Some Quadratic Quantum $\mathbb{P}^3$s with a Linear One-Dimensional Line Scheme

by

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this thesis is a well-polished version of my work and a small collection of everything I learned at UTA. What it lacks is a representation of my short-comings, mistakes, sign errors, etc... Learning and comprehension is non-linear, but it is increases with respect to time if you are persistent.

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ABSTRACT

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It is believed that quadratic Artin-Shelter regular (AS-regular) algebras of global dimension four (sometimes called quadratic quantum $\mathbb{P}^3$s) can be classified using a geometry similar to that developed in the 1980's by Artin, Tate, and Van den Bergh. Their geometry involved studying a scheme (later called the point scheme) that parametrizes the point modules over a graded algebra. The notion of line scheme (which parametrizes line modules) was introduced later by Shelton and Vancliff.

It is known that "generic" quadratic quantum $\mathbb{P}^3$s have a finite point scheme and one-dimensional line scheme. A family of algebras with these properties is presented herein where each member has a line scheme that is a union of lines. Moreover, we prove that if a quadratic quantum $\mathbb{P}^3$, denoted $\mathcal{A}$, is an Ore extension of a quadratic quantum $\mathbb{P}^2$, denoted $\mathcal{B}$, then the point variety of $\mathcal{B}$ is embedded in the line variety of $\mathcal{A}$. Indeed, this result is generalized to prove that, under certain conditions, if $\mathcal{A}$ is a quadratic quantum $\mathbb{P}^3$ that contains a subalgebra isomorphic to a quadratic quantum $\mathbb{P}^2$, then the point variety of the subalgebra is embedded in the line variety of $\mathcal{A}$.

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CHAPTER 1

Introduction

In commutative algebra, within the class of noetherian local rings, there are the following strict containments of subclasses:

$$\{\text{regular local rings}\} \subset \{\text{complete intersection rings}\} \subset \{\text{Gorenstein rings}\} \subset \cdots$$
$$\cdots \subset \{\text{Cohen-Macaulay rings}\} \subset \{\text{universally catenary rings}\}.$$

There are many avenues of research in each subclass and the connections with algebraic geometry are well known. In contrast, the heart of this dissertation seeks to study a certain class of non-commutative algebras and their connection to objects in algebraic geometry. In non-commutative algebra, there is not the structure pictured above, but there are still many interesting classes of algebras to investigate that have connections with algebraic geometry and even differential geometry.

Noetherian local rings are often quotients of polynomial rings or power series rings, so it seems reasonable to consider non-commutative algebras that are described by generators and relations. In the 1980's, Artin and Schelter considered a certain class of algebras, called regular algebras. In [1], they described regular algebras of global dimension three using generators and relations, and they provided a partial classification. In [2] and [3], Artin, Tate, and Van Den Bergh introduced a geometric technique to complete the classification of regular algebras of global dimension three that are generated by degree-one elements and proved they are noetherian.

This classification of regular algebras of global dimension three involved studying certain cyclic graded modules called point modules. To such a module one can associate a point in $\mathbb{P}^2$. The collection of all the points associated to the point modules of a quadratic regular algebra of global dimension three can be computed as a subscheme in $\mathbb{P}^2 \times \mathbb{P}^2$ and this scheme was later called the point scheme ([2, 20]). This scheme is viewed as a parametrization of the point modules associated to an algebra. In [2], it is shown for regular algebras of global dimension three, that the point scheme is the graph of an automorphsim and with this geometric information, Artin, Tate, and Van Den Bergh, were able to recover the description of the algebra from its point scheme.

Many algebraists consider regular algebras of global dimension $n+1$ to be non-commutative analogues of polynomial rings and often refer to such algebras as quantum $\mathbb{P}^n$s. The classification of quantum $\mathbb{P}^3$s is an open problem and it is believed that the geometric techniques of Artin, Tate, and Van den Bergh will be a central tool in the classification of such algebras. In [20], Shelton and Vancliff proved that, under certain conditions, the defining relations of a quadratic quantum $\mathbb{P}^3$ can be recovered from its point scheme, if the point scheme is finite. However, the method in which one retrieves the defining relations from the point scheme can be involved, so it seems that perhaps more tools or information is needed.

Line modules are a certain type of cyclic graded module that play the role of a line in projective space. In [20], Shelton and Vancliff show there exists a scheme that parametrizes the line modules over a quadratic quantum $\mathbb{P}^3$ and called it the line scheme. In [21], Shelton and Vancliff provided a method of computing the line scheme of a quadratic quantum $\mathbb{P}^3$ and provide various examples. Under certain conditions, if the line scheme of a quadratic quantum $\mathbb{P}^3$ is one-dimensional, then it determines the defining relations of that algebra. So further development of the theory seems to depend on the study of quadratic quantum $\mathbb{P}^3$s that possess a finite point scheme and a one-dimensional line scheme. A quadratic regular algebra with such associated geometries is considered to be a candidate for a generic quadratic quantum $\mathbb{P}^3$ (cf. [26]).

Conveniently, in many cases, there are some limitations on the number of components in the point scheme and line scheme of a quadratic quantum $\mathbb{P}^3$. An unpublished work by Van den Bergh, outlined in [26], proves that if the point scheme of a quadratic quantum $\mathbb{P}^3$ is finite, then it consists of twenty points counted with multiplicity. Additionally, in [9], it is proved that if the line scheme of a quadratic quantum $\mathbb{P}^3$ is one-dimensional, then it has degree twenty. This raises the question on whether or not there exists a quadratic quantum $\mathbb{P}^3$ that has a point scheme consisting of twenty distinct points and a one-dimensional line scheme consisting of twenty distinct lines. Unfortunately, methods for constructing quantum $\mathbb{P}^3$s with desirable geometric properties are quite limited.

In the early 2000s, it was still unknown whether a quadratic quantum $\mathbb{P}^3$ existed that had exactly twenty distinct points in its point scheme and a one-dimensional line scheme. In [19], Shelton and Tingey confirmed the existence of such an algebra with a computational trial-and-error method. They, and others, were unable to produce any more examples via this approach, so this discovery added only limited depth to the subject. In [5], Cassidy and Vancliff introduced the notion of a graded skew Clifford algebra and gave conditions when they are regular. This allowed for the construction of many candidates of generic quadratic quantum $\mathbb{P}^3$s, some of which possess a point scheme consisting of twenty distinct points.

There are not many candidates for generic quadratic quantum $\mathbb{P}^3$s with a detailed description of their line scheme (cf. [7, 8, 17, 25]). It is believed by many researchers that the study of line schemes will be essential in the classification of quantum $\mathbb{P}^3$s. Producing more examples with a one-dimensional line scheme will be necessary as geometric properties could yield hidden algebraic properties or vice versa. The algebras presented in [7, 8, 25] motivated the result in [9] which proves that if a quadratic quantum $\mathbb{P}^3$ has a one-dimensional line scheme, then it has degree twenty. In the same spirit, Theorem 3.3.1 provides an example of a line scheme that motivates Theorem 4.4.2.

In [23], Stephenson and Vancliff presented two families of quadratic quantum $\mathbb{P}^3$s that are infinite modules over their centers as a counterexample to a conjecture that aimed to generalize a result regarding quantum $\mathbb{P}^2$s. Both families were constructed via an Ore extension and have defining relations that depend on multiple parameters. All, but a select few members of each family, have a finite point scheme and a one-dimensional line scheme. In [17], Mastriania computed the line scheme of one of these families, and in Section 3.3, Theorem 3.3.1 computes the line scheme of the other family of algebras.

The discussion in Chapters 3 and 4 will contain a detailed analysis of the family of algebras previously mentioned. The family depends on four parameters and, in Section 3.2.1, we show that each member has a point scheme consisting of 2, 3, 4, or 5 distinct points where each case occurs depending on the value of each parameter. The discussion in Section 3.2, regarding Theorem 3.2.1, focuses on the case with five distinct points. The structure of the line scheme depends upon the four parameters, but there is a choice of parameters for which the line scheme is the union of four distinct lines, with various multiplicities; three of the lines are each counted with multiplicity six and the fourth line is counted with multiplicity two (see Theorem 4.1.1). We write $\mathfrak{L}_{bd}$ to denote this line scheme. Theorem 4.1.1 answers a long-standing open question on the possible existence of a quadratic quantum $\mathbb{P}^3$ possessing a line scheme that is a union of lines, so we focus on these algebras primarily.

The computational method, provided in [21], for producing the line scheme of a quadratic quantum $\mathbb{P}^3$ makes use of the Plücker embedding that takes a line in $\mathbb{P}^3$ and associates it to a point in $\mathbb{P}^5$. For the sake of completeness, in Section 4.2 we describe the lines in $\mathbb{P}^3$ parametrized by the points of $\mathfrak{L}_{bd}$ in $\mathbb{P}^5$. In [6, 24], Chandler and Tomlin each presented a family of graded skew Clifford algebras and their respective line schemes were computed. In both cases, the authors show that the intersection points of the line scheme seem to "highlight" certain normalizing sequences. So in Section 4.3, we undertake a comparable analysis.

At the end of Chapter 4, we prove Theorem 4.4.2, which is a general result regarding certain quadratic algebras. This result was motivated by our study of $\mathfrak{L}_{bd}$, since it is the line scheme of an Ore extension of a quantum $\mathbb{P}^2$. Theorem 4.4.2 proves that, under certain conditions, if a quadratic quantum $\mathbb{P}^3$, denoted $\mathcal{A}$, has a subalgebra isomorphic to a quadratic quantum $\mathbb{P}^2$, denoted $\mathcal{B}$, then the point variety of $\mathcal{B}$ is embedded in the line variety of $\mathcal{A}$. On the level of point modules and line modules, this means that each point module of $\mathcal{B}$ determines a line module of $\mathcal{A}$. This result could prove to be practical in the construction of quadratic regular algebras with desirable point schemes and line schemes.

CHAPTER 2

Preliminary Information

Throughout, it is assumed that all fields $\Bbbk$ are algebraically closed and $\text{char}(\Bbbk) = 0$ unless otherwise stated. The set $\mathbb{N}$ denotes the set of all positive integers. If V is a vector space, V^* denotes its dual space and $V^\times$ denotes its nonzero elements. We begin by defining terms in abstract algebra and then concepts from Artin, Tate, and Van den Bergh's geometry [2, 3].

2.1 Abstract Algebra

Definition 2.1.1 (cf. [11])**.** Let R be a commutative ring with unity. We say A is an *R-algebra* if it is a ring with unity that satisfies the following:

- there exists a ring homomorphism ϕ from R to the center of A, and
- the homomorphism ϕ maps the unity element of R to the unity element of A.

Remark. Regarding Definition 2.1.1, if R is a field, say $\Bbbk$, then A is also a vector space over $\Bbbk$. The reader should note that multiplication in an R-algebra is, by definition, associative.

Definition 2.1.2 (cf. [16])**.** We say a $\Bbbk$-algebra A is *$\mathbb{N}$-graded* if

- $A = \bigoplus_{i=0}^{\infty} A_i$ where each A_i is an abelian group, and
- $A_j A_k \subseteq A_{j+k}$ for all $j, k \in \mathbb{N} \cup \{0\}$.

If $A_0 = \Bbbk$, we say A is *connected*. An element $x \in A^\times$ is *homogeneous of degree i* if $x \in A_i$.

Definition 2.1.3 (cf. [16])**.** Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a connected $\mathbb{N}$-graded $\Bbbk$-algebra. A left A-module M is a *graded left A-module* if $M = \bigoplus_{j=0}^{\infty} M_j$ where M_j is a subspace of M, for all j, and $A_i M_j \subset M_{i+j}$ for all i, j. A graded right A-module is defined similarly.

Definition 2.1.4 ([4, Definition I.1.1.3])**.** A $\Bbbk$-algebra, A, is *finitely generated* if there exists a finite set of elements $a_1, \ldots, a_n \in A$ such that the set

$$\{a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} \mid m_i \in \mathbb{N} \cup \{0\} \text{ for all } i\} \cup \{1\}$$

spans A as a vector space.

Example 2.1.5 ([4, pg. 15])**.** We write $\Bbbk\langle x_1, x_2, \ldots, x_n\rangle$ for the free algebra generated by indeterminates $x_1, \ldots, x_n$. This algebra consists of linear combinations of words in $x_1, \ldots, x_n$ and 1, which is considered the empty word. Addition and scalar multiplication are defined the standard way, but multiplication is defined by concatenation of words and distribution across addition. The free algebra is a connected, $\mathbb{N}$-graded, associative, $\Bbbk$-algebra that is finitely generated by $x_1, \ldots, x_n$. One can assign degrees (or weights) to each x_i, but we always assume each x_i has degree one.

Moreover, any connected, $\mathbb{N}$-graded, associative, $\Bbbk$-algebra A is finitely generated if and only if there exists a degree-preserving surjective ring homomorphism $\Bbbk\langle x_1, \ldots, x_n\rangle \twoheadrightarrow A$ for some free algebra $\Bbbk\langle x_1, \ldots, x_n\rangle$ and suitable $n \in \mathbb{N}$. Thus, $A \cong \Bbbk\langle x_1, \ldots, x_n\rangle / I$ where I is a homogeneous ideal. In the case where $I = \langle f_1, \ldots, f_m\rangle$, for some homogeneous $f_1, \ldots, f_m \in \Bbbk\langle x_1, \ldots, x_n\rangle$, we say A is *finitely presented* with generators $x_1, \ldots, x_n$ and *defining relations* $f_1, \ldots, f_m$.

Definition 2.1.6 (cf. [18])**.** An $\mathbb{N}$-graded $\Bbbk$-algebra $A = \bigoplus_{i=0}^{\infty} A_i$ is *quadratic* if it is generated by homogeneous degree-one elements and each of its defining relations is homogeneous of degree two.

Definition 2.1.7 (cf. [18])**.** Let A be a finitely generated quadratic $\Bbbk$-algebra. It follows that $A \cong T(V)/\langle I \rangle$, where V is a finite-dimensional vector space and I is a subspace of $V \otimes V$. The *Koszul dual* of A is the $\Bbbk$-algebra $A^! = T(V^*)/\langle I^\perp \rangle$.

Definition 2.1.8 ([4, pg. 23])**.** Suppose A is a $\Bbbk$-algebra and M is an A-module. The *projective dimension* of M, denoted $\mathrm{pdim}(M)$, is the minimal $n \in \mathbb{N}$ such that there is a projective resolution of M of length n (and $\mathrm{pdim}(M) = \infty$ if no such n exists). That is:

$$0 \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0$$

is a projective resolution of M where each P_i is a projective module and no such shorter resolution exists.

Definition 2.1.9 ([4, pg. 27])**.** The *right global dimension* of a $\Bbbk$-algebra A is denoted $\mathrm{r.gldim}(A)$, and is defined to be the supremum of the projective dimensions of all right A-modules. The *left global dimension* of A is defined analogously.

Remark ([4, Proposition I.1.5.7])**.** Regarding Definition 2.1.9, for any $\Bbbk$-algebra A that is connected, $\mathbb{N}$-graded, and finitely generated, then

$$\mathrm{r.gldim}(A) = \mathrm{pdim}(\Bbbk_A) = \mathrm{pdim}({}_A\Bbbk) = \mathrm{l.gldim}(A).$$

For the sake of clarity, $\Bbbk_A$ is the one-dimensional graded right A-module, that is $\Bbbk \cong A/\bigoplus_{i>0} A_i$. Hence, left global dimension and right global dimension agree and we will refer to this homological invariant as the *global dimension* of an algebra.

Definition 2.1.10 ([1])**.** A $\Bbbk$-algebra $A = \bigoplus_{i=0}^{\infty} A_i$ that is connected and $\mathbb{N}$-graded is said to be *Artin-Schelter regular*, or *AS-regular*, if it satisfies the following three conditions:

- $\operatorname{gldim}(A) = n < \infty$,
- the algebra A has polynomial growth, meaning there exist positive real numbers c and δ such that $\dim_{\Bbbk}(A_i) \leq c i^{\delta}$, and
- the algebra A satisfies the *Gorenstein condition*, namely, a minimal projective resolution of the left trivial module ${}_A\Bbbk$ consists of finitely generated modules and dualizing this resolution yields a minimal projective resolution of the right trivial module $\Bbbk_A[\ell]$, where the grading is shifted by some degree $\ell \in \mathbb{Z}$.

We will refer to an algebra of global dimension $n+1$ satisfying this definition as a *quantum* $\mathbb{P}^n$.

Example 2.1.11 ([4, Examples I.1.5.3 & I.1.5.6])**.** To illustrate the third condition of Definition 2.1.10, consider the algebra $A = \Bbbk\langle x, y\rangle / \langle yx - qxy\rangle$ where $q \in \Bbbk^{\times}$. We will show that A satisfies the Gorenstein condition. The Gorenstein condition above is equivalent to another statement involving Ext groups. As right A-modules:

$$\operatorname{Ext}^i_A(\Bbbk_A, A_A) \cong \begin{cases} 0 & i \neq n, \\ {}_A\Bbbk[\ell] & i = n \end{cases}$$

where l is a shift in the grading and n is the global dimension of A.

To calculate the Ext groups of A, we first construct a minimal projective resolution of $\Bbbk_A$. Denoting this resolution as P, we see

$$P: \quad 0 \to A[-2] \xrightarrow{\begin{pmatrix} -qy \\ x \end{pmatrix}} A[-1]^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} A \to \Bbbk_A \to 0$$

is a minimal projective resolution where the maps are given by left multiplication of the matrices provided. The Ext groups are given by the homology of $\mathrm{Hom}_A(P, A_A)$ and this is

$$\mathrm{Hom}_A(P, A_A): \quad A[2] \xleftarrow{\begin{pmatrix} -qy \\ x \end{pmatrix}} A[1]^2 \xleftarrow{\begin{pmatrix} x & y \end{pmatrix}} A \leftarrow 0$$

where the maps are given by right multiplication of the same matrices. We can calculate homology in each position and see, $\mathrm{Ext}^i_A(\Bbbk_A, A_A) \cong 0$ for $i = 0, 1$ and $\mathrm{Ext}^i_A(\Bbbk_A, A_A) \cong {}_A\Bbbk[3]$ for $i = 2$.

Furthermore, if we recognize $\mathrm{Hom}_A(P, A_A)$ as

$$\mathrm{Hom}_A(P, A_A): \quad 0 \leftarrow {}_A\Bbbk[3] \leftarrow A[2] \xleftarrow{\begin{pmatrix} -qy \\ x \end{pmatrix}} A[1]^2 \xleftarrow{\begin{pmatrix} x & y \end{pmatrix}} A \leftarrow 0$$

we see that this resolution is, in a sense, a "reflection" of P. For this reason, the Gorenstein condition is viewed as imposing a symmetry condition on AS-regular algebras.

Definition 2.1.12 ([5])**.** For this definition, we temporarily allow $\Bbbk$ to denote an arbitrary field. Let $\mu = (\mu_{ij}) \in M(n, \Bbbk)$ be a matrix with the property that $\mu_{ij}\mu_{ji} = 1$ for all i, j such that $i \neq j$. A matrix $M = (M_{ij}) \in M(n, \Bbbk)$ is called *μ-symmetric* if $M_{ij} = \mu_{ij} M_{ji}$ for all $i, j = 1, \dots, n$.

Definition 2.1.13 ([5])**.** For this definition, we temporarily assume $\mathrm{char}(\Bbbk) \neq 2$. Let $\mu = (\mu_{ij}) \in M(n, \Bbbk)$ satisfy $\mu_{kk} = 1 = \mu_{ij}\mu_{ji}$ for all i, j, k, and suppose $M_1, \dots, M_n \in M(n, \Bbbk)$ are μ-symmetric matrices. A *graded skew Clifford algebra* $A(\mu, M_1, \dots, M_n)$, associated to $M_1, \dots, M_n$ and μ, is an associative $\mathbb{Z}$-graded $\Bbbk$-algebra on degree-1 generators $x_1, \dots, x_n$ and on degree-2 generators $y_1, \dots, y_n$ with defining relations given by:

(a) $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^{n} (M_k)_{ij} y_k$ for all $i, j = 1, \dots, n$, and

(b) the existence of a normalizing sequence $\{y'_1, \dots, y'_n\}$ consisting of homogeneous degree-2 elements of $A(\mu, M_1, \dots, M_n)$ that span $\Bbbk y_1 + \cdots + \Bbbk y_n$.

Definition 2.1.14 (cf. [12])**.** Let R be a ring and $\sigma : R \to R$ a ring homomorphism. A *left σ-derivation* on R is a linear map $\delta : R \to R$ such that

$$\delta(r_1 r_2) = \sigma(r_1)\delta(r_2) + \delta(r_1) r_2$$

for all $r_1, r_2 \in R$. An *Ore extension* of R is the free left R-module $R[x; \sigma, \delta]$ with basis $\{1, x, x^2, x^3, \dots\}$ subject to

$$xr = \sigma(r)x + \delta(r)$$

for all $r \in R$.

Definition 2.1.15 (cf. [4, Definition I.1.2.4])**.** Let A be a $\Bbbk$-algebra and let V be a finite-dimensional subspace of A. Let $V^0 = \Bbbk$, $V^1 = V$ and V^n be the span of all monomials of length n, and write $V^{\leq n} = \sum_{i=0}^{n} V^i$. The *Gelfand-Kirillov dimension* of A is

$$\operatorname{GKdim}(A) = \sup \{\overline{\lim_{n\to\infty}} log_n(\dim V^{\leq n}) \mid V \text{ is finite-dimensional}\}.$$

Definition 2.1.16 ([14])**.** A noetherian $\Bbbk$-algebra A is said to be *Auslander regular* if

- $\operatorname{gldim}(A) < \infty$, and
- for all finitely generated modules M, and for all $q \geq 0$, we have $j(N) \geq q$ for every A-submodule N of $\operatorname{Ext}^q_A(M, A)$ where $j(N)$ is the *grade* of N defined by

$$j(N) = \inf \{\ell \mid \operatorname{Ext}^\ell_A(N, A) \neq 0\}.$$

Remark ([14])**.** It is believed that the class of AS-regular algebras is more general than the class of graded Auslander-regular algebras as, in particular, the noetherian condition is part of the definition of Auslander regular. However, there is not a known example of an algebra that is AS-regular, but not Auslander regular. In addition to this, we will mention a few comments on properties of Auslander-regular, and AS-regular, algebras:

- if $A = \bigoplus_{i=0}^{\infty} A_i$ is $\mathbb{N}$-graded, connected and Auslander regular, then A is a domain ([14, Theorem 4.8]);
- if $A = \bigoplus_{i=0}^{\infty} A_i$ is $\mathbb{N}$-graded, connected and Auslander regular with polynomial growth, then A is Artin-Schelter regular ([14, pg. 278]).

Definition 2.1.17 ([14])**.** A noetherian $\Bbbk$-algebra A with GK-dimension $n \in \mathbb{N} \cup \{0\}$ is said to satisfy the *Cohen-Macaulay property* if $\mathrm{GKdim}(M) + j(M) = n$ for all nonzero finitely generated A-modules M where $j(M)$ is the grade of M as defined in Definition 2.1.16.

2.2 Artin, Tate, and Van den Bergh's Geometry

Definition 2.2.1 (cf. [10])**.** Consider an equivalence relation $\sim$ on $\Bbbk^{n+1} \setminus \{0\}$ where $(a_1, \ldots, a_{n+1}) \sim (b_1, \ldots, b_{n+1})$ if and only if there exists $\lambda \in \Bbbk^{\times}$ such that $a_i = \lambda b_i$ for all i. The definition of *n-dimensional projective space* is $\mathbb{P}^n = (\Bbbk^{n+1} \setminus \{0\}) / \sim$.

We will write e_i or E_i to be the point $(0, 0, \ldots, 1, \ldots, 0) \in \mathbb{P}^n$ where the i^{th} entry is nonzero and all others are zero.

Definition 2.2.2 (cf. [10])**.** Suppose $f_1, \ldots, f_m \in \Bbbk[x_1, \ldots, x_{n+1}]$ are homogeneous polynomials. The *projective variety* determined by $f_1, \ldots, f_m$ is

$$\mathcal{V}(f_1, \ldots, f_m) = \{p \in \mathbb{P}^n \mid f_i(p) = 0 \text{ for all } i\}.$$

Remark. A rigorous definition of a *scheme* can be found in [13], but in this work, a scheme can be considered as a projective variety that encodes multiplicity.

For example, let $V = \mathcal{V}(x)$ and $W = \mathcal{V}(x^2)$ be projective varieties in $\mathbb{P}^1$. As projective varieties, $V = W = \{(0,1)\}$, but as schemes, V is the point $(0,1)$ counted with multiplicity one and W is the same point counted with multiplicity two.

Definition 2.2.3 ([2])**.** Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a connected $\mathbb{N}$-graded $\Bbbk$-algebra generated by A_1 where $\dim_{\Bbbk}(A_1) = n < \infty$. A graded right A-module $M = \bigoplus_{i=0}^{\infty} M_i$ is a *point module* if:

- M is cyclic and generated by M_0 and
- $\dim_{\Bbbk}(M_i) = 1$ for all i.

Definition 2.2.4 ([3])**.** Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a connected $\mathbb{N}$-graded $\Bbbk$-algebra generated by A_1 where $\dim_{\Bbbk}(A_1) = n < \infty$. A graded right A-module $M = \bigoplus_{i=0}^{\infty} M_i$ is a *line module* if:

- M is cyclic and generated by M_0 and
- $\dim_{\Bbbk}(M_i) = i + 1$ for all i.

Example 2.2.5. If $A = \Bbbk\langle x, y\rangle / \langle yx - qxy\rangle$, where $q \in \Bbbk^{\times}$, then A has a right point module associated to every point in $\mathbb{P}^1$. Two examples are:

- $M = \dfrac{A}{xA}$ with associated point $(0,1) \in \mathbb{P}^1$,
- $N = \dfrac{A}{(x-y)A}$ with associated point $(1,1) \in \mathbb{P}^1$.

Remark. To justify the association between point modules and points, suppose A is a $\Bbbk$-algebra satisfying the hypothesis in Definition 2.2.3. Let M be a left point module of A, then one can associate to it a point in $\mathbb{P}^{n-1}$ as follows. Since M is graded and $\dim(M_i) = 1$ for all i, then $M = \bigoplus_{i=0}^{\infty} \Bbbk m_i$. Furthermore, since $am_0 = \alpha_a m_1$, where $\alpha_a \in \Bbbk$ for all $a \in A_1$, we can define a $\Bbbk$-linear epimorphism $\phi : A_1 \to \Bbbk$ where $\phi(a) = \alpha_a$. If we write $U = \ker(\phi)$, then $\Bbbk \cong A_1/U$ and $\dim(U) = n - 1$. Now, if we consider $U^\perp \subseteq A_1^*$, then $\dim(U^\perp) = 1$. Hence, $\mathbb{P}(U^\perp)$ is zero-dimensional in $\mathbb{P}(A_1^*)$ and so is a point in $\mathbb{P}^{n-1} = \mathbb{P}(A_1^*)$.

A similar argument can be made for a right line module of A. Using the same notation, one can say $\dim(U) = n - 2$ and $\mathbb{P}(U^\perp)$ is one-dimensional in $\mathbb{P}(A_1^*)$ and so is a line in $\mathbb{P}^{n-1} = \mathbb{P}(A_1^*)$.

In [2], Artin, Tate, and Van den Bergh proved that, under certain conditions, point modules are parametrized by a scheme, later called the *point scheme* in [27]. In [20], it was proved by Shelton and Vancliff that, under certain conditions, line modules are parametrized by a scheme called the *line scheme*.

CHAPTER 3

A Family of Algebras and Their Quantum Spaces

3.1 The Family of Algebras

From [23], let A be the $\Bbbk$-algebra generated by x_1, x_2, x_3, x_4 subject to the defining relations:

$$\begin{aligned} x_2x_1 &= -x_1x_2, & x_4x_1 &= -x_1x_4 + x_2^2 + ax_2x_3 + bx_3^2, \\ x_3x_1 &= x_1x_3, & x_4x_2 &= -x_2x_4 + x_1^2 + cx_1x_3 + dx_3^2, \\ x_3x_2 &= x_2x_3, & x_4x_3 &= x_3x_4 + x_1x_2, \end{aligned}$$

where $a, b, c, d \in \Bbbk$. The entirety of this chapter will consist of discussion around this algebra and associated geometry.

In [23], this family of algebras was constructed via an Ore extension. In addition, it is shown in that article that A has a finite point scheme when $b \neq 0$ or $d \neq 0$, and has a one-parameter family of line modules if at least one of a, b, c or d is nonzero. So, as previously discussed in Chapter 1, A is a candidate to be a generic quadratic quantum $\mathbb{P}^3$.

3.1.1 Symmetry on A

In non-commutative algebra, it is often convenient to recognize symmetries that exist. There is a hidden symmetry among the defining relations of A when $a = c = 0$ that we demonstrate before describing the geometry of A.

Theorem 3.1.1. *Let $A(b, d)$ denote a member of the subfamily of algebras A where $a = c = 0$ and $b, d \in \Bbbk$. For all $b, d \in \Bbbk$, $A(b, d) \cong A(d, b)$.*

Proof. Consider the map on $A(b, d)$ given by:

$$x_1 \mapsto x_2, \quad x_2 \mapsto x_1, \quad x_3 \mapsto -x_3, \quad x_4 \mapsto x_4.$$

The image of this map yields another quadratic algebra. If we apply the map to the defining relations of $A(b, d)$ we obtain:

$$\begin{aligned} x_1x_2 &= -x_2x_1, & x_4x_2 &= -x_2x_4 + x_1^2 + bx_3^2, \\ x_3x_2 &= x_2x_3, & x_4x_1 &= -x_1x_4 + x_2^2 + dx_3^2, \\ x_3x_1 &= x_1x_3, & -x_4x_3 &= -x_3x_4 + x_2x_1. \end{aligned}$$

We can recognize these relations as the defining relations of A where $a = c = 0$, except that the roles of b and d have been interchanged. Thus, $A(b, d)$ is isomorphic to $A(d, b)$. $\square$

3.2 Description of the Point Scheme

3.2.1 The Point Variety

The description of the point variety can be found in [23] but, for the sake of completeness, we present its computation below. In this computation, appropriate conditions on a, b, c, and d will be imposed to ensure that A has a finite point scheme. To begin, we factor the defining relations of A as the product Mx where M is a six-by-four matrix and $x = [x_1, x_2, x_3, x_4]^T$ and see:

$$M = \begin{bmatrix} x_2 & x_1 & 0 & 0 \\ x_3 & 0 & -x_1 & 0 \\ 0 & x_3 & -x_2 & 0 \\ x_4 & -x_2 & -ax_2 - bx_3 & x_1 \\ -x_1 & x_4 & -cx_1 - dx_3 & x_2 \\ 0 & -x_1 & x_4 & -x_3 \end{bmatrix}.$$

By [2], the point scheme of A can be identified with the zero locus in $\mathbb{P}(A_1^*)$ of the maximal minors of M. The 15 maximal minors of M are homogeneous degree-four polynomials and are listed in Section 6.1. To begin computing their zero locus, we observe that $2x_1^2x_2x_3 = 0$ and proceed by separating the argument into cases depending on whether or not x_1, x_2 or x_3 is zero.

Case 1: Assume $x_1 = 0$.

If $x_2 \neq 0$, we can restrict to an affine open subset by setting $x_2 = 1$. A Gröbner basis yields only two polynomials:

$$1 + ax_3 + bx_3^2 = 0,$$

$$-dx_3^2 + 2x_4 = 0.$$

From the first equation, x_3 has one or two values depending on the values of a and b, while $x_4 = \frac{d}{2}x_3^2$. That is, if $a \neq 0$ or $b \neq 0$, then the point scheme contains points of the form $(0, 2, 2\gamma_1, d\gamma_1^2) \in \mathbb{P}^3$ where $\gamma_1 \in \Bbbk^\times$ satisfies $b\gamma_1^2 + a\gamma_1 + 1 = 0$.

On the other hand, if $x_2 = 0$, a Gröbner basis yields only two polynomials:

$$bx_3^4 = 0,$$

$$dx_3^4 = 0.$$

If $b \neq 0$ or $d \neq 0$, we can conclude $x_3 = 0$ and the point scheme contains $(0, 0, 0, 1) \in \mathbb{P}^3$. If $b = d = 0$, the point scheme contains the line $\mathcal{V}(x_1, x_2)$ and would not be finite. So we will assume $b \neq 0$ or $d \neq 0$ when regarding the point scheme of A.

Case 2: Assume $x_1 \neq 0$.

The assumption implies that we may restrict to an affine open subset by taking $x_1 = 1$. A Gröbner basis yields only three polynomials:

$$x_2 = 0,$$

$$1 + cx_3 + dx_3^2 = 0,$$

$$-bx_3^2 + 2x_4 = 0.$$

From the second equation, we see x_3 has one or two values depending on the values of c and d, while $x_4 = \frac{b}{2}x_3^2$. It follows that if $c \neq 0$ or $d \neq 0$, then we obtain points that are of the form $(2, 0, 2\gamma_2, b\gamma_2^2) \in \mathbb{P}^3$ where $\gamma_2 \in \Bbbk^\times$ satisfies $1+c\gamma_2+d\gamma_2^2 = 0$.

Remark. Thus, for generic values of a, b, c, d, the point variety of A consists of the five distinct points:

(a) $(0, 2, 2\gamma_1, d\gamma_1^2) \in \mathbb{P}^3$ where $\gamma_1 \in \Bbbk^\times$ satisfies $b\gamma_1^2 + a\gamma_1 + 1 = 0$,

(b) $(2, 0, 2\gamma_2, b\gamma_2^2) \in \mathbb{P}^3$ where $\gamma_2 \in \Bbbk^\times$ satisfies $1 + c\gamma_2 + d\gamma_2^2 = 0$,

(c) $e_4 \in \mathbb{P}^3$.

On the other hand, if one were to set $a = b = 0$, the point variety then contains only three points: e_4 and two points of the form described in (b). Choosing other combinations of values for a, b, c, d, we now see the point scheme of A can consist of 2, 3, 4, or 5 distinct points. We will focus on the case when it consists of five distinct points.

3.2.2 The Point Scheme

For a complete description of the point scheme, it remains to determine the multiplicities of each component described previously. It is well known that if the point scheme is finite, then it contains twenty points counted with multiplicity, thanks to an unpublished work by M. Van den Bergh that was later outlined in [26]. It can be interesting to see the distribution of multiplicity among the components. In Section 3.3, we show it is interesting to study the subfamily of algebras where $a = c = 0$, so we will make this assumption now.

To find the multiplicity of each point, let $R = \Bbbk[x_1, x_2, x_3, x_4]/J$ where J is the ideal generated by the 15 polynomials described previously and listed in Section 6.1. If p is a point in the point scheme of A, we write R_p for the ring R localized at the point p. The dimension of R_p will be equal to the multiplicity of p in the point scheme. We will write $\bar{J}$ and $\tilde{x_i}$ for the images of J and x_i in R_p respectively.

Theorem 3.2.1. *For $A(b, d)$ as in Theorem 3.1.1, if $b, d \in \Bbbk^\times$, then the point scheme consists of:*

(a) two points of the form $(0, 2, 2\gamma_1, d\gamma_1^2)$, each with multiplicity one where $1+b\gamma_1^2 = 0$,

(b) two points of the form $(2, 0, 2\gamma_2, b\gamma_2^2)$, each with multiplicity one where $1+d\gamma_2^2 = 0$, and

(c) $e_4 \in \mathbb{P}^3$ with a multiplicity of 16.

Proof. In the ideal J, consider the polynomial:

$$x_1 x_2 x_3^2. \tag{1}$$

Let us first consider the points of the form $(0, 2, 2\gamma_1, d\gamma_1^2) \in \mathbb{P}^3$ where $1 + b\gamma_1^2 = 0$ and notice that the condition on γ_1 implies it is nonzero. Since $\tilde{x_2}$ and $\tilde{x_3}$ are nonzero at $(0, 2, 2\gamma_1, d\gamma_1^2)$, equation (1) implies $\tilde{x_1}$ is an element of $\bar{J}$. So we can compute a Gröbner basis with this information and restrict to an affine open subset by setting $\tilde{x_2} = 2$. This yields only the following four polynomials:

$$\tilde{x_1}, \qquad -2 + \tilde{x_2},$$

$$4 + b\tilde{x_3}^2, \qquad -d\tilde{x_3}^2 + 4\tilde{x_4}.$$

From these four polynomials, we see $\tilde{x_4} \in \Bbbk[\tilde{x_3}]$, $R_p \cong \Bbbk[x]/\langle x^2 + \alpha\rangle$ where $\alpha = \frac{4}{b}$ and $\dim_{\Bbbk}(R_p) = 2$. Since R_p is the localization at two distinct points, we conclude they each have multiplicity one.

Next, we consider the points listed in (b), but we will make use of the isomorphism in Theorem 3.1.1. Recall, $A(b,d) \cong A(d,b)$ via the map defined on $A(b,d)$ given by:

$$x_1 \mapsto x_2, \quad x_2 \mapsto x_1, \quad x_3 \mapsto -x_3, \quad x_4 \mapsto x_4.$$

The isomorphism applied to our point yields $(2,0,2\gamma_2, b\gamma_2^2) \mapsto (0,2,-2\gamma_2, b\gamma_2^2)$ where γ_2 satisfies $1+d\gamma_2^2 = 0$. That is, our point maps to the point $(0,2,2\gamma_1, b\gamma_1^2)$ where $1+d\gamma_1^2 = 0$, which is a point listed in (a) for $A(d,b)$. Since we already proved that such points have multiplicity one, it follows that the points in (b) have multiplicity one.

Lastly we consider $e_4 \in \mathbb{P}^3$. Since, as mentioned before, the sum of the multiplicities will be equal to 20, we wish to show this point has multiplicity 16. Firstly, we restrict to an affine open subset by setting $x_4 = 1$. A Gröbner basis is computed, and we make note of two polynomials it contains:

$$\tilde{x_1}^7(2d + b\tilde{x_1}),$$

$$\tilde{x_1}^2\tilde{x_3}^2(2d + b\tilde{x_1}).$$

As d is nonzero and $2d + b\tilde{x_1}$ is nonzero at e_4, it follows that $\tilde{x_1}^7$ and $\tilde{x_1}^2\tilde{x_3}^2$ are elements in $\bar{J}$. We compute another Gröbner basis with this information, and make note of the polynomial:

$$\tilde{x_3}^6(4b + d^2\tilde{x_3}^2).$$

As b is nonzero and $4b + d^2\tilde{x_3}^2$ is nonzero at e_4, we have $\tilde{x_3}^6$ is an element of $\bar{J}$. One last Gröbner basis is computed and it contains only the following nine polynomials:

$$\tilde{x_1}^7, \qquad \tilde{x_1}^5 + 2\tilde{x_1}^3\tilde{x_2}, \qquad \tilde{x_1}^2\tilde{x_2}^2,$$

$$\tilde{x_1}^4 + 2\tilde{x_1}^2\tilde{x_2} + \tilde{x_1}\tilde{x_2}^2, \qquad -\tilde{x_1}^5 + 4\tilde{x_1}\tilde{x_2}^2 + 2\tilde{x_2}^4, \qquad -2d\tilde{x_1}^5 + b\tilde{x_1}^6 + 8d\tilde{x_1}^2\tilde{x_3},$$

$$-\tilde{x_1}^3 - 2\tilde{x_1}\tilde{x_2} - \tilde{x_2}^3 + 2d\tilde{x_1}\tilde{x_3}^2, \qquad 12b\tilde{x_1}^4 - d\tilde{x_1}^6 + 16b\tilde{x_1}^2\tilde{x_2} - 16b\tilde{x_1}\tilde{x_3} + 8b^2\tilde{x_3}^3,$$

$$12bd^2\tilde{x_1}^4 + 6b^2d\tilde{x_1}^5 + (b^3 - d^3)\tilde{x_1}^6 + 16bd^2\tilde{x_1}^2\tilde{x_2} - 8b^2d\tilde{x_1}\tilde{x_2}^2 - 16bd^2\tilde{x_1}\tilde{x_3} + 16b^2d\tilde{x_2}\tilde{x_3}.$$

A package, Affine, in the program Maxima can apply Bergman's Diamond Lemma to determine a basis of R_p. The dimension of R_p is 16 as expected and, thus, the multiplicity of e_4 is 16. $\square$

3.3 Description of the Line Scheme

In this section, we describe in detail the line variety of the family of algebras, A, listed at the start of Section 3.1. We will refer to the line scheme of A as $\mathfrak{L}$ when the defining relations have generic values for a, b, c and d. Later in this section, we will also consider A when $a = c = 0$ and $bd \neq 0$, and in this case, we will refer to the line scheme as $\mathfrak{L}_{bd}$.

3.3.1 The Line Variety

In [21] Shelton and Vancliff describe a method for computing the line scheme of a quadratic AS-regular algebra of global dimension four with six defining relations. We will apply this method to A.

We begin by determining the Koszul Dual, $A^!$, of A. As A is a quadratic algebra with six defining relations, $A \cong T(V)/I$ where V is the vector space over $\Bbbk$ generated by $\{x_1, x_2, x_3, x_4\}$ and I is generated by $I_2 \subset V \otimes V$ with $\dim(I_2) = 6$. In this context, $A^! \cong T(V^*)/\langle I_2^\perp \rangle$, where V^* is the vector-space dual of V, $I_2^\perp \subset V^* \otimes V^*$, and $\dim(I_2^\perp) = 10$. Here, $I_2^\perp$ is the subspace of $V^* \otimes V^*$ consisting of those elements that vanish on I_2. We write $\{z_1, z_2, z_3, z_4\} \subset V^*$ for the dual basis to $\{x_1, x_2, x_3, x_4\}$.

For our algebra, we can consider the defining relations of $A^!$ to be:

$$\begin{aligned}
h_1 &= z_2z_1 - z_1z_2 - z_4z_3, & h_6 &= z_4z_3 + z_3z_4,\\
h_2 &= z_3z_1 + z_1z_3 + cz_4z_2, & h_7 &= z_1^2 + z_2z_4,\\
h_3 &= z_3z_2 + z_2z_3 + az_4z_1, & h_8 &= z_2^2 + z_1z_4,\\
h_4 &= z_4z_1 - z_1z_4, & h_9 &= z_3^2 + bz_1z_4 + dz_2z_4,\\
h_5 &= z_4z_2 - z_2z_4, & h_{10} &= z_4^2.
\end{aligned}$$

If $h = (h_1, h_2, \dots, h_{10})^T$ is a column vector containing the above defining relations of $A^!$, then $h = \tilde{M}z$, where $z = (z_1, \dots, z_4)^T$ and $\tilde{M}$ is a 10×4 matrix whose entries are linear combinations of the z_i. We form a 10×8 matrix by concatenating two 10×4 matrices constructed from $\tilde{M}$, the first of which is obtained by replacing each z_i in $\tilde{M}$ with $u_i \in \Bbbk$, and the second of which is obtained by replacing each z_i in $\tilde{M}$ with $v_i \in \Bbbk$. It is useful to assume $\sum_{i=1}^4 u_ix_i$ and $\sum_{i=1}^4 v_ix_i$ are linearly independent, in which case their common zero locus is a line in $\mathbb{P}(V^*)$.

For A, this process yields the following 10×8 matrix:

$$M = \begin{bmatrix}
u_2 & -u_1 & -u_4 & 0 & v_2 & -v_1 & -v_4 & 0\\
u_3 & cu_4 & u_1 & 0 & v_3 & cv_4 & v_1 & 0\\
au_4 & u_3 & u_2 & 0 & av_4 & v_3 & v_2 & 0\\
u_4 & 0 & 0 & -u_1 & v_4 & 0 & 0 & -v_1\\
0 & u_4 & 0 & -u_2 & 0 & v_4 & 0 & -v_2\\
0 & 0 & u_4 & u_3 & 0 & 0 & v_4 & v_3\\
u_1 & 0 & 0 & u_2 & v_1 & 0 & 0 & v_2\\
0 & u_2 & 0 & u_1 & 0 & v_2 & 0 & v_1\\
0 & 0 & u_3 & bu_1 + du_2 & 0 & 0 & v_3 & bv_1 + dv_2\\
0 & 0 & 0 & u_4 & 0 & 0 & 0 & v_4
\end{bmatrix}.$$

Each of the forty-five 8×8 minors of M is a bihomogeneous polynomial of bidegree $(4, 4)$ in u_i and v_i. In addition, each polynomial may be expressed as a linear combination of products of polynomials in $N_{ij} = u_i v_j - u_j v_i$ for $1 \leq i < j \leq 4$. Thus, M produces forty-five quartic polynomials in six variables $N_{12}, N_{13}, N_{14}, N_{23}, N_{24}$ and N_{34}. We next apply the orthogonality isomorphism on the generators as follows:

$$N_{12} \mapsto M_{34}, \quad N_{13} \mapsto -M_{24}, \quad N_{14} \mapsto M_{23},$$

$$N_{23} \mapsto M_{14}, \quad N_{24} \mapsto -M_{13}, \quad N_{34} \mapsto M_{12},$$

and thereby produce forty-five quartic polynomials in the Plücker coordinates, $M_{12}, M_{13}, M_{14}, M_{23}, M_{24}$ and M_{34}, on $\mathbb{P}^5$. The line scheme, $\mathcal{L}$, of A is isomorphic to the zero locus in $\mathbb{P}^5$ of these forty-five quartic polynomials in the M_{ij} coordinates along with the Plücker polynomial $P = M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}$. The addition of the Plücker polynomial ensures $\mathcal{L}$ contains points in $\mathbb{P}^5$ that correspond to lines in $\mathbb{P}^3$. For our algebra A, these polynomials are calculated by using Wolfram's Mathematica.

As mentioned in [21], one can apply a similar process to A rather than $A^!$. Doing so yields a collection of polynomials for which the zero locus is isomorphic to $\mathcal{L}$, but the polynomials produced cannot entirely be converted into Plücker coordinates. So the approach outlined above from [21], that makes use of $A^!$, yields a more user-friendly description of the line scheme.

In [23], it is shown that the line scheme of A is one-dimensional if at least one of a, b, c or d is nonzero, but a detailed description is not given, so we will calculate its line variety next.

Theorem 3.3.1. *For the family of algebras given at the start of Section 3.1, if at least one of a, b, c, d is nonzero, then the line variety is one-dimensional. Moreover, let*

- $L_1 = \mathcal{V}(M_{12}, M_{13}, M_{14}, M_{23})$,
- $L_2 = \mathcal{V}(M_{12}, M_{13}, M_{23}, M_{24})$,
- $L_3 = \mathcal{V}(M_{12}, M_{13}, M_{23}, M_{34})$,
- $L_4 = \mathcal{V}(M_{12},\ dM_{13} - bM_{23},\ dM_{14} - bM_{24},\ 2bd^2M_{34} - (b^3 + d^3)M_{23})$,
- $L_5 = \mathcal{V}(M_{12},\ \gamma M_{13} - \alpha M_{23},\ \gamma M_{14} - \alpha M_{24},\ 2\alpha\gamma^2 M_{34} - (\alpha^3 + \gamma^3)M_{23})$, *where* $\alpha, \gamma \in \Bbbk^\times$ *satisfy* $\alpha^2 = a$ *and* $\gamma^2 = c$,
- $L_6 = \mathcal{V}(M_{12},\ \alpha M_{23} + \gamma M_{13},\ \gamma M_{14} + \alpha M_{24},\ 2\alpha\gamma^2 M_{34} + (\gamma^3 - \alpha^3)M_{23})$, *where* $\alpha, \gamma \in \Bbbk^\times$ *satisfy* $\alpha^2 = a$ *and* $\gamma^2 = c$, *and*
- L_7 *be the nonlinear variety determined by the affine open subset:*

$$\mathcal{V}(M_{23}-1,\ M_{12}M_{34}-M_{13}M_{24}+M_{14},\ M_{12}^2-cM_{13}^2+a,\ M_{12}M_{13}-M_{13}^3+2M_{13}M_{34}-1,$$

$$cM_{12}M_{13} - dM_{13} - M_{12}^2 M_{13} + 2M_{12}M_{24} + b,$$

$$M_{12}^2 - M_{12}^3 M_{13} + bM_{13}^2 - dM_{13}^3 + cM_{12}M_{13}^3 + 2M_{12}M_{13}M_{14}).$$

If at least one of a, b, c, d is nonzero, then the line variety is:

(a) $L_1 \cup L_2 \cup L_3$ *if* $a = b = c = 0$ *or* $a = c = d = 0$;

(b) $L_1 \cup L_2 \cup L_3 \cup L_4$ *if* $a = c = 0 \neq bd$;

(c) $L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_7$ *if* $ad \neq 0$ *or* $cd \neq 0$;

(d) $L_1 \cup L_2 \cup L_3 \cup L_5 \cup L_6 \cup L_7$ *if* $b = 0 = d \neq ac$;

(e) $L_1 \cup L_2 \cup L_3 \cup L_7$ *otherwise.*

Proof. Using the process described above, the list of polynomials that give the line scheme is calculated; the list is included in Section 6.2.1. Among these polynomials, we see $M_{23}^2(M_{12}^2 - cM_{13}^2 + aM_{23}^2) = 0$, so we will continue the argument in two cases.

Case 1: Assume $M_{23} = 0$.

We can calculate a Gröbner basis after substituting $M_{23} = 0$. Among this list of polynomials, we see $M_{12}^4 = 0$ and $M_{13}^4 = 0$. So we set $M_{12} = 0$ and $M_{13} = 0$ and one last Gröbner basis tells us $M_{14}M_{24}M_{34}^2 = 0$, so either $M_{14} = 0$, $M_{24} = 0$, or $M_{34} = 0$. It follows that Case 1 yields the union of three lines, specifically L_1, L_2 and L_3 listed above.

Case 2: Assume $M_{23} \neq 0$.

We now restrict to an affine open subset by setting $M_{23} = 1$. With this assumption, we see that $M_{12}^2(a + M_{12}^2 - cM_{13}^2) = 0$, so we will continue Case 2 in two subcases depending on the value of M_{12}.

Case 2.1: Assume $M_{12} = 0$.

Calculating a Gröbner basis after substituting $M_{12} = 0$ and $M_{23} = 1$, we see that one of the polynomials obtained is $-b + dM_{13} = 0$. We will continue in two subcases depending on the value of d.

Case 2.1.1: Assume $d \neq 0$.

Calculating a Gröbner basis after setting $M_{13} = \frac{b}{d}$ yields four polynomials:

$$M_{12} = 0,$$

$$-b + dM_{13} = 0,$$

$$-b^3 - d^3 + 2bd^2M_{34} = 0,$$

$$dM_{14} - bM_{24} = 0.$$

If $b = 0$, then this subcase yields only the empty set. Hence we assume $b \neq 0$ in the remainder of Case 2.1.1. Homogenizing the latter four polynomials yields a line in $\mathbb{P}^5$. Any other polynomials obtained from homogenizing a different Gröbner basis (such as one using degree lexicographical ordering) cannot yield a proper one-dimensional subvariety of a line, as a line is irreducible and one-dimensional. Thus this component of $\mathfrak{L}$ is the line

$$L_4 = \mathcal{V}(M_{12},\ dM_{13} - bM_{23},\ dM_{14} - bM_{24},\ 2bd^2M_{34} - (b^3 + d^3)M_{23}),$$

which completes Case 2.1.1.

Case 2.1.2: Assume $d = 0$.

We can calculate a Gröbner basis after setting $M_{23} = 1$, $M_{12} = 0$ and $d = 0$, and, among the polynomials it returns, we see $a - cM_{13}^2 = 0$ and $b = 0$. From these two polynomials, we see that if $c = 0$, then $a = 0$ and this violates our initial hypothesis, so c must be nonzero for the zero locus to be nonempty. From the polynomial $a - cM_{13}^2 = 0$, we see that $M_{13} = \pm\frac{\alpha}{\gamma}$ where $\alpha^2 = a$ and $\gamma^2 = c$. When $M_{13} = \frac{\alpha}{\gamma}$, we can calculate a Gröbner basis and it yields only the four polynomials:

$$M_{12} = 0,$$

$$\alpha - \gamma M_{13} = 0,$$

$$\gamma M_{14} - \alpha M_{24} = 0,$$

$$-\alpha^3 - \gamma^3 + 2\alpha\gamma^2 M_{34} = 0.$$

If $a = 0 \neq c$, then this subcase yields only the empty set. Hence, we assume $a \neq 0 \neq c$ in the remainder of Case 2.1.2. Homogenizing the latter four polynomials yields a line in $\mathbb{P}^5$. Any other polynomials obtained from homogenizing a different Gröbner basis (such as one using degree lexicographical ordering) cannot yield a proper one-dimensional subvariety of a line, as a line is irreducible and one-dimensional. Thus this component of $\mathfrak{L}$ is the line

$$L_5 = \mathcal{V}(M_{12},\ \gamma M_{13} - \alpha M_{23},\ \gamma M_{14} - \alpha M_{24},\ 2\alpha\gamma^2 M_{34} - (\alpha^3 + \gamma^3)M_{23})).$$

Similarly, when $M_{13} = -\frac{\alpha}{\gamma}$ we can calculate a Gröbner basis and it yields only the four polynomials:

$$M_{12} = 0,$$

$$\alpha + \gamma M_{13} = 0,$$

$$\gamma M_{14} + \alpha M_{24} = 0,$$

$$\gamma^3 - \alpha^3 + 2\alpha\gamma^2 M_{34} = 0.$$

Homogenizing these four polynomials yields a line in $\mathbb{P}^5$. Any other polynomials obtained from homogenizing a different Gröbner basis (such as one using degree lexicographical ordering) cannot yield a proper one-dimensional subvariety of a line, as a line is irreducible and one-dimensional. Thus this component of $\mathfrak{L}$ is the line

$$L_6 = \mathcal{V}(M_{12},\ \alpha M_{23} + \gamma M_{13},\ \gamma M_{14} + \alpha M_{24},\ 2\alpha\gamma^2 M_{34} + (\gamma^3 - \alpha^3)M_{23}).$$

Case 2.1.2 is now complete.

Case 2.2: Assume $M_{12} \neq 0$.

The first polynomial in our collection in Section 6.2.1 allows us to conclude that M_{13} is nonzero. So we calculate a Gröbner basis by setting $M_{23} = 1$ and ensuring that our Mathematica calculations consider $M_{12} \neq 0$ and $M_{13} \neq 0$ by introducing variables y and z that satisfy $zM_{12} = 1$ and $yM_{13} = 1$. This yields the five polynomials listed in Section 6.2.3 which determine L_7. However, if $a = c = 0$, then $L_7 \subseteq L_4$. Case 2 is now complete and so is the proof. $\square$

We call the points of a one-dimensional line scheme that lie on two or more components the intersection points of the line scheme. The next result computes the intersection points of $\mathfrak{L}_{bd}$, but not their multiplicities.

Corollary 3.3.2. *With the hypotheses of (b) in Theorem 3.3.1, the four distinct lines of $\mathfrak{L}_{bd}$ intersect at four distinct points:*

$$L_1 \cap L_2 = \{E_6\}, \quad L_1 \cap L_3 = \{E_5\},$$

$$L_2 \cap L_3 = \{E_3\}, \quad L_3 \cap L_4 = \{(0,0,b,0,d,0)\}.$$

Proof. The result follows from direct computations using the polynomials from Theorem 3.3.1. □

Figure 3.3.3 is how one can imagine $\mathfrak{L}_{bd}$ in $\mathbb{P}^5$. When viewing the figure, the reader should note that components L_1, L_2 and L_3 lie in the plane where $M_{12} = M_{13} = M_{23} = 0$ and L_4 does not.

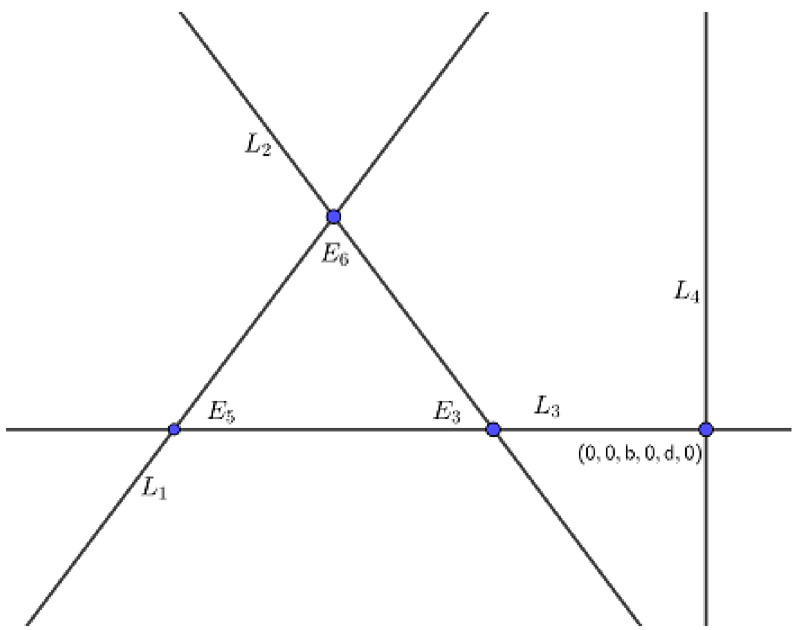

Figure 3.3.3. A depiction of the line variety of A when $a = c = 0$ and $b, d \in \Bbbk^{\times}$.

Theorem 3.3.1 is significant as it answers a long-standing open problem posed by S.P. Smith on whether or not there exists a quadratic quantum $\mathbb{P}^3$ that possesses a finite point scheme and a one-dimensional line scheme that is the union of lines. For this reason, Chapter 4 will consist of discussion of the case $a = c = 0$ and $b, d \in \Bbbk^{\times}$.

CHAPTER 4

Properties of $\mathfrak{L}_{bd}$

In this chapter, we denote members of the subfamily of algebras A where $a = c = 0$ and $b, d \in \Bbbk^\times$ by $A(b,d)$ and $\mathfrak{L}_{bd}$ will denote the line scheme of $A(b,d)$. Recall, from Theorem 3.3.1(b), $\mathfrak{L}_{bd}$ is the union of four distinct lines with various multiplicities. The multiplicities of the components of $\mathfrak{L}_{bd}$ will be determined in Theorem 4.1.1.

4.1 The Line Scheme $\mathfrak{L}_{bd}$

Similar to Section 3.2.2, to give a full description of $\mathfrak{L}_{bd}$ we must determine the multiplicity of each component. From [9], we know the degree of $\mathfrak{L}_{bd}$ is 20, and finding the multiplicity of each component corresponds to seeing how the degree is distributed among the lines of $\mathfrak{L}_{bd}$. To determine the multiplicity of each component, first consider the coordinate ring $\Bbbk[M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, M_{34}]/I$ of $\mathfrak{L}_{bd}$, where I is the ideal generated by the 46 polynomials in Section 6.2.2. By [20, Corollary 2.6] (cf. [9, Theorem 2.1]), to compute the multiplicity of each component, it suffices to find the dimension of the coordinate ring localized at one point lying on that component.

Consider any component $L_i \in \mathfrak{L}_{bd}$ and a point, p, on L_i such that p is not a point of intersection with any other component. We choose a homogeneous element f, in the coordinate ring of $\mathfrak{L}_{bd}$, such that $L_i \cap \mathcal{V}(f) = \{p\}$, $|\mathfrak{L}_{bd} \cap \mathcal{V}(f)| < \infty$ and no point of $\mathfrak{L}_{bd} \cap \mathcal{V}(f)$ is a point of intersection of the components of $\mathfrak{L}_{bd}$. Let $R = \Bbbk[M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, M_{34}]/J$ where $J = I + \langle f \rangle$, and let R_p denote R

localized at p. We wish to determine the dimension of R_p. In each case below, we will write m_{ij} and $\bar{J}$ for the images of M_{ij} and J (respectively) in R_p.

Theorem 4.1.1. *The line scheme $\mathfrak{L}_{bd}$ is the union of the following lines in $\mathbb{P}^5$:*

- $L_1 = \mathcal{V}(M_{12}, M_{13}, M_{14}, M_{23})$ *counted with multiplicity six,*
- $L_2 = \mathcal{V}(M_{12}, M_{13}, M_{23}, M_{24})$ *counted with multiplicity six,*
- $L_3 = \mathcal{V}(M_{12}, M_{13}, M_{23}, M_{34})$ *counted with multiplicity six, and*
- $L_4 = \mathcal{V}(M_{12}, dM_{13} - bM_{23}, dM_{14} - bM_{24}, 2bd^2M_{34} - (b^3 + d^3)M_{23})$ *counted with multiplicity two.*

Proof. We will begin with L_1 and consider the hyperplane determined by

$$f = M_{34} - xM_{24} + \beta_1 M_{12} + \beta_2 M_{13} + \beta_3 M_{14} + \beta_4 M_{23}$$

where $x, \beta_i \in \mathbb{k}^\times$ for all i and $\beta_3 \neq \frac{d}{b}x$. It follows that $\mathcal{V}(f) \cap L_1 = \{(0, 0, 0, 0, 1, x)\}$. The restrictions on x and the β_i ensure that $\mathcal{V}(f)$ does not contain any of the points in Corollary 3.3.2. Moreover, $|\mathcal{V}(f) \cap \mathfrak{L}_{bd}| = 4$. Using f to substitute for M_{34} and setting $M_{24} = 1$, we can compute a Gröbner basis of $\bar{J}$. This yields an element in the set:

$$m_{23}^5(2b^2\beta_3 d - 2bd^2x + \langle m_{23}\rangle)^2.$$

Since $\beta_3 \neq \frac{d}{b}x$, it follows m_{23}^5 is an element of $\bar{J}$. With this information, we compute a Gröbner basis and obtain the polynomial:

$$m_{14}^5(\beta_3 m_{14} - x)^5.$$

Since $\beta_3 m_{14} - x$ is nonzero at p, it follows that m_{14}^5 is an element of $\bar{J}$. With this, we can compute another Gröbner basis and obtain the polynomial:

$$m_{12}m_{14}^3x^2.$$

Since x is nonzero, it follows that $m_{12}m_{14}^3$ is an element of $\bar{J}$. Using this and introducing a variable y satisfying $xy = 1$, we compute one more Gröbner basis and obtain only five polynomials, three of which are:

$$m_{23}^5,$$

$$m_{23}(-\beta_3 m_{23}^3 - 2x\beta_4 m_{23}^3 - 2x^2 m_{23}^2 - bx^2\beta_3 m_{23}^3 + 4x^3 m_{13}),$$

$$-m_{23}^4 + 4x^2 m_{13}^2 - b^2 x^4 m_{23}^4,$$

and the other two polynomials belong to the sets:

$$8m_{12} + \Bbbk[m_{13}, m_{23}],$$

$$16m_{14} + \Bbbk[m_{13}, m_{23}].$$

The last two polynomials tell us $m_{12}, m_{14} \in \Bbbk[m_{13}, m_{23}]$. So $R_p \cong \Bbbk[m_{13}, m_{23}]/\bar{J}'$ where $\bar{J}'$ is generated by the first three of the latter five polynomials. Using Maxima with one of its packages, Affine, we can apply Bergman's Diamond Lemma to this quotient ring and find that R_p has dimension six. Thus L_1 has multiplicity six.

We will next consider the component L_3 and the hyperplane determined by

$$f = M_{14} - xM_{24} + \beta_1 M_{12} + \beta_2 M_{13} + \beta_3 M_{23} + \beta_4 M_{34}$$

where $x, \beta_i \in \Bbbk^\times$ for all i and $x \neq \frac{b}{d}$. It follows that $\mathcal{V}(f) \cap L_3 = \{(0, 0, x, 0, 1, 0)\}$. The restrictions on x and the β_i ensure that $\mathcal{V}(f)$ does not contain any of the points in Corollary 3.3.2. Moreover, $|\mathcal{V}(f) \cap \mathfrak{L}_{bd}| = 4$. Owing to computational limitations of Mathematica, we will begin by computing a Gröbner basis of I. Since $J = I + \langle f \rangle$, any information we find from a Gröbner basis of I can be considered for $\bar{J}$. We restrict to an affine open subset by setting $M_{24} = 1$, and, among the elements of a Gröbner basis for I, we see the two polynomials:

$$m_{14}^5(b - dm_{14})^2 m_{34}^5,$$

$$(b - dm_{14})^2 m_{23}^5.$$

Since $b - dm_{14}$ and m_{14} are nonzero at p, we have m_{34}^5 and m_{23}^5 are in $\bar{J}$. With this information, we are now able to compute another Gröbner basis for $\bar{J}$. In addition to using m_{34}^5 and m_{23}^5, we will substitute for m_{14} using f and we find that the polynomial

$$m_{12}m_{34}^3(\beta_4 m_{34} - x)$$

belongs to $\bar{J}$. Since $\beta_4 m_{34} - x$ is nonzero at p, it follows that $m_{12}m_{34}^3$ is an element of $\bar{J}$. We again introduce a variable y satisfying $xy = 1$ and compute one more Gröbner basis and obtain only five polynomials, three of which are:

$$m_{23}^5,$$

$$2m_{12}^2 - b^2 m_{23}^4 + 2bdxm_{23}^4 - d^2x^2m_{23}^4,$$

$$4xm_{12}m_{23} + 2bxm_{23}^3 - 2dx^2m_{23}^3 + \beta_4 dm_{23}^4 + 2\beta_3 dxm_{23}^4 + 2\beta_2 dx^2 m_{23}^4 + \beta_4 dx^3 m_{23}^4,$$

and the other two polynomials belong to the sets:

$$8m_{13} + \Bbbk[m_{23}],$$

$$16m_{34} + \Bbbk[m_{12}, m_{23}].$$

The last two polynomials imply $m_{13}, m_{34} \in \Bbbk[m_{12}, m_{23}]$. So $R_p \cong \Bbbk[m_{12}, m_{23}]/\bar{J}'$ where $\bar{J}'$ is generated by the first three of the latter five polynomials above. Using Maxima like before to apply Bergman's Diamond Lemma, we see R_p has dimension six so L_3 has multiplicity six.

We now consider L_4 and the hyperplane determined by:

$$f = bM_{23} - d(M_{13} + M_{14}) + \beta_1 M_{12} + \beta_2(dM_{13} - bM_{23}) + \beta_3(dM_{14} - bM_{24}) + \beta_4(2bd^2M_{34} - (b^3 + d^3)M_{23}),$$

where $\beta_i \in \Bbbk^\times$ for all i and $\beta_3 \neq 1$. It follows that

$$\mathcal{V}(f) \cap L_4 = \{(0, 2b^2d, 0, 2bd^2, 0, b^3 + d^3)\}.$$

The restrictions on the β_i ensure that $\mathcal{V}(f)$ does not contain any of the points in Corollary 3.3.2. Moreover, $|\mathcal{V}(f) \cap \mathfrak{L}_{bd}| = 4$. We restrict to an affine open subset by setting $M_{13} = 2b^2d$. Using f to substitute for M_{34}, a Gröbner basis yields the polynomial:

$$4b^4d^2m_{12}^2.$$

Since b and d are nonzero, m_{12}^2 is an element of $\bar{J}$. Using this information we compute one more Gröbner basis and obtain only four polynomials, two of which are:

$$m_{12}^2,$$

$$m_{23} - 2bd^2,$$

and the other polynomials belong to the sets:

$$2b^2dm_{24} + \Bbbk[m_{12}],$$

$$2bd^2m_{14} + \Bbbk[m_{12}].$$

The last two polynomials tell us $m_{24}, m_{14} \in \Bbbk[m_{12}]$. Thus, from the first two polynomials, we see $R_p \cong \Bbbk[x]/\langle x^2 \rangle$ which has dimension two. Therefore L_4 has multiplicity two.

By [9], $\deg(\mathfrak{L}_{bd}) = 20$, so it follows that L_2 has multiplicity six. However, we next compute the multiplicity of L_2 explicitly as an independent check. Recall from Theorem 3.1.1 that $A(b, d) \cong A(d, b)$. Thus, the line schemes of $A(b, d)$ and $A(d, b)$ are isomorphic, so $A(d, b)$ has a line variety that consists of four distinct lines. For clarity, we write L_i as $L_i(b, d)$ to show the dependence on the parameters $b, d \in \Bbbk^\times$.

The isomorphism map described in Theorem 3.1.1 corresponds to a map on each M_{ij} as follows:

$$M_{12} \mapsto -M_{12}, \quad M_{13} \mapsto -M_{23}, \quad M_{14} \mapsto M_{24},$$
$$M_{23} \mapsto -M_{13}, \quad M_{24} \mapsto M_{14}, \quad M_{34} \mapsto -M_{34}.$$

Notice that the line $L_1(b,d) = \mathcal{V}(M_{12}, M_{13}, M_{14}, M_{23}) \mapsto \mathcal{V}(M_{12}, M_{13}, M_{23}, M_{24}) = L_2(d,b)$. So we can conclude that, for all $b, d \in \Bbbk^{\times}$, the multiplicity of $L_2(d,b)$ is six. It follows that $L_2(b,d)$ has multiplicity six. $\square$

4.2 The Lines in $\mathbb{P}^3$ Parametrized by $\mathfrak{L}_{bd}$

In order to determine the lines in $\mathbb{P}^3$ that are parametrized by the line scheme, we first describe the Plücker embedding. Let l be any line in $\mathbb{P}^3$ and consider two points $a \neq b \in l$ and write $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$. We can represent l as a 2×4 matrix

$$M = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

of rank two such that the homogeneous coordinates of any point on l can be realized as a linear combination of the rows of M. We create coordinates for l using the 2×2 determinants of M, denoted $M_{ij} = a_i b_j - a_j b_i$ for $1 \leq i < j \leq 4$. Consequently, to describe the lines in $\mathbb{P}^3$ parametrized by $\mathfrak{L}_{bd}$, we will construct the 2×4 matrices associated to points on each component.

4.2.1 Lines in $\mathbb{P}^3$ Parametrized by L_1

For $L_1 = \mathcal{V}(M_{12}, M_{13}, M_{14}, M_{23})$, any line in $\mathbb{P}^3$ corresponding to a point on L_1 can be represented by:

$$\begin{pmatrix} 0 & a_2 & a_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $(a_2, a_3) \in \mathbb{P}^1$. Hence, L_1 parametrizes all lines on $\mathcal{V}(x_1)$ that contain e_4.

4.2.2 Lines in $\mathbb{P}^3$ Parametrized by L_2

For $L_2 = \mathcal{V}(M_{12}, M_{13}, M_{23}, M_{24})$, any line in $\mathbb{P}^3$ corresponding to a point on L_2 can be represented by:

$$\begin{pmatrix} a_1 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $(a_1, a_3) \in \mathbb{P}^1$. Hence, L_2 parametrizes all lines on $\mathcal{V}(x_2)$ that contain e_4.

4.2.3 Lines in $\mathbb{P}^3$ Parametrized by L_3

For $L_3 = \mathcal{V}(M_{12}, M_{13}, M_{23}, M_{34})$, any line in $\mathbb{P}^3$ corresponding to a point on L_3 can be represented by:

$$\begin{pmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $(a_1, a_2) \in \mathbb{P}^1$. Hence, L_3 parametrizes all lines on $\mathcal{V}(x_3)$ that contain e_4.

4.2.4 Lines in $\mathbb{P}^3$ Parametrized by L_4

For $L_4 = \mathcal{V}(M_{12}, dM_{13} - bM_{23}, dM_{14} - bM_{24}, 2bd^2M_{34} - (b^3 + d^3)M_{23})$, any line corresponding to a point on L_4 can be represented by:

$$\begin{pmatrix} 0 & 0 & a_3 & a_4 \\ 2b^2d & 2bd^2 & 0 & -(b^3 + d^3) \end{pmatrix}$$

where $(a_3, a_4) \in \mathbb{P}^1$. Hence, L_4 parametrizes all lines on $\mathcal{V}(dx_1 - bx_2)$ that contain $(2b^2d, 2bd^2, 0, -(b^3 + d^3))$.

4.3 Normalizing Sequences of $A(b,d)$

It was demonstrated in [6] and [24] that for two families of graded skew Clifford algebras that are also quadratic quantum $\mathbb{P}^3$s, there is a relationship between normalizing sequences in the algebra and the intersection points of the line scheme. By construction, the defining relations of a graded skew Clifford algebra determine a normalizing sequence consisting of four homogeneous elements of degree two. Now consider the ideal, N, generated by these four elements. In the examples of [6, 24], if I_p denotes the right ideal determined by an intersection point p of the line scheme, then $\dim_{\Bbbk}((I_p)_2 \cap N) = 2$. Furthermore, if $I_{p'}$ denotes the right ideal determined by a point p' of the line scheme that is *not* an intersection point, then $\dim_{\Bbbk}((I_{p'})_2 \cap N) = 1$. So in a way, distinguished points of the line scheme "highlight" distinguished elements of the algebra. Our algebra, $A(b,d)$, does not appear to be a graded skew Clifford algebra, but we wish to see if similar behavior is exhibited.

4.3.1 Normalizing Sequences of Degree-2 Elements

We begin by noting that $A(b,d)$ has two normalizing sequences consisting of four homogeneous elements of degree two. We will consider the two ideals generated by both normalizing sequences:

$$N_1 \quad \text{generated by} \quad \{x_1^2,\ x_2^2,\ dx_1x_3 - bx_2x_3,\ x_1x_2\},$$

$$N_2 \quad \text{generated by} \quad \{x_1^2,\ x_2^2,\ dx_1x_3 - bx_2x_3,\ x_3^2\}.$$

Next, we find the right ideals determined by each intersection point of $\mathfrak{L}_{bd}$. To outline the process, we will consider the point $L_2 \cap L_3 = \{E_3\}$ and denote its right ideal as I_{E_3}. Similar to our discussion in Section 4.2, the line in $\mathbb{P}^3$ to which this point corresponds is represented by the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence, we seek two linearly independent elements of $A(b,d)_1$ that vanish on e_1 and e_4. So $I_{E_3} = x_2A(b,d) + x_3A(b,d)$. We apply the same process to each of the intersection points listed in Corollary 3.3.2 and we write $p = (0,0,b,0,d,0)$.

$$E_5 \longrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow I_{E_5} = x_1A(b,d) + x_3A(b,d),$$

$$E_6 \longrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow I_{E_6} = x_1A(b,d) + x_2A(b,d),$$

$$p \longrightarrow \begin{pmatrix} b & d & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow I_p = (dx_1 - bx_2)A(b,d) + x_3A(b,d).$$

We are interested in elements in each right ideal that are homogeneous of degree two. Since each of these right ideals determines a line module over $A(b,d)$, their degree-two subspaces have dimension seven and can be described as:

$$(I_{E_3})_2 = \Bbbk x_1x_2 \oplus \Bbbk x_2^2 \oplus \Bbbk x_2x_3 \oplus \Bbbk x_2x_4 \oplus \Bbbk x_1x_3 \oplus \Bbbk x_3^2 \oplus \Bbbk x_3x_4,$$

$$(I_{E_5})_2 = \Bbbk x_1^2 \oplus \Bbbk x_1x_2 \oplus \Bbbk x_1x_3 \oplus \Bbbk x_1x_4 \oplus \Bbbk x_2x_3 \oplus \Bbbk x_3^2 \oplus \Bbbk x_3x_4,$$

$$(I_{E_6})_2 = \Bbbk x_1^2 \oplus \Bbbk x_1x_2 \oplus \Bbbk x_1x_3 \oplus \Bbbk x_1x_4 \oplus \Bbbk x_2^2 \oplus \Bbbk x_2x_3 \oplus \Bbbk x_2x_4,$$

$$(I_p)_2 = \Bbbk x_1x_3 \oplus \Bbbk x_2x_3 \oplus \Bbbk x_3^2 \oplus \Bbbk x_3x_4 \oplus \Bbbk(dx_1^2 + bx_1x_2) \oplus \Bbbk(dx_1x_2 - bx_2^2) \oplus \Bbbk(dx_1x_4 - bx_2x_4).$$

It follows that the intersection of N_1 and the degree-two subspaces of each right ideal are:

$$(I_{E_3})_2 \cap N_1 = \Bbbk x_1x_2 \oplus \Bbbk x_2^2 \oplus \Bbbk(dx_1x_3 - bx_2x_3),$$

$$(I_{E_5})_2 \cap N_1 = \Bbbk x_1^2 \oplus \Bbbk x_1x_2 \oplus \Bbbk(dx_1x_3 - bx_2x_3),$$

$$(I_{E_6})_2 \cap N_1 = \Bbbk x_1^2 \oplus \Bbbk x_1 x_2 \oplus \Bbbk x_2^2 \oplus \Bbbk(dx_1x_3 - bx_2x_3) = (N_1)_2,$$

$$(I_p)_2 \cap N_1 = \Bbbk(dx_1^2 + bx_1x_2) \oplus \Bbbk(dxy - by^2) \oplus \Bbbk(dx_1x_3 - bx_2x_3).$$

Hence, we find the dimensions of the subspaces to be

$$\dim_{\Bbbk}((I_{E_3})_2 \cap N_1) = \dim_{\Bbbk}((I_{E_5})_2 \cap N_1) = \dim_{\Bbbk}((I_p)_2 \cap N_1) = 3$$

and $\dim_{\Bbbk}((I_{E_6})_2 \cap N_1) = 4$.

The intersection of N_2 and the degree-two subspaces are:

$$(I_{E_3})_2 \cap N_2 = \Bbbk x_2^2 \oplus \Bbbk x_3^2 \oplus \Bbbk(dx_1x_3 - bx_2x_3),$$

$$(I_{E_5})_2 \cap N_2 = \Bbbk x_1^2 \oplus \Bbbk x_3^2 \oplus \Bbbk(dx_1x_3 - bx_2x_3),$$

$$(I_{E_6})_2 \cap N_2 = \Bbbk x_1^2 \oplus \Bbbk x_2^2 \oplus \Bbbk(dx_1x_3 - bx_2x_3),$$

$$(I_p)_2 \cap N_2 = \Bbbk(d^2x_1^2 + b^2x_2^2) \oplus \Bbbk x_3^2 \oplus \Bbbk(dx_1x_3 - bx_2x_3).$$

Each intersection has dimension three in this case.

Next, we will determine the intersection of N_1 and N_2 with right ideals determined by points that lie on only one component of the line scheme. We determine these right ideals in a similar manner as before and will outline the process for L_1. Consider any point on $L_1 \setminus \{E_5, E_6\}$. Such a point corresponds to a line in $\mathbb{P}^3$ given by the matrix:

$$\begin{pmatrix} 0 & 1 & \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\alpha \in \Bbbk^\times$. We denote the right ideal determined by any one of these points as J_{L_1}. Again, we need two elements that are homogeneous of degree one to determine this right ideal, so $J_{L_1} = x_1 A(b, d) + (\alpha x_2 - x_3)A(b, d)$. Applying this to the other components of the line scheme we find:

$$L_2 \setminus \{E_3, E_6\} \longrightarrow \begin{pmatrix} 1 & 0 & \beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow J_{L_2} = x_2 A(b,d) + (\beta x_1 - x_3) A(b,d)$$

$$L_3 \setminus \{E_3, E_5, p\} \longrightarrow \begin{pmatrix} 1 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow J_{L_3} = x_3 A(b,d) + (\gamma x_1 - x_2) A(b,d)$$

$$L_4 \setminus \{p\} \longrightarrow \begin{pmatrix} 0 & 0 & 1 & \delta \\ 2b^2d & 2bd^2 & 0 & -(b^3+d^3) \end{pmatrix}$$

$$J_{L_4} = (dx_1 - bx_2)A(b,d) + ((b^3+d^3)x_2 - 2bd^2(\delta x_3 - x_4))A(b,d)$$

where $\beta, \gamma, \delta \in \Bbbk^\times$ and $\gamma \neq \frac{d}{b}$. We now describe their degree-two subspaces, which, like previously, have dimension seven:

$$(J_{L_1})_2 = \Bbbk x_1^2 \oplus \Bbbk x_1x_2 \oplus \Bbbk x_1x_3 \oplus \Bbbk x_1x_4 \oplus \Bbbk(\alpha x_2^2 - x_2x_3) \oplus \Bbbk(\alpha x_2x_3 - x_3^2) \oplus \Bbbk(\alpha x_2x_4 - x_3x_4),$$

$$(J_{L_2})_2 = \Bbbk x_1x_2 \oplus \Bbbk x_2^2 \oplus \Bbbk x_2x_3 \oplus \Bbbk x_2x_4 \oplus \Bbbk(\beta x_1^2 - x_1x_3) \oplus \Bbbk(\beta x_1x_3 - x_3^2) \oplus \Bbbk(\beta x_1x_4 - x_3x_4),$$

$$(J_{L_3})_2 = \Bbbk x_1x_3 \oplus \Bbbk x_2x_3 \oplus \Bbbk x_3^2 \oplus \Bbbk x_3x_4 \oplus \Bbbk(\gamma x_1^2 + x_1x_2) \oplus \Bbbk(\gamma x_1x_2 - x_2^2) \oplus \Bbbk(\gamma x_1x_4 - x_2x_4),$$

$$\begin{aligned} (J_{L_4})_2 = \; & \Bbbk(dx_1^2 + bx_1x_2) \oplus \Bbbk(dx_1x_2 - bx_2^2) \oplus \Bbbk(dx_1x_3 - bx_2x_3) \oplus \Bbbk(dx_1x_4 - bx_2x_4) \oplus \\ & \oplus \Bbbk(2bd^2x_1^2 + (b^3+d^3)x_2^2 - 2\delta bd^2x_2x_3 + 2bd^3x_3^2 - 2bd^2x_2x_4) \oplus \\ & \oplus \Bbbk(2bd^2x_1x_2 + (b^3+d^3)x_2x_3 - 2\delta bd^2x_3^2 + 2bd^2x_3x_4) \oplus \\ & \oplus \Bbbk((b^3+d^3)x_2x_4 - 2\delta bd^2x_3x_4 + 2bd^2x_4^2). \end{aligned}$$

It follows that the intersection of N_1 with each degree-two subspace is:

$$(J_{L_1})_2 \cap N_1 = \Bbbk x_1^2 \oplus \Bbbk x_1x_2 \oplus \Bbbk(dx_1x_3 + \alpha bx_2^2 - bx_2x_3),$$

$$(J_{L_2})_2 \cap N_1 = \Bbbk x_1x_2 \oplus \Bbbk x_2^2 \oplus \Bbbk(d\beta x_1^2 - dx_1x_3 + bx_2x_3),$$

$$(J_{L_3})_2 \cap N_1 = \Bbbk(\gamma x_1^2 + x_1x_2) \oplus \Bbbk(\gamma x_1x_2 - x_2^2) \oplus \Bbbk(dx_1x_3 - bx_2x_3),$$

$$(J_{L_4})_2 \cap N_1 = \Bbbk(dx_1^2 + bx_1x_2) \oplus \Bbbk(dx_1x_2 - bx_2^2) \oplus \Bbbk(dx_1x_3 - bx_2x_3).$$

We see each intersection has dimension three.

On the other hand, the intersection of N_2 with each degree-two subspace is:

$$(J_{L_1})_2 \cap N_2 = \Bbbk x_1^2 \oplus \Bbbk(dx_1x_3 + \alpha bx_2^2 - bx_2x_3) \oplus \Bbbk(dx_1x_3 - bx_2x_3 + \alpha^{-1}bx_3^2),$$

$$(J_{L_2})_2 \cap N_2 = \Bbbk x_2^2 \oplus \Bbbk(\beta dx_1^2 - dx_1x_3 + bx_2x_3) \oplus \Bbbk(dx_1x_3 - bx_2x_3 - \beta^{-1}dx_3^2),$$

$$(J_{L_3})_2 \cap N_2 = \Bbbk x_3^2 \oplus \Bbbk(\gamma^2x_1^2 + x_2^2) \oplus \Bbbk(dx_1x_3 - bx_2x_3),$$

$$(J_{L_4})_2 \cap N_2 = \Bbbk(d^2x_1^2 + b^2x_2^2) \oplus \Bbbk(dx_1x_3 - bx_2x_3).$$

We see each intersection has dimension three except $\dim_{\Bbbk}((J_{L_4})_2 \cap N_2) = 2$. Unfortunately, there appears not to be a clear interplay between these normalizing sequences in $A(b, d)$ and intersection points of the line scheme like that demonstrated in [6] and [24]. One could consider a few more normalizing sequences of greater length, such as:

$$\{x_1^2,\ x_2^2,\ dx_1x_3 - bx_2x_3,\ x_1x_2,\ dx_1x_4 - bx_2x_4\},$$

$$\{x_1^2,\ x_2^2,\ dx_1x_3 - bx_2x_3,\ x_1x_2,\ x_3^2\},$$

$$\{x_1^2,\ x_2^2,\ dx_1x_3 - bx_2x_3,\ x_1x_2,\ x_3^2,\ dx_1x_4 - bx_2x_4\},$$

but we leave that to future work.

4.3.2 An Interesting Normalizing Sequence

To conclude this section, we will consider one last normalizing sequence. As previously stated, graded skew Clifford algebras of global dimension four are equipped, by definition, with a normalizing sequence consisting of four homogeneous elements of degree two. So in Section 4.3.1, we considered such normalizing sequences. What was not considered is that the normalizing sequence of a graded skew Clifford algebra satisfies a certain property that we will try to duplicate in this section. Suppose B is a quadratic AS-regular graded skew Clifford algebra of global dimension four and that $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is its normalizing sequence (provided by the definition of graded skew Clifford algebra) consisting of homogeneous elements of degree two. In [5], it is shown that $B/\langle \alpha_1, \ldots, \alpha_4 \rangle$ is the Koszul dual of a quadratic AS-regular algebra of global dimension four. So in this section, we will consider a normalizing sequence that mirrors this property.

Consider the elements $x_1^2,\ x_2^2,\ dx_1 - bx_2,\ x_3^2,\ x_4^2$ in $A(b,d)$. Indeed, these elements form a normalizing sequence of $A(b,d)$. Furthermore, if N is the ideal generated by these elements, then $A(b,d)/N$ is isomorphic to a factor ring of a quadratic quantum $\mathbb{P}^2$. We can write this factor ring using generators z_1, z_2, z_3 and defining relations:

$$\begin{aligned} z_2 z_1 &= z_1 z_2, & z_1^2 &= 0, \\ z_3 z_1 &= -z_1 z_3, & z_2^2 &= 0, \\ z_3 z_2 &= z_2 z_3, & z_3^2 &= 0. \end{aligned}$$

Thus, if C is the quadratic AS-regular algebra generated by $\{y_1,\ y_2,\ y_3\}$ with defining relations:

$$y_2 y_1 = -y_1 y_2, \quad y_3 y_1 = y_1 y_3, \quad y_3 y_2 = -y_2 y_3,$$

then $A(b,d)/N \cong C^!$.

Next, we will determine the intersection of N with the right ideals discussed in Section 4.3.1. The degree-two subspace of N is:

$$(N)_2 = \Bbbk x_1^2 \oplus \Bbbk x_1 x_2 \oplus \Bbbk(dx_1x_3 - bx_2x_3) \oplus \Bbbk(dx_1x_4 - bx_2x_4) \oplus \Bbbk x_2^2 \oplus \Bbbk x_3^2 \oplus \Bbbk x_4^2.$$

We can describe the intersection of $(N)_2$ with the right ideals associated to the intersection points of $\mathfrak{L}_{bd}$ by:

$$(I_{E_3})_2 \cap (N)_2 = \Bbbk x_1x_2 \oplus \Bbbk(dx_1x_3 - bx_2x_3) \oplus \Bbbk x_2^2 \oplus \Bbbk x_3^2,$$

$$(I_{E_5})_2 \cap (N)_2 = \Bbbk x_1^2 \oplus \Bbbk x_1x_2 \oplus \Bbbk(dx_1x_3 - bx_2x_3) \oplus \Bbbk x_3^2,$$

$$(I_{E_6})_2 \cap (N)_2 = \Bbbk x_1^2 \oplus \Bbbk x_1x_2 \oplus \Bbbk(dx_1x_3 - bx_2x_3) \oplus \Bbbk(dx_1x_4 - bx_2x_4) \oplus \Bbbk x_2^2,$$

$$(I_p)_2 \cap (N)_2 = \Bbbk(dx_1^2+bx_1x_2)\oplus\Bbbk(dx_1x_2-bx_2^2)\oplus\Bbbk(dx_1x_3-bx_2x_3)\oplus\Bbbk(dx_1x_4-bx_2x_4)\oplus\Bbbk x_3^2.$$

The first two subspaces have dimension four and the last two have dimension five.

The intersection of $(N)_2$ with all the other right ideals are:

$$(J_{L_1})_2 \cap (N)_2 = \Bbbk x_1^2 \oplus \Bbbk x_1x_2 \oplus \Bbbk(\alpha^2x_2^2 - x_3^2) \oplus \Bbbk(\alpha bx_2^2 + dx_1x_3 - bx_2x_3),$$

$$(J_{L_2})_2 \cap (N)_2 = \Bbbk x_1x_2 \oplus \Bbbk(\beta^2x_1^2 - x_3^2) \oplus \Bbbk x_2^2 \oplus \Bbbk(dx_1x_3 - bx_2x_3 - \beta dx_1^2),$$

$$(J_{L_3})_2 \cap (N)_2 = \Bbbk(\gamma x_1^2 + x_1x_2) \oplus \Bbbk(\gamma x_1x_2 - x_2^2) \oplus \Bbbk(dx_1x_3 - bx_2x_3) \oplus \Bbbk x_3^2.$$

The description of $J_{L_4} \cap (N)_2$ is not user friendly, so we will not write it explicitly. Nevertheless, the dimension of $\dim((J_{L_4})_2 \cap (N)_2) = 5$ since one can show $\dim((J_{L_4} + (N)_2) = 9$. Initial findings of this discussion seem to show that L_4 is "highlighting" this normalizing sequence.

4.4 A Subalgebra of $A(b,d)$

In this section, we will let B denote the quadratic quantum $\mathbb{P}^2$ generated by $\{x_1, x_2, x_3\}$ subject to the defining relations:

$$x_2x_1 = -x_1x_2, \qquad x_3x_1 = x_1x_3, \qquad x_3x_2 = x_2x_3.$$

In [23, Proposition 2.1], it is shown that $A(b,d)$ is an Ore extension of B, which allows us to conclude that B is a subalgebra of $A(b,d)$. As B is a quadratic quantum

$\mathbb{P}^2$, we begin by describing its point scheme in a manner similar to that in Section 3.2.1. We factor the defining relations as Mx where $x = [x_1,\ x_2,\ x_3]^T$ and M is the 3-by-3 matrix:

$$M = \begin{bmatrix} x_2 & x_1 & 0 \\ x_3 & 0 & -x_1 \\ 0 & x_3 & -x_2 \end{bmatrix}.$$

The point scheme of B can be computed as the zero locus in $\mathbb{P}^2$ where

$$\det(M) = 2x_1x_2x_3 = 0.$$

Thus, it is the union of three lines: $\mathcal{V}(x_1) \cup \mathcal{V}(x_2) \cup \mathcal{V}(x_3) \subset \mathbb{P}^2$. We remark that the point scheme of B consists of a "triangle" of three lines in $\mathbb{P}^2$ and that the line scheme of $A(b, d)$ contains a "triangle" of three lines in $\mathbb{P}^5$. This type of behavior is displayed in [17], where a quadratic quantum $\mathbb{P}^3$ has a subalgebra isomorphic to a quadratic quantum $\mathbb{P}^2$ and there are similar components in each respective scheme.

We now seek to show that if $\mathcal{A}$ is a quadratic quantum $\mathbb{P}^3$ with a subalgebra $\mathcal{B}$ isomorphic to a quadratic quantum $\mathbb{P}^2$, then the point variety of $\mathcal{B}$ is embedded in the line variety of $\mathcal{A}$. On the level of point modules and line modules, this would mean that a point module of $\mathcal{B}$ determines a line module of $\mathcal{A}$. We first consider the following lemma to compare certain subspaces of degree-two elements.

Lemma 4.4.1. *Suppose that* $\mathcal{A} = \bigoplus_{i=0}^{\infty} \mathcal{A}_i$ *is a quadratic algebra where* $\dim(\mathcal{A}_1) = 4$ *and that* $\mathcal{A}$ *contains a quadratic subalgebra* $\mathcal{B} = \bigoplus_{i=0}^{\infty} \mathcal{B}_i$ *where* $\dim(\mathcal{B}_1) = 3$ *such that* $\mathcal{B}_1 \subset \mathcal{A}_1$. *If* $u, v \in \mathcal{B}_1$ *satisfy* $\dim(u\mathcal{B}_1 + v\mathcal{B}_1) \leq 5$, *then dim*$(u\mathcal{A}_1 + v\mathcal{A}_1) \leq 7$.

Proof. Since $\mathcal{B}_1 \subset \mathcal{A}_1$ and $\dim(\mathcal{B}_1) = 3$, we may write $\mathcal{A}_1 = \mathcal{B}_1 \bigoplus \Bbbk a$ for some $a \in \mathcal{A}_1 \setminus \mathcal{B}_1$. It follows that $u\mathcal{A}_1 + v\mathcal{A}_1 = u\mathcal{B}_1 + v\mathcal{B}_1 + \Bbbk ua + \Bbbk va$. Thus

$$\dim(u\mathcal{A}_1 + v\mathcal{A}_1) \leq \dim(u\mathcal{B}_1 + v\mathcal{B}_1) + \dim(\Bbbk ua + \Bbbk va).$$

Since $\dim(\Bbbk ua + \Bbbk va) \leq 2$, it follows that $\dim(u\mathcal{A}_1 + v\mathcal{A}_1) \leq 7$. $\square$

Theorem 4.4.2. *Suppose $\mathcal{A} = \bigoplus_{i=0}^{\infty} \mathcal{A}_i$ is a quadratic Auslander-regular algebra of global dimension four that satisfies the Cohen Macaulay property with Hilbert series $(1-t)^{-4}$. If $\mathcal{A}$ contains a quadratic AS-regular subalgebra $\mathcal{B} = \bigoplus_{i=0}^{\infty} \mathcal{B}_i$ of global dimension three, where $\mathcal{B}_1 \subset \mathcal{A}_1$, then the point variety of $\mathcal{B}$ embeds in the line variety of $\mathcal{A}$.*

Proof. Let M be a right point module over $\mathcal{B}$. Since $\mathcal{B}$ is a quadratic AS-regular algebra, then $M \cong \frac{\mathcal{B}}{u\mathcal{B}+v\mathcal{B}}$ where $u, v \in \mathcal{B}_1$ are linearly independent and $\dim(u\mathcal{B}_1 + v\mathcal{B}_1) = 5$. The Hilbert series of $\mathcal{A}$ implies that $\dim(\mathcal{A}_1) = 4$, so the previous lemma implies that $\dim(u\mathcal{A}_1 + v\mathcal{A}_1) \leq 7$. Since $\mathcal{A}$ is connected and Auslander regular, it follows from [14, Theorem 4.8] that $\mathcal{A}$ is a domain. Thus, $u\mathcal{A}_1 \cap v\mathcal{A}_1 \neq 0$ and [15, Proposition 2.8] implies that $\frac{\mathcal{A}}{u\mathcal{A}+v\mathcal{A}}$ is a line module over $\mathcal{A}$. That is, point modules over the subalgebra $\mathcal{B}$ determine line modules over $\mathcal{A}$, or in other words, the point variety of $\mathcal{B}$ embeds in the line variety of $\mathcal{A}$. $\square$

Corollary 4.4.3. *Suppose $\mathcal{A} = \bigoplus_{i=0}^{\infty} \mathcal{A}_i$ is a quadratic Auslander-regular algebra of global dimension four that satisfies the Cohen Macaulay property with Hilbert series $(1-t)^{-4}$. If $\mathcal{A}$ is an Ore extension of a quadratic AS-regular algebra $\mathcal{B}$ of global dimension three, then the point variety of $\mathcal{B}$ embeds in the line variety of $\mathcal{A}$. In particular, the point variety of B embeds in the line variety of $A(b, d)$.*

Proof. The result follows from Theorem 4.4.2 and the Ore-extension construction. $\square$

Corollary 4.4.3 and the analyses in [6, 24] suggest that perhaps components or points in the line scheme of a quadratic quantum $\mathbb{P}^3$ that have multiplicity strictly greater than one are encoding algebraic properties of the quadratic quantum $\mathbb{P}^3$ (cf. Question 5.0.1).

CHAPTER 5

Closing Remarks

The classification of quantum $\mathbb{P}^3$s is a challenging open problem and detailed analyses, such as this work, will be paramount. Within this problem, there are many other interesting questions to ask, such as: does there exist a quadratic quantum $\mathbb{P}^3$ that is not noetherian? Does there exist a quadratic quantum $\mathbb{P}^3$ that is not Auslander-regular? Geometric methods have been shown to be fruitful in answering various analogous questions. In [22], geometric methods, motivated by [2, 3], were successful in proving that the universal enveloping algebra of the Witt algebra is not noetherian.

It is believed by the author, and many others, that the line scheme and point scheme of a quadratic quantum $\mathbb{P}^3$ will reveal hidden algebraic properties that could perhaps answer questions like those above. Indeed, stemming from this work directly, one could ask the following questions.

Question 5.0.1. Regarding Section 4.3, does there exist a relationship between normalizing sequences of a quadratic quantum $\mathbb{P}^3$ and its point scheme or line scheme? Based on our work in Section 4.3, if a component (respectively, point) of the line scheme of a quadratic quantum $\mathbb{P}^3$ has multiplicity two or more, does the component (respectively, point) necessarily determine right ideals that have substantial intersection with ideals determined by normalizing sequences?

Question 5.0.2. Regarding the algebras presented in Section 3.1 constructed via an Ore extension, $A \cong [B; \sigma, \delta]$, do σ and δ determine any properties of the point scheme or line scheme of A?

Question 5.0.3. Do the applications of Theorem 4.4.2 for an algebra that is not an Ore extension differ from those for an Ore extension?

Question 5.0.4. In [5], the notion of a graded skew Clifford algebra was introduced, together with necessary and sufficient conditions to determine when such an algebra is Auslander regular. Moreover, the construction of such an algebra sometimes allows one to control the number of point modules it possesses. So it seems reasonable to ask: what techniques, if any, could one use to control the components in the line scheme of a quadratic quantum $\mathbb{P}^3$?

Question 5.0.5. Related to the previous question, we can also ask: does there exist a quadratic quantum $\mathbb{P}^3$ that has a point scheme consisting of twenty distinct points and a one-dimensional line scheme consisting of twenty distinct lines?

CHAPTER 6

Appendix

In this appendix, we list the various polynomials that define the schemes of interest.

6.1 Point Scheme Polynomials

1. $2x_1^2x_2x_3$
2. $2x_1x_2^2x_3$
3. $-2x_1x_2x_3^2$
4. $-x_1(-ax_2^2x_3 - bx_2x_3^2 + cx_1^2x_3 + dx_1x_3^2 + x_1^3 + 2x_1x_2x_4 + x_2^3)$
5. $x_1(-ax_2x_3^2 - bx_3^3 + x_1^2x_2 + 2x_1x_3x_4 + x_2^2x_3)$
6. $-x_1(cx_1x_3^2 + dx_3^3 + x_1^2x_3 - x_1x_2^2)$
7. $-x_2(ax_2^2x_3 + bx_2x_3^2 - cx_1^2x_3 - dx_1x_3^2 + x_1^3 + 2x_1x_2x_4 + x_2^3)$
8. $x_2(ax_2x_3^2 + bx_3^3 + x_1^2x_2 + x_2^2x_3)$
9. $-x_2(-cx_1x_3^2 - dx_3^3 + x_1^2x_3 - x_1x_2^2 + 2x_2x_3x_4)$
10. $-ax_1^2x_2x_3 + ax_1x_2^3 - ax_2^2x_3x_4 - bx_1^2x_3^2 + bx_1x_2^2x_3 - bx_2x_3^2x_4 - cx_1^3x_2 - cx_1^2x_3x_4 - cx_1x_2^2x_3 - dx_1^2x_2x_3 - dx_1x_3^2x_4 - dx_2^2x_3^2 + x_1^3x_4 + 2x_1x_2x_4^2 + x_2^3x_4$
11. $x_3(-ax_2^2x_3 - bx_2x_3^2 + cx_1^2x_3 + dx_1x_3^2 + x_1^3 - x_2^3)$
12. $x_3(ax_2x_3^2 + bx_3^3 + x_1^2x_2 - 2x_1x_3x_4 + x_2^2x_3)$
13. $x_3(cx_1x_3^2 + dx_3^3 + x_1^2x_3 + x_1x_2^2 - 2x_2x_3x_4)$

14. $ax_1x_2^2x_3 - ax_2x_3^2x_4 + bx_1x_2x_3^2 - bx_3^3x_4 - cx_1^3x_3 - cx_1x_2x_3^2 - dx_1^2x_3^2 - dx_2x_3^3 - x_1^4 - x_1^2x_2x_3 - x_1^2x_2x_4 + 2x_1x_3x_4^2 + x_2^2x_3x_4$

15. $-ax_1x_2x_3^2 - bx_1x_3^3 - cx_1x_3^2x_4 - dx_3^3x_4 - x_1^3x_2 + x_1^2x_3x_4 - x_1x_2^2x_3 - x_1x_2^2x_4 + 2x_2x_3x_4^2$

6.2 Line Scheme Polynomials

6.2.1 Polynomials when a, b, c and d arbitrary

1. $-M_{12}(dM_{13}^3 - bM_{23}M_{13}^2 - 2M_{12}M_{14}M_{13} - aM_{12}M_{23}M_{13} - M_{12}^2M_{23})$
2. $-M_{13}(dM_{13}^3 - bM_{23}M_{13}^2 - 2M_{12}M_{14}M_{13} - aM_{12}M_{23}M_{13} - M_{12}^2M_{23})$
3. $-M_{12}^2(M_{12}^2 - cM_{13}^2 + aM_{23}^2)$
4. $M_{13}^2(M_{12}^2 - cM_{13}^2 + aM_{23}^2)$
5. $-M_{23}(dM_{13}^3 - bM_{23}M_{13}^2 - 2M_{12}M_{14}M_{13} - aM_{12}M_{23}M_{13} - M_{12}^2M_{23})$
6. $-M_{23}^2(M_{12}^2 - cM_{13}^2 + aM_{23}^2)$
7. $-M_{12}(-bM_{23}^3 + dM_{13}M_{23}^2 - cM_{12}M_{13}M_{23} - 2M_{12}M_{24}M_{23} + M_{12}^2M_{13})$
8. $M_{13}(-bM_{23}^3 + dM_{13}M_{23}^2 - cM_{12}M_{13}M_{23} - 2M_{12}M_{24}M_{23} + M_{12}^2M_{13})$
9. $-M_{23}(-bM_{23}^3 + dM_{13}M_{23}^2 - cM_{12}M_{13}M_{23} - 2M_{12}M_{24}M_{23} + M_{12}^2M_{13})$
10. $M_{14}M_{23} - M_{13}M_{24} + M_{12}M_{34}$
11. $M_{13}M_{12}^3 - M_{23}^2M_{12}^2 + M_{13}M_{34}M_{12}^2 - cM_{13}^3M_{12} - M_{13}M_{14}M_{23}M_{12} - M_{13}^2M_{24}M_{12} - bM_{13}^2M_{23}^2 + dM_{13}^3M_{23}$
12. $M_{12}^4 + aM_{13}M_{12}^3 + bM_{13}^2M_{12}^2 - cM_{13}^2M_{12}^2 + M_{14}M_{23}M_{12}^2 + aM_{13}M_{34}M_{12}^2 - acM_{13}^3M_{12} + 2M_{13}M_{14}^2M_{12} - aM_{13}^2M_{24}M_{12} - bcM_{13}^4 - dM_{13}^3M_{14} + adM_{13}^3M_{23} + bM_{13}^2M_{14}M_{23}$
13. $M_{23}M_{12}^3 - M_{13}^2M_{12}^2 + M_{23}M_{34}M_{12}^2 + aM_{23}^3M_{12} + M_{14}M_{23}^2M_{12} + M_{13}M_{23}M_{24}M_{12} + bM_{13}M_{23}^3 - dM_{13}^2M_{23}^2$

14. $M_{12}^4 - cM_{23}M_{12}^3 + aM_{23}^2M_{12}^2 + dM_{23}^2M_{12}^2 - M_{13}M_{24}M_{12}^2 - cM_{23}M_{34}M_{12}^2 - acM_{23}^3M_{12} - cM_{14}M_{23}^2M_{12} + 2M_{23}M_{24}^2M_{12} + adM_{23}^4 - bcM_{13}M_{23}^3 + bM_{23}^3M_{24} - dM_{13}M_{23}^2M_{24}$

15. $-M_{12}(-M_{13}^3 + M_{12}M_{23}M_{13} + 2M_{23}M_{34}M_{13} - M_{23}^3)$

16. $M_{12}M_{13}^3 - cM_{23}M_{13}^3 - M_{23}M_{24}M_{13}^2 + aM_{23}^3M_{13} + M_{14}M_{23}^2M_{13} - M_{12}M_{23}M_{34}M_{13} + M_{12}M_{23}^3$

17. $M_{13}M_{12}^3 + aM_{13}^2M_{12}^2 - cM_{13}M_{23}M_{12}^2 - M_{23}M_{24}M_{12}^2 + bM_{13}^3M_{12} + dM_{13}M_{23}^2M_{12} - acM_{13}^2M_{23}M_{12} + 2M_{13}M_{14}M_{24}M_{12} - aM_{13}M_{23}M_{24}M_{12} - bM_{13}M_{23}M_{34}M_{12} + adM_{13}^2M_{23}^2 + bM_{13}M_{14}M_{23}^2 - bcM_{13}^3M_{23} - dM_{13}^3M_{24}$

18. $M_{14}M_{12}^3 - dM_{13}^2M_{12}^2 + cM_{13}M_{23}M_{12}^2 - M_{23}M_{24}M_{12}^2 - M_{14}M_{34}M_{12}^2 - aM_{23}M_{34}M_{12}^2 + cM_{13}^2M_{14}M_{12} + acM_{13}^2M_{23}M_{12} - M_{14}^2M_{23}M_{12} - M_{13}M_{14}M_{24}M_{12} + dM_{13}^2M_{34}M_{12} - bM_{13}M_{23}M_{34}M_{12} - adM_{13}^2M_{23}^2 - bM_{13}M_{14}M_{23}^2 + bcM_{13}^3M_{23} + dM_{13}^2M_{14}M_{23}$

19. $-M_{23}M_{12}^3 + cM_{23}^2M_{12}^2 - M_{13}M_{14}M_{12}^2 - aM_{13}M_{23}M_{12}^2 - dM_{23}^3M_{12} + acM_{13}M_{23}^2M_{12} - bM_{13}^2M_{23}M_{12} + cM_{13}M_{14}M_{23}M_{12} - 2M_{14}M_{23}M_{24}M_{12} + dM_{13}M_{23}M_{34}M_{12} - adM_{13}M_{23}^3 - bM_{14}M_{23}^3 + bcM_{13}^2M_{23}^2 + dM_{13}^2M_{23}M_{24}$

20. $M_{13}(M_{13}^3 - M_{12}M_{23}M_{13} - 2M_{23}M_{34}M_{13} + M_{23}^3)$

21. $M_{23}M_{12}^3 + M_{13}M_{14}M_{12}^2 + aM_{13}M_{23}M_{12}^2 - dM_{13}^3M_{12} + M_{14}M_{23}^2M_{12} + bM_{13}^2M_{23}M_{12} - M_{13}M_{23}M_{24}M_{12} - M_{13}M_{14}M_{34}M_{12} + cM_{13}^3M_{14} + M_{13}M_{14}^2M_{23} - M_{13}^2M_{14}M_{24} - aM_{13}^2M_{23}M_{24} + dM_{13}^3M_{34} - bM_{13}^2M_{23}M_{34}$

22. $bM_{13}^4 + aM_{12}M_{13}^3 - dM_{34}M_{13}^3 + M_{12}^2M_{13}^2 + dM_{23}^2M_{13}^2 - bM_{12}M_{23}M_{13}^2 - bM_{23}M_{34}M_{13}^2 - aM_{12}^2M_{23}M_{13} - M_{12}M_{23}M_{24}M_{13} + 2M_{12}M_{14}M_{34}M_{13} - aM_{12}M_{23}M_{34}M_{13} - M_{12}M_{14}M_{23}^2 - M_{12}^3M_{23}$

23. $-M_{14}M_{13}^3 + dM_{23}M_{13}^3 - bM_{23}^2M_{13}^2 - aM_{12}M_{23}^2M_{13} - M_{12}M_{14}M_{23}M_{13} + 2M_{14}M_{23}M_{34}M_{13} - M_{14}M_{23}^3 - M_{12}^2M_{23}^2$

24. $-M_{23}(M_{13}^3 - M_{12}M_{23}M_{13} - 2M_{23}M_{34}M_{13} + M_{23}^3)$

25. $bM_{23}M_{13}^3+M_{12}M_{14}M_{13}^2+cM_{12}M_{34}M_{13}^2-dM_{23}M_{34}M_{13}^2+dM_{23}^3M_{13}-bM_{12}M_{23}^2M_{13}-cM_{12}M_{23}^2M_{13} + M_{12}^2M_{23}M_{13} - M_{12}^2M_{24}M_{13} - bM_{23}^2M_{34}M_{13} - M_{12}M_{23}^2M_{24}$

26. $-M_{12}M_{14}M_{13}^2 + cM_{14}M_{23}M_{13}^2 + dM_{23}M_{34}M_{13}^2 - aM_{23}^2M_{24}M_{13} - bM_{23}^2M_{34}M_{13} - M_{12}M_{23}^2M_{24}$

27. $-bM_{23}M_{13}^3 - M_{12}M_{14}M_{13}^2 - aM_{12}M_{23}M_{13}^2 + dM_{12}M_{23}M_{13}^2 + dM_{23}M_{34}M_{13}^2 - dM_{23}^3M_{13}-M_{12}^2M_{23}M_{13}+bM_{23}^2M_{34}M_{13}-M_{12}^2M_{14}M_{23}+M_{12}M_{23}^2M_{24}+aM_{12}M_{23}^2M_{34}$

28. $M_{24}M_{12}^3+bM_{23}^2M_{12}^2-M_{13}M_{14}M_{12}^2+aM_{13}M_{23}M_{12}^2-cM_{13}M_{34}M_{12}^2-M_{24}M_{34}M_{12}^2-acM_{13}M_{23}^2M_{12}+M_{13}M_{24}^2M_{12}-aM_{23}^2M_{24}M_{12}+M_{14}M_{23}M_{24}M_{12}-bM_{23}^2M_{34}M_{12}+dM_{13}M_{23}M_{34}M_{12} + adM_{13}M_{23}^3 - bcM_{13}^2M_{23}^2 + bM_{13}M_{23}^2M_{24} - dM_{13}^2M_{23}M_{24}$

29. $dM_{23}^4-cM_{12}M_{23}^3-bM_{34}M_{23}^3+M_{12}^2M_{23}^2+bM_{13}^2M_{23}^2-dM_{12}M_{13}M_{23}^2-dM_{13}M_{34}M_{23}^2+cM_{12}^2M_{13}M_{23}+M_{12}M_{13}M_{14}M_{23}+cM_{12}M_{13}M_{34}M_{23}-2M_{12}M_{24}M_{34}M_{23}-M_{12}^3M_{13}+M_{12}M_{13}^2M_{24}$

30. $M_{13}M_{12}^3-cM_{13}M_{23}M_{12}^2-M_{23}M_{24}M_{12}^2-bM_{23}^3M_{12}+dM_{13}M_{23}^2M_{12}+M_{13}M_{14}M_{23}M_{12}-M_{13}^2M_{24}M_{12} + M_{23}M_{24}M_{34}M_{12} - cM_{13}M_{14}M_{23}^2 + M_{13}M_{23}M_{24}^2 + aM_{23}^3M_{24} - M_{14}M_{23}^2M_{24} + bM_{23}^3M_{34} - dM_{13}M_{23}^2M_{34}$

31. $M_{24}M_{13}^3 - M_{12}^2M_{13}^2 - dM_{23}^2M_{13}^2 + cM_{12}M_{23}M_{13}^2 + bM_{23}^3M_{13} + M_{12}M_{23}M_{24}M_{13} - 2M_{23}M_{24}M_{34}M_{13} + M_{23}^3M_{24}$

32. $-M_{34}M_{12}^3 - aM_{13}M_{34}M_{12}^2 + cM_{23}M_{34}M_{12}^2 + M_{13}M_{14}^2M_{12} + M_{23}M_{24}^2M_{12} - bM_{13}^2M_{34}M_{12} - dM_{23}^2M_{34}M_{12} + acM_{13}M_{23}M_{34}M_{12} - M_{14}M_{24}M_{34}M_{12} - M_{13}M_{14}M_{24}^2 + aM_{13}M_{23}M_{24}^2 - cM_{13}M_{14}^2M_{23} + M_{14}^2M_{23}M_{24} - adM_{13}M_{23}^2M_{34} - bM_{14}M_{23}^2M_{34}+bcM_{13}^2M_{23}M_{34}-dM_{13}M_{14}M_{23}M_{34}+dM_{13}^2M_{24}M_{34}+bM_{13}M_{23}M_{24}M_{34}$

33. $M_{14}M_{13}^3 - aM_{23}M_{13}^3 + cM_{34}M_{13}^3 - cM_{23}^2M_{13}^2 - M_{12}M_{24}M_{13}^2 - M_{24}M_{34}M_{13}^2 + aM_{12}M_{23}^2M_{13} + M_{12}M_{34}^2M_{13} + M_{23}^2M_{24}M_{13} + aM_{23}^2M_{34}M_{13} - M_{14}M_{23}M_{34}M_{13} + M_{12}^2M_{23}^2 - M_{12}M_{23}^2M_{34}$

34. $-M_{24}M_{12}^3 - M_{14}^2M_{12}^2 - aM_{34}^2M_{12}^2 - aM_{13}M_{24}M_{12}^2 + cM_{13}M_{34}M_{12}^2 - M_{24}M_{34}M_{12}^2 + M_{13}M_{24}^2M_{12} - bM_{13}M_{34}^2M_{12} + dM_{13}^2M_{14}M_{12} - bM_{13}^2M_{24}M_{12} - M_{14}M_{23}M_{24}M_{12} + acM_{13}^2M_{34}M_{12} - cM_{13}^2M_{14}^2 + aM_{13}^2M_{24}^2 + bcM_{13}^3M_{34} - adM_{13}^2M_{23}M_{34} - bM_{13}M_{14}M_{23}M_{34} + bM_{13}^2M_{24}M_{34}$

35. $cM_{13}M_{23}^3 - M_{24}M_{23}^3 - aM_{34}M_{23}^3 + aM_{13}^2M_{23}^2 + M_{12}M_{14}M_{23}^2 + M_{14}M_{34}M_{23}^2 - cM_{12}M_{13}^2M_{23} + M_{12}M_{34}^2M_{23} - M_{13}^2M_{14}M_{23} - cM_{13}^2M_{34}M_{23} + M_{13}M_{24}M_{34}M_{23} + M_{12}^2M_{13}^2 - M_{12}M_{13}^2M_{34}$

36. $-M_{12}(M_{12}^3 + aM_{13}M_{12}^2 - cM_{23}M_{12}^2 + bM_{13}^2M_{12} + dM_{23}^2M_{12} - acM_{13}M_{23}M_{12} + M_{14}M_{23}M_{12} - M_{13}M_{24}M_{12} + M_{14}M_{24}M_{12} + M_{13}M_{14}^2 + adM_{13}M_{23}^2 + bM_{14}M_{23}^2 - cM_{14}M_{23}^2 + M_{23}M_{24}^2 + dM_{13}M_{34}^2 + bM_{23}M_{34}^2 - bcM_{13}^2M_{23} - aM_{13}^2M_{24} - dM_{13}^2M_{24} - bM_{13}^2M_{34} - dM_{23}^2M_{34} + cM_{13}M_{14}M_{34} - 2M_{14}M_{24}M_{34} + aM_{23}M_{24}M_{34})$

37. $-M_{14}M_{12}^3 + M_{24}^2M_{12}^2 - cM_{34}^2M_{12}^2 + cM_{14}M_{23}M_{12}^2 - M_{14}M_{34}M_{12}^2 + aM_{23}M_{34}M_{12}^2 - dM_{14}M_{23}^2M_{12} + dM_{23}M_{34}^2M_{12} - M_{14}^2M_{23}M_{12} + bM_{23}^2M_{24}M_{12} + M_{13}M_{14}M_{24}M_{12} - acM_{23}^2M_{34}M_{12} + cM_{14}^2M_{23}^2 - aM_{23}^2M_{24}^2 + adM_{23}^3M_{34} - bcM_{13}M_{23}^2M_{34} + dM_{14}M_{23}^2M_{34} - dM_{13}M_{23}M_{24}M_{34}$

38. $-M_{12}M_{13}^3 + cM_{23}M_{13}^3 + M_{34}M_{13}^3 + M_{23}M_{24}M_{13}^2 - aM_{23}^3M_{13} - M_{14}M_{23}^2M_{13} - 2M_{23}M_{34}^2M_{13} - M_{12}M_{23}^3 + M_{23}^3M_{34}$

39. $-M_{13}(M_{12}^3 + aM_{13}M_{12}^2 - cM_{23}M_{12}^2 + bM_{13}^2M_{12} + dM_{23}^2M_{12} - acM_{13}M_{23}M_{12} + M_{14}M_{23}M_{12} - M_{13}M_{24}M_{12} + M_{14}M_{24}M_{12} + M_{13}M_{14}^2 + adM_{13}M_{23}^2 + bM_{14}M_{23}^2 - cM_{14}M_{23}^2 + M_{23}M_{24}^2 + dM_{13}M_{34}^2 + bM_{23}M_{34}^2 - bcM_{13}^2M_{23} - aM_{13}^2M_{24} - dM_{13}^2M_{24} - bM_{13}^2M_{34} - dM_{23}^2M_{34} + cM_{13}M_{14}M_{34} - 2M_{14}M_{24}M_{34} + aM_{23}M_{24}M_{34})$

40. $-bM_{34}M_{13}^3+M_{14}^2M_{13}^2+dM_{34}^2M_{13}^2-dM_{14}M_{23}M_{13}^2+bM_{23}M_{24}M_{13}^2-aM_{12}M_{34}M_{13}^2+bM_{23}M_{34}^2M_{13}+aM_{12}M_{23}M_{24}M_{13}-M_{12}^2M_{34}M_{13}-dM_{23}^2M_{34}M_{13}-M_{14}M_{24}M_{34}M_{13}-M_{12}M_{14}M_{34}^2+aM_{12}M_{23}M_{34}^2+M_{12}M_{14}^2M_{23}+M_{14}M_{23}^2M_{24}+M_{12}^2M_{23}M_{24}-M_{14}^2M_{23}M_{34}+M_{12}M_{23}M_{24}M_{34}$

41. $M_{14}(M_{12}^3+aM_{13}M_{12}^2-cM_{23}M_{12}^2+bM_{13}^2M_{12}+dM_{23}^2M_{12}-acM_{13}M_{23}M_{12}+M_{14}M_{23}M_{12}-M_{13}M_{24}M_{12}+M_{14}M_{24}M_{12}+M_{13}M_{14}^2+adM_{13}M_{23}^2+bM_{14}M_{23}^2-cM_{14}M_{23}^2+M_{23}M_{24}^2+dM_{13}M_{34}^2+bM_{23}M_{34}^2-bcM_{13}^2M_{23}-aM_{13}^2M_{24}-dM_{13}^2M_{24}-bM_{13}^2M_{34}-dM_{23}^2M_{34}+cM_{13}M_{14}M_{34}-2M_{14}M_{24}M_{34}+aM_{23}M_{24}M_{34})$

42. $-M_{24}^2M_{13}^2+cM_{34}^2M_{13}^2+M_{12}M_{14}M_{13}^2-cM_{14}M_{23}M_{13}^2-aM_{23}M_{34}M_{13}^2+aM_{23}^2M_{24}M_{13}-cM_{23}^2M_{34}M_{13}+M_{14}^2M_{23}^2+aM_{23}^2M_{34}^2+M_{12}M_{23}^2M_{24}$

43. $M_{23}(M_{12}^3+aM_{13}M_{12}^2-cM_{23}M_{12}^2+bM_{13}^2M_{12}+dM_{23}^2M_{12}-acM_{13}M_{23}M_{12}+M_{14}M_{23}M_{12}-M_{13}M_{24}M_{12}+M_{14}M_{24}M_{12}+M_{13}M_{14}^2+adM_{13}M_{23}^2+bM_{14}M_{23}^2-cM_{14}M_{23}^2+M_{23}M_{24}^2+dM_{13}M_{34}^2+bM_{23}M_{34}^2-bcM_{13}^2M_{23}-aM_{13}^2M_{24}-dM_{13}^2M_{24}-bM_{13}^2M_{34}-dM_{23}^2M_{34}+cM_{13}M_{14}M_{34}-2M_{14}M_{24}M_{34}+aM_{23}M_{24}M_{34})$

44. $dM_{34}M_{23}^3-M_{24}^2M_{23}^2-bM_{34}^2M_{23}^2+dM_{13}M_{14}M_{23}^2-bM_{13}M_{24}M_{23}^2-cM_{12}M_{34}M_{23}^2-dM_{13}M_{34}^2M_{23}-cM_{12}M_{13}M_{14}M_{23}+M_{12}^2M_{34}M_{23}+bM_{13}^2M_{34}M_{23}+M_{14}M_{24}M_{34}M_{23}-M_{12}M_{13}M_{24}^2+cM_{12}M_{13}M_{34}^2-M_{12}M_{24}M_{34}^2+M_{12}^2M_{13}M_{14}-M_{13}^2M_{14}M_{24}+M_{13}M_{24}^2M_{34}+M_{12}M_{13}M_{14}M_{34}$

45. $-M_{24}(M_{12}^3+aM_{13}M_{12}^2-cM_{23}M_{12}^2+bM_{13}^2M_{12}+dM_{23}^2M_{12}-acM_{13}M_{23}M_{12}+M_{14}M_{23}M_{12}-M_{13}M_{24}M_{12}+M_{14}M_{24}M_{12}+M_{13}M_{14}^2+adM_{13}M_{23}^2+bM_{14}M_{23}^2-cM_{14}M_{23}^2+M_{23}M_{24}^2+dM_{13}M_{34}^2+bM_{23}M_{34}^2-bcM_{13}^2M_{23}-aM_{13}^2M_{24}-dM_{13}^2M_{24}-bM_{13}^2M_{34}-dM_{23}^2M_{34}+cM_{13}M_{14}M_{34}-2M_{14}M_{24}M_{34}+aM_{23}M_{24}M_{34})$

46. $-M_{34}(M_{12}^3+aM_{13}M_{12}^2-cM_{23}M_{12}^2+bM_{13}^2M_{12}+dM_{23}^2M_{12}-acM_{13}M_{23}M_{12}+M_{14}M_{23}M_{12}-M_{13}M_{24}M_{12}+M_{14}M_{24}M_{12}+M_{13}M_{14}^2+adM_{13}M_{23}^2+bM_{14}M_{23}^2-$

$cM_{14}M_{23}^2+M_{23}M_{24}^2+dM_{13}M_{34}^2+bM_{23}M_{34}^2-bcM_{13}^2M_{23}-aM_{13}^2M_{24}-dM_{13}^2M_{24}-$ $bM_{13}^2M_{34}-dM_{23}^2M_{34}+cM_{13}M_{14}M_{34}-2M_{14}M_{24}M_{34}+aM_{23}M_{24}M_{34})$

6.2.2 Polynomials when $a=c=0$

1. $-M_{12}^4$
2. $M_{12}^2M_{13}^2$
3. $-M_{12}^2M_{23}^2$
4. $-M_{12}(dM_{13}^3-2M_{12}M_{13}M_{14}-M_{12}^2M_{23}-bM_{13}^2M_{23})$
5. $-M_{13}(dM_{13}^3-2M_{12}M_{13}M_{14}-M_{12}^2M_{23}-bM_{13}^2M_{23})$
6. $M_{12}^4+bM_{12}^2M_{13}^2-dM_{13}^3M_{14}+2M_{12}M_{13}M_{14}^2+M_{12}^2M_{14}M_{23}+bM_{13}^2M_{14}M_{23}$
7. $-M_{23}(dM_{13}^3-2M_{12}M_{13}M_{14}-M_{12}^2M_{23}-bM_{13}^2M_{23})$
8. $-M_{12}(M_{12}^2M_{13}+dM_{13}M_{23}^2-bM_{23}^3-2M_{12}M_{23}M_{24})$
9. $M_{13}(M_{12}^2M_{13}+dM_{13}M_{23}^2-bM_{23}^3-2M_{12}M_{23}M_{24})$
10. $-M_{23}(M_{12}^2M_{13}+dM_{13}M_{23}^2-bM_{23}^3-2M_{12}M_{23}M_{24})$
11. $M_{12}M_{13}^2M_{14}+M_{14}^2M_{23}^2+M_{12}M_{23}^2M_{24}-M_{13}^2M_{24}^2$
12. $M_{12}^4+dM_{12}^2M_{23}^2-M_{12}^2M_{13}M_{24}-dM_{13}M_{23}^2M_{24}+bM_{23}^3M_{24}+2M_{12}M_{23}M_{24}^2$
13. $M_{14}M_{23}-M_{13}M_{24}+M_{12}M_{34}$
14. $M_{12}^3M_{13}+dM_{13}^3M_{23}-M_{12}M_{13}M_{14}M_{23}-M_{12}^2M_{23}^2-bM_{13}^2M_{23}^2-M_{12}M_{13}^2M_{24}+$ $M_{12}^2M_{13}M_{34}$
15. $-M_{12}^2M_{13}^2+M_{12}^3M_{23}-dM_{13}^2M_{23}^2+M_{12}M_{14}M_{23}^2+bM_{13}M_{23}^3+M_{12}M_{13}M_{23}M_{24}+$ $M_{12}^2M_{23}M_{34}$
16. $-M_{12}(-M_{13}^3+M_{12}M_{13}M_{23}-M_{23}^3+2M_{13}M_{23}M_{34})$

17. $M_{12}M_{13}^3 + M_{13}M_{14}M_{23}^2 + M_{12}M_{23}^3 - M_{13}^2M_{23}M_{24} - M_{12}M_{13}M_{23}M_{34}$

18. $M_{12}^3M_{13}+bM_{12}M_{13}^3+dM_{12}M_{13}M_{23}^2+bM_{13}M_{14}M_{23}^2-dM_{13}^3M_{24}+2M_{12}M_{13}M_{14}M_{24}-M_{12}^2M_{23}M_{24} - bM_{12}M_{13}M_{23}M_{34}$

19. $-dM_{12}^2M_{13}^2 + M_{12}^3M_{14} + dM_{13}^2M_{14}M_{23} - M_{12}M_{14}^2M_{23} - bM_{13}M_{14}M_{23}^2 -M_{12}M_{13}M_{14}M_{24}-M_{12}^2M_{23}M_{24}+dM_{12}M_{13}^2M_{34}-M_{12}^2M_{14}M_{34}-bM_{12}M_{13}M_{23}M_{34}$

20. $-M_{12}^2M_{13}M_{14}-M_{12}^3M_{23}-bM_{12}M_{13}^2M_{23}-dM_{12}M_{23}^3-bM_{14}M_{23}^3+dM_{13}^2M_{23}M_{24}-2M_{12}M_{14}M_{23}M_{24} + dM_{12}M_{13}M_{23}M_{34}$

21. $M_{13}(M_{13}^3 - M_{12}M_{13}M_{23} + M_{23}^3 - 2M_{13}M_{23}M_{34})$

22. $-dM_{12}M_{13}^3+M_{12}^2M_{13}M_{14}+M_{12}^3M_{23}+bM_{12}M_{13}^2M_{23}+M_{13}M_{14}^2M_{23}+M_{12}M_{14}M_{23}^2-M_{13}^2M_{14}M_{24} - M_{12}M_{13}M_{23}M_{24} + dM_{13}^3M_{34} - M_{12}M_{13}M_{14}M_{34} - bM_{13}^2M_{23}M_{34}$

23. $M_{12}^2M_{13}^2 + bM_{13}^4 - M_{12}^3M_{23} - bM_{12}M_{13}^2M_{23} + dM_{13}^2M_{23}^2 - M_{12}M_{14}M_{23}^2 - M_{12}M_{13}M_{23}M_{24} - dM_{13}^3M_{34} + 2M_{12}M_{13}M_{14}M_{34} - bM_{13}^2M_{23}M_{34}$

24. $-M_{13}^3M_{14} + dM_{13}^3M_{23} - M_{12}M_{13}M_{14}M_{23} - M_{12}^2M_{23}^2 - bM_{13}^2M_{23}^2 - M_{14}M_{23}^3 + 2M_{13}M_{14}M_{23}M_{34}$

25. $-M_{23}(M_{13}^3 - M_{12}M_{13}M_{23} + M_{23}^3 - 2M_{13}M_{23}M_{34})$

26. $M_{12}M_{13}^2M_{14}+M_{12}^2M_{13}M_{23}+bM_{13}^3M_{23}-bM_{12}M_{13}M_{23}^2+dM_{13}M_{23}^3-M_{12}^2M_{13}M_{24}-M_{12}M_{23}^2M_{24} - dM_{13}^2M_{23}M_{34} - bM_{13}M_{23}^2M_{34}$

27. $-M_{12}M_{13}^2M_{14} - M_{12}M_{23}^2M_{24} + dM_{13}^2M_{23}M_{34} - bM_{13}M_{23}^2M_{34}$

28. $-M_{12}M_{13}^2M_{14}-M_{12}^2M_{13}M_{23}+dM_{12}M_{13}^2M_{23}-bM_{13}^3M_{23}-M_{12}^2M_{14}M_{23}-dM_{13}M_{23}^3+M_{12}M_{23}^2M_{24} + dM_{13}^2M_{23}M_{34} + bM_{13}M_{23}^2M_{34}$

29. $-M_{12}^2M_{13}M_{14} + bM_{12}^2M_{23}^2 + M_{12}^3M_{24} - dM_{13}^2M_{23}M_{24} + M_{12}M_{14}M_{23}M_{24} + bM_{13}M_{23}^2M_{24} + M_{12}M_{13}M_{24}^2 + dM_{12}M_{13}M_{23}M_{34} - bM_{12}M_{23}^2M_{34} - M_{12}^2M_{24}M_{34}$

30. $-M_{12}^3M_{13} + M_{12}M_{13}M_{14}M_{23} + M_{12}^2M_{23}^2 - dM_{12}M_{13}M_{23}^2 + bM_{13}^2M_{23}^2 + dM_{23}^4 + M_{12}M_{13}^2M_{24} - dM_{13}M_{23}^2M_{34} - bM_{23}^3M_{34} - 2M_{12}M_{23}M_{24}M_{34}$

31. $M_{12}^3M_{13}+M_{12}M_{13}M_{14}M_{23}+dM_{12}M_{13}M_{23}^2-bM_{12}M_{23}^3-M_{12}M_{13}^2M_{24}-M_{12}^2M_{23}M_{24}-M_{14}M_{23}^2M_{24} + M_{13}M_{23}M_{24}^2 - dM_{13}M_{23}^2M_{34} + bM_{23}^3M_{34} + M_{12}M_{23}M_{24}M_{34}$

32. $-M_{12}^2M_{13}^2 - dM_{13}^2M_{23}^2 + bM_{13}M_{23}^3 + M_{13}^3M_{24} + M_{12}M_{13}M_{23}M_{24} + M_{23}^3M_{24} - 2M_{13}M_{23}M_{24}M_{34}$

33. $M_{12}M_{13}M_{14}^2+M_{14}^2M_{23}M_{24}-M_{13}M_{14}M_{24}^2+M_{12}M_{23}M_{24}^2-M_{12}^3M_{34}-bM_{12}M_{13}^2M_{34}-dM_{13}M_{14}M_{23}M_{34}-dM_{12}M_{23}^2M_{34}-bM_{14}M_{23}^2M_{34}+dM_{13}^2M_{24}M_{34}-M_{12}M_{14}M_{24}M_{34}+bM_{13}M_{23}M_{24}M_{34}$

34. $M_{13}^3M_{14}+M_{12}^2M_{23}^2-M_{12}M_{13}^2M_{24}+M_{13}M_{23}^2M_{24}-M_{13}M_{14}M_{23}M_{34}-M_{12}M_{23}^2M_{34}-M_{13}^2M_{24}M_{34} + M_{12}M_{13}M_{34}^2$

35. $dM_{12}M_{13}^2M_{14}-M_{12}^2M_{14}^2-M_{12}^3M_{24}-bM_{12}M_{13}^2M_{24}-M_{12}M_{14}M_{23}M_{24}+M_{12}M_{13}M_{24}^2-bM_{13}M_{14}M_{23}M_{34} - M_{12}^2M_{24}M_{34} + bM_{13}^2M_{24}M_{34} - bM_{12}M_{13}M_{34}^2$

36. $M_{12}^2M_{13}^2 - M_{13}^2M_{14}M_{23} + M_{12}M_{14}M_{23}^2 - M_{23}^3M_{24} - M_{12}M_{13}^2M_{34} + M_{14}M_{23}^2M_{34} + M_{13}M_{23}M_{24}M_{34} + M_{12}M_{23}M_{34}^2$

37. $-M_{12}(M_{12}^3+bM_{12}M_{13}^2+M_{13}M_{14}^2+M_{12}M_{14}M_{23}+dM_{12}M_{23}^2+bM_{14}M_{23}^2-M_{12}M_{13}M_{24}-dM_{13}^2M_{24} + M_{12}M_{14}M_{24} + M_{23}M_{24}^2 - bM_{13}^2M_{34} - dM_{23}^2M_{34} - 2M_{14}M_{24}M_{34} + dM_{13}M_{34}^2 + bM_{23}M_{34}^2)$

38. $-M_{12}^3M_{14} - M_{12}M_{14}^2M_{23} - dM_{12}M_{14}M_{23}^2 + M_{12}M_{13}M_{14}M_{24} + bM_{12}M_{23}^2M_{24} + M_{12}^2M_{24}^2 - M_{12}^2M_{14}M_{34} + dM_{14}M_{23}^2M_{34} - dM_{13}M_{23}M_{24}M_{34} + dM_{12}M_{23}M_{34}^2$

39. $-M_{12}M_{13}^3-M_{13}M_{14}M_{23}^2-M_{12}M_{23}^3+M_{13}^2M_{23}M_{24}+M_{13}^3M_{34}+M_{23}^3M_{34}-2M_{13}M_{23}M_{34}^2$

40. $-M_{13}(M_{12}^3+bM_{12}M_{13}^2+M_{13}M_{14}^2+M_{12}M_{14}M_{23}+dM_{12}M_{23}^2+bM_{14}M_{23}^2-M_{12}M_{13}M_{24}-dM_{13}^2M_{24}+M_{12}M_{14}M_{24}+M_{23}M_{24}^2-bM_{13}^2M_{34}-dM_{23}^2M_{34}-2M_{14}M_{24}M_{34}+dM_{13}M_{34}^2+bM_{23}M_{34}^2)$

41. $M_{13}^2M_{14}^2-dM_{13}^2M_{14}M_{23}+M_{12}M_{14}^2M_{23}+M_{12}^2M_{23}M_{24}+bM_{13}^2M_{23}M_{24}+M_{14}M_{23}^2M_{24}-M_{12}^2M_{13}M_{34}-bM_{13}^3M_{34}-M_{14}^2M_{23}M_{34}-dM_{13}M_{23}^2M_{34}-M_{13}M_{14}M_{24}M_{34}+M_{12}M_{23}M_{24}M_{34}+dM_{13}^2M_{34}^2-M_{12}M_{14}M_{34}^2+bM_{13}M_{23}M_{34}^2$

42. $M_{14}(M_{12}^3+bM_{12}M_{13}^2+M_{13}M_{14}^2+M_{12}M_{14}M_{23}+dM_{12}M_{23}^2+bM_{14}M_{23}^2-M_{12}M_{13}M_{24}-dM_{13}^2M_{24}+M_{12}M_{14}M_{24}+M_{23}M_{24}^2-bM_{13}^2M_{34}-dM_{23}^2M_{34}-2M_{14}M_{24}M_{34}+dM_{13}M_{34}^2+bM_{23}M_{34}^2)$

43. $M_{23}(M_{12}^3+bM_{12}M_{13}^2+M_{13}M_{14}^2+M_{12}M_{14}M_{23}+dM_{12}M_{23}^2+bM_{14}M_{23}^2-M_{12}M_{13}M_{24}-dM_{13}^2M_{24}+M_{12}M_{14}M_{24}+M_{23}M_{24}^2-bM_{13}^2M_{34}-dM_{23}^2M_{34}-2M_{14}M_{24}M_{34}+dM_{13}M_{34}^2+bM_{23}M_{34}^2)$

44. $M_{12}^2M_{13}M_{14}+dM_{13}M_{14}M_{23}^2-M_{13}^2M_{14}M_{24}-bM_{13}M_{23}^2M_{24}-M_{12}M_{13}M_{24}^2-M_{23}^2M_{24}^2+M_{12}M_{13}M_{14}M_{34}+M_{12}^2M_{23}M_{34}+bM_{13}^2M_{23}M_{34}+dM_{23}^3M_{34}+M_{14}M_{23}M_{24}M_{34}+M_{13}M_{24}^2M_{34}-dM_{13}M_{23}M_{34}^2-bM_{23}^2M_{34}^2-M_{12}M_{24}M_{34}^2$

45. $-M_{24}(M_{12}^3+bM_{12}M_{13}^2+M_{13}M_{14}^2+M_{12}M_{14}M_{23}+dM_{12}M_{23}^2+bM_{14}M_{23}^2-M_{12}M_{13}M_{24}-dM_{13}^2M_{24}+M_{12}M_{14}M_{24}+M_{23}M_{24}^2-bM_{13}^2M_{34}-dM_{23}^2M_{34}-2M_{14}M_{24}M_{34}+dM_{13}M_{34}^2+bM_{23}M_{34}^2)$

46. $-M_{34}(M_{12}^3+bM_{12}M_{13}^2+M_{13}M_{14}^2+M_{12}M_{14}M_{23}+dM_{12}M_{23}^2+bM_{14}M_{23}^2-M_{12}M_{13}M_{24}-dM_{13}^2M_{24}+M_{12}M_{14}M_{24}+M_{23}M_{24}^2-bM_{13}^2M_{34}-dM_{23}^2M_{34}-2M_{14}M_{24}M_{34}+dM_{13}M_{34}^2+bM_{23}M_{34}^2)$

6.2.3 L_7 Polynomials

1. $M_{12}^2 - cM_{13}^2 + a$
2. $M_{12}M_{13} - M_{13}^3 + 2M_{13}M_{34} - 1$
3. $cM_{12}M_{13} - dM_{13} - M_{12}^2M_{13} + 2M_{12}M_{24} + b$
4. $M_{12}^2 - M_{12}^3M_{13} + bM_{13}^2 - dM_{13}^3 + cM_{12}M_{13}^3 + 2M_{12}M_{13}M_{14}$
5. $M_{12}M_{34} - M_{13}M_{24} + M_{14}$

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BIOGRAPHICAL STATEMENT

Ian Lim grew up in Aliquippa, Pennsylvania, where he graduated from Central Valley High School with high honors in 2012. He is one of four children to his parents Joseph and Anita Lim.

Beginning in the Fall of 2012, Ian studied at Youngstown State University where he was an active member of the Sigma Alpha Epsilon Fraternity and the Pi Mu Epsilon Honor Society. He graduated in the Spring of 2016, *magna cum laude*, with a Bachelor of Science in Applied Mathematics.

Ian attended the University of Texas at Arlington beginning in the Fall of 2016 to pursue his doctorate in mathematics. His research in non-commutative algebra was overseen by Dr. Michaela Vancliff. In the Summer of 2020, Ian participated in a 10-week internship with NASA at their Glenn Research Center located in Cleveland, Ohio, where he conducted research on machine learning via topological data analysis.

Ian graduated with his Ph.D. in May of 2021 and is employed at NASA Glenn Research Center as a Computer Scientist, applying his mathematical knowledge to cutting-edge problems.