningful homework assignments. Because these homework assignments were not simply an exercise in reproducing known procedures, some participants found value in working with their classmates; this, in turn, gave them a perspective on different ways to approach the same problem. The characteristics participants describe of their first proof-based course often mirror important characteristics of powerful thinking classrooms as described in problem solving literature (e.g. Schoenfeld, 2014; Liljedahl, 2016). The result that participants did not experience such characteristics *until* this time, however, is important to note. Furthermore, many participants were influenced by their first proof-based course even if the instructor's pedagogical methods were still relatively prescriptive (i.e., rigid two-column proofs); an emphasis on explaining *why* something works appeared to motivate a shift in mathematical beliefs regardless of the day-to-day mechanics of the course.

5.5 Limitations of the Study

The questionnaire used for this study, while useful as a tool to guide questions during the interviews, was not especially reliable as an indicator of participants' beliefs when considered outside of additional explanation that participants may have provided. For example, there were drastically different interpretations of the meaning of "success in mathematics" in part two; these included discovering new mathematical principles in cutting-edge research settings, being satisfied with one's own understanding of a given topic, and simply passing a class. Accordingly, unless the opportunity arose to discuss specific responses during the interviews, it is difficult to interpret the underlying beliefs that informed them. In a related way, the categories of problem solving strategy in part one were broad and thus few in number so as not to create an unreasonably long questionnaire; however, this led to certain problem solving behaviors that could reasonably be interpreted to belong to multiple categories. For example, if a participant reviews the text of a problem in the middle of their work, is that behavior indicative of *Check Work* or *Read Carefully* or both? If a participant explains that their diagram will likely lead to a solution, are they employing *Justify Work*, *Draw Diagram*, or *Imagine Solution*? Participants might interpret their own behavior as belonging to a subset of problem solving behavior different from the one to which I ultimately attributed it. It is thus difficult to gauge whether or not participants' problem solving behavior matched their questionnaire responses, since it cannot be guaranteed that my interpretation of their work matched their interpretation of the prompts.

Another limitation was simply many participants' unfamiliarity with the speak-aloud format of the interviews. Several different participants remarked that they were uncomfortable voicing their thoughts while being recorded in the presence of an observer. Many others, even if they made no such explicit claims, had to be prompted by the interviewer to comment on their thought process; otherwise, they tended to work quietly until they had come to a conclusion, at which point they would retrospectively summarize the steps that led to that conclusion. It was fairly uncommon for this summary to include considerations that did not actively contribute to their discovery of the conclusion at hand (e.g., they seldom referenced the approaches they abandoned). This is not particularly unusual given that one does not typically include all the ways *not* to proceed when one writes a proof, and proof-writing was most participants' closest analog for the interview task commentary. However, much problem solving behavior involves parsing the things which one chooses *not* do while solving a problem, and I suspect that much useful evidence of subtler strategies such as *Plan Steps* or *Imagine Solution* was not evinced in the recorded interviews even if they did occur.

Finally, there is an argument to be made that one cannot actually expect participants to accurately recognize precisely what experiences led to the development of their problem solving strategies, especially when those strategies (or the changes brought about in them) are especially subtle. Some participants remarked that they had been taught to underline or circle important key words in problem texts and did so during interviews; none of them commented on when they had learned to assess the qualitative relationships between key quantities in a problem, although many of them did. This limitation is magnified given that I asked participants to reflect on their problem solving development over their entire mathematical career. Subtle changes in mathematical beliefs or behaviors that occurred many years ago were not likely to be described in sufficient detail (if at all) during interviews.

5.6 Directions for Future Research

Future research involving a longitudinal examination of the same students over a year or more of their mathematical education would allow further exploration of my findings, given that many participants in this study appeared to have learned a new problem solving strategy but not had sufficient time or opportunity to learn to apply it. An extended study of one group might also alleviate the issue of participants' initial discomfort with the interview process by giving them several opportunities to familiarize themselves with speak-aloud procedures. Instead of attempting to capture the development of problem solving holistically, future research might instead focus exclusively on the development of one problem solving strategy in particular. This would allow researchers to build a more focused and descriptive questionnaire that might circumvent issues caused by differing participant interpretations.

Many participants' experiences as mathematics educators, in some capacity, motivated a change in their problem solving strategy. Further study is warranted in which the anticipated results of teaching experience on problem solving are laid out as a primary interest, rather than an emergent theme, of the project. In particular, a pre/post study centered around a teaching intervention that promotes peer-to-peer explanations could be enlightening. One aspect of teaching that may influence problem solving development that was not described by any participants in this study is the creation of meaningful mathematical problems (as opposed to exercises). Do educators who must generate problems with the express intent of challenging their students' problem solving capabilities simultaneously develop their own problem solving strategies? In what way?

Finally, almost every participant cited their first (or other times, most recent) proof-based course as that which most clearly guided the development of their problem solving processes. However, mathematical problem solving is not a skill that should only be developed in those students (such as the mathematics majors from this study) who are interested in mathematical theory; in fact, I might argue it is *more* important to develop problem solving strategies in courses that reach students who might not otherwise take a proof-based course (e.g. discrete mathematics, linear algebra, mathematical analysis). How can the aspects of these courses that influenced problem solving (such as an emphasis on mathematical agency and creativity) be transferred to those mathematics courses that are traditionally structured around computational exercises?

Appendix A

Questionnaire

Questionnaire

Please indicate to what extent you agree with the following statements by marking the appropriate number in the right hand column.

1 = Strongly Disagree
2 = Disagree
3 = Somewhat Disagree
4 = Somewhat Agree
5 = Agree
6 = Strongly Agree

	When solving a difficult math problem, it is helpful to...						
1.	Read the problem carefully	1	2	3	4	5	6
2.	Try and imagine what a solution might look like	1	2	3	4	5	6
3.	Consider multiple different ways in which the problem might be solved	1	2	3	4	5	6
4.	Plan the steps I will take in advance	1	2	3	4	5	6
5.	Draw a diagram or sketch	1	2	3	4	5	6
6.	Check my work at every step	1	2	3	4	5	6
7.	Justify each step of my approach	1	2	3	4	5	6
8.	Verify my answer at the end	1	2	3	4	5	6

	In general, success in mathematics requires...						
1.	Natural talent	1	2	3	4	5	6
2.	Hard Work	1	2	3	4	5	6
3.	Creativity	1	2	3	4	5	6
4.	Persistence	1	2	3	4	5	6
5.	Memorization	1	2	3	4	5	6

	In mathematics...						
1.	An answer is either right or it is wrong	1	2	3	4	5	6
2.	There is often only one correct approach to a particular problem	1	2	3	4	5	6
3.	It is possible to discover how to do a problem you have never seen before	1	2	3	4	5	6
4.	I can solve any homework problem assigned to me if I do not give up	1	2	3	4	5	6
5.	Getting an answer correct is often a matter of knowing the correct formula	1	2	3	4	5	6
6.	I often consider whether my solution was the best way to solve a given problem	1	2	3	4	5	6

Appendix B

Demographic Surveys

Demographic Survey (Graduate)

1. Please indicate your age:
    a. 18 – 22 years old
    b. 23 – 28 years old
    c. 29 – 33 years old
    d. 33 years or older
    e. Prefer not to say

2. Please indicate your preferred gender:
    a. Male
    b. Female
    c. Other
    d. Prefer not to say

3. Please list your last three mathematics courses and indicate your grades in those courses, if applicable:

	A or B	C or lower	Prefer not to say
	A or B	C or lower	Prefer not to say
	A or B	C or lower	Prefer not to say

4. List any mathematics courses in which you are currently enrolled.

5. What degree(s) do you hold, if any?

6. Which best describes your undergraduate university?
    a. Small Public (Less than 10,000 student population)
    b. Medium Public (10,000 to 20,000 student population)
    c. Large Public (More than 20,000 student population)
    d. Small Private (Less than 2,000 student population)
    e. Medium Private (2,000 to 8,000 student population)
    f. Large Private (More than 8,000 student population)

7. If you are interested in participating in the interview portion of this project, please provide your:
    g. Email:

    h. MAV ID:

Demographic Survey (Undergraduate)

1. Please indicate your age:
    a. 18 – 22 years old
    b. 23 – 28 years old
    c. 29 – 33 years old
    d. 33 years or older
    e. Prefer not to say

2. Please indicate your preferred gender:
    a. Male
    b. Female
    c. Other
    d. Prefer not to say

3. Please list your last three mathematics courses and indicate your grades in those courses, if applicable:

	A or B	C or lower	Prefer not to say
	A or B	C or lower	Prefer not to say
	A or B	C or lower	Prefer not to say

4. List any mathematics courses in which you are currently enrolled.

5. Do you currently plan on attending a graduate school program? If yes, in what subject area? If no, in what field do you plan to seek employment?

6. If you are interested in participating in the interview portion of this project, please provide your:
    a. Email:

    b. MAV ID:

Appendix C

Interview Tasks

First Interview[1]

Problem 1.1. The entire population of a small town has an 8:7 ratio of men to women. After 90 men and 80 women move to the town, the ratio of men to women is now 17:15. Assuming no other changes in population have occurred, what is the new population of the town?

Problem 1.2. What value(s) of α maximize the number of solutions to the following system of equations?

$$x^2 - y^2 = 0$$
$$(x - \alpha)^2 + y^2 = 1$$

Problem 1.3. Is it always possible to cut an arbitrary right triangle into two isosceles triangles? Explain why or why not.

Second Interview[1]

Problem 2.1. At a particular boarding school, every student plays at least one of two sports: tennis or golf. A survey finds that $\frac{1}{7}$ of the tennis-players also play golf, while $\frac{1}{9}$ of the golf-players also play tennis. What fraction of the student body plays tennis?

Problem 2.2. For which positive integer(s) α does the number of solutions to the following system of equations equal α?

$$y = x$$
$$\mathrm{y} = \sin(\alpha x)$$

Problem 2.3. Is it always possible to inscribe a square in an arbitrary triangle? Explain why or why not.

Auxiliary Problem[1]

Problem 3.1. Is it possible to use precisely 100 U.S. coins, none of which are nickels, in order to make precisely $5? If so, how?

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[1] For all interviews, participants were given each question on a separate sheet of paper to allow them room to work.

Appendix D

Interview Protocol

Interview Protocol

During the course of the interview process, the subject will work up to four non-traditional mathematics problems (as time allows). The researcher's main goal during the interview is to elicit the subject to "think aloud" as they are solving the problem; to that end, the researcher may employ prompts such as

- Give me a summary of your approach to this problem, explaining the rationale behind your decisions when appropriate.
- Tell me why you chose to (draw that diagram/ use that formula/ write that expression).
- What aspect of this problem do you find most interesting/ challenging? Why?
- What strategies, if any, did you consider using but ultimately rejected? Why?
- In what way have you chosen to solve this problem (like/ unlike) other mathematics problems you have worked in the past?

Another facet of the interview is to prompt the subject to reflect on the development of their problem-solving strategies, where applicable. Questions the interviewer may use to encourage such introspection could be

- How has your approach to problem-solving changed (over time/ **in the past semester**)?
    - Have your problem solving strategies changed in a significant way? How?
    - Has your attitude about problem solving changed in a significant way? How?
- Has any singular (class/ person/ event) significantly affected your approach to problem-solving (**in the past semester**)? How?

Bolded text in the above questions will be part of the interview procedure only during the second interview at the end of the participants' fall semester.

If the subject's questionnaire responses do not appear to align with their actual problem-solving strategies during the interview, the researcher may ask for elaboration by stating

- You indicated that when solving a difficult math problem, you are likely to do _____. In the (first/ …/ fourth) interview question, you did not _____. Tell me more about why that problem did not prompt you to use that particular problem-solving strategy.

Finally, only during the second interview, the researcher may prompt the subject to elaborate on any observed changes in the subject's approaches to problem solving or questionnaire responses:

- Last time we spoke, you said ______ in response to questionnaire prompt #_. This time, you have responded #_. Tell me more about why this change occurred.

- During the first interview, you (did/ did not) use ____ as a problem-solving strategy. During this interview, you (did not/ did). Tell me more about why this change occurred.

Additional question based on literature review:

- How would you describe your usual relationship with your mathematics teachers/ professors? Describe any outliers to this trend.
- Describe the particular qualities of your favorite math course that made it stand out.

Appendix E

Codebook

Codebook

Attitudes: Participant describes the way that they tend to react to (i.e. their disposition towards) a certain type of mathematical experience.

Beliefs: Participant describes something they hold to be true, usually (implicitly or explicitly stated to be) based on a sequence of experiences that reinforce the belief.

- About Math: The belief is about the nature of mathematics in general (e.g. its place in education, how it should be done, what makes it difficult, etc.).
- Important Qualities: The belief is about personal qualities that may or may not contribute to one's ability to succeed in mathematics.
- About Self: The belief is about themselves as mathematicians (e.g. how competent they are at (some subset of) mathematics).

Emotions: Participant reacts to an immediate circumstance with an emotional response. These are fleeting and directly triggered by some at-hand stimulus.

Commentary on Problem Solving Strategy: Participant describes the importance, usefulness, or validity of a particular problem solving strategy.

- Read Carefully: Attending closely to some aspect of the problem text.
- Imagine Solution: Using the anticipated form of the solution to motivate their work.
- Consider Approaches: Generating various alternative approaches.
- Plan Steps: Weighting the validity/ expediency of alternative approaches.
- Draw Diagram: In some way, visualizing some aspect of the problem setting.
- Check Work: Validating intermediary computations while solving a problem.
- Justify Work: Validating conceptual claims or problem solving moves while solving a problem.
- Verify Solution: Checking the ultimate solution to a problem.
- Miscellaneous Heuristic Strategies
    - Try Specific Example(s): Using representative values or examples to understand the problem in isolated instances.
    - Similar Problem(s): Considering a different, previously-encountered problem that bears some structural or conceptual similarities to the current problem.
    - Simpler Problem(s): Attending to cases or relaxing constraints in order to make headway in a problem.
    - Don't Hesitate: Not thinking too much about a problem before beginning along some approach in a concrete way.
    - Find the Trick: Knowing a theorem, algorithm, or way of thinking that, once remembered, would render the problem trivially easy.

Favorite Class: Participant describes what aspect of the favorite mathematics course affected them most.

- Emotional: The course appealed to their emotional state or facilitated their emotional development with respect to mathematics.
- Intellectual: The course appealed to their intellectual curiosity or understanding of mathematics.

- Social: The course presented opportunities to experience mathematics as part of a relationship with one or more person.

Formative Experience: Participant describes a mathematical experience that they perceived as formative with respect to the development of their problem solving strategy.

- Course Content: The experience was centered around on a course's structure or mathematical content.
- Person: The experience was centered around an instructor and their personality or pedagogical style.
- Technique: The experience was centered around the acquisition of a specific strategy or way of thinking about solving problems.
- None: The participant does not report a formative experience with respect to problem solving development.

Last Semester: Participant describes a mathematical experience *from the last semester* that they perceived as formative with respect to the development of their problem solving strategy.

- Course Content: The experience was centered around on a course's structure or mathematical content.
- Person: The experience was centered around an instructor and their personality or pedagogical style.
- Technique: The experience was centered around the acquisition of a specific strategy or way of thinking about solving problems.
- None: The participant does not report a formative experience with respect to problem solving development.

Benefit of Experience: Participant comments on the problem solving advantage conferred by familiarity or experience with mathematical concepts/ material.

Conceptual v. Procedural: Participant comments on the difference (if any) between understanding material at different levels of conceptual depth.

Proofs: Participant uses proofs or proof-writing as a lens through which to examine problem solving or mathematics in general.

Relationship with Instructor: Participant describes their normal pattern of interaction with instructors of their mathematics courses.

Teaching: Participant explains how an experience in which they taught someone about mathematics has affected their own understanding of the topic or how one might solve problems in that topic.

Reliance on Memory: Participant comments specifically on how their problem solving hinges on their ability to remember theorems or reproduce known techniques.

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