The Natural Middle of a Complete Resolution

by

REBEKAH J. ADUDDELL

Presented to the Faculty of the Graduate School of

The University of Texas at Arlington in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT ARLINGTON

August 2021

This thesis is hereby dedicated to:

The universe. Because it is Awesome.

ACKNOWLEDGEMENTS

I am a part of all that I have met;

Yet all experience is an arch wherethro'

Gleams that untravell'd world whose margin fades

For ever and forever when I move.

Ulysses, Lord Alfred Tennyson

Perhaps, in some small way, I might have affected many throughout my life thus far; yet, certainly the greater impression has been made upon me. For this reason, I cannot take sole credit for the words and math which follow– rather, it is a culmination of what began long ago and, more so, what has brewing for quite some time now. Assuredly, for this, my gratitude knows no measure; for I am part of all that I have met and they, too, are very much a part of me. While I may not name all names whom are interwoven throughout the very threads of this manuscript, please know that if you have played a role in my life, you have indeed left an impression for which I am certainly thankful for.

First and foremost, an absurdly abundant amount of thanks goes to my advisor, David Jorgensen, for putting up with me these five past years. To you, I say: I know I have been the black sheep in your flock, but even a black sheep needs a shepherd willing to care for them and that you have done. I came to the University of Texas at Arlington intending to work with you and although, admittedly, there exist an epsilon

of occasions I have questioned this choice, there have been infinitely more occasions in which I have realized it was, in fact, the best choice. I would not have grown into the mathematician I am today without your guidance, wisdom, and support. I hope you can be proud of this dissertation even a fraction of the amount I am proud to call you my advisor[1]. The ideas throughout this thesis were gifted to me by you, and for that I am eternally grateful.

Second, I would like to thank my committee members: Michaela Vancliff, Dimitar Grantcharov, Minerva Cordero, and Theresa Jorgensen. You have served as great teachers and mentors to me over the past five years; it is impossible for me to put into words the appreciation I have for all that you have taught me, as well as the wisdom and support you have afforded me throughout my graduate studies. A special thanks goes to Theresa Jorgensen and her NSF grant (joint with Ashley and Elizabeth Griffith), which allowed me to spend a year working on the GeoPaths research in lieu of teaching; this did additionally enable me to focus more of my energy on my own research, part of which is presented in this dissertation. I would also like to give special thanks to Michaela Vancliff, whose comments have allowed me to present a polished version of this dissertation, with fewer inconsistencies and grammatical lapses in judgment.

Third, I owe an infinitely deep amount of appreciation to my family, who have supported me throughout all my endeavors. To my parents, Ed and Kathie Aduddell, I say– thank you, for all you have done; I would not be here without your wisdom, love, support, and many, *many* prayers. And to my sister, Maggie Aduddell: I would not be nearly as creative and innovative if not for your (unintentional) mentorship throughout our formative years. Unsurprisingly, these are both characteristics useful

[1] which is, of course, homotopically equivalent to the amount I am proud to hold the universal record on *"Greatest Number of Demerits a PhD student has been awarded by their advisor"*.

when conducting research; the remaining quirks of my personality I contribute in part to you serve the sole purpose of making said research more fun, which is indeed quite a necessity for the world we live in.[2]

Last, and quite contrary to the least, immense gratitude is given to the one and only Tyler D. Anway for being a substantial partner-in-~~crime~~-math. To the great diagram-aficionado, all-star algebra-wizard, and my *favorite* mathematician, I say– *I'd do the stars with you, any time.* This manuscript is every bit of yours as it is mine and, although I maintain a distinct disappointment in everything I do in life, you have made me realize this disappointment is quite trivial after all. My life never quite made sense until you arrived, and while some might consider the Doctor of Philosophy the highest of achievements, my greatest accomplishment throughout this journey is, undeniably, that I met you.

July 19th, 2021

[2]These acknowledgments would not be complete if I did not mention my furry companions, Mo and Suki. Mona Lisa– you have been with me every step of the way throughout these past five years, and I hope to now be able to devote more time for walks and taking you to the dog park so that you may frolic with fellow furry friends. And, of course, I could not leave out Suki, the newest member of my family who has somehow been able to console me throughout the most stressful times this past year. The both of you bring so much joy into my life, and I would not have been able to write this manuscript if I did not know that joy.

ABSTRACT

The Natural Middle of a Complete Resolution

Rebekah J. Aduddell, Ph.D.

The University of Texas at Arlington, 2021

Supervising Professor: David A. Jorgensen

It is widely known that minimal free resolutions of a module over a complete intersection ring have nice patterns that arise in their Betti sequences. In the late 1990's Avramov, Gasharov and Peeva defined a new class of modules over more general types rings that would exhibit similar patterns in their free resolutions. In doing so, they additionally defined the notion of critical degree for a module, which serves as a "flag" for when such patterns arise in the Betti sequence. The main purpose of this thesis is to present an extension of critical degree to the category of totally acyclic complexes, $\mathbf{K}_{\mathrm{tac}}(R)$, where $(R, \mathfrak{m}, k)$ is a commutative noetherian, local ring. Furthermore, we will provide an appropriate dual analogue and then look towards realizing the cohomological characterization for these notions, utilizing the original such characterization given by the aforementioned authors. With regard to this topic, our attention will predominantly turn towards when R is further assumed to be a complete intersection ring of the form $R = Q/(f_1, \ldots, f_c)$ where $(Q, \mathfrak{m}, \Bbbk)$ is a regular local ring and $f_1, \ldots, f_c$ a Q-regular sequence in $\mathfrak{m}$.

We then investigate how the critical and cocritical degrees of an R-complex may change under certain operations of complexes; such as translations, direct sums,

and tensoring with a bounded complex. Lastly, we introduce a new invariant of R-complexes and R-modules called the *critical width*, or *diameter*, which we define to be the "distance" between the critical and cocritical degrees of an R-complex.

TABLE OF CONTENTS

CHAPTER 0

INTRODUCTION

0.1 Prelude: Historical Connections

0.1.1 *Commutative Algebra: An Origin Story*

The three pillars of commutative algebra[1], on which the subject was built, are none other than the Queen of Mathematics herself (Number Theory), the classical study of algebraic geometry (now modernized by many techniques rooted in commutative algebra), and the never-dying Ghost of Invariant Theory[2]. Mathematical intrigue which motivated the earliest seeds of commutative algebra is said to have likely began with number-theoretic problems. After questions about characteristics of the earliest algebraic structures ($\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$) had been exhausted, interest developed with the ring of Gaussian integers $\mathbb{Z}[i]$, introduced by Gauss in his 1828 paper in which he proved that its elements satisfied a unique prime factorization characteristic, much like $\mathbb{Z}$ itself.

Many number theorists (Euler, Dirichlet, Kummer) then looked towards utilizing this idea of "adjoining" solutions for polynomial equations to $\mathbb{Z}$ as a method for solving Fermat's Last Theorem, or at least proving special cases of it. In fact, it was such endeavors which led to Dedekind's introduction of *ideals* of a ring, with respect

[1]Much of the historical background discussed throughout this introduction has been motivated by the much more thorough (and superior) treatment in Chapter 1 of Eisenbud's *Commutative Algebra* text ([Ei2]); nevertheless, we provide a brief synopsis of the origins of commutative algebra, with special attention to the objects of significance to this thesis.

[2]Although Hilbert was said to have "killed" the theory by solving its fundamental problem, it has since seen a resurgence in activity in the form of geometric invariant theory.

to the search for some kind of generalization of unique factorization. Consequently, it was Kronecker (Kummer's student) who established the idea of adjoining the root of a polynomial equation to a field $\Bbbk$. Henceforth, interest in rings of the form $\Bbbk[x]/(f(x))$ emerged and, moreover, in 1905, Lasker finally succeeded in giving a generalization of unique factorization, known as primary decomposition. It was, in fact, Emmy Noether who later reformulated and axiomatized these theories in the 1920s.

Around the same time as some of these number-theoretic discoveries, mathematicians also working in algebraic geometry (Kronecker, Weierstrass, Dedekind, Weber) realized that many of these techniques could actually be applied to geometrically defined fields. In modern times, the interaction between geometry and commutative algebra is well known but interestingly enough, these connections did not begin to take form until the work of Dedekind and Weber in the late 19^{th} century. And, of course, it was the end of this century in which Hilbert published two phenomenal papers advancing early commutative algebra ideas, motivated himself by Invariant Theory. At its core, the study invariants began also with a geometric perspective, posing the question about which geometric properties of plane curves were "invariant" under certain classes of transformations. It was later realized that this question could be reformulated with respect to (usually polynomial) ring elements invariant under the action of a group (typically $\mathrm{SL}_n(\Bbbk)$ or $\mathrm{GL}_n(\Bbbk)$).

The general problem of Invariant Theory was focused on finding finite systems of generators of rings of invariants, and it was Hilbert who solved this fundamental problem. Of course, in doing so, he proved many significant results for commutative algebra, two of which we choose to direct our attention towards. First, Hilbert's idea was that, given a polynomial ring in finitely many variables over a field, every ideal of the ring can in fact be generated by finitely many elements. Although the theorem which encapsulates this phenomenon is called *Hilbert's Basis Theorem*, it is actually

Emmy Noether who received credit for rings with such a property; for it was she who demonstrated (in 1921) how to use this property as a basic axiom in commutative algebra and how Dedekind's Primary Decomposition could be derived quite simply under this axiom. As such rings and the theorem in question serve as particular focal point throughout this thesis (as they do throughout much of commutative algebra), we include them here.

Definition. A commutative ring R is called *noetherian* if every ideal in R is finitely generated. Equivalently, if R satisfies the *ascending chain condition* on ideals (e.g. every strictly ascending chain of ideals in R must terminate).

Theorem (Hilbert's Basis Theorem)**.** *If R is noetherian, then $R[x_1, \ldots, x_n]$ is noetherian.*

We may also at times refer to noetherian R-modules (for which the analogous definition uses submodules in lieu of ideals), since our focus is on such algebraic structures; in which case, the following statement also holds.

Fact. *If R is noetherian and M is a finitely generated module, then M is noetherian.*

The second significant result from Hilbert's 1888-93 papers which we will discuss is his *Syzygy Theorem*. As mentioned, at some point algebraists turned towards studying *modules* over a given ring R, which are structures fairly similar to vector spaces. An object which is, in some sense, at the center-point of the thematic elements for this thesis is what is called a *free resolution*. These objects can intuitively be thought as a manner of "linking together" systems of generators. Throughout Chapter 1 of this thesis we provide definitions of many of these constructions, which play a significant role in the the topics discussed in later chapters. For now, we simply include Hilbert's syzygy theorem:

Theorem. *If $R = \Bbbk[x_1, \ldots, x_n]$ then every finitely generated module over R has a finite free resolution of length at most n.*

Our interest in this thesis actually deals with *infinite* free resolutions, as described in §1.3 of this thesis. In Chapter 2, we will make mention of the progress made in better understanding patterns which may occur in these infinite sequences. For now, we move on to briefly discuss the other dominant broad algebraic area from which we pull many techniques and methods.

0.1.2 *When Two Algebraic Theories Collide*

Just as its sister, Homological Algebra has its origins beginning in the 19^{th} century.[3] The subject itself is derived from Algebraic Topology, which is why much of the language descends from topological intuition. It was actually the work of Riemann in the 1850s and Betti in the 1870s on "homology numbers" which planted the seeds for homological interest by later mathematicians, including Poincaré who provided a more rigorous treatment in 1895. Nonetheless, it was once again Emmy Noether who stepped in and shifted the focus to *homology groups* of a space, opening the gates for algebraic techniques to be developed in the 1930s.

However, topology remained the main motivation for this development until the mid-20^{th} century when Cartan and Eilenberg's *Homological Algebra* text ([CaEi]) broadened the use for such methods to commutative algebra. Their systematic approach to derived functors, via resolutions of R-modules, served as a pathway to the questions explored throughout commutative algebra, which had only surfaced as a field of study shortly beforehand. Henceforth, many algebraists dealing with questions in commutative algebra were then able to utilize the strength of homological methods, which had predominately only been utilized within problems of a topological nature.

[3] Much of the history discussed on Homological Algebra here is pulled from Weibel's (again, far, *far* superior rendition of) *History of Homological Algebra*, for which the reader may refer to [We2].

Of course, after much of such homological methods were developed for algebraists, an attempt to generalize the methods to a broader setting was made. MacLane made the first attempt (in 1948), defining what he called an "abelian category" (see [Ma2]), though his definition is now known as only an *additive category.* The next attempt was made by Eilenberg's student Buchsbaum, whose 1955 thesis (see [Bu]) introduced *exact categories*, with the additional axiom of existence of direct sums imposed. It was actually Grothendieck and Heller who developed work involving *abelian categories* as we know them today (see [Gr]). And so, just as algebraic topology spawned homological algebra, the latter in turn motivated the development of category theory via the attempt to generalize such methods.

As is typical in modern-day commutative algebra, we make much use of homological methods throughout this thesis, with particular application of the Ext functor. Furthermore, we continually acknowledge the categorical structure related to our objects of interest in order to both provide insight and build motivation for our work. With this in mind, we now relay the ordering of topics enclosed within this thesis.

0.2 A Tale of Two Degrees

The main goal of this thesis is to transfer a notion previously developed in the category of modules to the category of totally acyclic complexes. We additionally aim to develop the appropriate dual notion, as is natural to both categories. Throughout the majority of thesis, it should be assumed that R is a commutative, local noetherian ring as is indicated in §1.1. At times, we will impose additional assumptions upon R and will make clear when this is the case. Chapter 1 builds background on not only the categorical setting of R-modules, but also on many of the constructions of R-modules we routinely use throughout this manuscript. Of special significance are the functors on $\mathscr{R}$-$\mathscr{mod}$ described in §1.1.3, along with the derived functors of Hom presented in §1.2.3. We also include definitions of free resolutions and Betti Sequences in §1.3, along with definitions of R-complexes and R-complex chain maps in §1.2.2.2.

In Chapter 2, we provide the pertinent background for our main definitions. We present Eisenbud's special class of chain endomorphisms on minimal free resolutions in §2.1.1 and give proof of his main result from [Ei] (eventual surjectivity of a chain endomorphism), with respect to Ext rather than Tor. Then, we provide the original definition of *critical degree* for R-modules in §2.2.1 with inclusion of results from [AvGaPe] most relevant to this thesis. Towards the end of chapter 2, in §2.3.1 we provide the authors' cohomological characterization of critical degree and then, in §2.3.2.1 give reasoning for why a linear form of CI operators realizes the critical degree of a module over a complete intersection ring.

Chapter 3 turns towards the categorical setting of totally acyclic complexes, for which we point the well-informed reader towards §3.5 on a depiction of the connection between $\mathscr{R}\text{-}\mathscr{mod}$ and $\mathbf{K}_{\text{tac}}(R)$. Part II of this thesis begins with Chapter 4, in which we present our main definitions (see §4.1, Definitions 25 and 26). We also give two equivalent characterizations of these definitions in Sections 4.3 and 4.4, with §4.4.1 focusing attention on development of the dual cohomological notion to critical degree.

It is then in Chapter 5 where we investigate boundedness problems, with respect to how the critical and cocritical degrees might change under different operations of complexes. After first presenting our initial general approach to these questions, we explore "basic" operations on complexes guaranteed by the categorical structure of $\mathbf{K}_{\text{tac}}(R)$ in §5.1. Then, afterwards in §5.2 we look towards operations defined by two endofunctors on $\mathbf{K}_{\text{tac}}(R)$: $\text{Hom}(\text{-},\mathscr{B})$ (or $\text{Hom}(\mathscr{B},\text{-})$) and $(- \otimes \mathscr{B})$ (or $(\mathscr{B} \otimes -)$).

Finally, in Chapter 6 of this thesis we explore some additional boundedness questions related to critical degree of a given module. One of our primary goals with our extension is to address the lack in ability of bounding the critical degree over all modules of a given complexity $d > 1$. While our main definitions do not solve this particular problem, we present a new invariant of R-modules (and R-complexes) in §6.2 with hope that it might be possible to bound *this* value over modules of a given complexity.

Part I

The Essentials

CHAPTER 1

The Category of R-Modules

We begin with laying the groundwork for understanding the initial categorical setting from which the main definitions of this thesis are derived. Undoubtedly, there is a fascinating relationship between R-modules and R-complexes of a particular form; yet, to fully comprehend such a connection between these objects, we must first explore the categorical structure of finitely-generated R-modules, along with the main constructions and invariants used to better understand such objects. Of course, we work towards this understanding with a nod towards the structural similarities we will uncover in Chapter 3 of this thesis.

1.1 Categorical Structure of $\mathscr{R}\text{-}\mathscr{M}od$

Let R be a ring and denote M as an R-module. In some sense, such algebraic structures represent a generalization of vector spaces, though certainly not as well-behaved. For this reason, we can describe the study of R-modules as an exploration of what could go wrong when the underlying abelian group in a vector space is instead imposed with an action via scalars that are in a ring which does not hold such rigid structure as a field. This is a significant topic; for instance, we may be interested in an action by *polynomials* over a given abelian group, and the ring $R = \Bbbk[x_1, \ldots, x_n]$ is certainly not a field (any indeterminant x_i lacks a multiplicative inverse). Such a ring will arise commonly throughout this thesis, as commutative algebra has close ties to geometric questions and such questions precipitated the study of commutative algebra, serving as the main motivation for many early topics explored within the subject.

Rather than studying these early topics, we instead turn our attention to viewing the underlying categorical structure of R-modules, which will in turn highlight powerful characteristics that can arise in other settings as well. Take as objects all left R-modules and take the left R-module homomorphisms as arrows. Clearly, associative function composition is guaranteed by the nature of R-module homomorphisms and the identity map $\mathrm{Id}: M \to M$ for any R-module M is well defined. Denote this category $\mathcal{R}$-$\mathcal{Mod}$. Note that we may also consider all right R-modules with right R-module homomorphisms, which also constitutes a category, which we shall denote $\mathcal{Mod}$-$\mathcal{R}$. For any ${}_RM, {}_RN \in \mathcal{R}$-$\mathcal{Mod}$ define $\mathrm{Hom}_R({}_RM, {}_RN)$ as the set of all left R-module homomorphisms $\phi\colon M \to N$ and we may define $\mathrm{Hom}_R(M_R, N_R)$ similarly for right R-modules. In either case, $\mathrm{Hom}_R(M, N)$ has an abelian group structure, but will only have an additional module structure if either ${}_RM_S$ is a left R- and right S- bimodule for another ring S, (in which case, $\mathrm{Hom}_R({}_RM_S, {}_RN)$ will be a left S-module) or ${}_RN_T$ is a left R- and right T-module for another ring T (in which case, $\mathrm{Hom}_R({}_RM, {}_RN_T)$ will be a right T-module).

Fact. *If R is a commutative ring then $\mathcal{R}$-$\mathcal{Mod}$ and $\mathcal{Mod}$-$\mathcal{R}$ represent the same category.*

The above fact is easy to see since for a commutative ring, there is no distinction between left and right modules, as they coincide.

> From this point on, let R be a commutative noetherian ring and denote $\mathcal{R}$-$\mathcal{Mod}$ as the category of R-modules and R-module homomorphisms.

It is worth also noting that in this case $\mathrm{Hom}_R(M, N)$ has an R-module structure as well since both M and N are R-bimodules.

We now consider the full subcategory of $\mathscr{R}\text{-}\mathscr{Mod}$ comprised of finitely-generated R-modules and the R-module homomorphisms between them; denote this category $\mathscr{R}\text{-}\mathscr{mod}$. The focus of Chapter 1 is distinctly on finitely-generated R-modules, and so we will phrase many of the subsequent structural properties in terms of $\mathscr{R}\text{-}\mathscr{mod}$ but note that such properties are also true for $\mathscr{R}\text{-}\mathscr{Mod}$. Moreover, from here on out an R-module is always assumed to be finitely-generated unless otherwise stated. Many of the results in this chapter can be found in numerous sources (for instance, see [Wa] or [Ma]), and are only given for completeness of the discussion.

Proposition 1.1 (cf. [HoJoRo]). *$\mathscr{R}\text{-}\mathscr{mod}$ is an additive category.*

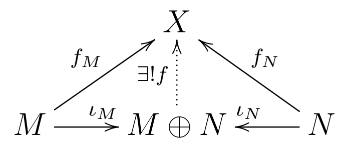
Proof. As it has already been discussed, we shall take as fact that $\mathrm{Hom}_R(M, N)$ is an abelian group, and so only need to justify that composition of morphisms is bilinear. For any $f,g,h \in \mathrm{Hom}_R(M, N)$ note that

$$f \circ (g + h)(x) = f \circ (g(x) + h(x)) = f(g(x)) + f(h(x)) = (f \circ g)(x) + (f \circ h)(x)$$

and

$$((f + g) \circ h)(x) = (f + g)(h(x)) = f(h(x)) + g(h(x)) = (f \circ h)(x) + (g \circ h)(x)$$

for any $x \in M$. Additionally, it should be clear that $0 \in \mathscr{R}\text{-}\mathscr{mod}$ is the Zero object such that $\mathrm{Hom}_R(M, 0)$ and $\mathrm{Hom}_R(0, M)$ each contain a single element: the zero map. Thus, $\mathscr{R}\text{-}\mathscr{mod}$ is preadditive and it remains to show only that for every pair M, $N \in \mathscr{R}\text{-}\mathscr{mod}$ there exists a coproduct $M \oplus N$ in $\mathscr{R}\text{-}\mathscr{mod}$. Let X be any R-module, along with any R-module homomorphisms $f_M : M \to X$ and $f_N : N \to X$ as given in the diagram



where ι_M and ι_N are the natural inclusions. Define $f\colon M \oplus N \to X$ as $(f_M \circ \pi_M) \oplus (f_N \circ \pi_N)$. First note that for any $m \in M$ we have that $[((f_M \circ \pi_M) \oplus (f_N \circ \pi_N)) \circ \iota_M](m) = ((f_M \circ \pi_M) \oplus (f_N \circ \pi_N))(m, 0) = f_M(m)$ and for any $n \in N$ we have that $[((f_M \circ \pi_M) \oplus (f_N \circ \pi_N)) \circ \iota_N](n) = ((f_M \circ \pi_M) \oplus (f_N \circ \pi_N))(0, n) = f_N(n)$. Thus, the diagram is commutative and we only need to show uniqueness of f. Suppose there exists a map $g\colon M \oplus N \to X$ such that the above diagram commutes. Then, by definition $f_M = g \circ \iota_M$ and $f_N = g \circ \iota_N$ so that for any $(m, n) \in M \oplus N$ we have that

$$\begin{aligned} f(m,n) &= [(f_M \circ \pi_M) \oplus (f_N \circ \pi_N)](m,n) \\ &= [((g \circ \iota_M) \circ \pi_M) \oplus ((g \circ \iota_N) \circ \pi_N)](m,n) \\ &= g(\iota_M \pi_M)(m,n) \oplus g(\iota_N \pi_N)(m,n) \\ &= g(m,0) \oplus g(0,n) = g(m,n). \end{aligned}$$

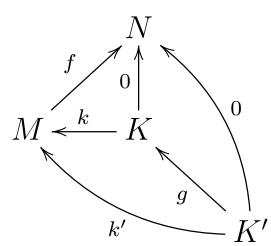
Thus, we have shown the equality $f = g$, which in turn demonstrates the uniqueness of f, as desired. $\square$

1.1.1 $\mathscr{R}$-$\mathscr{mod}$ is an Abelian Category

Informally, an additive category can be thought of as one in which objects and the morphisms between them can be *added*. Moreover, such a category is the first step towards recognizing the categorical structure inspired by abelian groups. As in the case of this prototypical example, abelian categories are those that are not only additive, but that which also have the additional existence of kernels and cokernels. Such objects within the category yield a rich theory of various constructions, such as short exact sequences, and we now look towards recognizing these constructions within $\mathscr{R}$-$\mathscr{mod}$.

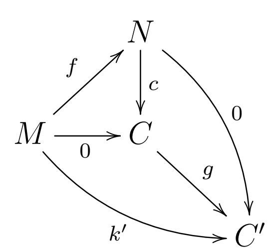
For any morphism $f\colon M \to N \in \mathscr{R}$-$\mathscr{mod}$, the *kernel* of f is an object K together with a morphism $k\colon K \to M$ such that (i) $f \circ k = 0$ and (ii) the following universal property is satisfied.

Universal Property (cf. [HoJoRo])**.** For every $k' : K' \to M$ such that $f \circ k' = 0$ there is a unique morphism $g\colon K' \to K$ making the following diagram commute:



Roughly, we are saying that any other such k' satisfying the first condition must *factor through* k. If such a K exists, denote it as $\ker(f)$, and, moreover, it is unique up to isomorphism. The dual notion is the *cokernel* of f, defined as an object C together with a morphism $c\colon C \to M$ such that (i) $c \circ f = 0$ and (ii) the following universal property is satisfied.

Universal Property (cf. [HoJoRo])**.** For every $c' : N \to C'$ such that $c' \circ f = 0$ there is a unique morphism $g\colon C \to C'$ making the following diagram commute:



If such a C exists, denote it $\operatorname{coker}(f)$, and, moreover, it is unique up to isomorphism. We define the *image* of a morphism f to be $\ker(c)$ where $c\colon N \to \operatorname{coker}(f)$ and denote it $\operatorname{im}(f)$. Similarly, we define the *coimage* of f to be the cokernel of

$k\colon \ker(f) \to M$ and denote it $\mathrm{coim}(f)$. With respect to these categorical definitions in $\mathcal{R}$-$\mathcal{mod}$, the usual definitions of kernel and cokernel do in fact satisfy the above universal properties. That is, $\ker(f) = \{x \in M \mid f(x) = 0\}$ and we may view $\mathrm{coker}(f) \cong \frac{N}{\mathrm{im}(f)}$ where $\mathrm{im}(f) = \{f(x) \in N \mid x \in M\}$.

While it is true that for a category to be Abelian, kernels and cokernels must exist for each morphism, one more condition must also be satisfied: the kernels and cokernels must have "desirable" properties. One way of describing such properties is to say that every monomorphism is the kernel of some morphism and every epimorphism is the cokernel of some morphism. We will explore these special types of morphisms shortly, but for now we will simply use the equivalent requirement (given in [HoJoRo]) that for every $f\colon M \to N$ the natural morphism $\phi\colon \mathrm{coim}(f) \to \mathrm{im}(f)$, guaranteed by the existence of $\ker(f)$ and $\mathrm{coker}(f)$, is an isomorphism.

That ϕ is an isomorphism follows directly from the First Isomorphism Theorem for R-modules: $\mathrm{im}(f) \cong M/\ker(f)$. Now, it is worth noting that while $\mathcal{R}$-$\mathcal{Mod}$ is an abelian category, $\mathcal{R}$-$\mathcal{mod}$ is not always necessarily so; kernels of R-module homomorphisms between finitely generated modules may not be finitely generated themselves. However, whenever R is noetherian this does not occur, in which case $\mathcal{R}$-$\mathcal{mod}$ is an abelian category. As we generally assume R is noetherian throughout this thesis, our focus on $\mathcal{R}$-$\mathcal{mod}$ as an abelian category is not deterred.

1.1.2 Morphisms in $\mathcal{R}$-$\mathcal{mod}$

Special types of morphisms are interwoven throughout the thematic elements of this thesis, and thus we present the common definitions of such morphisms in $\mathcal{R}$-$\mathcal{mod}$ as well as their categorical counterparts. The following definitions are the typical ones given in any elementary algebra text (e.g. [Hu] or [DuFo]) discussing R-module homomorphisms. A morphism $f\colon M \to N$ is called *surjective* (or *is a surjection*) if for each $n \in N$ there exists an $m \in M$ such that $f(m) = n$. Alternatively, f is called *injective* (or *is an injection*) if $f(m) = f(m')$ in N implies $m = m'$ in M. It is an elementary proof to show that this definition is equivalent to saying $\ker(f) = 0$. If f is both injective and surjective, then we call f a *bijection* and say it is both one-to-one (injective) and onto (surjective).

We now introduce the notions of *monomorphism* and *epimorphism*, which are the categorical analogues of injective and surjective functions. A morphism $f\colon X \to Y$ in a category is a monomorphism if it is *left-cancellative*:

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2$$

for any other morphisms $g_1 : Z \to X$ and $g_2 : Z \to Y$. In the same vein, f is an epimorphism if it is *right-cancellative*:

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

A morphism that is both a monomorphism and epimorphism is an *isomorphism*. In $\mathcal{R}$-*mod* monomorphisms coincide with injections and epimorphisms coincide with surjections; therefore, an R-module homomorphism that is bijective is an isomorphism. This is because $\mathcal{R}$-*mod* is both an abelian and a *concrete* category, which is described in the remark below. Note further that we make no use of left- or right- inverses in our definitions above. It does hold that a morphism which has a *left* inverse will imply it is left cancellative and thus a monomorphism. Likewise, a morphism which has a *right* inverse will imply it is right cancellative and thus an epimorphism. However, the existence of a left or right inverse is actually a bit stronger than the notions of mono- and epimorphisms; hence, we consider morphisms as *split* monomorphisms or epimorphisms if they have either a left or right inverse, respectively.

Remark. *One distinction between $\mathcal{R}$-mod and the category we will explore in Chapter 3 is that $\mathcal{R}$-mod is an abelian category. This particular characteristic, or lack thereof, will have an intriguing role to play in the main topics of this thesis. Another aspect about $\mathcal{R}$-mod to point out is that it is a* concrete *category, meaning that there exists a faithful functor $\mathcal{F} : \mathcal{R}\text{-mod} \to \text{Set}$ (where* Set *denotes the category of sets).*

1.1.3 Functors on $\mathcal{R}$-mod

The remark above indicates that we should explore the primary functors which take objects in $\mathcal{R}$-mod to other objects in $\mathcal{R}$-mod, or objects from another category, such as $\mathcal{Ab}$ (the category of abelian groups). A discussion of these functors can be found in any text on homological algebra (e.g. see [Ro] or [CaEi]), and we merely present such well-known data with respect to commutative rings. As mentioned previously, a *functor* is a structure-preserving map from one category to another; or, we might consider an *endo*functor, which is a functor from a category to itself. In particular, functors not only take objects to objects and morphisms to morphisms, but they also preserve the identity morphism and compositions of morphisms.

There exist notions of covariant and contravariant functors, where the latter type "flips" arrows and reverses compositions. We may define a wide variety of functors; for example, we may define a basic type called a *forgetful* functor in which we map objects from a category to their underlying sets and the morphisms to the underlying functions on those sets. Essentially, the idea is that we "forget" some structure in a particular category as the functor takes objects and morphisms to a category with less structure. As it turns out, there exists such a functor $\mathcal{U} : \mathcal{R}\text{-mod} \to \text{Set}$ which turns out to be *faithful*, meaning that $\mathcal{R}$-mod is a concrete category (as previously mentioned).

Of particular importance are the two quintessential types of functors studied in homological algebra, especially with respect to $\mathscr{R}$-$\mathscr{mod}$: Hom and $\otimes$. We have already discussed the fact that for any two R-modules M and N, $\mathrm{Hom}_R(M, N)$ will be an R-module under the given assumptions. For this reason, we can actually define a covariant functor $\mathrm{Hom}_R(M, -)\colon \mathscr{R}\text{-}\mathscr{mod} \to \mathscr{R}\text{-}\mathscr{mod}$ for each R-module M and it should be clear that this functor takes R-modules to the Hom-sets representing all morphisms between M and the R-modules. To see how the functor acts on morphisms, note that for an R-module homomorphism $f\colon N \to N'$ and any R-module homomorphism $g\colon M \to N$ the following diagram commutes

$$\begin{array}{ccc} N & \xrightarrow{\;f\;} & N' \\ {\scriptstyle g}\big\uparrow & \nearrow {\scriptstyle f\circ g := \mathrm{Hom}_R(M,f)(g)} & \\ M & & \end{array}$$

That is, the functor $\mathrm{Hom}_R(M, -)$ takes morphisms to their respective composite morphisms. Likewise, we may define a contravariant functor $\mathrm{Hom}_R(-, M)\colon \mathscr{R}\text{-}\mathscr{mod}$ $\to \mathscr{R}\text{-}\mathscr{mod}$ for each R-module M and it should be clear that this functor takes R-modules to the Hom-sets representing all morphisms between the R-modules and M. And, for any R-module homomorphism $f\colon N \to N'$ there exists an R-module homomorphism $\mathrm{Hom}_R(f, M) : \mathrm{Hom}_R(N', M) \to \mathrm{Hom}_R(N, M)$ such that for any R-module homomorphism $g\colon N' \to M$ the following diagram commutes

$$\begin{array}{ccc} N' & \xleftarrow{\;f\;} & N \\ {\scriptstyle g}\big\downarrow & \swarrow {\scriptstyle g\circ f := \mathrm{Hom}_R(f,M)(g)} & \\ M & & \end{array}$$

It should be an easy check that both the given covariant and contravariant functors preserve the identity morphism and composition, noting that the latter reverses the composition of morphisms. Moreover, it is worth noting that we may combine the actions of these two functors since the diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(M,N) & \xrightarrow{\mathrm{Hom}_R(f,N)} & \mathrm{Hom}_R(M',N) \\ \downarrow{\scriptstyle \mathrm{Hom}_R(M,g)} & & \downarrow{\scriptstyle \mathrm{Hom}_R(M',g)} \\ \mathrm{Hom}_R(M,N') & \xrightarrow{\mathrm{Hom}_R(f,N')} & \mathrm{Hom}_R(M',N') \end{array}$$

commutes for any pair of R-module homomorphisms $f\colon M' \to M$ and $g\colon N \to N'$. This implies that $\mathrm{Hom}_R(-,-)\colon \mathcal{R}\text{-}\mathcal{mod} \times \mathcal{R}\text{-}\mathcal{mod} \to \mathcal{R}\text{-}\mathcal{mod}$ is actually a bifunctor. On this note, we now give more abstract definitions of monomorphisms and epimorphisms with respect to the Hom functor and which will be commonly utilized within this thesis.

Definition 1.2 (cf. [KaSc])**.** A morphism $f\colon M \to N$ in $\mathcal{R}\text{-}\mathcal{mod}$ is a *monomorphism* if for every R-module L the functor $\mathrm{Hom}_R(L,-)$ takes f to an injective function between Hom-sets

$$\mathrm{Hom}_R(L,M) \xhookrightarrow{\mathrm{Hom}(L,f)} \mathrm{Hom}_R(L,N).$$

It is quite easy to understand why the above definition is equivalent to the usual one. For any R-module L and any $g_1,\ g_2 \in \mathrm{Hom}_R(L,M)$, note that the statement $fg_1 = fg_2$ implies $g_1 = g_2$ is true if and only if $\mathrm{Hom}_R(L,f)(g_1) = \mathrm{Hom}_R(L,f)(g_2)$ implies $g_1 = g_2$ via the action of $\mathrm{Hom}(L,-)$ on R-module homomorphisms. Similarly, we can consider, for $g_1,\ g_2 \in \mathrm{Hom}_R(L,M)$, the statement $g_1 f = g_2 f$ implies $g_1 = g_2$, noting that this is true if and only if the contravariant Hom yields an injective map $\mathrm{Hom}_R(f,L)\colon \mathrm{Hom}_R(N,L) \to \mathrm{Hom}_R(M,L)$.

Definition 1.3 (cf. [KaSc])**.** A morphism $f\colon M \to N$ in $\mathcal{R}\text{-}\mathcal{mod}$ is a *epimorphism* if for every R-module L the functor $\mathrm{Hom}_R(-,L)$ takes f to an injective function between Hom-sets

$$\mathrm{Hom}_R(N,L) \xhookrightarrow{\mathrm{Hom}(f,L)} \mathrm{Hom}_R(M,L).$$

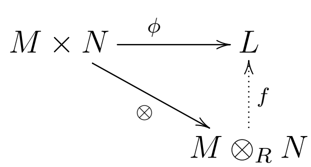
Tensor Product of R-Modules and the Tensor Functor

In addition to Hom, we may also consider the *tensor product* of R-modules, which will again be an R-module (see [Ro], [Hu], or any other algebra text, for the following definition and properties discussed). Recall that for R-modules M and N we define $M \otimes_R N = \frac{\mathcal{F}(M \times N)}{X}$ where $\mathcal{F}(M \times N)$ is the free R-module with basis $M \times N$ and X is the submodule generated by the usual relations:

1. $(m_1 + m_2, n) - (m_1, n) - (m_2, n)$
2. $(m, n_1 + n_2) - (m, n_1) - (m, n_2)$
3. $(rm, n) - r(m, n)$
4. $(m, rn) - r(m, n)$

with $m_1, m_2, m \in M$, $n_1, n_2, n \in N$, and $r \in R$. Categorically, we say that $M \otimes_R N$ together with the bilinear map $\otimes$ solves the following universal mapping problem:

Universal Property. For any R-module L with a bilinear map $\phi\colon M \times N \to L$ there exists a unique homomorphism $f\colon M \otimes_R N \to L$ making the diagram



commute; that is, $f\otimes = \phi$.

Properties. Let M, N be R-modules, $\{B_i \mid i \in \Gamma\}$ be a family of R-modules, and F, G be free R-modules. Then recall the following properties for tensor products of R-modules:

1. $M \otimes_R N \cong N \otimes_R M$
2. $M \otimes_R (\bigoplus_i B_i) \cong \bigoplus_i (M \otimes_R B_i)$
3. $\mathrm{rk}(F \otimes_R G) = \mathrm{rk}(F) \cdot \mathrm{rk}(G)$

At this point, we now transition into viewing $\otimes$ as a *functor*. Similarly to Hom, we may fix any R-module M to see that $M \otimes_R -$ and $- \otimes_R M$ are endofunctors which take any R-module N to the tensor product R-modules $M \otimes_R N$ or $N \otimes_R M$. From the first property given above, we see that there is no distinction between these two products. Moreover, we have a similar naturality as demonstrated with Hom so that we may also consider the bifunctor $- \otimes_R - \colon \mathscr{R}\text{-}\mathscr{mod} \times \mathscr{R}\text{-}\mathscr{mod} \to \mathscr{R}\text{-}\mathscr{mod}$. We only need to ascertain how $\otimes$ acts on morphisms: for any R-module homomorphisms $f : M \to M'$ and $g : N \to N'$ we define $f \otimes g : M \otimes_R N \to M' \otimes_R N'$ so that $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$ for any $m \in M$, $n \in N$, and extend by linearity. It is easy to check that $- \otimes -$ preserves the identity and compositions, as a special case demonstrates in Chapter 4 of this thesis.

1.2 Special Constructions of R-Modules

Now that we have developed the blueprint for the categorical structure of $\mathscr{R}\text{-}\mathscr{mod}$ it is paramount to discuss particular constructions of the objects and morphisms in the category, which turn out to be vital towards better understanding the structure of the objects themselves. Once again, many of these constructions can be found in any homological text; for instance, the reader may refer to [Ro], [CaEi], or [Ve3].

1.2.1 Short Exact Sequences

First recall that $\mathscr{R}\text{-}\mathscr{mod}$ is an abelian category, resulting in the existence of kernels and cokernels of morphisms with the category. What arises due to this is the notion of a *short exact sequence.*

Example 1.4. Given an R-module homomorphism $f \colon M \to N$ note that $\ker(f)$ is a submodule of M, $\operatorname{im}(f)$ is a submodule of N, and $\operatorname{coker}(f)$ is a quotient module of N. Thus consider the following two sequences of R-modules and R-module homomorphisms,

$$0 \to \ker(f) \xrightarrow{k} M \xrightarrow{\hat{f}} \operatorname{im}(f) \to 0 \qquad \text{and} \qquad 0 \to \operatorname{im}(f) \xrightarrow{k_c} N \xrightarrow{c} N/\operatorname{im}(f) \to 0,$$

where k is the natural embedding of $\ker(f)$ into M, $\hat{f}$ is the restriction of f to $\operatorname{im}(f)$, c is the natural surjection from N onto $\operatorname{coker}(f)$, and k_c is the natural embedding of $\operatorname{im}(f) = \ker(c)$ into N. First note that any element in $\ker f \supseteq \ker(\hat{f})$ is trivially in the image of k since $\ker(f)$ just gets mapped to itself in M. Moreover, since $f \circ k = 0$ by definition of the kernel, we will have that $\hat{f} \circ k = 0$ and thus $\operatorname{im}(k) \subseteq \ker(f)$ as well. In other words, the image of k is *exactly* the kernel of f. When this occurs, we say that the sequence is *exact* at that spot. We see the the same occurs for the latter sequence since $c \circ k_c = 0$ as $\operatorname{im}(f)$ is just the kernel of c. Note further that in both cases we have *exactness* at the tails too– k is injective, thus having only 0 in its kernel; f is surjective, thus $\operatorname{im}(f) = N$ which is then subsequently mapped to 0; likewise for k_c and c. Consequently, both sequences above are examples of what we call *short exact sequences.*

Definition 1.5 (cf. [Hu]). A *short exact sequence* of R-modules and R-module homomorphisms is an *exact* sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

That is, $\operatorname{im}(f) = \ker(g)$, f is a monomorphism, and g is an epimorphism. We may think of $A \overset{f}{\hookrightarrow} B$ as embedding A into B and C as isomorphic to the quotient module $B/\operatorname{im}(f)$.

Example 1.6. Given a *surjective* R-module homomorphism $f\colon M \to N$, the following sequence of R-modules and R-module homomorphisms is a short exact sequence:

$$0 \to \ker(f) \overset{k}{\hookrightarrow} M \overset{f}{\twoheadrightarrow} N \to 0.$$

Here, k is the natural embedding of $\ker(f)$ into M meaning that any element in the kernel of f is trivially in the image of k since $\ker(f)$ just gets mapped to itself in M. Moreover, since $f \circ k = 0$ by definition of the kernel, we will have that $\operatorname{im}(k) \subseteq \ker(f)$ as well. In other words, the image of k is *exactly* the kernel of f. Similarly, if $f\colon M \to N$ is an *injective* R-module homomorphism then note that we may form the short exact sequence

$$0 \to M \overset{f}{\hookrightarrow} N \overset{c}{\twoheadrightarrow} \operatorname{coker}(f) \to 0$$

where c is the surjection of N onto $\operatorname{coker}(f) \cong N/\operatorname{im}(f)$.

Definition 1.7 (cf. [Hu])**.** We say that the short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *splits* if f is a split monomorphism or g is a split epimorphism.

Recall from the end of Section 1.1.2 that if f is a split monomorphism, there exists a left inverse; likewise, if g is a split epimorphism it has a right inverse. Hence, by definition, a *split* short exact sequence is one in which there exists a map $h\colon C \to B$ such that $g \circ h = \mathrm{Id}_C$, or there exists a map $k\colon B \to A$ such that $k \circ f = \mathrm{Id}_A$. As $\mathscr{R}$-$\mathscr{mod}$ is an abelian category, the splitting lemma holds so that these two conditions are in fact equivalent. Furthermore, these conditions are equivalent to the statement that $B \cong A \oplus C$, where $A \cong \operatorname{im}(f)$ and $C \cong \ker(k)$.

Note that equivalently, we may characterize the split conditions as saying that every map $l_1 : A \to \operatorname{im}(f)$ must factor through f and every map $l_2 : \ker(g) \to C$ must factor through g. As indicated above, the quintessential example of a *split* short exact sequence is one of the form

$$0 \to A \xrightarrow{f} A \oplus C \xrightarrow{g} C \to 0$$

with A and C as previously given. However, not every short exact sequence splits as demonstrated in the following example.

Example 1.8. Let $R = \Bbbk[x]/(x^2)$ and $I = (x) \subseteq R$. Now consider the sequence

$$0 \to I \hookrightarrow R \twoheadrightarrow \Bbbk \to 0.$$

Note that $I \cong \Bbbk$ since multiplication by x annihilates any x multiple in R. But clearly, $R \ncong \Bbbk \oplus \Bbbk \cong I \oplus \Bbbk$. Hence the short exact sequence does *not* split.

Regardless of whether they are split or not, short exact sequences are prevalent in $\mathcal{R}$-$\mathcal{mod}$ and are guaranteed by the existence of kernels and cokernels; furthermore, they are necessary building blocks for *long exact sequences*, which are our next construction of interest.

1.2.2 Resolutions and Long Exact Sequences

A *long exact sequence* is a sequence of R-modules and R-module homomorphisms that is *exact* at each degree. That is, the sequence

$$\cdots \to A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \to \cdots$$

is a long exact sequence if $\ker(f_n) = \operatorname{im}(f_{n+1})$ for each $n \in \mathbb{Z}$. Here, it could be that $A_n \neq 0$ for each $n \in \mathbb{Z}$, $A_n = 0$ for $n \gg 0$, $A_n = 0$ for $n \ll 0$, or $A_n = 0$ for $n \gg 0$ *and* $n \ll 0$. We denote the latter three cases by calling the sequence *bounded above*, *bounded below*, or *bounded*, respectively. Roughly speaking, we can view a long exact sequence as finite (bounded), infinite in one direction (bounded above or below), or doubly infinite (neither bounded above nor below). It is important to note that each long exact sequence of R-modules can actually be broken down into a sequence of short exact sequences. There is also a significant connection between short exact sequences and a special type of long exact sequence, which will be presented shortly. One interesting question is whether or not a particular functor preserves *exactness* of a sequence. For instance, if we apply the covariant Hom functor to a short exact sequence, is the sequence

$$0 \to \mathrm{Hom}_R(M,A) \xrightarrow{\mathrm{Hom}_R(M,f)} \mathrm{Hom}_R(M,B) \xrightarrow{\mathrm{Hom}_R(M,g)} \mathrm{Hom}_R(M,C) \to 0,$$

where M is any R-module, also exact? As it turns out $\mathrm{Hom}_R(M,-)$ is only *left*-exact, meaning that only the sequence

$$0 \to \mathrm{Hom}_R(M,A) \xrightarrow{\mathrm{Hom}_R(M,f)} \mathrm{Hom}_R(M,B) \xrightarrow{\mathrm{Hom}_R(M,g)} \mathrm{Hom}_R(M,C)$$

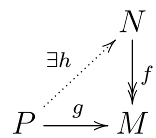
is exact. Nonetheless, in the next section we will explore a particular class of modules for which the first sequence is exact too. Likewise, the contravariant functor $\mathrm{Hom}_R(-,M)$ is also left-exact in general, but there exist special cases for which it is exact as well. Alternatively, the tensor functor is *right*-exact so given the exact sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the sequence

$$M \otimes_R A \xrightarrow{M\otimes f} M \otimes_R B \xrightarrow{M\otimes g} M \otimes_R C \to 0,$$

for any R-module M is also exact. A significant point to note here is that if we merely consider the exact sequence $A \to B \to C$, applying $M \otimes_R -$ will not yield an exact sequence, meaning that the surjectivity of g is a necessary assumption for the right-exactness of $\otimes$. However, we will later explore certain assumptions which lead to the exactness of $\otimes$ as well. For now, we will begin a brief discussion of *resolutions* of R-modules.

1.2.2.1 Projective Modules and Resolutions

Definition 1.9 (cf. [Ro], [Hu])**.** An R-module P is called *projective* if (and only if) for every surjection $f\colon N \to M$ and every homomorphism $g\colon P \to M$ there exists an R-module homomorphism $h\colon P \to N$ such that $fh = g$, meaning that the diagram



commutes. We may characterize this "lifting" property by stating that any morphism from P to M *factors through* an epimorphism on M.

As it turns out, $\mathrm{Hom}_R(P, -)$ is exact if and only if P is projective. Similarly, we obtain exactness of $P \otimes_R -$ for any projective R-module P, since all finitely generated projective modules (over a commutative ring) are flat (see §2.5 of [Wa]). However, projective modules do not yield exactness with the contravariant functor Hom, but rather the dual notion of *injective* module is needed here. The topics of this thesis do not involve such modules, so we will omit a discussion, but their definition and properties are commonly discussed in any algebra text involving R-modules (e.g. [Hu], [Ro], [DuFo]).

Instead, we turn towards long exact sequences of projective modules. A *projective resolution* of an R-module M is an exact sequence of the form

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0$$

where P_i is a projective R-module for each $i \in \mathbb{Z}$. We refer to the *deleted* projective resolution as the above sequence written without M, noting that this sequence will be exact for each $i > 0$ but not *at* $i = 0$. If there exists such a sequence (deleted or not) which is finite, then the *projective dimension* of M, denoted $\mathrm{pd}_R(M)$, is the minimal length of all such finite sequences.

Example 1.10. Let M be an R-module such that $\mathrm{pd}_R(M) = 0$ so that there exists a projective resolution of the form $0 \to P_0 \to M \to 0$. Since this sequence is exact, note then that $P \cong M$ and thus M must be projective itself.

As for the existence of projective resolutions, this depends on whether or not there are *enough projectives* in the category. Seemingly a vague statement, this is actually a rigorous characteristic. Projective modules in $\mathscr{R}$-$\mathscr{mod}$ satisfy the categorical notion of projective objects, which are in fact a generalization of such modules for any other category (just use the term *epimorphism* in place of *surjection*). We say that an abelian category $\mathscr{A}$ has *enough projectives* if for every object $A \in \mathscr{A}$ there exists a projective object $P \in \mathscr{A}$ and an epimorphism $P \to A$; that is, if there exists a short exact sequence of the form $0 \to K \to P \to A \to 0$ (See [KaSc, 8.4.1]). It does in fact hold that $\mathscr{R}$-$\mathscr{mod}$ has enough projectives, and we will discuss this further in Section 1.3 of this chapter.

1.2.2.2 R-Complexes

We now consider one more construction involving the objects and arrows in $\mathscr{R}$-$\mathscr{mod}$ that is a bit more general than a resolution. This type of construction will of course reappear in Chapter 3 of this thesis as a quintessential building block for the category introduced then. For now, we simply make precise the definition and a significant relationship between morphisms of such constructions.

Definition 1.11 (cf. [Ro]). An R-complex is a sequence of R-modules and R-module homomorphisms such that each homomorphism maps the preceding module into the kernel of the directly subsequent map. That is, a sequence of the form

$$\mathrm{C} : \cdots \to C_{n+2} \xrightarrow{\partial^{\mathrm{C}}_{n+2}} C_{n+1} \xrightarrow{\partial^{\mathrm{C}}_{n+1}} C_n \xrightarrow{\partial^{\mathrm{C}}_{n}} C_{n-1} \xrightarrow{\partial^{\mathrm{C}}_{n-1}} C_{n-2} \to \cdots$$

where each $C_n \in \mathscr{R}$-$\mathscr{mod}$, each ∂^{C}_n is a morphism in $\mathscr{R}$-$\mathscr{mod}$, and $\mathrm{Im}(\partial^{\mathrm{C}}_{n+1}) \subseteq \mathrm{Ker}(\partial^{\mathrm{C}}_n)$ for each $n \in \mathbb{Z}$.

Note that a deleted projective resolution of an R-module M (where $M \cong \operatorname{coker} \partial_1$) is a *type* of R-complex where $C_n = 0$ for all $n < 0$, as is any long exact sequence of R-modules. However, R-complexes are not exact sequences themselves, as we would have to impose the additional condition that $\mathrm{Ker}(\partial_n^{\mathrm{C}}) \subseteq \mathrm{Im}(\partial_{n+1}^{\mathrm{C}})$ for each $n \in \mathbb{Z}$ in order for an R-complex to be exact. It is common to refer to an R-complex as $(\mathrm{C}, \partial^{\mathrm{C}})$ since the R-module homomorphisms in the sequence, called the *differentials*, are an important piece of the complex. As mentioned, we will explore more characteristics of R-complexes in Chapter 3, but for now we merely define a few notions necessary for the discussions included in the remainder of this Chapter.

Definition 1.12 (cf. [Ro]). An R-complex *morphism* (also called a *chain map*) $\mathrm{f}\colon \mathrm{C} \to \mathrm{D}$ is a family of R-module homomorphisms $\{f_n\}_{n\in\mathbb{Z}}$ such that $\partial_n^D f_n = f_{n-1}\partial_n^C$ for each $n \in \mathbb{Z}$. Equivalently, we say that all squares in the following diagram commute:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}^C} & C_n & \xrightarrow{\partial_n^C} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & D_{n+1} & \xrightarrow{\partial_{n+1}^D} & D_n & \xrightarrow{\partial_n^D} & D_{n-1} & \longrightarrow & \cdots
\end{array}$$

Since R-complexes are not exact, it makes sense to explore to what degree the complex is not exact; meaning, *how much* of the kernel of the subsequent map is *not* contained in the image of the preceding map? The notion of *homology* helps us explore this question.

Definition 1.13 (cf. [Ro]). Let $(\mathrm{C}, \partial^{\mathrm{C}})$ be an R-complex and define the *homology* of C, denoted H(C), as the graded R-module where

$$\mathrm{H}(\mathrm{C}) = \bigoplus_n \mathrm{H}(\mathrm{C})_n$$

and

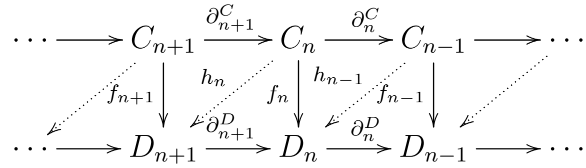
$$\mathrm{H}_n(\mathrm{C}) := \frac{\mathrm{Ker}(\partial_n^{\mathrm{C}})}{\mathrm{Im}(\partial_{n+1}^{\mathrm{C}})}$$

for all $n \in \mathbb{Z}$. Call $\mathrm{H}_n(\mathrm{C})$ the n^{th} homology of $(\mathrm{C}, \partial^{\mathrm{C}})$.

First note that given a morphism of complexes $\mathrm{f} : \mathrm{C} \to \mathrm{D}$, we may define a graded R-module homomorphism $\mathrm{H}(\mathrm{f}) : \mathrm{H}(\mathrm{C}) \to \mathrm{H}(\mathrm{D})$ where $\mathrm{H}_n(\mathrm{f})(z + \mathrm{Im}\partial^{\mathrm{C}}_{n+1}) = f_n(z) + \mathrm{Im}\partial^{\mathrm{D}}_{n+1}$ for any $z \in \mathrm{Ker}(\partial^{\mathrm{C}}_n)$. Moreover, it is clear that the n^{th} homology represents the *equivalence classes* of the kernel elements in the n^{th} degree. Of course, when C is exact, $\mathrm{H}(\mathrm{C}) = 0$ at every degree.

By nature of R-complexes, it is rather uninteresting when two R-complexes are completely the same, in the sense of an R-complex chain map being comprised of R-module *isomorphisms* at each degree. Instead, we present two alternative "equivalencies" of R-complexes, which yield a richer theory. First, we might consider when the *homologies* of two complexes are the same; that is to say, $\mathrm{H}(\mathrm{C}) \cong \mathrm{H}(\mathrm{D})$ at each degree. In this case, we define a morphism of R-complexes $\mathrm{f} : \mathrm{C} \to \mathrm{D}$ to be a *quasi-isomorphism* (or, shorthand, a "quism") if $\mathrm{H}_n(\mathrm{f})$ is an isomorphism for each $n \in \mathbb{Z}$. While this might provide some powerful insight into similarities of complexes that are *not* exact, it turns out to be quite useless in examining structural similarities whenever two R-complexes are both exact, since the homologies are all zero in this case. Of course, this thesis will deal predominantly with long exact sequences, and so we focus on the latter type of "equivalency" for R-complexes, known as *homotopy equivalence.*

Definition 1.14 (cf. [Ro])**.** A morphism of R-complexes $\mathrm{f}\colon \mathrm{C} \to \mathrm{D}$ is called *null-homotopic* if there exist a family of R-module homomorphisms $h_n\colon C_n \to D_{n+1}$ such that $f_n = h_{n-1}\partial^C_n + \partial^D_{n+1}h_n$ for each $n \in \mathbb{Z}$. When f is null-homotopic, we write $f \sim 0$ and visually it means that each appropriate parallelogram in the diagram below is equal to the appropriate vertical arrows.



If $\mathrm{f} : \mathrm{C} \to \mathrm{D}$ and $\mathrm{g} : \mathrm{C} \to \mathrm{D}$ such that $\mathrm{f} - \mathrm{g} \sim 0$, then we say that f and g are *homotopic*, writing $\mathrm{f} \sim \mathrm{g}$.

Remark. *Note that homotopy implies a quasi-isomorphism; that is, if* $\mathrm{f} \sim \mathrm{g}$*, then* $\mathrm{H(f)} = \mathrm{H(g)}$*. Of course, this is trivially true for exact complexes.*

Lastly, we say that two R-complexes C and D are *homotopically equivalent* if there exist chain maps $\mathrm{f}\colon \mathrm{C} \to \mathrm{D}$ and $\mathrm{g}\colon \mathrm{D} \to \mathrm{C}$ such that $\mathrm{fg} \sim \mathrm{Id_D}$ and $\mathrm{gf} \sim \mathrm{Id_C}$. In this case, we write $\mathrm{C} \simeq \mathrm{D}$.

1.2.3 Derived Functors on $\mathscr{R}$-*mod*

With regard to the previously discussed constructions of R-modules, we might ask the question: is there a deeper connection between short and long exact sequences? In particular, consider the fact that $\mathrm{Hom}_R(-, M)$ is left exact for any M in $\mathscr{R}$-*mod*. Hence, given a short exact sequence $0 \to A \to B \to C \to 0$, we have that the sequence $0 \to \mathrm{Hom}_R(C, M) \to \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(C, M)$ is exact. One might ponder whether or not we may extend this sequence to a long exact sequence. While in general, short exact sequences of objects may lead to long exact sequences in any number of ways, some categories that are particularly "nice" actually yield a canonical way to extend such a short left exact sequence to a long exact sequence. Unsurprisingly, $\mathscr{R}$-*mod* is such a category and this topic carries us to the discussion of the derived functors of Hom.

In essence, derived functors are just those derived from other functors; the predominant derived functors are called Ext and Tor, which are derived from the functors Hom and $\otimes$, respectively. The topics of this thesis do not rely upon Tor to any great extent, and so we will predominantly discuss Ext with only the comment that Tor is computed similarly, except for using $\otimes$ in place of Hom. Additionally, the description given for Ext will make use of the contravariant Hom and projective resolutions, but note that there is a dual description using the covariant Hom and *injective* resolutions.

Construction (Ext of R-Modules, [Ro])**.** Given a fixed R-module N, define $\operatorname{Ext}^i_R(-,N)$ as the *right* derived functor of $\operatorname{Hom}_R(-,N)$ for each $i \in \mathbb{N}$. That is, for any R-module M, define

$$\operatorname{Ext}^i_R(M,N) = \mathrm{R}^i \operatorname{Hom}_R(M,N)$$

where we compute the i^{th} right derived functor as follows[1]. Let $\mathbf{P}$ be any projective resolution of M (recall since $\mathscr{R}$-$\mathscr{mod}$ has enough projectives, such a resolution exists):

$$\mathbf{P}\colon \cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

Then apply the functor $\operatorname{Hom}_R(-,M)$ to the *deleted* projective resolution to obtain

$$0 \to \operatorname{Hom}_R(P_0,M) \to \operatorname{Hom}_R(P_1,M) \to \cdots \to \operatorname{Hom}_R(P_n,M) \to \cdots \qquad (\dagger)$$

and note that this will not be an exact sequence for $i > 0$. Meaning that we can compute its cohomology at the i^{th} spot and it will almost never be 0. Given a sequence $\mathbf{S}$, its *cohomology* is just the quotient submodule (or, subgroup in a more general scenario) defined by $\mathrm{H}^i(\mathbf{S}) = \ker(s_i)/\operatorname{im}(s_{i-1})$. Define $\operatorname{Ext}^i_R(M,N)$ as the i^{th} cohomology of the Hom sequence given above in ($\dagger$) so that, in this case,

$$\operatorname{Ext}^i_R(M,N) = \mathrm{H}^i(\operatorname{Hom}_R(\mathbf{P},M)) = \ker(p_i^*)/\operatorname{im}(p_{i-1}^*)$$

where $p_i^* = \operatorname{Hom}_R(p_{i+1},M)$ and $p_{i-1}^* = \operatorname{Hom}_R(p_i,M)$.

Based upon this construction, it should be clear that in a way, Ext measures Hom's failure to be exact. Interestingly enough, Ext functors yield a rich theory significant for understanding differences in R-module structure and are an integral

[1]Here, R is an operator representing the right derived functor and not to be confused with the ring R.

part of Homological Algebra. More on Ext will be discussed in subsequent sections, but for now we conclude this discussion with the following statement for short exact sequences in $\mathscr{R}$-$\mathscr{mod}$.

Proposition 1.15 (cf. [Ro]). *For any R-module A, every short exact sequence* $0 \to K \to L \to M \to 0$ *of R-modules induces a long exact sequence of the form*

$$0 \to \mathrm{Hom}_R(A, K) \to \mathrm{Hom}_R(A, L) \to \mathrm{Hom}_R(A, M) \to \mathrm{Ext}^1_R(A, K) \to \mathrm{Ext}^1_R(A, L) \to \cdots$$

and for any R-module B, every short exact sequence $0 \to K \to L \to M \to 0$ *of R-modules induces a long exact sequence of the form*

$$0 \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(L, B) \to \mathrm{Hom}_R(K, B) \to \mathrm{Ext}^1_R(M, B) \to \mathrm{Ext}^1_R(L, B) \to \cdots$$

1.2.3.1 Extensions

Before moving on to discuss the constructions which are a focal point for the theory presented in Chapter 2, we provide a definition related to the Ext functors and, in fact, served as original motivation for these functors. Given R-modules M and N, we can form an *extension of* M *by* N as the short exact sequence of the form

$$0 \to N \to E \to M \to 0$$

and we say two extensions are *equivalent* if there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

implying that the middle arrow is an isomorphism. Any extension equivalent to the split short exact sequence

$$0 \to N \to M \oplus N \to M \to 0$$

is defined to be the trivial extension. As it turns out, the equivalence classes of extensions are in one-to-one correspondence with elements in $\operatorname{Ext}_R^1(M,N)$ and we can generalize this notion to equivalence classes of *i-extensions*, which are exact sequences of the form $\xi\colon 0 \to N \to X_i \to \cdots \to X_1 \to M \to 0$. Unsurprisingly, we can view elements in each $\operatorname{Ext}_R^i(M,N)$ group as such $[\xi]$. Of course, there is also a way of "adjoining" extensions, called *Yoneda multiplication*, defined by a bilinear map $\operatorname{Ext}_R^i(M,N) \times \operatorname{Ext}_R^j(N,L) \to \operatorname{Ext}_R^{i+j}(M,L)$ where we simply concatenate the i- and j-extensions

$$\begin{array}{ccccccccccccccc} & & & & & & 0 & \longrightarrow & N & \longrightarrow & X_i & \longrightarrow \cdots \longrightarrow & X_1 & \longrightarrow & M \longrightarrow 0 \\ & & & & & & & & \| & & & & & & \\ 0 & \longrightarrow & L & \longrightarrow & Y_j & \longrightarrow \cdots \longrightarrow & Y_1 & \longrightarrow & N & \longrightarrow & 0 & & & & \end{array}$$

to obtain the $(i+j)$-extension

$$\begin{array}{ccccccccccccc} 0 \longrightarrow L \longrightarrow Y_j \longrightarrow \cdots \longrightarrow & Y_1 & & \longrightarrow & & X_i & \longrightarrow \cdots \longrightarrow X_1 \longrightarrow M \longrightarrow 0. \\ & & \searrow & & \nearrow & & \\ & & & N & & & \end{array}$$

The language of extensions will be used briefly in Chapter 2 of this thesis, but the topics will not involve a deeper understanding of extensions past what is explained above. For a more fulfilling coverage of extensions, the reader should refer to Chapter 14 of [CaEi], §3.4 of [We], or §4 of [Ma4].

1.3 Free Resolutions and Betti Sequences

By definition, all objects in $\mathscr{R}$-*mod* have a finite generating set; however, the majority of the time this set is not *linearly independent*, a characteristic lost by the looser structure of R-modules as compared to vector spaces. As we previously saw though, a special type of R-module yields more rigid structure regarding functors and those derived from them. Moreover, such modules are intimately connected with computing the Ext functors, which are in turn utilized to better understand the structure of R-modules. We now consider another special type of R-module that will turn out to be of special significance for this thesis: one that is *most like* a vector space.

Definition 1.16 (cf. [Ro])**.** A *free* R-module F is an R-module with generating set X such that X is a linearly independent subset of F. We call X the *basis* of F.

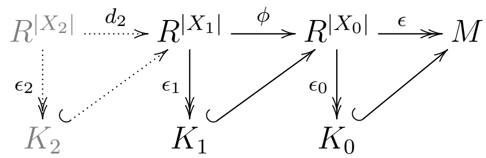
Such a module can be viewed as $F = \bigoplus_{i=1}^{n} R_i$, often denoted R^n, for some $n \in \mathbb{N}$, where each $R_i = R$ and n is the rank of X (F has n linearly independent elements in its generating set). This is because we can map the generating set of the module to each $e_i \in R^n$ (where e_i is just the column vector with the multiplicative identity of R in the i^{th} row).

For a module that is *not* free, we can make a comparison between itself and a sequence of free modules, along with the maps between them. The idea is that this sequence represents the relations on the generating set of M, the relations on those relations, ad infinitum– thus, as a whole the sequence describes M's failure to be free. This sequence is what we call a *free resolution* of the R-module M and its construction is described below.

Construction (cf. [Ro])**.** Given a finitely-generated R-module, M, let X_0 represent its generating set, which is *not* linearly independent. Consider the map $\epsilon\colon R^{|X_0|} \twoheadrightarrow X_0$ where each $e_i \mapsto x_i$ for each $x_i \in X_0$.

0. Compute $K_1 = \ker(\epsilon) \subseteq R^{|X_0|}$ and identify the set of generators for this submodule, denoting the set X_1. Note that by computing K_1, we are computing the *relations* on the generators of M, and since K_1 is likely to not be a free module itself, we may repeat this process.

1. Map the basis elements of $R^{|X_1|}$ to the generators of K_1, and extend this to obtain a map $\phi\colon R^{|K|} \to R^{|X_0|}$. We call ϕ the *free presentation* of M and note that the following diagram represents the given process:



2. As indicated above, we may continue the outlined process where we denote $K_2 = \ker(\phi)$ and, letting X_2 be the set of generators for K_2, define $\epsilon_2\colon R^{|X_2|} \twoheadrightarrow K_2$. Extending this map, we obtain a free module map $d_2\colon R^{|X_2|} \to R^{|X_1|}$.
3. Now, for any $i > 2$ denote $K_i = \ker(d_{i-1}) \subseteq R^{|X_{i-1}|}$ and X_i as the generating set. Then define $\epsilon_i\colon R^{|X_i|} \to K_i$ and extend to get $d_i\colon R^{|X_i|} \to R^{|X_{i-1}|}$.

At this point, we should make a few observations, the first being that at each degree $i > 0$ this sequence of R-modules and R-module homomorphisms is *exact* by design since $\operatorname{im}(d_i) = K_i = \ker(d_{i-1})$. Therefore, we obtain a long exact sequence of the following form

$$\mathbf{F} : \cdots \to F_n \xrightarrow{d_n} F_{n-1} \to \cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0$$

where each $F_i = R^{|X_i|}$ is a free R-module. Secondly, it should be clear that every finitely generated module has a free resolution since we can always map the basis of a free module to the generating set of a finitely-generated module that is not free, as described above. Lastly, we can construct a free resolution *minimally* by choosing a minimal generating set at each step in the construction described above. Whenever R is a local or graded ring, this *minimal* construction is unique. Recall that a *local ring* $(R, \mathfrak{m}, \Bbbk)$ is a ring with a unique maximal ideal $\mathfrak{m}$ and residue field $\Bbbk \cong R/\mathfrak{m}$. When R is *local*, we can define the resolution $\mathbf{F}$ to be minimal precisely when the differentials map into the maximal ideal: $d(\mathbf{F}) \subseteq \mathfrak{m}\mathbf{F}$.

> For the remainder of this thesis, $(R, \mathfrak{m}, \Bbbk)$ is assumed to be a *local* ring unless it is specifically stated otherwise. Note that this is in addition to our prior assumptions that R is a commutative noetherian ring.

Hence, every object in $\mathcal{R}$-$\mathcal{mod}$ has a minimal free resolution associated to it, and we will commonly denote this resolution as $\mathbf{F}$. If at any point $K_i = 0$, then note that this implies that we obtain an injective map between free modules at the i^{th} step and, moreover, $\mathbf{F}$ is finite in this case. However, it should be noted that $\mathbf{F}$ is only *sometimes* finite; for instance, Hilbert's Syzygy Theorem states that the minimal free resolution of a module over a polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$ in n indeterminants will be *at most* n. But if we instead consider $\mathbf{F}$ when M is a module over a different type of ring (such as a quotient R/I for some ideal $I \subseteq R$) we will often obtain an infinite free resolution, as the following examples demonstrate.

Example 1.17. Let $R = k[x, y, z]/(x^2, y^2, y - z)$, then the free resolution associated to the quotient module $R/(x, y, z)$ is given by:

$$\cdots \to R^6 \xrightarrow{\begin{pmatrix} z & -x & 0 & 0 & 0 & 0 \\ 0 & z & x & 0 & 0 & 0 \\ 0 & 0 & z & x & 0 & 0 \\ 0 & 0 & 0 & z & -x & 0 \\ 0 & 0 & 0 & 0 & z & x \end{pmatrix}} R^5 \xrightarrow{\begin{pmatrix} z & x & 0 & 0 & 0 \\ 0 & z & -x & 0 & 0 \\ 0 & 0 & z & -x & 0 \\ 0 & 0 & 0 & z & x \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} 0 & 0 & 0 & 0 \\ z & -x & 0 & 0 \\ 0 & z & x & 0 \\ 0 & 0 & z & x \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} 0 & 0 & -z & x \\ -1 & z & x & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}} R^3 \xrightarrow{(x\ z\ z)} R$$

Example 1.18. Let $R = k[x, y, z]/(xy, xz, yz)$, then the free resolution associated to the quotient module $R/(x, y, z)$ is given by:

$$\cdots \to R^{24} \longrightarrow R^{12} \xrightarrow{\begin{pmatrix} y & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & y \end{pmatrix}} R^6 \xrightarrow{\begin{pmatrix} 0 & 0 & z & 0 & y & 0 \\ z & 0 & 0 & 0 & 0 & x \\ 0 & y & 0 & x & 0 & 0 \end{pmatrix}} R^3 \xrightarrow{(x\ y\ z)} R$$

As exhibited in the examples above, the power of free resolutions is that we can represent the maps in the sequence as *matrices* with entries in R, and thus understanding any module $M \in \mathcal{R}$-$\mathcal{mod}$ reduces to a linear algebra problem (more or less). Now, we recall the notion of projective R-modules and consider their connection to free modules.

Proposition 1.19 (cf. [Hu]). *If F is a free R-module, then it is a projective R-module.*

Proof. Let F be a free R-module. Additionally, let M, N be any R-modules such that there exists a surjection $f\colon N \to M$ and a homomorphism $g\colon F \to M$ as indicated in the diagram below:

$$\begin{array}{ccc} & & F \\ & {\scriptstyle \exists h}\swarrow & \downarrow {\scriptstyle g} \\ N & \xrightarrow{f} & M \end{array}$$

Note first that for each $e_i \in \mathscr{E}$, where $\mathscr{E}$ is the basis of F, there exists an $x_i \in M$ such that $g(e_i) = x_i$. Next, since f is surjective, we know there exists a $y_i \in N$ for each i such that $f(y_i) = x_i$. So define $h\colon F \to N$ as $h(e_i) = y_i$ for each $e_i \in \mathscr{E}$ and clearly the diagram above commutes since $fh(e_i) = f(y_i) = x_i = g(e_i)$. Therefore, by definition F is a projective R-module. $\square$

The significance of this statement is that we know every R-module has a free module associated to it by its free presentation ϕ and since free modules *are* projective, this guarantees that $\mathscr{R}\text{-}\mathscr{mod}$ has enough projectives, as previously mentioned. Therefore, every $M \in \mathscr{R}\text{-}\mathscr{mod}$ has a projective resolution since we can just take a free resolution of M as a projective resolution. In particular, we can just use a free resolution when we compute the Ext functors of a given module. Now, in the above proposition and these statements, we make no use of the fact that R is local, meaning that these observations apply even if we relax that condition. However, in the local (or graded) case, we can actually make the stronger statement that the free R-modules are *precisely* the projective R-modules. Meaning that *any* projective resolution of an R-module is going to be free and this is why our focus rests upon free modules.

This is especially nice because in $\mathscr{R}$-$\mathscr{mod}$ under the local case, understanding objects in part relies upon understanding sequences of *matrices* and their *sizes.* Note also that as $\mathrm{pd}_R(M)$ is equivalent to the length of the minimal free resolution of M, understanding the structure of a module's minimal free resolution gives us information about how complicated the homological nature of the module is. As previously mentioned, though, not all modules have finite free resolutions and, in fact, there are many interesting R-modules that yield infinite minimal free resolutions as in the example given above. When this occurs, the tactic for studying these *infinite* sequences is to inquire about the patterns that may arise in the sizes of the matrices, which are equivalent to the ranks of the free modules, in the sequence. We include the following definitions to help make precise the study of patterns in infinite free resolutions.

Definition 1.20 (cf. [Ei2])**.** Let M be an R-module with minimal free resolution $\mathbf{F}$. The i^{th} *syzygy module* of M, denoted $\Omega^i M$, is defined to be the *image* of the i^{th} differential or, equivalently, the *kernel* of the $(i-1)^{st}$ differential; that is,

$$\Omega^i M = \mathrm{im} d_i = \ker d_{i-1} \cong \mathrm{coker}\, d_i.$$

Definition 1.21 (cf. [Ei2])**.** The i^{th} *Betti number* of an R-module M is the rank of the i^{th} free module in the minimal free resolution of M, and we denote it $\beta_i^R(M) = \mathrm{rk}(F_i)$. We call the *Betti sequence* the sequence of Betti numbers of M and denote it $\{\beta_i^R(M)\}_{i\in\mathbb{N}}$.

While the structure of finite free resolutions is (to some degree) well understood, that of *infinite* free resolutions can be quite daunting. Utilizing the Betti sequence of an R-module and studying possible patterns that can occur in the infinite case can help, though. Given an R-module and its minimal free resolution, note that we may write $F_i \cong R^{\beta_i}$. Then, if we consider $\mathrm{Ext}^i_R(M, \Bbbk) = H_i(\mathrm{Hom}_R(\mathbf{F}, \Bbbk))$ it should be clear that $\ker(\mathrm{Hom}_R(d_i), \Bbbk)/\mathrm{im}(\mathrm{Hom}_R(d_{i-1}), \Bbbk)$ yields exactly $\Bbbk^{\beta_i}$ and hence we may alternatively write $\{b_n^R(M)\} = \dim_{\Bbbk}(\mathrm{Ext}^i_R(M, \Bbbk))$.

Depending on the class of modules, typically defined by the type of ring (or at least by some aspect of the ring), different patterns can occur in the Betti sequence. Many algebraists have made great strides in understanding these patterns; for example, if R is a type of ring called a *complete intersection*, it has been shown that the free resolution of the residue field $k \cong R/\mathfrak{m}$ has a Betti sequence that is eventually given by a polynomial (see [Ta, 6], cf. [AvGaPe]). We will explore this type of ring and the known patterns for Betti sequences of modules over this type of ring in the next section.

1.4 An Important Invariant and a Special Type of Ring

To understand the additional structure a complete intersection ring provides, we must first understand the notion of a *regular sequence*. Roughly speaking, such a sequence can be thought of as a sequence in a commutative ring that is as *independent as possible*. Recall that an element $r \in R$ is a *non zero-divisor* if for $s \in R$, $rs = 0$ implies $s = 0$. We also have this notion for an R-module M: $r \in R$ is called M-regular if $rx = 0$ for $x \in M$ implies $x = 0$. Therefore, we may compose a sequence of successively M-regular elements, as described in the following definition.

Definition 1.22 (cf. [BrHe]). A sequence $\mathbf{f} = f_1, \ldots, f_c \in R$ is called an M-regular sequence if the following conditions hold:

1. f_i is a regular element on $M/(f_1, \ldots, f_{i-1})M$ for each $i = 1, \ldots, c$; and[2]
2. $M/\mathbf{f}M \neq 0$.

[2]By convention, when $i = 1$ it is assumed that f_1 is a regular element (non zero-divisor) on M.

It is important to note just a few things; first and foremost, we may consider a regular sequence *on a ring* rather than a module, in which case we would just replace M in the definition above with R. Moreover, an R-regular element is just a non zero-divisor in the ring and, rather than referring to an R-regular sequence, we would call $\mathbf{f}$ a *regular sequence*. Secondly, in the case where R is local, as we have assumed, if we take the regular sequence to be in the maximal ideal, $\mathbf{f} \in \mathfrak{m}$, then the latter condition in the definition is guaranteed by Nakayama's Lemma.

The notion of regular sequences gives rise to two significant concepts: an invariant in $\mathscr{R}$-$\mathscr{mod}$ and the primary step towards defining the type of ring known as a *complete intersection*. We will begin by exploring the former topic, along with related facts and results which will be employed in later sections of this thesis; then, we shall unravel the structure of a complete intersection ring. The diligent reader should refer to [BrHe] or [Ei2] for a more thorough coverage of these topics.

1.4.1 Depth of an R-Module

By definition, if $\mathbf{f}$ is an M-regular sequence, it should be clear that there exists a strictly ascending chain of ideals in R: $(f_1) \subsetneq (f_1, f_2) \subsetneq \cdots \subsetneq (f_1, f_2, \ldots, f_c)$. For an ideal $I \subseteq R$ such that $IM \neq M$, call an M-sequence $\mathbf{f} \in I$ *maximal in* I if $f_1, \ldots, f_{c+1}$ is not an M-sequence for any $f_{c+1} \in I$. It actually holds (for noetherian rings) that all maximal M-sequences have the *same* length, given by

$$c = \min\{i \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}$$

and this is called the *grade of I on M*, denoted $\operatorname{grade}(I, M)$. If $IM = M$, then we set $\operatorname{grade}(I, M) = \infty$ which is true if and only if $\operatorname{Ext}^i_R(R/I, M) = 0$ for all i. When R is a *local ring*, as we have assumed, then we can consider the grade of the maximal ideal $\mathfrak{m}$ on $M \in \mathscr{R}$-$\mathscr{mod}$:

Definition 1.23 (cf. [BrHe]). Let $(R, \mathfrak{m}, \Bbbk)$ be a noetherian local ring and M a (nonzero) finitely-generated R-module. Then the *depth of M* is defined to be:

$$\operatorname{depth}_R M = \min\{i \mid \operatorname{Ext}^i_R(\Bbbk, M) \neq 0\}.$$

We might also consider the *depth of a ring*, where we just consider maximal regular sequences on the ring itself. Depth serves as an important invariant of rings and modules, since in some sense it gives us a measure of *size* of the ring or module. Moreover, it plays a significant role in better understanding these algebraic structures; for example, it is used to define an interesting class of modules (Cohen-Macaulay) and is connected with the projective dimension of an R-module, as given in the Auslander-Buchbaum formula. Our interest in depth will arise in Chapter 2, as we will discover how it can be connected with the main topic of this thesis.

1.4.2 Modules over Complete Intersection Rings

A regular local ring is a Notherian local ring in which the minimum number of generators of its maximal ideal is equivalent to its Krull dimension[3]. That is, if R is a noetherian local ring then it is additionally regular if and only if $\dim_{\Bbbk} \mathfrak{m}/\mathfrak{m}^2 = \dim R$. For example, any field $\Bbbk$ is a regular local ring and, more generally, the ring of formal power series $\Bbbk[\![x_1, \ldots, x_d]\!]$ is a regular local ring. This type of ring, together with regular sequences, will render the type of ring known as a complete intersection.

Definition 1.24 (cf. [BrHe]). A ring R is a *complete intersection* if it is the completion of a regular local ring Q modulo an ideal generated by a Q-regular sequence: $R \cong \widehat{Q/(\mathbf{f})}$.

[3]The *Krull dimension* of a ring R is the supremum of the lengths of all chains of prime ideals in R (cf. [Ei2]).

Viewing complete intersections as the *completion* of a factor of the given form is more of a technical detail than an imperative necessity to understanding what complete intersection rings are. In an informal manner, we can think of these rings as the subset of local rings that are defined with the *minimum possible number of relations* due to the fact that the process of "factoring out" an ideal generated by a regular sequence "cuts down" on the possible relations which can be constructed from the regular sequence.

As it turns out, modules over these types of rings yield a tractable study of the associated minimal free resolutions. In 1954, Tate showed that $b_n^R(\Bbbk)$ is eventually given by a polynomial (see [Ta]). Subsequently, in 1974, Gulliksen proved that each $b_n^R(M)$ is a quasi-polynomial of period 2 and degree smaller than the codimension (see [Gu]). Later, Avramov demonstrated that $b_n^R(\Bbbk)$ has exponential growth *unless* R is a complete intersection (see [Av, 1.8]). Finally, in 1997, Avramov, Gasharov, and Peeva published a paper (see [AvGaPe]) highlighting that although the beginning of a free resolution (over a complete intersection) is often unstable, patterns do emerge at infinity. In particular, they proved that $\{\beta_n^R(M)\}$ is eventually either strictly increasing or constant. Of essential significance is their generalization of modules over a complete intersection to the notion of modules of finite *CI-dimension*, for which the same statement holds.

In the next chapter, we will discuss these topics in further detail, with the goal of introducing the notion, originally defined in $\mathscr{R}$-$\mathscr{mod}$, which inspired the main definitions of this thesis.

CHAPTER 2

Endomorphisms on Free Resolutions and Critical Degree

We now turn towards the main topic of this thesis: the critical degree of an R-module. As a reminder, $(R, \mathfrak{m}, \Bbbk)$ is a commutative noetherian local ring and we will denote M for a finitely generated R-module. The notion of critical degree was originally introduced in Section 7 of the paper *Complete Intersection Dimension* ([AvGaPe]), in which the authors introduced a new type of dimension for finitely generated modules. This dimension, in some sense, gives a generalization for modules over complete intersection rings and is motivated by the nice structure of such modules. As discussed in Chapter 1, patterns in the Betti sequence of a module over a complete intersection eventually arise; this, of course, results in the study of such infinite free resolutions becoming slightly less daunting.

One of the original papers which gave way to observation of these patterns is Eisenbud's *Homological Algebra on a Complete Intersection, with an Application to Group Representations* (see [Ei]). In this paper, he gives a different approach to Gulliksen's work in [Gu], focusing on a special case which we include in this Chapter. After presenting the origins for the theory further established in [AvGaPe], we introduce the definition which pinpoints the location of where patterns in free resolutions over complete intersections are guaranteed to arise. This definition, in turn, aids in the proof of eventual nondecreasing growth of the Betti sequence. All results presented in this chapter can be found with the same or more generality in the aforementioned papers.

2.1 Endomorphisms on Resolutions

In Chapter 1, we presented the essential tools for understanding $\mathscr{R}$-$\mathscr{mod}$ and the theory of free resolutions (which are intimately connected with the topics of this thesis). To carry these topics forward, we will begin with a discussion of endomorphisms on (necessarily minimal) free resolutions. Let R be as previously indicated (a commutative local noetherian ring), M a finitely generated R-module, and $\mathbf{F}$ its minimal free resolution. Then a (*degree* $-q < 0$) *chain endomorphism* on $\mathbf{F}$ is defined to be a family of R-module homomorphisms $\{\mu_n\}_{n=q}^{\infty}$, with $\mu_n \colon F_n \to F_{n-q}$, and such that $d_{n-q}\mu_n = \mu_{n-1}d_n$ for each $n \in \mathbb{Z}$. For the remainder of this section, assume R is additionally a complete intersection ring; we will explore a special class of degree -2 endomorphisms on $\mathbf{F}$ called the *CI operators.*[1]

2.1.1 CI Operators

In 1980, Eisenbud published the aforesaid paper ([Ei]), which served as motivation for the main topics relayed in this chapter, and thus, the primary thematic elements of this thesis. The idea of his paper is to study how homological algebra over a hypersurface ring $R = Q/(f)$ differs from that over Q, where Q itself is a quotient of a regular local ring by an ideal generated by a nonunit. In particular, when we consider a free resolution over the hypersurface ring there is a natural manner in which we can identify a well-defined morphism on the resolution.

Let Q be a regular local ring and $\mathbf{f} = f_1, \ldots, f_c$ a Q-regular sequence. Furthermore, let $R = Q/(\mathbf{f})$ and suppose M is a finitely generated R-module with $(\mathbf{F}, d)$ as its minimal free resolution. Since there exists the ring surjection $Q \twoheadrightarrow R$, we can lift $(\mathbf{F}, d)$ to Q so that we obtain a sequence of Q-modules and Q-module homomorphisms. Denote $\tilde{d}$ as the lifting of the differentials to Q and note it holds that $d = R \otimes_R \tilde{d}$.

[1] also sometimes called "Eisenbud operators".

Moreover, keep in mind that while $d^2 = 0$, after the lifting $\tilde{d}^2 \neq 0$; meaning, $(\mathbf{F}, d, R)$ is an R-complex but it should be clear that $(\mathbf{F}, \tilde{d}, Q)$ is *not* a Q-complex. It turns out, though, that we can identify maps $\tilde{t}_j : (\mathbf{F}, \tilde{d}, Q) \to (\mathbf{F}[2], \tilde{d}, Q)$ for each $1 \leq j \leq c$ such that

$$\tilde{d}^2 = \sum_{j=1}^{c} f_j \tilde{t}_j.$$

Essentially, each $\tilde{t}_j$ is the component preventing $\widetilde{\mathbf{F}}$ from being a Q-complex. Define the *CI operators* on $\mathbf{F}$ associated to the sequence $f_1, \ldots, f_c$ as $t_j := R \otimes_R \tilde{t}_j$. We may denote each operator as $t_j(Q, \{f_i\}, \mathbf{F})$ in order to specify the dependency on the ring, regular sequence, and free resolution but whenever these choices are unambiguous we simply write t_j. Additionally, note that in the case where R is a hypersurface (when $c = 1$), we have just one operator where $\tilde{d}^2 = f\tilde{t}$.

Proposition 2.1 (Properties of CI Operators, [Ei])**.** *Suppose $\mathbf{f} = f_1, \ldots, f_c$ is a Q-regular sequence, $\alpha \colon (\mathbf{F}, d_F) \to (\mathbf{G}, d_G)$ is a chain map of resolutions over $R = Q/(\mathbf{f})$, and $\tilde{d}_F$,$\tilde{d}_G$ are the appropriate liftings to Q. Then each of the following hold:*

1. *Each $t_j(Q, \{f_i\}, \mathbf{F})$ is a degree -2 chain endomorphism $\mathbf{F} \to \mathbf{F}[2]$.*
2. *Each $t_j(Q, \{f_i\}, \mathbf{F})$ is uniquely determined up to homotopy by choice of Q, $(\mathbf{F}, \tilde{d}_F)$, and $\mathbf{f}$.*
3. *Each $t_j(Q, \{f_i\}, \mathbf{F})$ is independent of the choice of the lifting $\tilde{d}_F$, up to homotopy.*
4. *The CI operators commute up to homotopy with α. That is, let $s_j = s_j(Q, \{f_i\}, \mathbf{G})$; then $\alpha t_j \sim s_j \alpha$ for each $j = 1, \ldots, c$.*

For proof of the above properties, see Propositions 1.1, 1.2, and 1.3, along with Corollary 1.4, in [Ei]. As these proofs are rather straightforward, so they will be omitted here. Note that we may apply the last part of the above proposition with $\alpha = 1 \colon \mathbf{F} \to \mathbf{F}$ to see that the following statement holds.

Corollary 2.2 ([Ei], 1.5)**.** *The CI operators commute with each other, up to homotopy.*

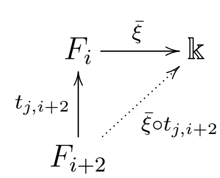
2.1.2 The Cohomology Operators on the Graded $\mathcal{S}$-Module, $\operatorname{Ext}^*_R(M, \Bbbk)$

Now that it has been established $\{t_j\}_{j=1}^{c}$ is a well-defined class of degree -2 endomorphisms on $(\mathbf{F}, d, R)$, we consider a related class of operators called the *cohomology operators*. First and foremost, note that $\operatorname{Ext}^*_R(M, \Bbbk)$ is a graded R-module, with the grading defined by the homological degree of $\mathbf{F}$ so that we may write

$$\operatorname{Ext}^*_R(M, \Bbbk) = \bigoplus_i \operatorname{Ext}^i_R(M, \Bbbk).$$

Recall that each $\operatorname{Ext}^i_R(M, \Bbbk) = \operatorname{H}^i(\operatorname{Hom}_R(\mathbf{F}, \Bbbk))$, which is a factor submodule of the R-module $\operatorname{Hom}_R(F_{i+1}, \Bbbk)$. Thus, each $\operatorname{Ext}^i_R(M, \Bbbk)$ is an R-module itself and so if we set $\deg(\xi) = i$ for any $\xi \in \operatorname{Ext}^i_R(M, \Bbbk)$ then it should be clear why $\operatorname{Ext}^*_R(M, \Bbbk)$ is a graded R-module. However, it turns out that we may impose additional module structure on $\operatorname{Ext}^*_R(M, \Bbbk)$. Given $t_j = (Q, \{f_i\}, \mathbf{F})$ such that $\mathbf{F}$ is a free resolution of M over $R = Q/\mathbf{f}$ and Q is a regular local ring, define $\chi_j := \operatorname{Hom}_R(t_j, \Bbbk)$. By definition, each χ_j will actually be a degree 2 endomorphism on the R-complex $\operatorname{Hom}_R(\mathbf{F}, \Bbbk)$ and so for each $i \in \mathbb{N}$ we obtain an induced map $\chi_{ji} : \operatorname{Ext}^i_R(M, \Bbbk) \to \operatorname{Ext}^{i+2}_R(M, \Bbbk)$ of R-modules.[2]

To make explicit the action of each χ_j on $\operatorname{Ext}^*_R(M, \Bbbk)$, note that for any $\xi \in \operatorname{Ext}^i_R(M, \Bbbk)$ we may consider an appropriate representative $\bar{\xi} \in \ker(\operatorname{Hom}_R(d_{i+1}, \Bbbk)) \setminus \operatorname{im}(\operatorname{Hom}_R(d_i, \Bbbk)) \subseteq \operatorname{Hom}_R(F_i, \Bbbk)$. That is, we may view $\bar{\xi}$ as an R-module map $\bar{\xi} \colon F_i \to \Bbbk$ and thus $\chi_{ji}(\bar{\xi}) = \bar{\xi} \circ t_{j,i+2}$ as depicted in the following diagram.



[2]Note that since $\mathbf{F}$ is minimal, we have that $\operatorname{Ext}^i_R(M, \Bbbk) = \operatorname{Hom}_R(F_i, \Bbbk)$ for each integer $i > 0$.

Therefore, we can view the action of each χ_j as composition of R-module morphisms, taking the perspective that elements in $\operatorname{Ext}^i_R(M, \Bbbk)$ are maps from F_i to $\Bbbk$ which are in the kernel of $\operatorname{Hom}_R(d_{i+1}, \Bbbk)$ but not in the image of $\operatorname{Hom}_R(d_i, \Bbbk)$. Another manner in which we may view elements in $\operatorname{Ext}^i_R(M, \Bbbk)$ is as equivalence classes of i-extensions of M by $\Bbbk$. From this perspective, it is easier to see the action of the graded algebras $\operatorname{Ext}^*_R(M, M)$ or $\operatorname{Ext}^*_R(\Bbbk, \Bbbk)$ on $\operatorname{Ext}^*_R(M, \Bbbk)$, given by Yoneda multiplication, with reduction to action by the χ_j's explained subsequently in Section 2.3.1 of this chapter. Regardless of how elements in $\operatorname{Ext}^i_R(M, \Bbbk)$ are viewed, $\operatorname{Ext}^*_R(M, \Bbbk)$ has a clear module structure over $R[\chi_1, \dots, \chi_c]$ and we call this polynomial ring the *ring of cohomology operators*, denoting it $\mathcal{S} = R[\chi_1, \dots, \chi_c] = R[\boldsymbol{\chi}]$ throughout this thesis.

Remark. *Coupled with the fact that each* $\operatorname{Ext}^i_R(M, \Bbbk)$ *is additionally a* $\Bbbk$*-vector space, we even have that* $\operatorname{Ext}^*_R(M, \Bbbk)$ *is a graded module over the polynomial ring* $\Bbbk[\chi_1, \dots, \chi_c]$ *which we will also commonly denote as the ring of cohomology operators,* $\mathcal{S}$*. In many cases the reader may assume either definition of* $\mathcal{S}$*, but in some scenarios taking* $\operatorname{Ext}^*_R(M, \Bbbk)$ *as a module over the latter polynomial ring is necessary; it should be clear via context when this is the case. Note, for example, if we refer to the maximal ideal* $(\chi_1, \dots, \chi_c) \subseteq \mathcal{S}$ *it should be clear that we mean the latter definition since this ideal is not maximal in* $R[\boldsymbol{\chi}]$*. We may at times also denote the ideal* $\mathfrak{X} = (\chi_1, \dots, \chi_c)$*.*

In [Ei], Eisenbud proves $\operatorname{Tor}^R_*(M, \Bbbk)$ is unambiguously an $R[t_1, \dots, t_c]$-module and uses *this* module in the proof of his main theorem. While we will present proof of the same result at the end of this section, we will instead use the graded Ext-module. A few years after the CI operators were first introduced in Eisenbud's paper, Avramov gave an analogue for injective complexes in [Av2] and was able to show that $\operatorname{Ext}^*_R(M, \Bbbk)$ is unambiguously an $\mathcal{S}$-module. Specifically, the cohomology operators will coincide for any free (projective) resolution of M used *or* any injective resolution of $\Bbbk$ used in the computation of $\operatorname{Ext}^i_R(M, \Bbbk)$, thus yielding the same $\mathcal{S}$-modules.

Remark. *Both Eisenbud and Avramov prove* $\mathrm{Tor}_R^*(M,N)$ *and* $\mathrm{Ext}_R^*(M,N)$ *are unambiguously* $R[t_1,\ldots,t_c]$*- and* $\mathcal{S}$*-modules, where* N *is any* R*-module. For reasons that will become apparent throughout this thesis, our focus is on when* $N = \Bbbk$*, but we will keep* N *general for the following clarification of what is meant by "unambiguously".*

Proposition 2.3 ([Av2], 1.4)**.** *Let* $\mathbf{F}$ *be a free resolution of the* R*-module* M *and* $\mathbf{I}$ *any injective resolution of the* R*-module* N*. Then the following diagram is commutative:*

$$\begin{array}{ccccc}
\mathrm{H}^*(\mathrm{Hom}_R(\mathbf{F},N)) & \xrightarrow{\eta_*} & \mathrm{H}^*(\mathrm{Hom}_R(\mathbf{F},\mathbf{I})) & \xleftarrow{\epsilon^*} & \mathrm{H}^*(\mathrm{Hom}_R(M,\mathbf{I})) \\
\downarrow{\scriptstyle \mathrm{H}^*(\mathrm{Hom}_R(t_j(Q,\{f_i\},\mathbf{F}),N))} & & \downarrow{\scriptstyle \mathrm{H}^*(u_j(Q,\{f_i\},\mathrm{Hom}_R(\mathbf{F},\mathbf{I}))} & & \downarrow{\scriptstyle \mathrm{H}^*(\mathrm{Hom}_R(M,u_j(Q,\{f_i\},\mathbf{I})))} \\
\mathrm{H}^*(\mathrm{Hom}_R(\mathbf{F},N)) & \xrightarrow{\eta_*} & \mathrm{H}^*(\mathrm{Hom}_R(\mathbf{F},\mathbf{I})) & \xleftarrow{\epsilon^*} & \mathrm{H}^*(\mathrm{Hom}_R(M,\mathbf{I}))
\end{array}$$

After choosing the appropriate operators $\tilde{t}_j$ and $\tilde{u}_j$, with the process for choosing the latter outlined in [Av2], the proof of the above proposition boils down to checking that $\mathrm{Hom}(\tilde{t}_j,\tilde{\mathbf{I}}) + \mathrm{Hom}(\tilde{\mathbf{F}},\tilde{u}_j)$ works as the operators $\tilde{u}_j(Q,\{f_i\},\mathrm{Hom}_R(\tilde{\mathbf{F}},\tilde{\mathbf{I}}))$ on the complex $\mathrm{Hom}_R(\tilde{\mathbf{F}},\tilde{\mathbf{I}})$. The point of all this is that by Proposition 2.1 along with analogous statements discussed in [Av2], the χ_j are first independent of the choice of $\mathbf{F}$ as they are independent of the choice of $\mathbf{I}$. The proposition given above states that if we were instead to identify $\mathrm{Ext}_R^*(M,N)$ with $\mathrm{H}^*(\mathrm{Hom}_R(M,\mathbf{I}))$ for some injective resolution $\mathbf{I}$ of N, then we obtain the $\mathrm{H}^*(\mathrm{Hom}(M,u_j))$ which agree with the χ_j constructed from starting with a free resolution $\mathbf{F}$ of M.

Furthermore, similarly to the CI operators, the cohomology operators commute with each other; that is, $\chi_j\chi_i = \chi_i\chi_j$ for each $1 \le i,j \le c$. Additionally, note that $\mathcal{S} = R[\chi_1,\ldots,\chi_c]$ is a *graded* polynomial ring with $\deg(\chi_j) = 2$ for each $j = 1,\ldots,c$ and $\deg(r) = 0$ for all $r \in R$. Therefore, $\mathrm{Ext}_R^*(M,N)$ has a well-defined module structure over the graded ring $\mathcal{S}$ and this module is functorial, meaning that for any $M' \to M$ and $N \to N'$ the following square commutes:

$$\begin{array}{ccc} \operatorname{Ext}^*_R(M,N) & \longrightarrow & \operatorname{Ext}^*_R(M,N') \\ \downarrow & & \downarrow \\ \operatorname{Ext}^*_R(M',N) & \longrightarrow & \operatorname{Ext}^*_R(M',N') \end{array}$$

The functoriality of $\operatorname{Ext}^*_R(M,N)$, like $\operatorname{Tor}^*_R(M,N)$, is guaranteed by the naturality of the χ_j for each j. An important note is that, in [Av2] Avramov gives the finiteness theorem analogue to Gullisken's main result in [Gu] with respect to the cohomology operators. That is, he shows $\operatorname{Ext}^*_R(M,N)$ is finitely generated as a graded $\mathcal{S}$-module if $\operatorname{Ext}^i_Q(M,N) = 0$ for $i \gg 0$.[3] In the next section, we present a proof analogous to the main theorem of [Ei], but with respect to Ext and the cohomology operators.

2.1.3 Surjectivity of a CI Operator on the Minimal Free Resolution

Let R be a complete intersection of the form $Q/(\mathbf{f})$ and consider any finitely generated R-module M. As discussed previously, $\operatorname{Ext}^*_R(M,\Bbbk)$ is a graded module over the graded polynomial ring $\mathcal{S} = R[\chi_1,\ldots,\chi_c]$. It should be clear that nonzero-divisors of a particular degree in $\operatorname{Ext}^*_R(M,\Bbbk)$ correlate to surjective R-module homomorphisms at a particular homological degree from $\mathbf{F}$ to $\mathbf{F}[\mathbf{2}]$ (refer to Definition 2 in Chapter 1). In [Ei], Theorem 3.1 is the result showcasing this, but Eisenbud proves the statement utilizing the graded module $\operatorname{Tor}^*_R(M,\Bbbk)$ over the graded ring $R[t_1,\ldots,t_c]$. We include the proof with respect to the graded Ext module, since the same argument echoes throughout later results in this thesis.

Theorem 2.4 (Main Theorem from [Ei], §3)**.** *Let Q be a regular local ring with infinite residue class and let $(\mathbf{f}) \subseteq Q$ represent an ideal generated by a regular Q-*

[3] In [Av2], Avramov shows the statement with the condition that either the projective dimension of M as a Q-module is finite or the injective dimension of N as a Q-module is finite. However, it is also true under the more general assumption given.

sequence. Given $R = Q/(\mathbf{f})$ and a minimal R-free resolution $\mathbf{F}$ of a finitely generated R-module M, there exists a Q-sequence $f_1, \ldots, f_c$ generating $(\mathbf{f})$ such that

$$t_1(Q, \{f_j\}, \mathbf{F})\colon F_{n+2} \to F_n$$

is an epimorphism for all sufficiently large n.

Before giving the proof of this result, we make a few notes of importance, which will distinguish the differences in our argument from that given in [Ei]. Eisenbud provides two lemmas which aid his goal in proving his main theorem; first, he justifies that $\mathrm{Tor}^*_R(M, \Bbbk)$ is an *artinian* module over $R[t_1, \ldots, t_c]$ (and even $\Bbbk[t_1, \ldots, t_c]$). The purpose of this is to demonstrate that the dual of this module will be noetherian, so that he can attain his goal by making use of the ACC being satisfied in the dual. Then, dualizing again, the result is preserved. However, now that it is well known $\mathrm{Ext}^*_R(M, \Bbbk)$ is a module over $\mathcal{S}$, it behooves us to use this module instead as it is noetherian (see [Av2], [AvBu]), thus allowing us to disregard the purpose of Lemma 3.2 in [Ei]. This, of course, enables us to present a more straightforward proof (albeit less general), following a similar argument to Lemma 3.3 in [Ei].

Proof of Theorem 1. Let $g_1, \ldots, g_c$ be *any* Q-sequence generating $(\mathbf{f})$ and denote $t_j = t_j(Q, \{g_i\}, \mathbf{F})$. Our goal is to show that there exist elements $a_j \in Q$ for $j = 1, \ldots, c$ such that the degree -2 map

$$t = t_1 + \Sigma_{j=2}^{n} a_j t_j$$

from $\{F_{n+2}\}$ to $\{F_n\}$ is an epimorphism for all sufficiently large n. Then, if we set

$$f_1 = g_1; \quad f_j = g_j - a_j g_1, \;\; 2 \leq j \leq c$$

it should be clear that $f_1, \ldots, f_c$ generate $\mathbf{f}$ and we may apply the change of rings proposition to see that $t_1(Q, \{f_i\}, \mathbf{F}) = t_1 + \Sigma a_i t_i$. To prove existence of such a_j, first denote E as the largest artinian submodule of $\operatorname{Ext}_R^*(M, \Bbbk)$ and note that since it is also noetherian, it will have finite length and thus must be finitely generated. Moreover, since E is the largest artinian submodule, it must contain all socle elements of $\operatorname{Ext}_R^*(M, \Bbbk)$. Meaning, E is generated by only finitely many degrees in $\operatorname{Ext}_R^*(M, \Bbbk)$ and so there must exist a positive integer $N_0 < \infty$ such that

$$E^{\geq N_0} = \operatorname{Ext}_R^{\geq N_0}(M, \Bbbk)$$

contains no nonzero element annihilated by the maximal ideal $(\chi_1, \ldots, \chi_c) \subseteq \mathcal{S}$. Now, denote $P_1, \ldots, P_r$ as the associated primes of the zero module in $E^{\geq N_0}$ so that $\bigcup_{k=1}^r P_k$ represents the set of zero-divisors on $E^{\geq N_0}$. Consider the set

$$\chi_1 + \sum_{i=2}^{c} \Bbbk \chi_i$$

and note that this set generates the maximal ideal $\mathfrak{X}$. But since there is no zero-divisor of $E^{\geq N_0}$ contained in $(\chi_1, \ldots, \chi_c)$, this set cannot be contained in any P_k. Furthermore, since $\Bbbk$ is infinite there exists a subspace of $\Bbbk[\chi_1, \ldots, \chi_c]$ for which $\chi_1 + \sum_{i=2}^{c} \Bbbk \chi_i$ is a translation; and so, because there are only finitely many P_k, it must hold that

$$\chi_1 + \sum_{i=2}^{c} \Bbbk \chi_i \not\subseteq \bigcup_{k=1}^{r} P_k$$

meaning there exists a linear form

$$\chi = \chi_1 + \sum_{i=2}^{c} \alpha_j \chi_j, \quad \alpha_j \in \Bbbk$$

such that χ is a non zero-divisor on $E^{\geq N_0}$. Now, for each $j = 1, \ldots, c$ set a_j equal to a pre-image of α_j in R so that $\hat{\chi} = \chi_1 + \sum_{i=2}^{c} a_j \chi_j \in \mathcal{S}$ and note that Nakayama's Lemma tells us that $\hat{\chi}$ is also a non zero-divisor on $E^{\geq N_0}$. Lastly, $\hat{\chi}$ is a non zero-divisor on $E^{\geq N}$ if and only if $\hat{\chi}_n : \operatorname{Ext}_R^n(M, \Bbbk) \to \operatorname{Ext}_R^{n+2}(M, \Bbbk)$ is injective for all $n > N$; equivalently,

$$\hat{t} = t_1 + \sum_{i=2}^{c} a_j t_j$$

is surjective on F_n for all $n > N$, where $\hat{\chi} = \operatorname{Hom}_R(\hat{t}, \Bbbk)$. □

2.2 Critical Degree and Finiteness

Given Eisenbud's main theorem and proof, we see that under certain conditions, such as when R is a complete intersection, we are guaranteed an endomorphism for which the R-module homomorphisms eventually become surjective for all higher homological degrees. This of course motivates the question, first, of whether there exist more general conditions for which similar behavior occurs; secondly, given the proof of the statement over a complete intersection, it seems that we can in fact identify where these surjections begin in a free resolution.

Moreover, it should not be too much of a jump to understand the significance of these surjections; guaranteed surjections of the form identified by Eisenbud lead to guaranteed growth of every other Betti number. In fact, as we shall see shortly, it is possible to ascertain that the Betti sequence eventually becomes nondecreasing in the case of a complete intersection. Thus, *complexity* of an R-module is very deeply connected with these topics, as it captures the growth of a module's Betti numbers. Recall that the complexity of an R-module is defined to be

$$\operatorname{cx}_R M = \inf\{d \mid \lim_{n\to\infty} \frac{b_n^R(M)}{n^d} = 0\}$$

and hence, in some sense, complexity of M measures the "size" of its minimal free resolution $\mathbf{F}$. For example, when $\operatorname{pd}_R M < \infty$, it should be clear that $\operatorname{cx}_R M = 0$ by definition. On the other hand if $\mathbf{F}$ is a resolution with constant Betti numbers, then $\operatorname{cx}_R M = 1$. Linear growth is depicted by $\operatorname{cx}_R M = 2$, quadratic growth by $\operatorname{cx}_R M = 3$, and so on. As it turns out, whenever R is *not* a complete intersection, $\operatorname{cx}_R \Bbbk = \infty$, since $\{b_n^R(\Bbbk)\}$ eventually has exponential growth [Av]. However, when R *is* a complete intersection, $\operatorname{cx}_R \Bbbk < \infty$ since $\{b_n^R(\Bbbk)\}$ is eventually given by a polynomial, first shown in [Ta]. In fact, it will be demonstrated shortly that for any R-module M over a complete intersection it holds that $\operatorname{cx}_R M < \infty$, as it was originally proven in [Gu] and noted by [AvGaPe]. We now move on to understanding the answers to both of the questions posed at the beginning of this section.

2.2.1 Critical Degree of an R-module

Although we do not make explicit use of complete intersection dimension throughout this thesis, we begin with presenting the definition so that we may reference it throughout. We also discuss some significant characteristics, making it a meaningful measure of R-modules.

Definition 2.5 ([AvGaPe], 1.2)**.** Given a nonzero finitely generated module M over R, the *CI-dimension* (shorthand for *complete intersection dimension*) of M is

$$\text{CI-dim}_R M = \inf\{\operatorname{pd}_Q M' - \operatorname{pd}_Q R' \mid R \to R' \leftarrow Q \text{ is a quasi-deformation}\}$$

and set $\text{CI-dim}_R 0 = 0$.

This dimension can help generalize the class of modules over complete intersection rings, as seen in the following theorem.

Theorem 2.6 ([AvGaPe], 1.3)**.** *If R is a complete intersection ring then each R-module M has finite CI-dimension. Conversely, if $\text{CI-dim}_R \Bbbk < \infty$, then R is a complete intersection ring.*

We also include the connection between CI-dimension of an R-module and that of its syzygy modules.

Lemma 2.7 ([AvGaPe], 1.9)**.** *If $M \neq 0$ is a finitely generated module over R, then*

$$\text{CI-dim}_R \Omega^n(M) = \max\{\text{CI-dim}_R M - n, 0\}$$

for $n \geq 0$. If furthermore, $\text{CI-dim}_R M < \infty$ then it holds that

$$\operatorname{depth}_R \Omega^n M = \min\{\operatorname{depth}_R M + n, \operatorname{depth} R\}$$

for $0 \leq n \leq \operatorname{pd}_R M$*.*

Thus, the depth lemma (cf. [BrHe, 1.2.9]) generalizes to modules of finite CI-dimension, and we have an analogous reduction of CI-dimension for the syzygy sequence as well. Lastly, we may also characterize the CI-dimension of a module as intermediary between Gorenstein dimension[4] and projective dimension.

Theorem 2.8 ([AvGaPe], 1.4)**.** *For each finitely generated R-module M, it holds that*

$$\text{G-dim}_R M \leq \text{CI-dim}_R M \leq \operatorname{pd}_R M$$

and if one of these dimensions is finite then it equals the dimension(s) to the left.

Corollary 2.9 ([AvGaPe], 1.4)**.** *If* $\text{CI-dim}_R M < \infty$ *then* $\text{CI-dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M$*.*

The above assertion states that, when finite, CI-dimension satisfies the Auslander-Buchsbaum formula (cf. [BrHe, 1.3.3]), just as Gorenstein dimension does. We will commonly refer to $g = \operatorname{depth} R - \operatorname{depth}_R M$ since, at minimum, we tend to assume that G-dimension is finite (e.g. when R is Gorenstein). However, the topics we discuss are predominantly devoted to modules for which $\operatorname{pd}_R M = \infty$, since the goal is to better understand infinite free resolutions, and thus their modules. In particular, we further focus on R-modules with finite CI-dimension, since we will see in subsequent sections that realizable patterns in $\{b_n^R(M)\}$ are then guaranteed to occur. The notion of critical degree for a finitely generated R-module was originally introduced in [AvGaPe], along with these results, and gives a name to the homological degree at which patterns may arise in a module's Betti sequence. The definition below makes this notion more precise.

[4] For the definition of *Gorenstein dimension*, the reader may refer to [AvMa].

Definition 2.10 ([AvGaPe], 7.1[5]). An R-module M has *critical degree* of at most s, denoted by $\mathrm{crdeg_R M} \leq s$, if its minimal resolution $\mathbf{F}$ has a chain endomorphism μ of degree $-q < 0$ such that $\mu_{n+q} : F_{n+q} \to F_n$ is surjective for all $n > s$. If no such s exists, then set $\mathrm{crdeg_R M} = \infty$. Note that $\mathrm{crdeg_R 0} = -\infty$ and for any $M \neq 0$, $-1 \leq \mathrm{crdeg_R M} \leq \infty$.

It should be obvious that the critical degree is mainly useful when it is finite. For example, when $M \neq 0$ and $\mathrm{pd}_R\, M$ is finite, $\mathrm{crdeg_R M}$ is equivalent to the projective dimension (which is in turn equivalent to $\text{CI-dim}_R\, M = \text{G-dim}_R\, M$, by the previous theorem). Of course, if we only have finiteness of $\text{CI-dim}_R\, M$, then is the critical degree of M finite too? As it turns out, $\mathrm{crdeg_R M} < \infty$ whenever $\text{CI-dim}_R\, M < \infty$ and, more significantly, the Betti sequence $\{\beta_n^R(M)\}$ is non-decreasing *after* $\mathrm{crdeg_R M}$ steps. This result will be discussed in Section 2.3.3 of this chapter; first, we explore the patterns that may occur in the Betti sequence if we only assume finiteness of the critical degree itself (but not necessarily the CI-dimension).

2.2.2 Finiteness of Critical Degree

Since the critical degree of an R-module communicates the existence of an endomorphism on $\mathbf{F}$ for which it becomes certain that $\mathrm{rk}(F_{n+q}) \geq \mathrm{rk}(F_n)$ for all n sufficiently large, there is a very clear connection between patterns in $\{b_n^R(M)\}$, as outlined by the complexity, and $\mathrm{crdeg_R M}$.

Theorem 2.11 ([AvGaPe], 7.8). *Suppose M is a finitely generated R-module such that* $\mathrm{crdeg_R M} = s < \infty$ *and $\mu : \mathbf{F} \to \mathbf{F}$ is an endomorphism, of degree q, which*

[5]In the original paper, the authors specified that $q < 0$; yet, given how they defined μ degree-wise, it must hold that $q > 0$ for the definition to make sense. We have altered only this typo in the definition presented in this thesis.

realizes s. Additionally, denote $g = \operatorname{depth} R - \operatorname{depth}_R M$. Then, one of the following cases occurs:

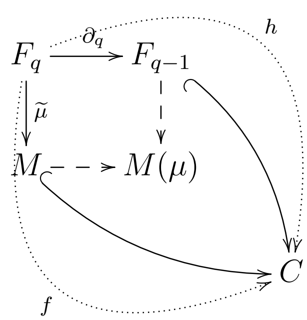
(i) The $\operatorname{cx}_R M \leq 1$ and M has period q after g steps.

(ii) The $\operatorname{cx}_R M > 1$ and $b_n^R M < b_{n+q}^R M$ for $n > s$.

If furthermore $q \leq 2$, then $b_n^R M < b_{n+1}^R M$ for $n > s$, with equality when $\operatorname{cx}_R M \leq 1$.

Before presentation of the proof, we re-emphasize that we may take g to be the Gorenstein dimension of M *if* $\text{G-dim}_R M < \infty$. However, there is no assumption of this, nor any assumption of finite CI-dimension. Theorem 7.3, listed in Section 2.3.3, focuses on what can be said with the additional assumption that M has finite CI-dimension. We now present the proof of Theorem 7.8, as given in [AvGaPe], with full detail below.

Proof. Let μ be an endomorphism on $\mathbf{F}$ that realizes $\mathrm{crdeg_R M}$. Furthermore, denote M_n the n^{th} syzygy module $\partial(F_n) \subseteq F_{n-1}$ for all $n \in \mathbb{Z}$, and let $\widetilde{\mu} : F_q \to M$ be the composition of $\mu_q : F_q \to F_0$ with the augmentation map $F_0 \to M$. Now, we can form the pushout of ∂_q and $\widetilde{\mu}$ in the following manner. Consider the diagram



where $M(\mu)$ is the coequalizer of ∂_q and $\widetilde{\mu}$, which is just a quotient of the coproduct of M and F_{q-1}. Recall that in $\mathcal{R}$-$\mathcal{mod}$ the coproduct is equivalent to the direct sum, so $M(\mu) = \frac{M \oplus F_{q-1}}{\operatorname{im}(f-h)}$. (See [Ma] for further details regarding the pushout.) Thus, we obtain the pushout diagram:

$$
\begin{array}{ccccccccccccccc}
F_{q+1} & \xrightarrow{\partial_{q+1}} & F_q & \xrightarrow{\partial_q} & F_{q-1} & \xrightarrow{\partial_{q-1}} & F_{q-2} & \xrightarrow{\partial_{q-2}} & \cdots & \xrightarrow{\partial_1} & F_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle\widetilde{\mu}} & & \big\downarrow & & \big\| & & & & \big\| & & \big\| & & \\
0 & \longrightarrow & M & \longrightarrow & M(\mu) & \longrightarrow & F_{q-2} & \xrightarrow{\partial_{q-2}} & \cdots & \xrightarrow{\partial_1} & F_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

Since $M(\mu)$ is a quotient of $M \oplus F_{q-1}$, we can consider the following exact sequences

$$0 \to M \to M(\mu) \to M_{q-1} \to 0 \tag{2.1}$$

and

$$0 \to M_{q-1} \to F_{q-2} \xrightarrow{\partial_{q-2}} \cdots \xrightarrow{\partial_1} M \to 0 \tag{2.2}$$

where (2.1) can be viewed as the extension of M_{q-1} by M and (2.2) is the $(q-1)$-extension defining an equivalence class in $\operatorname{Ext}_R^{q-1}(M, M_{q-1})$. Thus, we can consider the Yoneda product $\operatorname{Ext}_R^{q-1}(M, M_{q-1}) \times \operatorname{Ext}_R^1(M_{q-1}, M) \to \operatorname{Ext}_R^q(M, M)$ and notice that the bottom row in the pushout diagram above is just the Yoneda splice of (2.1) and (2.2). Now note that (2.1) induces a long exact sequence in cohomology of the form

$$0 \to \operatorname{Hom}_R(M_{q-1}, \Bbbk) \to \operatorname{Hom}_R(M(\mu), \Bbbk) \to \operatorname{Hom}_R(M, \Bbbk) \xrightarrow{\alpha^0} \operatorname{Ext}_R^1(M_{q-1}, \Bbbk) \to \cdots$$

$$\cdots \to \operatorname{Ext}_R^n(M_{q-1}, \Bbbk) \to \operatorname{Ext}_R^n(M(\mu), \Bbbk) \to \operatorname{Ext}_R^n(M, \Bbbk) \xrightarrow{\alpha^n} \operatorname{Ext}_R^{n+1}(M_{q-1}, \Bbbk) \to \cdots$$

with connecting homomorphism α^n for each $n \in \mathbb{Z}$. Moreover, note that (2.2) and the bottom row in the pushout diagram each induce long exact sequences with iterated connecting homomorphisms $\mu^n : \operatorname{Ext}_R^n(M, \Bbbk) \to \operatorname{Ext}_R^{n+q}(M, \Bbbk)$ and $\beta^{n+1} \colon \operatorname{Ext}_R^{n+1}(M_{q-1}, \Bbbk) \to \operatorname{Ext}_R^{(n+1)+(q-1)}(M, \Bbbk)$ for each $n \in \mathbb{Z}$. And since the bottom row is just the splice of (2.1) and (2.2), it holds that $\mu^n = \pm\beta^{n+1}\alpha^n$. Now it should be clear that $\operatorname{Ext}_R^n(M_{q-1}, \Bbbk) \cong \operatorname{Ext}_R^{n+q}(M, \Bbbk)$ since $M_{q-1} = \Omega^{q-1}M$ and so we can use $\Sigma^{q-1}\mathrm{F}$ when computing the former n^{th} Ext module to see that it coincides with the latter.[6] Thus, β^{n+1} is bijective for all $n \in \mathbb{Z}$ and so we may rewrite the long exact sequence in the form

$$\cdots \to \operatorname{Ext}_R^{n+q-1}(M, \Bbbk) \to \operatorname{Ext}_R^n(M(\mu), \Bbbk) \to \operatorname{Ext}_R^n(M, \Bbbk) \xrightarrow{\mu^n} \operatorname{Ext}_R^{n+q}(M, \Bbbk) \to \operatorname{Ext}_R^{n+1}(M(\mu), \Bbbk) \to \cdots$$

[6]Refer to Chapter 3 for definition of Σ.

Consider now that we may write $\mu^n = \operatorname{Hom}_R(\mu_{n+q}, \Bbbk)$ and by definition of an epimorphism in $\mathscr{R}$-$\mathscr{mod}$, it should be clear that μ^n is injective for all $n > s$ since $\mu_{n+q} \colon F_{n+q} \to F_n$ is an epimorphism for all $n > s$ by assumption. Moreover, since each Ext module in the sequence above is additionally a $\Bbbk$-vector space, μ^n splits. Therefore, the long exact sequence above can be broken up into split short exact sequences of the form

$$0 \to \operatorname{Ext}_R^n(M, \Bbbk) \xrightarrow{\mu^n} \operatorname{Ext}_R^{n+q}(M, \Bbbk) \twoheadrightarrow \operatorname{Ext}_R^{n+1}(M(\mu), \Bbbk) \to 0$$

for all $n > s$. Hence, $\operatorname{Ext}_R^{n+q}(M, \Bbbk) \cong \operatorname{Ext}_R^n(M, \Bbbk) \bigoplus \operatorname{Ext}_R^{n+1}(M(\mu), \Bbbk)$ for all $n > s$ and so

$$b_{n+q}^R(M) = b_n^R(M) + b_{n+1}^R(M(\mu))$$

for all such n. It should be clear from this statement that $\operatorname{cx}_R M \leq 1$ precisely when $\operatorname{pd}_R M(\mu) < \infty$ since then $b_{n+1}^R(M(\mu)) = 0$ for all $n \gg s$. If however, $\operatorname{cx}_R M > 1$ then note that $b_{n+1}^R(M(\mu)) \neq 0$ since otherwise there would be equality of $b_n^R(M)$ and $b_{n+q}^R(M)$ for all $n \gg 0$. Thus, part (2.11) of the theorem has been shown. To justify the remainder of part (2.11), suppose $\operatorname{cx}_R M \leq 1$ and denote $r = \operatorname{pd}_R M(\mu) = \operatorname{depth} R - \operatorname{depth}_R M(\mu) \geq 0$. First note that in this case, μ^n must be an isomorphism for all $n > r$ since $\operatorname{Ext}_R^{n+1}(M(\mu), \Bbbk)$ will then be 0. Meaning that for $n > r$, μ_n is a surjective homomorphism of free R-modules with the same rank and thus, as a consequence of Nakayama's lemma, μ_n must be an isomorphism.

Now that we have shown M has period q after r steps, it remains to show that $g \geq r$; that is, $\operatorname{depth}_R M \leq \operatorname{depth}_R M(\mu)$. For the sake of contradiction, first assume $\operatorname{depth}_R M \gneq \operatorname{depth} R$ and so, since M_{q-1} is a syzygy of M, we see that $\operatorname{depth}_R M_{q-1} \geq \operatorname{depth} R$ by the depth formula $\operatorname{depth}_R \Omega^n M \geq \min\{\operatorname{depth}_R M + n, \operatorname{depth} R\}$. Then note that (2.1) implies $\operatorname{depth}_R M(\mu) \geq \min\{\operatorname{depth}_R M, \operatorname{depth}_R M_{q-1}\}$ so it must hold that $\operatorname{depth}_R M(\mu) \geq \operatorname{depth} R$. But $\operatorname{depth}_R M(\mu)$ cannot be strictly greater than depth R since $r \geq 0$, so it must be that $\operatorname{depth}_R M(\mu) = \operatorname{depth} R$ implying $M(\mu)$ is a free R-module. And so this means that we may indefinitely repeat the bottom row of the original pushout diagram to construct a free resolution of M; namely, M is periodic and thus an infinite syzygy. Of course, this implies that $\operatorname{depth}_R M = \operatorname{depth} R$, which contradicts the original assumption. So now that we see $\operatorname{depth}_R M \leq \operatorname{depth} R$, note also that $\operatorname{depth}_R M \leq \operatorname{depth}_R M_(q-1)$, implying that $\operatorname{depth}_R M_{q-1} \geq \operatorname{depth}_R M$ again by the depth formula. This together with (2.1) gives the implication that $\operatorname{depth}_R M(\mu) \geq \operatorname{depth}_R M$, as desired.

Hence, it only remains to show the last assertion of the theorem. Assume $q = 2$ so that μ_{n_2} is surjective for $n > s$, inducing a surjection from M_{n+2} onto M_n for all such n. Now choose a minimal prime ideal $\mathfrak{p}$ of R so that localizing at $\mathfrak{p}$ yields an artinian ring guaranteeing that all modules over $R_\mathfrak{p}$ have finite length. Since localization is an exact functor, note that

$$0 \to (M_{n+1})_\mathfrak{p} \to (R^{b_n})_\mathfrak{p} \to (M_n)_\mathfrak{p} \to 0$$

and

$$0 \to (M_{n+2})_\mathfrak{p} \to (R^{b_{n+1}})_\mathfrak{p} \to (M_{n+1})_\mathfrak{p} \to 0$$

will also be short exact sequences. Thus, we have that $\operatorname{length}(M_n)_\mathfrak{p} + \operatorname{length}(M_{n+1})_\mathfrak{p} = b_n \operatorname{length}(R_\mathfrak{p})$ and $\operatorname{length}(M_{n+1})_\mathfrak{p} + \operatorname{length}(M_{n+2})_\mathfrak{p} = b_{n+1} \operatorname{length}(R_\mathfrak{p})$. Combining these statements we obtain

$$(b_{n+1} - b_n) \operatorname{length}(R_\mathfrak{p}) = \operatorname{length}(M_{n+1})_\mathfrak{p} - \operatorname{length}(M_n)_\mathfrak{p}$$

and note that since there exists a surjection $M_{n+2} \twoheadrightarrow M_n$, the right-hand side of the above statement must be non-negative. Therefore, $b_{n+1} \geq b_n$ for $n > s$. And, by the previous argument given for the first statement of Theorem 2.11, if $\operatorname{cx}_R M \leq 1$ we have that $M_{n+2} \cong M_n$ for $n > r$ so that $b_{n+1} = b_n$ for all such n. □

The proof of Theorem 7.8 gives great insight into the relationship between the critical degree and the emergence of patterns in the Betti sequence of a module. Furthermore, the seeds for the connection between critical degree and the $\mathcal{S}$-module $\operatorname{Ext}^*_R(M, \Bbbk)$ are sown in this proof; in fact, we return to many of these ideas in Chapter 4 of this thesis. For now, we move to further explore this connection which yields a cohomological characterization of critical degree in $\mathscr{R}$-$\mathscr{mod}$. In some sense, this will provide more precision, and even tangibility, to the notion of critical degree.

2.3 Critical Degree and Modules of Finite CI-Dimension

In this section we focus first on finitely generated R-modules with finite CI-dimension and then later reduce to the case where R is a complete intersection of the form $Q/(\mathbf{f})$, with $\mathbf{f} = f_1, \ldots, f_c$ a regular Q-sequence. Before we present Proposition 7.2 in [AvGaPe], however, we must first aim to understand $\operatorname{Ext}^*_R(M, \Bbbk)$ as a module over two special subalgebras in the more general setting.

2.3.1 Cohomological Characterization of Critical Degree

Recall that for an R-module M, the graded R-module $\operatorname{Ext}^*_R(M, \Bbbk)$ can be viewed both as a *left* $\operatorname{Ext}^*_R(\Bbbk, \Bbbk)$- and a *right* $\operatorname{Ext}^*_R(M, M)$-bimodule via the action induced by Yoneda multiplication. Moreover, there exist natural homomorphisms $\operatorname{Ext}^*_R(\Bbbk, \Bbbk) \leftarrow \mathcal{S} \to \operatorname{Ext}^*_R(M, M)$ with images lying within the centers of the graded algebras. In particular, the authors of [AvGaPe] first prove that if M is a finitely generated module with $\text{CI-dim}_R\, M < \infty$, then $\operatorname{Ext}^*_R(M, \Bbbk)$ is finitely generated over $\mathscr{Z}^*$, where we denote $\mathscr{Z}^*$ as the R-subalgebra of $\operatorname{Ext}^*_R(M, M)$ generated by the central elements in $\operatorname{Ext}^2_R(M, M)$. Then, noting that $\operatorname{Ext}^*_R(\Bbbk, \Bbbk)$ is actually the universal enveloping algebra[7] of the homotopy Lie algebra $\pi^*(R)$, it also holds that $\operatorname{Ext}^*_R(M, \Bbbk)$

[7]since we have assumed R to be a noetherian local ring.

is finitely generated over the $\Bbbk$-subalgebra $\mathscr{P}^* \subseteq \operatorname{Ext}_R^*(\Bbbk,\Bbbk)$ generated by the central elements in $\pi^*(R)$.[8] Equipped with the definitions of $\mathscr{Z}^*$ and $\mathscr{P}^*$, we are now ready to present the cohomological characterization of critical degree, as originally described.

Proposition 2.12 ([AvGaPe], 7.2)**.** *Let $M \neq 0$ be a finitely generated R-module with* $\text{CI-dim}_R M < \infty$*. Then the critical degree of M is finite, say $\mathrm{crdeg_R M} = s$, and the following statements hold*

(i) There exist equalities

$$\begin{aligned} s &= \sup\{r \in \mathbb{N} \cup \{0\} \mid \operatorname{depth}_{\mathscr{Z}^*} \operatorname{Ext}_R^{\geq r}(M,\Bbbk) = 0\} \\ &= \sup\{r \in \mathbb{N} \cup \{0\} \mid \operatorname{depth}_{\mathscr{P}^*} \operatorname{Ext}_R^{\geq r}(M,\Bbbk) = 0\} \end{aligned}$$

(ii) There is a codimension 1 quasi-deformation $R \to R' \leftarrow Q$ such that $\mathrm{crdeg_{R'} M'} = s$ and the CI operator on the minimal resolution of M' is surjective in degrees $n > s$.

We will only discuss and use Part (i) of this proposition throughout this thesis; we simply include the Part (ii) for completeness of the statement (see [AvGaPe] for more details). To give an idea of why the former statement is true, we will now present a brief sketch of the proof.

Sketch of Proof. First, the authors note that finiteness of critical degree follows from the given equalities and the finiteness of $\operatorname{Ext}_R^*(M,\Bbbk)$ over $\mathscr{Z}^*$ (and $\mathscr{P}^*$). Next, note that $\tilde{\mu}$ from the proof of Theorem 7.8 can be viewed as a q-extension of M by M, and thus $[\tilde{\mu}] \in \operatorname{Ext}_R^q(M,M)$. Specifically, $[\tilde{\mu}]$ represents the equivalence class of q-extensions represented by

$$\tilde{\mu}\colon \quad 0 \to M \to M(\mu) \to F_{q-2} \to \cdots \to F_0 \to M \to 0$$

[8] See Theorems (4.9) and (4.10) in [AvGaPe].

and μ^n can be given (up to sign) by Yoneda multiplication of $\operatorname{Ext}_R^n(M,\Bbbk)$ with $[\tilde{\mu}]$. That is, $\mu^n(\xi) = \xi\tilde{\mu}$ for any n-extension in $\xi \in \operatorname{Ext}_R^n(M,\Bbbk)$. Moreover, the splitting of the long exact sequence in cohomology for all $n > s$ demonstrates that $\xi\tilde{\mu} = 0$ only if $\xi = 0$; thus, $\tilde{\mu}$ is a non zero-divisor on the truncation $\operatorname{Ext}_R^{>s}(M,\Bbbk)$. And since $\operatorname{depth}_{\mathscr{Z}^*} \operatorname{Ext}_R^{>s}(M,\Bbbk) = \operatorname{depth}_{\mathscr{Z}^*[\tilde{\mu}]} \operatorname{Ext}_R^{>s}(M,\Bbbk) > 0$, then s must be greater than or equal to the maximum r such that $\operatorname{depth}_{\mathscr{Z}^*} \operatorname{Ext}_R^{\geq r}(M,\Bbbk) = 0$.

Conversely, it should be clear that any $[\xi] \in \mathscr{Z}^* \subseteq \operatorname{Ext}_R^*(M,M)$ originates from a chain endomorphism on $\mathbf{F}$ (in the same way $[\mu]$ did, depicted in the proof of Theorem 7.8). Therefore, $s \leq \sup\{r \in \mathbb{N} \cup \{0\} \mid \operatorname{depth}_{\mathscr{Z}^*} \operatorname{Ext}_R^{\geq r}(M,\Bbbk) = 0\}$, proving the first equality. Lastly, the authors give reasoning for the second equality via the fact that the depth of $\operatorname{Ext}_R^{\geq r}(M,\Bbbk)$ over $\mathscr{Z}^*$ coincides with that over $\mathscr{P}^*$ (see construction in (6.2)). $\square$

As it turns out, when R is a complete intersection ring, $\pi^2(R) = \operatorname{Hom}_{\Bbbk}(L_{\mathbf{f}},\Bbbk)$ where $L_{\mathbf{f}}$ is the c-dimensional $\Bbbk$-vector space $(\mathbf{f})/\mathfrak{m}(\mathbf{f})$. That is to say, the $\Bbbk$-subalgebra $\mathscr{P}^* \subseteq \operatorname{Ext}_R^*(\Bbbk,\Bbbk)$ reduces to $\mathscr{S}$ in this case (see (3.8) and (6.1.3) in [Av2] for more details). In the next section, we focus on the case of a complete intersection ring and give an intuitive explanation for what insight Proposition 7.2 provides.

2.3.2 Critical Degree of Modules over Complete Intersection Rings

For this section we assume R is a complete intersection of the form $Q/(\mathbf{f})$, with $\mathbf{f} = f_1, \ldots, f_c$ a regular Q-sequence, and $M \neq 0$ a finitely generated R-module. First, note that by Proposition 7.2 modules over a complete intersection ring *always* have finite critical degree, since such modules have finite CI-dimension. However, we can also gain this fact from just the definition of critical degree; recall that the proof of Theorem 3.1 in [Ei] guaranteed an endomorphism $\hat{t}$ which is eventually surjective for all homological degrees large enough. Therefore, by definition, $\mathrm{crdeg_R M} \leq N < \infty$ for the sufficiently large N identified in the proof. Of course, the same argument holds for *any* module over a complete intersection; and so, we have finiteness of the critical degree for all such modules.

Specifically, the additional insight gained from Proposition 7.2 is not necessarily the finiteness of the critical degree in this case, but rather the cohomological characterization of it. We now provide this within the context of a module over a complete intersection, with a slight correction to the original statement.

Proposition 2.13 (cf. [AvGaPe], 7.2 and Proposition 2.12)**.** *Let R be a complete intersection of the form $Q/(\mathbf{f})$, with $\mathbf{f} = f_1, \dots, f_c$ a regular Q-sequence. If $M \neq 0$ is a finitely generated R-module then the critical degree of M is finite, say* $\mathrm{crdeg_R M} = s$, *and if we set*

$$r^* = \sup\{r \in \mathbb{N} \cup \{0\} \mid \operatorname{depth}_{\mathcal{S}} \operatorname{Ext}_R^{\geq r}(M, \Bbbk) = 0\}$$

the following equality holds

$$s = \max\{r^*, -1\}.$$

Discussion of Statement. First note that the correction in the above proposition should also be applied to the general version listed in the previous section. Essentially, we have to allow for the case when $\mathrm{crdeg_R M} = -1$ and, since it is assumed that $M \neq 0$, then $\mathrm{crdeg_R M} \neq -\infty = \sup\{\emptyset\}$. Next, if we know that the critical degree is realized by some endomorphism $\hat{t} \in R[\mathbf{t}]$, and hence a non zero-divisor $\hat{\chi} \in \mathcal{S}$, then the equality above is quite easy to see.

Denote $E^{\geq n} = \operatorname{Ext}_R^{\geq n}(M, \Bbbk)$ for any integer $n \in \mathbb{N}$, and note that on one hand the existence of such a $\hat{\chi}$ implies $\operatorname{depth}_{\mathcal{S}} E^{\geq s+1} \neq 0$. Thus, since $\operatorname{depth}_{\mathcal{S}} E^{\geq r^*} = 0$ by definition (and, in fact, is the *greatest* such degree) then we know $r^* \leq s$. However, on the other hand, the definition of critical degree implies $s + 1$ is the *least* degree such that there exists a non zero-divisor on $E^{\geq s+1}$ implying that $\operatorname{depth}_{\mathcal{S}} E^{\geq s} = 0$. Therefore, we must have that $s \geq r^*$, proving equality of the two.

The difficulty in proving this equality (and, accordingly, the more general one given in the previous section) is in knowing that there exists an endomorphism *from the ring of cohomology operators* which realizes the critical degree. Note that the construction of $\tilde{\mu}$ depended solely on the assumption that μ was an endomorphism which realized the critical degree; then, we were able to view it as an extension $[\tilde{\mu}] \in \operatorname{Ext}^q_R(M, M)$. In fact, this technique can be done for any endomorphism $\mu\colon \mathbf{F} \to \Sigma^{q_\mu}\mathbf{F}$ where we can define $\tilde{\mu}$ in the same way as in the proof of Theorem 7.8 so that $[\tilde{\mu}] \in \operatorname{Ext}^{q_\mu}_R(M, M)$. Whether or not this element is a non zero-divisor on some $\operatorname{Ext}^{\geq n}_R(M, M)$ depends on whether μ is surjective for all $i \geq n$.

The point of this discussion is to emphasize that the key to the argument for Proposition 7.2, and thus the special case given above, is that the depth of $\operatorname{Ext}^*_R(M, \Bbbk)$ coincides over the sub-algebras $\mathscr{Z}^*$ and $\mathscr{P}^*(= \mathcal{S}$ when R is a complete intersection). Given this fact, we only need to demonstrate the construction of $\hat{\chi}$ since the least degree for which there exists a non zero-divisor in $\mathcal{S}$ on $\operatorname{Ext}^{\geq n}_R$ will coincide with the least degree for which there exists a non zero-divisor in $\mathscr{Z}^*$. □

2.3.2.1 Construction of $\hat{\chi}$ *à la* Eisenbud

With the guarantee that there exists some $\hat{\chi} \in \mathcal{S}$ which realizes the critical degree of an R-module, we now demonstrate the construction of this element. In essence, this element is derived from the proof of Eisenbud's surjectivity theorem, but with a slight modification.

Recall from Eisenbud's proof that E denotes the largest artinian submodule of $\operatorname{Ext}^*_R(M, \Bbbk)$, so E must both contain $\operatorname{Soc}(\operatorname{Ext}^*_R(M, \Bbbk))$ and have finite length. However, it actually holds that $S = \operatorname{Soc}(\operatorname{Ext}^*_R(M, \Bbbk))$ has finite-length since it is both Artinian and semisimple. Hence, denote $E^{\geq n} = \mathrm{Ext}_R^{\geq n}(M, \Bbbk)$ for any integer $n \in \mathbb{N} \cup \{0\}$ and set

$$N_0^* = \inf\left\{n \in \mathbb{N} \cup \{0\} \mid E^{\geq n} \cap S = 0\right\}.$$

Note that since S has finite length, there must exist some truncation such that there are no nonzero elements annihilated by $\mathfrak{X}$, meaning $N_0^* < \infty$. And so, if we apply Eisenbud's argument to $E^{\geq N_0^*}$, then there must exist an element $\hat{\chi}^* \in \mathfrak{X}$ such that it is a non zero-divisor on $E^{> N_0^*}$. Moreover, this must be the largest truncation for which such a non zero-divisor exists, since by construction $E^{\geq N_0^* - 1} \cap S \neq 0$ implying there exists some nonzero element of degree $(N_0^* - 1)$ which is annihilated by each χ_j. Therefore, by definition $N_0^* - 1 = s$ and consequently, we see that the critical degree is precisely where the highest degree socle element lives in $\mathrm{Ext}_R^*(M, \Bbbk)$.

2.3.3 Patterns in the Betti Sequence and an Illuminating Example

Recall from Theorem 7.8 in [AvGaPe] that if $\mathrm{crdeg_R M} < \infty$, then either M has period q after $s = g$ steps *or* $b_n^R(M) \lneq b_{n+q}^R(M)$ where q is the (magnitude of) the degree of the endomorphism which realizes the critical degree. Given our previous discussion, note that $\deg(\hat{\chi}) = 2$ and so, by Theorem 7.8, we see that when R is a complete intersection the Betti numbers of M are eventually either constant or strictly increasing after the critical degree. In fact, the authors demonstrated that the same holds even if we relax our conditions to the class of R-modules with finite CI-dimension, as indicated by the following theorem.

Theorem (7.3 from [AvGaPe]). *Suppose first that* $\text{CI-dim}_R M < \infty$ *and thus equivalent to* $g = \operatorname{depth} R - \operatorname{depth}_R M$. *Then, one of the following three cases occurs:*

1. *The* $\mathrm{cx}_R M = 0$ *(meaning* $\mathrm{pd}_R M < \infty$*), so* $\mathrm{crdeg_R M} = g = \mathrm{pd}_R M$ *and* $b_n^R M = 0$ *for* $n > g$.

2. *The* $\operatorname{cx}_R M = 1$ *implies* $\mathrm{crdeg_R M} \leq g$*; in which case,* $b_n^R M = b$ *for* $b \in \mathbb{N}$ *and* M *has period* 2 *after* $\mathrm{crdeg_R M} + 1$ *steps.*
3. *When* $\operatorname{cx}_R M \geq 2$*, the* $\mathrm{crdeg_R M} < \infty$ *and* $b_n^R M < b_{n+1}^R M$ *after* $\mathrm{crdeg_R M} + 1$ *steps.*

Since we have discussed why this pattern holds in the case of modules over a complete intersection, we omit the authors' more general proof. Rather, we move on to discuss an illuminating example. It is clear from the results discussed in this section that when $\operatorname{cx}_R M \leq 1$, the critical degree of M is bounded above by depth R. However, in the more interesting cases of $\operatorname{cx}_R M \geq 2$ no bound for all R-modules of a given complexity $d > 1$ exists.

Example 2.14 (7.5 in [AvGaPe])**.** Suppose $\mathrm{crdeg_R M} = s < \infty$ and denote $M' = \Omega^{s+1} M$. Then note that $\operatorname{cx}_R M = \operatorname{cx}_R M'$ but $\mathrm{crdeg_R M'} = 0$.

While the critical degree is a significant concept, serving as the "marker" or "flag" for when growth is guaranteed to occur in the Betti sequence of an R-module, it is quite disappointing that we cannot hope to bound this marker for all modules over a given complexity (greater than one) and, even more so, for all modules in a given syzygy sequence. One goal of this thesis is to make an attempt at rectifying this misfortune.

CHAPTER 3

The Category of Totally Acyclic Complexes

While the notion of critical degree in $\mathscr{R}$-*mod* is an interesting and useful invariant, it does have the disadvantage described at the end of the previous chapter. Furthermore, although it is possible to give bounds on this degree for a *specific* module of complexity strictly less than 3, and possibly of a higher complexity, our hope to give bounds over *all* modules of a given complexity, or even particular types of modules, becomes all the more futile with this disadvantage in mind. To alleviate these concerns, we turn towards viewing the notion over a different category– one that is, in some sense, analogous to $\mathscr{R}$-*mod*. In this chapter we build the necessary components for understanding the motivation for this chosen category and highlight its connection to $\mathscr{R}$-*mod* in order to provide intuition for the reasoning behind Chapter 4 of this thesis. It should be noted that many of the definitions and results in this chapter can be found in [Ch] or [HoJoRo].[1]

3.1 Basics of R-Complexes

We now return to the topic of R-complexes, introduced in Chapter 1, and relay a few common characteristics of them. Recall that an R-complex is a sequence of R-modules and R-module homomorphisms such that the image of each preceding morphism maps into the kernel of each subsequent morphism. Commonly, we call such sequences *chain* (or *cochain*) complexes and, we may call the morphisms between

[1] A good reference has not been identified for the lemmas and proposition discussed in Section 3.1.2; hence, the name and provided proofs of statements.

them *chain* (respectively, *cochain*) *maps.* One may view chain complexes as finite sequences (bounded), infinite in one direction (e.g. projective resolutions), or as doubly-infinite sequences (such as the following example).

Example 3.1. Let $R = \Bbbk[\![x, y]\!]/(x^5 - y^5, xy)$ then the following sequence represents an R-complex:

$$\cdots \to R \xrightarrow{(x-y)} R \xrightarrow{(x^4-y^4)} R \xrightarrow{(x-y)} R \xrightarrow{(x^4-y^4)} R \xrightarrow{(x-y)} R \to \cdots$$

It should be clear that $\operatorname{im}(x - y) \subseteq \ker(x^4 - y^4)$ and $\operatorname{im}(x^4 - y^4) \subseteq \ker(x - y)$. In fact, for this example, we have that $\operatorname{im}\partial_n^{\mathrm{C}} = \ker \partial_{n-1}^{\mathrm{C}}$ for all $n \in \mathbb{Z}$.

3.1.1 Equivalency of R-Complexes

In Chapter 1, we mentioned that rather than studying equivalency in the sense of isomorphisms between two chain complexes at every degree, we study a looser form called *homotopy equivalency.* Recall that two chain maps f and g are *homotopic* if there exist degree 1 maps $\{h_n\}$, called *homotopy maps*, such that $f_n - g_n = h_{n-1}\partial_n^{\mathrm{C}} + \partial_{n+1}^{\mathrm{D}} h_n$ for all $n \in \mathbb{Z}$. Essentially, the chain maps f and g are similar *enough* that we can view their difference at each degree as being "equivalent" to zero. Unsurprisingly, we find a rich theory if we consider homotopy the "isomorphism" for R-complexes, as there is interesting structure we gain from this type of equivalence. However, before discussing such structure further, we review a few important characteristics of R-complexes.

3.1.2 Common Folklore of R-Complexes

Just as we predominantly focus on *minimal* resolutions, we have an appropriate analogue for R-complexes. A complex (C, ∂) is called *minimal* if every homotopy equivalence $e \colon \mathrm{C} \to \mathrm{C}$ is an isomorphism; here, we mean R-module isomorphisms at each degree. With the assumption that R is local, we actually have that this is true if and only if $\partial(\mathrm{C}) \subseteq \mathfrak{m}\mathrm{C}$. Hence, we will use this characterization for minimal complexes throughout the remainder of this thesis. Alternatively, an R-complex is called *contractible*, or (*homotopically*) *trivial*, if the identity morphism 1^{C} is null-homotopic. A homotopy between 1^{C} and 0 is called a *contraction*, and note that, by definition, if C is contractible, then it is homotopically equivalent to the zero complex. It should be clear that a contractible complex cannot be minimal. If C was both contractible and minimal, there would exist homotopy maps $h_n \colon C_n \to C_{n+1}$ such that $1^{\mathrm{C}} = h_{n-1}\partial_n^{\mathrm{C}} + \partial_{n+1}^{\mathrm{C}} h_n \subseteq \mathfrak{m}C_n$. Thus, if C is contractible, then note that $\partial(\mathrm{C}) \nsubseteq \mathfrak{m}\mathrm{C}$.

Moreover, there exists a decomposition of all R-complexes such that $\mathrm{C} = \overline{\mathrm{C}} \oplus \mathrm{T}$ where $\overline{\mathrm{C}}$ is a *unique* minimal subcomplex of C, and T is contractible. It also holds that if $\mathrm{C} \simeq \mathrm{D}$ then $\overline{\mathrm{C}} \cong \overline{\mathrm{D}}$ where $\overline{\mathrm{C}}$ is the minimal subcomplex of C and $\overline{\mathrm{D}}$ is that of D. Lastly, since contractible complexes are homotopically equivalent to the zero complex, it is quite natural that each R-complex would, in some sense, be "equivalent" to its minimal subcomplex. We demonstrate this result in the next lemma.

Lemma 3.2. *Let* (C, ∂) *be an R-complex with the decomposition* $\overline{\mathrm{C}} \oplus \mathrm{T}$ *such that* $\overline{\mathrm{C}}$ *is minimal and* T *is contractible. Then,* $\mathrm{C} \simeq \overline{\mathrm{C}}$.

Proof. Our goal is to show that the natural projection $\pi : \mathrm{C} \to \overline{\mathrm{C}}$ is in fact a homotopy equivalence; that is, we wish to show $\pi\iota - \mathrm{Id}^{\overline{\mathrm{C}}} \sim 0$ and $\iota\pi - \mathrm{Id}^{\mathrm{C}} \sim 0$, where $\iota : \overline{\mathrm{C}} \to \mathrm{C}$ is the natural inclusion. First note that $\pi\iota = \mathrm{Id}^{\overline{\mathrm{C}}}$ since for any $a \in C_n$ the composition $\pi_n\iota_n$ sends $a \mapsto (a, 0) \mapsto a$ and this is the case for each $n \in \mathbb{Z}$, so that $\pi\iota$ is trivially homotopic to $\mathrm{Id}^{\overline{\mathrm{C}}}$. And so, it only remains to show that there exists a homotopy map $\{h_n\}_{n\in\mathbb{Z}}$ such that $\iota_n\pi_n - \mathrm{Id}_n^C = h_{n-1}\partial_n^C + \partial_{n+1}^C h_n$ for each $n \in \mathbb{Z}$.

Since $\mathrm{C} = \overline{\mathrm{C}} \oplus \mathrm{T}$ with T contractible, there exist homotopy maps $\{k_n\}_{n\in\mathbb{Z}}$ such that $0 - \mathrm{Id}_n^{\mathrm{T}} = k_{n-1}\partial_n^{\mathrm{T}} + \partial_{n+1}^{\mathrm{T}} k_n$ for each $n \in \mathbb{Z}$. Hence, we may define the homotopy maps

$$h_n = \begin{pmatrix} 0 & 0 \\ 0 & k_n \end{pmatrix}$$

noting that for any $(a, b) \in C_n$ we have

$$(h_{n-1}\partial_n^{\mathrm{C}} + \partial_{n+1}^{\mathrm{C}} h_n)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & k_{n-1} \end{pmatrix}\begin{pmatrix} \partial_n^{\overline{\mathrm{C}}} & 0 \\ 0 & \partial_n^{\mathrm{T}} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} \partial_{n+1}^{\overline{\mathrm{C}}} & 0 \\ 0 & \partial_{n+1}^{\mathrm{T}} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & k_n \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

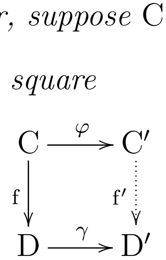
$$= \begin{pmatrix} 0 & 0 \\ 0 & k_{n-1} \end{pmatrix}\begin{pmatrix} \partial_n^{\overline{\mathrm{C}}}(a) \\ \partial_n^{\mathrm{T}}(b) \end{pmatrix} + \begin{pmatrix} \partial_{n+1}^{\overline{\mathrm{C}}} & 0 \\ 0 & \partial_{n+1}^{\mathrm{T}} \end{pmatrix}\begin{pmatrix} 0 \\ k_n(b) \end{pmatrix} = \begin{pmatrix} 0 \\ k_{n-1}\partial_n^{\mathrm{T}}(b) \end{pmatrix} + \begin{pmatrix} 0 \\ \partial_{n+1}^{\mathrm{T}} k_n(b) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ (k_{n-1}\partial_n^{\mathrm{T}} + \partial_{n+1}^{\mathrm{T}} k_n)(b) \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathrm{Id}_n^{\mathrm{T}}(b) \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} = (i_n\pi_n - \mathrm{Id}_n^{\mathrm{C}})\begin{pmatrix} a \\ b \end{pmatrix}.$$

Thus, by definition, we see that $\iota\pi \sim \mathrm{Id}^{\mathrm{C}}$, implying $\mathrm{C} \simeq \overline{\mathrm{C}}$. □

We now consider the connection between homotopically equivalent complexes and chain maps. Whenever we have some sort of equivalency between objects, the natural consideration is to ask what maps may be *induced* by this equivalency. Already a well-known phenomenon, the subsequent lemma plays an important role within the definitions presented in Chapter 4 of this thesis, as well as the methodology discussed in Chapter 5; hence, we include proof for the statement as well.

Lemma 3.3. *Let* C*,* D*,* C′ *and* D′ *be R-complexes for which there exist chain maps* $\mathrm{f}\colon \mathrm{C} \to \mathrm{D}$ *and* $\gamma\colon \mathrm{D} \to \mathrm{D}'$*. Moreover, suppose* $\mathrm{C} \simeq \mathrm{C}'$*. Then there exists an induced chain map* $\mathrm{f}'\colon \mathrm{C}' \to \mathrm{D}'$ *such that the square*



commutes (up to homotopy); that is, $\gamma \mathrm{f} \sim \mathrm{f}'\varphi$*. Furthermore, this map is unique up to homotopy.*

Proof. Our goal is to first show existence of a map f' such that $\gamma\mathrm{f} - \mathrm{f}'\varphi \sim 0$. Note that since $\varphi\colon \mathrm{C} \to \mathrm{C}'$ is a homotopy equivalence, there exists a map $\varphi^{-1} : \mathrm{C}' \to \mathrm{C}$ such that $\varphi\varphi^{-1} \sim \mathrm{Id}^{\mathrm{C}'}$ and $\varphi^{-1}\varphi \sim Id^{\mathrm{C}}$. Define $\mathrm{f}' = \gamma\mathrm{f}\varphi^{-1}$ and note that $\varphi^{-1}\varphi - \mathrm{Id}^{\mathrm{C}} \sim 0$ implies existence of some $k_n\colon C_n \to C_{n+1}$ such that $\varphi_n^{-1}\varphi_n - \mathrm{Id}_n^{\mathrm{C}} = k_{n-1}\partial_n^{\mathrm{C}} + \partial_{n+1}^{\mathrm{C}}k_n$ for each $n \in \mathbb{Z}$. Hence, it holds that

$$\begin{aligned}
f'_n\varphi_n - \gamma_n f_n &= (\gamma_n f_n \varphi_n^{-1})\varphi_n - \gamma_n f_n \\
&= (\gamma_n f_n)\varphi_n^{-1}\varphi_n - (\gamma_n f_n) \\
&= \gamma_n f_n(\varphi_n^{-1}\varphi_n - \mathrm{Id}_n^C) \\
&= \gamma_n f_n(k_{n-1}\partial_n^C + \partial_{n+1}^C k_n) \\
&= \gamma_n f_n k_{n-1}\partial_n^C + \gamma_n(f_n\partial_{n+1}^C)k_n \\
&= \gamma_n f_n k_{n-1}\partial_n^C + \gamma_n(\partial_{n+1}^D f_{n+1})k_n \\
&= \gamma_n f_n k_{n-1}\partial_n^C + (\gamma_n\partial_{n+1}^D) f_{n+1}k_n \\
&= \gamma_n f_n k_{n-1}\partial_n^C + (\partial_{n+1}^{D'}\gamma_{n+1}) f_{n+1}k_n \\
&= (\gamma_n f_n k_{n-1})\partial_n^C + \partial_{n+1}^{D'}(\gamma_{n+1} f_{n+1}k_n)
\end{aligned}$$

and defining $h_n : C_n \to D'_{n+1}$ as $h_n = \gamma_{n+1}f_{n+1}k_n$, the above equation yields a homotopy equivalence proving $\gamma\mathrm{f} \sim \mathrm{f}'\varphi$. Now note that since $\varphi\varphi^{-1} - \mathrm{Id}^{\mathrm{C}} \sim 0$ there exist homotopy maps $k'_n\colon C'_n \to C'_{n+1}$ such that $\varphi_n\varphi_n^{-1} - \mathrm{Id}_n^{\mathrm{C}'} = k'_{n-1}\partial_n^{\mathrm{C}'} + \partial_{n+1}^{\mathrm{C}'}k'_n$ for each $n \in \mathbb{Z}$. To show uniqueness, suppose $\mathrm{g}\colon \mathrm{C}' \to \mathrm{D}'$ such that $\mathrm{g}\varphi \sim \gamma\mathrm{f}$ and so there exist maps $l_n\colon C_n \to D'_{n+1}$ where $\gamma_n f_n - g_n\varphi_n = l_{n-1}\partial_n^{\mathrm{C}} + \partial_{n+1}^{\mathrm{D}'}l_n$ for each $n \in \mathbb{Z}$. This in turn implies that

$$(\gamma_n f_n - g_n\varphi_n)\varphi_n^{-1} = (l_{n-1}\partial_n^{\mathrm{C}} + \partial_{n+1}^{\mathrm{D}'}l_n)\varphi_n^{-1}$$

$$\gamma_n f_n \varphi_n^{-1} - g_n(\varphi_n \varphi_n^{-1}) = l_{n-1}\partial_n^{\mathrm{C}}\varphi_n^{-1} + \partial_{n+1}^{\mathrm{D}'} l_n \varphi_n^{-1}$$

$$f'_n - g_n(k'_{n-1}\partial_n^{\mathrm{C}'} + \partial_{n+1}^{\mathrm{C}'} k'_n + \mathrm{Id}_n^{\mathrm{C}}) = l_{n-1}(\partial_n^{\mathrm{C}}\varphi_n^{-1}) + \partial_{n+1}^{\mathrm{D}'} l_n \varphi_n^{-1}$$

$$(f'_n - g_n \,\mathrm{Id}_n^{\mathrm{C}}) - (g_n k'_{n-1}\partial_n^{\mathrm{C}'} + (g_n \partial_{n+1}^{\mathrm{C}'})k'_n) = l_{n-1}(\varphi_{n-1}^{-1}\partial_n^{\mathrm{C}'}) + \partial_{n+1}^{\mathrm{D}'} l_n \varphi_n^{-1}$$

$$f'_n - g_n = (l_{n-1}\varphi_{n-1}^{-1})\partial_n^{\mathrm{C}'} + \partial_{n+1}^{\mathrm{D}'} l_n \varphi_n^{-1} + (g_n k'_{n-1})\partial_n^{\mathrm{C}'} + (\partial_{n+1}^{\mathrm{D}'} g_{n+1})k'_n$$

$$f'_n - g_n = (l_{n-1}\varphi_{n-1}^{-1})\partial_n^{\mathrm{C}'} + (g_n k'_{n-1})\partial_n^{\mathrm{C}'} + \partial_{n+1}^{\mathrm{D}'}(l_n \varphi_n^{-1}) + \partial_{n+1}^{\mathrm{D}'}(g_{n+1}k'_n)$$

$$f'_n - g_n = (l_{n-1}\varphi_{n-1}^{-1} + g_n k'_{n-1})\partial_n^{\mathrm{C}'} + \partial_{n+1}^{\mathrm{D}'}(l_n \varphi_n^{-1} + g_{n+1}k'_n)$$

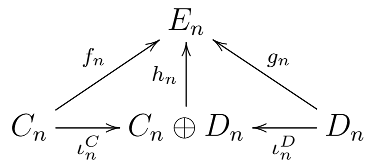
for each $n \in \mathbb{Z}$. Hence, we see the existence of a map $h'_n : C'_n \to D'_{n+1}$, defined as $h'_n = l_n\varphi_n^{-1} + g_{n+1}k'_n$, such that $f'_n - g_n = h'_{n-1}\partial_n^{C'} + \partial_{n+1}^{\mathrm{D}'}h'_n$, which of course demonstrates that $f' \sim g$, as needed. □

Therefore, for any chain map between R-complexes, we obtain an induced chain map between any two complexes that are homotopically equivalent to them, whilst respecting homotopy equivalences. Combining the two lemmas, we realize the following statement, for which the significance will become apparent in Chapter 4.

Proposition 3.4. *Let* C *and* D *be* R*-complexes with chain map* $\mathrm{f}\colon \mathrm{C} \to \mathrm{D}$*. If* $\overline{\mathrm{C}}$ *and* $\overline{\mathrm{D}}$ *are the respective minimal subcomplexes, then there exists an induced chain map* $\bar{f}\colon \overline{\mathrm{C}} \to \overline{\mathrm{D}}$*, which is unique (up to homotopy).*

3.1.3 Category of R-Complexes

Similar to R-modules, we can take as objects R-complexes and then take the set of chain maps to be the morphisms. Because $\mathscr{R}$*-mod* is additive, this forms a category called the *category of complexes over* $\mathscr{R}$*-mod* , or the *category of* R*-complexes*, and we may denote it $\mathcal{C}(R)$. Unsurprisingly, $\mathcal{C}(R)$ is again additive where the coproduct is described with degree-wise direct sums: $\mathrm{C} \oplus \mathrm{D} = (C_n \oplus D_n, \partial_n)_{n\in\mathbb{Z}}$ where $\partial_n(c,d) = (\partial_n^{\mathrm{C}}(c), \partial_n^{\mathrm{D}}(d))$ for $c \in C_n$ and $d \in D_n$. And so, for any R-complex E such that there exist chain maps $f\colon \mathrm{C} \to \mathrm{E}$ and $g\colon \mathrm{D} \to \mathrm{E}$, we have the following commutative diagram



where $h_n(c, d) = f_n(c) + g_n(d)$ for $c \in C_n$ and $d \in D_n$. Furthermore, $\mathcal{C}(R)$ is an abelian category; we will present the existence of short exact sequences in Section 3.2, but leave the reader to refer to Proposition 2.5 in [HoJoRo] for proof that $\mathcal{C}(R)$ is abelian (existence of kernels, cokernels, etc.). So the advantage of $\mathcal{C}(R)$ is that it has clear structural similarities to $\mathscr{R}$-$\mathscr{mod}$. However, there is one downside– equivalence in $\mathcal{C}(R)$ is determined by degree-wise isomorphism, which turns out to be too rigid of a structure to study. Hence, we use $\mathcal{C}(R)$ as the foundation for a category with a less rigid type of equivalency, discussed in Section 3.3.

3.2 Special Constructions of R-Complexes

3.2.1 The Suspension Endofunctor

Let C be an R-complex and note that we may consider "shifting" the degrees of C such that the module and differential at degree i may be redefined as degree $i-1$, or even degree $i-q$ for some $q \in \mathbb{Z}$. This process is made precise via functors on $\mathcal{C}(R)$, as well as other R-complex categories discussed shortly, and is described in the definition below.

Definition 3.5 (cf. [HoJoRo])**.** The *Suspension Endofunctor*, often denoted Σ, is a functor on the category of complexes such that for any R-complex (C, ∂), we obtain an R-complex $(\Sigma\mathrm{C}, \partial^{\Sigma\mathrm{C}})$ with

$$(\Sigma C)_n = C_{n-1} \text{ and } \partial_n^{\Sigma C} = -\sigma_{n-2}\partial_{n-1}^{C}\sigma_{n-1}^{-1}$$

for each $n \in \mathbb{Z}$ where $\sigma_n : C_n \to (\Sigma C)_{n+1}$ is the natural isomorphism (described below). Moreover, Σ acts on chain morphisms as follows: if $\mathrm{f}\colon \mathrm{C} \to \mathrm{D}$, then $\Sigma\mathrm{f}$ is a chain map from $\Sigma\mathrm{C}$ to $\Sigma\mathrm{D}$ with $(\Sigma\mathrm{f})_n = f_{n-1}\sigma_{n-1}^{-1}$ for each $n \in \mathbb{Z}$.

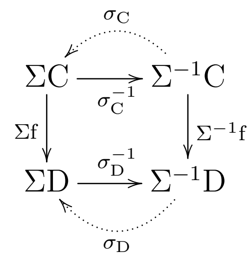
For simplicity's sake, and because the added rigor of including σ notationally does not contribute any necessity to the general theory itself, we will commonly just write $\partial_n^{\Sigma\mathrm{C}} = -\partial_{n-1}^{C}$ to denote the n^{th} differential of $\Sigma\mathrm{C}$ and will only utilize σ when it is necessary to ascertain technical details. Moreover, we may apply this functor any number of times and denote this action on C as the complex $\Sigma^q\mathrm{C}$ with $(\Sigma^q\mathrm{C})_n = C_{n-q}$ and $\partial_n^{\Sigma^q\mathrm{C}} = (-1)^q\partial_{n-q}^{\mathrm{C}}$ where $q \in \mathbb{Z}^+$. Note that this functor essentially "shifts" a complex 1 (or q) degrees to the left, which is equivalent to reassigning the homological degrees of the R-modules and the differentials. Therefore, it should be clear that $\Sigma^q\mathrm{C}$ is again a complex since

$$\partial_n^{\Sigma^q\mathrm{C}} \circ \partial_{n+1}^{\Sigma^q\mathrm{C}} = (-1)^q\partial_{n-q}^{\mathrm{C}} \circ (-1)^q\partial_{n+1-q}^{\mathrm{C}} = (-1)^{2q}\partial_{n-q}^{\mathrm{C}}\partial_{n+1-q}^{\mathrm{C}} = 0.$$

Given a chain map between R-complexes $\mathrm{f}\colon \mathrm{C} \to \mathrm{D}$, applying the suspension endofunctor q times to f yields the morphism $\Sigma^q\mathrm{f}\colon \Sigma^q\mathrm{C} \to \Sigma^q\mathrm{D}$ where $(\Sigma^q\mathrm{f})_n(x) = f_{n-q}(x)$ for any $x \in C_{n-q}$ and each $n \in \mathbb{Z}$. As it turns out Σ is both *additive* and an *automorphism*. That is, Σ respects direct sums:

$$\Sigma^q(\mathrm{C} \oplus \mathrm{D}) = \Sigma^q\mathrm{C} \oplus \Sigma^q\mathrm{D}.$$

Moreover, there exists a functor Σ^{-1} so that $\Sigma^{-1} \circ \Sigma$ and $\Sigma \circ \Sigma^{-1}$ are the identity functors on $\mathcal{C}(R)$. Here, we may view Σ^{-1} as "shifting" degrees to the right, as opposed to the left, or as the normal suspension functor on the opposite category of R-complexes, $\mathcal{C}(R)^{\mathrm{op}}$. The natural isomorphism listed in the definition above is a natural transformation for which there exists a two-sided inverse; in particular $\sigma^{-1} : \Sigma \to \Sigma^{-1}$ is a natural transformation such that the diagram



commutes; that is, $\Sigma^{-1}\mathrm{f} \circ \sigma_{\mathrm{C}}^{-1} = \sigma_{\mathrm{D}}^{-1}\Sigma\mathrm{f}$. Essentially, σ and σ^{-1} give a formal manner in which to map elements of a particular degree in C (e.g. n) to another degree in ΣC (e.g. $n+1$).

3.2.2 Mapping Cones

We now consider another special type of construction of R-complexes which plays a pivotal role in the main topic of Section 3.3.3. This construction, called the *mapping cone*, is in some manner similar to direct sums of R-complexes; however, the significance of this type of complex will not become apparent until we view it in a different type of category. For now, we merely present its definition so that we may discuss the existence of short exact sequences in $\mathcal{C}(R)$.

Definition 3.6 (cf. [HoJoRo])**.** If $\mathrm{f}\colon \mathrm{C} \to \mathrm{D}$ is a morphism of R-complexes in $\mathcal{C}(R)$, then the *mapping cone of* f, denoted either Cone (f) or M (f), is the R-complex with n^{th} module $\mathrm{M}(\mathrm{f})_n = (\Sigma\mathrm{C})_n \oplus D_n$ and n^{th} differential

$$\partial_n^{\mathrm{M}(\mathrm{f})} = \begin{pmatrix} \partial_n^{\Sigma C} & 0 \\ \Sigma\mathrm{f}_n & \partial_n^D \end{pmatrix}.$$

Note that we may informally write

$$\partial_n^{\mathrm{M}(\mathrm{f})} = \begin{pmatrix} -\partial_{n-1}^{C} & 0 \\ f_{n-1} & \partial_n^D \end{pmatrix}$$

and $(\mathrm{M}(\mathrm{f}))_n = C_{n-1} \oplus D_n$, which we will commonly do so that we may write

$$\cdots \to C_n \oplus D_{n+1} \xrightarrow{\begin{pmatrix} -\partial_n^C & 0 \\ f_n & \partial_{n+1}^D \end{pmatrix}} C_{n-1} \oplus D_n \xrightarrow{\begin{pmatrix} -\partial_{n-1}^C & 0 \\ f_{n-1} & \partial_n^D \end{pmatrix}} C_{n-2} \oplus D_{n-1} \to \cdots$$

as the R-complex, $\mathrm{M}(\mathrm{f})$.

3.2.3 Short Exact Sequences in $\mathcal{C}(R)$

Given any morphism $\mathrm{f}\colon \mathrm{C} \to \mathrm{D}$ of R-complexes, note that we may identify the natural inclusion $\iota_n\colon D_n \to \mathrm{M}(\mathrm{f})_n$ and natural projection $\pi_{n-1}\colon \mathrm{M}(\mathrm{f})_n \to C_{n-1}$ for each degree $n \in \mathbb{Z}$. Define the chain maps $\iota: \mathrm{D} \to \mathrm{M}(\mathrm{f})$ as the family of R-module homomorphisms $\{\iota_n\}_{n\in\mathbb{Z}}$ and $\pi: \mathrm{M}(\mathrm{f}) \to \Sigma\mathrm{C}$ as the family of R-module homomorphisms $\{\pi_n\}_{n\in\mathbb{Z}}$. Note that the latter is indeed a morphism of complexes since $\partial^{\Sigma\mathrm{C}}$ carries the sign. Furthermore, we may view $\iota_n = \begin{pmatrix} 0 & \mathrm{id}_n^{\mathrm{D}} \end{pmatrix}$ and $\pi_n = \begin{pmatrix} \mathrm{id}_n^{\Sigma\mathrm{C}} & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{n-1}^{\mathrm{C}} & 0 \end{pmatrix}$ for each $n \in \mathbb{Z}$. It should be clear that $\pi_n \circ \iota_n = 0$ at each degree. Hence, we may construct the following short exact sequence in $\mathcal{C}(R)$

$$0 \to \mathrm{D} \xrightarrow{\iota} \mathrm{Cone}(\mathrm{f}) \xrightarrow{\pi} \Sigma\mathrm{C} \to 0$$

since $\pi \circ \iota = 0$. It holds that this short exact sequence splits if and only if $f \sim 0$. In this case, there exist homotopy maps $s_n\colon C_n \to D_{n+1}$ such that $f_n = s_{n-1}\partial_n^{\mathrm{C}} + \partial_{n+1}^{\mathrm{D}} s_n$ for each $n \in \mathbb{Z}$ and so we may construct the map $\sigma\colon \Sigma\mathrm{C} \to \mathrm{M}(\mathrm{f})$ where $\sigma_n(x) = \begin{pmatrix} x & -s_n(x) \end{pmatrix}$ for any $x \in C_{n-1}$.

For any R-complexes C and D, the mapping cone of the zero map $0^{\mathrm{C}}\colon \mathrm{C} \to 0$ is just $\mathrm{M}(0^{\mathrm{C}}) = \Sigma\mathrm{C}$ and the mapping cone of $0^{\mathrm{D}}\colon 0 \to \mathrm{D}$ is $\mathrm{M}(0^{\mathrm{D}}) = D$. Lastly, the mapping cone of the identity morphism Id^{C} is the R-complex $\mathrm{M}\,(\mathrm{Id}^{\mathrm{C}})$ with R-modules $(\mathrm{M}\,(\mathrm{Id}^{\mathrm{C}}))_n = C_{n-1} \oplus C_n$ and differentials

$$\begin{pmatrix} \partial^{\mathrm{C}}_{n-1} & 0 \\ \mathrm{id}^{\mathrm{C}}_{n-1} & \partial^{\mathrm{C}}_{n} \end{pmatrix}$$

such that $\partial_n^{\mathrm{M}\,(\mathrm{Id}^{\mathrm{C}})} \colon C_{n-1} \oplus C_n \to C_{n-2} \oplus C_{n-1}$. And the identity morphism $\mathrm{Id}^{\mathrm{M}\,(\mathrm{Id}^{\mathrm{C}})} \colon \mathrm{M}\,(\mathrm{Id}^{\mathrm{C}}) \to \mathrm{M}\,(\mathrm{Id}^{\mathrm{C}})$ is actually homotopic to 0 with the homotopy maps

$$\begin{pmatrix} 0 & \mathrm{id}^{\mathrm{C}}_{n} \\ 0 & 0 \end{pmatrix}$$

meaning that $\mathrm{M}\,(\mathrm{Id}^{\mathrm{C}}) \simeq 0$.

3.2.4 Hom and Tensor of R-Complexes

Just as we can apply the Hom and $\otimes$ functors to R-modules, we can do so with R-complexes. Interestingly enough, we obtain an R-complex in both cases. As we make use of the Hom of R-complexes at the end of this chapter, as well as in Chapters 4 and 5 of this thesis, along with the $\otimes$ of R-complexes in Chapter 5, we present the general definitions to which the reader may refer.

Definition 3.7 (cf. [Ro], pg. 321)**.** If C and D are R-complexes, then the *tensor product of the complexes C and D* is the complex with R-modules

$$(\mathrm{C} \otimes_R \mathrm{D})_n = \coprod_{i+j=n} (C_i \otimes_R D_j)$$

and differentials ∂_n which map pure tensors as follows

$$\partial_n(c \otimes d) = \partial_i^C(c) \otimes d + (-1)^i a \otimes \partial_j^D(d)$$

where $c \in C_i$ and $d \in D_j$.

Note that the complex $\mathrm{C} \otimes \mathrm{D}$ is the total complex of the bicomplex of C and D. Similarly, we may consider the R-complex $\mathrm{Hom}(\mathrm{C},\mathrm{D})$, defined as follows.

Definition 3.8 (cf. [Ro], pg. 323)**.** If C and D are R-complexes, then the Hom *of the complexes C and D* is the complex with R-modules of the form

$$\mathrm{Hom}(\mathrm{C},\mathrm{D})_n = \prod_{i+j=n} \mathrm{Hom}_R(C_{-i}, D_j)$$

for which any element $\mathrm{f} \in \mathrm{Hom}(\mathrm{C},\mathrm{D})_n$ is a family of maps $\{f_{i,j}\}$ where $f_{i,j} : C_{-i} \to D_j$ with $i + j = n$. Here, the differentials are given by $\partial_n(\mathrm{f}) = \{g_{i,j}\}$ where

$$g_{i,j} = (-1)^{i+j} f_{i+1,j}\partial^C_{-i} + \partial^D_{j+1} f_{i,j+1}$$

with $i + j = n - 1$.

Note that for the n^{th} R-module we may also write

$$\mathrm{Hom}(\mathrm{C},\mathrm{D})_n = \prod_{j-i=n} \mathrm{Hom}_R(C_i, D_j) = \prod_i \mathrm{Hom}_R(C_i, D_{i+n}) = \prod_j \mathrm{Hom}_R(C_{j-n}, D_j)$$

where the third form clearly represents precisely the degree n chain maps $\mathrm{C} \to \mathrm{D}$, as one would expect. With respect to the second form of $\mathrm{Hom}(\mathrm{C},\mathrm{D})_n$ given above, for any $f_i \colon C_i \to D_{i+n}$ the differential is given by $\partial_n(f) = (-1)^n f_{i-1}\partial^{\mathrm{C}}_i + \partial^{\mathrm{D}}_{i+n} f_i$.

3.3 Building the Homotopy Category of R-Modules

As previously mentioned, while $\mathcal{C}(R)$ has an abelian structure, the equivalency in this category is too rigid to produce meaningful theory; therefore, we return to the notion of homotopy in order to build a more interesting category of R-complexes. First and foremost, we see that it makes sense to view homotopy as a type of equivalence on the objects in $\mathcal{C}(R)$.

Proposition 3.9 (cf. [Ha])**.** *Homotopy is an equivalence relation.*

Proof. It should be clear that for any chain map $\mathrm{f} \colon \mathrm{C} \to \mathrm{D}$, $\mathrm{f} \sim \mathrm{f}$ since $\mathrm{f} - \mathrm{f} = 0 \sim 0$. Secondly, it should also be clear that $\mathrm{f} \sim \mathrm{g}$ implies $\mathrm{g} \sim \mathrm{f}$. If $\{h_n\}_{\mathfrak{n}\in\mathbb{Z}}$ are the homotopy maps such that $f_n - g_n = \partial^D_{n+1} h_n + h_{n-1}\partial^C_n$ then $g_n - f_n = \partial^D_{n+1}(-h_n) + (-h_{n-1})\partial^C_n$ for each $n \in \mathbb{Z}$. Lastly, if $f \sim g$ and $g \sim e$ for chain maps $f, g, e\colon \mathrm{C} \to \mathrm{D}$ then there exist homotopy maps $\{h_n\}_{\mathfrak{n}\in\mathbb{Z}}$ and $\{h^{'}_n\}_{\mathfrak{n}\in\mathbb{Z}}$ such that $f_n - g_n = \partial^D_{n+1} h_n + h_{n-1}\partial^C_n$ and $g_n - e_n = \partial^D_{n+1} h^{'}_n + h^{'}_{n-1}\partial^C_n$ for each $n \in \mathbb{Z}$. Thus, we have that

$$f_n - e_n = (f_n - g_n) - (e_n - g_n) = \partial^D_{n+1} h_n + h_{n-1}\partial^C_n - (\partial^D_{n+1}(-h^{'}_n) + (-h^{'}_{n-1})\partial^C_n)$$

$$= \partial^D_{n+1}(h_n + h^{'}_n) + (h_{n-1} + h^{'}_{n-1})\partial^C_n$$

for each $n \in \mathbb{Z}$ and so, by definition, $f \sim e$. □

Moreover, we actually can see that homotopy respects chain map composition; that is, if f,g: C → D are homotopic and e: D → E, then ef ∼ eg. Specifically, if $h_n\colon C_n \to D_{n+1}$ are the homotopy maps with respect to f, g then $h_n e_n\colon C_n \to E_{n+1}$ are the homotopy maps with respect to the compositions. Likewise, if $e\colon \mathrm{E} \to \mathrm{C}$ instead, then fe ∼ ge. Hence, if we take morphisms equal to each other whenever they are homotopic, then composition of morphisms is well defined. Meaning, we can define a category similar to $\mathcal{C}(R)$, but with equivalence based upon homotopies rather than isomorphisms at each degree.

Definition 3.10 (cf. [HoJoRo]). The *homotopy category* of R-complexes, denoted $\mathcal{K}(\mathrm{R})$, is a category whose objects are the same as $\mathcal{C}(R)$ but whose morphisms are the equivalence classes of morphisms in $\mathcal{C}(R)$ *modulo homotopy*; that is:

$$\mathrm{Hom}_{\mathcal{K}(\mathrm{R})}(\mathrm{C}, \mathrm{D}) = \mathrm{Hom}_{\mathcal{C}(R)}(\mathrm{C}, \mathrm{D})/\sim$$

As it turns out, $\mathcal{K}(\mathrm{R})$ inherits the additive structure from $\mathcal{C}(R)$, thus yielding a category with similar structure to $\mathcal{R}$-*mod*, albeit a more intriguing type of equivalence; this, in turn, leads to development of a deeper theory.[2] Lamentably, $\mathcal{K}(\mathrm{R})$ does *not*

[2]Surprisingly, or perhaps unsurprisingly, some algebraic structures are "too boring" to say anything meaningful!

inherit the abelian structure from $\mathcal{C}(R)$ and so we lose some structural similarities to $\mathcal{R}$-mod when we move to study objects and morphisms in $\mathcal{K}(\mathrm{R})$. We will look at a remedy for this shortly, but first we present a subcategory of $\mathcal{K}(\mathrm{R})$ which will become our main category of interest for the remainder of this thesis.

3.3.1 A Subcategory of the Homotopy Category

We begin this section by defining a special type of R-complex:

Definition 3.11 (cf. [BeJoMo])**.** A *totally acyclic complex* over a ring R is an R-complex (C, ∂) of projective R-modules such that

$$\mathrm{H}(\mathrm{C}) = 0 = \mathrm{H}(\mathrm{C}^*)$$

where C^* is the algebraic dual of C, defined as $\mathrm{Hom}_R(\mathrm{C}, R)$. In other words, if C is totally acyclic then both C and C^* are *exact* at each homological degree.

Note that in the definition above, the term *acyclic* is a synonym for “exact”. Much of homological algebra originated from algebraic topology which used the language of *boundaries* and *cycles*. The elements at each degree of an R-complex are called *chains*, which is why we also call objects in $\mathcal{C}(R)$ *chain* or *cochain* complexes. Furthermore, we may refer to the differentials as *boundary operators*, the elements of each kernel as *cycles* (sometimes also called *closed* elements), and elements contained in each image as a *boundary* (or *exact*) element. When a chain (or cochain) complex is exact, any cycle (closed element) is additionally an exact element (boundary). Hence, the term *acyclic* alludes to the fact that there are “no cycles” in this case, since we consider the cycles modulo the boundaries when we take homology at each degree.

Now, since totally acyclic complexes are just a specific type of R-complex, we can consider the sub-collection of objects from $\mathcal{C}(R)$– and, thus $\mathcal{K}(\mathrm{R})$– made up of all such complexes. Therefore, maps which are homotopic in $\mathcal{K}(\mathrm{R})$ are also homotopic on this restricted class of objects. Denote $\mathbf{K}_{\mathrm{tac}}(R)$ as the collection of objects which are totally acyclic R-complexes together with the same morphisms from $\mathscr{K}(\mathrm{R})$. We call $\mathbf{K}_{\mathrm{tac}}(R)$ a *full subcategory* of $\mathscr{K}(\mathrm{R})$, which means that it consists of a subcollection of objects from $\mathscr{K}(\mathrm{R})$ such that $\mathrm{Hom}_{\mathbf{K}_{\mathrm{tac}}(R)}(\mathrm{C},\mathrm{D}) = \mathrm{Hom}_{\mathscr{K}(\mathrm{R})}(\mathrm{C},\mathrm{D})$ for any two totally acyclic complexes C and D.

Now that we have recognized $\mathbf{K}_{\mathrm{tac}}(R)$ as a meaningful subcategory of $\mathscr{K}(\mathrm{R})$, we turn towards better understanding its categorical structure. Topics discussed in the remainder of Section 3.3.1 can also be applied to $\mathscr{K}(\mathrm{R})$ but, for the purposes of this thesis, we solely focus on structural properties of $\mathbf{K}_{\mathrm{tac}}(R)$.

3.3.2 Constructions of R-Complexes Viewed in $\mathbf{K}_{\mathrm{tac}}(R)$

We now return to the constructions defined in Sections 3.2.1 and 3.2.2. As it turns out, the functor Σ preserves homotopies; that is, if $\mathrm{f} \sim \mathrm{g}$ then $\Sigma^q\mathrm{f} \sim \Sigma^q\mathrm{g}$. In particular, if $\mathrm{h} = \{h_n\}_{n\in\mathbb{Z}}$ is the homotopy map associated to $\mathrm{f} - \mathrm{g} \sim 0$, then $\Sigma^q\mathrm{h}$ is the homotopy map associated to $\Sigma^q\mathrm{f} - \Sigma^q\mathrm{g} \sim 0$. We further demonstrate the following fact for homotopically equivalent R-complexes. It should be noted that we include proofs for the following three propositions, due to the lack of a better reference.

Proposition 3.12. *Given two R-complexes* C *and* D *in* $\mathbf{K}_{\mathrm{tac}}(R)$*, if* $\mathrm{C} \simeq \mathrm{D}$ *then* $\Sigma^q\mathrm{C} \simeq \Sigma^q\mathrm{D}$ *for any* $q \in \mathbb{Z}^+$.

Proof. This statement follows directly from the definition of homotopy equivalence. Since $\mathrm{C} \simeq \mathrm{D}$, there exist morphisms $\alpha\colon \mathrm{C} \to \mathrm{D}$ and $\beta\colon \mathrm{D} \to \mathrm{C}$ such that $\alpha\beta \sim \mathrm{Id}^{\mathrm{D}}$ and $\beta\alpha \sim \mathrm{Id}^{\mathrm{C}}$. Then note that we have induced morphisms $\Sigma^q\alpha\colon \Sigma^q\mathrm{C} \to \Sigma^q\mathrm{D}$ and $\Sigma^q\beta\colon \Sigma^q\mathrm{D} \to \Sigma^q\mathrm{C}$. If $\mathrm{h} = \{h_n\}_{n\in\mathbb{Z}}$ are the homotopy maps induced from $\alpha\beta - \mathrm{Id}^{\mathrm{D}} \sim 0$ and $\mathrm{k} = \{k_n\}_{n\in\mathbb{Z}}$ are the homotopy maps induced from $\beta\alpha - \mathrm{Id}^{\mathrm{C}} \sim 0$, then the appropriate homotopy maps showing $\Sigma^q\beta\Sigma^q\alpha - \mathrm{Id}^{\Sigma^q \mathrm{C}} \sim 0$ and $\Sigma^q\alpha\Sigma^q\beta - \mathrm{Id}^{\Sigma^q \mathrm{D}} \sim 0$ will be $\Sigma^q \mathrm{k}$ and $\Sigma^q \mathrm{h}$, respectively. □

Therefore, the action of this functor on morphisms in $\mathbf{K}_{\mathrm{tac}}(R)$ is well defined; specifically, mapping cones of morphisms in $\mathbf{K}_{\mathrm{tac}}(R)$ exist. However, it remains to verify that mapping cones are again objects in $\mathbf{K}_{\mathrm{tac}}(R)$ and that they are a well-defined notion within the category.

Proposition 3.13. *Given two chain maps of R-complexes* $\mathrm{f}\colon \mathrm{C} \to \mathrm{D}$ *and* $\mathrm{g}\colon \mathrm{C} \to \mathrm{D}$, *if* $\mathrm{f} \sim \mathrm{g}$, *then* $\mathrm{Cone}\,(\mathrm{f}) \simeq \mathrm{Cone}\,(\mathrm{g})$.

Proof. Our goal is to show that there exist morphisms $\alpha : \mathrm{M}\,(\mathrm{f}) \to \mathrm{M}\,(\mathrm{g})$ and $\beta\colon \mathrm{M}\,(\mathrm{g}) \to \mathrm{M}\,(\mathrm{f})$ such that $\alpha\beta \sim \mathrm{Id}^{\mathrm{M}\,(\mathrm{g})}$ and $\beta\alpha \sim \mathrm{Id}^{\mathrm{M}\,(\mathrm{f})}$. First, since $\mathrm{f} \sim \mathrm{g}$ there exist homotopy maps $h_n : C_n \to D_{n+1}$ such that $f_n - g_n = \partial^{\mathrm{D}}_{n+1} h_n + h_{n-1}\partial^{\mathrm{C}}_n$ for all $n \in \mathbb{Z}$. Now consider the diagram

$$\begin{array}{ccccc}
\cdots \to & C_{n-1} \oplus D_n & \xrightarrow{\begin{pmatrix} -\partial^{\mathrm{C}}_{n-1} & 0 \\ f_{n-1} & \partial^{\mathrm{D}}_n \end{pmatrix}} & C_{n-2} \oplus D_{n-1} & \to \cdots \\
& \beta_n \uparrow\downarrow \alpha_n & & \beta_{n-1} \uparrow\downarrow \alpha_{n-1} & \\
\cdots \to & C_{n-1} \oplus D_n & \xrightarrow[\begin{pmatrix} -\partial^{\mathrm{C}}_{n-1} & 0 \\ g_{n-1} & \partial^{\mathrm{D}}_n \end{pmatrix}]{} & C_{n-2} \oplus D_{n-1} & \to \cdots
\end{array}$$

where α and β are defined degree-wise as

$$\alpha_n = \begin{pmatrix} 1 & 0 \\ h_{n-1} & 1 \end{pmatrix}, \quad \beta_n = \begin{pmatrix} 1 & 0 \\ -h_{n-1} & 1 \end{pmatrix}.$$

Note that α is indeed a chain map from $\mathrm{M}\,(\mathrm{f})$ to $\mathrm{M}\,(\mathrm{g})$ since

$$\alpha_{n-1} \circ \partial^{\mathrm{M}\,(\mathrm{f})}_n = \begin{pmatrix} 1 & 0 \\ h_{n-2} & 1 \end{pmatrix} \begin{pmatrix} -\partial^{\mathrm{C}}_{n-1} & 0 \\ f_{n-1} & \partial^{\mathrm{D}}_n \end{pmatrix} = \begin{pmatrix} -\partial^{\mathrm{C}}_{n-1} & 0 \\ -h_{n-2}\partial^{\mathrm{C}}_{n-1} + f_{n-1} & \partial^{\mathrm{D}}_n \end{pmatrix}$$
$$= \begin{pmatrix} -\partial^{C}_{n-1} & 0 \\ g_{n-1} + \partial^{D}_n h_{n-1} & \partial^{D}_n \end{pmatrix} = \begin{pmatrix} -\partial^{C}_{n-1} & 0 \\ g_{n-1} & \partial^{D}_n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ h_{n-1} & 1 \end{pmatrix} = \partial^{\mathrm{M}\,(\mathrm{g})}_n \circ \alpha_n$$

by the homotopy relation between f_{n-1} and g_{n-1}. An identical argument can be given to demonstrate that β is a chain map as well. Finally, observe that

$$\alpha_n \circ \beta_n = \begin{pmatrix} 1 & 0 \\ h_{n-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -h_{n-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\beta_n \circ \alpha_n = \begin{pmatrix} 1 & 0 \\ -h_{n-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ h_{n-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

proving $\mathrm{M}\,(\mathrm{f}) \simeq \mathrm{M}\,(\mathrm{g})$, as needed. □

Lastly, we note that if both C and D are totally acyclic, then for any $f \in \mathrm{Hom}_{\mathbf{K}_{\mathrm{tac}}(R)}(\mathrm{C}, \mathrm{D})$ the R-complex $\mathrm{M}\,(\mathrm{f})$ is totally acyclic as well.

Proposition 3.14. *Given a chain map of totally acyclic complexes* $\mathrm{f}\colon \mathrm{C} \to \mathrm{D}$*, the* R*-complex* $\mathrm{M}\,(\mathrm{f})$ *is acyclic.*

Proof. As $\mathrm{M}\,(\mathrm{f})$ is an R-complex by definition, we only need show that $\ker(\partial_n^{\mathrm{M}\,(\mathrm{f})}) \subseteq \mathrm{im}(\partial_{n+1}^{\mathrm{M}\,(\mathrm{f})})$. Suppose $(x, y) \in \ker(\partial_n^{\mathrm{M}\,(\mathrm{f})})$ so that

$$\begin{pmatrix} -\partial_{n-1}^{\mathrm{C}} & 0 \\ f_{n-1} & \partial_n^{\mathrm{D}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\partial_{n-1}^{\mathrm{C}}(x) \\ f_{n-1}(x) + \partial_n^{\mathrm{D}}(y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

And since $x \in \ker(-\partial_{n-1}^{\mathrm{C}}) = \mathrm{im}(-\partial_n^{\mathrm{D}})$, there exists some $\alpha \in C_n$ such that $-\partial_n^{\mathrm{C}}(\alpha) = x$. Moreover, $f_{n-1}(x) = -\partial_n^{\mathrm{D}}(y)$ and so we have that

$$0 = (f_{n-1}\partial_n^{\mathrm{C}} - \partial_n^{\mathrm{D}} f_n)(\alpha) = f_{n-1}\partial_n^{\mathrm{C}}(\alpha) - \partial_n^{\mathrm{D}} f_n(\alpha) =$$

$$-f_{n-1}(x) - \partial_n^{\mathrm{D}} f_n(\alpha) = \partial_n^{\mathrm{D}}(y) - \partial_n^{\mathrm{D}} f_n(\alpha) = \partial_n^{\mathrm{D}}(y - f_n(\alpha)).$$

Thus $y - f_n(\alpha) \in \ker(\partial_n^{\mathrm{D}}) = \mathrm{im}(\partial_{n+1}^{\mathrm{D}})$ implying existence of a $\beta \in D_{n+1}$ such that $\partial_{n+1}^{\mathrm{D}}(\beta) = y - f_n(\alpha)$.

Hence, we have shown that there exists $(\alpha, \beta) \in C_n \oplus D_{n+1}$ such that

$$\partial_{n+1}^{\mathrm{M}\,(\mathrm{f})} \begin{pmatrix} \alpha \\ \\ \beta \end{pmatrix} = \begin{pmatrix} -\partial_n^{\mathrm{C}}(\alpha) \\ \\ f_n(\alpha) + \partial_{n+1}^{\mathrm{D}}(\beta) \end{pmatrix} = \begin{pmatrix} x \\ \\ f_n(\alpha) + (y - f_n(\alpha)) \end{pmatrix} = \begin{pmatrix} x \\ \\ y \end{pmatrix}$$

proving that $\mathrm{im}(\partial_{n+1}^{\mathrm{M}\,(\mathrm{f})}) = \ker(\partial_n^{\mathrm{M}\,(\mathrm{f})})$ and thus $\mathrm{M}\,(\mathrm{f})$ is acyclic. □

It also follows that $\mathrm{M}\,(\mathrm{f})^* = \mathrm{Hom}_R(\mathrm{M}\,(\mathrm{f}), R)$ is acyclic since $\mathrm{M}\,(\mathrm{f}^*) \cong \mathrm{M}\,(\mathrm{f})$, where f^* is the induced chain map from $\mathrm{Hom}_R(\mathrm{D}, R)$ to $\mathrm{Hom}_R(\mathrm{C}, R)$. Both C and D are acyclic by definition, meaning that we may apply a similar argument as given above to see that $\mathrm{M}\,(\mathrm{f}^*)$, and thus $\mathrm{M}\,(\mathrm{f})^*$, is acyclic as well. Therefore, mapping cones are well-defined objects in $\mathbf{K}_{\mathrm{tac}}(R)$. Recall that $\mathscr{K}(\mathrm{R})$, and thus $\mathbf{K}_{\mathrm{tac}}(R)$, is not abelian; to rectify this misfortune, we begin with consideration of another special construction in $\mathbf{K}_{\mathrm{tac}}(R)$ called a *candidate (or standard) triangle*. The manner in which this construction aids the structural disadvantage of $\mathbf{K}_{\mathrm{tac}}(R)$ not being abelian will be explained in the subsequent section.

3.3.3 The Triangulated Structure of $\mathbf{K}_{\mathrm{tac}}(R)$

We now have established the constructions in $\mathbf{K}_{\mathrm{tac}}(R)$ necessary for understanding an appropriate categorical analogue to abelian categories. The reader should note that typically this analogue is presented generally and then it is shown that $\mathscr{K}(\mathrm{R})$ has this structure (see [HoJoRo]); it does not require too much work thereafter to demonstrate that $\mathbf{K}_{\mathrm{tac}}(R)$ inherits the triangulated structure from $\mathscr{K}(\mathrm{R})$. However, in this thesis we choose instead to present $\mathbf{K}_{\mathrm{tac}}(R)$ as the prototype for a *triangulated category*, since our focus is solely on this subcategory of $\mathscr{K}(\mathrm{R})$. We leave any proof that $\mathbf{K}_{\mathrm{tac}}(R)$ is a triangulated subcategory of $\mathscr{K}(\mathrm{R})$ to the reader, though it is sure to be found elsewhere (e.g. [St]).

In Section 3.2.3, short exact sequences in $\mathcal{C}(R)$ were discussed; recall that they were constructed from the mapping cone of a chain map. Since we established in the last section that mapping cones are well-defined objects in $\mathbf{K}_{\text{tac}}(R)$, we can consider a construction of a similar nature to short exact sequences. First, let $\iota\colon \mathrm{D} \to \mathrm{M}(\mathrm{f})$ and $\pi : \mathrm{M}(\mathrm{f}) \to \Sigma\mathrm{C}$ represent the natural inclusion and projection morphisms (respectively) for a given R-complex morphism $\mathrm{f} : \mathrm{C} \to \mathrm{D}$. These morphisms respect homotopies, meaning that if $\mathrm{g} : \mathrm{C} \to \mathrm{D}$ such that $\mathrm{f} \sim \mathrm{g}$, then $\iota_f \sim \iota_g$ and $\pi_f \sim \pi_g$, so that both ι and π are well defined in $\mathbf{K}_{\text{tac}}(R)$. Moreover, we see in the following propositions that the given compositions are null-homotopic.

Proposition 3.15 (cf. [HoJoRo])**.** *If ι and π are as defined above, then $\pi\iota \sim 0$ and $\iota f \sim 0$.*

Proof. It is trivial that $\pi\iota = 0 \sim 0$. Our work lies in demonstrating that ιf is null-homotopic. First define $\sigma = \{\begin{pmatrix}\sigma_n & 0\end{pmatrix}\}_{n\in\mathbb{Z}} \colon C_n \to \mathrm{M}(\mathrm{f})_{n+1}$ where $\sigma_n \colon C_n \to (\Sigma\mathrm{C})_{n+1}$ is the natural isomorphism at each degree. Then for any $x \in C_n$ we have that

$$\begin{pmatrix}\sigma_{n-1}\\ 0\end{pmatrix}\partial_n^{\mathrm{C}}(x) + \partial_{n+1}^{\mathrm{M}(\mathrm{f})}\begin{pmatrix}\sigma_n\\ 0\end{pmatrix}(x) = \begin{pmatrix}\sigma_{n-1}\partial_n^{\mathrm{C}}(x)\\ 0\end{pmatrix} + \begin{pmatrix}\partial_{n+1}^{\Sigma\mathrm{C}} & 0\\ f_n\sigma_n^{-1} & \partial_{n+1}^{\mathrm{D}}\end{pmatrix}\begin{pmatrix}\sigma_n(x)\\ 0\end{pmatrix}$$

$$= \begin{pmatrix}\sigma_{n-1}\partial_n^{\mathrm{C}}(x)\\ 0\end{pmatrix} + \begin{pmatrix}(-\sigma_{n-1}\partial_n^{\mathrm{C}}\sigma_n^{-1})\sigma_n(x)\\ f_n\sigma_n^{-1}\sigma_n(x)\end{pmatrix} = \begin{pmatrix}\sigma_{n-1}\partial_n^{\mathrm{C}}(x) + (-\sigma_{n-1}\partial_n^{\mathrm{C}}(x))\\ f_n(x)\end{pmatrix}$$

$$= \begin{pmatrix}0\\ f_n(x)\end{pmatrix} = \begin{pmatrix}0 & \mathrm{id}_n^{\mathrm{D}}\end{pmatrix} f_n(x) = \iota_n f_n(x)$$

proving $\iota \circ f \sim 0$, as needed. □

Hence, there exists an "exact" (in $\mathbf{K}_{\text{tac}}(R)$) sequence of morphisms

$$\mathrm{C} \xrightarrow{\mathrm{f}} \mathrm{D} \xrightarrow{\iota} \mathrm{M}(\mathrm{f}) \xrightarrow{\pi} \Sigma\mathrm{C}$$

which yields a categorical analogue to the short exact sequences in $\mathcal{C}(R)$ (and any abelian category). In this vein, mapping cones play the roles of *kernels* and *cokernels* in $\mathbf{K}_{\mathrm{tac}}(R)$. Our goal now is to better understand the additional structure provided by sequences of the form above. Formally, we call such a sequence a *standard triangle.*

Definition 3.16 (cf. [HoJoRo])**.** A *candidate triangle* (sometimes just called a *triangle*) is a sequence of objects and morphisms in $\mathbf{K}_{\mathrm{tac}}(R)$ of the form $\mathrm{X} \xrightarrow{x} \mathrm{Y} \xrightarrow{y} \mathrm{Z} \xrightarrow{z} \Sigma\mathrm{X}$. A *distinguished triangle* is a candidate triangle which is isomorphic in $\mathbf{K}_{\mathrm{tac}}(R)$ to a standard triangle.

Note that a morphism of triangles is defined to be a triple (f, g, h) of morphisms in $\mathbf{K}_{\mathrm{tac}}(R)$ such that the diagram

$$\begin{array}{ccccccc} \mathrm{X} & \xrightarrow{x} & \mathrm{Y} & \xrightarrow{y} & \mathrm{Z} & \xrightarrow{z} & \Sigma\mathrm{X} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma\mathrm{f}} \\ \mathrm{X}' & \xrightarrow{x'} & \mathrm{Y}' & \xrightarrow{y'} & \mathrm{Z}' & \xrightarrow{z'} & \Sigma\mathrm{X}' \end{array}$$

is commutative in $\mathbf{K}_{\mathrm{tac}}(R)$. Thus, an isomorphism of triangles is a triangle morphism in which f, g, and h are homotopy equivalencies. The class of distinguished triangles, along with triangle isomorphisms, help us define the initial axiom identifying the triangulated structure of $\mathbf{K}_{\mathrm{tac}}(R)$.

(TR0) *Any triangle isomorphic to a distinguished triangle is again a distinguished triangle.*

Recall that for any $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$, $\mathrm{M}\left(\mathrm{Id}^{\mathrm{C}}\right) \simeq 0$. Thus, there exists a clear isomorphism of triangles

$$\begin{array}{ccccccc} \mathrm{C} & \xrightarrow{\mathrm{Id}^{\mathrm{C}}} & \mathrm{C} & \longrightarrow & \mathrm{M}\left(\mathrm{Id}^{\mathrm{C}}\right) & \longrightarrow & \Sigma\mathrm{C} \\ \| & & \| & & \downarrow{\scriptstyle \simeq} & & \| \\ \mathrm{C} & \xrightarrow{\mathrm{Id}^{\mathrm{C}}} & \mathrm{C} & \longrightarrow & 0 & \longrightarrow & \Sigma\mathrm{C} \end{array}$$

demonstrating that the bottom triangle must also be distinguished by (TR0).

(TR1) *For each* $\mathrm{C} \in \mathbf{K}_{\text{tac}}(R)$*, the triangle* $\mathrm{C} \xrightarrow{\mathrm{Id}^{\mathrm{C}}} \mathrm{C} \to 0 \to \Sigma\mathrm{C}$ is a distinguished triangle.

And we have already justified the following statement, taking $\mathrm{E} = \mathrm{M}(\mathrm{f})$.

(TR2) *For every morphism* $\mathrm{f}\colon \mathrm{C} \to \mathrm{D}$ *in* $\mathbf{K}_{\text{tac}}(R)$*, there is a distinguished triangle of the form*

$$\mathrm{C} \xrightarrow{\mathrm{f}} \mathrm{D} \to \mathrm{E} \to \Sigma\mathrm{C}.$$

Next, consider any standard triangle $\mathrm{C} \xrightarrow{\mathrm{f}} \mathrm{D} \xrightarrow{\iota(f)} \mathrm{M}(\mathrm{f}) \xrightarrow{\pi(f)} \Sigma\mathrm{C}$ in $\mathbf{K}_{\text{tac}}(R)$ and the candidate triangle

$$\mathrm{D} \xrightarrow{\iota(f)} \mathrm{M}(\mathrm{f}) \xrightarrow{\pi(f)} \Sigma\mathrm{C} \xrightarrow{-\Sigma\mathrm{f}} \Sigma\mathrm{D}$$

which represents a rotation of the former triangle. As it turns out, this candidate triangle is actually distinguished, since it is isomorphic to the standard triangle

$$\mathrm{D} \xrightarrow{\iota(f)} \mathrm{M}(\mathrm{f}) \xrightarrow{\iota(\iota(f))} \mathrm{M}(\iota(\mathrm{f})) \xrightarrow{\pi(\iota(f))} \Sigma\mathrm{D}$$

and, although the proof will be omitted here, the reader may refer to Theorem 6.7 in [HoJoRo] for a proof with respect to $\mathscr{K}(\mathrm{R})$. The significance, however, is the following rotational property of distinguished triangles:

(TR3) *If* $\mathrm{X} \xrightarrow{x} \mathrm{Y} \xrightarrow{y} \mathrm{Z} \xrightarrow{z} \Sigma\mathrm{X}$ *is a distinguished triangle, then* $\mathrm{Y} \xrightarrow{y} \mathrm{Z} \xrightarrow{z} \Sigma\mathrm{X} \xrightarrow{-\Sigma\mathrm{x}} \Sigma\mathrm{Y}$ *is too, and vice versa.*

Now consider a diagram of standard triangles of the form

$$\begin{array}{ccccccc}
\mathrm{C} & \xrightarrow{f} & \mathrm{D} & \xrightarrow{\iota(f)} & \mathrm{M}(\mathrm{f}) & \xrightarrow{\pi(f)} & \Sigma\mathrm{C} \\
\downarrow u & & \downarrow v & & & & \downarrow \Sigma\mathrm{u} \\
\mathrm{E} & \xrightarrow{g} & \mathrm{F} & \xrightarrow{\iota(g)} & \mathrm{M}(\mathrm{g}) & \xrightarrow{\pi(g)} & \Sigma\mathrm{E}
\end{array}$$

with the left square commuting (up to homotopy). That is, there exist homotopy maps $s_n \colon C_n \to F_{n+1}$ such that $v_n f_n - g_n u_n = \partial^{\mathrm{F}}_{n+1} s_n + s_{n-1}\partial^{\mathrm{C}}_n$ for each $n \in \mathbb{Z}$. We can then define $\mathrm{h} \colon \mathrm{M}\,(\mathrm{f}) \to \mathrm{M}\,(\mathrm{g})$ by setting

$$h_n = \begin{pmatrix} u_{n-1} & 0 \\ s_{n-1} & v_n \end{pmatrix}$$

for each $n \in \mathbb{Z}$ so that $h_n \colon C_{n-1} \oplus D_n \to E_{n-1} \oplus F_n$ and $\mathrm{h} := \{h_n\}_{n\in\mathbb{Z}}$. By definition of h, it should be clear that $\mathrm{h} \circ \iota(f) = \iota(g) \circ \mathrm{v}$ and $\pi(g) \circ \mathrm{h} = \Sigma\mathrm{u} \circ \pi(f)$; in fact, we even have proper equalities here. Moreover, the homotopy property of $\{s_n\}$ guarantees that h is a morphism in $\mathbf{K}_{\mathrm{tac}}(R)$, and so we obtain a map such that the diagram

$$\begin{array}{ccccccc}
\mathrm{C} & \xrightarrow{f} & \mathrm{D} & \xrightarrow{\iota(f)} & \mathrm{M}\,(\mathrm{f}) & \xrightarrow{\pi(f)} & \Sigma\mathrm{C} \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \vdots{\scriptstyle \mathrm{h}} & & \downarrow{\scriptstyle \Sigma\mathrm{u}} \\
\mathrm{E} & \xrightarrow{g} & \mathrm{F} & \xrightarrow{\iota(g)} & \mathrm{M}\,(\mathrm{g}) & \xrightarrow{\pi(g)} & \Sigma\mathrm{E}
\end{array}$$

commutes up to homotopy. This characteristic can be extended to *all* distinguished triangles so that we have the following property on the class of distinguished triangles in $\mathbf{K}_{\mathrm{tac}}(R)$.

(TR4) *Given distinguished triangles* $\mathrm{X} \xrightarrow{x} \mathrm{Y} \xrightarrow{y} \mathrm{Z} \xrightarrow{z} \Sigma\mathrm{X}$ *and* $\mathrm{X}' \xrightarrow{x'} \mathrm{Y}' \xrightarrow{y'} \mathrm{Z}' \xrightarrow{z'} \Sigma\mathrm{X}'$, *then each commutative diagram*

$$\begin{array}{ccccccc}
\mathrm{X} & \xrightarrow{x} & \mathrm{Y} & \xrightarrow{y} & \mathrm{Z} & \xrightarrow{z} & \Sigma\mathrm{X} \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle \Sigma\mathrm{f}} \\
\mathrm{X}' & \xrightarrow{x'} & \mathrm{Y}' & \xrightarrow{y'} & \mathrm{Z}' & \xrightarrow{z'} & \Sigma\mathrm{X}'
\end{array}$$

can be completed to a morphism of triangles (but not necessarily uniquely).

Although it will not be described here with respect to standard triangles, we also have the following property of distinguished triangles, which can be given in two variations other than what is given below. See [HoJoRo] for these variations.

(TR5) (**Octahedral axiom**) *Given distinguished triangles* $\mathrm{X} \xrightarrow{x} \mathrm{Y} \to \mathrm{Z}' \to \Sigma\mathrm{X}$, $\mathrm{Y} \xrightarrow{y} \mathrm{Z} \to \mathrm{X}' \to \Sigma\mathrm{Y}$ *and* $\mathrm{X} \xrightarrow{yx} \mathrm{Z} \to \mathrm{Y}' \to \Sigma\mathrm{X}$, *there exists a distinguished triangle* $\mathrm{Z}' \to \mathrm{Y}' \to \mathrm{X}' \to \Sigma\mathrm{Z}'$ *making the following diagram commute:*

$$\begin{array}{ccccccc}
\mathrm{X} & \xrightarrow{x} & \mathrm{Y} & \longrightarrow & \mathrm{Z}' & \longrightarrow & \Sigma\mathrm{X} \\
\downarrow{\scriptstyle \mathrm{id}_{\mathrm{X}}} & & \downarrow{\scriptstyle y} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\Sigma\mathrm{X}}} \\
\mathrm{X} & \xrightarrow{yx} & \mathrm{Z} & \longrightarrow & \mathrm{Y}' & \longrightarrow & \Sigma\mathrm{X}' \\
\downarrow{\scriptstyle x} & & \downarrow{\scriptstyle \mathrm{id}_{\mathrm{C}}} & & \downarrow & & \downarrow{\scriptstyle \Sigma x} \\
\mathrm{Y} & \xrightarrow{y} & \mathrm{Z} & \longrightarrow & \mathrm{X}' & \longrightarrow & \Sigma\mathrm{Y} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathrm{X}'}} & & \downarrow \\
\mathrm{Z}' & \longrightarrow & \mathrm{Y}' & \longrightarrow & \mathrm{X}' & \longrightarrow & \Sigma\mathrm{Z}'
\end{array}$$

Any category with a translation functor and a class of distinguished triangles which satisfy the above properties is called *triangulated* and (T0)-(T5) are the axioms of a triangulated category. The homotopy category $\mathcal{K}(\mathrm{R})$ is triangulated, and $\mathbf{K}_{\mathrm{tac}}(R)$ is a triangulated subcategory of $\mathcal{K}(\mathrm{R})$. Another common type of triangulated category, and in fact one of the archetypal examples, is called the Derived category of R-modules, often denoted $\mathcal{D}(R)$. Triangulated categories were introduced independently by Dieter Puppe in 1962 ([Pu]), with motivation given by the stable homotopy category, and by John-Louis Verdier in his 1963 thesis ([Ve2]). It was Verdier's thesis which outlined the five axioms, using his definition of the derived category of an abelian category, extending the work of his advisor Alexander Grothendieck, as his primary motivation.

3.3.3.1 Triangles Induce Long Exact Sequences in Ext

Now that we have discussed the main properties which contribute to the triangulated structure of a category, we will discuss two important consequences of such structure. It should be clear that the distinguished triangles in $\mathbf{K}_{\mathrm{tac}}(R)$, in some sense, generalize the notion of short exact sequences in abelian categories. While a category can be both abelian and triangulated, this only occurs if the category is semisimple (every short exact sequence splits). And so, there is only a slight overlap between these two types of categories. Hence, the class of distinguished triangles in a (non-abelian) triangulated category truly is the appropriate analogue to the class of short exact sequences in an abelian (non-triangulated) category.

For this reason, we might expect some similarities between distinguished triangles and short exact sequences. We will now discuss two consequences of distinguished triangles, in particular, which will highlight some of these similarities.

Proposition 3.17 (Composition of morphisms, cf. [HoJoRo])**.** *If* $\mathrm{X} \xrightarrow{u} \mathrm{Y} \xrightarrow{v} \mathrm{Z} \xrightarrow{w} \Sigma\mathrm{X}$ *is a distinguished triangle in* $\mathbf{K}_{\mathrm{tac}}(R)$*, then* $v \circ u \sim 0$ *and* $w \circ v \sim 0$*.*

Proof. We have already seen that this holds for the standard triangles in $\mathbf{K}_{\mathrm{tac}}(R)$. Moreover, by the rotational axiom (TR3) it only needs to be shown that $v \circ u \sim 0$ and proof of this is given by [HoJoRo, 4.1]. □

Thus, we have seen that *all* distinguished triangles are indeed the "short exact sequences" in $\mathbf{K}_{\mathrm{tac}}(R)$. Now, we might hope that, similarly to $\mathscr{R}$-$\mathscr{mod}$ or any other abelian category, such sequences can induce long exact sequences.

Proposition 3.18 (Long Exact Sequences, cf. [HoJoRo])**.** *Let* $\mathrm{X} \xrightarrow{u} \mathrm{Y} \xrightarrow{v} \mathrm{Z} \xrightarrow{w} \Sigma\mathrm{X}$ *be a distinguished triangle in* $\mathbf{K}_{\mathrm{tac}}(R)$*. For any object* $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$*, there exists two long exact sequences of abelian groups:*

$$\cdots \to \mathrm{Hom}_{\mathscr{K}}(\mathrm{C}, \Sigma^i\mathrm{X}) \to \mathrm{Hom}_{\mathscr{K}}(\mathrm{C}, \Sigma^i\mathrm{Y}) \to \mathrm{Hom}_{\mathscr{K}}(\mathrm{C}, \Sigma^i\mathrm{Z}) \to \mathrm{Hom}_{\mathscr{K}}(\mathrm{C}, \Sigma^{i+1}\mathrm{X}) \to \mathrm{Hom}_{\mathscr{K}}(\mathrm{C}, \Sigma^{i+1}\mathrm{Y}) \to \cdots$$

and

$$\cdots \to \mathrm{Hom}_{\mathscr{K}}(\mathrm{Z}, \Sigma^i\mathrm{C}) \to \mathrm{Hom}_{\mathscr{K}}(\mathrm{Y}, \Sigma^i\mathrm{C}) \to \mathrm{Hom}_{\mathscr{K}}(\mathrm{X}, \Sigma^i\mathrm{C}) \to \mathrm{Hom}_{\mathscr{K}}(\mathrm{Z}, \Sigma^{i+1}\mathrm{C}) \to \mathrm{Hom}_{\mathscr{K}}(\mathrm{Y}, \Sigma^{i+1}\mathrm{C}) \to \cdots$$

where $\mathrm{Hom}_{\mathscr{K}}(-,-)$ *is shorthand notation for* $\mathrm{Hom}_{\mathscr{K}(\mathrm{R})}(-,-) = \mathrm{Hom}_{\mathbf{K}_{\mathrm{tac}}(R)}(-,-)$*.*

Equivalently, Proposition 3.18 states that the functors $\mathrm{Hom}(\mathrm{C},\text{-})\colon \mathbf{K}_{\mathrm{tac}}(R) \to \mathbb{Z}\text{-mod}$ and $\mathrm{Hom}(\text{-},\mathrm{C})\colon \mathbf{K}_{\mathrm{tac}}(R)^{op} \to \mathbb{Z}\text{-mod}$ are both cohomological, as with $\mathscr{R}$-$\mathscr{mod}$. Furthermore, for any abelian category $\mathscr{Ab}$, one has $\mathrm{Ext}^i_{\mathscr{Ab}}(A,B) = \mathrm{Hom}_{\mathscr{D}(\mathscr{Ab})}(A, \Sigma^i\mathrm{B})$.

Since the *objects* of $\mathbf{K}_{\text{tac}}(R)$ coincide with a subset of those from the derived category of $\mathcal{R}$-$\mathcal{mod}$, we can actually rewrite the sequences given in Proposition 3.18 as

$$\cdots \to \operatorname{Ext}^i_R(\mathrm{C},\mathrm{X}) \to \operatorname{Ext}^i_R(\mathrm{C},\mathrm{Y}) \to \operatorname{Ext}^i_R(\mathrm{C},\mathrm{Z}) \to \operatorname{Ext}^{i+1}_R(\mathrm{C},\mathrm{X}) \to \operatorname{Ext}^{i+1}_R(\mathrm{C},\mathrm{Y}) \to \cdots$$

and

$$\cdots \to \operatorname{Ext}^i_R(\mathrm{Z},\mathrm{C}) \to \operatorname{Ext}^i_R(\mathrm{Y},\mathrm{C}) \to \operatorname{Ext}^i_R(\mathrm{X},\mathrm{C}) \to \operatorname{Ext}^{i+1}_R(\mathrm{Z},\mathrm{C}) \to \operatorname{Ext}^{i+1}_R(\mathrm{Y},\mathrm{C}) \to \cdots$$

so that any distinguished triangle induces long exact sequences in Ext, just as short exact sequences do in $\mathcal{R}$-$\mathcal{mod}$. We will utilize this fact in a significant manner in Chapter 4 of this thesis; for now, though, we move onto a discussion of the CI and Cohomological Operators on an object in $\mathbf{K}_{\text{tac}}(R)$.

3.4 Endomorphisms of R-Complexes

We now turn to the study of endomorphisms on R-complexes in hopes that they shed some light upon the structural patterns within such objects. We find that Eisenbud's approach to identifying even-degree endomorphisms given in [Ei] generalizes nicely to R-complexes of free modules, and thus objects in $\mathbf{K}_{\text{tac}}(R)$. To start, let μ be a negative-degree chain endomorphism on an R-complex $\mathrm{C} \in \mathcal{C}(R)$ so that $\mu_{n+q}\colon C_{n+q} \to C_n$ for some integer $q > 0$ and $\mu_{n-1}\partial^{\mathrm{C}}_n = \partial^{\mathrm{C}}_{n-q}\mu_n$ for each $n \in \mathbb{Z}$. In a similar vein, we may consider an "endomorphism" on a totally acyclic complex from $\mathbf{K}_{\text{tac}}(R)$ in the following manner. Let $\mu\colon \mathrm{C} \to \Sigma^q\mathrm{C}$ be a morphism in $\mathbf{K}_{\text{tac}}(R)$ so that for each $n \in \mathbb{Z}$, there exist R-module homomorphisms $\mu_{n+q}\colon C_{n+q} \to C_n$ for some integer $q > 0$ and $\mu_{n-1}\partial^{\mathrm{C}}_n \sim \partial^{\Sigma^q\mathrm{C}}_n\mu_n$. Note that although C and Σ^qC are distinct objects in $\mathbf{K}_{\text{tac}}(R)$, we will still informally refer to μ as a $-q$ degree endomorphism on C, as we can always view a representative of μ's equivalence class on the R-complex C in $\mathcal{C}(R)$.

3.4.1 Generalization of CI/Cohomology Operators to R-complexes

Now that we have established what is meant by an "endomorphism" on a complex in $\mathbf{K}_{\text{tac}}(R)$, we can specify the generalization of the CI, or Eisenbud, operators to an R-complex. Since a description of these operators has already been given for free resolutions and the process generalizes quite naturally, we exploit an example to give demonstration of finding the operators for an R-complex.

Example 3.19. Let $R = \dfrac{\Bbbk[\![x,y]\!]}{(x^2,y^2)}$ and $Q = \dfrac{\Bbbk[\![x,y]\!]}{(y^2)}$. Consider the R-complex $\mathrm{C} \in \mathbf{K}_{\text{tac}}(R)$ given in the diagram below and note that since there exists a natural ring surjection $Q \twoheadrightarrow R$, we can view all C_i as Q-modules. Additionally, since the differentials are represented by matrices, we can view the entries of each differential matrix as if they are elements in Q, not R. Meaning, we can naively "lift" the R-complex C to a sequence of free Q-modules and Q-module homomorphisms, denoting this sequence $\widetilde{C}$ as written in the diagram below.

$$\widetilde{C}: \ \cdots \to Q^2 \xrightarrow{\left[\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right]} Q^2 \xrightarrow{\left[\begin{smallmatrix} x & -y \\ 0 & x \end{smallmatrix}\right]} Q^2 \xrightarrow{\left[\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right]} Q^2 \xrightarrow{\left[\begin{smallmatrix} x & -y \\ 0 & x \end{smallmatrix}\right]} Q^2 \to \cdots$$

$$\Uparrow \quad \Uparrow \quad \Uparrow$$

$$\mathrm{C}: \ \cdots \to R^2 \xrightarrow{\left[\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right]} R^2 \xrightarrow{\left[\begin{smallmatrix} x & -y \\ 0 & x \end{smallmatrix}\right]} R^2 \xrightarrow{\left[\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right]} R^2 \xrightarrow{\left[\begin{smallmatrix} x & -y \\ 0 & x \end{smallmatrix}\right]} R^2 \to \cdots$$

Now, C itself is a periodic complex, so it is easy to observe that $\tilde{\partial}_n \circ \tilde{\partial}_{n+1} \neq 0$ for any n since

$$\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)\left(\begin{smallmatrix} x & -y \\ 0 & x \end{smallmatrix}\right) = \left(\begin{smallmatrix} x^2 & 0 \\ 0 & x^2 \end{smallmatrix}\right) \neq 0 \in Q^2 \qquad \text{and} \qquad \left(\begin{smallmatrix} x & -y \\ 0 & x \end{smallmatrix}\right)\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) = \left(\begin{smallmatrix} x^2 & 0 \\ 0 & x^2 \end{smallmatrix}\right) \neq 0 \in Q^2$$

Hence, $\widetilde{C}$ is *not* a Q-complex. We can actually see by "how much" $\widetilde{C}$ is not so by observing that for each composition $\tilde{\partial}_n \circ \tilde{\partial}_{n+1} = x^2\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. And since $x^2 \neq 0 \in Q$, the failure of the composition to be 0 relies solely on the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which turns out to be I_2 in this example. Moreover, once we factor x^2 out, this remaining piece preventing $\tilde{\partial}^2$ from being 0 is actually a map from $\widetilde{C}_n \to \widetilde{C}_{n-2}$. This produces a family of maps $\{\tilde{t}_n\}_{n\in\mathbb{Z}}$ and tensoring "back down" to R (by applying $R \otimes_Q -$), we obtain a -2 degree endomorphism $\mathrm{t} = \{t_n\}_{n\in\mathbb{Z}}$: $\mathrm{C} \to \Sigma^{-2}\mathrm{C}$. For this example,

$$t_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for each $n \in \mathbb{Z}$, which turns out to be fairly unsurprising since C is periodic.

Notice that in the above example, $\operatorname{codim}(R, Q) = 1$ (where the *codimension* is defined to be $\dim Q - \dim R$, which follows from the definition given in [BrHe, Pg. 413]) and so we obtain a single -2 degree endomorphism on C. Suppose now that $\operatorname{codim}(R, Q) = c$, so that R is a complete intersection ring of the form $Q/\mathbf{f}$ where $\mathbf{f} = f_1, \ldots, f_c$ is a Q-regular sequence. "Lifting" an R-complex C to a sequence $\widetilde{C}$, it turns out that $\tilde{\partial}^2 \in (f_1, \ldots, f_n)$ since $\partial^2 = 0$ in R. Therefore, the lifted composition actually takes on the form

$$\tilde{\partial}^2 = \sum_{j=1}^{c} f_j \tilde{t}_j$$

where $\tilde{t}_j{:}\widetilde{C} \to \Sigma^{-2}\widetilde{C}$ for each $j = 1, \ldots, c$. Similarly to the example above, we may define $t_j = R \otimes_Q \tilde{t}_j$ to obtain a -2 degree endomorphism $\mathrm{C} \to \Sigma^2\mathrm{C}$ *for each* j. That is, $\mathfrak{t} = \{\mathrm{t_j}\}$ is a family of c degree -2 chain endomorphisms on C where each $\mathrm{t_j} = \{t_n\}_{n\in\mathbb{Z}}$. As indicated in the example above, $\mathfrak{t}$ is a measure of how much $\widetilde{C}$ is *not* a Q-complex. Formally, we denote each $\mathrm{t_j} = (Q, \{f_i\}, \mathrm{C})$ as it was denoted in Chapter 1 for free resolutions of R-modules. Of course, we have the same statements discussed in Chapter 1 for the CI operators on R-complexes.

Most importantly, we have the naturality of each $\mathrm{t_j}$; that is, if $\mathrm{g}\colon \mathrm{C} \to \mathrm{D}$, $\mathrm{t_j}$ is defined for C as it is above and $\mathrm{s_j} = (Q, \{f_i\}, \mathrm{D})$, then $\mathrm{gt_j} \sim \mathrm{s_j g}$. Equivalently, the square

$$\begin{array}{ccc} \mathrm{C} & \xrightarrow{f} & \mathrm{D} \\ \downarrow{\scriptstyle \mathrm{t_j}} & & \downarrow{\scriptstyle \mathrm{s_j}} \\ \Sigma\mathrm{C} & \xrightarrow{\Sigma f} & \Sigma\mathrm{D} \end{array}$$

commutes up to homotopy. Furthermore, the operators commute up to homotopy themselves: $\mathrm{t_i t_j} \sim \mathrm{t_j t_i}$ for each $1 \leq i \neq j \leq c$. The culmination of these characteristics is that the Eisenbud operators are well-defined morphisms $\mathrm{C} \to \Sigma^2\mathrm{C}$ for any $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$.

Recall from Chapter 1 that a linear form of the CI operators eventually becomes surjective on any minimal free resolution of M. In fact, the same occurs for any complex $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$ since we can associate it to a complete resolution, a construction that will be introduced shortly in Section 3.5. However, a similar phenomenon occurs on the right-hand, or "negative-degree", side of C as well. If we set $M = \operatorname{Im} \partial_0^{\mathrm{C}}$, then note that a linear form ℓ of CI operators $(Q, \{f_i\}, \mathbf{F}^*)$ will eventually become surjective where $\mathbf{F}^*$ is the (minimal) free resolution of M^*. Dualizing, we see that $\operatorname{Hom}_R(\mathbf{F}^*, R)$ is equivalent to $\mathrm{C}^{\leq 0}$ and, more importantly, that $\ell^* = \operatorname{Hom}_R(\ell, R)$ will eventually become *injective* for $n \ll 0$. One interesting question is whether or not there exists a linear form which realizes eventual surjection and injection on an R-complex C. This question will be addressed in the conclusion of Chapter 4 in this thesis.

3.5 Complete Resolutions

So far we have discovered structural similarities between the categories $\mathcal{R}$-$\mathcal{mod}$ and $\mathbf{K}_{\mathrm{tac}}(R)$; they are both additive, and while $\mathbf{K}_{\mathrm{tac}}(R)$ is not abelian, its triangulated structure in some sense mimics many of the properties and constructions we gain from an abelian category. We have also just discussed how we can realize the CI and Cohomology Operators on an (acyclic) R-complex, just as we introduced the concept for (minimal) free resolutions of R-modules in Chapter 1. We conclude this chapter with one remaining construction, which will help fuse the theories of the two categories, motivating the work presented in Chapter 4 of this thesis. This construction is called a *complete resolution*.

First, for the following definition we specify one of two conditions: either (1) R is a Gorenstein ring, or (2) M is an R-module with *finite* G-dimension.[3] Note that all examples in this thesis and much of the theory (e.g. CI operators) centers upon when R is a complete intersection ring; and, of course, in this case the former condition listed is satisfied. If however, the only assumption given is that R is a local, noetherian ring then the following definition exists only if the latter condition is assumed. Although the main definitions in Chapter 4 will make sense as long as R is a local noetherian ring, it behooves us conceptually to assume additionally one of the aforementioned conditions on either R or M so that the connection between $\mathscr{R}\text{-}\mathscr{mod}$ and $\mathbf{K}_{\text{tac}}(R)$ is not severed.

3.5.1 Connection between Free R-resolutions and Totally Acyclic R-complexes

Definition 3.20 (cf. [AvMa]). Let M be a finitely-generated R-module such that either (1) R is a Gorenstein ring, or (2) $\text{G-dim}_R M < \infty$. A *complete resolution* of M is a diagram

$$\mathcal{U} \xrightarrow{\rho} \mathbb{P} \xrightarrow{\pi} M$$

where $\mathcal{U} \in \mathbf{K}_{\text{tac}}(R)$, $\mathbb{P}$ is a projective resolution of M, and ρ is a morphism of complexes such that ρ_n is bijective for all $n \gg 0$.

[3]Recall *Gorenstein dimension* from Theorem 2.8 of this thesis.

One of the first papers discussing this construction was [AvMa], in which the authors employed the notion to give an adequate treatment of functoriality in the contravariant argument and naturality of comparison maps with respect to Tate cohomology. Shortly, we will introduce Tate cohomology, as it plays a significant role in making our theory precise, but for now we continue our discussion of complete resolutions.

Whenever it does not affect accuracy of the theory, we may abuse notation and refer to $\mathcal{U}$ as the complete resolution, rather than the formal diagram. We now make the connection between $\mathscr{R}$-$\mathscr{mod}$ and $\mathbf{K}_{\text{tac}}(R)$ explicit.

Proposition 3.21 (cf. [AvMa])**.** *There is a one-to-one correspondence between objects in $\mathscr{R}$-$\mathscr{mod}$ and objects in $\mathbf{K}_{\text{tac}}(R)$.*

Proof. For any $\mathrm{C} \in \mathbf{K}_{\text{tac}}(R)$, denote $\mathrm{C}^{\geq n}$ as the truncated complex of C, meaning that

$$(\mathrm{C}^{\geq n})_i = \begin{cases} C_i & i \geq n \\ 0 & i < n \end{cases}$$

and consider the diagram

$$\mathrm{C} \xrightarrow{\rho} \mathrm{C}^{\geq n} \xrightarrow{\pi} \mathrm{Im}\partial_n^C$$

where it should be clear that ρ is a morphism of complexes with ρ_n bijective for all $n \gg 0$. Moreover, it holds that $\mathrm{C}^{\geq n}$ is a projective resolution of $\mathrm{Im}\partial_n^C$, since it is an acyclic complex of free modules. Therefore, any object in $\mathbf{K}_{\text{tac}}(R)$ can be realized as a complete resolution of an R-module.

Next, there exists a (unique) minimal free resolution $\mathbf{F}$ for every finitely-generated R-module and we may take this resolution as a projective resolution of M for the first part of the diagram. Furthermore, there is a manner in which we may construct a *minimal* totally acyclic complex $\mathcal{U}$ such that the bijectivity condition between $\mathcal{U}$ and $\mathbf{F}$ holds. This manner is described below; and so, we have a one-to-one correspondence between objects in $\mathbf{K}_{\text{tac}}(R)$ and objects in $\mathscr{R}\text{-}\mathscr{mod}$. □

The use of an R-module's minimal free resolution in the construction of a complete resolution provides reason as to why we might call $\mathcal{U}$ a complete resolution of M. Additionally, over a local ring R, we can informally think of complete resolutions as "doubly infinite" free resolutions, since a projective module over a local ring is always free. We now provide the construction of a complete resolution for any finitely-generated R-module; the reader may also refer to [AvMa] for a reasonable coverage as well.

3.5.2 Construction of a Complete Resolution

For now, let M be a finitely-generated, maximal Cohen-Macaulay R-module (so that $\operatorname{depth}_R M = \dim R$). Then we can find the complete resolution of M as follows:

1. Compute the minimal free resolution of M:
   $\cdots \to R^{\beta_n} \xrightarrow{\partial_n} R^{\beta_{n-1}} \xrightarrow{\partial_{n-1}} \cdots \to R^{\beta_1} \xrightarrow{\partial_1} R^{\beta_0} \xrightarrow{\epsilon} M \to 0$
2. Take the dual of M, denoted $M^* = \operatorname{Hom}_R(M, R)$, and compute the minimal free resolution of M^*:
   $\cdots \to R^{b_n} \xrightarrow{\delta_n} R^{b_{n-1}} \xrightarrow{\delta_{n-1}} \cdots \to R^{b_1} \xrightarrow{\delta_1} R^{b_0} \xrightarrow{\epsilon} M^* \to 0$
3. (Dualizing) Apply the $\operatorname{Hom}_R(-, R)$ to the resolution from Step (2) in order to obtain a *co*chain complex bounded below and note that $M^{**} = \operatorname{Hom}_R(M^*, R) \cong M$ (since M is maximal Cohen-Macaulay, and thus reflexive; meaning, the biduality map is an isomorphism):
   $0 \to M \xrightarrow{\epsilon^T} R^{b_0} \xrightarrow{\delta_1^T} R^{b_1} \to \cdots \xrightarrow{\delta_{n-1}^T} R^{b_{n-1}} \xrightarrow{\delta_n^T} R^{b_n} \to \cdots$
4. Concatenate the two resolutions from Steps (1) and (3):
   $\cdots \to R^{\beta_n} \xrightarrow{\partial_n} \cdots \to R^{\beta_1} \xrightarrow{\partial_1} R^{\beta_0} \xrightarrow{\epsilon^T \circ \epsilon} R^{b_0} \xrightarrow{\delta_1^T} R^{b_1} \to \cdots \xrightarrow{\delta_n^T} R^{b_n} \to \cdots$

5. Setting $\partial_0 = \epsilon^T \circ \epsilon$ and re-indexing $\delta_i^T = \partial_{-i}$, we can rewrite the complex from Step (4) as:

$$\mathcal{U}: \quad \cdots \to R^{\beta_n} \xrightarrow{\partial_n} \cdots \to R^{\beta_1} \xrightarrow{\partial_1} R^{\beta_0} \xrightarrow{\partial_0} R^{\beta_{-1}} \xrightarrow{\partial_{-1}} R^{\beta_{-2}} \to \cdots \xrightarrow{\partial_{-n}} R^{\beta_{-(n+1)}} \to \cdots$$

Note that each δ_i^T represents the transpose of the matrix representing δ_i. Now, it should be clear that, via the process outlined above, we obtain a diagram

$$\mathcal{U} \xrightarrow{\rho} \mathbf{F} \twoheadrightarrow M$$

such that ρ_n is bijective for all $n > 0$ and so the diagram is, in fact, a complete resolution. Now, suppose M is *not* maximal Cohen-Macaulay, so that the "concatenation" step is not possible at homological degree 0. In this case, we must "cut out" the portion of $\mathbf{F}$ in which $\Omega^i M$ are not maximal Cohen-Macaulay modules. Note that for some $n < \infty$, the n^{th} syzygy module will have depth equal to the Krull dimension of the ring (by the depth lemma). Therefore, after (1) we will include an intermediary step:

(1.5) Let $n \in \mathbb{Z}$ be the homological degree such that $\operatorname{depth}_R(\operatorname{Im}(\partial_n)) = \dim R$ and set $M' = \Omega^n M$. Then note that $\mathbf{F}^{\geq n}$ will be the minimal free resolution of M' and that $(M')^*$ will also be maximal Cohen-Macaulay. Now complete (2)-(5) with M' in place of M.

We now present an example of constructing a complete resolution of a finitely-generated R-module:

Example 3.22. Let $R = \dfrac{\Bbbk[\![x,y]\!]}{(x^2,y^2)}$ and $M = \Bbbk \cong \dfrac{R}{(x,y)}$.

1. Compute the (minimal) free resolution of M:

$$\mathbf{F}^{\Bbbk}: \quad \cdots R^5 \xrightarrow{\begin{bmatrix} y & x & 0 & 0 & 0 \\ 0 & y & -x & 0 & 0 \\ 0 & 0 & y & -x & 0 \\ 0 & 0 & 0 & y & x \end{bmatrix}} R^4 \xrightarrow{\begin{bmatrix} y & -x & 0 & 0 \\ 0 & y & x & 0 \\ 0 & 0 & y & x \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} 0 & -y & x \\ y & x & 0 \end{bmatrix}} R^2 \xrightarrow{[x \ y]} R \to \Bbbk \to 0$$

2. Noting that the dual of M is $M^* = \mathrm{Hom}_R(\Bbbk, R) = R(xy) \cong \Bbbk$, compute the free resolution of M^*:

$$\mathbf{F}^{\Bbbk}: \quad \cdots R^5 \xrightarrow{\begin{bmatrix} y & x & 0 & 0 & 0 \\ 0 & y & -x & 0 & 0 \\ 0 & 0 & y & -x & 0 \\ 0 & 0 & 0 & y & x \end{bmatrix}} R^4 \xrightarrow{\begin{bmatrix} y & -x & 0 & 0 \\ 0 & y & x & 0 \\ 0 & 0 & y & x \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} 0 & -y & x \\ y & x & 0 \end{bmatrix}} R^2 \xrightarrow{[x\ y]} R \to \Bbbk \to 0$$

3. And dualizing:

$$\mathrm{Hom}_R(\mathbf{F}^{\Bbbk}, R): \quad 0 \to \Bbbk \to R \xrightarrow{\begin{bmatrix} x \\ y \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} 0 & y \\ -y & x \\ x & 0 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} y & 0 & 0 \\ -x & y & 0 \\ 0 & x & y \\ 0 & 0 & x \end{bmatrix}} R^4 \xrightarrow{\begin{bmatrix} y & 0 & 0 & 0 \\ x & y & 0 & 0 \\ 0 & -x & y & 0 \\ 0 & 0 & -x & y \\ 0 & 0 & 0 & x \end{bmatrix}} R^5 \to \cdots$$

4. Finally, we concatenate:

$$\cdots \to R^5 \xrightarrow{\begin{bmatrix} y & x & 0 & 0 & 0 \\ 0 & y & -x & 0 & 0 \\ 0 & 0 & y & -x & 0 \\ 0 & 0 & 0 & y & x \end{bmatrix}} R^4 \xrightarrow{\begin{bmatrix} y & -x & 0 & 0 \\ 0 & y & x & 0 \\ 0 & 0 & y & x \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} 0 & -y & x \\ y & x & 0 \end{bmatrix}} R^2 \xrightarrow{[x\ y]} R \xrightarrow{[xy]} R \xrightarrow{\begin{bmatrix} x \\ y \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} 0 & y \\ -y & x \\ x & 0 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} y & 0 & 0 \\ -x & y & 0 \\ 0 & x & y \\ 0 & 0 & x \end{bmatrix}} R^4 \xrightarrow{\begin{bmatrix} y & 0 & 0 & 0 \\ x & y & 0 & 0 \\ 0 & -x & y & 0 \\ 0 & 0 & -x & y \\ 0 & 0 & 0 & x \end{bmatrix}} R^5 \to \cdots$$

Note that in the above example, since we computed our complete resolution from the module $\Bbbk$, $\mathrm{C}_0 = R$ and $\partial_0 = [xy]$.

3.5.3 A Brief Encounter with Tate Ext

At this time, it becomes necessary to introduce *Tate cohomology* in order to present analogous constructs discussed in Chapter 1 with respect to complete resolutions. Tate cohomology was studied initially by Buchweitz in [Bu2], additionally in [AvMa], with respect to complexes in [Ve], and used in [ChJo] with respect to the authors' definitions of pinched complexes. We simply recall the definition of *"Tate Ext"* with respect to R-complexes, as given in both [Ve] and [ChJo].

Definition 3.23 (cf. [Ve], [ChJo])**.** Let M be an R-complex with complete projective resolution $\mathcal{U} \xrightarrow{\rho} \mathbb{P} \to \mathrm{M}$. For any R-complex N the *Tate cohomology of* M *with coefficients in* N is

$$\widehat{\mathrm{Ext}}_R^i(M, N) := \mathrm{H}_{-i}(\mathrm{Hom}_R(\mathcal{U}, \mathrm{N})) = \mathrm{H}^i(\mathrm{Hom}_R(\mathcal{U}, N))$$

Note that we may apply the above definition whenever M and N are R-modules, in which case we view them as R-complexes concentrated in degree 0. Further note that in this case, we have that

$$\widehat{\mathrm{Ext}}^i_R(M,N) \cong \mathrm{Ext}^i_R(M,N)$$

for $i > g = \text{G-dim}_R M$. With respect to R-modules, we can view the computation of $\widehat{\mathrm{Ext}}$ as analogous to the computation of Ext, with the exception of using a *complete resolution* $\mathcal{U}$ of M in lieu of a projective resolution. We now relate some of the topics discussed in Chapter 1, with respect to totally acyclic complexes and Tate cohomology.

Definition 3.24. Let M be a finitely-generated R-module with complete resolution $C \to \mathbf{F} \to M$ where $\mathbf{F}$ is the minimal free resolution of M. Then define the *complete Betti sequence of* M to be $\{\hat{b}^R_n(M)\}$ where each $\hat{b}^R_n(M) := \mathrm{rk}(C_n)$ or, equivalently,

$$\hat{b}^R_n(M) = \dim_{\Bbbk} \widehat{\mathrm{Ext}}^n_R(M,\Bbbk)$$

for each $n \in \mathbb{Z}$.

One may also refer to the complete syzygy sequence $\{\hat{\Omega}^n M\}_{n\in\mathbb{Z}}$ to mean all syzygy and cosyzygy modules, in which case it becomes clear that our goal in this thesis is to additionally understand the dual picture of a module's free resolution. The next definition clarifies the direction we intend to take moving forward.

Definition 3.25. Let M, $\mathbf{F}$ and C be as in Definition 3.24. Furthermore, let R be a complete intersection ring cut out by the regular Q-sequence $\mathbf{f} = f_1, \ldots, f_c$. Then, define the *complete cohomology operators* to be $\hat{\chi}_j := \mathrm{Hom}_R(\hat{t}_j, \Bbbk)$ where $\hat{t}_j$ are the complete CI operators on C.

As one might expect, $\widehat{\mathrm{Ext}}_R(M,\Bbbk)$ is unambiguously a module over the "complete" ring of cohomology operators $\hat{\mathcal{S}} = R[\hat{\chi}_1, \ldots, \hat{\chi}_c]$. We will often not use notation marks to distinguish between $\mathcal{S}$ and $\hat{\mathcal{S}}$, or the like; rather, we will rely on context with the note that for the remainder of this thesis, our primary focus is on complete resolutions, and thus we consider when patterns arise in the tails of their *complete* Betti sequences.

To conclude this chapter, we simply note that our goal in the next chapter will be to shift perspective from patterns in $\{b_n^R(M)\}$ to patterns in $\{\hat{b}_n^R(M)\}$, with emphasis on filling in the dual half of the picture. We will work towards extending the work done in [AvGaPe], presented in Chapter 2 of this thesis, so that the notion of critical degree and what it communicates augments our understanding of these patterns. Without further ado, we persist onward.

Part II

The Elements

CHAPTER 4

The Critical and Cocritical Degrees of a Totally Acyclic Complex

We now find ourselves in the nucleus of this thesis, having built up the necessary components to understand the motivation behind the definitions presented in Section 4.1 of this chapter. As alluded to in previous chapters, our intention is to transplant the notion of critical degree into $\mathbf{K}_{\text{tac}}(R)$. Our hope is that, with this shifted perspective, we may gain additional insight and, at minimum, relate the entire syzygy sequence via the critical degree. In fact, one major theme indicated in [AvGaPe] is to study the sequence of syzygy modules as a whole, as opposed to just one module at a time (called *asymptotic homological algebra*). Hence, our approach to using critical degree with respect to complete resolutions will certainly aid us in this endeavor. Of special importance is the ability to capture the "unstable phenomena" which may occur at the beginning of a free resolution within a portion of the associated *complete* resolution. This particular motivation will be discussed in further detail in Chapter 6 of this thesis. For now, we introduce the main definitions and work to uncover both their connection to the original definition in $\mathscr{R}$-$\mathscr{mod}$ as well as the graded $\hat{\mathcal{S}}$-module $\widehat{\text{Ext}}_R(M, \Bbbk)$.

4.1 The Main Definitions

Let C be any R-complex in $\mathbf{K}_{\text{tac}}(R)$, and further denote $\overline{\text{C}}$ as the minimal subcomplex of C so that we may write $\text{C} = \overline{\text{C}} \oplus \text{Z}$ where Z is some contractible R-complex. If $\mu\colon \text{C} \to \Sigma^q\text{C}$ is a morphism in $\mathbf{K}_{\text{tac}}(R)$, we will (loosely) refer to μ as a $-q < 0$ degree chain endomorphism on C. Then denote $\bar{\mu}\colon \overline{\text{C}} \to \Sigma^q\overline{\text{C}}$ as the induced endomorphism on $\overline{\mathrm{C}}$. We will make use of this notation for the remainder of the thesis. Now, we present the definition of critical degree, from the perspective of totally acyclic complexes.

Definition 4.1. An R-complex $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$ has *critical degree relative to* μ (or μ*-critical degree*) equal to s, denoted $\mathrm{crdeg}_{\mathrm{R}}^{\mu}\mathrm{C} = s$, if μ is a degree $-q < 0$ chain endomorphism of C and s is the least integer such that $\bar{\mu}_{n+q} : \overline{\mathrm{C}}_{n+q} \to \overline{\mathrm{C}}_n$ is surjective for all $n > s$; that is,

$$\mathrm{crdeg}_{\mathrm{R}}^{\mu}\mathrm{C} = \inf\{n \mid \bar{\mu}_{i+q} : \overline{\mathrm{C}}_{i+q} \twoheadrightarrow \overline{\mathrm{C}}_i \; \forall\, i > n\}$$

where $\bar{\mu}$ is the induced endomorphism on the minimal subcomplex $\overline{\mathrm{C}}$. Note that if no such integer s exists and $s \neq -\infty$, then $\mathrm{crdeg}_{\mathrm{R}}^{\mu}\mathrm{C} = \infty$ by definition.

The **critical degree** of $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$ is defined to be the infimum over all μ-critical degrees:

$$\mathrm{crdeg}_{\mathrm{R}}\mathrm{C} = \inf\{\mathrm{crdeg}_{\mathrm{R}}^{\mu}\mathrm{C} \mid \mu : \mathrm{C} \to \Sigma^{q_\mu}\mathrm{C}\}$$

where, once again, if all such relative critical degrees are infinite, then $\mathrm{crdeg}_{\mathrm{R}}\mathrm{C} = \infty$ by definition.

Remark. *Note that if we take* $M = \mathrm{Im}\,\partial_0^{\mathrm{C}}$ *for some* $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$*, then the above definition of critical degree for* C *will indeed agree with the definition for* M *in* $\mathcal{R}$*-mod as long as* $\mathrm{crdeg}_{\mathrm{R}}\mathrm{M} \geq 0$*. We will later discuss a special case in which* $\mathrm{crdeg}_{\mathrm{R}}\mathrm{M} \geq -1$ *but* $-\infty \leq \mathrm{crdeg}_{\mathrm{R}}\mathrm{C} \lneq -1$.

It is also significant to note the distinction of using the intermediary definition of *relative* critical degree. The reasons for doing so mainly involve ease of discussion– for example, we can more definitively refer to an endomorphism which *realizes* the critical degree whenever $\mathrm{crdeg}_{\mathrm{R}}\mathrm{C} < \infty$. In particular, the methodology presented with regard to the topics covered in Chapter 5 of this thesis makes explicit use of this intermediary definition.

Now recall that the construction of a complete resolution for an R-module involves its dual; hence, this is indication that it is only natural to develop a dual notion to critical degree.

Definition 4.2. An R-complex $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$ has *cocritical degree relative to* μ (or μ*-cocritical degree*) equal to t, denoted $\mathrm{cocrdeg}_{\mathrm{R}}^{\mu}\mathrm{C} = t$, if μ is a degree $-q < 0$ chain endomorphism of C and t is the greatest integer such that $\bar{\mu}_n : \overline{\mathrm{C}}_n \to \overline{\mathrm{C}}_{n-q}$ is split injective for all $n < t$; that is,

$$\mathrm{cocrdeg}_{\mathrm{Q}}^{\mu}\mathrm{C} = \sup\{n \mid \bar{\mu}_i : \overline{\mathrm{C}}_i \hookrightarrow \overline{\mathrm{C}}_{i-q} \text{ splits } \ \forall\, i < n\}$$

where $\hat{\mu}$ is the induced endomorphism on the minimal subcomplex $\overline{\mathrm{C}}$. Note that if no such integer t exists and $t \neq \infty$, then $\mathrm{crdeg}_{\mathrm{R}}^{\mu}\mathrm{C} = -\infty$ by definition.

The **cocritical degree** of $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$ is defined to be the supremum over all μ-cocritical degrees

$$\mathrm{cocrdeg}_{\mathrm{R}}\mathrm{C} = \sup\{\mathrm{cocrdeg}_{\mathrm{R}}^{\mu}\mathrm{C} \mid \mu : \mathrm{C} \to \Sigma^{q_\mu}\mathrm{C}\}$$

where, once again, if all such relative cocritical degrees are negatively infinite, then $\mathrm{crdeg}_{\mathrm{R}}\mathrm{C} = -\infty$ by definition.

Remark. *Note that if we take* $M = \operatorname{Im} \partial_0^{\mathrm{C}}$ *for some* $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$*, then the above definition of cocritical degree for* C *actually agrees negatively (up to a one degree shift to the right) with the definition for* $M^* = \mathrm{Hom}_R(M, R)$ *in* $\mathcal{R}$*-mod as long as* $\mathrm{crdeg}_{\mathrm{R}}\mathrm{M}^* \geq 0$*. That is, if* $\mathrm{crdeg}_{\mathrm{R}}\mathrm{M}^* = s$ *for some* $0 \leq s < \infty$*, then* $\mathrm{cocrdeg}_{\mathrm{R}}\mathrm{C} = -s - 1$*. The negative one-degree shift is due to the relabeling of the degrees of* $\mathrm{Hom}_R(\mathbf{F}^*, R)$ *within the concatenation step of constructing a complete resolution.*

Therefore, we have reasonable consistency with the definition of critical degree of M and M^* in $\mathcal{R}$-*mod* with the definitions of critical and cocritical degrees of a complex C in $\mathbf{K}_{\mathrm{tac}}(R)$. Additionally, it should be clear that, by definition, $\mathrm{crdeg}_{\mathrm{R}}^{\mu}\mathrm{C} \geq \mathrm{crdeg}_{\mathrm{R}}\mathrm{C}$

and $\mathrm{cocrdeg}^{\mu}_{\mathrm{R}}\mathrm{C} \leq \mathrm{cocrdeg}_{\mathrm{R}}\mathrm{C}$, for any chain endomorphism $\mu : \mathrm{C} \to \Sigma^q\mathrm{C}$. This fact will be used in Chapter 5 of this thesis.

We must now check that these definitions make sense in $\mathbf{K}_{\mathrm{tac}}(R)$, since the equivalency in this category is up to homotopy. The next lemma and proposition assure that this is the case.

Lemma 4.3. *Let* C *and* D *be isomorphic as R-complexes in $\mathcal{C}(R)$. Then there is a one-to-one correspondence between chain endomorphisms on* C *and those on* D. *Moreover, an endomorphism on* C *is degree-wise surjective (split injective) if the corresponding endomorphism on* D *is surjective (split injective) at the same degrees, and vice versa.*

Proof. Take $f : \mathrm{C} \to \mathrm{D}$ to be an R-complex chain map such that $f_n : C_n \to D_n$ is isomorphic for each $n \in \mathbb{Z}$ and consider the following diagram

$$\begin{array}{ccc} \mathrm{C} & \xrightarrow{f} & \mathrm{D} \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle \exists!\, \nu} \\ \Sigma^q\mathrm{C} & \xrightarrow{\Sigma^q\mathrm{f}} & \Sigma^q\mathrm{D} \end{array}$$

where $\mu : \mathrm{C} \to \Sigma^q\mathrm{C}$ is a chain endomorphism. Now define $\nu = (\Sigma^q\mathrm{f})\mu f^{-1}$ as a $-q$ degree chain endomorphism $\mathrm{D} \to \Sigma^q\mathrm{D}$ making the square commute; that is, $\nu f = ((\Sigma^q\mathrm{f})\mu f^{-1})f = (\Sigma^q\mathrm{f})\mu$. Uniqueness of ν follows from the fact that f is an isomorphism, since any other $\nu' : \mathrm{D} \to \Sigma^q\mathrm{D}$ such that $\nu' f = (\Sigma^q\mathrm{f})\mu$ can be rewritten as $\nu' = (\Sigma^q\mathrm{f})\mu f^{-1} = \nu$.

To show the latter part of the lemma, first note that since $\Sigma^q\mathrm{f}$ is an isomorphism at each degree, we have that for any $d \in D_{n-q}$ there exists some $c \in C_{n-q}$ such that $(\Sigma^q\mathrm{f})_n(c) = d$. Now suppose $\mu_n : C_n \to C_{n-q}$ is surjective for some $n \in \mathbb{Z}$. Then for any $d \in (\Sigma^q\mathrm{D})_n = D_{n-q}$, there exists some $c' \in C_n$ such that $(\Sigma^q\mathrm{f})_n\mu_n(c') = d$. Lastly, setting $d' = f(c') \in D_n$ we see that for any $d \in (\Sigma^q\mathrm{D})_n$ there exists a $d' \in D_n$ where $\nu_n(d') = (\Sigma^q\mathrm{f})_n\mu_n f_n^{-1}(d') = (\Sigma^q\mathrm{f})_n\mu_n(c') = f_{n-q}(c) = d$ as desired. Hence,

$\nu_n \colon D_n \to D_{n-q}$ must also be surjective. Conversely, we may apply the exact same argument as above, reversing the roles of C and μ with D and ν, to see that if ν_n is surjective then μ_n must be as well.

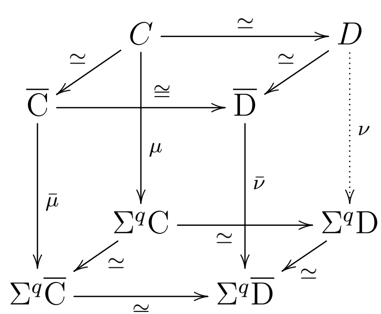
Similarly, suppose now that μ_n is split injective so that there exists a left inverse $\mu_n^{-1} \colon C_{n-q} \to C_n$ so that $\mu_n^{-1}\mu_n = \mathrm{Id}_n^C$. Then define $\nu_n^{-1} = f_n\mu_n^{-1}f_{n-q}^{-1} \colon D_{n-q} \to D_n$ and note that

$$\nu_n^{-1}\nu_n = (f_n\mu_n^{-1}f_{n-q}^{-1})(f_{n-q}\mu_n f_n^{-1}) = f_n\mu_n^{-1}\mu_n f_n^{-1} = f_n f_n^{-1} = \mathrm{Id}_n^D$$

showing ν_n has a left inverse. Hence, ν_n must also be split injective and we can apply the same argument switching the roles of ν and μ to see the other direction holds as well. $\square$

Proposition 4.4. *Let* C *and* D *be homotopically equivalent R-complexes. Then the (co)critical degree of* C *equals the (co)critical degree of* D*; that is, if* $\mathrm{C} \simeq \mathrm{D}$ *then* $\mathrm{crdeg_R C} = \mathrm{crdeg_R D}$ *and* $\mathrm{cocrdeg_R C} = \mathrm{cocrdeg_R D}$.

Proof. Suppose $\mathrm{C} \simeq \mathrm{D}$, implying that $\overline{\mathrm{C}} \cong \overline{\mathrm{D}}$. Now take $f : \overline{\mathrm{C}} \to \overline{\mathrm{D}}$ to be an R-complex chain map such that $f_n : \overline{C}_n \to \overline{D}_n$ is isomorphic for each $n \in \mathbb{Z}$ and consider the following diagram



where $\bar{\nu} \colon \overline{\mathrm{D}} \to \Sigma^q\overline{\mathrm{D}}$ is the endomorphism induced by $\bar{\mu}$ from Lemma 4.3. Note that front face of the diagram commutes, whereas all other faces commute *up to homotopy*. Moreover, we may apply Lemma 2 from Chapter 3 upon the right-most face of the diagram to see that we can extend $\bar{\nu}$ to an endomorphism $\nu\colon \mathrm{D} \to \Sigma^q \mathrm{D}$ where D is not necessarily minimal. Now, by the argument given in Lemma 4.3, it should be clear that $\mathrm{crdeg}_{\mathrm{R}}^{\bar{\mu}} \overline{\mathrm{C}} = \mathrm{crdeg}_{\mathrm{R}}^{\bar{\nu}} \overline{\mathrm{D}}$ and thus $\mathrm{crdeg}_{\mathrm{R}}^{\mu} \mathrm{C} = \mathrm{crdeg}_{\mathrm{R}}^{\nu} \mathrm{D}$ also. Therefore, supposing that $\mathrm{crdeg}_{\mathrm{R}} \mathrm{C} = s = \mathrm{crdeg}_{\mathrm{R}}^{\mu} \mathrm{C}$ where $s < \infty$ and $\mu\colon \mathrm{C} \to \Sigma^{q_\mu} \mathrm{C}$, further assume for the sake of contradiction that $\mathrm{crdeg}_{\mathrm{R}} \mathrm{D} = t \lneq s$. This would imply that there exists some $\bar{\nu}'\colon \overline{\mathrm{D}} \to \Sigma^{q'_\nu} \overline{\mathrm{D}}$ which is surjective for all $n > t$. But then by the previous Lemma, we must obtain an induced endomorphism $\bar{\mu}'\colon \overline{\mathrm{C}} \to \Sigma^{q'_\mu} \overline{\mathrm{C}}$ surjective for all $n > t \lneq s$, contradicting the assumption that s is the critical degree of C. Thus, $t \geq s$ and so now suppose $t \neq s$. Then note that $\bar{\mu}$ induces an endomorphism $\bar{\nu}''\colon \overline{\mathrm{D}} \to \Sigma^{q''_\nu} \overline{\mathrm{D}}$ which is surjective for all $n > s \lneq t$, contradicting the assumption that t is the critical degree of D. Hence, $t = s$ whenever both $\mathrm{crdeg}_{\mathrm{R}} \mathrm{C}$ and $\mathrm{crdeg}_{\mathrm{R}} \mathrm{D}$ are assumed to be (positively) finite.

If we assume at least one is not, say $\mathrm{crdeg}_{\mathrm{R}} \mathrm{C} = \infty$ then note that there does not exist any endomorphism on C satisfying the definition. This would then imply that there cannot be *any* endomorphism on D that is eventually surjective (on $\overline{\mathrm{D}}$), since otherwise, we would obtain an induced endomorphism on C which is eventually surjective, thus contradicting the assumption that $\mathrm{crdeg}_{\mathrm{R}} \mathrm{C}$ is (positively) infinite. Therefore, the equality holds even in the case of infinite critical degrees. Lastly, we can apply an analogous argument to that above for the case of cocritical degree in order to see that $\mathrm{cocrdeg}_{\mathrm{R}} \mathrm{C} = \mathrm{cocrdeg}_{\mathrm{R}} \mathrm{D}$ as well. □

Given the above proposition, we see that the critical and cocritical degrees are stable under homotopy, making these concepts well defined in $\mathbf{K}_{\mathrm{tac}}(R)$. One topic we have not specifically addressed yet is the critical and cocritical degrees of 0 in $\mathbf{K}_{\mathrm{tac}}(R)$. The next corollary gives the answer to this question.

Corollary 4.5. *If* $\mathrm{C} \simeq 0$, *then* $\mathrm{crdeg}_{\mathrm{R}} \mathrm{C} = -\infty$ *and* $\mathrm{cocrdeg}_{\mathrm{R}} \mathrm{C} = \infty$.

With the above corollary in mind, it should be clear that for any $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$, we have $-\infty \leq \mathrm{crdeg}^{\mu}_{\mathrm{R}}\mathrm{C} \leq \infty$ and $-\infty \leq \mathrm{cocrdeg}^{\mu}_{\mathrm{R}}\mathrm{C} \leq \infty$. In the next section, we discuss precisely when, other than the zero complex, $\mathrm{crdeg}^{\mu}_{\mathrm{R}}\mathrm{C} = -\infty$ and $\mathrm{cocrdeg}^{\mu}_{\mathrm{R}}\mathrm{C} = \infty$.

4.2 Some Patterns and Properties

Given a finitely-generated R-module, we can associate to it a totally acyclic complex via the construction of its complete resolution. We have already addressed the connection between critical degree in $\mathcal{R}\text{-}\mathit{mod}$ and $\mathbf{K}_{\mathrm{tac}}(R)$ in one direction; recall that given a complex in the latter category, the critical degree agrees with that of the image of its differential at degree zero as long as $\mathrm{crdeg}_{\mathrm{R}}\mathrm{C} \geq 0$. However, we have yet to discuss the connection between these notions if we instead start from the module and build its complete resolution. We will first look to understand this case; afterwards, we will move to answering a natural question about the critical degree and its dual notion cocritical degree.

4.2.1 Critical Degree in $\mathcal{R}\text{-}\mathit{mod}$ versus $\mathbf{K}_{\mathrm{tac}}(R)$

Let $\mathrm{C} \to \mathbf{F} \twoheadrightarrow M$ be a complete resolution and note that if we begin with a *maximal Cohen-Macaulay* (MCM) module M, then it should be clear that $\mathrm{crdeg}_{\mathrm{R}}\mathrm{C} = \mathrm{crdeg}_{\mathrm{R}}\mathrm{M}$ and $\mathrm{cocrdeg}_{\mathrm{R}}\mathrm{C} = -\mathrm{crdeg}_{\mathrm{R}}\mathrm{M}^* - 1$, whenever the critical degrees of M and its dual are non-negative. On the other hand, if M is *not* MCM, then

$$\begin{cases} \mathrm{crdeg}_{\mathrm{R}}\mathrm{C} = s - g & \text{if } s \geq 0 \\ \mathrm{cocrdeg}_{\mathrm{R}}\mathrm{C} = s^* - g^* & \text{if } s^* \geq 0 \end{cases}$$

where $\mathrm{crdeg}_{\mathrm{R}}\mathrm{M} = s$ and $g = \dim R - \mathrm{depth}_R\, M$ (with s^* and g^* denoted analogously for M^*). Again, of special significance is that these statements hold only in the cases that $s \geq 0$ or $s^* \geq 0$. Observe that, in these cases, $\mathrm{crdeg_R C} > 0$ if $g < s$ and $\mathrm{cocrdeg_R C} < 0$ if $g^* > s^*$. If we did not consider the alternative to these cases, one might mistakenly think that the critical degree of an R-complex always occurs on the positive side and the cocritical degree always occurs on the negative side; however, this is not necessarily true. Whenever we consider an R-module M such that $\mathrm{crdeg_R M} = -1$ it could be the case that $\mathrm{crdeg_R C} \lneq -1$, as demonstrated in the following example.

Example 4.6. Let M be the R-module with complete resolution given at the beginning of this subsection and further suppose that $0 \leq \mathrm{crdeg_R M} = s < \infty$. For simplicity, assume also that M is MCM. Now set $N = \Omega^{s+k}M$ for some fixed integer $k > 1$, so that $\mathrm{crdeg_R N} = -1$. This is because $\mu(F_{n+q}) = F_n$ for some $\mu\colon \mathbf{F} \to \mathbf{\Sigma^q}\mathrm{F}$ and for all $n > s$ by assumption, but $\mathbf{G} := \mathbf{F}^{>s+k}$ is the minimal free resolution of N. Thus, $G_n = F_{n+s+k}$ so it should be clear that $\mu(G_{n+q}) = G_n$ for all $n > 0$ since $n + s + k > s$. However, note that $N^* = \mathrm{Hom}_R(\Omega^{s+k}M, R) \cong \Omega^{s+k}\,\mathrm{Hom}_R(M, R)$ and, furthermore, we can complete the chain endomorphism on $\mathbf{F}$ realizing the critical degree of M to a chain map on C (see (1.5) in [BeJoMo]). Of course, there also exists a complete resolution of N of the form $\mathrm{C} \to \mathbf{G} \twoheadrightarrow N$, which is equivalent to $\mathrm{C} \to \mathbf{F^{>s+k}} \twoheadrightarrow \Omega^{s+k}M$ (up to isomorphism in the first two components and up to homotopy in the last). Therefore, we see that $\mathrm{crdeg_R C} \leq -k$ where $-k \lneq -1$ by assumption.

Note that in the example above, we should technically distinguish between the complete resolution of M and N in the following manner: if $\mathrm{C} \to \mathbf{G} \twoheadrightarrow N$ is the complete resolution of N, then $\Sigma^{\mathrm{s+k}}\mathrm{C} \to \mathbf{F} \twoheadrightarrow M$ is the complete resolution of M. Regardless of this technicality, the point remains, allowing us to see that critical degree in $\mathscr{R}$-$\mathscr{mod}$ is not always in accordance with critical degree in $\mathbf{K}_{\mathrm{tac}}(R)$. This is due to the distinction between $\mathbf{F}$ being a bounded below R-complex and C not being so. Nevertheless, both are useful and can be used to complement each other; one topic in particular about what we gain from the notion in $\mathbf{K}_{\mathrm{tac}}(R)$ will be discussed in Chapter 6 of this thesis.

4.2.2 Can Cocritical Degree be Greater than Critical Degree?

In relation to the previous section, we might wonder whether or not the cocritical degree of a given complex can be *greater than* its critical degree and, if this is possible, what types of complexes might exhibit this behavior? The next proposition explores these ideas.

Proposition 4.7. *Let* $\mu\colon \mathrm{C} \to \Sigma^q\mathrm{C}$ *be a given endomorphism of an R-complex in* $\mathbf{K}_{\mathrm{tac}}(R)$*. Then* $\operatorname{crdeg}^{\mu}_{\mathrm{R}}\mathrm{C} \le \operatorname{cocrdeg}^{\mu}_{\mathrm{R}}\mathrm{C} - (2+q)$ *if and only if the minimal subcomplex* $\overline{\mathrm{C}}$ *is periodic. In which case, it is then necessarily true that* $\operatorname{crdeg}_{\mathrm{R}}\mathrm{C} = -\infty$ *and* $\operatorname{cocrdeg}_{\mathrm{R}}\mathrm{C} = \infty$*.*

Proof. For the sake of simplicity, assume $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$ is minimal. First note that if C is periodic, then we may define an obvious endomorphism of degree $-q$, where the period C is equal to q. Denote $\rho_n = \mathrm{id}^C_n\colon C_n \to C_{n-q} \cong C_n$ for all $n \in \mathbb{Z}$ and note that $\{\rho_n\}_{n\in\mathbb{Z}}$ will be a well-defined chain endomorphism on C. Therefore, $\operatorname{crdeg}^{\rho}_{\mathrm{R}}\mathrm{C} = -\infty$ and $\operatorname{cocrdeg}^{\rho}_{\mathrm{R}}\mathrm{C} = \infty$, forcing the critical and cocritical degrees of C to be $-\infty$ and ∞, respectively.

Now our goal is to show that an endomorphism with an isomorphic map in at least one degree will force the complex to be periodic. Let $\mu\colon \mathrm{C} \to \Sigma^q\mathrm{C}$ be a chain endomorphism and suppose that $\operatorname{crdeg}^{\mu}_{\mathrm{R}}\mathrm{C} = s = \operatorname{cocrdeg}^{\mu}_{\mathrm{R}}\mathrm{C} - (2+q)$ so that μ_{n+q} is split injective for all $n < s+2$ and μ_{n+q} is surjective for all $n > s$. In particular, $\mu_{(s+1)+q}\colon C_{(s+1)+q} \to C_{(s+1)}$ is bijective, as depicted in the following diagram.

$$
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & C_{(s+3)+q} & \xrightarrow{\partial_{(s+3)+q}} & C_{(s+2)+q} & \xrightarrow{\partial_{(s+2)+q}} & C_{(s+1)+q} & \xrightarrow{\partial_{(s+1)+q}} & C_{s+q} & \xrightarrow{\partial_{s+q}} & C_{(s-1)+q} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle u_{(s+3)+q}} & & \downarrow{\scriptstyle \mu_{(s+2)+q}} & & \downarrow{\scriptstyle \mu_{(s+1)+q}\,\cong} & & \downarrow{\scriptstyle \mu_{s+q}} & & \downarrow{\scriptstyle u_{s-1+q}} & & \\
\cdots & \longrightarrow & C_{s+3} & \xrightarrow{\partial_{s+3}} & C_{s+2} & \xrightarrow{\partial_{s+2}} & C_{s+1} & \xrightarrow{\partial_{s+1}} & C_{s} & \xrightarrow{\partial_{s}} & C_{s-1} & \longrightarrow & \cdots
\end{array}
$$

Note that since $C_{(s+1)+q} \cong C_{s+1}$ and $\mu_{(s+2)+q}$ is surjective, $\operatorname{Ker}\partial_{(s+1)+q} \cong \operatorname{Ker}\partial_{s+1}$. First, by commutativity of the diagram, we have $\mu_{(s+1)+q}(\operatorname{Im}\partial_{(s+2)+q}) = \partial_{s+2}\mu_{(s+2)+q} \subseteq \operatorname{Im}\partial_{s+2}$. Then we obtain $\operatorname{Im}\partial_{s+2} \subseteq \mu_{(s+1)+q}(\operatorname{Im}\partial_{(s+2)+q})$ since for any $x \in \operatorname{Im}\partial_{s+2}$ there exists a $y \in C_{(s+2)+q}$ such that $\partial_{s+2}\mu_{(s+2)+q}(y) = x = \mu_{(s+1)+q}\partial_{(s+2)+q}$. Therefore, $\operatorname{Im}\partial_{(s+2)+q} \cong \operatorname{Im}\partial_{s+2}$, as stated.

Consider now the (necessarily minimal) complete resolution $\mathrm{C} \to \mathbf{F} \to M$ where $M = \operatorname{Im}\partial_{s+1}$ and note that, by definition, the truncated complex $\mathrm{C}^{>s}$ is degree-wise bijective with $\mathbf{F}$. Furthermore note that $\Omega^1(M) = \operatorname{Im}(\partial_{(s+2)}) \cong \operatorname{Im}(\partial_{(s+2)+q}) = \Omega^{1+q}(M)$; and thus, by uniqueness of $\mathbf{F}$, it must hold that $\Omega^{1+q+n}(M) \cong \Omega^{1+n}(M)$ for any $n \in \mathbb{N}$.[1] Now apply the same argument to the minimal free resolution of $M^* = \operatorname{Hom}_R(M, R)$ to get a periodic resolution and, dualizing back, $\mathrm{C}^{\leq s}$ is still periodic. Then note that concatenation of the two truncated complexes at degree s yields C. Therefore, C is a periodic complex of period q; moreover, it should be clear that $\mathrm{crdeg}^{\mu}_{\mathrm{R}}\mathrm{C} = -\infty = \mathrm{crdeg}_{\mathrm{R}}\mathrm{C}$ and $\mathrm{cocrdeg}^{\mu}_{\mathrm{R}}\mathrm{C} = \infty = \mathrm{cocrdeg}_{\mathrm{R}}\mathrm{C}$. It is easy to see how the same argument applies to any number of sequential degrees with isomorphic maps in a chain endomorphism. $\square$

[1] Roughly speaking, we mean to say that one can "replace" $\Omega^{1+q}M$ with $\Omega^1 M$ in the reconstruction of $\mathbf{F}$, and can do so ad infimum; it is of course the unique construction of minimal resolutions which allows us to say that this reconstruction must be the same as the original $\mathbf{F}$, and thus the truncated complex $\mathrm{C}^{>\mathrm{s}}$.

Corollary 4.8. *If* $\mathrm{crdeg_R C}$ *and* $\mathrm{cocrdeg_R C}$ *are realized by the same degree* $-q$ *endomorphism, then* $\mathrm{crdeg_R C} \leq \mathrm{cocrdeg_R C} - (2+q)$ *if and only if* $\mathrm{crdeg_R C} = -\infty$ *and* $\mathrm{cocrdeg_R C} = \infty$.

Note that the above corollary is just a restatement of the proposition, where we replace the *relative* critical and critical degrees with the requirement that the critical and cocritical degrees are realized by the same endomorphism. Essentially, what the preceding discussion tells us is that, under this requirement, a *non*-periodic complex has a cocritical degree that *can* be larger than the critical degree, but only by a limited amount. In particular, if $-q < 0$ is the degree of the endomorphism realizing both the critical and cocritical degrees of a given complex, then its cocritical degree can only be *at most* $1+q$ larger than its critical degree. If the critical and cocritical degrees of a given complex are *not* realized by the same endomorphism, it is currently unknown whether there is any such restriction for a non-periodic complex.

4.3 A Shift in Perspective: Triangulated Definitions

We now introduce an alternative manner of viewing the critical and cocritical degrees for a complex in $\mathbf{K}_{\mathrm{tac}}(R)$. This perspective will mimic many of the ideas from the proof of Theorem 7.8 in [AvGaPe], presented in Chapter 2 of this thesis. Rather than include it as a proof of an analogous theorem, though, we choose to provide what we will call the *triangulated* definition of the critical and cocritical degrees. Then, after demonstrating equivalence of the definitions, we give proof for the cohomological characterization in Section 4.4 of this chapter.

Recall that for any triangulated category $\mathscr{T}$, distinguished triangles induce long exact sequences of Ext groups, which are in fact also R-modules under our given assumptions. The following definition involves this process to give an equivalent, yet alternative perspective of the critical and cocritical degrees. More importantly, this definition will be an intermediary step for developing the cohomological characterization of these degrees in $\mathbf{K}_{\text{tac}}(R)$.

Definition 4.9. Denote $\mathrm{K} \to \mathbf{F}^{\Bbbk} \twoheadrightarrow \Bbbk$ as the *minimal* complete resolution of the residue field $\Bbbk = \frac{R}{\mathfrak{m}}$. For any endomorphism $\mu \colon \mathrm{C} \to \Sigma^q \mathrm{C}$, the distinguished triangle $\mathrm{C} \xrightarrow{\mu} \Sigma^{\mathrm{q}}\mathrm{C} \xrightarrow{\iota} \mathrm{M}\,(\mathrm{u}) \xrightarrow{\pi} \Sigma\mathrm{C}$ yields the long exact sequence of abelian groups

$$\cdots \to \mathrm{Ext}^n_{\mathcal{K}}(\mathrm{M}\,(\mu), \mathrm{K}) \to \mathrm{Ext}^n_{\mathcal{K}}(\Sigma^q \mathrm{C}, \mathrm{K}) \xrightarrow{\mu^n} \mathrm{Ext}^n_{\mathcal{K}}(\mathrm{C}, \mathrm{K}) \to \mathrm{Ext}^{n+1}_{\mathcal{K}}(\mathrm{M}\,(\mu), \mathrm{K}) \to \mathrm{Ext}^{n+1}_{\mathcal{K}}(\Sigma^q \mathrm{C}, \mathrm{K}) \to \cdots$$

where $\mu^n = \mathrm{Hom}_{\mathcal{K}}(\mu_n, \mathrm{K})$. The *$\mu$-critical degree of* C is the least homological degree s^μ for which μ^{n+q} is (split) injective for all $n > s^\mu$; that is,

$$\mathrm{crdeg}^{\mu}_{\mathrm{R}}\mathrm{C} := \inf\{i \,|\, \mu^{n+q} \colon \mathrm{Ext}^{n+q}_{\mathcal{K}}(\Sigma^q \mathrm{C}, \mathrm{K}) \hookrightarrow \mathrm{Ext}^{n+q}_{\mathcal{K}}(\mathrm{C}, \mathrm{K}) \text{ for all } n > i\}.$$

Likewise, the *u-cocritical degree of* C is the greatest homological degree t^μ for which μ^n is (split) surjective for all $n < t^\mu$; that is,

$$\mathrm{cocrdeg}^{u}_{\mathrm{R}}\mathrm{C} := \sup\{i \,|\, \mu^{n} \colon \mathrm{Ext}^{n}_{\mathcal{K}}(\Sigma^q \mathrm{C}, \mathrm{K}) \twoheadrightarrow \mathrm{Ext}^{n}_{\mathcal{K}}(\mathrm{C}, \mathrm{K}) \text{ for all } n < i\}.$$

Note that if there exists no such infimum s^μ, then, by definition, $\mathrm{crdeg}^{\mu}_{\mathrm{R}}\mathrm{C} = \infty$ and, similarly, if there exists no such supremum t^μ then $\mathrm{crdeg}^{\mu}_{\mathrm{R}}\mathrm{C} = -\infty$. Further note that we may define the *critical degree of* C in the same way as Definition 4.1 and we may also define the *cocritical degree of* C as such. Therefore, the only distinction here is how we are defining *relative* critical and cocritical degrees for a given R-complex and chain endomorphism in $\mathbf{K}_{\text{tac}}(R)$.

4.3.1 Equivalency of the Definitions

Understanding the equivalency of Definitions 4.1 and 4.2 with Definition 4.9 is not too much of a jump, reflecting the simplicity in $\mathcal{R}$-$\mathcal{mod}$. The bulk of the work is actually involved in describing the connection between $\mathrm{Ext}^n_{\mathcal{K}}(\mathrm{C}, \mathrm{K})$ and $\widehat{\mathrm{Ext}}^n_R(M, \Bbbk)$ for any complete resolution $\mathrm{C} \to M \twoheadrightarrow \Bbbk$. Afterward, it is quite simple to see the equivalency of the different formats for the definitions of critical and cocritical degree.

Thus, we begin with a lemma to show the necessary connection between Ext of complexes and $\widehat{\mathrm{Ext}}$ of modules.

Lemma 4.10. *Given the complete resolutions* $\mathrm{K} \to \mathbf{F}^{\Bbbk} \to \Bbbk$ *and* $\mathrm{C} \to \mathbf{F}^{\mathbf{M}} \to M$*, where* M *is maximal Cohen-Macaulay, there exists an isomorphism*

$$\mathrm{Ext}^n_{\mathscr{K}}(\mathrm{C}, \mathrm{K}) \cong \widehat{\mathrm{Ext}}^n_R(M, \Bbbk)$$

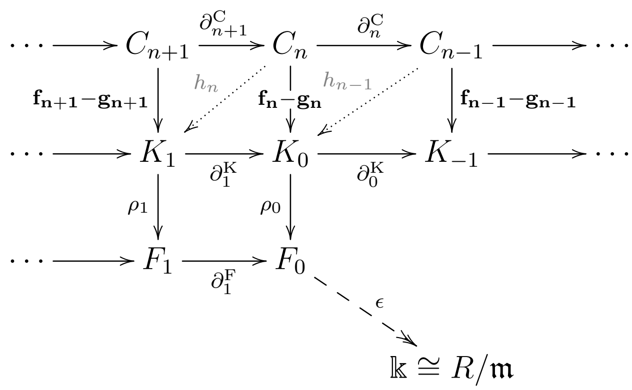
for all $n \in \mathbb{Z}$.

Proof. We begin with noting that for any two R-complexes A and B, $\mathrm{Hom}_{\mathscr{K}}(\mathrm{A}, \Sigma^n \mathrm{B}) \cong \mathrm{Ext}^n_{\mathscr{K}}(\mathrm{A}, \mathrm{B})$ (see Chapter 3 for a description of this). Then recognize that for any $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$ with minimal subcomplex $\overline{\mathrm{C}}$, we have that $\mathrm{Hom}_{\mathscr{K}}(\mathrm{C}, \Sigma^n \mathrm{K}) \cong \mathrm{Hom}_{\mathscr{K}}(\overline{\mathrm{C}}, \Sigma^n \mathrm{K})$ since C and $\overline{\mathrm{C}}$ are equivalent objects in $\mathscr{K}$ (and thus $\mathbf{K}_{\mathrm{tac}}(R)$). So, without loss of generality, suppose C is a minimal R-complex in $\mathbf{K}_{\mathrm{tac}}(R)$. We now aim to demonstrate the isomorphism

$$\mathrm{Hom}_{\mathscr{K}}(\mathrm{C}, \Sigma^n \mathrm{K}) \cong \widehat{\mathrm{Ext}}^n_R(M, \Bbbk)$$

(for any $n \in \mathbb{Z}$) in order to attain the appropriate reduction of $\mathrm{Ext}^n_{\mathscr{K}}(\mathrm{C}, \mathrm{K})$ to $\widehat{\mathrm{Ext}}^n_R(M, \Bbbk)$, where $M = \mathrm{Im}\,\partial^{\mathrm{C}}_0$. Then, we are able to use either form above to show the statement in the subsequent proposition.

Define a map $\Phi_n \colon \mathrm{Hom}_{\mathscr{K}}(\mathrm{C}, \Sigma^n \mathrm{K}) \to \widehat{\mathrm{Ext}}^n_R(M, \Bbbk)$ where $\Phi([f]) = \epsilon \rho_0 f_n$ for any R-complex chain map $\mathrm{f} \colon \mathrm{C} \to \Sigma^n \mathrm{K}$. First, we check that this map is well defined; suppose $\mathrm{f} \sim \mathrm{g}$ so that there exist homotopy maps $h_j \colon C_j \to K_{j+1-n}$ for each $j \in \mathbb{Z}$, as indicated in the following diagram.



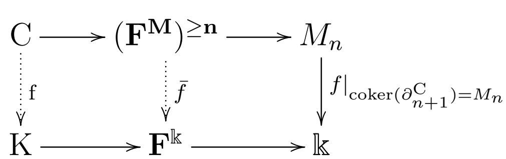
Note that

$$\epsilon\rho_0 \circ (f_n - g_n) = \epsilon\rho_0 \circ (\partial^{\Sigma^n \mathrm{K}}_{n+1} h_n + h_{n-1}\partial^{\mathrm{C}}_n)$$

$$= \epsilon(\rho_0 \partial^{\mathrm{K}}_1)h_n + \epsilon\rho_0 h_{n-1}\partial^{\mathrm{C}}_n$$

$$= \epsilon(\partial^{\mathrm{F}}_1 \rho_0)h_n + \epsilon\rho_0 h_{n-1}\partial^{\mathrm{C}}_n = 0$$

since $\epsilon\partial^{\mathrm{F}}_1 = 0$ by minimality of F and, similarly, $\rho_0 h_{n-1}\partial^{\mathrm{C}}_n(C_n) \subseteq \mathfrak{m}C_n$ so that the latter composition must be 0 as well. Now, for any cycle $f: C_n \to \Bbbk$, define $\Theta_n\colon \widehat{\mathrm{Ext}}^n_R(M,\Bbbk) \to \mathrm{Hom}_{\mathscr{K}}(\mathrm{C}, \Sigma^n\mathrm{K})$ in the following manner. Set $f_n = \epsilon^{-1} \circ f$ where ϵ^{-1} is a preimage map guaranteed by the surjectivity of ϵ. And so, we have an R-module map $f_n\colon C_n \to K_0$. Now let $\Phi(f) = [\mathrm{f}]$ where $\mathrm{f}\colon \mathrm{C} \to \Sigma^n\mathrm{K}$ represents the completion of $f_n\colon C_n \to K_0$ guaranteed by (1.5) in [BeJoMo], indicated in the following diagram.



Moreover, f is unique up to homotopy, ensuring that Θ is well defined. Finally, it should be clear that $\Phi_n\Theta_n = \mathrm{Id}^{\widehat{\mathrm{Ext}}^n_R(M,\Bbbk)}$ and $\Theta_n\Phi_n = \mathrm{Id}^{\mathrm{Hom}_{\mathscr{K}}(\mathrm{C},\Sigma^n\mathrm{K})}$ for each $n \in \mathbb{Z}$ by construction. □

Proposition 4.11. *The two definitions of relative μ-(co)critical degree are equivalent.*

Proof. Without loss of generality, suppose C is a minimal R-complex and, furthermore, let $\mu\colon \mathrm{C} \to \Sigma^q\mathrm{C}$ be a chain endomorphism in $\mathbf{K}_{\mathrm{tac}}(R)$. To show the equivalency of definitions, we only need prove that $\hat{\mu}^{n+q}$ (as described above) is (split) injective if and only if μ_{n+q} is surjective. Likewise, given $\nu\colon \mathrm{C} \to \Sigma^r\mathrm{C}$, we must prove that $\hat{\nu}^n$ is surjective if and only if ν_n is (split) injective. To see the former statement, note that μ_{n+q} surjective yields the (split) short exact sequence

$$0 \to \ker(\mu_{n+q}) \to C_{n+q} \xrightarrow{\mu_{n+q}} C_n \to 0$$

and since $\mathrm{Hom}_R(-, \Bbbk)$ is an additive functor on $\mathcal{R}\text{-}\mathcal{mod}$, we have preservation of the split short exact sequence. This gives

$$0 \to \mathrm{Hom}_R(C_n, \Bbbk) \xrightarrow{\mathrm{Hom}(\mu_{n+q}, \Bbbk)} \mathrm{Hom}_R(C_{n+q}, \Bbbk) \to \mathrm{Hom}_R(\ker(\mu_{n+q}), \Bbbk) \to 0$$

where we observe that $\mathrm{Hom}(\mu_{n+q}, \Bbbk)$ is in fact split injective. And since, $\widehat{\mathrm{Ext}}_R^n(M, \Bbbk) \subseteq \mathrm{Hom}_R(C_n, \Bbbk)$ for any $n \in \mathbb{Z}$ the same holds for $\hat{\mu}^{n+q}\colon \widehat{\mathrm{Ext}}_R^{n+q}(M^q, \Bbbk) \to \widehat{\mathrm{Ext}}_R^{n+q}(M, \Bbbk)$ where $M^q := \mathrm{Im}\, \partial_0^{\Sigma^q\mathrm{C}} = M_{-q}$, according to the notation used in Chapter 2. To show the other direction, first note that for any cycle $\alpha \in \mathrm{Hom}_R((\Sigma^q\mathrm{C})_{n+q}, \Bbbk) = \mathrm{Hom}_R(C_n, \Bbbk)$, the action is given by $\hat{\mu}^{n+q}(\alpha) = \alpha\mu_{n+q} \in \mathrm{Hom}_R(C_{n+q}, \Bbbk)$. If $\hat{\mu}^{n+q}$ is assumed to be split injective, then there exists a left inverse, say $(\hat{\mu}^{n+q})^{-1}$: $\widehat{\mathrm{Ext}}_R^{n+q}(M, \Bbbk) \to \widehat{\mathrm{Ext}}_R^{n+q}(M^q, \Bbbk)$, so that

$$\hat{\mu}^{n+q}(\alpha) = \hat{\mu}^{n+q}(\beta)$$
$$(\hat{\mu}^{n+q})^{-1}\hat{\mu}^{n+q}(\alpha) = (\hat{\mu}^{n+q})^{-1}\hat{\mu}^{n+q}(\beta)$$
$$\alpha = \beta$$

for any two cycles α, $\beta \in \mathrm{Hom}_R(C_n \Bbbk)$. Of course this means for any such morphisms $\alpha, \beta\colon C_n \to \Bbbk$ we have that $\alpha\mu_{n+q} = \beta\mu_{n+q}$ implies $\alpha = \beta$; equivalently, μ_{n+q} is right-cancellative, and thus surjective.

Now, to show equivalency of the definitions for cocritical degree, we start with the split short exact sequence

$$0 \to C_n \overset{\mu_n}{\hookrightarrow} C_{n-q} \to A \to 0$$

where A is the summand such that $C_{n-q} = A \oplus \mathrm{im}(\mu_n)$. Once again, applying $\mathrm{Hom}_R(-, \Bbbk)$ to this sequence yields the desired result since

$$0 \to \mathrm{Hom}_R(A, \Bbbk) \to \mathrm{Hom}_R(C_{n-q}, \Bbbk) \xrightarrow{\hat{\mu}^n} \mathrm{Hom}_R(C_n, \Bbbk) \to 0$$

is split exact, implying that $\hat{\mu}^n$ is surjective. Now suppose $\hat{\mu}^n$ is surjective so that for any cycle $\alpha \in \mathrm{Hom}(C_n, \Bbbk)$ there exists a cycle $\beta \in \mathrm{Hom}(C_{n-q}, \Bbbk)$ with $\alpha = \beta\mu_n$. Since C_n is free, we can denote $\{e_i\}$ as a basis of C_n. Define maps $\pi_j : C_n \to \Bbbk$ where $\pi_j(e_i) = \delta_{ij}$ and note that each π_j is a cycle in $\mathrm{Hom}_R(C_n, \Bbbk)$ because $\partial(\mathrm{C}) \subseteq \mathfrak{m}\mathrm{C}$. Hence, by surjectivity of $\hat{\mu}^n$, there must exist associated maps $\rho_j : C_{n-q} \to \Bbbk$ such that $\pi_j = \rho_j\mu_n$, implying that $\mu_n(e_i)$ forms a linearly independent subset of a basis $\mathcal{E}$ for C_{n-q}. (If we take $\mu_n(e_i)$ as the appropriate image in the $\Bbbk$-vector space $C_{n-q}/\mathfrak{m}C_{n-q}$ and assume $0 = \sum_i a_i\mu_n(e_i)$ for some $a_i \in \Bbbk$, then note for any j we must have that $0 = \rho_j(0) = \sum_i a_i\rho_j\mu_n(e_i) = \sum_i a_i\pi_j(e_i)$. And so we see that $a_j = 0$ for each j, entailing that $\mu_n(e_i)$ is a linearly independent set on $C_{n-q}/\mathfrak{m}C_{n-q}$. Thus, we can complete this set to a $\Bbbk$-basis $\bar{\mathcal{E}}$ and, by Nakayama's Lemma, the preimage $\mathcal{E}$ will be a basis for C_{n-q} as well.)

Since a linearly independent sub-basis of C_{n-q} defined by $\mathrm{im}\mu_n$ is in one-to-one correspondence with a basis for C_n, it must hold that μ_n is injective. Moreover, $\mathcal{E} = (\mathcal{E} \setminus \{\mu_n(e_i)\}) \cup \{\mu_n(e_i)\}$ and if we denote A as the subspace generated by $\mathcal{E} \setminus \{\mu_n(e_i)\}$, then we may write $C_{n-q} = A \oplus \mathrm{im}\mu_n$ and μ_n is split injective, as needed. □

4.3.2 Consequences of the Long Exact Sequence

We now discuss a few consequences of Definition 4.9, which will have a similar flavor to what was uncovered in the proof of Theorem 7.8 in [AvGaPe]. Note that since the sequence of abelian groups is exact, an injective map $\mathrm{Ext}^n_R(\Sigma^q \mathrm{C}, \mathrm{K}) \hookrightarrow \mathrm{Ext}^n_R(\mathrm{C}, \mathrm{K})$ implies that the previous map $\mathrm{Ext}^n_R(\mathrm{M}\,(\mathrm{u}), \mathrm{K}) \to \mathrm{Ext}^n_R(\Sigma^q \mathrm{C}, \mathrm{K})$ is the zero map. And, consequently, the map $\mathrm{Ext}^n_R(\mathrm{C}, \mathrm{K}) \to \mathrm{Ext}^n_R(\mathrm{M}\,(\mathrm{u}), \mathrm{K})$ is surjective. Thus, for all $n > s$ the long exact sequence becomes

$$\cdots \to \mathrm{Ext}^{n-1}_{\mathscr{K}}(\mathrm{C},\mathrm{K}) \twoheadrightarrow \mathrm{Ext}^{n}_{\mathscr{K}}(\mathrm{M}\,(\mu),\mathrm{K}) \xrightarrow{0} \mathrm{Ext}^{n}_{\mathscr{K}}(\Sigma^q\mathrm{C},\mathrm{K}) \hookrightarrow \mathrm{Ext}^{n}_{\mathscr{K}}(\mathrm{C},\mathrm{K}) \twoheadrightarrow \mathrm{Ext}^{n+1}_{\mathscr{K}}(\mathrm{M}\,(\mu),\mathrm{K}) \xrightarrow{0} \mathrm{Ext}^{n+1}_{\mathscr{K}}(\Sigma^q\mathrm{C},\mathrm{K}) \hookrightarrow \cdots$$

and, moreover, we can rewrite the sequence as

$$\cdots \to \widehat{\mathrm{Ext}}^{n-1}_R(M,\Bbbk) \twoheadrightarrow \widehat{\mathrm{Ext}}^{n}_R(M(\mu),\Bbbk) \xrightarrow{0} \widehat{\mathrm{Ext}}^{n-q}_R(M,\Bbbk) \hookrightarrow \widehat{\mathrm{Ext}}^{n}_R(M,\Bbbk) \twoheadrightarrow \widehat{\mathrm{Ext}}^{n+1}_R(M(\mu),\Bbbk) \xrightarrow{0} \widehat{\mathrm{Ext}}^{n-q+1}_R(M,\Bbbk) \hookrightarrow \cdots$$

where $M = \mathrm{Im}\,\partial^{\mathrm{C}}_0$ and $M(\mu) = \mathrm{Im}\,\partial^{\mathrm{M}\,(\mu)}_0$. Note the distinction that here, $M(\mu)$ is not the same as the pushout presented in Chapter 2. However, we find that there exists an analogous relationship between the Betti numbers after the critical degree. If $\mathrm{crdeg}_{\mathrm{R}}\mathrm{C} = s < \infty$ is realized by $\mu\colon \mathrm{C} \to \Sigma^q\mathrm{C}$, then the long exact sequence above breaks up into *split* short exact sequences of the form

$$0 \to \widehat{\mathrm{Ext}}^{n}_R(M,\Bbbk) \xhookrightarrow{\hat{\mu}^n} \widehat{\mathrm{Ext}}^{n+q}_R(M,\Bbbk) \twoheadrightarrow \widehat{\mathrm{Ext}}^{n+q+1}_R(M(\mu),\Bbbk) \to 0$$

for all $n > s$. Therefore, since each $\widehat{\mathrm{Ext}}$ is in fact a $\Bbbk$-vector space, we obtain the following relationship

$$\hat{b}^R_{n+q}(M) = \hat{b}^R_n(M) + \hat{b}^R_{n+q+1}(M(\mu))$$

mirroring the results in the proof of Theorem 7.8 in [AvGaPe]. One observation here is that C is periodic (of period q) *precisely when* $\mathrm{M}\,(\mu) \simeq 0$, since $\hat{b}^R_{n+q+1}(M(\mu)) = 0$ for any n implies this is the case.

In the same vein, suppose $\mathrm{cocrdeg_R C} = t > -\infty$ is realized by $\nu\colon \mathrm{C} \to \Sigma^r \mathrm{C}$ so that the long exact sequence breaks up into split short exact sequences of the form

$$0 \to \widehat{\operatorname{Ext}}^n_R(M(\nu), \Bbbk) \hookrightarrow \widehat{\operatorname{Ext}}^{n-q}_R(M, \Bbbk) \xrightarrow{\hat{\nu}^n} \widehat{\operatorname{Ext}}^n_R(M, \Bbbk) \to 0$$

for all $n < t$. This yields a similar relationship of the complete Betti numbers

$$\hat{b}^R_{n-q}(M) = \hat{b}^R_n(M(\mu)) + \hat{b}^R_n(M)$$

in which case we find the same observation for when C is periodic (of period q).

4.4 Cohomological Characterization in $\mathbf{K}_{\mathrm{tac}}(R)$

Given Definition 4.9, we are now ready to present the cohomological characterizations of the critical and cocritical degrees in $\mathbf{K}_{\mathrm{tac}}(R)$. The former turns out to be as expected: a natural extension from the cohomological characterization of critical degree in $\mathcal{R}$-$\mathcal{mod}$. The latter involves dual notions to some of the ideas involved with critical degree in $\mathcal{R}$-$\mathcal{mod}$; once we equip ourselves with such notions, the characterization for cocritical degree aligns nicely with what we understand about critical degree.

For the sake of simplicity, assume that R is a complete intersection ring for the remainder of this chapter.

Proposition 4.12. *If* $\mathrm{C} \not\simeq 0$ *is a totally acyclic* R*-complex, then* $\mathrm{crdeg_R C} = s < \infty$ *and the following equalities hold:*

$$\begin{aligned}\mathrm{crdeg_R C} &= \sup\{r \in \mathbb{Z} \mid \operatorname{depth}_{\hat{\mathcal{S}}} \operatorname{Ext}^{\geq r}_{\mathcal{K}}(\mathrm{C}, \mathrm{K}) = 0\} \\ &= \sup\{r \in \mathbb{Z} \mid \operatorname{depth}_{\hat{\mathcal{S}}} \widehat{\operatorname{Ext}}^{\geq r}_R(\operatorname{im}\partial^{\mathrm{C}}_0, \Bbbk) = 0\}.\end{aligned}$$

Remark. *In the above proposition, note that we can take* $\operatorname{Ext}^*_{\mathcal{K}}(\mathrm{C}, \mathrm{K})$ *to be an* $\hat{\mathcal{S}}$*-module; refer to [St] for a general description of the action of* $\hat{\mathcal{S}}$ *on* $\operatorname{Hom}_{\mathcal{K}}(\mathrm{C}, \mathrm{D})$ *for two* R*-complexes in* $\mathbf{K}_{\mathrm{tac}}(R)$.

The proof of Proposition 4.12 is somewhat simplified by the connection to the critical degree's cohomological characterization with respect to $\mathcal{R}$-$\mathcal{mod}$. In particular, we must handle the cases for $\mathrm{crdeg_R C} \geq 0$ and $\mathrm{crdeg_R C} < 0$ separately, with the simplification occurring in the former case.

Proof. Finiteness of s is guaranteed by proof of Theorem 3.1 in [Ei], since R is a complete intersection and thus there exists a linear form $\ell \in \mathcal{S}$ which is eventually a non zero-divisor on the truncation $\mathrm{Ext}_R^{\geq N}(M, \Bbbk)$ for some $N \gg 0$. To proceed, denote $M = \mathrm{im}\partial_0^{\mathrm{C}}$, $\mathrm{crdeg_R C} = s$, and $r^* = \sup\{r \in \mathbb{Z} \mid \mathrm{depth}_{\mathcal{S}} \widehat{\mathrm{Ext}}_R^{\geq r}(\mathrm{im}\partial_0^{\mathrm{C}}, \Bbbk) = 0\}$. First note that for each $r \geq 0$, $\widehat{\mathrm{Ext}}_R^{\geq r}(M, \Bbbk) = \mathrm{Ext}_R^{\geq r}(M, \Bbbk)$; therefore, if $r^* \geq 0$, the above characterization reduces to that for $\mathrm{crdeg_R M}$ in $\mathcal{R}$-$\mathcal{mod}$. Additionally, note that if $s \geq 0$ then $s = \sup\{r \in N \cup 0 \mid \mathrm{depth}_{\mathcal{S}} \mathrm{Ext}_R^{\geq r}(M, \Bbbk) = 0\}$ and so $r^* = s$ since any negative value of r^* would contradict that we took the supremum of all truncations of Ext with depth 0.

Now assume that $r^* \lneq 0$ and $s \lneq 0$. Noting that we cannot use $\mathrm{crdeg_R M}$ under these assumptions, our approach is to instead use $\mathrm{crdeg_R M_n}$ for some smart choice of $n \in \mathbb{N}$ where $M_n = \mathrm{Im}\,\partial_n^{\mathrm{C}}$. Suppose first that $r^* \leq s < 0$ and consider the R-complex $\Sigma^{|r^*|}\mathrm{C}$ which has non-negative critical degree since $\mathrm{crdeg_R}\Sigma^{|\mathrm{r}^*|}\mathrm{C} = s + \mid r^* \mid$ (see Proposition 5.5). Note that $\mathrm{Im}\,\partial_0^{\Sigma^{|r^*|}\mathrm{C}} = \mathrm{Im}\,\partial_{r^*}^{\mathrm{C}}$ and so

$$s + \mid r^* \mid = \{r \in \mathbb{N} \cup \{0\} \mid \mathrm{depth}_{\mathcal{S}} \mathrm{Ext}_R^{\geq r}(M_{r^*}, \Bbbk) = 0\}.$$

Hence there exists some $\chi \in \mathfrak{X}$ that is a non zero-divisor on $\mathrm{Ext}_R^{>s+|r^*|}(M_{r^*}, \Bbbk)$ but there exist no non zero-divisors on $\mathrm{Ext}_R^{\geq s+|r^*|}(M_{r^*}, \Bbbk)$. Since there exists an isomorphism $\widehat{\mathrm{Ext}}_R^{s+|r^*|+i}(M_{r^*}, \Bbbk) \cong \widehat{\mathrm{Ext}}_R^{s+i}(\Omega^{|r^*|}M_{r^*}, \Bbbk)$ (see Lemma 4.3 in [Ta]), we obtain the isomorphism $\mathrm{Ext}_R^{s+|r^*|+i}(M_{r^*}, \Bbbk) \cong \widehat{\mathrm{Ext}}_R^{s+i}(M, \Bbbk)$ for all $i \in \mathbb{Z}$. Note that χ is a non zero-divisor on $\mathrm{Ext}_R^{\geq s+|r^*|+i}(M_{r^*}, \Bbbk)$ for all $i > 0$, so it must also be a non zero-divisor on $\widehat{\mathrm{Ext}}_R^{\geq s+i}(M, \Bbbk)$ for all $i > 0$ as well. Therefore, $\mathrm{depth}_{\mathcal{S}} \widehat{\mathrm{Ext}}_R^{\geq s+i}(M, \Bbbk) \neq 0$

for all $i > 0$ but it must hold that $\operatorname{depth}_{\mathcal{S}} \widehat{\operatorname{Ext}}_R^{\geq s}(M, \Bbbk) = 0$ because otherwise $\operatorname{depth}_{\mathcal{S}} \operatorname{Ext}_R^{\geq s+|r^*|}(M_{r^*}, \Bbbk)$ would not be 0; and so, $r^* = s$.

If we instead suppose $s \leq r^* < 0$ and consider the R-complex $\Sigma^{|s|}\mathrm{C}$, we can apply the same argument as above since $\mathrm{crdeg}_{\mathrm{R}}\Sigma^{|s|}\mathrm{C} = 0$ and there exists an isomorphism $\widehat{\operatorname{Ext}}_R^{s+|s|+i}(M_s, \Bbbk) \cong \widehat{\operatorname{Ext}}_R^{s+i}(\Omega^{|s|}M_s, \Bbbk)$ so that $\widehat{\operatorname{Ext}}_R^{i}(M_s, \Bbbk) \cong \widehat{\operatorname{Ext}}_R^{s+i}(M, \Bbbk)$ for all $i > 0$. □

4.4.1 Time to Dual

We now move to introducing dual notions to regular elements (non zero-divisors), regular sequences, and the socle of an R-module. These notions are not as well known or discussed as their counterparts, but provide the necessary machinery in order to give a notion of cohomological characterization for cocritical degree.

It is commonly known that the *socle* of an R-module M is the largest semisimple submodule of M (cf. [BrHe, 1.2.18], [Ei2, Pg. 526], [AnFu, §9]). We now give an equivalent definition for semisimple in terms of the socle.

Definition 4.13 (cf. [AnFu])**.** Let M be an R-module. Then M is *semisimple* if and only if $\operatorname{Soc}(M) = M$.

The *radical* of M is often referred to as the "dual" notion to socle. However, there is a more natural dual notion, given by the following definition which can be found in [AnFu] named *capital* of a module, among other sources referred to as below.

Definition 4.14 (cf. [AnFu])**.** Let M be an R-module. The *cosocle* of M, denoted $\operatorname{Cosoc}(M)$, is defined to be the largest (or, maximal) semisimple quotient module of M.

Recall R is a local ring with maximal ideal $\mathfrak{m}$ and that, in this case, $\operatorname{Soc}(M) = \{x \in M \mid x\mathfrak{m} = 0\}$. That is, a socle element in M is an element annihilated by

$\mathfrak{m}$; equivalently, an element for which each generator of $\mathfrak{m}$ is a zero-divisor. Now, note that if $\mathrm{Cosoc}(M) = M/N$ such that M/N is the *largest* semisimple quotient of M, then N must be the *smallest* submodule such that M/N is semisimple. And, combining this with Definition 4.13 above, we see that in the case of R local, $N = \mathfrak{m}M$. Thus, a cosocle element has the form $\bar{x} = x + \mathfrak{m}M$ implying that its preimage $x \in M$ must not be of the form mx' where $m \in \mathfrak{m}$ and $x' \in M$. That is, we can denote the *preimage* of $\mathrm{Cosoc}(M)$ as

$$\mathrm{preim}(\mathrm{Cosoc}(M)) = \{x \in M \mid x \notin \mathfrak{m}M\}.$$

One manner of viewing the preimage of a cosocle element is as a minimal generator of the module, M. We now relay a fact which will maintain a certain significance for the next section.

Proposition 4.15 (Theorem 2 in [Ma3]). *Let E be an S-module satisfying the descending chain condition (i.e. artinian) and let $\mathfrak{a}$ be an ideal of S (with S a commutative, noetherian ring). Then $\mathfrak{a}E = E$ if and only if there exists some $a \in \mathfrak{a}$ such that $aE = E$.*

Returning to our local ring R, this fact implies that if $\mathrm{Cosoc}(M) = 0$, then there exists some $a \in \mathfrak{m}$ such that the submodule generated by a returns the module M. Unsurprisingly, this notion is connected to the dual analogues to regular sequences and depth; these analogues were first defined in [Ma3].

Definition 4.16 (See [Ma3], cf. [HaPo]). Let $\mathfrak{a} \subseteq R$ be an ideal and let M be a nonzero R-module. Then a coregular sequence in M, or a M-cosequence, is a sequence $\tilde{\mathbf{a}} = a_1, \ldots, a_d$ such that

(1) $a_1 M = M$, and

(2) $a_i(0 :_M (a_1, \ldots, a_{i-1})) = (0 :_M (a_1, \ldots, a_{i-1}))$ for each $i = 2, \ldots, d$.

As the authors of [HaPo] note, a coregular sequence $(\tilde{\mathbf{a}})$ is a sequence such that multiplication by a_1 is surjective on M, multiplication by a_2 is surjective on the *kernel* of multiplication by a_1, and so on. Accordingly, we call $a \in R$ a *coregular* element if $aM = M$ (multiplication by a is surjective). Lastly, we call $a_1, \ldots, a_d$ a *maximal* M-cosequence *in* $\mathfrak{a}$ if $a_1, \ldots, a_d, a_{d+1}$ is *not* an M-cosequence for any other element $a_{d+1} \in \mathfrak{a}$. And now, we are ready to give a dual notion to depth of an R-module.

Definition 4.17 (See [Ma3], cf. [HaPo] and [Oo])**.** The $\mathfrak{a}$-*codepth* of M, denoted $\operatorname{codepth}_{\mathfrak{a}} M$, is defined to be the maximal length of a M-cosequence in $\mathfrak{a}$. If $(R, \mathfrak{m}, \Bbbk)$ is local, then set $\operatorname{codepth}_R M = \operatorname{codepth}_{\mathfrak{m}} M$.

From the above definitions, and our earlier discussion, it should be clear that $\operatorname{codepth}_R M = 0$ implies $\operatorname{Cosoc}(M) \neq 0$, so that existence of some nonzero element $x \notin \mathfrak{m}M$ is guaranteed. In the next section, we utilize these definitions in order to accomplish our goal of providing a dual analogue for the cohomological characterization of critical degree.

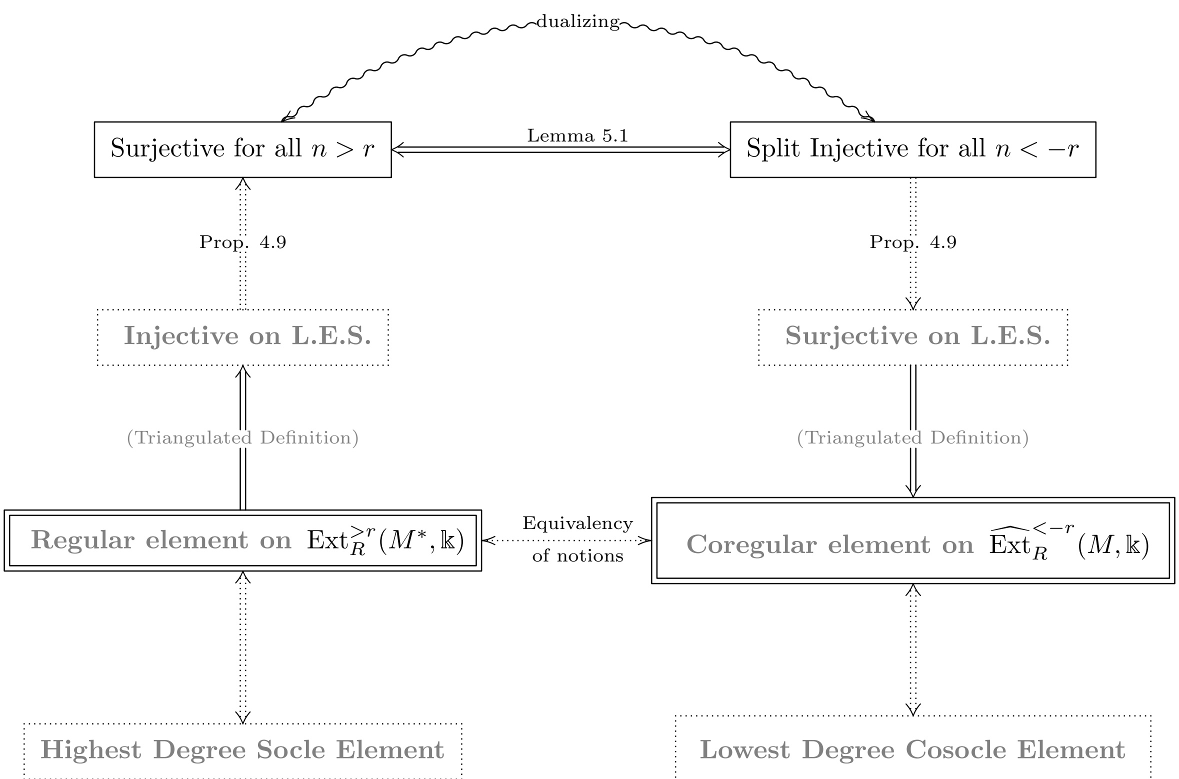
4.4.2 The Cohomological Characterization of Cocritical Degree

Let $\mathrm{C} \to \mathbf{F} \twoheadrightarrow M$ denote the minimal complete resolution of $M = \operatorname{im}\partial_0^{\mathrm{C}}$ so that $\Sigma^{-1}\mathrm{C}^* \to (\boldsymbol{\Sigma}^{-1}\mathrm{C}^*)^{\geq 1} \twoheadrightarrow M^*$ is the (minimal) complete resolution of $M^* \cong \operatorname{im}\partial_1^*$. Before proving the cohomological characterization for cocritical degree, we first establish a significant relationship between the depth of $\widehat{\operatorname{Ext}}_R^{\geq r}(M^*, \Bbbk)$ and the codepth of $\widehat{\operatorname{Ext}}_R^{\leq -r}(M, \Bbbk)$ for any non-negative integer r.

Lemma 4.18. *Let* $0 \not\simeq \mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$ *with* M *and* M^* *be defined as above. Then* $\operatorname{depth}_{\mathcal{S}} \widehat{\operatorname{Ext}}_R^{\geq r}(M^*, \Bbbk) = 0$ *if and only if* $\operatorname{codepth}_{\mathcal{S}} \widehat{\operatorname{Ext}}_R^{\leq -r}(M, \Bbbk) = 0$.

Proof. For ease of notation, denote $\widehat{\operatorname{Ext}}_R^{>r}(M^*, \Bbbk) = E^r(M^*)$ and $\widehat{\operatorname{Ext}}_R^{<-r}(M, \Bbbk) = E^{-r}(M)$. First note that if $\operatorname{depth}_{\mathcal{S}} E^r(M^*) \neq 0$ then there exists some $\chi \in \mathfrak{X}$ which is a non zero-divisor on $E^r(M^*)$; by definition, this corresponds directly with a degree 2 endomorphism χ on $\operatorname{Ext}_R^*(M^*, \Bbbk)$ that is injective for all $n \geq r$. This in turn corresponds with a -2 degree endomorphism μ on C^* which is surjective for all $n \geq r$. By Corollary 2 in Chapter 5, we obtain an endomorphism μ^T on $(\mathrm{C}^*)^* = \mathrm{C}$ that is split injective for all degrees $n \leq -r$ and, again by definition (refer to Proposition 4.11), this corresponds directly with a degree 2 map χ on $\widehat{\operatorname{Ext}}_R^*(M, \Bbbk)$ that is surjective for all $n \leq r$ thus implying χ is surjective on $E^{-r}(M)$. Hence, there exists some $\chi \in \mathfrak{X}$ such that $\chi E^{-r}(M) = E^{-r}(M)$ and so $\mathfrak{X} E^{-r}(M) = E^{-r}(M)$ by Proposition 4.15. Therefore, $\operatorname{depth}_{\mathcal{S}} E^r(M^*) \neq 0$ if and only if $\operatorname{codepth}_{\mathcal{S}} E^{-r}(M) \neq 0$, proving the lemma. □

We provide the following diagram of correspondences to demonstrate a visual interpretation of the proof above.



Note that if we take $s+1$ to be the *lowest* degree such that there exists a non zero-divisor on $\widehat{\operatorname{Ext}}_R^{>r}(M^*,\Bbbk)$, then s is the highest degree of a nonzero element in $\operatorname{Soc}(\widehat{\operatorname{Ext}}_R^*(M^*,\Bbbk))$. And since this correlates to the *highest* degree for which there exists a generating, or (formally) coregular element, apply Proposition 4.15 to see that $-s$ should be the *lowest* degree for which there exists a nonzero element in $\operatorname{Cosoc}(\widehat{\operatorname{Ext}}_R^*(M,\Bbbk))$. Further note that since $\operatorname{depth}_{\mathscr{Z}^*}\operatorname{Ext}_R(M^*,\Bbbk)$ coincides with $\operatorname{depth}_{\mathscr{S}}\operatorname{Ext}_R(M^*,\Bbbk)$ by Proposition 7.2 in [AvGaPe], there must exist a nonzero coregular element from $\mathfrak{X}$ on the greatest truncation of $\widehat{\operatorname{Ext}}_R^{<-r}(M,\Bbbk)$ for which there exists such a generating element. That is to say, there exists some $\chi\in\mathfrak{X}$ such that $\chi\,\widehat{\operatorname{Ext}}_R^{\leq t}(M,\Bbbk)=\widehat{\operatorname{Ext}}_R^{\leq t}(M,\Bbbk)$ where $t=\operatorname{cocrdeg_R}\mathrm{C}$. Thus, the cocritical degree of a complex over a complete intersection is (negatively) finite[2]; moreover, the following proposition holds.

Proposition 4.19. *If* C *is a totally acyclic R-complex (not homotopically equivalent to the zero complex), then* $\operatorname{cocrdeg_R}\mathrm{C}=t>-\infty$ *and the following equalities hold:*

$$\operatorname{cocrdeg_R}\mathrm{C}=\inf\{r\in\mathbb{Z}\mid \operatorname{codepth}_{\hat{\mathscr{S}}}\operatorname{Ext}_{\mathscr{K}}^{\leq r}(\mathrm{C},\mathrm{K})=0\}$$

$$=\inf\{r\in\mathbb{Z}\mid \operatorname{codepth}_{\hat{\mathscr{S}}}\widehat{\operatorname{Ext}}_R^{\leq r}(\operatorname{im}\partial_0^{\mathrm{C}},\Bbbk)=0\}.$$

In the next section, we explore the connection between an endomorphism on an R-complex C which is eventually surjective and one that is eventually (split) injective.

4.4.3 Realizability of Cocritical Degree, Given Critical Degree

Our goal in this section is to demonstrate that there exists some $\chi\in\mathfrak{X}$ such that χ is both a non zero-divisor on $\widehat{\operatorname{Ext}}_R^{>s}(M,\Bbbk)$ for some $s\in\mathbb{Z}$ and such that $\chi\,\widehat{\operatorname{Ext}}_R^{<t}(M,\Bbbk)=\widehat{\operatorname{Ext}}_R^{<t}(M,\Bbbk)$ for some $t\in\mathbb{Z}$.

[2]This holds even for a complex with $\operatorname{CI-dim}(\operatorname{im}\partial_0^{\mathrm{C}})<\infty$.

First note that $\widehat{\operatorname{Ext}}_R^*(M, \Bbbk)$ and $\widehat{\operatorname{Ext}}_R^*(M^*, \Bbbk)$ are both graded modules over $\mathcal{S}$; specifically, the distinct cohomology operators are dependent upon the complete resolutions, but the action of this ring remains the same on each of the modules. Denote $E_1 = \widehat{\operatorname{Ext}}_R^{\geq 0}(M, \Bbbk) = \operatorname{Ext}_R^{\geq 0}(M, \Bbbk)$ and $E_2 = \widehat{\operatorname{Ext}}_R^{\geq 0}(M^*, \Bbbk) = \operatorname{Ext}_R^{\geq 0}(M^*, \Bbbk)$ so that E_1 and E_2 are both noetherian modules over $\mathcal{S}$. Thus, let A_i represent the largest artinian submodule of E_i and note that $A_i \supseteq \operatorname{Soc}(E_i)$ for each $i = 1, 2$, since each submodule $\operatorname{Soc}(E_i)$ must itself be artinian.

Since each A_i is both artinian and noetherian, it is of finite length; this, of course, implies that there exists some $N_i = \sup\{n \in \mathbb{N} \cup \{0\} \mid \deg(x) = n \text{ for } x \in A_i\}$ for each $i = 1, 2$. Take $E_1^{>N_0} = \operatorname{Ext}_R^{>N_0}(M, \Bbbk)$ and $E_2^{>N_0} = \operatorname{Ext}_R^{>N_0}(M^*, \Bbbk)$ where $N_0 = \max\{N_1, N_2\}$ so that neither truncation contains a nonzero element annihilated by $\mathfrak{X}$. Then denote $P_1, \ldots, P_q$ the associated primes of $0 \in E_1^{>N_0}$ and $Q_1, \ldots, Q_r$ the associated primes of $0 \in E_2^{>N_0}$ so that $P_1 \cup \cdots \cup P_q \cup Q_1 \cup \cdots \cup Q_r$ is the set of zero-divisors on both $E_1^{>N_0}$ and $E_2^{>N_0}$. Note that the set

$$\chi_1 + \sum_{i=2}^{c} \Bbbk \chi_i,$$

generates $\mathfrak{X}$, so it cannot be contained in any P_k or Q_k since there is no element of $\mathfrak{X}$ which is a zero-divisor on $E_i^{\geq N_0}$.

And since $\Bbbk$ is infinite, there exists a translation of the set $\chi_1 + \sum_{i=2}^{c} \Bbbk \chi_i$ which is a subspace of $\Bbbk[\chi_1, \ldots, \chi_c]$, implying that $\chi_1 + \sum_{i=2}^{c} \Bbbk \chi_i \not\subseteq P_1 \cup \cdots \cup P_q \cup Q_1 \cup \cdots \cup Q_r$. Hence, there exists a linear form

$$\hat{\chi} = \chi_1 + \sum_{i=2}^{c} \alpha_j \chi_j$$

with $\alpha_j \in \Bbbk$ such that χ is a non zero-divisor on both $E_1^{\geq N_0}$ and $E_2^{\geq N_0}$. Now, for each $j = 1, \ldots, c$ set a_j equal to a pre-image of α_j in R so that $\chi = \chi_1 + \sum_{i=2}^{c} a_j \chi_j \in \mathcal{S}$.

Lastly, note that χ is a non zero-divisor on $E_1^{\geq N_0}$ if and only if $\chi_n \colon \operatorname{Ext}_R^n(M, \Bbbk) \to \operatorname{Ext}_R^{n+2}(M, \Bbbk)$ is injective for all $n > N_0$ if and only if

$$t = t_1 + \sum_{i=2}^{c} a_j t_j$$

is surjective for all $n > N_0$, where $\chi = \operatorname{Hom}_R(t, \Bbbk)$. Similarly, we see that χ is additionally a non zero-divisor on $E_2^{\geq N_0}$.

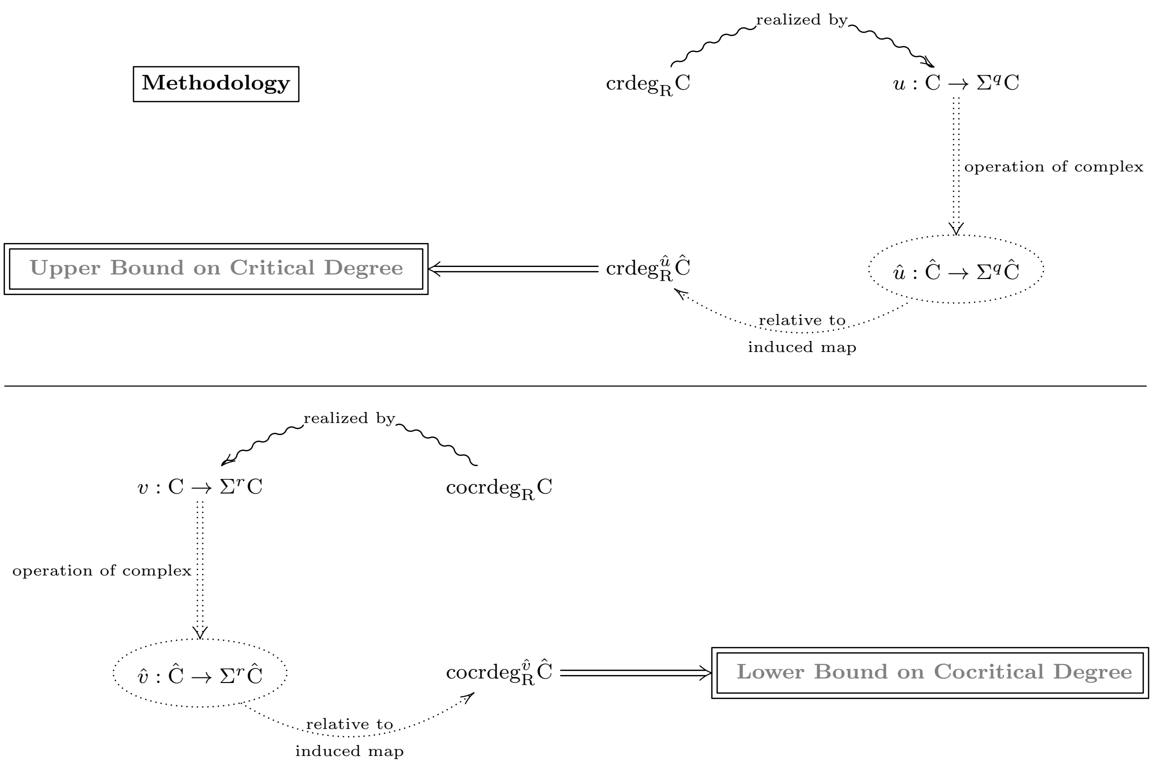
Hence, *à la* Eisenbud, we have shown that there exists some linear form $\chi \in \mathcal{S}$ such that χ is a non zero-divisor on some truncation of $\operatorname{Ext}_R^{\geq N_0}(M, \Bbbk)$ as well as $\operatorname{Ext}_R^{\geq N_0}(M^*, \Bbbk)$ for $N_0 \gg 0$. Now note, as we have demonstrated in Lemma 4.18, there is a one-to-one correspondence between regular elements on $\operatorname{Ext}_R^{\geq N_0}(M^*, \Bbbk)$ and coregular elements on $\widehat{\operatorname{Ext}}_R^{\leq -N_0}(M, \Bbbk)$. Consequently, there exists some linear form of CI operators which is both eventually surjective on the left and eventually injective on the right of an R-complex $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$. The question remains on whether $N_1 = N_2$ in the proof above, implying that the critical and cocritical degrees are realized by the same endomorphism.

CHAPTER 5

Operations on Totally Acyclic Complexes

Now that we have introduced the notion of critical degree (along with its dual) in $\mathbf{K}_{\text{tac}}(R)$, we move on to discuss "boundedness" problems. Our primary goal in this chapter will be to understand how the critical and cocritical degrees are altered through operations of totally acyclic R-complexes. We will begin with special attention towards basic operations which ensue from the categorical structure of $\mathbf{K}_{\text{tac}}(R)$. Afterwards, we will turn our focus to operations which involve applying the Hom and $\otimes$ functors with a specific type of R-complex called a *perfect complex*. With each operation discussed, we will first describe how such manipulations result in well-defined objects in $\mathbf{K}_{\text{tac}}(R)$; then, we will explore how the critical and cocritical degrees of an R-complex might change under these actions.

Preceding our discussion, we make clear the methodology used in consideration of each operation studied throughout this chapter. Our hope is to obtain sufficient bounds for the critical and cocritical degrees after some manipulation is made to an R-complex. Hence, given a chain endomorphism realizing the critical and cocritical degrees on the original complex, our main technique is to then take note of the endomorphism induced by the operation on this complex. If we are able to say what the critical and cocritical degrees *relative* to this induced endomorphism are, then we will have attained a sufficient upper and lower bound, respectively. That is, if u and v are the endomorphisms which realize the critical and cocritical degrees on the original complex, then we will examine what the critical and cocritical degrees degrees of the induced endomorphisms $\hat{u}$ and $\hat{v}$ on the new complex are. This will then provide an upper-bound for the critical degree and lower-bound for the cocritical degree of the new complex, as depicted in the diagram below.



The shortcoming of this method is the reliance upon understanding the induced endomorphism on the *minimal* subcomplex after the operation. While some of the operations explored throughout this chapter preserve minimality, not all do; in particular, the mapping cone of a chain map is almost never minimal and it is impossible to give any generalized form for the minimal subcomplex of a mapping cone. Under these circumstances, such methodology as outlined above is not conducive and, for this reason, we will look towards using the *cohomological characterization* of critical and cocritical degrees. Here, the drawback is that we only have this characterization of critical degree when R is a complete intersection ring, or at minimum when $\text{CI-dim}_R \operatorname{Im} \partial_0^{\mathrm{C}} < \infty$. However, this still covers a wide class of R-complexes and so such a drawback is rather minimal.

5.1 Basic Operations of Complexes

Throughout this section, we build operations of R-complexes in $\mathbf{K}_{\text{tac}}(R)$ via its categorical structure, starting with the condition that the dual of an R-complex is again in the category. Then, we will explore the operations guaranteed by the additive and triangulated structure.

5.1.1 Dualizing

Given a complex $\mathrm{C} \in \mathbf{K}_{\text{tac}}(R)$, its dual $\mathrm{C}^* = \mathrm{Hom}_R(\mathrm{C}, R)$ is also an object in $\mathbf{K}_{\text{tac}}(R)$ as indicated by the definition of a totally acyclic complex, and thus the process of applying $\mathrm{Hom}_R(-, R)$ to any object in $\mathbf{K}_{\text{tac}}(R)$ can be viewed as a basic operation within the category. We can informally refer to this operation as "dualizing" and, in this case, we will see that the critical and cocritical degrees of an R-complex C completely determine the critical and cocritical degrees of its dual, as described below.

We begin with making explicit the form C^* takes on, given a totally acyclic complex C. If (C_n, ∂_n^C) is a family of projective R-modules and R-module maps, then $(C_n^*, \partial_n^{C^*})$ is an R-complex with

$$C_n^* = \mathrm{Hom}_R(C_{-n}, R)$$

$$\partial_n^{C^*} = \mathrm{Hom}_R(\partial_{1-n}^C, R).$$

If, furthermore $\mathrm{H}(C_n) = 0$ for all $n \in \mathbb{Z}$, then $\mathrm{H}(C_n^*) = 0$ for all n as well, in the case that R is Gorenstein. If R is not Gorenstein, then we only consider R-complexes for which $\mathrm{H}(\mathrm{C}^*) = 0$ by definition of $\mathbf{K}_{\text{tac}}(R)$. Moreover, any degree q chain endomorphism on C will induce a degree q chain endomorphism on C^* with

$u_n^* = \mathrm{Hom}_R(u_{-n-q}, R)$. Furthermore, note that when C_n is free, we have the following relationships:

$$C_n^* \cong C_{-n}$$

$$\partial_n^{C^*} = (\partial_{1-n}^C)^T$$

$$u_n^* = (u_{-n+q})^T$$

where the R-module maps are given by matrices and $(-)^T$ represents the transpose. Therefore, it should be clear that if $\partial_{n-q}^C u_n = u_{n-1}\partial_n^C$ then

$$(\partial_{n-q}^C u_n)^T = (u_{n-1}\partial_n^C)^T \implies (u_n)^T(\partial_{n-q}^C)^T = (\partial_n^C)^T(u_{n-1})^T$$

which yields the equality $u_{-n+q}^*\partial_{-n+1+q}^{C^*} = \partial_{-n+1}^{C^*}u_{-n+1+q}^*$ for all $n \in \mathbb{Z}$. Now we present the following lemma, which will be needed to prove the appropriate bounds for the critical and cocritical degrees of C*.

Lemma 5.1. *Let $f\colon R^n \to R^m$ be a map between free R-modules. Then f is (split) surjective if and only if f^T is split injective.*

Proof. Consider the split short exact sequence $0 \to \ker(f) \hookrightarrow R^n \overset{f}{\twoheadrightarrow} R^m \to 0$ and note that applying the additive functor $\mathrm{Hom}_R(-, R)$ yields the split exact sequence

$$0 \to \mathrm{Hom}(R^m, R) \xrightarrow{\mathrm{Hom}_R(f,R)=f^T} \mathrm{Hom}(R^n, R) \to \mathrm{Hom}(\ker(f), R) \to 0$$

thus proving that $f^T : R^m \to R^n$ is a split injection. Likewise, if we first assume $f^T : R^m \to R^n$ is split injective, then the split exact sequence $0 \to R^m \overset{f^T}{\hookrightarrow} R^n \to \mathrm{coker}(f^T) \to 0$ induces the split exact sequence

$$0 \to \mathrm{Hom}(\mathrm{coker}(f^T), R) \to \mathrm{Hom}(R^n, R) \xrightarrow{(f^T)^T} \mathrm{Hom}(R^m, R) \to 0$$

and since $(f^T)^T = f$, it holds that $f\colon R^n \to R^m$ is (split) surjective if and only if $f^T\colon R^m \to R^n$ is split injective. $\square$

Corollary 5.2. *Let $g\colon R^n \to R^m$ be a map between free R-modules. Then g is split injective if and only if g^T is (split) surjective.*

Proof. Apply the previous lemma with $g = (f^T)$ and $g^T = (f^T)^T = f$. $\square$

Remark. *Note that the contrapositives of both statements given above also hold:*

$$f^T \text{ is } \textbf{not} \text{ (split) injective} \iff f \text{ is } \textbf{not} \text{ surjective}$$

$$f^T \text{ is } \textbf{not} \text{ surjective} \iff f \text{ is } \textbf{not} \text{ (split) injective}$$

Now suppose for $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$, $u\colon \mathrm{C} \to \Sigma^q\mathrm{C}$ is the chain endomorphism which realizes $\mathrm{crdeg_R C} = s$ and $v : \mathrm{C} \to \Sigma^r\mathrm{C}$ is the chain endomorphism which realizes $\mathrm{cocrdeg_R C} = t$.

Proposition 5.3. *Suppose $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$, as described above, and denote $\mathrm{C}^* = \mathrm{Hom}_R(C, R)$. Then $\mathrm{crdeg_R C^*} \leq t$ and $\mathrm{cocrdeg_R C^*} \geq q - s$.*

Proof. First, apply the previous corollary above to see that $v^*_{n+r} = (v_{-n})^T$ is surjective whenever v_{-n} is split injective, which occurs for all $-n < t$ or all $n > t$. Furthermore, note that since $\mathrm{cocrdeg^v_R C} = t$, v_{-t} cannot be split injective and so v^*_{t+r} cannot be surjective. Hence, $\mathrm{crdeg^{v^*}_R C^*} = t \geq \mathrm{crdeg_R C^*}$. Now apply the previous lemma to see that $u^*_n = (u_{-n+q})^T$ is split injective whenever u_{-n+q} is surjective, which occurs for all $-n + q > s$ or all $n < q - s$. Moreover, since $\mathrm{crdeg^u_R C} = s$, it must hold that u_{s+q} is not surjective, meaning u^*_t cannot be split injective and thus $\mathrm{cocrdeg^{u^*}_R C^*} = q - s \leq \mathrm{cocrdeg_R C^*}$. $\square$

Corollary 5.4. *The critical and cocritical degrees of C^* are completely determined by that of C. That is, $\mathrm{crdeg_R C^*} = -\mathrm{cocrdeg_R C}$ and $\mathrm{cocrdeg_R C^*} = q - \mathrm{crdeg_R C}$ (where q is as above).*

Proof. There is a one-to-one correspondence between endomorphisms on C and endomorphisms on C^*, leading to such a correspondence between surjections on C_n

and (split) injections on C_n^* as well as (split) injections on C_n and surjections on C_n^*, given by the prior lemma. □

In the above case, we see that due to the relationship between a complex C and its dual C*, we can say *explicitly* what the critical and cocritical degrees of one will be in terms of the other. However, as we consider other operations, this is not always the case; under these circumstances, we only gain sufficient bounds on the critical and cocritical degrees.

5.1.2 Translations

We now consider the operation imposed on totally acyclic complexes via the translation endofunctor, which is guaranteed by the triangulated nature of $\mathbf{K}_{\text{tac}}(R)$. As discussed previously in Chapter 3, the translation functor, or shift, is an additive automorphism on a category, usually denoted as Σ or $[-]$. Recall that for any object $\mathrm{C} \in \mathbf{K}_{\text{tac}}(R)$, $\Sigma^n\mathrm{C}$ denotes the R-complex with R-modules $(\Sigma^n\mathrm{C})_n = C_0$ and differentials $\partial_n^{\Sigma^n\mathrm{C}} = (-1)^n\partial_0^{\mathrm{C}}$. It is quite easy to understand how the critical and cocritical degrees change under the suspension endofunctor, as described in the next proposition.

Proposition 5.5. *If* $\mathrm{crdeg_R C} = s$ *and* $\mathrm{cocrdeg_R C} = t$*, then* $\mathrm{crdeg_R \Sigma^n C} = s + n$ *and* $\mathrm{cocrdeg_R \Sigma^n C} = t + n$*.*

Proof. We begin with assuming C is minimal, since it is clear that the minimal subcomplex of any complex would coincide with a shift of itself under the translation functor. Furthermore note that there is a one-to-one correspondence between endomorphisms on C and those on $\Sigma^n\mathrm{C}$. Therefore, if $\mathrm{u}\colon \mathrm{C} \to \Sigma^q\mathrm{C}$ is the endomorphism which realizes the critical degree on C, then $\Sigma^n\mathrm{u}\colon \Sigma^n\mathrm{C} \to \Sigma^{n+q}\mathrm{C}$ will be the endomorphism that realizes the critical degree on $\Sigma^n\mathrm{C}$. And since $(\Sigma^n\mathrm{u})_i = u_{i-n}\colon C_{i-n} \to C_{i-n-q}$

note that $u_{i+q}\colon C_{i+q} \to C_i$ surjective for all $i > s$ implies $(\Sigma^n \mathrm{u})_{i+q}\colon C_{i+q-n} \to C_{i-n}$ will be surjective for all $i > n+s$. Moreover, since s is the least degree such that u_{i+q} is surjective for all $i > s$, $s+n$ will be the least degree such that $(\Sigma^n \mathrm{u}_{i+q})$ is surjective for all $i > s+n$. Hence, $\mathrm{crdeg_R}\Sigma^{\mathrm{n}}\mathrm{C} = s+n$. Likewise, if $\mathrm{v}\colon \mathrm{C} \to \Sigma^r\mathrm{C}$ is the endomorphism which realizes the cocritical degree on C, then $\Sigma^n\mathrm{v}\colon \Sigma^n\mathrm{C} \to \Sigma^{n+r}\mathrm{C}$ will be such that it realizes the cocritical degree on $\Sigma^n\mathrm{C}$. Therefore, $(\Sigma^n\mathrm{v})_i$ will be split injective for all $i < t+n$ and $\mathrm{cocrdeg_R}\Sigma^{\mathrm{n}}\mathrm{C} = t+n$. □

5.1.3 Direct Sums

Given that $\mathbf{K}_{\mathrm{tac}}(R)$ is a triangulated category, the underlying category is *additive*; therefore, given two complexes C, D $\in \mathbf{K}_{\mathrm{tac}}(R)$ the direct sum $\mathrm{C} \oplus \mathrm{D}$ is also a totally acyclic complex. Conversely, if a sum of complexes $\mathrm{C} \oplus \mathrm{D} \in \mathbf{K}_{\mathrm{tac}}(R)$ then each summand C and D is a totally acyclic complex as well, since $\mathbf{K}_{\mathrm{tac}}(R)$ is a *thick* subcategory of the homotopy category, $\mathbb{K}(\mathrm{R})$.

When we consider the *sum* of two complexes, it is quite easy to see how the critical and cocritical degrees are affected. However, it is necessary to make particular assumptions in consideration of this operation. Let C, D $\in \mathbf{K}_{\mathrm{tac}}(R)$ where $\mathrm{C} \oplus \mathrm{D}$ is the R-complex with R-modules

$$(\mathrm{C} \oplus \mathrm{D})_n = C_n \oplus D_n$$

and R-module homomorphisms

$$\partial_n^{\mathrm{C}\oplus\mathrm{D}} = \begin{pmatrix} \partial_n^{\mathrm{C}} & 0 \\ 0 & \partial_n^{\mathrm{D}} \end{pmatrix}.$$

It should be clear that this is in fact a totally acyclic R-complex since

$$\mathrm{H}_n(\mathrm{C}\oplus\mathrm{D}) = \frac{\ker\begin{pmatrix}\partial_n^{\mathrm{C}} & 0\\ 0 & \partial_n^{\mathrm{D}}\end{pmatrix}}{\mathrm{im}\begin{pmatrix}\partial_{n-1}^{\mathrm{C}} & 0\\ 0 & \partial_{n-1}^{\mathrm{D}}\end{pmatrix}} = \begin{pmatrix}\frac{\ker(\partial_n^{\mathrm{C}})}{\mathrm{im}(\partial_{n-1}^{\mathrm{C}})} & 0\\ 0 & \frac{\ker(\partial_n^{\mathrm{D}})}{\mathrm{im}(\partial_{n-1}^{\mathrm{D}})}\end{pmatrix} = 0 = \mathrm{H}_n((\mathrm{C}\oplus\mathrm{D})^*).$$

Furthermore, suppose that $\mathrm{C} = \overline{\mathrm{C}}\oplus\mathrm{T}'$ and $\mathrm{D} = \overline{\mathrm{D}}\oplus\mathrm{T}''$ where $\overline{\mathrm{C}}$, $\overline{\mathrm{D}}$ are the respective minimal subcomplexes of C, D (with $\mathrm{T}'\simeq 0\simeq\mathrm{T}''$). Then we have that

$$\mathrm{C}\oplus\mathrm{D} = (\overline{\mathrm{C}}\oplus\mathrm{T}')\oplus(\overline{\mathrm{D}}\oplus\mathrm{T}'') = (\overline{\mathrm{C}}\oplus\overline{\mathrm{D}})\oplus(\mathrm{T}'\oplus\mathrm{T}'')$$

where $\overline{\mathrm{C}}\oplus\overline{\mathrm{D}} = \overline{\mathrm{C}\oplus\mathrm{D}}$ is the minimal subcomplex of $\mathrm{C}\oplus\mathrm{D}$ since

$$\mathrm{im}(\partial_n^{\overline{\mathrm{C}}\oplus\overline{\mathrm{D}}}) = \begin{pmatrix}\mathrm{im}(\partial_n^{\overline{\mathrm{C}}}) & 0\\ 0 & \mathrm{im}(\partial_n^{\overline{\mathrm{D}}})\end{pmatrix} \subseteq \begin{pmatrix}\mathfrak{m}(\overline{C}_{n-1}) & 0\\ 0 & \mathfrak{m}(\overline{D}_{n-1})\end{pmatrix} \subseteq \mathfrak{m}(\overline{\mathrm{C}}\oplus\overline{\mathrm{D}})_{n-1}$$

and similarly, $\mathrm{im}(\partial_n^{\mathrm{T}'\oplus\mathrm{T}''}) \nsubseteq \mathfrak{m}(\mathrm{T}'\oplus\mathrm{T}'')_{n-1}$. Meaning that $\oplus$ preserves minimality and, therefore, we may assume for simplicity that C, D are minimal complexes for the remainder of this section.

Now suppose that $\mathrm{crdeg_R C} = s_1$ is realized by a degree q_μ endomorphism $\mu\colon \mathrm{C}\to\Sigma^{q_\mu}\mathrm{C}$ and $\mathrm{crdeg_R D} = s_2$ is realized by a degree q_υ endomorphism $\upsilon\colon\mathrm{D}\to\Sigma^{q_\upsilon}\mathrm{D}$. Without loss of generality, assume $q_\mu\ge q_\upsilon$ and set $m = q_\mu - q_\upsilon$. In this case, we explore the critical degree of $\mathrm{C}\oplus\Sigma^{\mathrm{m}}\mathrm{D}$, noting that if $q_\upsilon > q_\mu$ we would have a similar statement for $\Sigma^{\mathrm{m}'}\mathrm{C}\oplus\mathrm{D}$ (with $m' = -m$).

Proposition 5.6. *Suppose* $\mathrm{crdeg_R C}$, $\mathrm{crdeg_R D}$, μ, υ *and* m *are given as above. Then* $\mathrm{crdeg_R}(\mathrm{C}\oplus\Sigma^{\mathrm{m}}\mathrm{D}) \le \max(s_1, s_2)$.

Proof. Define an endomorphism $\mu \oplus \upsilon \colon \mathrm{C} \oplus \Sigma^{\mathrm{m}}\mathrm{D} \to \Sigma^{q_\mu}(\mathrm{C} \oplus \Sigma^{\mathrm{m}}\mathrm{D})$ where the n^{th} degree maps are given as

$$(\mu \oplus \upsilon)_n = \begin{pmatrix} \mu & 0 \\ 0 & \Sigma^m \upsilon \end{pmatrix}$$

and note that this is an R-complex chain endomorphism since

$$\partial_n^{\Sigma^{q_\mu}(\mathrm{C}\oplus\Sigma^{\mathrm{m}}\mathrm{D})}(\mu \oplus \upsilon)_n = \begin{pmatrix} \partial^{\mathrm{C}}_{n-q_\mu} & 0 \\ 0 & \partial^{\Sigma^m\mathrm{D}}_{n-q_\mu} \end{pmatrix} \begin{pmatrix} \mu_n & 0 \\ 0 & (\Sigma^m\upsilon)_n \end{pmatrix} = \begin{pmatrix} \partial^{\mathrm{C}}_{n-q_\mu}\mu_n & 0 \\ 0 & \partial^{\Sigma^m\mathrm{D}}_{n-q_\mu}(\Sigma^m\upsilon)_n \end{pmatrix}$$

$$= \begin{pmatrix} \mu_{n-1}\partial^{\mathrm{C}}_n & 0 \\ 0 & (\Sigma^m\upsilon)_{n-1}\partial^{\Sigma^m\mathrm{D}}_n \end{pmatrix} = \begin{pmatrix} \mu_{n-1} & 0 \\ 0 & (\Sigma^m\upsilon)_{n-1} \end{pmatrix} \begin{pmatrix} \partial^{\mathrm{C}}_n & 0 \\ 0 & \partial^{\Sigma^m\mathrm{D}}_n \end{pmatrix} = (\mu \oplus \upsilon)_{n-1}\partial_n^{(\mathrm{C}\oplus\Sigma^{\mathrm{m}}\mathrm{D})}.$$

Moreover, it is well defined since each of μ and $\Sigma^m\upsilon$ are well defined in $\mathbf{K}_{\mathrm{tac}}(R)$. Lastly, it should be clear that $(\mu \oplus \Sigma^{\mathrm{m}}\upsilon)_n$ is surjective if and only if both μ_n and $(\Sigma^m\upsilon)_n$ are surjective. Therefore, $(\mu \oplus \upsilon)_n$ will be surjective for all $n > \max(s_1, s_2)$ and $\mathrm{crdeg}_{\mathrm{R}}^{\mu\oplus\upsilon}(\mathrm{C} \oplus \Sigma^{\mathrm{m}}\mathrm{D}) = \max(s_1, s_2) \geq \mathrm{crdeg}_{\mathrm{R}}(\mathrm{C} \oplus \Sigma^{\mathrm{m}}\mathrm{D})$. □

Now assume $\mathrm{cocrdeg}_{\mathrm{R}}\mathrm{C} = t_1$ is realized by a degree q_ν endomorphism $\nu \colon \mathrm{C} \to \Sigma^{q_\nu}\mathrm{C}$ and $\mathrm{cocrdeg}_{\mathrm{R}}\mathrm{D} = t_2$ is realized by a degree q_ω endomorphism $\omega \colon \mathrm{D} \to \Sigma^{q_\omega}\mathrm{D}$. Again, suppose $q_\nu \geq q_\omega$ and set $m = q_\nu - q_\omega$ (not necessarily same as above). In this case, we explore the cocritical degree of $\mathrm{C} \oplus \Sigma^{\mathrm{m}}\mathrm{D}$, noting that if $q_\omega > q_\nu$ we would have a similar statement for $\Sigma^{\mathrm{m}'}\mathrm{C} \oplus \mathrm{D}$ (with $m' = -m$).

Proposition 5.7. *Suppose* $\mathrm{cocrdeg}_{\mathrm{R}}\mathrm{C}$*,* $\mathrm{cocrdeg}_{\mathrm{R}}\mathrm{D}$*,* ν*,* ω *and* m *are given as above. Then* $\mathrm{cocrdeg}_{\mathrm{R}}(\mathrm{C} \oplus \Sigma^{\mathrm{m}}\mathrm{D}) \geq \min(t_1, t_2)$.

Proof. We define the endomorphism $\nu \oplus \omega \colon \mathrm{C} \oplus \Sigma^{\mathrm{m}}\mathrm{D} \to \Sigma^{q_\nu}(\mathrm{C} \oplus \Sigma^{\mathrm{m}}\mathrm{D})$ as in the previous proof, so that the n^{th} degree maps are given as

$$(\nu \oplus \omega)_n = \begin{pmatrix} \nu & 0 \\ 0 & \Sigma^m \omega \end{pmatrix}$$

and it has already been justified that this will be a well-defined endomorphism. Once again, it should be clear that $(\nu \oplus \Sigma^{\mathrm{m}}\omega)_n$ is (split) injective if and only if both ν_n

and $(\Sigma^m\omega)_n$ are (split) injective. Therefore, $(\nu \oplus \omega)_n$ will be (split) injective for all $n < \min(t_1, t_2)$ and $\mathrm{cocrdeg}_{\mathrm{R}}^{\nu\oplus\omega}(\mathrm{C} \oplus \Sigma^{\mathrm{m}}\mathrm{D}) = \min(t_1, t_2) \leq \mathrm{cocrdeg}_{\mathrm{R}}(\mathrm{C} \oplus \Sigma^{\mathrm{m}}\mathrm{D})$. □

Corollary 5.8. *If* $\deg(\mu) = \deg(\upsilon)$*, then* $\mathrm{crdeg}_{\mathrm{R}}\mathrm{C} \oplus \mathrm{D} \leq \max(s_1, s_2)$*. Likewise, if* $\deg(\nu) = \deg(\omega)$*, then* $\mathrm{cocrdeg}_{\mathrm{R}}\mathrm{C} \oplus \mathrm{D} \geq \min(s_1, s_2)$*.*

Remark. *We can also consider the result with a slightly different assumption: given* s_1*, if there exists some* $\upsilon' : \mathrm{D} \to \Sigma^q\mathrm{D}$ *such that* $\mathrm{crdeg}_{\mathrm{R}}^{\upsilon'}\mathrm{D} < \infty$ *(not necessarily* s_2*), then we can use* $\max(s_1, \mathrm{crdeg}_{\mathrm{R}}^{\upsilon'}\mathrm{D})$ *as the bound. Similarly for the cocritical degree. However, this result will be a bit weaker and there is no guarantee that such an endomorphism exists.*

5.1.4 Retracts

Now, we consider the question centered upon taking summands, otherwise known as retracts. First suppose that $\mathrm{E} \in \mathbf{K}_{\mathrm{tac}}(R)$ such that it can actually be written as a direct sum of R-complexes, say $\mathrm{E} = \mathrm{C} \oplus \mathrm{D}$. Then, in this case, each of C and D must be totally acyclic as well. Now suppose that we know the critical and cocritical degrees of E, say $\mathrm{crdeg}_{\mathrm{R}}\mathrm{E} = s$ and $\mathrm{cocrdeg}_{\mathrm{R}}\mathrm{E} = t$, then can we say anything about $\mathrm{crdeg}_{\mathrm{R}}\mathrm{C}$ and $\mathrm{crdeg}_{\mathrm{R}}\mathrm{D}$? To simplify the problem, let us assume that E is *minimal* so that $\partial(\mathrm{C} \oplus \mathrm{D}) \subseteq \mathfrak{m}(\mathrm{C} \oplus \mathrm{D})$. Thus, the subcomplexes C and D must be minimal as well since

$$\partial_n^{\mathrm{E}} = \begin{pmatrix} \partial_n^{\mathrm{C}} & 0 \\ 0 & \partial_n^{\mathrm{D}} \end{pmatrix}$$

for each $n \in \mathbb{Z}$. However, if we consider an endomorphism $\mu\colon \mathrm{E} \to \Sigma^q\mathrm{E}$ then note that we get *four* induced maps where

$$\mu_1\colon \mathrm{C} \to \Sigma^q\mathrm{C}$$

$$\mu_2\colon \mathrm{D} \to \Sigma^q\mathrm{C}$$

$$\mu_3 \colon \mathrm{C} \to \Sigma^q \mathrm{D}$$

$$\mu_4 \colon \mathrm{D} \to \Sigma^q \mathrm{D}.$$

If $\mu_{n+q} : C_{n+q} \oplus D_{n+q} \to C_n \oplus D_n$ is surjective for all $n > s$, then note that the surjectivity onto one summand, take C_n for example, could not just be gained from $\mu_{1,(n+q)}$ alone; meaning that the map $\mu_{2,(n+q)}$ may contribute in part to the surjectivity. The same could occur with injectivity for $n < t$. In either case, it becomes difficult to use our previous approach under the operation of "taking summands".

So now, assume R is a complete intersection of the form $Q/(\mathbf{f})$, with $\mathbf{f} = f_1, \dots, f_c$ a regular Q-sequence and $\chi_1, \dots, \chi_c$ the cohomological operators associated to E. Furthermore, denote $M \oplus N = \operatorname{Im} \partial_0^{\mathrm{E}}$ where $M = \mathrm{im}\partial_0^{\mathrm{C}}$ and $N = \mathrm{im}\partial_0^{\mathrm{D}}$. By the results presented in §4.4 of this thesis, the maximal degree of a nonzero element $(\mathfrak{x}, \mathfrak{z}) \in \widehat{\mathrm{Ext}}_R^*(M \oplus N, \Bbbk) = \hat{E}$ such that $(\mathfrak{x}, \mathfrak{z}) \in (0 :_{\hat{E}} \mathfrak{X})$ is s, so that $\mathfrak{x} \in \widehat{\mathrm{Ext}}_R^s(M, \Bbbk)$ and $\mathfrak{z} \in \widehat{\mathrm{Ext}}_R^s(N, \Bbbk)$ (with at least one nonzero). For contradiction's sake, suppose $\mathrm{crdeg_R C} = s' \gneq s$ so that there exists some nonzero element $\mathfrak{x}' \in \widehat{\mathrm{Ext}}_R^{s'}(M, \Bbbk) \subset \widehat{\mathrm{Ext}}_R^{>s}(M, \Bbbk)$ such that $\mathfrak{x}'\mathfrak{X} = 0$. But then note that this would imply the element $(\mathfrak{x}', 0) \in \widehat{\mathrm{Ext}}_R^{s'}(M \oplus N, \Bbbk)$ is annihilated by $\mathfrak{X}$ thus contradicting s as the highest degree socle element. The same argument can be applied to $\widehat{\mathrm{Ext}}_R^*(N, \Bbbk)$, so that we see both $\mathrm{crdeg_R C}$ and $\mathrm{crdeg_R D}$ must be *bounded above* by s.

Lastly, it should be easy to see that *at least one* of $\mathrm{crdeg_R C}$ or $\mathrm{crdeg_R D}$ must be *equal to* s. If we suppose that both $\mathrm{crdeg_R C}$, $\mathrm{crdeg_R D} \lneq s$ so that the highest degree nonzero element $\mathfrak{x}' \in \mathrm{Soc}(\widehat{\mathrm{Ext}}_R^*(M, \Bbbk))$ has degree s' and the highest degree nonzero element $\mathfrak{z}' \in \mathrm{Soc}(\widehat{\mathrm{Ext}}_R^*(N, \Bbbk))$ has degree s''. Then note that $(\mathfrak{x}, \mathfrak{z}) \notin \mathrm{Soc}(\widehat{\mathrm{Ext}}_R^*(M \oplus N, \Bbbk))$ since otherwise either $\mathfrak{x} \in \mathrm{Soc}(\widehat{\mathrm{Ext}}_R^*(M, \Bbbk))$ or $\mathfrak{z} \in \mathrm{Soc}(\widehat{\mathrm{Ext}}_R^*(N, \Bbbk))$ by our previous argument. But $\deg(\mathfrak{x}) = s \gneq s' = \deg(\mathfrak{x}')$ and $\deg(\mathfrak{z}) = s \gneq s'' = \deg(\mathfrak{z}')$, contradicting the assumption that both s' and s'' are strictly less than s. We are now able to present the following conclusion to this argument, along with the appropriate dual notion.

Proposition 5.9. *Let R be a complete intersection of the form $Q/(\mathbf{f})$, with $\mathbf{f} = f_1, \ldots, f_c$ a regular Q-sequence, and further suppose* $\mathrm{C} \oplus \mathrm{D} \in \mathbf{K}_{\mathrm{tac}}(R)$. *If* $\mathrm{crdeg}_{\mathrm{R}}\mathrm{C} \oplus \mathrm{D} = s$, *then* $\mathrm{crdeg}_{\mathrm{R}}\mathrm{C} \leq s$ *and* $\mathrm{crdeg}_{\mathrm{R}}\mathrm{D} \leq s$, *with at least one being an equality.*

Proposition 5.10. *Let R be a complete intersection of the form $Q/(\mathbf{f})$, with $\mathbf{f} = f_1, \ldots, f_c$ a regular Q-sequence, and further suppose* $\mathrm{C} \oplus \mathrm{D} \in \mathbf{K}_{\mathrm{tac}}(R)$. *If* $\mathrm{cocrdeg}_{\mathrm{R}}\mathrm{C} \oplus \mathrm{D} = t$, *then* $\mathrm{cocrdeg}_{\mathrm{R}}\mathrm{C} \geq t$ *and* $\mathrm{cocrdeg}_{\mathrm{R}}\mathrm{D} \geq t$, *with at least one being an equality.*

Proof. For ease of discussion, denote $E(M \oplus N) = \widehat{\mathrm{Ext}}^*_R(M \oplus N, \Bbbk)$, $E(M) = \widehat{\mathrm{Ext}}^*_R(M, \Bbbk)$, and $E(N) = \widehat{\mathrm{Ext}}^*_R(N, \Bbbk)$ (noting that $E(M \oplus N) \cong E(M) \oplus E(N)$). The argument for Proposition 5.10 is completely analogous to that given for Proposition 5.9. First, there must exist a lowest degree nonzero element $(\bar{\mathfrak{x}}, \bar{\mathfrak{z}}) \in \mathrm{Cosoc}(E(M \oplus N))$ with $\mathfrak{x} \in E^t(M)$ and $\mathfrak{z} \in E^t(N)$ (at least one nonzero). That is, there exists some nonzero element of degree t such that $(\mathfrak{x}, \mathfrak{z}) \notin \mathfrak{X}E^{\leq t}(M \oplus N)$. Suppose that $\mathrm{crdeg}_{\mathrm{R}}\mathrm{C} = t' \lneq t$ so that there exists some $0 \neq \mathfrak{x}' \notin \mathfrak{X}E^{\leq t'}(M)$, implying $(\mathfrak{x}', 0) \notin \mathfrak{X}E^{\leq t'}(M \oplus N)$ and contradicting the assumption that t is the lowest degree of such an element. The same argument can be applied to $\mathrm{crdeg}_{\mathrm{R}}\mathrm{D}$, so we see that the cocritical degrees of C and D are bounded below by t. On the other hand, since there exists some nonzero element of degree t such that $(\mathfrak{x}, \mathfrak{z}) \notin \mathfrak{X}E^{\leq t}(M \oplus N)$, either $0 \neq \mathfrak{x} \notin \mathfrak{X}E^{\leq t}(M)$ or $0 \neq \mathfrak{z} \notin \mathfrak{X}E^{\leq t}(N)$, demonstrating that we must have equality of at least one of the cocritical degrees. □

5.1.4.1 A Retract to Sums

In Section 5.1.3, our conclusions using the original methodology were only valid under certain circumstances; for example, when $\mathrm{crdeg}_{\mathrm{R}}\mathrm{C}$ and $\mathrm{crdeg}_{\mathrm{R}}\mathrm{D}$ are realized by an endomorphism of the same degree. Note first that due to the result presented in Section 4.4.3, when R is a complete intersection we know there exists a linear form of CI operators that will have both positively finite and negatively finite relative critical and cocritical degrees, respectively. For this reason, we know there always exists a degree -2 endomorphism in this scenario which we can use to give an upper bound for critical degree and a lower bound for cocritical degree of the individual complexes, as well as their sum. However, the question still arises if we can use the technique described in the last section to perfect our statement.

Let R be a complete intersection and note that we use the same notation as the previous section where $M = \operatorname{Im} \partial_0^{\mathrm{C}}$ and $N = \operatorname{Im} \partial_0^{\mathrm{D}}$, so that $M \oplus N = \operatorname{Im} \partial_0^{\mathrm{C} \oplus \mathrm{D}}$. Furthermore, denote $s_1 = \mathrm{crdeg_R C}$, $s_2 = \mathrm{crdeg_R D}$, and $s = \mathrm{crdeg_R C \oplus D}$. Then if we assume $0 \neq \mathfrak{x} \in \operatorname{Soc}(\widehat{\operatorname{Ext}}^*_R(M, \Bbbk))$ such that $\deg(\mathfrak{x}) = s_1$ and $0 \neq \mathfrak{z} \in \operatorname{Soc}(\widehat{\operatorname{Ext}}^*_R(N, \Bbbk))$ such that $\deg(\mathfrak{z}) = s_2$ (i.e. $\mathfrak{x}$ is highest degree nonzero socle element, etc.), note that $(\mathfrak{x}, 0) \in \widehat{\operatorname{Ext}}^{s_1}_R(M \oplus N, \Bbbk)$ and $(0, \mathfrak{z}) \in \widehat{\operatorname{Ext}}^{s_2}_R(M \oplus N, \Bbbk)$ must both be annihilated by $\mathfrak{X}$. Hence, $s \geq \max\{s_1, s_2\}$. Now note that there exists some $0 \neq (\mathfrak{x}', \mathfrak{z}') \in \operatorname{Soc}(\widehat{\operatorname{Ext}}^*_R(M\oplus, \Bbbk))$ with $\mathfrak{x}' \in \widehat{\operatorname{Ext}}^s_R(M, \Bbbk)$ and $\mathfrak{z}' \in \widehat{\operatorname{Ext}}^s_R(N, \Bbbk)$ such that $(\mathfrak{x}', \mathfrak{z}') \in (0 :_{\hat{E}} \mathfrak{X})$. Thus, either $\mathfrak{x}'$ or $\mathfrak{z}'$ must be nonzero and annihilated by χ; that is, $s = \max\{s_1, s_2\}$.

Proposition 5.11. *Let R be a complete intersection of the form $Q/(\mathbf{f})$, with $\mathbf{f} = f_1, \ldots, f_c$ a regular Q-sequence. Further suppose $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$ and $\mathrm{D} \in \mathbf{K}_{\mathrm{tac}}(R)$, so that $\mathrm{C} \oplus \mathrm{D} \in \mathbf{K}_{\mathrm{tac}}(R)$. If $\mathrm{crdeg_R C} = s_1$ and $\mathrm{crdeg_R D} = s_2$, then $\mathrm{crdeg_R (C \oplus D)} = \max\{s_1, s_2\}$.*

And, of course, we have the following analogous statement for cocritical degree.

Proposition 5.12. *Let R be a complete intersection of the form $Q/(\mathbf{f})$, with $\mathbf{f} = f_1, \ldots, f_c$ a regular Q-sequence. Further suppose $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$ and $\mathrm{D} \in \mathbf{K}_{\mathrm{tac}}(R)$, so that $\mathrm{C} \oplus \mathrm{D} \in \mathbf{K}_{\mathrm{tac}}(R)$. If $\mathrm{cocrdeg}_{\mathrm{R}}\mathrm{C} = t_1$ and $\mathrm{cocrdeg}_{\mathrm{R}}\mathrm{D} = t_2$, then*

$$\mathrm{cocrdeg}_{\mathrm{R}}(\mathrm{C} \oplus \mathrm{D}) = \min\{t_1, t_2\}.$$

Proof. Using the same notation from the proof of Proposition 5.10, first note that if there exist some nonzero $\mathfrak{x} \notin \mathfrak{X}E^{\leq t_1}(M)$ and $\mathfrak{z} \notin \mathfrak{X}E^{\leq t_2}(M)$, then $0 \neq (\mathfrak{x}, 0) \notin \mathfrak{X}E^{\leq t_1}(M \oplus N)$ and $0 \neq (0, \mathfrak{z}) \notin \mathfrak{X}E^{\leq t_2}(N)$. Hence, $t \leq t_1$ and $t \leq t_2$. However, note that the existence of a nonzero element $(\mathfrak{x}', \mathfrak{z}') \notin \chi E^{\leq t}(M \oplus N)$ implies that either $0 \neq \mathfrak{x}' \notin \chi E^{\leq t}(M)$ or $0 \neq \mathfrak{z}' \notin \chi E^{\leq t}(N)$, thereby proving equality of $t = \min\{t_1, t_2\}$. □

Remark. *Given the arguments for these Propositions, along with Propositions 5.9 and 5.10, notice that we did not deal with the case of infinite critical or cocritical degrees. Recall that whenever R is a complete intersection the critical degree of any R-complex (and R-module) will be positively finite and, likewise, the cocritical degree will always be negatively finite. If at least one of* C *or* D *is periodic, then the given statements (and arguments) still apply.*

5.1.5 Cones

The triangulated structure of $\mathbf{K}_{\mathrm{tac}}(R)$ guarantees us one more operation: mapping cones. As discussed in Chapter 3, these constructs serve a significant role within the category, helping to define the class of distinguished triangles within $\mathbf{K}_{\mathrm{tac}}(R)$ thus making the category triangulated. Recall that if $\mathrm{f} : \mathrm{C} \to \mathrm{D}$ is a morphism between complexes, then the *mapping cone* $\mathrm{M}(\mathrm{f})$ is a complex in $\mathbf{K}_{\mathrm{tac}}(R)$ with R-modules and R-module homomorphisms defined (informally) as

$$\mathrm{M}(\mathrm{f})_n = \mathrm{C}_{\mathrm{n}-1} \oplus \mathrm{D}_{\mathrm{n}} \text{ and } \partial_n^{\mathrm{M}(\mathrm{f})} = \begin{pmatrix} -\partial_{n-1}^C & 0 \\ f_{n-1} & \partial_n^D \end{pmatrix}.$$

Lamentably, it turns out to be more difficult to say anything about how the critical and cocritical degrees change under this operation. The reason for this is that taking the mapping cone of a chain map almost never preserves minimality, thus invalidating the methodology presented at the beginning of this chapter. However, there is some hope, in the way of using the cohomological characterizations for the critical and cocritical degrees. Although, if this approach proves useful it is certainly not as straightforward as the approach for direct sums and taking summands portrayed in the previous section. For now, we leave the reader with only a conjecture.

Conjecture 1. *Let* $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$ *and let* $\mathrm{f} : \mathrm{C} \to \Sigma^j \mathrm{C}$ *be a chain endomorphism which does not realize* $\mathrm{crdeg_R C} = s$*. Then* $\mathrm{crdeg_R M}\,(\mathrm{f}) \leq s+1$*.*

5.2 Operations with Perfect Complexes

In this section, we will explore manipulations of complexes in $\mathbf{K}_{\mathrm{tac}}(R)$ which result from the Hom and $\otimes$ functors. Recall that R is a local ring and, for the remainder of this chapter, it should additionally be assumed that R is Gorenstein. Now first, let us recollect the definitions of Hom and $\otimes$ with regard to complexes, as we will use them. The tensor product of R-complexes C and D is the R-complex $\mathrm{C} \otimes \mathrm{D}$ with R-modules

$$(\mathrm{C} \otimes_R \mathrm{D})_n = \bigoplus_{i+j=n} (C_i \otimes_R D_j)$$

and differentials ∂_n where $\partial_n(c \otimes d) = \partial_i^C(c) \otimes d + (-1)^i a \otimes \partial_j^D(d)$ for $c \in C_i$ and $d \in D_j$. Note that the complex $\mathrm{C} \otimes \mathrm{D}$ is the total complex of the bicomplex of C and D. Similarly, the Hom of C and D is the R-complex $\mathrm{Hom}(C, D)$ with R-modules

$$\mathrm{Hom}(C, D)_n = \bigoplus_i \mathrm{Hom}(C_i, D_{i+n}) = \bigoplus_j \mathrm{Hom}(C_{j-n}, D_j)$$

and any element in $\mathrm{Hom}(C,D)_n$ can be viewed as a family of chain maps where $\mathrm{f}^i : C_i \to D_{i+n} = (\Sigma^{-n}\mathrm{D})_i$ (*or* $\mathrm{f}^j : C_{j-n} = (\Sigma^n\mathrm{C})_j \to D_j$). Here, the differentials are given by $\partial_n(\mathrm{f}^i) = (-1)^n f_{i-1}\partial_i^C + \partial_{i+n+1}^D f_{i+1}$ on each degree.

The issue that arises with these operations is that, in both cases, the resulting R-modules are *not* finitely generated. In particular, if C, $D \in \mathbf{K}_{\mathrm{tac}}(R)$ then each C_i and D_j will be nonzero (otherwise they would be 0 in the category) and then there is no possibility of these complexes having finitely generated modules. Unsurprisingly, the notions of critical and cocritical degrees are a bit moot in the case of infinitely generated modules. Henceforth, in order to discuss interesting operations on objects in $\mathbf{K}_{\mathrm{tac}}(R)$ which yield complexes of *finitely generated* modules, we turn towards *perfect* (or *bounded*) complexes.

5.2.1 Perfect Complexes, A Thick Subcategory of $\mathscr{D}(R)$

Let R be a commutative ring. A perfect complex can, more or less, be thought of as a bounded complex. If $\mathscr{D}(R)$ represents the derived category of R-modules, then:

Definition. A *perfect complex* of R-modules is an object in $\mathscr{D}(R)$ that is quasi-isomorphic to a bounded complex of finite projective R-modules. An R-module is a *perfect module* if it is perfect when it is viewed as a complex concentrated in degree zero.

If R is noetherian, then an R-module is perfect if and only if its projective dimension is finite. Recall that two complexes are *quasi-isomorphic* if there exists a chain map $f\colon \mathrm{C} \to \mathrm{D}$ that induces an isomorphism of homology groups, $H_n(\mathrm{C}) \xrightarrow{\cong} H_n(\mathrm{D})$. Note that this implies if P is perfect, then for $n \gg 0$ and $n \ll 0$, $H_n(\mathrm{P}) = 0$. Of course, in $\mathscr{D}(R)$ the quasi-isomorphisms are isomorphisms so perfect complexes can just be thought of as bounded complexes, since by definition they are equivalent in the category $\mathscr{D}(R)$.

The purpose of considering perfect complexes is that when we consider applying Hom(-, P) or $-\otimes$ P with P perfect, then the resulting complex *will* have finitely generated R-modules since P is equivalent to a bounded complex. To make these two operations rigorous, one must define bifunctors $\mathscr{F}, \mathscr{G}\colon \mathbf{K}_{\mathrm{tac}}(R) \times \mathscr{P}(R) \to \mathbf{K}_{\mathrm{tac}}(R)$ and $\mathscr{F}', \mathscr{G}'\colon \mathscr{P}(R) \times \mathbf{K}_{\mathrm{tac}}(R) \to \mathbf{K}_{\mathrm{tac}}(R)$, where $\mathscr{P}(R)$ is the thick subcategory of perfect complexes in $\mathscr{D}(R)$. Our goal will be to understand what happens to the critical and cocritical degrees of C $\in \mathbf{K}_{\mathrm{tac}}(R)$ under these functors. However, to simplify this exploration, we will reduce to endofunctors with *fixed* bounded complexes, and it should then be clear how the results extend when one instead considers bifunctors.

Definition. An R-complex $\mathscr{B}$ is called *bounded above* if $B_n = 0$ for $n \gg 0$ and is called *bounded below* if $B_n = 0$ for $n \ll 0$. If $\mathscr{B}$ is bounded above and below, $\mathscr{B}$ is called *bounded*.

Let $\mathscr{P}(R)$ denote the category with objects as perfect complexes and morphisms the homotopy equivalence classes of chain maps between them. The category of perfect complexes $\mathscr{P}(R)$ is a thick subcategory of the derived category $\mathscr{D}(R)$, meaning that it is closed under shifts, triangles, and retracts. Since $P \in \mathscr{P}(R)$ is defined such that $P \cong \mathscr{B}$ for some bounded complex $\mathscr{B}$, for any fixed P we can instead consider the fixed bounded complex $\mathscr{B}$ guaranteed by the isomorphism. In the subsequent sections, we define two different endofunctors, $-\otimes \mathscr{B} : \mathbf{K}_{\mathrm{tac}}(R) \to \mathbf{K}_{\mathrm{tac}}(R)$ and $\mathrm{Hom}(-, \mathscr{B}) : \mathbf{K}_{\mathrm{tac}}(R) \to \mathbf{K}_{\mathrm{tac}}(R)$, where $\mathscr{B} \in \mathscr{P}(R)$ is a fixed bounded complex.

5.2.2 Tensor with a Perfect Complex

Let $\mathcal{B}$ be a bounded complex of projective modules such that $B_i = 0$ for all $i < 0$ and $i > \ell$.[1] Then we may define endofunctors $- \otimes \mathcal{B} : \mathbf{K}_{\text{tac}}(R) \to \mathbf{K}_{\text{tac}}(R)$ and $\mathcal{B} \otimes - : \mathbf{K}_{\text{tac}}(R) \to \mathbf{K}_{\text{tac}}(R)$. In order to justify that these functors make sense, we first must show that for any $\mathrm{C} \in \mathbf{K}_{\text{tac}}(R)$, $\mathrm{C} \otimes \mathcal{B}$ and $\mathcal{B} \otimes \mathrm{C}$ will again be totally acyclic complexes. Note first that the n^{th} R-module will have form either

$$(\mathrm{C} \otimes_R \mathcal{B})_n = \bigoplus_{j=0}^{\ell} (C_{n-j} \otimes_R B_j)$$

with differentials $\partial_n(c \otimes b) = \partial^C_{n-j}(c) \otimes b + (-1)^{n-j} c \otimes \partial^{\mathcal{B}}_j(b)$ where $c \in C_{n-j}$ and $b \in B_j$ or

$$(\mathcal{B} \otimes_R \mathrm{C})_n = \bigoplus_{i=0}^{\ell} (B_i \otimes_R C_{n-i})$$

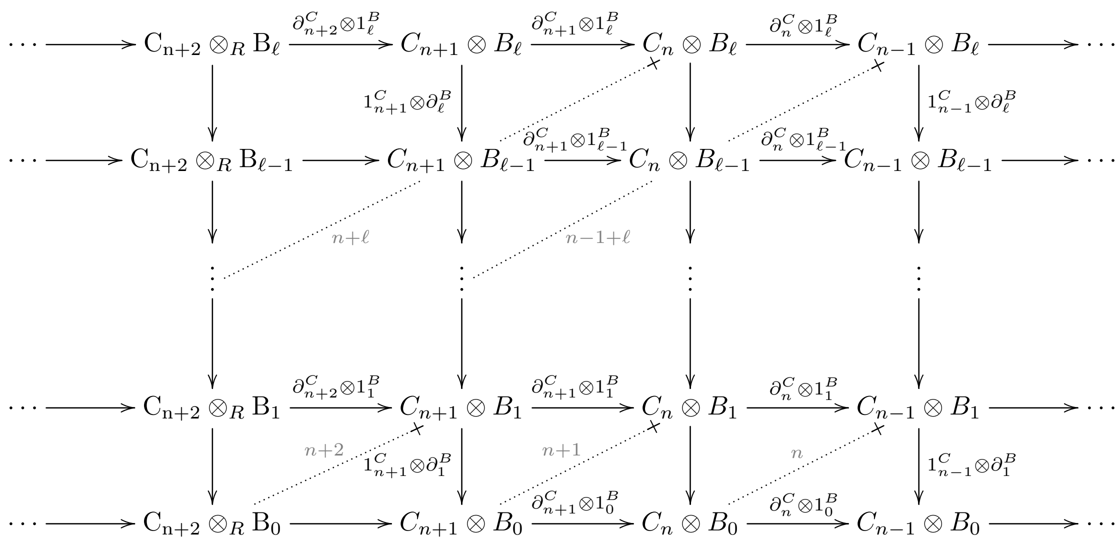
with differentials $\partial_n(b \otimes c) = \partial^{\mathcal{B}}_i(b) \otimes c + (-1)^i b \otimes \partial^C_{n-i}(c)$ where $c \in C_{n-i}$ and $b \in B_i$. It should be clear from the definition of the differentials that a map on the tensor product of R-modules written $f \otimes g$ acts as $(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$ on pure tensors, and then extend by linearity. We will show that $- \otimes \mathcal{B} : \mathbf{K}_{\text{tac}}(R) \to \mathbf{K}_{\text{tac}}(R)$ is a functor, and note that the arguments for $\mathcal{B} \otimes - : \mathbf{K}_{\text{tac}}(R) \to \mathbf{K}_{\text{tac}}(R)$ will be the same, except for Proposition 5.14 where an additional argument will be given.

Proposition 5.13. *For any* $\mathrm{C} \in \mathbf{K}_{\text{tac}}(R)$*, the* R*-complexes* $\mathrm{C} \otimes \mathcal{B}$ *and* $(\mathrm{C} \otimes \mathcal{B})^* = \mathrm{Hom}_R(\mathrm{C} \otimes \mathcal{B}, R)$ *are both acyclic.*

Proof. We will prove that $\mathrm{C} \otimes \mathcal{B}$ is acyclic by induction on the length of $\mathcal{B}$. First, when $\ell = 0$ each R-module in $\mathrm{C} \otimes \mathcal{B}$ has the form $(\mathrm{C} \otimes \mathcal{B})_n = C_n \otimes_R B_0$ which is equivalent to the n^{th} R-module in the complex $\mathrm{C} \otimes_R B_0$ and the differential $\partial^{\mathrm{C} \otimes \mathcal{B}} = \partial^{\mathrm{C}} \otimes B_0$. Since the funtor $- \otimes_R B_0$ is an exact functor on R-modules, we have exactness for each $n \in \mathbb{Z}$. Now, we apply the inductive step: assume that the complex B has length $\ell + 1$ and suppose that the tensor complex $\mathrm{C} \otimes_R \mathrm{B}'$ is exact, where B' is the complex

[1] Note that we can write any bounded complex in this form, after an appropriate shift.

with R-modules $B'_i = B_i$ and differentials $\partial_i^{B'} = \partial_i^{B}$ for $i = 0, \dots \ell - 1$. Now consider the following diagram representing the nonzero portion of the bicomplex formed from the bigraded module $M_{(n-i),i} = \mathrm{C_{n-i}} \otimes_R \mathrm{B_i}$ and defining $\partial'_{(n-i),i} = \partial^{\mathrm{C}}_{n-i} \otimes 1^{\mathcal{B}}_i$, $\partial''_{(n-i),i} = (-1)^{n-i} 1^{\mathrm{C}}_{n-i} \otimes \partial^{\mathcal{B}}_i$ as the bidegree maps:



Then, the total complex of M, $\mathrm{Tot}(M)$ is formed by taking direct sums along the diagonals. Note that for each diagonal, there will be exactly $\ell + 1$ summands at each degree and this direct sum coincides with the n^{th} module of the complex $\mathrm{C} \otimes_R \mathrm{B}$. Consider any element z in the kernel of $\partial_n^{\mathrm{C} \otimes \mathrm{B}}$ and note that we may represent z as $\sum_{k=0}^{\ell} z_k$ where each z_k has some part in $\ker(\partial^C_{n-k} \otimes 1^{\mathcal{B}}_k)$ as well as some part in $\ker((-1)^{n-k} 1^{\mathrm{C}}_{n-k} \otimes \partial^{\mathcal{B}}_k)$. Now observe that, in particular, $z_\ell \in \ker(\partial^C_{n-\ell} \otimes 1^{\mathcal{B}}_\ell) = \mathrm{im}(\partial^C_{n+1-\ell} \otimes 1^{\mathcal{B}}_\ell)$ by exactness of the top row. Hence $z_\ell = (\partial^C_{n+1-\ell} \otimes 1^{\mathcal{B}}_\ell)(y)$ for some $y \in C_{n+1-\ell} \otimes B_\ell$ and accordingly we may write $\sum_{k=0}^{\ell-1} z_k = \sum_{k=0}^{\ell} z_k - (\partial^C_{n+1-\ell} \otimes 1^{\mathcal{B}}_\ell)(y)$. Now apply the inductive hypothesis with the fact that each $z_k \in \ker(\partial^C_{n-k} \otimes 1^{\mathcal{B}}_k)$ to see that $\sum_{k=0}^{\ell-1} z_k \in \ker(\partial_n^{\mathrm{C} \otimes \mathcal{B}'}) = \mathrm{im}(\partial_{n+1}^{\mathrm{C} \otimes \mathcal{B}'})$. Meaning, there exists some $\sum_{k=0}^{\ell-1} x_k \in (\mathrm{C} \otimes$

$\mathcal{B}')_{n+1} = \bigoplus_{i=0}^{\ell-1}(C_{n+1-i} \otimes_R B_i)$ such that $\partial_{n+1}^{\mathrm{C}\otimes\mathcal{B}'}(\sum_{k=0}^{\ell-1} x_k) = \sum_{k=0}^{\ell-1} z_k$ and consequently, we may write

$$\sum_{k=0}^{\ell} z_k = (\partial_{n+1-\ell}^{C} \otimes 1_\ell^B)(y) + (\partial_{n+1}^{\mathrm{C}\otimes\mathcal{B}'})(\sum_{k=0}^{\ell-1} x_k)$$

which is clearly in $\mathrm{im}(\partial_{n+1}^{\mathrm{C}\otimes\mathcal{B}})$. Therefore, $\mathrm{C} \otimes \mathcal{B}$ is acyclic. Furthermore, since R is Gorenstein, $\mathrm{Im}(\partial_n^{\mathrm{C}\otimes\mathcal{B}})$ is totally reflexive for each $n \in \mathbb{Z}$ and hence $\mathrm{H}_n(\mathrm{Hom}_R(\mathrm{C} \otimes \mathcal{B}, R)) = 0$ (by Lemmas 2.4-5 in [AvMa]). And so it follows that $\mathrm{C} \otimes \mathcal{B} \in \mathbf{K}_{\mathrm{tac}}(R)$. $\square$

We now want to understand how $- \otimes \mathcal{B}$ affects morphisms in $\mathbf{K}_{\mathrm{tac}}(R)$ and justify that this action is well-define; meaning, the action preserves equivalencies in the category. We consider the natural choice: for any $\mathrm{f} \in \mathbf{K}_{\mathrm{tac}}(R)$, define $\mathrm{f} \otimes \mathcal{B}$ as the R-complex chain map where $(\mathrm{f} \otimes \mathcal{B})_n$ can be represented by a diagonal matrix with the nonzero entries as $f_{n-i} \otimes 1_i^{\mathcal{B}}$ for $i = 0, \dots, \ell$.

Proposition 5.14. *Given a chain map* $\mathrm{f} : \mathrm{C} \to \mathrm{D}$ *in* $\mathbf{K}_{\mathrm{tac}}(R)$, $\mathrm{f} \otimes \mathcal{B}$ *is a well-defined chain map from* $\mathrm{C} \otimes \mathcal{B}$ *to* $\mathrm{D} \otimes \mathcal{B}$.

Proof. Let f and $\mathrm{f} \otimes \mathcal{B}$ be defined as given above. We will show that for any $\hat{x} \in (\mathrm{C} \otimes \mathcal{B})_n$ the equality $(\mathrm{f} \otimes \mathcal{B})_{n-1}\partial_n^{\mathrm{C}\otimes\mathrm{B}}(\hat{x}) = \partial_n^{\mathrm{D}\otimes\mathrm{B}}(\mathrm{f} \otimes \mathcal{B})_n(\hat{x})$ holds. That is, we wish to show that the following square commutes:

$$\begin{array}{ccccc}
\bigoplus_{j=0}^{\ell}(\mathrm{C_{n-j}} \otimes_R \mathrm{B_j}) & & \xrightarrow{\quad \partial_n^{\mathrm{C}\otimes\mathcal{B}} \quad} & & \bigoplus_{j=0}^{\ell}(\mathrm{C_{n-1-j}} \otimes_R \mathrm{B_j}) \\
\Big\downarrow (\mathbf{f}\otimes\mathcal{B})_n & c_{n-j} \otimes b_j & \longmapsto & \partial_n^{\mathrm{C}\otimes\mathcal{B}}(c_{n-j} \otimes b_j) & \Big\downarrow (\mathbf{f}\otimes\mathcal{B})_{n-1} \\
 & \big\downarrow & & \big\downarrow & \\
 & f_{n-j}(c_{n-j}) \otimes b_j & \longmapsto & ? & \\
\bigoplus_{j=0}^{\ell}(\mathrm{D_{n-j}} \otimes_R \mathrm{B_j}) & & \xrightarrow[\quad \partial_n^{\mathrm{D}\otimes\mathcal{B}} \quad]{} & & \bigoplus_{j=0}^{\ell}(\mathrm{D_{n-1-j}} \otimes_R \mathrm{B_j})
\end{array}$$

where $\hat{x} = \sum_{j=0}^{\ell}(c_{n-j} \otimes b_j)$ with $c_in - j \in C_{n-j}$ and $b_j \in B_j$ for each $j = 0, \dots, \ell$. Specifically, we only need show commutativity on each summand $c_{n-j} \otimes b_j$:

$$\partial_n^{\mathrm{D}\otimes_R\mathrm{B}}(f_{n-j}(c_{n-j}) \otimes b_j) = \partial_{n-j}^{D} f_{n-j}(c_{n-j}) \otimes b_j + (-1)^{n-j} f_{n-j}(c_{n-j}) \otimes \partial_j^{B}(b_j)$$

$$= f_{n-j-1}\partial^C_{n-j}(c_{n-j}) \otimes b_j + f_{n-j}((-1)^{n-j}c_{n-j}) \otimes \partial^B_j(b_j)$$

$$= (\mathrm{f} \otimes \mathcal{B})_{n-1}[\partial^C_{n-j}(c_{n-j}) \otimes b_j + (-1)^{n-j}c_{n-j} \otimes \partial^B_j(b_j)]$$

$$= (\mathrm{f} \otimes \mathcal{B})_{n-1}(\partial^{\mathrm{C}\otimes_R\mathrm{B}}_n(x \otimes y))$$

by commutativity of the differentials with f and by definition of $\mathrm{f} \otimes \mathcal{B}$. Thus, the square commutes for all $\hat{x} \in (\mathrm{C} \otimes \mathcal{B})_n$ and $\mathrm{f} \otimes \mathcal{B}$ is an R-complex chain map. Now, to show that this map is well defined in $\mathbf{K}_{\mathrm{tac}}(R)$, we will show that if $\mathrm{f} \sim \mathrm{g}$ then $\mathrm{f} \otimes \mathcal{B} \sim \mathrm{g} \otimes \mathcal{B}$.

If $\mathrm{f} \sim \mathrm{g}$ then there exist homotopy maps $k_n \colon C_n \to D_{n+1}$ and $f_n - g_n = k_{n-1}\partial^C_n + \partial^D_{n+1}k_n$ for each $n \in \mathbb{Z}$. Define a family of R-module maps $h_n \colon (\mathrm{C}\otimes_R\mathrm{B})_n \to (\mathrm{D}\otimes_R\mathrm{B})_{n+1}$ where each map can be represented by the diagonal matrix $h_n = [k_{n-j} \otimes 1^B_j]$ where $j = 0, \dots, \ell$ and $n \in \mathbb{Z}$. Then note that for any $c \otimes b$ with $c \in C_{n-j}$ and $b \in B_j$ it holds that

$$h_{n-1}\partial^{\mathrm{C}\otimes\mathcal{B}}_n(c \otimes b) + \partial^{\mathrm{D}\otimes\mathcal{B}}_{n+1}h_n(c \otimes b) = [k_{n-j-1} \otimes 1^B_j](\partial^C_{n-j}(c) \otimes b + (-1)^{n-j}c \otimes \partial^B_j(b))$$

$$+\partial^{\mathrm{D}\otimes\mathcal{B}}_{n+1}([k_{n-j} \otimes 1^B_j](c \otimes b))$$

$$= (k_{n-j-1}\partial^C_{n-j}(c) \otimes b + k_{n-j}((-1)^{n-j}c) \otimes \partial^B_j(b))$$

$$+(\partial^D_{n-j+1}k_{n-j}(c) \otimes b + (-1)^{n-j+1}k_{n-j}(c) \otimes \partial^B_j(b))$$

$$= k_{n-j-1}\partial^C_{n-j}(c) \otimes b + (-\mathbf{1})^{\mathbf{n-j}}k_{n-j}(c) \otimes \partial^B_j(b)$$

$$+\partial^D_{n-j+1}k_{n-j}(c) \otimes b + (-\mathbf{1})^{\mathbf{n-j+1}}k_{n-j}(c) \otimes \partial^B_j(b)$$

$$= (k_{n-j-1}\partial^C_{n-j}(c) + \partial^D_{n-j+1}k_{n-j}(c)) \otimes b$$

$$= ((f_{n-j} - g_{n-j})(c)) \otimes b$$

$$= ((\mathrm{f} \otimes \mathcal{B})_n - (\mathrm{g} \otimes \mathcal{B})_n)(c \otimes b).$$

Hence, by definition $\mathrm{f} \otimes \mathcal{B} \sim \mathrm{g} \otimes \mathcal{B}$. □

Proposition 5.15. *Given a chain map* $\mathrm{f} : \mathrm{C} \to \mathrm{D}$ *in* $\mathbf{K}_{\mathrm{tac}}(R)$*,* $\mathcal{B} \otimes \mathrm{f}$ *is a well-defined chain map from* $\mathcal{B} \otimes \mathrm{C}$ *to* $\mathcal{B} \otimes \mathrm{D}$*.*

Proof. The proof that $\partial_n^{\mathscr{B}\otimes \mathrm{D}}(\mathscr{B}\otimes \mathrm{f})_n = (\mathscr{B}\otimes \mathrm{f})_{n-1}\partial_n^{\mathscr{B}\otimes \mathrm{C}}$ is almost identical (except for a possible difference in sign) to the argument given for $\mathrm{f}\otimes\mathscr{B}$ in the previous proposition. We focus on the justification that $\mathrm{f}\sim\mathrm{g}$ implies $\mathscr{B}\otimes\mathrm{f}\sim\mathscr{B}\otimes\mathrm{g}$, so let $k_n\colon C_n\to D_{n+1}$ be homotopy maps such that $f_n - g_n = k_{n-1}\partial_n^C + \partial_{n+1}^D k_n$ for each $n\in\mathbb{Z}$. Define a family of R-module maps $h_n\colon (\mathrm{C}\otimes_R \mathrm{B})_n \to (\mathrm{D}\otimes_R\mathrm{B})_{n+1}$ where each map can be represented by the diagonal matrix $h_n = [1_i^{\mathscr{B}}\otimes k_{n-i}]$ where $i=0,\ldots,\ell$ and $n\in\mathbb{Z}$. Then note that for any $b\otimes c$ with $c\in C_{n-i}$ and $b\in B_i$ it holds that

$$\begin{aligned}
h_{n-1}\partial_n^{\mathscr{B}\otimes\mathrm{C}}(b\otimes c)+\partial_{n+1}^{\mathscr{B}\otimes\mathrm{D}}h_n(b\otimes c) &= [1_i^B\otimes k_{n-i-1}](\partial_i^{\mathscr{B}}(b)\otimes c+(-1)^i b\otimes\partial_{n-i}^{\mathrm{C}}(c))\\
&\quad +\partial_{n+1}^{\mathscr{B}\otimes\mathrm{D}}([1_i^B\otimes k_{n-i}](b\otimes c))\\
&= (1_{i-1}^B\otimes k_{n-i})(\partial_i^{\mathscr{B}}(b)\otimes c)+(-1)^i(1_i^B\otimes k_{n-i-1})(b\otimes\partial_{n-i}^{\mathrm{C}}(c))\\
&\quad +\partial_{n+1}^{\mathscr{B}\otimes\mathrm{D}}((-1)^{|k_{n-i}|\,|b|}(b\otimes k_{n-i}(c)))\\
&= ((-1)^{|k_{n-i}|\,|\partial_i^{\mathscr{B}}(b)|}\partial_i^{\mathscr{B}}(b)\otimes k_{n-i}(c)+(-1)^{|k_{n-i-1}|\,|b|+i}b\otimes k_{n-i-1}\partial_{n-i}^{\mathrm{C}}(c))\\
&\quad +(-1)^{|k_{n-i}|\,|b|}(\partial_i^{\mathscr{B}}(b)\otimes k_{n-i}(c)+(-1)^i b\otimes\partial_{n-i+1}^{\mathrm{D}}k_{n-i}(c))\\
&= (-\mathbf{1})^{|\mathbf{k_{n-i}}|\,|\partial_{\mathbf{i}}^{\mathscr{B}}(\mathbf{b})|}\partial_i^{\mathscr{B}}(b)\otimes k_{n-i}(c)+(-1)^{|k_{n-i-1}|\,|b|+i}b\otimes k_{n-i-1}\partial_{n-i}^{\mathrm{C}}(c)\\
&\quad +(-\mathbf{1})^{|\mathbf{k_{n-i}}|\,|\mathbf{b}|}\partial_i^{\mathscr{B}}(b)\otimes k_{n-i}(c)+(-1)^{|k_{n-i}|\,|b|+i}b\otimes\partial_{n-i+1}^{\mathrm{D}}k_{n-i}(c)\\
&= (-1)^{|b|+i}b\otimes k_{n-i-1}\partial_{n-i}^{\mathrm{C}}(c)+(-1)^{|b|+i}b\otimes\partial_{n-i+1}^{\mathrm{D}}k_{n-i}(c)\\
&= (-1)^{|b|+i}(b\otimes(k_{n-i-1}\partial_{n-i}^{C}(c)+\partial_{n-i+1}^{D}k_{n-i}(c)))\\
&= (-1)^{|b|+i}(b\otimes((f_{n-i}-g_{n-i})(c)))\\
&= ((\mathscr{B}\otimes\mathrm{f})_n-(\mathscr{B}\otimes\mathrm{g})_n)(b\otimes c).
\end{aligned}$$

Hence, by definition $\mathscr{B}\otimes\mathrm{f}\sim\mathscr{B}\otimes\mathrm{g}$.[2] □

Lastly, we will verify that $-\otimes\mathscr{B}$ does indeed preserve the identity morphism on an R-complex and compositions of morphisms, making $-\otimes\mathscr{B}$ into an endofunctor. Given the composition $\mathrm{C}\xrightarrow{\mathrm{f}}\mathrm{D}\xrightarrow{\mathrm{g}}\mathrm{E}$, it is easy to see that

$$((\mathrm{g}\otimes\mathscr{B})_n)((\mathrm{f}\otimes\mathscr{B})_n)(c\otimes b) = ((\mathrm{g}\otimes\mathscr{B})_n)(f_{n-j}(c)\otimes b)$$

[2] Note that if $|\,b\,| = 2j$ (for some $j\in\mathbb{N}$) then $|\,\partial_i^{\mathscr{B}}(b)\,| = 2j-1$ since $\partial^{\mathscr{B}}$ is a -1 degree map. Furthermore, note that $|\,k_{n-i-1}\,| = 1 = |\,k_{n-i}\,|$. We are applying Koszul's Law of Duality here; for the proof of Proposition 5.13, we did not include this step since $\deg(1^{\mathscr{B}})=0$.

$$= g_{n-j}f_{n-j}(c) \otimes b = (\mathrm{gf} \otimes \mathcal{B})_n(c \otimes b)$$

where $c \in C_{n-j}$ and $b \in B_j$ for each $j = 0, \ldots, \ell$. It is even simpler to see that for each $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$, $1^{\mathrm{C}} \otimes \mathcal{B} = 1^{\mathrm{C}\otimes\mathcal{B}}$ since $(1^{\mathrm{C}} \otimes \mathcal{B})(c \otimes b) = 1^{\mathrm{C}}_{n-j}(c) \otimes 1^{\mathcal{B}}_j(b) = c \otimes b$ for any $c \in C_{n-j}$ and $b \in B_j$. Therefore, $- \otimes \mathcal{B}$ is a well-defined endofunctor on $\mathbf{K}_{\mathrm{tac}}(R)$ so this is a meaningful operation on totally acyclic complexes. Moreover, we may find the following additional structure useful:

Proposition 5.16. *The endofunctor* $- \otimes \mathcal{B} : \mathbf{K}_{\mathrm{tac}}(R) \to \mathbf{K}_{\mathrm{tac}}(R)$ *is a* triangle *functor.*

Proof. To justify this last claim, we need to show that $-\otimes\mathcal{B}$ is additive (preserves the zero map and direct sums) as well as triangulated (preserves distinguished triangles). It should be obvious that $0 \otimes \mathcal{B} = 0$ and that for any null-homotopic map f, the map $\mathrm{f} \otimes \mathcal{B}$ will be null-homotopic too (by the previous proposition). Next, it is easy to see that direct sums are preserved by the bilinearity of the tensor product on R-modules:

$$((\mathrm{C} \oplus \mathrm{D}) \otimes \mathcal{B})_n = \bigoplus_{i+j=n} (C_i \oplus D_i) \otimes_R B_j = \bigoplus_{i+j=n} ((C_i \otimes_R B_j) \oplus (D_i \otimes_R B_j))$$

$$= (\bigoplus_{i+j=n} (C_i \otimes_R B_j)) \oplus (\bigoplus_{i+j=n} (D_i \otimes_R B_j)) = (\mathrm{C} \otimes \mathrm{B})_n \oplus (\mathrm{D} \otimes \mathrm{B})_n$$

for $j = 0, \ldots, \ell$ and each $n \in \mathbb{Z}$. Now, we only need to show the triangulated structure for $- \otimes \mathcal{B}$.

Consider any distinguished triangle in $\mathbf{K}_{\mathrm{tac}}(R)$ and note that it will have the form:

$$\mathrm{C} \xrightarrow{\mathrm{f}} \mathrm{D} \xrightarrow{\iota} \mathrm{M}(\mathrm{f}) \xrightarrow{\pi} \Sigma\mathrm{C}$$

where $(\mathrm{M}(\mathrm{f}))_n = (\Sigma\mathrm{C} \oplus \mathrm{D})_n = C_{n-1} \oplus D_n$ is the mapping cone of $\mathbf{f}$. We will show that the triangle

$$\mathrm{C} \otimes \mathcal{B} \xrightarrow{\mathrm{f}\otimes\mathcal{B}} \mathrm{D} \otimes \mathcal{B} \xrightarrow{\iota\otimes\mathcal{B}} \mathrm{M}(\mathrm{f}) \otimes \mathcal{B} \xrightarrow{\pi\otimes\mathcal{B}} \Sigma\mathrm{C} \otimes \mathcal{B}$$

is distinguished as well. First note that the induced map $\mathrm{C} \otimes \mathcal{B} \xrightarrow{\mathrm{f} \otimes \mathcal{B}} \mathrm{D} \otimes \mathcal{B}$ yields a distinguished triangle of the form

$$\mathrm{C} \otimes \mathcal{B} \xrightarrow{\mathrm{f} \otimes \mathcal{B}} \mathrm{D} \otimes \mathcal{B} \xrightarrow{\widehat{\iota}} \mathrm{M}\,(\mathrm{f} \otimes \mathcal{B}) \xrightarrow{\widehat{\pi}} \Sigma \mathrm{C} \otimes \mathcal{B}$$

and so if we show the former triangle is isomorphic to this one, we are done. Consider the diagram of triangles

$$\begin{array}{ccccccc} \mathrm{C} \otimes \mathcal{B} & \xrightarrow{\mathrm{f} \otimes \mathcal{B}} & \mathrm{D} \otimes \mathcal{B} & \xrightarrow{\widehat{\iota}} & \mathrm{M}\,(\mathrm{f} \otimes \mathcal{B}) & \xrightarrow{\widehat{\pi}} & \Sigma(\mathrm{C} \otimes \mathcal{B}) \\ \downarrow 1^{\mathrm{C} \otimes \mathcal{B}} & & \downarrow 1^{\mathrm{D} \otimes \mathcal{B}} & & \downarrow \cong & & \downarrow \cong \\ \mathrm{C} \otimes \mathcal{B} & \xrightarrow{\mathrm{f} \otimes \mathcal{B}} & \mathrm{D} \otimes \mathcal{B} & \xrightarrow{\iota \otimes \mathcal{B}} & \mathrm{M}\,(\mathrm{f}) \otimes \mathcal{B} & \xrightarrow{\pi \otimes \mathcal{B}} & \Sigma \mathrm{C} \otimes \mathcal{B} \end{array}$$

where it should be clear that the first square commutes and since the first two downward maps are the identities, they are isomorphisms. We obtain the final two necessary isomorphisms via properties of tensor products. First note that $\Sigma(\mathrm{C} \otimes \mathrm{D}) \cong \Sigma \mathrm{C} \otimes \mathrm{D}$ for any R-complexes. Then, $\mathrm{M}\,(\mathrm{f} \otimes \mathcal{B})_n = (\Sigma(\mathrm{C} \otimes \mathcal{B}))_n \oplus ((\mathrm{D} \otimes \mathcal{B}))_n \cong (\Sigma \mathrm{C} \otimes \mathrm{B})_n \oplus (\mathrm{D} \otimes \mathrm{B})_n = (\Sigma \mathrm{C} \oplus \mathrm{D})_{\mathrm{n}} \otimes \mathrm{B}_{\mathrm{n}} = (\mathrm{M}\,(\mathrm{f}))_{\mathrm{n}} \otimes \mathrm{B}_{\mathrm{n}}$ as R-modules for each $n \in \mathbb{Z}$. Additionally, it should be obvious that the differentials $\partial^{\mathrm{M}\,(\mathrm{f} \otimes \mathcal{B})}$ and $\partial^{\mathrm{M}\,(\mathrm{f}) \otimes \mathcal{B}}$ act on each n^{th}-degree R-module in the same manner. And, clearly the first square commutes, so we only need check that the latter two commute as well. Since $\bar{\iota}$ is simply the inclusion of $\mathrm{D} \otimes_R \mathrm{B}$ into $\mathrm{M}\,(\mathrm{f} \otimes 1^{\mathrm{B}})$ it acts the same as $\iota \otimes 1^B$, so the square commutes. Similarly for $\bar{\pi}$ and the last square. Therefore, the second triangle is isomorphic to a distinguished triangle, and is thus distinguished itself. □

Now that it has been justified that $- \otimes \mathcal{B}$ is a triangle endofunctor on $\mathbf{K}_{\text{tac}}(R)$, we will examine how the critical and cocritical degrees of a complex might change under this functor. Suppose $\mathrm{crdeg}_{\mathrm{R}}\mathrm{C} = s < \infty$ and $\mathrm{cocrdeg}_{\mathrm{R}}\mathrm{C} = t > -\infty$ such that $\mu : \mathrm{C} \to \Sigma^q \mathrm{C}$ is a chain endomorphism which realizes the critical degree and $\nu : \mathrm{C} \to \Sigma^r \mathrm{C}$ is a chain endomorphism which realizes the cocritical degree. Furthermore, we will impose the additional condition that $\mathcal{B}$ is minimal for the remainder of this section. We begin with a simple lemma that is needed to discuss the injectivity or surjectivity at each homological degree of the R-complex $\mathrm{C} \otimes \mathscr{B}$.

Lemma 5.17. *Let M, M', N be free R-modules and $f : M \to M'$ an R-module homomorphism. Then the map $f \otimes \mathrm{id}^N$ is surjective (split injective) if and only if f is surjective (split injective).*

Proof. Any map f from M to M' yields an exact sequence $M \xrightarrow{f} M' \to \mathrm{coker}(f) \to 0$ and applying the functor $- \otimes_R \mathrm{N}$ preserves exactness of the sequence

$$M \otimes_R N \xrightarrow{f \otimes \mathrm{id}^N} M' \otimes_R N \to \mathrm{coker}(f) \otimes_R N \to 0$$

since $- \otimes_R \mathrm{N}$ exact whenever N is flat[3]. But surjectivity of f means that $\mathrm{coker}(f) = 0$, which implies $\mathrm{coker}(f) \otimes_R N = 0$ (recall N is free) and so consequently $f \otimes 1^N$ is surjective by exactness of the latter sequence given above. We obtain the "only if" direction from the fact that $\frac{M' \otimes_R N}{\mathrm{im}(f \otimes 1^N)} \cong \frac{M' \otimes_R N}{\mathrm{im}(f) \otimes 1^N}$ meaning that $f \otimes 1^N$ surjective implies $\mathrm{im}(f) \otimes 1^N = M' \otimes_R N$.

Now consider the exact sequence $0 \to \ker(f) \hookrightarrow M \xrightarrow{f} M'$ and apply the functor $- \otimes_R N$ to obtain the exact sequence:

$$0 \to \ker(f) \otimes_R N \to M \otimes_R N \xrightarrow{f \otimes 1^N} M' \otimes_R N$$

If $\ker(f) = 0$, then $\ker(f) \otimes_R N = 0$ and thus $\ker(f \otimes 1^N) = 0$ by exactness of the sequence. The "only if" direction follows from the fact that the R-modules are free and thus $\ker(f \otimes 1^N) \cong \ker(f) \otimes_R N$. Furthermore, note that if f additionally splits, then there exists some $\epsilon \colon N \to M$ such that $\epsilon f = \mathrm{Id}^M$. If this holds, then consider $\epsilon \otimes 1^N \colon M \otimes_R N \to M' \otimes_R N$ and note $(\epsilon \otimes 1^N)(f \otimes 1^N) = \epsilon f \otimes 1^N = 1^M \otimes 1^N = 1^{M \otimes_R N}$ since $- \otimes \mathrm{N}$ is a well-defined functor on $\mathscr{R}$-$\mathscr{mod}$. And thus, $f \otimes 1^N$ must be split as well. ☐

[3]Recall any free module is flat, and finitely generated flat modules are precisely free modules over a local ring.

Proposition 5.18. *Given the assumptions above,* $\operatorname{crdeg}_{\mathrm{R}}^{\mu\otimes\mathcal{B}}(\mathrm{C}\otimes\mathcal{B}) = s+\ell$ *and* $\operatorname{cocrdeg}_{\mathrm{R}}^{\nu\otimes\mathcal{B}}(\mathrm{C}\otimes\mathcal{B}) = t$*, which yields the upper and lower bounds:* $\operatorname{crdeg}_{\mathrm{R}}(\mathrm{C}\otimes\mathcal{B}) \leq s+\ell$ *and* $\operatorname{cocrdeg}_{\mathrm{R}}(\mathrm{C}\otimes\mathcal{B}) \geq t$.

Proof. Assuming the former part of the statement holds, it is clear by the definition of critical and cocritical degrees that $s+\ell$ and t will be the upper and lower bounds, respectively. First, note from the previous proposition that we obtain the induced maps $\mu\otimes\mathcal{B}\colon \mathrm{C}\otimes\mathcal{B} \to \Sigma^q(\mathrm{C}\otimes\mathcal{B})$ and $\nu\otimes\mathcal{B}\colon \mathrm{C}\otimes\mathcal{B} \to \Sigma^r(\mathrm{C}\otimes\mathcal{B})$, which can be represented as diagonal matrices with nonzero entries $\mu_{n-j}\otimes 1_j^{\mathcal{B}}$ and $\nu_{n-j}\otimes 1_j^{\mathcal{B}}$ for $j=0,\ldots,\ell$. Apply the lemma to see that $(\mu\otimes\mathcal{B})_n$ is surjective for all n such that $n-\ell > s$ and $(\mu\otimes\mathcal{B})_n$ is precisely not surjective whenever $n-\ell \leq s$. Similarly, we can apply the lemma to see that $(\mu\otimes\mathcal{B})_n$ is split injective for all n such that $n<t$ (in which case, $n-j<t$ for $j=1,\ldots,\ell$) and $(\mu\otimes\mathcal{B})_n$ is precisely not injective whenever $n\geq t$. □

5.2.3 Hom with a Perfect Complex

Let $\mathcal{B}$ be a fixed bounded complex of projective modules such that $B_i = 0$ for all $i<0$ and all $i>\ell$. Now consider the complex $\operatorname{Hom}(\mathcal{B},\mathrm{C})$ for any $C\in\mathbf{K}_{\mathrm{tac}}(R)$ with n^{th} module

$$\operatorname{Hom}(\mathcal{B},C)_n = \bigoplus_{i=0}^{\ell}\operatorname{Hom}_R(B_i, C_{i+n})$$

and, additionally, the contravariant functor Hom(-, $\mathcal{B}$) yields a complex with n^{th} module

$$\operatorname{Hom}(C,\mathcal{B})_n = \bigoplus_{j=0}^{\ell}\operatorname{Hom}_R(C_{j-n}, B_j)$$

First, we want to justify that both Hom($\mathcal{B}$, -) and Hom(-, $\mathcal{B}$) are endofunctors on $\mathbf{K}_{\mathrm{tac}}(R)$. A similar argument as given in the proof of Proposition 5.13 can be applied to see that $\operatorname{Hom}(\mathcal{B},\mathrm{C}), \operatorname{Hom}(\mathrm{C},\mathcal{B}) \in \mathbf{K}_{\mathrm{tac}}(R)$ for any $\mathrm{C}\in\mathbf{K}_{\mathrm{tac}}(R)$. However, such an argument can only be given after the observation that $\mathrm{H}_n(\mathrm{Hom}(\mathcal{B},\mathrm{C})) \cong \mathrm{H}_n(\mathrm{Tot}(M))$ and $\mathrm{H}_n(\mathrm{Hom}(\mathrm{C},\mathcal{B})) \cong \mathrm{H}_n(\mathrm{Tot}(M'))$ where M, M' are the bigraded complexes with modules $M_{p,q} = \mathrm{Hom}(B_{-p}, C_q)$ and $M'_{p,q} = \mathrm{Hom}(C_{-p}, B_q)$. The isomorphisms hold precisely because $\mathcal{B}$ is a bounded complex. Rather than presenting such arguments here, we will instead refer to [Ch] (Propositions A.3 and A.6) when proving the following proposition.

Proposition 5.19. *For any* $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$, $\mathrm{Hom}(\mathrm{C},\mathcal{B})$ *and* $\mathrm{Hom}(\mathcal{B},\mathrm{C})$ *are totally acyclic.*

Proof. We will justify the statement for $\mathrm{Hom}(\mathrm{C},\mathcal{B})$ and note that the statement for $\mathrm{Hom}(\mathcal{B},\mathrm{C})$ is identical (if not simpler), with application of [Ch, A.6] in lieu of [Ch, A.3]. The first condition of [Ch, A.3] that must be satisfied is $\mathrm{Hom}_R(\mathrm{C}, B_n)$ is acyclic for each $n \in \mathbb{Z}$. This is evident by the fact that B_n is free, $\mathrm{Hom}(\mathrm{C},\mathrm{R})$ is acyclic, and $\mathrm{Hom}_R(M, R^n) \cong \Pi_{i=1}^n \mathrm{Hom}_R(M,R)$ for any R-module M. The latter condition of [Ch, A.3] holds since $\mathcal{B}$ is bounded and thus the boundary submodules $\mathfrak{B}_{\mathfrak{n}}(\mathcal{B}) = 0$ for all $n > \ell$. Hence, $\mathrm{Hom}(\mathrm{C}, \mathfrak{B}_{\mathfrak{n}}(\mathcal{B}))$ is acyclic trivially for infinitely many integers $n > 0$. Lastly, note that $(\mathrm{Hom}(\mathrm{C},\mathcal{B}))^*$ and $(\mathrm{Hom}(\mathcal{B},\mathrm{C}))^*$ are both acyclic since R is Gorenstein. □

Now, we ask how morphisms are altered under these functors, and look towards ensuring that the proposed actions are well defined. Afterward, we show the conditions that the identity and compositions are respected,thereby making $\mathrm{Hom}(\mathcal{B},\text{-})$ and $\mathrm{Hom}(\text{-},\mathcal{B})$ into endofunctors on $\mathbf{K}_{\mathrm{tac}}(R)$. Let f be an R-complex chain map from C to D. Again, we consider the natural choice for defining $\mathrm{Hom}(\mathcal{B},\mathrm{f})$ and $\mathrm{Hom}(\mathrm{f},\mathcal{B})$. Define $\mathrm{f}_* = \mathrm{Hom}(\mathcal{B},\mathrm{f})$ to be $\bigoplus_{i=0}^{\ell} \mathrm{Hom}(B_i, f_{i+n})$ and $\mathrm{f}^* = \mathrm{Hom}(\mathrm{f},\mathcal{B})$ to be $\oplus_{j=0}^{\ell} \mathrm{Hom}(f_{j-n}, B_j)$. That is, for any $\alpha \in \mathrm{Hom}_R(B_i, C_{i+n})$, $\mathrm{f}_*(\alpha) = f_{i+n}\alpha \in \mathrm{Hom}_R(B_i, D_{i+n})$. Note that by nature of $\mathrm{Hom}(\text{-},\mathcal{B})$, f^* will instead correlate to a map from $\mathrm{Hom}(\mathrm{D},\mathscr{B})$ to $\mathrm{Hom}(\mathrm{C},\mathscr{B})$: for any $\alpha \in \mathrm{Hom}_R(D_{j-n}, B_j)$, $\mathrm{f}^*(\alpha) = \alpha f_{j-n} \in \mathrm{Hom}_R(D_{j-n}, B_j)$. Note further that we may also represent f^* and f_* as diagonal matrices, with $\mathrm{Hom}(f_{j-n}, B_j)$ and $\mathrm{Hom}(B_i, f_{i+n})$ as the appropriate nonzero entries for each map, similar to $\mathrm{f} \otimes \mathscr{B}$ and $\mathscr{B} \otimes \mathrm{f}$.

Proposition 5.20. *Given a chain map* $\mathrm{f} : \mathrm{C} \to \mathrm{D}$ *in* $\mathbf{K}_{\mathrm{tac}}(R)$, $\mathrm{Hom}(\mathscr{B},\mathrm{f})$ *and* $\mathrm{Hom}(\mathrm{f},\mathscr{B})$ *are well-defined chain maps from* $\mathrm{Hom}(\mathscr{B},\mathrm{C})$ *to* $\mathrm{Hom}(\mathscr{B},\mathrm{D})$ *and* $\mathrm{Hom}(\mathrm{D},\mathscr{B})$ *to* $\mathrm{Hom}(\mathrm{C},\mathscr{B})$, *respectively.*

Proof. First we must show that both $\mathrm{Hom}(\mathscr{B},\mathrm{f})$ and $\mathrm{Hom}(\mathrm{f},\mathscr{B})$, as defined above, commute with the differentials on $\mathrm{Hom}(\mathscr{B},\mathrm{C})$ and $\mathrm{Hom}(\mathrm{C},\mathscr{B})$, respectively. For $\mathrm{Hom}(\mathscr{B},\mathrm{f})$, note that this reduces to the task of showing that the following square commutes:

$$\begin{array}{ccccc}
\bigoplus_{i=0}^{\ell} \mathrm{Hom}_R(B_i, C_{i+n}) & & \xrightarrow{\partial_n^{\mathrm{Hom}(\mathscr{B},\mathrm{C})}} & & \bigoplus_{i=0}^{\ell} \mathrm{Hom}_R(B_i, C_{i+n-1}) \\
\Big\downarrow {\scriptstyle (\mathrm{Hom}(\mathbf{f},\mathscr{B}))_n} & \alpha_i & \longmapsto & \partial_n^{\mathrm{Hom}(\mathscr{B},\mathrm{C})}(\alpha_i) & \Big\downarrow {\scriptstyle (\mathrm{Hom}(\mathbf{f},\mathscr{B}))_{n-1}} \\
 & \downarrow & & \downarrow & \\
 & f_{i+n}\alpha_i & \longmapsto & ? & \\
\bigoplus_{i=0}^{\ell} \mathrm{Hom}_R(B_i, D_{i+n}) & & \xrightarrow[\partial_n^{\mathrm{Hom}(\mathscr{B},\mathrm{D})}]{} & & \bigoplus_{i=0}^{\ell} \mathrm{Hom}_R(B_i, D_{i+n-1})
\end{array}$$

To start, note that $\partial_n^{\mathrm{Hom}(\mathscr{B},\mathrm{C})}(\alpha_i) = \partial_{i+n}^{\mathrm{C}} \circ \alpha_i + (-1)^{2i+n}\alpha_i \circ \partial_{i+1}^{\mathscr{B}}$ for any $\alpha_i \in \mathrm{Hom}_R(B_i, C_{i+n})$ with each $i = 0, \dots, \ell$. Therefore, for any such α_i and any $n \in \mathbb{Z}$, we have

$$\begin{aligned}
(\mathrm{f}_{*,n-1} \circ \partial_n^{\mathrm{Hom}(\mathscr{B},\mathrm{C})})(\alpha_i) &= \mathrm{f}_{*,n-1}(\partial_{i+n}^{\mathrm{C}} \circ \alpha_i + (-1)^{2i+n}\alpha_i \circ \partial_{i+1}^{\mathscr{B}}) \\
= f_{i+n-1} \circ (\partial_{i+n}^{\mathrm{C}}\alpha_i) + (-1)^n f_{i+n} \circ (\alpha_i \partial_{i+n}^{\mathscr{B}}) &= (f_{i+n-1}\partial_{i+n}^{\mathrm{C}})\alpha_i + (-1)^n f_{i+n}\alpha_i\partial_{i+n}^{\mathscr{B}} \\
= (\partial_{i+n}^{\mathrm{D}} f_{i+n})\alpha_i + (-1)^n f_{i+n}\alpha_i\partial_{i+n}^{\mathscr{B}} &= \partial_n^{\mathrm{Hom}(\mathscr{B},\mathrm{D})}(f_{i+n}\alpha_i) \\
&= (\partial_n^{\mathrm{Hom}(\mathscr{B},\mathrm{D})} \circ \mathrm{f}_{*,n})(\alpha_i).
\end{aligned}$$

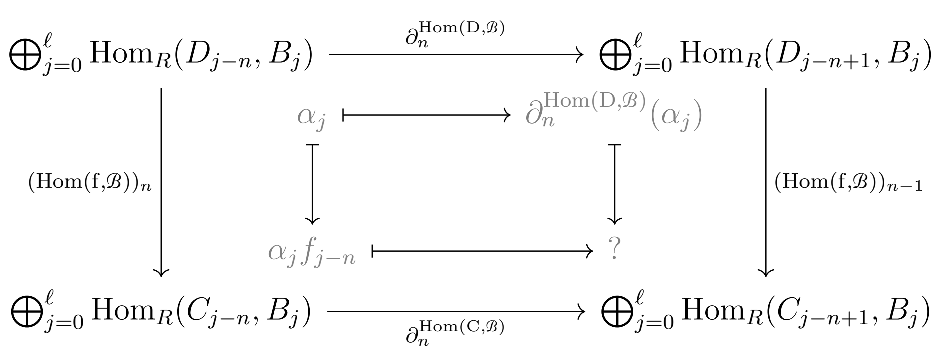
Thus, $\mathrm{Hom}(\mathcal{B},\mathrm{f})$ commutes with $\partial^{\mathrm{Hom}(\mathcal{B},\mathrm{C})}$ and $\partial^{\mathrm{Hom}(\mathcal{B},\mathrm{D})}$, but it still remains to be shown that $\mathrm{Hom}(\mathcal{B},\mathrm{f})$ is well defined in $\mathbf{K}_{\mathrm{tac}}(R)$. Suppose $\mathrm{f} \sim \mathrm{g}$, and so there exists a family of homotopy maps $h_n\colon C_n \to D_{n+1}$ such that $(f_n - g_n) = h_{n-1}\partial^C_n + \partial^D_{n+1}h_n$ for each $n \in \mathbb{Z}$. Consider now the family of R-module maps defined by $\{h_{*,n}\}_{n\in\mathbb{Z}}$ where $h_{*,n} = \oplus_{i=0}^{\ell} \mathrm{Hom}(B_i, h_{i+n})$ and observe:

$$(h_{*,n-1}\partial_n^{\mathrm{Hom}(\mathcal{B},\mathrm{C})} + \partial_{n+1}^{\mathrm{Hom}(\mathcal{B},\mathrm{D})}h_{*,n})(\alpha_i) = h_{*,n-1}(\partial^{\mathrm{C}}_{i+n}\alpha_i + (-1)^n\alpha_i\partial^{\mathcal{B}}_{i+1}) + \partial_{n+1}^{\mathrm{Hom}(\mathcal{B},\mathrm{D})}(h_{i+n}\alpha_i)$$

$$= h_{i+n-1}\partial^{\mathrm{C}}_{i+n}\alpha_i + (-1)^n h_{i+n}\alpha_i\partial^{\mathcal{B}}_{i+1} + \partial^{\mathrm{D}}_{i+n+1}(h_{i+n}\alpha_i) + (-1)^{n-1}h_{i+n}\alpha_i\partial^{\mathcal{B}}_{i+1}$$

$$= h_{i+n-1}\partial^{\mathrm{C}}_{i+n}(\alpha_i) + \partial^{\mathrm{D}}_{i+n+1}h_{i+n}(\alpha_i) = (f_{i+n} - g_{i+n})(\alpha_i) = (f_{*,n} - g_{*,n})(\alpha_i)$$

for any $\alpha_i \in \mathrm{Hom}_R(B_i, C_{i+n})$. Therefore, $\mathrm{Hom}(\mathcal{B},\mathrm{f}) \sim \mathrm{Hom}(\mathcal{B},\mathrm{g})$ by definition. Now, we give an identical argument for $\mathrm{Hom}(\text{-},\mathcal{B})$ and, to begin, we shall establish that the following square commutes for all $n \in \mathbb{Z}$:



By definition of the differential, $\partial_n^{\mathrm{Hom}(\mathrm{D},\mathcal{B})}(\alpha_j) = \partial^{\mathcal{B}}_j \circ \alpha_j + (-1)^{2j-n}\,\alpha_j \circ \partial^{\mathrm{D}}_{j-n+1}$ for any $\alpha_j \in \mathrm{Hom}_R(D_{j-n}, B_j)$ with each $j = 0, \ldots, \ell$. And so, for any such α_j we have

$$(\mathrm{f}^*_{n-1}\circ\partial_n^{\mathrm{Hom}(\mathrm{D},\mathcal{B})})(\alpha_j) = \mathrm{f}^*_{n-1}(\partial^{\mathcal{B}}_j\alpha_j + (-1)^{2j-n}\,\alpha_j\partial^{\mathrm{D}}_{j-n+1}) = (\partial^{\mathcal{B}}_j\alpha_j)f_{j-n} + (-1)^n(\alpha_j\partial^{\mathrm{D}}_{j-n+1})f_{j-n+1}$$

$$= \partial^{\mathcal{B}}_j\alpha_j f_{j-n} + (-1)^n\alpha_j(f_{j-n}\partial^{\mathrm{C}}_{j-n+1}) = \partial^{\mathcal{B}}_j(\alpha_j f_{j-n}) + (-1)^n(\alpha_j f_{j-n})\partial^{\mathrm{C}}_{j-n+1} = \partial_n^{\mathrm{Hom}(\mathrm{C},\mathcal{B})}(\alpha_j f_{j-n})$$

$$= (\partial_n^{\mathrm{Hom}(\mathrm{C},\mathcal{B})} \circ \mathrm{f}^*_n)(\alpha_j).$$

Meaning, $\mathrm{Hom}(\mathrm{f},\mathcal{B})$ commutes with the differentials on $\mathrm{Hom}(\text{-},\mathcal{B})$ and is thus an R-complex chain map. It remains to show that $\mathrm{Hom}(\mathrm{f},\mathcal{B})$ is a well-defined morphism in

$\mathbf{K}_{\mathrm{tac}}(R)$. Suppose f $\sim$ g and take $h_n \colon C_n \to D_{n+1}$ to be the associated homotopy maps such that $(f_n - g_n) = h_{n-1}\partial_n^C + \partial_{n+1}^D h_n$ for each $n \in \mathbb{Z}$. Take $\{h_n^*\}_{n\in\mathbb{Z}}$ to be the family of R-module maps defined as $h_n^* = \oplus_{j=0}^{\ell} \operatorname{Hom}(h_{j-n}, B_j) \colon \operatorname{Hom}(\mathrm{D}, \mathcal{B})_n \to \operatorname{Hom}(\mathrm{C}, \mathcal{B})_{n+1}$ and note:

$$(h_{n-1}^* \partial_n^{\operatorname{Hom}(\mathrm{D},\mathcal{B})} + \partial_{n+1}^{\operatorname{Hom}(\mathrm{C},\mathcal{B})} h_n^*)(\alpha_j) = h_{n-1}^*(\partial_j^{\mathcal{B}}\alpha_j +$$
$$(-1)^{2j-n}\,\alpha_j\partial_{j-n+1}^{\mathrm{D}}) + \partial_{n+1}^{\operatorname{Hom}(\mathrm{C},\mathcal{B})}(-1)^{|\alpha_j|+1}(\alpha_j h_{j-n-1})$$
$$= (-1)^{|\partial_j^{\mathcal{B}}\alpha_j|+1}\partial_j^{\mathrm{B}}\alpha_j h_{j-n-1} + (-1)^{|\alpha_j\partial_{j-n+1}^{\mathrm{D}}|+1}(-1)^{2j-n}\alpha_j\partial_{j-n+1}^{\mathrm{D}}h_{j-n}$$
$$+(-1)^{|\alpha_j|+1}\partial_j^{\mathcal{B}}(\alpha_j h_{j-n-1}) + (-1)^{n-1}(-1)^{|\alpha_j|+1}(\alpha_j h_{j-n-1})\partial_{j-n}^{\mathrm{C}}$$
$$= (-1)^{|\partial_j^{\mathcal{B}}\alpha_j|+1}\partial_j^{\mathcal{B}}\alpha_j h_{j-n-1} + (-1)^{|\alpha_j\partial_{j-n+1}^{\mathrm{D}}|+1+(2j-n)}\alpha_j\partial_{j-n+1}^{\mathrm{D}}h_{j-n}$$
$$+(-1)^{|\alpha_j|+1}\partial_j^{\mathcal{B}}\alpha_j h_{j-n-1} + (-1)^{n+|\alpha_j|}(\alpha_j h_{j-n-1})\partial_{j-n}^{\mathrm{C}}$$
$$= (-1)^{|\alpha_j\partial_{j-n+1}^{\mathrm{D}}|+1+n}\alpha_j\partial_{j-n+1}^{\mathrm{D}}h_{j-n} + (-1)^{n+|\alpha_j|}(\alpha_j h_{j-n-1})\partial_{j-n}^{\mathrm{C}}$$
$$= (\alpha_j)h_{j-n-1}\partial_{j-n}^{\mathrm{C}} + (\alpha_j)\partial_{j-n+1}^{\mathrm{D}}h_{j-n} = (f_{j-n} - g_{j-n})(\alpha_j) = (f_n^* - g_n^*)(\alpha_j)$$

for any $\alpha_j \in \operatorname{Hom}_R(D_{j-n}, B_j)$. Therefore, $\operatorname{Hom}(\mathrm{f}, \mathcal{B}) \sim \operatorname{Hom}(\mathrm{g}, \mathcal{B})$ by definition.[4] □

Proposition 5.21. *Let* C, D, *and* E *be totally acyclic complexes. Furthermore, let* 1^{C} *be the identity morphism on* C, $\mathrm{f} \colon \mathrm{C} \to \mathrm{D}$, *and* $\mathrm{g} \colon \mathrm{D} \to \mathrm{E}$. *Then the following hold:*

1. $\operatorname{Hom}(\mathcal{B}, 1^{\mathrm{C}})$ *and* $\operatorname{Hom}(1^{\mathrm{C}}, \mathcal{B})$ *are the identity morphisms on* $\operatorname{Hom}(\mathcal{B}, \mathrm{C})$ *and* $\operatorname{Hom}(\mathrm{C}, \mathcal{B})$, *respectively.*
2. $\operatorname{Hom}(\mathrm{fg}, \mathcal{B}) = \operatorname{Hom}(\mathrm{g}, \mathcal{B}) \operatorname{Hom}(\mathrm{f}, \mathcal{B})$ *and* $\operatorname{Hom}(\mathcal{B}, \mathrm{gf}) = \operatorname{Hom}(\mathcal{B}, \mathrm{g}) \operatorname{Hom}(\mathcal{B}, \mathrm{f})$.

Proof. It is easy to see that the first part of the proposition holds, since for any $\alpha_i \in \operatorname{Hom}_R(B_i, C_{i+n})$ we have that $\operatorname{Hom}(\mathcal{B}, 1^{\mathrm{C}})(\alpha_i) = 1^{C_{i+n}} \circ \alpha_i = \alpha_i = 1^{\operatorname{Hom}(\mathcal{B},\mathrm{C})}(\alpha_i)$.

[4]Note here that we apply the sign convention $\operatorname{Hom}(f, \mathcal{B})(\alpha) = (-1)^{|\alpha|+1}\alpha \circ f$ as is necessary when applying a map to a graded structure, such as an R-complex. However, we omit inclusion of this sign convention for all proofs whenever omission does not alter the result.

Similarly, for any $\alpha_j \in \mathrm{Hom}_R(D_{j+n}, B_j)$ note $\mathrm{Hom}(1^{\mathrm{D}}, \mathcal{B})(\alpha_j) = \alpha_j \circ 1^{\mathrm{D}_{j+n}} = \alpha_j = 1^{\mathrm{Hom(C,}\mathcal{B})}(\alpha_j)$. For the latter part of the proposition, first note that if $\alpha_i \in \mathrm{Hom}_R(B_i, C_{i+n})$ then $\mathrm{Hom}(\mathcal{B}, \mathrm{g})\,\mathrm{Hom}(\mathcal{B}, \mathrm{f})(\alpha_i) = \mathrm{Hom}(\mathcal{B}, \mathrm{g})(f_{i+n}\alpha_i) = g_{i+n}f_{i+n}\alpha_i = (gf)_{i+n}\alpha_i = \mathrm{Hom}(\mathcal{B}, \mathrm{gf})(\alpha_i)$. Likewise, if $\alpha_j \in \mathrm{Hom}_R(E_{j+n}, B_j)$ then

$$\mathrm{Hom}(\mathrm{f}, \mathcal{B})\,\mathrm{Hom}(\mathrm{g}, \mathcal{B})(\alpha_j) = \mathrm{Hom}(\mathrm{f}, \mathcal{B})(\alpha_j g_{j+n})$$
$$= \alpha_j g_{j+n} f_{j+n} = (gf)_{j+n}\alpha_j = \mathrm{Hom}(\mathrm{fg}, \mathcal{B})(\alpha_j).$$

□

Our goal is to use $\mathrm{crdeg_R C}$ and $\mathrm{cocrdeg_R C}$ to give sufficient bounds for the critical and cocritical degrees of the R-complexes $\mathrm{Hom}(\mathcal{B}, \mathrm{C})$ and $\mathrm{Hom}(\mathrm{C}, \mathcal{B})$. For this discussion let $\mathrm{crdeg_R C} = s$, $\mathrm{cocrdeg_R C} = t$ and suppose $u\colon C \to \Sigma^q \mathrm{C}$, $v\colon C \to \Sigma^r \mathrm{C}$ are the endomorphisms which realize the critical and cocritical degrees of C, respectively. We first consider $\mathrm{Hom}(\mathcal{B}, \mathrm{C})$ and denote $u_* : \mathrm{Hom}(\mathcal{B}, \mathrm{C}) \to \Sigma^q \mathrm{Hom}(\mathcal{B}, \mathrm{C})$ as the induced endomorphism on $\mathrm{Hom}(\mathcal{B}, \mathrm{C})$ and recall from above that this map acts at each degree, on each summand, as follows: for $\alpha_i \in \mathrm{Hom}(B_i, C_{i+n+q})$, $u_*(\alpha_i) = u_{i+n+q}\alpha_i \in \mathrm{Hom}(B_i, C_{i+n})$. Likewise, denote $v_*\colon \mathrm{Hom}(\mathcal{B}, \mathrm{C}) \to \Sigma^r \mathrm{Hom}(\mathcal{B}, \mathrm{C})$ as the endomorphism induced by v. The following proposition gives sufficient bounds on $\mathrm{crdeg_R Hom}(\mathcal{B}, \mathrm{C})$ and $\mathrm{cocrdeg_R Hom}(\mathcal{B}, \mathrm{C})$:

Proposition 5.22. *If* $\mathrm{crdeg_R C} = s$ *and* $\mathrm{cocrdeg_R C} = t$, *then* $\mathrm{cocrdeg_R Hom}(\mathcal{B}, \mathrm{C}) \geq t - \ell$ *and* $\mathrm{crdeg_R Hom}(\mathcal{B}, \mathrm{C}) \leq s$.

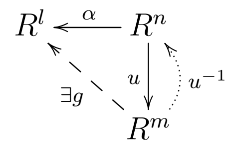
Proof. First note that since $v_n : C_n \to C_{n-q}$ is injective for all $n < t$, any such v_n is a monomorphism in R-mod. By definition, the Hom functor takes v_{i+n} to an injective function $v_{*,i+n} = \mathrm{Hom}(B_i, v_{i+n})$ between Hom sets and this happens for each $i = 0, \ldots, \ell$. Thus, $\mathrm{v}_{*,n} = (\mathrm{Hom}(\mathcal{B}, \mathrm{v}))_n = \oplus_{i=0}^{\ell} v_{*,i+n}$ is injective for all $n < t - \ell$ and note that if v_n splits for any such n, then there exists a map $\gamma\colon C_{n-q} \to C_n$ such that $\gamma v_n = \mathrm{Id}(C_n)$. Here we see that $\mathrm{v}_{*,n}$ splits for all $n < t - \ell$ as well since we may define $\mathrm{Hom}(\mathcal{B},\gamma)_{i+n} = \oplus_{i=0}^{\ell} \mathrm{Hom}(B_i, \gamma_{i+n})$ where $\mathrm{Hom}(B_i, \gamma_{i+n})\,\mathrm{Hom}(B_i, v_{i+n}) = \mathrm{Hom}(B_i, \gamma_{i+n} v_{i+n}) = \mathrm{Hom}(B_i, 1_{i+n}^{C}) = 1^{\mathrm{Hom}(B_i, C_{i+n})}$ for each $i = 0, \ldots, \ell$ due to $\mathrm{Hom}_R(B_i, -)$ being a well-defined functor on $\mathcal{R}$-$\mathcal{mod}$. Thus, each $v_{*,i+n}$ splits for $n < t - \ell$ and so we have that $(\oplus_{i=0}^{\ell}\gamma_{*,i+n})(\oplus_{i=0}^{\ell} v_{*,i+n}) = (\oplus_{i=0}^{\ell}\gamma_{*,i+n} v_{*,i+n}) = (\oplus_{i=0}^{\ell} 1^{\mathrm{Hom}(B_i, C_{i+n})}) = 1_n^{\mathrm{Hom}(\mathcal{B},\mathrm{C})}$. Lastly, it should be clear that $v_{*,n}$ will not be injective for $n > t$ since v_n is not injective for $n > t$. Therefore, v_n is split injective for all $n < t$ if and only if $v_{*,n}$ is split injective for all $n < t - \ell$, meaning $\mathrm{cocrdeg}_{\mathrm{R}}^{\mathrm{v}_*}\mathrm{Hom}(\mathcal{B},\mathrm{C}) = t - \ell \leq \mathrm{cocrdeg}_{\mathrm{R}}\mathrm{Hom}(\mathcal{B},\mathrm{C})$.

Now, to show the appropriate bound for critical degree of $\mathrm{Hom}(\mathcal{B},\mathrm{C})$, we shall first justify that each $u_{*,i+n+q} = \mathrm{Hom}(B_i, u_{i+n+q})$ is surjective on each summand $\mathrm{Hom}_R(B_i, C_{i+n})$ for $i = 0, \ldots, \ell$. Note that since B_i is free, and thus projective, there exists an $h_i\colon B_i \to C_{i+n}$ such that $u_{n+i+q} h_i = g_i$ for any $g_i\colon B_i \to C_{i+n+q}$ whenever u_{n+i+q} is surjective. Meaning, for any $g_i \in \mathrm{Hom}_R(B_i, C_{i+n+q})$, there exists such an $h_i \in \mathrm{Hom}_R(B_i, C_{i+n})$ where $u_{*,n+i+q}(h_i) = u_{n+i+q} h_i = g_i$ for all $n > s$. Thus, for any $\mathrm{g} \in (\mathrm{Hom}(\mathcal{B},\mathrm{C}))_n$ note that $\mathrm{u}_{*,n+q}(\oplus_{i=0}^{\ell} h_i) = \mathrm{g}$ where $\oplus_{i=0}^{\ell} h_i \in \mathrm{Hom}(\mathrm{B_i},\mathrm{C})_{n+q}$ and so, by definition, $\mathrm{u}_{*,n+q}$ is surjective for all $n > s$. Therefore, we have that $s \geq \mathrm{crdeg}_{\mathrm{R}}^{u_*}\mathrm{Hom}(\mathcal{B},\mathrm{C}) \geq \mathrm{crdeg}_{\mathrm{R}}\mathrm{Hom}(\mathcal{B},\mathrm{C})$. $\square$

Now, denote $\mathrm{u}^*\colon \mathrm{Hom}(\mathrm{C},\mathcal{B}) \to \Sigma^q\mathrm{Hom}(\mathrm{C},\mathcal{B})$ as the induced endomorphism on $\mathrm{Hom}(\mathrm{C},\mathcal{B})$ and note that this map acts at each degree, on each summand, as follows: for $\alpha_j \in \mathrm{Hom}(C_{j-(n+q)}, B_j)$, $u_n^*(\alpha_j) = \alpha_j u_{j-n} \in \mathrm{Hom}(C_{j-n}, B_j)$. Likewise, denote $\mathrm{v}^*\colon \mathrm{Hom}(\mathrm{C},\mathcal{B}) \to \Sigma^r\mathrm{Hom}(\mathrm{C},\mathcal{B})$ as the endomorphism induced by v. The following lemma and proposition give sufficient bounds on $\mathrm{crdeg}_{\mathrm{R}}\mathrm{Hom}(\mathrm{C},\mathcal{B})$ and $\mathrm{cocrdeg}_{\mathrm{R}}\mathrm{Hom}(\mathrm{C},\mathcal{B})$:

Lemma 5.23. *If* $u : R^n \to R^m$ *is split injective then* $u^* : \mathrm{Hom}_R(R^m, R^l) \to \mathrm{Hom}_R(R^n, R^l)$ *is (split) surjective.*

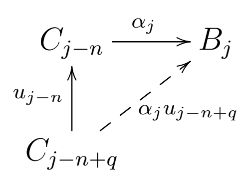
Proof. First note that since u splits, there exists a left inverse $u^{-1}\colon R^m \to R^n$ such that $u^{-1}u = \mathrm{Id}^{R^n}$. So now consider any $\alpha \in \mathrm{Hom}_R(R^n, R^l)$ and define $g = \alpha u^{-1}\colon R^m \to R^l$.



It should be clear from the fact that u^{-1} is a left inverse of u that the diagram above commutes: $gu = (\alpha u^{-1})u = \alpha$. Furthermore, since $u^*(g) = gu$, the surjectivity of u^* holds, as desired. ☐

Proposition 5.24. *If* $\mathrm{crdeg_R C} = s$ *and* $\mathrm{cocrdeg_R C} = t$*, then* $\mathrm{cocrdeg_R Hom(C, \mathcal{B})} \geq q - s$ *and* $\mathrm{crdeg_R Hom(C, \mathcal{B})} \leq \ell - t$*.*

Proof. First consider the R-module map $u^*_{j-n+q}\colon \mathrm{Hom}_R(C_{j-n}, B_j) \to \mathrm{Hom}(C_{j-n+q}, B_j)$ for any $j = 0, \ldots, \ell$ and note that this map is defined as taking any $\alpha_j \in \mathrm{Hom}(C_{j-n}, B_j)$ to the composition $\alpha_j u_{j-n+q}$:



If $-n + q > s$, then u_{j-n+q} is surjective for each $j = 0, \ldots, \ell$ and since surjections are epimorphisms in $\mathcal{R}$-$\mathcal{mod}$ this is equivalent to u^*_{j-n+q} being injective for each $j = 0 \ldots, \ell$. Thus, whenever $n < q - s$, $\mathrm{u^*}_n = \mathrm{Hom(u, \mathcal{B})} = \oplus_{j=0}^{\ell} \mathrm{Hom}(u_{j-n}, B_j)$ is injective and furthermore splits since

$$(u^*_{j-n+q})^{-1}\, u^*_{j-n+q} = \mathrm{Hom}(u^{-1}_{j-n+q}, B_j)\,\mathrm{Hom}(u_{j-n+q}, B_j) = \mathrm{Hom}(u_{j-n+q}u^{-1}_{j-n+q}, B_j)$$

$$= \mathrm{Hom}(1^C_{j-n}, B_j) = 1_n^{\mathrm{Hom(C,\mathcal{B})}}$$

where $(u^*_{j-n+q})^{-1}$ is the right inverse of u^*_{j-n+q} guaranteed by the fact that all free module surjections split. And so we see that $\mathrm{cocrdeg_R Hom(C,\mathscr{B})} \geq \mathrm{cocrdeg_R^{u^*} Hom(C,\mathscr{B})} \geq q - s$.

Now, apply the above lemma with $R^l = B_j$, $R^m = C_{j-(n+q)}$, and $R^n = C_{j-n}$ to see that u^*_{j-n} will be surjective for each $j = 0, \ldots, m$ whenever $n > \ell - t$ so that $\mathrm{u}^* = \oplus_{j=0}^{\ell}(u^*_{j-n})$ will be (split) surjective for all such n. Hence, $\ell - t \geq \mathrm{crdeg_R^{u^*} Hom(B,C)} \geq \mathrm{crdeg_R Hom(B,C)}$ as desired.

□

Now that we have explored how the critical and cocritical degrees might change under a myriad of operations of R-complexes, we will turn towards boundedness problems of a different nature. In the next chapter, we present a proposal focused on addressing the inability to provide a universal bound for critical degree (and thus, cocritical degree) over all R-complexes (or modules) of a given complexity greater than one.

CHAPTER 6

Application: Critical Width of an R-Complex

And now, we are at the crux of this thesis; all that remains is a proper conclusion to the argument for how viewing the notion of critical degree (along with its dual notion) in $\mathbf{K}_{\text{tac}}(R)$ might benefit us. In this chapter we will examine additional insights, motivated by previous work by the authors of [AvGaPe], and further in [AvBu], on boundedness problems with regard to critical degree in $\mathcal{R}$-$\mathcal{mod}$. These insights will motivate a new definition, which we present in Section 6.2 of this chapter. It is this definition that will establish our conclusion and we then close with an interesting question that, for now, remains unanswered.

6.1 Some More Boundedness Problems

For this chapter, we will maintain the base assumption that $(R, \mathfrak{m}, \Bbbk)$ is a local, Gorenstein ring with additional assumptions imposed frequently. Recall that given a finitely-generated R-module, we can associate to it a totally acyclic complex via the construction of its complete resolution. In Chapter 4, we addressed the connection between critical degree in $\mathcal{R}$-$\mathcal{mod}$ and $\mathbf{K}_{\text{tac}}(R)$ with some detail. For example, if the module is maximal Cohen-Macaulay then its critical degree agrees with that of any associated totally acyclic complex, as long as the critical degrees are non-negative. However, distinctions between the two notions of critical degree arise when they are negative or the module is not maximal Cohen-Macaulay.

In this section, we aim to understand the comparison between critical degree in $\mathcal{R}$-$\mathcal{mod}$ and the analogous notion (along with its dual) in $\mathbf{K}_{\text{tac}}(R)$. We give particular attention to the consequences of this comparison and the results in [AvGaPe] and [AvBu], which provide bounds for the critical degree of an R-module of certain complexities.

6.1.1 A Continued Comparison: Critical Degree in $\mathscr{R}$-$\mathscr{mod}$ versus $\mathbf{K}_{\mathrm{tac}}(R)$

Let $\mathrm{C} \to \mathbf{F} \twoheadrightarrow M$ be a complete resolution of a finitely-generated R-module M such that $\mathbf{F}$ is the minimal free resolution of M. Further assume C is minimal, $\text{CI-dim}_R M < \operatorname{pd}_R M = \infty$, and set $\mathrm{crdeg_R M} = s$ for some integer $-1 \leq s < \infty$. Denote $g = \text{CI-dim}_R M = \dim R - \operatorname{depth}_R M$, noting that $\dim R = \operatorname{depth} R$ (since R is Gorenstein and thus Cohen-Macaulay) so this value is precisely the same g mentioned in Chapter 2.

Since $\text{CI-dim}_R M < \infty$, if $\operatorname{cx}_R M = 1$ then $s \leq g$ as stated in Theorem 7.2 from [AvGaPe] (restated in Chapter 2 of this thesis). Note that it then must hold that $\mathrm{crdeg_R C} \leq 0$ since construction of the complete resolution requires $C_0 := F_g$. On the other hand, if $\operatorname{cx}_R M^* = 1$ then $\mathrm{crdeg_R M^*} \leq g^* = \dim R - \operatorname{depth}_R M^* = \text{CI-dim}_R M^*$. And so $\mathrm{cocrdeg_R C} \geq -1$ since $C_{-1} := \operatorname{Hom}_R(F^*_{g^*}, R)$. Of course, in either of these cases we already know that C is periodic, so that $\mathrm{crdeg_R C} = -\infty$ and $\mathrm{cocrdeg_R C} = \infty$ as described in Proposition 2 of Chapter 4.

In [AvBu], Avramov and Buchweitz additionally give the bound

$$\mathrm{crdeg_R M} \leq \max\{2b^R_g(M) - 1, 2b^R_{g+1}(M)\} + g - 1$$

for a given R-module M whenever $\operatorname{cx}_R M = 2$. So, assuming linear growth of $\{b^R_n(M)\}$, it should hold that

$$\mathrm{crdeg_R C} \leq \max\{2\hat{b}^R_0(M) - 1, 2\hat{b}^R_1(M)\} - 1$$

and

$$\mathrm{cocrdeg_R C} \geq \max\{2\hat{b}^R_{-1}(M) - 1, 2\hat{b}^R_0(M)\} - 2$$

for a given R-module M. This then leads to a bound on the *distance* between $\mathrm{crdeg_R C}$ and $\mathrm{cocrdeg_R C}$, which must be

$$\max\{2(\hat{b}_0^R(M) - \hat{b}_{-1}^R(M)) + 1,\ 2(\hat{b}_1^R(M) - \hat{b}_{-1}^R(M) + 1),\ 2(\hat{b}_1^R(M) - \hat{b}_0^R(M)) + 1,\ 0\}.$$

Unfortunately, this bound is given in terms of a specific module, and, moreover, such bounds are currently unknown for R-modules with $\operatorname{cx}_R M \geq 3$. Nevertheless, this observation leads to the next natural question– what does this "distance" tell us about M itself? We explore this question in greater detail with the next section.

6.2 A New Measurement of R-Complexes and R-Modules

One disadvantage of critical degree in $\mathscr{R}$-$\mathscr{mod}$ discussed at the end of Chapter 2 is that critical degree is entirely dependent upon the homological degree of a free resolution, which is determined by the R-module one begins with. While we can define the relationship between the critical degree of an R-module and that of one of its syzygy modules ($\mathrm{crdeg_R M} = \max\{\mathrm{crdeg_R \Omega^n M} - n, 0\}$), starting with another syzygy module yields a different value. Moreover, the example at the end of Chapter 2 demonstrates that there is no hope for finding a bound for critical degree of modules of a given complexity, as taking a higher syzygy (or cosyzygy) will always yield a higher (resp. lower) critical degree.

Part of our motivation for generalizing this notion to $\mathbf{K}_{\mathrm{tac}}(R)$ involved addressing this boundedness issue. The critical degree of an R-complex describes when growth to the left occurs with respect to the entire syzygy sequence; similarly, the cocritical degree describes when growth occurs to the right. Moreover, we have established a more precise theory with respect to the entire syzygy sequence, as we set out to do, in view of the fact that we are able to distinguish between the R-complexes C and $\Sigma^q\mathrm{C}$ in $\mathbf{K}_{\mathrm{tac}}(R)$. We provide the following example to elucidate this idea.

Example 6.1. Let R be a local complete intersection ring. Denote $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$ the minimal R-complex with $\operatorname{Im} \partial_0^{\mathrm{C}} = M$ and $\mathrm{D} \in \mathbf{K}_{\mathrm{tac}}(R)$ the minimal R-complex with $\operatorname{Im} \partial_0^{\mathrm{D}} = N$, so that there exist complete resolutions $\mathrm{C} \to \mathbf{F} \twoheadrightarrow M$ and $\mathrm{D} \to \mathbf{G} \twoheadrightarrow N$. But now suppose M and N are syzygies of each other, say $N = \Omega^{-n} M$ and $M = \Omega^n N$. Then note that $\mathrm{D} \simeq \Sigma^n \mathrm{C}$ (in fact, they are isomorphic!). If $\operatorname{crdeg_R} \mathrm{C} = s$ and $\operatorname{cocrdeg_R} \mathrm{C} = t$, then $\operatorname{crdeg_R} \mathrm{D} = s + n$ and $\operatorname{cocrdeg_R} \mathrm{D} = t + n$ by Proposition 2 in Chapter 5. However, suppose we instead start with the assumption that $\operatorname{crdeg_R} \mathrm{D} = s'$ and $\operatorname{cocrdeg_R} \mathrm{D} = t'$, viewing $\mathrm{C} = \Sigma^{-n} \mathrm{D}$. Once again applying Proposition 2, it is not difficult to see that $\operatorname{crdeg_R} \mathrm{C} = s' - n$ and $\operatorname{cocrdeg_R} \mathrm{C} = t' - n$. Note that in doing so, $s' - n = s$ and $t' - n = t$.

However, we have not yet realized our goal. While C and $\Sigma^q \mathrm{C}$ are theoretically distinct complexes in $\mathbf{K}_{\mathrm{tac}}(R)$, in practice this is not always useful. In particular, one might only care to understand patterns in a syzygy sequence, void of translations on such sequences. This does indeed create a bit of an ambiguity, for any shifted complex will have shifted critical and cocritical degrees, as is expected, but the Betti or syzygy sequences are inherently the *same*, albeit with a translated indexing set imposed.

Despite this observation, the critical and cocritical degrees of a totally acyclic complex still provide powerful insight. In the example above, notice that while the critical and cocritical degrees change under the action of translations, the *difference* between these two degrees does not alter: $\operatorname{crdeg_R} \mathrm{C} - \operatorname{cocrdeg_R} \mathrm{C} = s - t = (s + n) - (t + n) = \operatorname{crdeg_R} \mathrm{D} - \operatorname{cocrdeg_R} \mathrm{D}$. This of course sparks motivation for introducing a new invariant of R-complexes, which will remain unchanged under any translation of the syzygy sequence.

6.2.1 Width of a Totally Acyclic Complex

Now that we have discussed motivation for the following definition– both in the way of boundedness problems and introducing a measure which will remain invariant under translations– we now present what we shall call the *diameter* of an R-complex.

Definition 6.2. Let R be a local, Gorenstein ring and C a totally acyclic complex with minimal subcomplex $\overline{\mathrm{C}}$. Furthermore assume $\text{CI-dim}_R \operatorname{Im} \partial_0^{\overline{\mathrm{C}}} < \infty$ and $\operatorname{cx}_R \overline{\mathrm{C}} > 1$. Then the *diameter* of C is the distance between the critical and cocritical degrees of C. That is,

$$\operatorname{diam}_R(\mathrm{C}) = \mathrm{crdeg_R C} - \mathrm{cocrdeg_R C}.$$

Define $\operatorname{diam}_R(\mathrm{C}) = -\infty$ for any C with $\operatorname{cx}_R \overline{\mathrm{C}} = 1$ or if $\mathrm{C} \simeq 0$. Moreover, note that under the assumptions given, $\operatorname{diam}_R(\mathrm{C}) < \infty$ but, relaxing these assumptions, $\operatorname{diam}_R(\mathrm{C}) = \infty$ if and only if $\mathrm{crdeg_R C} = \infty$ or $\mathrm{cocrdeg_R C} = -\infty$.

It should be clear that

$$\operatorname{diam}_R(\mathrm{C}) \leq \inf\{\mathrm{crdeg_R^\mu C} - \mathrm{cocrdeg_R^\mu C} \ : \ \mu \in \operatorname{End}_{\mathscr{K}}(C)\}$$

with equality whenever μ realizes both the critical and cocritical degrees. Furthermore, if $\mathrm{crdeg_R^\mu C} = \mathrm{crdeg_R C}$ *and* $\mathrm{cocrdeg_R^\mu C} = \mathrm{cocrdeg_R C}$, then note that whenever $\operatorname{cx}_R C > 1$ it must hold that $-q < \operatorname{diam}_R(\mathrm{C}) < \infty$ by Proposition 2 in Chapter 4. That is, for a *non*-periodic R-complex the maximal amount for which the critical degree can be smaller than the cocritical degree is $q - 1$. However, if the critical and cocritical degrees are realized by different endomorphisms, it is currently unknown whether there exists a lower bound on $\operatorname{diam}_R(\mathrm{C})$ for non-periodic, nonzero complexes. We now list the following assertion that this measurement is well defined in $\mathbf{K}_{\mathrm{tac}}(R)$.

Proposition 6.3. *If* $\mathrm{C} \simeq \mathrm{D}$, *then* $\operatorname{diam}_R(\mathrm{C}) = \operatorname{diam}_R(\mathrm{D})$.

Proof. It is already known that $\mathrm{crdeg_R C} = \mathrm{crdeg_R D}$ and $\mathrm{cocrdeg_R C} = \mathrm{cocrdeg_R D}$. Thus, the statement follows directly from these observations. □

Therefore, the diameter of an R-complex is stable under homotopy. We are also easily able to see how this measurement changes under some of the operations discussed in Chapter 5. As already mentioned, $\mathrm{diam}_R(\mathrm{C}) = \mathrm{diam}_R(\Sigma^{\mathrm{n}}\mathrm{C})$ for any $n \in \mathbb{Z}$. We may also consider the relationship between the diameter of C and C^*, noting that

$$\begin{aligned}\mathrm{diam}_R(\mathrm{C}^*) &= \mathrm{crdeg_R}\mathrm{C}^* - \mathrm{cocrdeg_R}\mathrm{C}^* \\ &= -\mathrm{cocrdeg_R}\mathrm{C} - (q - \mathrm{crdeg_R}\mathrm{C}) \\ &= \mathrm{diam}_R(\mathrm{C}) - q\end{aligned}$$

where $q = \deg(\mu)$ with μ the endomorphism which realizes $\mathrm{crdeg_R}\mathrm{C}$. Now, supposing R is a complete intersection ring, we may also say what $\mathrm{diam}_R(\mathrm{C} \oplus \mathrm{D})$ is, given $\mathrm{diam}_R(\mathrm{C})$ and $\mathrm{diam}_R(\mathrm{D})$. Note that

$$\begin{aligned}\mathrm{diam}_R(\mathrm{C} \oplus \mathrm{D}) &= \mathrm{crdeg_R}(\mathrm{C} \oplus \mathrm{D}) - \mathrm{cocrdeg_R}(\mathrm{C} \oplus \mathrm{D}) \\ &= \max\{\mathrm{crdeg_R}\mathrm{C}, \mathrm{crdeg_R}\mathrm{D}\} - \min\{\mathrm{cocrdeg_R}\mathrm{C}, \mathrm{cocrdeg_R}\mathrm{D}\} \\ &= \max\{\mathrm{diam}_R(\mathrm{C}),\ \mathrm{diam}_R(\mathrm{D}),\ \mathrm{diam}_R(\mathrm{C}) + (t_C - t_D),\ \mathrm{diam}_R(\mathrm{D}) + (t_D - t_C)\}\end{aligned}$$

where $t_C = \mathrm{cocrdeg_R}\mathrm{C}$ and $t_D = \mathrm{cocrdeg_R}\mathrm{D}$. Moreover, when we consider taking summands, we see that the following inequalities must hold

$$\begin{cases}\mathrm{diam}_R(\mathrm{C}) \leq \mathrm{diam}_R(\mathrm{C} \oplus \mathrm{D}) \\ \mathrm{diam}_R(\mathrm{D}) \leq \mathrm{diam}_R(\mathrm{C} \oplus \mathrm{D})\end{cases}$$

aligning with what was observed in the previous equation. Lastly, we may also consider bounds for the diameter on the manipulations of complexes discussed in the second part of Chapter 5. Note first that for the functors $- \otimes \mathcal{B}$ and $\mathcal{B} \otimes -$, if $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$ then $\mathrm{diam}_R(\mathcal{B} \otimes \mathrm{C}) = \mathrm{diam}_R(\mathrm{C} \otimes \mathcal{B}) \leq \mathrm{diam}_R(\mathrm{C}) + \ell$ with ℓ as defined previously. Moreover, for the functors $\mathrm{Hom}(\text{-}, \mathcal{B})$ and $\mathrm{Hom}(\mathcal{B}, \text{-})$, if $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$ then

$\operatorname{diam}_R(\operatorname{Hom}(\mathcal{B},\mathrm{C})) \leq \operatorname{diam}_R(\mathrm{C}) + \ell$ and $\operatorname{diam}_R(\operatorname{Hom}(\mathrm{C},\mathcal{B})) \leq \operatorname{diam}_R(\mathrm{C}) + (\ell - q)$, where $q = \deg(\mu)$ for $\operatorname{crdeg}_{\mathrm{R}}\mathrm{C} = \operatorname{crdeg}^{\mu}_{\mathrm{R}}\mathrm{C}$. Now that we have discussed some of the boundedness results for the operations from Chapter 5, we continue by discussing the notion of diameter with respect to $\mathcal{R}$-$\mathcal{mod}$.

6.2.2 Width of a Finitely-Generated Module

Definition 6.4. Let R be a local, Gorenstein ring and M a finitely-generated R-module with $\operatorname{CI-dim}_R M < \infty$ and minimal complete resolution $\mathrm{C} \to \mathbf{F} \twoheadrightarrow M$. Then the *diameter* of M is the distance between the critical and cocritical degrees of C. That is,

$$\operatorname{diam}_R(M) = \operatorname{crdeg}_{\mathrm{R}}\mathrm{C} - \operatorname{cocrdeg}_{\mathrm{R}}\mathrm{C}$$

and we define $\operatorname{diam}_R(M) = -\infty$ for any R-module with $\operatorname{cx}_R M = 1$. Moreover, define $\operatorname{diam}_R(0) = 0$ and note that $\operatorname{diam}_R(M) < \infty$ under the specified conditions.

Of course, if we relax the condition that $\operatorname{CI-dim}_R M < \infty$, then note that $\operatorname{diam}_R(M) = \infty$ if and only if $\operatorname{crdeg}_{\mathrm{R}}\mathrm{C} = \infty$ or $\operatorname{cocrdeg}_{\mathrm{R}}\mathrm{C} = -\infty$. Note that we specifically use the definition critical degree with respect to the complete resolution. Therefore, $\operatorname{diam}_R(M)$ coincides with $\operatorname{diam}_R(\mathrm{C})$ and the only distinction is whether our focus is on the complex or the module itself.

6.3 The Natural Middle of a Complete Resolution

Let M be a finitely-generated R-module with $\operatorname{CI-dim}_R M < \infty$ and denote its minimal complete resolution $\mathrm{C} \to \mathbf{F} \twoheadrightarrow M$. In general, we view the "middle" of the R-complex $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$ (informally, complete resolution) to be $M = \operatorname{Im}\partial_0^{\mathrm{C}}$. However, suppose $N = \operatorname{Im}\partial_0^{\Sigma^n\mathrm{C}}$ with minimal complete resolution $\Sigma^{\mathrm{n}}\mathrm{C} \to \mathbf{F}^{\geq \mathbf{n}} \twoheadrightarrow N$. In essence, the doubly-infinite sequences represented by C and $\Sigma^n\mathrm{C}$ are the same, as they are just translations of one another. Hence, it does not entirely make sense that what is viewed as the "middle" of each sequence is different, dependent upon an arbitrary labeling of homological degrees. It is for this reason, we present what we shall call the *canonical* complex $\mathbf{C}^\star$ of an R-complex C, with a standardized indexing of homological degrees.

First, we present a notion of the *natural middle* of a complete resolution. Given the assumptions defined above, denote $\omega = \operatorname{diam}_R(\mathrm{C})$, $s = \mathrm{crdeg_R C} < \infty$, and $t = \mathrm{cocrdeg_R C} > -\infty$.

Definition 6.5. The *natural middle* of $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$ is defined to be $\Omega^{d^\star} M = \mathrm{im}\partial^{\mathrm{C}}_{d^\star}$ where

$$d^\star = s - \left\lceil \frac{\omega}{2} \right\rceil = t + \left\lceil \frac{\omega}{2} \right\rceil .$$

We may also refer to the *natural middle sequence* of C as the portion of C which lies between the critical and cocritical degrees.[1] Specifically, if $s > t$ then the natural middle sequence of C is

$$C_s \xrightarrow{\partial_s} C_{s-1} \xrightarrow{\partial_{s-1}} C_{s-2} \to \cdots \to C_{d^\star} \xrightarrow{\partial_{d^\star}} C_{d^\star-1} \to \cdots \to C_{t+2} \xrightarrow{\partial_{t+2}} C_{t+1} \xrightarrow{\partial_{t+1}} C_t$$

or, equivalently

$$C_{d^\star+\lceil\frac{\omega}{2}\rceil} \xrightarrow{\partial_{d^\star+\lceil\frac{\omega}{2}\rceil-1}} C_{d^\star+\lceil\frac{\omega}{2}\rceil-1} \to \cdots \to C_{d^\star} \xrightarrow{\partial_{d^\star}} C_{d^\star-1} \to \cdots \to C_{\partial_{d^\star-\lceil\frac{\omega}{2}\rceil+1}} \xrightarrow{\partial_{d^\star-\lceil\frac{\omega}{2}\rceil+1}} C_{d^\star-\lceil\frac{\omega}{2}\rceil}.$$

If instead $s < t$, then the natural middle is

$$C_t \xrightarrow{\partial_t} C_{t-1} \xrightarrow{\partial_{t-1}} C_{t-2} \to \cdots \to C_{d^\star} \xrightarrow{\partial_{d^\star}} C_{d^\star-1} \to \cdots \to C_{s+2} \xrightarrow{\partial_{s+2}} C_{s+1} \xrightarrow{\partial_{s+1}} C_s$$

or, equivalently

$$C_{d^\star+\lceil\frac{\lfloor\omega\rfloor}{2}\rceil} \xrightarrow{\partial_{d^\star+\lceil\frac{\lfloor\omega\rfloor}{2}\rceil-1}} C_{d^\star+\lceil\frac{\lfloor\omega\rfloor}{2}\rceil-1} \to \cdots \to C_{d^\star} \xrightarrow{\partial_{d^\star}} C_{d^\star-1} \to \cdots \to C_{\partial_{d^\star-\lceil\frac{\lfloor\omega\rfloor}{2}\rceil+1}} \xrightarrow{\partial_{d^\star-\lceil\frac{\lfloor\omega\rfloor}{2}\rceil+1}} C_{d^\star-\lceil\frac{\lfloor\omega\rfloor}{2}\rceil}.$$

And of course if $s = t$, then the natural middle is just $C_{d^\star}$. It should be clear that in the case where we relax our conditions and $\mathrm{crdeg_R C} = \infty$ or $\mathrm{cocrdeg_R C} = -\infty$, then

[1] We may informally refer to the natural middle of C as the sequence, as well as the degree $d^\star$.

we say that the natural middle of C is infinite. Furthermore, if $\operatorname{diam}_R(\mathrm{C}) = -\infty$, then we shall say that the natural middle of C is nonexistent.

Note that we can discuss the natural middle *relative* to an endomorphism, in which case, it is easy to see that any μ-critical and μ-cocritical degree "bound" the natural middle sequence as long as $s \geq t$. In other words, any *relative* natural middle contains the natural middle of an R-complex C, with respect to the first case listed above. In the latter case, it is unknown whether or not any relative natural middle will contain the natural middle sequence. We now consider a prior example to help clarify these definitions.

Example 6.6. Let $R = \dfrac{\Bbbk[x,y]}{(x^2,y^2)}$ and $Q = \dfrac{\Bbbk[\![x,y]\!]}{(x^2)}$. Recall that we may naively "lift" the complex $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$ to the sequence of q-modules and Q-module homomorphisms, $\widetilde{\mathrm{C}}$:

$$\widetilde{\mathrm{C}}: \cdots \to Q^4 \xrightarrow{\left[\begin{smallmatrix} y & -x & 0 & 0 \\ 0 & y & x & 0 \\ 0 & 0 & y & x \end{smallmatrix}\right]} Q^3 \xrightarrow{\left[\begin{smallmatrix} 0 & -y & x \\ y & x & 0 \end{smallmatrix}\right]} Q^2 \xrightarrow{\left[\begin{smallmatrix} x & y \end{smallmatrix}\right]} Q \xrightarrow{\left[\begin{smallmatrix} xy \end{smallmatrix}\right]} Q \xrightarrow{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]} Q^2 \xrightarrow{\left[\begin{smallmatrix} 0 & y \\ -y & x \\ x & 0 \end{smallmatrix}\right]} Q^3 \xrightarrow{\left[\begin{smallmatrix} y & 0 & 0 \\ -x & y & 0 \\ 0 & x & y \\ 0 & 0 & x \end{smallmatrix}\right]} Q^4 \to \cdots$$

$$\Uparrow \quad \Uparrow \quad \Uparrow$$

$$\mathrm{C}: \cdots \to R^4 \xrightarrow{\left[\begin{smallmatrix} y & -x & 0 & 0 \\ 0 & y & x & 0 \\ 0 & 0 & y & x \end{smallmatrix}\right]} R^3 \xrightarrow{\left[\begin{smallmatrix} 0 & -y & x \\ y & x & 0 \end{smallmatrix}\right]} R^2 \xrightarrow{\left[\begin{smallmatrix} x & y \end{smallmatrix}\right]} R \xrightarrow{\left[\begin{smallmatrix} xy \end{smallmatrix}\right]} R \xrightarrow{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]} R^2 \xrightarrow{\left[\begin{smallmatrix} 0 & y \\ -y & x \\ x & 0 \end{smallmatrix}\right]} R^3 \xrightarrow{\left[\begin{smallmatrix} y & 0 & 0 \\ -x & y & 0 \\ 0 & x & y \\ 0 & 0 & x \end{smallmatrix}\right]} R^4 \to \cdots$$

Moreover, computation of the Eisenbud operator yields:

- $\partial_2^{\widetilde{C}}\partial_3^{\widetilde{C}} = \left[\begin{smallmatrix} 0 & -y & x \\ y & x & 0 \end{smallmatrix}\right]\begin{bmatrix} y & -x & 0 & 0 \\ 0 & y & x & 0 \\ 0 & 0 & y & x \end{bmatrix} = y^2\left[\begin{smallmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix}\right]$
- $\partial_1^{\widetilde{C}}\partial_2^{\widetilde{C}} = \left[\begin{smallmatrix} x & y \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & -y & x \\ y & x & 0 \end{smallmatrix}\right] = y^2\left[\begin{smallmatrix} 1 & 0 & 0 \end{smallmatrix}\right]$
- $\partial_0^{\widetilde{C}}\partial_1^{\widetilde{C}} = \left[\begin{smallmatrix} xy \end{smallmatrix}\right]\left[\begin{smallmatrix} x & y \end{smallmatrix}\right] = y^2\left[\begin{smallmatrix} 0 & x \end{smallmatrix}\right]$
- $\delta_{-1}^{\widetilde{C}}\delta_0^{\widetilde{C}} = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\left[\begin{smallmatrix} xy \end{smallmatrix}\right] = y^2\left[\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right]$
- $\partial_{-2}^{\widetilde{C}}\partial_{-1}^{\widetilde{C}} = \begin{bmatrix} 0 & y \\ -y & x \\ x & 0 \end{bmatrix}\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = y^2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
- $\partial_{-3}^{\widetilde{C}}\partial_{-2}^{\widetilde{C}} = \begin{bmatrix} y & 0 & 0 \\ -x & y & 0 \\ 0 & x & y \\ 0 & 0 & x \end{bmatrix}\begin{bmatrix} 0 & y \\ -y & x \\ x & 0 \end{bmatrix} = y^2\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

We will sacrifice rigor for demonstrative purposes. As it turns out, for this particular example, t_i is surjective for $i \geq 2$ and injective for $i \leq -2$, implying $\operatorname{crdeg}_{\mathrm{R}}\mathrm{C} \leq 2$

and $\mathrm{cocrdeg_R C} \geq -2$. Hence, the the natural middle of C is contained within the sequence

$$R^2 \xrightarrow{[x\ y]} R \xrightarrow{[xy]} R \xrightarrow{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]} R^2$$

Therefore, it should be clear that, as depicted in this example, if R is a complete intersection ring, then the natural middle of a complete resolution contains the free modules of *lowest* rank. The above example is, of course, very well behaved as it is the complete resolution of the residue field $\Bbbk$. Note that in some cases, it could be possible that not all free modules contained within the natural middle are of rank lower than those on the periphery; rather, we are guaranteed strict growth to the left of the critical degree and to the right of the cocritical degree (whenever $\operatorname{cx}_R M > 1$). Hence, there exists $N_1, N_2 \in \mathbb{Z}$ such that $\beta_n^R > \beta_k^R$ for all $n \geq N_1$ and for all $n \leq N_2$, where C_k is any module contained in the natural middle sequence. We are now ready to define the *canonical* complex of $\mathrm{C} \in \mathbf{K}_{\mathrm{tac}}(R)$.

Definition 6.7. The *canonical* complex of an R-complex C, denoted $\mathbf{C}^{\star}$, is defined to be $\Sigma^{-d^{\star}}\mathrm{C}$. That is, it is the R-complex in $\mathbf{K}_{\mathrm{tac}}(R)$ with R-modules

$$(\mathbf{C}^{\star})_0 = C_{d^{\star}}\ ;\quad (\mathbf{C}^{\star})_n = C_{n+d^{\star}} \quad \text{for any} \quad n \in \mathbb{Z}$$

and differentials

$$\partial_0^{\mathbf{C}^{\star}} = \partial_{d^{\star}}^{\mathrm{C}}\ ;\quad \partial_n^{\mathbf{C}^{\star}} = \partial_{n+d^{\star}}^{\mathrm{C}} \quad \text{for any} \quad n \in \mathbb{Z}.$$

Note that we may also view $(\mathbf{C}^{\star})_n = (\Sigma^{-d^{\star}}\mathrm{C})_n$ where $d^{\star} > 0$ correlates to shifting C to the right and $d^{\star} < 0$ correlates to a leftward shift of C. And, with this notation, we may write $\partial_n^{\mathbf{C}^{\star}} = \partial_n^{\Sigma^{-d^{\star}}\mathrm{C}}$ as well.

6.4 A Question Remains

It is still unclear what the diameter of an R-complex (or module) communicates about the structure itself. To some degree, it seems reasonable that this measurement captures some characteristic of the structure's "size" or, for lack of a better word, "complexity". In that regard, the actual complexity of an R-module communicates a measure of growth of the module's Betti sequence. This notion of *diameter*, although tied to complexity, seems to communicate a different characteristic related to growth of $\{\hat{b}_n^R(M)\}$. Intrinsically, it seems as if modules with similar structures should not have wildly different diameters. Should there not be some similarities, not only in the type of growth the modules' syzygy sequences maintain, but also with respect to the amount of "time" it takes them to start growing?

Even if this new measure for R-modules, and complexes, provides no additional insight to the structural similarities between such objects, perhaps the reader might find some small comfort in knowing that they are now able to compute the cylindrical volume of an R-module. All humor laid aside, we do leave the reader with the following example and question.

Example 6.8. Let R be a complete intersection ring and denote M a finitely-generated R-module with minimal complete resolution $\mathrm{C} \to \mathbf{F} \twoheadrightarrow M$. Further suppose that $\operatorname{cx}_R M \gneq 1$ and denote $\operatorname{diam}_R(M) = \omega_M = \operatorname{diam}_R(\mathrm{C})$. Then consider the R-complex $\mathrm{C} \oplus \Sigma^{\mathrm{n}}\mathrm{C}$ with associated (minimal) complete resolution $\mathrm{C} \oplus \Sigma^{\mathrm{n}}\mathrm{C} \to \mathbf{F} \oplus \boldsymbol{\Sigma}^{\mathbf{n}}\mathrm{F} \twoheadrightarrow M \oplus \Omega^n M$ for some fixed integer $n \in \mathbb{Z}$. It should be clear that for any such integer $\operatorname{cx}_R(M \oplus \Omega^n M) = \operatorname{cx}_R M$; however, $\omega_{M \oplus \Omega^n M} < \omega_{M \oplus \Omega^{n+1} M}$ for all $n \in \mathbb{Z}$.

The example above demonstrates that the diameter is not necessarily bounded for all modules (or complexes) of a given complexity larger than one. Nevertheless, the example itself seems rather unfair; hence, one might wonder what happens when we impose an additional condition upon C which would render the given example inapplicable.

Open Problem. Let R be a complete intersection ring and denote M a finitely-generated R-module with minimal complete resolution $\mathrm{C} \to \mathbf{F} \twoheadrightarrow M$ *where* C *is indecomposable*. Further suppose that $\operatorname{cx}_R M \gneq 1$ and denote $\operatorname{diam}_R(M) = \omega_M = \operatorname{diam}_R(\mathrm{C})$. Then does it hold that there exists some $\ell_c \in \mathbb{N}$ such that $\omega_M \leq \ell_c$ for all finitely-generated R-modules M with $\operatorname{cx}_R M = c$?

REFERENCES

[Al] Y. A. Alkhezi, (2014). *Properties of the Pinched Tensor Product*[Unpublished Doctoral Dissertation]. University of Texas at Arlington.

[AnFu] F. W. Anderson, K. R. Fuller, (1998). *Rings and Categories of Modules*, Graduate Texts in Mathematics, Second Edition.

[Av] Avramov, L.L. (1981). *Obstructions to the Existence of Multiplicative Structures on Minimal Free Resolutions*. American Journal of Mathematics, 103(1), 1-31. https://doi.org/10.2307/2374187.

[Av2] Avramov, L.L. (1989). *Modules of finite virtual projective dimension.* Invent Math 96, 71–101. https://doi.org/10.1007/BF01393971.

[AvBu] Avramov, L.L., Buchweitz, R.O. (1999). *Homological Algebra Modulo A Regular Sequence With Special Attention To Codimension Two.* Journal of Algebra, 230(1), 24–67. https://doi.org/10.1006/jabr.1999.7953.

[AvGaPe] Avramov, L.L., Gasharov, V.N., Peeva, I.V. *Complete intersection dimension*. Publications Mathématiques de l'IHÉS, Tome 86 (1997), 67-114. https://doi.org/10.1007/BF02698901

[AvMa] Avramov, L. L., Martsinkovsky, A. (2002). *Absolute, Relative, and Tate Cohomology of Modules of Finite Gorenstein Dimension*. Proceedings of the London Mathematical Society, Volume 85, Issue 2, 393–440. https://doi.org/10.1112/S0024611502013527.

[BeJoMo] Bergh, P. A., Jorgensen, D. A., Moore, W. F., (2021). *Totally Acyclic Approximations*. Appl Categor Struct. 29, 729–745. https://doi.org/10.1007/s10485-021-09633-1.

[BrHe] Bruns, W., Herzog, J., (1998). *Cohen-Macaulay rings.* (2nd ed., Cambridge Studies in Advanced Mathematics). Cambridge: Cambridge University Press.

[Bu] Buchsbaum, D. A. (1955). *Exact categories and duality.* Transactions of the American Mathematical Society, 80(1), 1. https://doi.org/10.1090/s0002-9947-1955-0074407-6.

[Bu2] Buchweitz, R. O. (1987). *Maximal Cohen-Macaulay Modules and Tate-Cohomology over Gorenstein Rings.*

[CaEi] Cartan, H., Eilenberg, S. (1999). *Homological Algebra* (PMS-19) (n Reprint ed.). Princeton University Press.

[Ch] Christensen, L. W., *Derived Category Methods in Homological Algebra*, Unpublished Manuscript, https://www.math.ttu.edu/ lchriste/download/dcmca.pdf.

[ChJo] Christensen, L. W., Jorgensen, D. A. (2013). *Tate (co)homology via pinched complexes.* Transactions of the American Mathematical Society, 366(2), 667–689. https://doi.org/10.1090/s0002-9947-2013-05746-7.

[DuFo] Dummit, D. S., Foote, R. M. (1991). *Abstract algebra.* Prentice Hall.

[Ei] Eisenbud, D. (1980). *Homological Algebra on a Complete Intersection, with an Application to Group Representations.* Transactions of the American Mathematical Society, 260(1), 35-64. https://doi.org/10.1090/s0002-9947-1980-0570778-7

[Ei2] Eisenbud, D. (1995). *Commutative Algebra: with a View Toward Algebraic Geometry* (Graduate Texts in Mathematics, 150) (1995. Corr. 3rd ed.). Springer.

[Gr] Grothendieck, A., (1957). *Sur quelques points d'algèbre homologique*, Tohoku Math. J., II. Ser. 9, 119-221 ZBL0118.26104.

[Gu] Gulliksen, T. H. (1974). *A Change of Ring Theorem with Applications to Poincaré Series and Intersection Multiplicity.* MATHEMATICA SCANDINAVICA, 34, 167-183. https://doi.org/10.7146/math.scand.a-11518.

[HaPo] Hartshorne, R., Polini, C. (2021). *Quasi-cyclic modules and coregular sequences*. Math. Z. https://doi.org/10.1007/s00209-020-02676-5.

[Ha] Hatcher, A. (2001). *Algebraic Topology* (1st ed.). Cambridge University Press.

[HoJoRo] Holm, T., Jorgensen, P., Rouquier, R., *Triangulated Categories* (London Mathematical Society Lecture Note Series, pp. 1-51), Cambridge: Cambridge University Press.

[Hu] Hungerford, T. (2003). *Algebra*. Springer. ISBN: 0030860784 9780030860782 0387905189 9780387905181

[KaSc] Kashiwara, M., Schapira, P. (2006). *Categories and Sheaves* (Grundlehren der mathematischen Wissenschaften, 332) (1st ed.). Springer.

[Ma] MacLane, S. (1998). *Categories for the Working Mathematician* (Graduate Texts in Mathematics, 5) (2nd ed.). Springer.

[Ma2] MacLane, S., (1950). *Duality for groups*. Bull. Am. Math. Soc. 56, 485-516 ZBL0041.36306.

[Ma3] Matlis, E. (1960).*Modules with descending chain condition*. Transactions of the American Mathematical Society, 97(3), 495. https://doi.org/10.1090/s0002-9947-1960-0169879-4.

[Ma4] May, P., *An Introduction to Homological Algebra*.

[Oo] Ooishi, A., (1976). *Matlis duality and the width of a module.* Hiroshima Mathematical Journal, 6(3) 573-587, https://doi.org/10.32917/hmj/1206136213

[Pu] Puppe, D., (1962). *On the formal structure of stable homotopy theory.* Colloquium on algebraic topology. Aarhus Universitet Matematisk Institute. pp. 65–71. Zbl 0139.41106

[Ro] Rotman, J., (1979). *Introduction to Homological Algebra* (Pure and Applied Mathematics, No. 85). 1st edition Hardcover (1st ed.). Academic Press.

[St] Steele, N. T., (2016). *Support and Rank Varieties of Totally Acyclic Complexes*[Unpublished Doctoral Dissertation]. University of Texas at Arlington.

[Ta] Tate, J. (1957). *Homology of Noetherian rings and local rings.* Illinois J. Math. 1, no. 1, 14–27. doi:10.1215/ijm/1255378502.

[Ve] Veliche, O. (2005). *Gorenstein projective dimension for complexes.* Transactions of the American Mathematical Society, 358(03), 1257–1283. https://doi.org/10.1090/s0002-9947-05-03771-2.

[Ve2] Verdier, J.L. (1963). *Des Catégories Dérivées des Catégories Abéliennes*, Astérisque (Paris: Société Mathématique de France), 239.

[Ve3] Vermani, L. R. (2003). *An Elementary Approach to Homological Algebra* (Monographs and Surveys in Pure and Applied Mathematics) (1st ed.). CRC Press.

[Wa] Wang, F., Kim, H. (2017). *Foundations of Commutative Rings and Their Modules.* Springer Publishing.

[We] Weibel, C. A. (1995). *An Introduction to Homological Algebra* (Cambridge Studies in Advanced Mathematics, Series Number 38). Cambridge University Press.

[We2] Weibel, C. (1999). Chapter 28 – *History of Homological Algebra.*

BIOGRAPHICAL STATEMENT

Rebekah Aduddell was born a unicorn in Orange County, California far, *far* too long ago. She received her B.S. in *How to Occasionally Forget the Important Stuff, But Always Remember the Good Stuff* at some point in the Time-Space Continuum. In the Fall of 2016, she began her adventures as a Mathematics Ph.D. student at The University of Texas, Arlington, at which point she met the devilishly good-looking T-Rex who goes by the name of Thor. Her research interests lie within the areas of Being-A-Math-Ninja and How-to-Tensor-Product-Non-Math-Related-Objects, which is actually quite difficult. She did not write this statement, but finds it to be in good humor and of the utmost top-notch quality. That is all.

To follow knowledge like a sinking star,

Beyond the utmost bound of human thought.

...

To sail beyond the sunset, and the baths

Of all the western stars, until I die.

...

To strive, to seek, to find, and not to yield.

ULYSSES, LORD ALFRED TENNYSON