AN APPROXIMATE SOLUTION TO BUCKLING OF PLATES BY THE GALERKIN METHOD

by

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Presented to the Faculty of the Graduate School of

The University of Texas at Arlington in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE IN MECHANICAL ENGINEERING

THE UNIVERSITY OF TEXAS AT ARLINGTON

August 2005

ACKNOWLEDGEMENTS

I would like to acknowledge with great pleasure and appreciation the support and encouragement by Dr. Seiichi Nomura throughout the thesis work. I am also thankful to Dr. Wen S. Chan and Dr. Dereje Agonafer for serving on my thesis committee.

Special appreciation is extended to my parents Dr. Chandra Prakash Dubey, Mrs. Sarita Dubey, and my sibling Ms. Swati Dubey. I appreciate the encouragement and help that I have received from my roommates and all my friends.

July 25, 2005

ABSTRACT

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Publication No. ______

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The University of Texas at Arlington, 2005

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This thesis presents a semi-analytical method to solve the governing differential equation of plates using the Galerkin method. Symbolic algebra software is extensively used to perform necessary calculations.

In this thesis, the lateral deflection of a plate is expressed in a series of polynomials each of which satisfies the given homogeneous boundary conditions. The coefficients of these polynomials are found out by the Galerkin method. Symbolic algebra software works best while handling necessary algebra to generate admissible polynomials and build required matrices for solving for the coefficients.

This thesis demonstrates the calculation of the lateral load for different plates under different homogeneous boundary conditions and initial condition. As this analysis

involves very complicated computation, it is almost impossible to handle all the calculations without the help of symbolic algebra software.

Numerical examples are presented and the results are compared with the known analytical solutions. It was shown that a reasonable level of convergence is achieved with the present method.

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CHAPTER 1

INTRODUCTION

1.1 Overview

The first significant analysis of plates began in the early 1800s, with much of the work attributed to Cauchy, Poisson, Navier, Lagrange, and Kirchoff. The first mathematical approach to the membrane theory of plates was formulated by Euler (1766), who solved the problem of free vibrations of rectangular and circular elastic membranes using the analogy of two systems of stretched strings perpendicular to each other [17]. The great engineer, Navier (1785-1836), can be considered the real originator of the modern theory of elasticity. His numerous scientific activities included the solution of various plate problems. Navier derived the correct differential equation of rectangular plates with flexural resistance. Navier's method is based on the use of trigonometric series introduced by Fourier in the same decade. Poisson extended (1829) the use of the governing plate equation, derived by Navier, to the lateral vibration of circular plates. The boundary conditions of the problem, as formulated by Poisson, however, are applicable only to thick plates. Kirchoff (1824-1887) [18] is considered the founder of the extended plate theory which takes into account the combined bending and stretching. The research done by these early "Engineers" was extremely significant, and many of the techniques which they developed are still used in engineering analysis today.

However, in the last 30 years, with the advent of the digital computer, other methods such as finite differences and finite elements have become practical. In 1956 Turner, Clough, Martin and Topp [15] introduced the finite element method, which permits the numerical solution of complex plate and shell problems in an economical way. The use of the finite element method, however, anticipates the availability of high-speed computers with considerable storage capacity. Another computer-oriented method for the static and dynamic analysis of plates of arbitrary shape subjected to arbitrary loads is based on improved finite difference techniques developed by Stussi and Collatz.

The equation defining small lateral deflections of the middle surface of a thin plate subjected to lateral loads may be formulated in different ways. The most general method involves the elimination of the unimportant terms from the equations of the three-dimensional elasticity as the thickness is made small compared with other dimensions. A second method, which requires less mathematical rigor but more physical interpretation, utilizes the basic assumptions made in the theory of beams to generate directly the equations for thin plates. Both methods lead to the same results.

The development of the governing partial differential equation defining small lateral deflections of the middle surface of thin plates, as well as the development of the companion relationships, is based on certain assumptions adopted because of prior knowledge about the behavior of beams. Because of the limitation of small deflections,

1. The middle surface is assumed to remain unstrained. Because of the limitation of thinness,

2. Normals to the middle surface before deformation remain normal after deformation, and

3. Normal stresses in the direction transverse to the plate are neglected. A fourth assumption based on the material properties is that

4. The material is homogeneous, isotropic, continuous, and linearly elastic.

1.2 Galerkin Method

Prior to the development of the Finite Element Method, there existed an approximation technique for solving differential equations called the Method of Weighted Residuals (MWR).The basic idea of MWR is to use a *trial function* with a number of unknown parameters to approximate the solution. Then a weighted average over the interior and boundary is set to zero. The idea is to approximate the solution with a polynomial involving a set of parameters. The polynomial is made to satisfy both the differential equation and the associated boundary conditions.

The Method of Weighted Residuals is broadly classified into:

1.2.1 Collocation Method

1.2.2 Least Squares Method

1.2.3 Galerkin Method

The Galerkin method has been used to solve problems in mechanical engineering such as structural mechanics, dynamics, fluid flow, heat and mass transfer, acoustics, neutron transport and others. The Galerkin method can be used to approximate the solution to ordinary differential equations, partial differential equations and integral equations.

The origin of the method is generally associated with a paper published by Galerkin in 1915 on the elastic equilibrium of rods and thin plates. The use of Galerkin method increased rapidly during the 1950's when it was used for analyzing dynamics of aeronautical structures. The link with the Galerkin method permitted the finite element techniques to be extended into areas where no variational principle was available for example, such as many aspects of fluid mechanics, solid mechanics and heat transfer.

The finite element and Galerkin methods are currently the standard numerical technique in use to solve various nonlinear problems. The methods retain the advantages of weak formulations, which lower the continuity requirements of matching elements and permits to use simple basis functions. However, these methods demand a great amount of numerical integration effort in updating Jacobian matrix of each Newton-Raphson iteration.

1.3 Symbolic algebra software

Since the 1960's, research work on the design of algorithms and systems for performing symbolic mathematics (now commonly known as "computer algebra") was underway at institutions such as MIT. The systems which were used during the 1970's (such as MACSYMA and REDUCE) were very large LISP-based systems requiring many megabytes of RAM and extensive CPU time to perform routine mathematical computations. Consequently, only a tiny number of researchers with access to large mainframe computers (and only if they didn't have to time-share with many other users) were able to exploit this technology. With the emergence of modern symbolic algebra software packages such as MATHEMATICA, MATLAB and MAPLE, which were written in the C language and its variations, the megabytes of RAM and CPU running time was considerably reduced.

Mathematica is the one of the most useful general computation system. Ever since the 1960s individual packages had existed for specific numerical, algebraic, graphical, and other tasks. But the visionary concept of *Mathematica* was to create once and for all a single system that could handle all the various aspects of technical computing--and beyond--in a coherent and unified way. The generality of symbolic expressions allows *Mathematica* to cover a wide variety of applications. The symbolic function is built-in and is very simple and robust and does not require any additional commands to activate it. Programs written in *Mathematica* have better readability.

This thesis demonstrates how to find the critical buckling load value of square plates under different constrained conditions using Galerkin method. It takes advantage of both numerical techniques and symbolic algebra software. In this thesis, it is assumed that the lateral deflection is expressed by a series of polynomials each of which satisfies the given boundary conditions. The Galerkin method is used to determine the coefficients of these polynomials.

Chapter 2 focuses on the buckling theory of plates and derivation of the governing equations for buckling of square plates with associated boundary conditions. The application of Galerkin method is also illustrated in this chapter.

Chapter 3 discusses solving the buckling load equation for a square plate subjected to uniform compressive load with different boundary conditions. There are two types of boundary conditions that are being discussed .Examples include –

a) Simply supported boundary conditions

b) Mixed boundary conditions

Chapter 3 will also illustrate the use of Galerkin method to solve for the buckling load using *Mathematica*. Since *Mathematica* handles all the expressions symbolically, a major advantage it has is the ability to carry out calculations and substitutions with ease. After finding the buckling load using the same approximating function as the square plate, a higher order approximating function will be used to show the convergence of this method.

In Chapter 4 numerical results obtained from *Mathematica* programs will be compared with standard numerical examples.

Chapter 5 discusses conclusion and future recommendations to the existing work.

CHAPTER 2

BUCKLING THEORY

2.1 Historical Background

The initial theoretical research into elastic flexural-torsional buckling was preceded by Euler's 1759 treatise[12] on column flexural buckling, which gave the first analytical method of predicting the reduced strengths of slender columns, and by Saint-Venant's 1855 memoir [13]on uniform torsion, which gave the first reliable description of the twisting response of members to torsion.

However, it was not until 1899 that the first treatments were published of flexural-torsional buckling by Michell and Prandtl, who considered the lateral buckling of beams of narrow rectangular cross-section. Their work was extended by Timoshenko[14] to include the effects of warping torsion in I-section beams. Most recently the invention of high-speed electronic computers (1950) exerted a considerable influence on the static and dynamic analysis of plates. Although in 1941 Hrennikoff had already developed an equivalent grid work system for the static analysis of complex plate problem, his fundamental work in discretization of continua could not be fully utilized due to the lack of high-speed computers. In 1956 Turner, Clough, Martin and Topp [15] introduced the finite element method, which permits the numerical solution of computers with considerable storage capacity.

The extension of the general finite element method of structural analysis to flexural-torsional buckling studies, since almost any particular situation can now be analyzed using a general purpose computer program. This development is similar to that which occurred in the in-plane analysis of plane rigid-jointed frames, in which the tabulations of solutions used in the 1930s were replaced by general purpose frame computer analysis programs.

2.2 What is Buckling?

When a slender structure is loaded in compression, for small loads it deforms with hardly any noticeable change in geometry and load carrying ability. On reaching a critical load value, the structure suddenly experiences a large deformation and it may lose its ability to carry the load. At this stage, the structure is considered to have buckled.

Buckling, also known as structural instability may be classified into two categories:

a) Bifurcation buckling

b) Limit load buckling

In bifurcation buckling, the deflection under compressive load changes from one direction to a different direction (e.g., from axial shortening to lateral deflection). In limit load buckling, the structure attains a maximum load without any previous bifurcation, i.e., with only a single mode of deflection.

2.3 Derivation of the Governing equation

We consider the classical thin plate theory (CPT) in deriving the governing equation, which is based on the Kirchhoff hypothesis:

(a) Straight lines perpendicular to the mid-surface (i.e., transverse normals) before deformation remain straight after deformation

(b) The transverse normals do not experience elongation (i.e., they are inextensible).

(c)The transverse normals rotate such that they remain perpendicular to the mid-surface after deformation.

The condition of equilibrium is met by considering a rectangular differential element of dimensions dx, dy and h as shown in figure. For simplicity, only the middle surface of the plate is shown. The stress resultants, which are internal reactions per unit of length, exist because of the transverse distributed load of intensity p(x,y) on the upper surface of the plate.

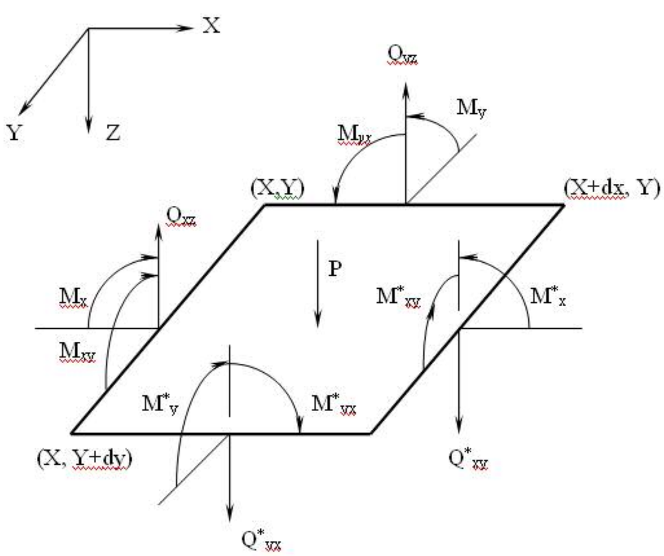

Fig 2.1 Differential plate element with stress resultants

$$M_x = -D(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial y^2}) \tag{2.1}$$

$$M_y = -D(\upsilon \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}) \tag{2.2}$$

$$M_{xy} = M_{yx} = D(1-\upsilon)\frac{\partial^2 w}{\partial x \partial y} \tag{2.3}$$

Expressions for the stress resultants acting on the transverse faces at (x+ dx, y) and at (x, y +dy) are obtained by expanding each into a Taylor's series about (x, y). The higher order terms are considered negligible since dx and dy are quantities of infinitesimal value.

$$Q_{yz}* = Q_{yz} + \frac{\partial Q_{yz}}{\partial y} dy \tag{2.4}$$

$$Q_{xz}* = Q_{xz} + \frac{\partial Q_{xz}}{\partial x} dx$$

$$M_y* = M_y + \frac{\partial M_y}{\partial y} dy$$

$$M_x* = M_x + \frac{\partial Mx}{\partial x} dx \tag{2.5}$$

$$M_{yx}* = M_{yx} + \frac{\partial M_{yz}}{\partial y} dy$$

$$M_{xy}* = M_{xy} + \frac{\partial M_{xy}}{\partial x} dx$$

The condition of a vanishing resultant force in the z-direction results in the equation

$$\frac{\partial Q_{yz}}{\partial y} + \frac{\partial Q_{xz}}{\partial x} + p = 0. \tag{2.6}$$

If the resultant moment about an edge parallel to the x-axis is set equal to zero while neglecting higher order terms, the resulting equation is

$$\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} - Q_{yz} = 0. \tag{2.7}$$

Similarly, the equilibrium equation with respect to rotation about an edge parallel to the y-axis is

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{yx}}{\partial y} - Q_{xz} = 0. \tag{2.8}$$

if the expression for Q_{yz} from Eq.(2.7) and the expression for Q_{xz} from Eq. (2.8) is substituted into Eq. (2.6), the resulting equation is

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p \tag{2.9}$$

Substitution of Eqs. (2.1), (2.2), and (2.3) into Eqs. (2.7) and (2.8) gives the expressions for Q_{xz} and Q_{yz} in terms of the deflection of the middle surface

$$Q_{xz} = -D(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}) = -D\frac{\partial}{\partial x}(\nabla^2 w) \tag{2.10}$$

and

$$Q_{yz} = -D(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y}) = -D\frac{\partial}{\partial y}(\nabla^2 w) \tag{2.11}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2.12}$$

2.3.1 Governing Equation

The governing partial differential equation defining the lateral deflection of the middle surface of the plate in terms of the applied transverse load is obtained by direct substitution of Eqs. (2.1), (2.2), and (2.3) into the equilibrium Eq.(2.9). The result of this is

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}\frac{\partial^2 w}{\partial x^2} \tag{2.13}$$

or

$$\nabla^4 w = \frac{p}{D}\frac{\partial^2 w}{\partial x^2} \tag{2.14}$$

or

$$\nabla^2 \nabla^2 w = \frac{p}{D}\frac{\partial^2 w}{\partial x^2} \tag{2.15}$$

Where

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \tag{2.16}$$

The fourth-order partial differential Eq. (2.13) can be reduced to two second-order partial differential equations which are sometimes preferred, depending upon the method of solution to be used. This reduction is accomplished as follows. The addition of Eqs. (2.1) and (2.2) gives

$$M_x + M_y = -D(1+\upsilon)(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}) \tag{2.17}$$

or

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{M_x + M_y}{D(1+\upsilon)} \tag{2.18}$$

or

$$\nabla^2 w = \frac{M}{D} \tag{2.19}$$

where $$M = -\frac{M_x + M_y}{1+\upsilon} \tag{2.20}$$

Substitution of Eq. (2.19) into (2.15) gives

$$\nabla^2 (\frac{M}{D}) = \frac{p}{D} \tag{2.21}$$

or $$\nabla^2 M = p \tag{2.22}$$

Thus, the fourth order Eq. (2.13) has been reduced to the two second-order Eqs, (2.19) and (2.22). If boundary conditions and the transverse load p are known, Eq. (2.22) can be solved for M(x,y). Then Eq. (2.19) can be solved for *w(x,y).*

A complete solution of the governing Eq. (2.13) depends upon the knowledge of the conditions of the plate at the boundaries in terms of the lateral deflection of the middle surface w(x,y). Thus, expressions for these conditions must be developed. In general, there are four types of mathematically "exact" solutions available for plate problems:

The rigorous solution of plate problems is essentially a boundary value problem of mathematical physics. Since the fulfillment of the boundary conditions usually presents considerable mathematical difficulties, in general, rigorous solutions of plate problems are rare. The most general form of the rigorous solution of the governing differential equation can be written as

$$w(x,y) = w_H(x,y) + w_P(x,y) \tag{2.23}$$

where w_H represents the solution of the homogeneous equation $\nabla^2 \nabla^2 w = 0$, and w_P is a particular solution of the non-homogeneous differential equation of the plate. There are few cases, however, when the solution can be obtained directly, without employing the above mentioned superposition principle.

Certain boundary conditions permit the use of special solutions, such as the Navier solution. In the Navier solution $w_H = 0$; thus

$$w(x, y) = w_P(x, y). \tag{2.24}$$

2.3.1.1 Solution of the Homogeneous Equation.

The physical interpretations of the solution of the biharmonic equation $(\nabla^4 w = 0)$ is to obtain the deflection of the plate $w_H(x, y)$ when only edge forces are acting. Consequently, the solution of the homogeneous equation fulfills the prescribed boundary conditions and maintains the equilibrium with the external boundary forces. The fundamental difficulty of the rigorous solution is the proper choice of functions $w_H(x, y)$ for a given problem.

The biharmonic equation $(\nabla^4 w = 0)$ permits the use of the solutions of the Laplace equation $\nabla^2 w = 0$, which are

$$x,\ xy,\ \cos\alpha x,\ \cosh\alpha y,\ x^2 - y^2,\ \text{and}\ x^3 - 3xy^2, \tag{2.25}$$

Where α, represents an arbitrary constant. In Eq. (2.25) we may interchange x and y and replace cos by sin and cosh by sinh, respectively. If $w_1(x, y)$ and $w_2(x, y)$ are solutions of the Laplace equation, then

$$w_1 + xw_2,\ w_1 + yw_2,\ \text{and}\ w_1 + (x^2 + y^2)w_2 \tag{2.26}$$

are solutions of the biharmonic equation $\nabla^2\nabla^2 w = 0$.

2.3.1.2 Particular Solution

Although the solution w_H of the biharmonic equation effectively describes the equilibrium conditions of the plate subjected to edge forces, the expression of the deflection *w(x, y)* is not complete without also considering the equilibrium of the lateral forces $P_{z.}$ for this purpose, a particular solution w_P of the nonhomogeneous differential equation (2.13) must also be determined. We require from the particular solution that it satisfy the differential equation of the plate (2.13), but the fulfillment of the boundary conditions is not mandatory. In the case of an infinite series solution, however, a more rapid convergence can be obtained if the particular solution at least fulfills the boundary conditions of two opposite edges of the plate.

For rectangular plates with fixed, or partially fixed, boundary conditions at certain or all edges, the expression of deflections of the simply supported plate can be used as a particular solution. This can be obtained with relative ease using Navier's method. Finally, the possibility of expressing the true singularity of the problem, in the case of a concentrated lateral load, must be mentioned briefly. If such a force acts at the center of a rectangular plate, for instance, the particular solution can be written as

$$w_P(x, y) = \frac{P_z}{16\pi D}(x^2 + y^2)\ln\frac{x^2 + y^2}{a^2}. \qquad (2.27)$$

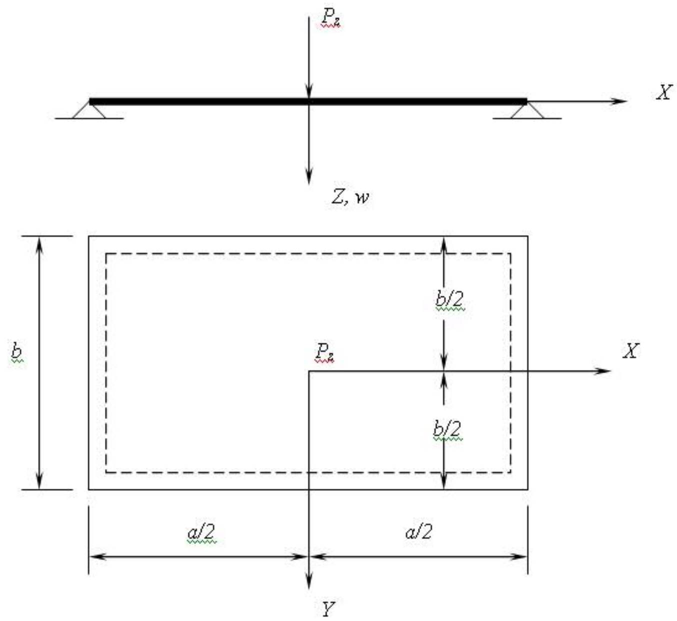

Fig. 2.2 Rectangular plate under concentrated load

These types of solutions coupled with Maxwell's law of reciprocity are used for obtaining influence surfaces for plates.

2.3.1.3 Fourier Series

The Fourier series are indispensable instruments in the analytical treatment of many problems in the field of applied mechanics, such as the solution of partial differential equations of the theory of elasticity, vibrations, flow of heat, transmission of electricity, and electromagnetic waves. The extension of the Fourier series leads to the Fourier integrals and to the Fourier transforms.

2.3.1.4 Single Fourier series

Fourier's theorem states that any arbitrary function $y = f(x)$ can be expressed by an infinite series, consisting of sine and cosine terms. Thus the original function is replaced by the superposition of numerous sine and cosine waves. If $f(x)$ is a periodic function, the Fourier's theorem states that

$$f(x) = \frac{1}{2}A_0 + A_1\cos\frac{2\pi x}{T} + A_2\cos\frac{4\pi x}{T} + \dots\dots\dots + A_n\cos\frac{2n\pi x}{T} + \dots\dots\dots \quad (2.28)$$
$$+ B_1\sin\frac{2\pi x}{T} + B_2\sin\frac{4\pi x}{T} + \dots\dots\dots\dots + B_n\sin\frac{2n\pi x}{T} + \dots\dots\dots$$

or, in more concise form,

$$f(x) = \frac{1}{2}A_0 + \sum_1^\infty A_n\cos n\omega x + \sum_1^\infty B_n\sin n\omega x, \quad (2.29)$$

where A_0 , A_n , and B_n (n=1,2,3….) are the coefficients of Fourier expansion;

ω represents

$$\omega = \frac{2\pi}{T} \quad (2.30)$$

and T is the period of the function

Equation (2.29) is valid for any piecewise regular periodic function, which might also have discontinuities, and represents the arbitrary periodic function *f(x)* in the full range from $x = -\infty$ to $x = +\infty$; thus it is called *full-range expansion*.

The coefficients A_0 , A_n , and B_n are obtained from (2.28)(2.29)(2.30)

$$A_0 = \frac{2}{T}\int_0^T f(x)dx \quad (2.31)$$

$$A_n = \frac{2}{T}\int_0^T f(x)\cos n\omega x dx \qquad (n = 1,2,3,\dots\dots) \quad (2.32)$$

$$B_n = \frac{2}{T}\int f(x)\sin n\omega x dx, \quad (2.33)$$

When the function *f(x)* is not given in analytical form or it is too complicated to perform the integrations prescribed, then the so-called harmonic analysis, which replaces the integrals by summations, can be advantageously utilized. Dividing the period T into 2m equals intervals, the Fourier coefficients are determined from

$$A_0 = \frac{1}{m}\sum_{k=0}^{2m-1} y_k \tag{2.34}$$

$$A_n = \frac{1}{m}\sum_{k=0}^{2m-1} y_k \cos\frac{kn\pi}{m} \tag{2.35}$$

$$B_n = \frac{1}{m}\sum_{k=0}^{2m-1} y_k \sin\frac{kn\pi}{m}, \tag{2.36}$$

(k=0, 1, 2… 2*m* and *n* = 1, 2, 3…*m)*

Another approximate method for evaluating the constant of the Fourier expansion consists of plotting $f(x), f(x)\cos(2n\pi x/T) and f(x)\sin(2n\pi x/T)$ curves and determining the area of the respective curves by planimeter.

2.3.1.5 Double Fourier series

In the static and dynamic analysis of plates, as discussed later, a given function *f(x, y)* is often expanded into sine series of two variables, *x* and *y*, using the following expression:

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{2.37}$$

The above equation represents a half range sine expansion in *x*, multiplied by a half-range sine expansion in *y*, using for the period of expansion *T=2a* and *T=2b*, respectively. To obtain the coefficient F_{mn}, we first multiply the equation by $\sin(k\pi y/b)dy$ and then integrate between the limits 0 and b. Thus, we can write,

$$\int_0^b f(x,y)\sin\frac{k\pi y}{b}dy = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn}\sin\frac{m\pi x}{a}\int_0^b \sin\frac{n\pi y}{b}\sin\frac{k\pi y}{b}dy. \tag{2.38}$$

if $n \neq k$, then

$$\int_0^b \sin\frac{n\pi y}{b}\sin\frac{k\pi y}{b}dy = 0. \tag{2.39}$$

if $n = k$, then

$$\int_0^b \sin^2\frac{n\pi y}{b}dy = \frac{b}{2}. \tag{2.40}$$

Utilizing a similar approach for the variable x, we obtain

$$\int_0^a \sin^2\frac{m\pi x}{a}dx = \frac{a}{2}. \tag{2.41}$$

Thus Eq. (2.37) becomes

$$F_{mn}\frac{a}{2}\frac{b}{2} = \int_0^a\int_0^b f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy. \tag{2.42}$$

Hence the coefficient of the double Fourier expansion is

$$F_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy. \tag{2.43}$$

Three types of boundary are considered at this time: simply supported, clamped, and mixed.

2.3.2 Boundary Conditions

A complete solution of the governing Eq. (2.13) depends upon the knowledge of the conditions of the plate at the boundaries in terms of the lateral deflection of the middle surface *w(x, y).* Thus, expressions for these conditions must be developed. Three types of boundary are considered at this time: simply supported, clamped, and free.

2.3.2.1 Simply Supported Edge Conditions

A plate boundary that is prevented from deflecting but free to rotate about a line along the boundary edge, such as a hinge, is defined as a simply supported edge. The conditions on a simply supported edge paralled to the y-axis at $x = a$, are.

$$w|_{X=a} = 0 \tag{2.44}$$

$$M_x|_{X=a} = -D\left(\frac{\partial^2 w}{\partial x^2} + \upsilon\frac{\partial^2 w}{\partial y^2}\right)_{X=a} = 0\,. \tag{2.45}$$

Since the change of *w* with respect to the y coordinate vanishes along this edge, these conditions become

$$w|_{X=a} = 0 \tag{2.46}$$

$$\frac{\partial^2 w}{\partial x^2}|_{X=a} = 0\,. \tag{2.47}$$

On a simply supported edge parallel to the x-axis at y = b, the change of *w* with respect to the x-coordinate vanishes; thus, the condition along this boundary are

$$w|_{Y=b} = 0 \tag{2.48}$$

$$M_y|_{Y=b} = -D\left(\frac{\partial^2 w}{\partial y^2} + \upsilon\frac{\partial^2 w}{\partial x^2}\right)_{Y=b} = 0 \tag{2.49}$$

$$= -D\frac{\partial^2 w}{\partial y^2}|_{Y=b} = 0 \tag{2.50}$$

2.3.2.2 Clamped Edge Conditions

If a plate is clamped, the deflection and the slope of the middle surface must vanish at the boundary. On a clamped edge parallel to the y-axis at $x=a$, the boundary conditions are

$$w|_{x=a} = 0 \tag{2.51}$$

$$\frac{\partial w}{\partial x}|_{x=a} = 0 \tag{2.52}$$

The boundary conditions on a clamped edge parallel to the x- axis at $y=b$ are

$$w|_{y=b} = 0 \tag{2.53}$$

$$\frac{\partial w}{\partial y}|_{y=b} = 0. \tag{2.54}$$

2.3.2.3 Mixed Edge Conditions

If a plate is simply supported along two opposite sides and clamped on the other two sides at y=*0*, and y=*b.* The boundary conditions are

$$w = 0, \tag{2.55}$$

$$\frac{\partial w}{\partial y} = 0 \qquad \text{at } y=0,\ b \tag{2.56}$$

also, on the simply supported edge parallel to the y-axis , the boundary conditions at $x=0$, and $x=a$ are

$$w=0, \tag{2.57}$$

and $$M|_{x=a} = -D(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial y^2})_{x=a} = 0. \quad (2.58)$$

According to Kirchoff, only two boundary conditions are sufficient for the complete determination of the deflection *w* satisfying the governing equation of the plate and the three conditions derived from physical reasoning are too many. This inconsistency is due to the assumption that the normals of the middle plane before bending are deformed into the normals of the middle plane after bending. Without using an assumption, a sixth-order differential equation can be obtained for which all the three boundary conditions can be satisfied. It can be shown, however, that, except in the immediate region of the boundary, the stress distribution given by this new equation is substantially the same as that given by the *governing equation*. If this plate is thin, the sixth-order terms can be neglected and this new equation reduces to *governing equation*. This justifies the use of *governing equation* in the study of the bending of thin plates.

2.4 Galerkin method

The Galerkin method is a member of the methods of weighted residuals (MWR). This concept of weighted residuals was introduced by Crandall [16].

Consider a differential equation that can be represented in the following form,

$$L\,(T) = f, \quad (2.59)$$

in a domain *D(x, y),* with boundary conditions T = 0 on ∂D, the boundary of *D*. Where *L* is a differential operator in the Sturm-Liouville system defined as

$$Lf = (k\,(x)f')'. \quad (2.60)$$

The Galerkin method assumes that T can be approximately represented as

$$T_a(x, y) = \sum_{j=1}^{N} a_j \phi_j(x, y), \qquad (2.61)$$

Where,

ϕ_j = Known analytic functions (trial functions),

a_j = Coefficients yet to be determined,

If we have a partial differential equation with a non-homogeneous boundary condition, then we can rewrite eq. (2.16) as

$$T_a(x, y) = T_0(x, y) + \sum_{j=1}^{N} a_j \phi_j(x, y), \qquad (2.62)$$

where T_0 was introduced to satisfy the inhomogeneous boundary conditions.

The residual, R, is defined as,

$$R(a_0, a_1, \ldots\ldots\ldots\ldots, a_n, x, y) = L(u_a) - f$$

$$= L(u_0) + \sum_{j=1}^{N} a_j L(\phi_j) - f. \qquad (2.63)$$

if an inner product, *(f,g)* , between two functions, *f(x, y)* and *g(x, y),* can be defined as

$$(f, g) = \iint_D fg dx dy \qquad (2.64)$$

then the unknown coefficients, a_j ,are obtained by solving the following systems of equations,

$$(R, \phi_k) = 0 \qquad k=1,\ldots\ldots,N \qquad (2.65)$$

where ϕ_k's are the same trial functions. The above equation can be written as a matrix equation for the coefficient a_j as,

$$\sum_{j=1}^{N} a_j (L(\phi_j), \phi_k) = -(L(u_0), \phi_k) \tag{2.66}$$

So, by solving the above equation, we can find out a_j. we can substitute, a_j in eq (2.15) to obtain $T_a(x, y)$.

The trial functions, $\phi_j(x)s$, should be chosen from a complete set of polynomials of the N^{th} order. It is necessary condition for convergence to the exact solution as $N \to \infty$.

The conditions required in applying the traditional Galerkin method are as follows:

(1) The functions ϕ_k , are chosen from the same family as the trial functions $\phi_{j.}$

(2) The trial functions must be linearly independent.

(3) The trial functions should satisfy the homogeneous boundary conditions exactly.

The accuracy of the Galerkin method is influenced by the order of polynomials and the choice of trial functions. The efficiency of the Galerkin method can be defined in terms of the solution accuracy per unit of computer execution time.

There exists a close connection between the Rayleigh-Ritz method and the Galerkin method in connection with the finite-element method. The importance of the equivalence of the Galerkin method and the Rayleigh-Ritz method is that the convergence of the Rayleigh-Ritz solution to the exact solution as N, the number of terms, approaches to infinity is well established as long as the trial functions are members of a complete set of functions. The convergence properties associated with the Rayleigh-Ritz method carry over to the Galerkin method and the equivalence is also exploited in obtaining error estimates.

CHAPTER 3

METHOD OF WEIGHTED RESIDUALS

3.1 Introduction

Prior to the development of the finite element method, there existed an approximation technique for solving differential equations called the method of weighted residuals (MWR).The basic idea of MWR is to use a *trial function* with a number of unknown parameters to approximate the solution. Then a weighted average over the interior and boundary is set to zero. The idea is to approximate the solution with a polynomial involving a set of parameters. The polynomial is made to satisfy both the differential equation and the associated boundary conditions.

The main objective is to solve general linear equations in the form

$$L\,u = c \tag{3.1}$$

Where L is a linear operator (differential operators, matrices etc.), u is the unknown function and c is a given function.

An approximate solution to eq. (3.1) is sought by a linear combination of N base vectors in the linear space as

$$\tilde{U} = \sum_{i=1}^{N} u_i e_i , \tag{3.2}$$

where u_i is the unknown coefficient and $\boldsymbol{e_i}$ is the base vector in the linear space.

The residual (error), $\boldsymbol{R}$, between the approximate solution and the exact solution is defined as

$$\boldsymbol{R} \equiv L\,\tilde{u} - c \tag{3.3}$$

$$= L\sum_{i=1}^{N} u_i e_i - c \tag{3.4}$$

$$= \sum_{i=1}^{N} u_i L e_i(x) - c(x) \tag{3.5}$$

Note that for a function space, $\boldsymbol{R}$ is a function of the position, i.e.

$$\boldsymbol{R}(\mathrm{x}) = \sum_{\mathbf{i=1}}^{N} u_i\, L\, \boldsymbol{e_i(x)} - \boldsymbol{c(x)}. \tag{3.6}$$

The Method of Weighted Residuals is broadly classified into the following methods

3.1.1 Collocation method

Choose u_i so that the residual (error) vanishes at N selected points, i.e.

$$\boldsymbol{R}\,(\mathrm{x_i}) = 0,\ \mathrm{i} = 1, \ldots\ldots N \tag{3.7}$$

Although this method gives the exact values at the selected points, there is no guarantee that the approximation behaves nicely between the selected points.

3.1.2 Least Square method

Choose *ui* so that the magnitude of residual (error) becomes the minimum, i.e.

$$\|\,\boldsymbol{R}(\mathrm{x})\,\| \rightarrow min. \tag{3.8}$$

This method is expected to give an overall well-behaved approximation.

3.1.3 Galerkin method

Choose u_i so that $\boldsymbol{R}$ is orthogonal to N base functions $(\boldsymbol{e_i})$, i.e.

$$(\boldsymbol{R}, \boldsymbol{e_i}) = 0 \quad i = 1, \ldots, N. \tag{3.9}$$

The idea of the Galerkin's method is that if $\boldsymbol{e_i}$*'s* span the entire linear space, a vector that is perpendicular to all the base vectors must be a zero vector.

The Galerkin method has been used to solve problems in mechanical engineering such as structural mechanics, dynamics, fluid flow, heat and mass transfer, acoustics and other related fields. The Galerkin method can be used to approximate the solution to ordinary differential equations, partial differential equations and integral equations.

3.2 Application of Galerkin method

Initially the finite-element method was used for constructing matrix solutions to stress and displacement calculations in structural analysis. The finite-element method is a special case of the Galerkin method in which the base functions are chosen such that each base function becomes 1 at the corresponding nodes but otherwise 0 at other nodes. The link with the Galerkin method permitted finite-element techniques to be extended into areas such as fluid mechanics and heat transfer. At the present time, the Galerkin finite element formulation is the most popular finite-element method.

Consider a rectangular plate as shown in figure with dimensions of a and b.

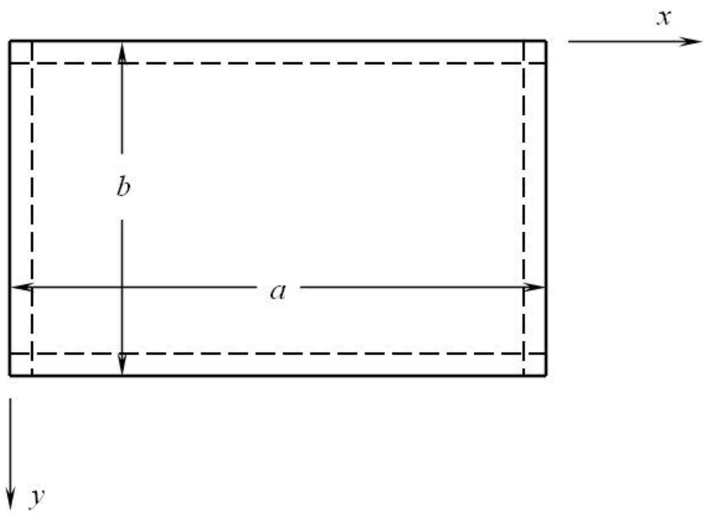

Fig 3.1 Rectangular plate

The governing differential equation for a plate can be written as:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p_z(x,y)}{D}\frac{\partial^2 w}{\partial x^2} \qquad (3.10)$$

Where p_z is the external load acting on the plate surface and D is the bending or *flexural rigidity* of the plate. The exact solution of the governing plate equation (3.10) must simultaneously satisfy the differential equation and the boundary conditions of any given plate problem. Since Eq (3.10) is a fourth-order differential equation, two boundary conditions, either for the displacements or for the internal forces, are required at each boundary.

The displacement components to be used in formulating the boundary conditions are lateral deflections and slope. At fixed edges, for instance, the deflection and the slope of the deflected plate surface are zero.

$$(w)_x = 0, \qquad (\frac{\partial w}{\partial x})_x = 0 \qquad (x=0 \text{ or } x=a) \qquad (3.11)$$

$$(w)_y = 0, \qquad (\frac{\partial w}{\partial y})_y = 0 \qquad (y=0 \text{ or } y=b) \qquad (3.12)$$

Equation (3.10) can be rewritten using the two-dimensional Laplacian operator $(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ as :

$$\nabla^2 \nabla^2 w = \frac{p_z}{D} \frac{\partial^2 w}{\partial x^2} \qquad (3.13)$$

Consider a differential operator L, defined as

$$L = \left[\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right].$$

So, eq.(3.10) becomes,

$$Lw = \frac{p_z}{D} \qquad (3.14)$$

We can express the solution to the above equation in terms of the eigenfunction and eigenvalues, which are defined as:

$$Le_{nm}(x, y) = \lambda_{nm} Le_{nm}(x, y) \qquad (3.15)$$

Where $e_{nm}(x, y)$ are the eigenfunctions and λ_{nm} are the corresponding eigenvalues. Once the eigenfunctions and eigenvalues are known, it is possible to express $w(x, y)$ as,

$$w(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} w_{nm} e^{\lambda_{nm}} e_{nm}(x, y) \qquad (3.16)$$

Where, w_{nm} are unknown coefficients

The eigenfunctions, $e_{nm}(x, y)$, are expressed as a linear combination of base functions, $f_i(x, y)$, as

$$e_{nm}(x,y) = \sum_{i}^{N} c_i^{nm} f_i(x,y) \tag{3.17}$$

We have to select the base functions, $f_i(x, y)$, which satisfy the boundary conditions. So substituting Eq. (3.17) in Eq. (3.15) gives,

$$\sum_{i}^{N} c_i^{nm} Lf_i(x,y) = \sum_{i}^{N} \lambda_{nm} c_i^{nm} Lf_i(x,y). \tag{3.18}$$

now, multiplying Eq.(3.18) with another base function $f_j(x, y)$ which also satisfies the boundary conditions as

$$\sum_{i}^{N}\sum_{j}^{N} c_i^{nm} Lf_i(x,y) f_j(x,y) = \sum_{i}^{N}\sum_{j}^{N} \lambda_{nm} c_i^{nm} Lf_i(x,y) f_j(x,y) \tag{3.19}$$

We can write the above equation in the following form,

$$A\bar{c} = \lambda_{nm} B\bar{c}, \tag{3.20}$$

Where

$$\left.\begin{aligned} a_{ij} &= \int_0^a\int_0^b LLf_i(x,y) f_j(x,y)\,dydx \\ b_{ij} &= \int_0^a\int_0^b Lf_i(x,y) f_j(x,y)\,dydx \end{aligned}\right| \tag{3.21}$$

The quantities A and B are N x N square matrices as shown below:

$$A=\begin{bmatrix} a_{11}..............a_{1N} \\ .... \\ .... \\ a_{NN}................a_{NN} \end{bmatrix} \qquad B=\begin{bmatrix} b_{11}..............b_{1N} \\ .... \\ .... \\ b_{NN}...................b_{NN} \end{bmatrix}$$

Where λ_{nm}, is the eigenvalue and $\bar{c}$ will be the corresponding eigenvectors.

For calculating the eigenvalues and corresponding eigenvectors, we used the Cholesky decomposition.

Applying the Cholesky decomposition on matrix B as

$$B = U^T U, \tag{3.22}$$

where, U is the upper triangular matrix and U^T is the transpose of matrix U.

So Eq. (3.20) becomes,

$$A\bar{c} = \lambda_{nm} U^T U \bar{c}. \tag{3.23}$$

Substituting,

$$U\bar{c} = \bar{x}$$

$$\bar{c} = U^{-1}\bar{x}$$

Eq. (3.22) then becomes,

$$AU^{-1}\bar{x} = \lambda_{nm} U^T \bar{x} \tag{3.24}$$

$$U^{T^{-1}} AU^{-1}\bar{x} = \lambda_{nm} U^{T^{-1}} U^T \bar{x} \tag{3.25}$$

Finally Eq. (3.25) becomes,

$$\Psi \bar{x} = \lambda_{nm} \bar{x},$$

Where,

$$\Psi = U^{T^{-1}} AU^{-1}. \tag{3.26}$$

We can calculate the eigenvalues of λ_{nm} and corresponding eigenvectors $\bar{c}$ with the help of symbolic algebra software *Mathematica*. The matrix B is positive definite and symmetric. Cholesky decomposition is a faster and stable method than any other alternative methods for solving linear equations.

We can also calculate the values of unknown coefficients w_{nm} used in the calculation of approximate solution as follows:

$$w_{nm} = \int_0^a \int_0^b f(x,y) e_{nm}(x,y) dy dx \tag{3.27}$$

Where $f(x, y)$, is the initial condition.

So substituting the calculated values in Eq. (3.16) we can get the lateral deflection of the rectangular plate.

It was mentioned in Chapter 2 that two types of boundary conditions are being considered here.

3.2.1 Simply Supported Boundary Condition

To study each of the boundary conditions stated before, we need to begin with the governing differential equation of the plate. At first, considering the simply supported boundary condition for a square plate subjected to lateral loads.

The governing differential equation of the plate subjected to lateral loads is:

$$\frac{\partial^4 w}{\partial x^4} + \frac{2\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p_z(x,y)}{D} \frac{\partial^2 w}{\partial x^2} \tag{3.28}$$

where,

p_z is the lateral load that is being applied

D is the bending or flexural rigidity of the plate

A plate boundary that is prevented from deflecting but free to rotate about a line along the boundary edge, such as a hinge, is defined as a simply supported edge. The conditions on a simply supported edge paralled to the y-axis at $x = a$, are.

$$w|_{X=a} = 0 \tag{3.29}$$

$$M_x|_{X=a} = -D\left(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial y^2}\right)_{X=a} = 0 \tag{3.30}$$

Since the change of w with respect to the y coordinate vanishes along this edge, these conditions become

$$w|_{X=a} = 0 \tag{3.31}$$

$$\frac{\partial^2 w}{\partial x^2}|_{X=a} = 0\,. \tag{3.32}$$

On a simply supported edge parallel to the x-axis at $y = a$, the change of w with respect to the x-coordinate vanishes; thus, the conditions along this boundary are

$$w|_{Y=a} = 0 \tag{3.33}$$

$$M_y|_{Y=a} = -D\left(\frac{\partial^2 w}{\partial y^2} + \upsilon \frac{\partial^2 w}{\partial x^2}\right)_{Y=a} = 0 \tag{3.34}$$

By assuming a solution in the form

$$w(x, y) = \sum_i c_i e_i(x, y) \tag{3.35}$$

Eq. (3.28) can be solved using the standard Galerkin method. Substituting Eq. (3.35) into Eq. (3.28) and integrating over the entire plate produces the following eigenvalue problem.

$$[A]\bar{c} = \lambda[B]\bar{c} \tag{3.36}$$

by introducing the notation

$$\bar{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \tag{3.37}$$

also, $$\Delta = \bar{\nabla}.\bar{\nabla} \tag{3.38}$$

Using the Galerkin method, the elements of [A] and [B] matrices are represented as:

$$a_{ij} = \int_0^b \int_0^a \Delta\Delta e_i e_j dxdy \tag{3.39}$$

$$b_{ij} = \int_0^b \int_0^a \Delta e_i e_j dxdy \tag{3.40}$$

These expressions can be simplified by integration by parts, thus distributing the operator over both e_i and e_j.

$$a_{ij} = \int_0^b \int_0^a \Delta\Delta e_i e_j dxdy \tag{3.41}$$

$$= \int_0^b \int_0^a (\bar{\nabla}.\bar{\nabla})\bar{\nabla}.\bar{\nabla}\, e_i e_j dxdy \tag{3.42}$$

When applying surface integral over the entire surface of the plate and homogeneous boundary conditions a_{ij} becomes

$$a_{ij} = -\int_0^b \int_0^a \bar{\nabla}(\bar{\nabla}.\bar{\nabla})e_i \,\bar{\nabla}\, e_j dxdy \qquad (3.43)$$

Using another homogeneous boundary condition gives the expression:

$$a_{ij} = \int_0^b \int_0^a (\nabla.\nabla)e_i (\nabla.\nabla)e_j dxdy \qquad (3.44)$$

By substituting equation (3.37) into equation (3.44), the final expression for members a_{ij} is revealed as.

$$a_{ij} = \int_0^b \int_0^a \left[\frac{\partial^2 e_i}{\partial x^2} + \frac{\partial^2 e_i}{\partial y^2} \right] . \left[\frac{\partial^2 e_j}{\partial x^2} + \frac{\partial^2 e_j}{\partial y^2} \right] dxdy \qquad (3.45)$$

Deriving the expression for b_{ij} in the similar way by starting with equation (3.40) and using integration by parts, we get.

$$b_{ij} = -\int_0^b \int_0^a \Delta e_i e_j dxdy \qquad (3.46)$$

The first step in solving this problem is to systematically choose a trial function e that satisfies the plate's boundary conditions. Polynomial approximating functions will be used to represent the lateral displacement of the plate. In this discussion, the trial function $e_i\,(x, y)$ will be represented as:

$$\phi_i(x, y) = \sum_{i=1}^{N} \sum_{j=1}^{N} a[i, j] u_j(x, y) \qquad (3.47)$$

Where

$$u_j(x, y) = x^{L_j} . y^{M_j} \qquad (3.48)$$

and, L_j and M_j are positive integers and ϕ_i are coefficients to be determined.

In the simply supported boundary condition example, it is found that an eight order polynomial is the lowest order possible to satisfy the boundary conditions.

$$\phi_i = a[1,1] + a[2,1]x + a[3,1]y + a[4,1]\ xy$$

$$\begin{aligned} &+ a[5,1]x^2 + a[6,1]y^2 + a[7,1]xy^2 + ... \\ &+ a[42,1]x^3y^5 + a[43,1]x^2y^6 + a[44,1]xy^7 + a[45,1]y^8 \end{aligned} \tag{3.49}$$

where $a[1,1]$ through $a[45,1]$ are unknown coefficients of ϕ_i.

Once a general Nth order polynomial is defined as mentioned in Eq.(3.49), the next step is to apply the boundary conditions and solve for the unknown coefficients. This is carried out by developing eight new equations using *Mathematica.*

Each equation represents a different boundary condition.

For Deflection

$$\text{BC (1):}\quad \phi(0,y) = 0 \tag{3.50}$$

$$\text{BC (2):}\quad \phi(a,y) = 0 \tag{3.51}$$

$$\text{BC (3):}\quad \phi(x,0) = 0 \tag{3.52}$$

$$\text{BC (4):}\quad \phi(x,b) = 0 \tag{3.53}$$

For Moments

$$\text{BC (5):}\quad \left(\frac{\partial^2\phi}{\partial x^2} + \upsilon\frac{\partial^2\phi}{\partial y^2}\right)_{x=0} = 0 \tag{3.54}$$

$$\text{BC (6):}\quad \left(\frac{\partial^2\phi}{\partial x^2} + \upsilon\frac{\partial^2\phi}{\partial y^2}\right)_{x=a} = 0 \tag{3.55}$$

$$\text{BC (7):}\quad \left(\upsilon\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)_{y=0} = 0 \tag{3.56}$$

$$\text{BC (8):} \quad \left(\upsilon \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right)_{y=b} = 0 \tag{3.57}$$

This results in an undetermined system of one hundred forty five equations with forty-five unknowns. Flattening out all the nested eight equations above using the Flatten [Table]. command in *Mathematica*. Tabulating all the eight equations and solving, we get values of coefficients *a* [i,j].

Substituting these values of the coefficients back in the generalized polynomial function we end up with an eighth order polynomial (3.49), and in this case, one independent equation results in the following polynomial.

$$\phi_1(x,y) = \{x(1-2x^2+x^3)y(1-2y^2+y^3)\} \tag{3.58}$$

Using *Mathematica* to evaluate the elements of matrices [A] and [B] wherein integration operations are carried over the polynomial. The eigenvalue is evaluated using the Eigenvalue[] command in *Mathematica,* which matches with the value found from Eq. (3.36).

The general eighth order polynomial that is generated from the generalized polynomial Eq. [3.49] is as follows:

$$\begin{aligned} \phi_1(x,y) = {} & a[1] + xa[2] + ya[3] + x^2a[4] + xya[5] + y^2a[6] + \\ & x^3a[7] + x^2ya[8] + xy^2a[9] + y^3a[10] + x^4a[11] + \\ & x^3ya[12] + x^2y^2a[13] + xy^3a[14] + y^4a[11] + x^3ya[12] \\ & + x^2y^2a[13] + xy^3a[14] + y^4a[15] + \dots\dots\dots\dots\dots\dots + x^3y^3a[25] \\ & + x^2y^4a[26] + xy^5a[27] + y^6a[28] + \dots\dots\dots\dots\dots\dots + x^3y^5a[42] \\ & + x^2y^6a[43] + xy^7a[44] + y^8a[45]. \end{aligned} \tag{3.59}$$

The coefficients a[i,j] which are obtained by solving the equations of boundary conditions are as follows :

Table 3.1 – Coefficients of 8th order polynomial for simply supported plate with aspect ratio 1.0

a[5,1]	a[12,1]	a[14,1]	a[17,1]	a[20,1]	a[25,1]	a[32,1]	a[33,1]
1.00	-2.00	-2.00	1.00	1.00	4.00	-2.00	-2.00

Once the eigenvalue problem is solved using the generalized method, the entire procedure is carried out by calculating the eigenvalues and corresponding eigenvectors using the Cholesky decomposition. Wherein the Cholesky decomposition is applied to matrix [B] as

$$B = U^T U,$$

where, U is the upper triangular matix and U^T is the transpose of U.

The result obtained using the Cholesky decomposition was very accurate and compatible with the direct eigenvalue method.

In order to improve on the percent error and get good convergence, a higher order approximating polynomial must be considered. A ninth order polynomial is formed by adding terms to equation (3.49).

$$\begin{aligned}\phi_i = {} & a[1,1] + a[2,1]x + a[3,1]y + a[4,1]\ xy \\ & + a[5,1]x^2 + a[6,1]y^2 + a[7,1]xy^2 + ... \\ & + a[42,1]x^3y^5 + a[43,1]x^2y^6 + a[44,1]xy^7 + a[45,1]y^8 \\ & + a[46]x^9 + a[47]x^8y + a[48]x^7y^2 + a[49]x^6y^3 + ... \\ & + a[52]x^3y^6 + a[53]x^2y^7 + a[54]xy^8 + a[55]y^9\end{aligned} \tag{3.60}$$

In order to evaluate the result for a ninth order polynomial, it is necessary to change the eight order polynomial to a ninth order polynomial. By applying the boundary conditions to this new approximating function and solving for all the coefficients of the ninth order using the *Mathematica* program, we get a system of three trial functions $\phi_1, \phi_2 and \phi_3$.

The corresponding coefficients to these three trial functions are:

Table 3.2 – Coefficients of 9th order polynomial for simply supported plate with aspect ratio 1.0

a[5,1]	a[12,1]	a[14,1]	a[17,1]	a[20,1]	a[25,1]	a[32,1]	a[33,1]
1.00	-2.00	-2.00	1.00	1.00	4.00	-2.00	-2.00

a[5,2]	a[12,2]	a[14,2]	a[20,2]	a[23,2]	a[25,2]	a[33,2]	a[40,2]
2.33	-3.33	-4.66	2.33	1.00	6.66	-3.33	-2.00

a[5,3]	a[12,3]	a[14,3]	a[17,1]	a[25,1]	a[27,1]	a[32,3]	a[42,3]
2.33	-4.66	-3.33	2.33	6.66	1.00	-3.33	-2.00

Substituting these values into Eq. (3.60) we get the following trial functions:

$$\phi_1(x, y) = \{x(1-2x^2+x^3)y(1-2y^2+y^3)\} \tag{3.61}$$

$$\phi_2(x, y) = \{\frac{x}{3}(7-10x^2+3x^4)y(1-2y^2+y^3)\} \tag{3.62}$$

$$\phi_3(x, y) = \{\frac{x}{3}(1-2x^2+x^3)y(7-10y^2+3y^4)\} \tag{3.63}$$

While the first order eigenvalue system was solved by simply multiplying the single element matrix [A] by the single element inverse matrix [B], this third order system is solved using the *Mathematica* eigenvalue function. The elements of matrices

[A] and [B] found by the definite integral command in *Mathematica* are then operated upon using the eigenvalue function in *Mathematica.*

The convergence of the critical buckling load from as overestimated value toward the exact values is a property of the Galerkin method. Such a property occurs when the order of the approximating function is increased. It was found that when the order of the approximating function was increased, the lower order polynomial was kept. In the current example ϕ_1 is the same for the eighth and ninth order polynomials. If this did not occur, the Galerkin system would not be complete and the calculated value would be incorrect.

3.2.2 Mixed Boundary Conditions

In the mixed boundary conditions, we consider two opposite edges to be simply supported and the other two edges of the plate to be fixed. If a plate is simply supported along two opposite sides and clamped on the other two sides at y=*0*, and y=*b.* The boundary conditions are

$$w = 0, \tag{3.64}$$

$$\frac{\partial w}{\partial y} = 0 \qquad \text{at } y=0,\ b \tag{3.65}$$

Also, on the simply supported edge parallel to the y-axis, the boundary conditions at *x=0,* and *x=a* are

$$w=0, \tag{3.66}$$

and

$$M\,|_{x=a} = -D\left(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial y^2}\right)_{x=a} = 0. \tag{3.67}$$

In order to find the buckling load of a plate subjected to mixed boundary conditions as mentioned above, we need to begin with the governing differential equation of a plate i.e. Eq. (3.28).

We start with assuming the solution to be in the form:

$$w(x,y) = \sum_i c_i e_i(x,y)$$

Using the standard Galerkin method to solve the governing differential Eq. (3.28) and then generating an eigenvalue problem by integrating over the entire plate. The elements of matrices [A] and [B] are evaluated in the same way as shown for the case of simply supported boundary condition.

Once the expression for the elements of matrices [A] and [B] are obtained, the next step is to choose a trial function that satisfies the plate's boundary conditions. As before, the trial function is represented as in Eq. (3.47)

$$\phi_i(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{N} a[i,j] u_j(x,y)$$

Where $$u_j(x,y) = x^{L_j} . y^{M_j}$$

For the Mixed boundary condition example, it is found that a sixth order polynomial is the lowest order possible to satisfy the boundary conditions.

$$\begin{aligned}\phi_i &= a[1,1] + a[2,1]x + a[3,1]y + a[4,1]\ x^2 \\ &+ a[5,1]xy + a[6,1]y^2 + a[7,1]x^3 + \ldots.. \\ &+ a[14,1]xy^3 + a[15,1]y^4 + a[16,1]x^5 + \ldots\ldots. \\ &+ a[25,1]x^3y^3 + a[26,1]x^2y^4 + a[27,1]xy^5 + a[28,1]y^6.\end{aligned} \tag{3.68}$$

Where $a[1,1]$ to a [28, 1] are unknown coefficients of ϕ_i.

Once a general N^{th} order polynomial is defined as mentioned in equation (3.68), the next step is to apply the boundary conditions and solve for the unknown coefficients. This is carried out by developing eight new equations using *Mathematica.* Each equation represents a different boundary condition.

For Deflection

$$\text{BC (1):} \quad \phi(0, y) = 0 \tag{3.69}$$

$$\text{BC (2):} \quad \phi(a, y) = 0 \tag{3.70}$$

$$\text{BC (3):} \quad \phi(x,0) = 0 \tag{3.71}$$

$$\text{BC (4):} \quad \phi(x,b) = 0 \tag{3.72}$$

For Moments

$$\text{BC (5):} \quad \left(\frac{\partial^2\phi}{\partial x^2} + \upsilon\frac{\partial^2\phi}{\partial y^2}\right)_{x=0} = 0 \tag{3.73}$$

$$\text{BC (6):} \quad \left(\frac{\partial^2\phi}{\partial x^2} + \upsilon\frac{\partial^2\phi}{\partial y^2}\right)_{x=a} = 0 \tag{3.74}$$

For Slope

$$\text{BC (7):} \quad \left[\frac{\partial\phi}{\partial x}\right]_{y=0} = 0 \tag{3.75}$$

$$\text{BC (8):} \quad \left[\frac{\partial\phi}{\partial x}\right]_{y=a} = 0 \tag{3.76}$$

Having solved all the eight boundary condition equations and tabulating them together using the Table[] command in *Mathematica*, we get values of coefficients a[i,j]. Substituting these values of the coefficients back into the generalized polynomial function we end up at a sixth order polynomial which results in one independent equation in the following polynomial.

$$\phi_1 = x(1-2x^2+x^3)(-1+y)y \tag{3.77}$$

Using *Mathematica,* we evaluate the elements of matrices [A] and [B]. The eigenvalue is evaluated using the EigenValue[] command in *Mathematica,* which matches with the value found from Eq. (3.36).

The general sixth order polynomial that is generated from the generalized polynomial Eq. (3.68) is as follows:

$$\begin{aligned}\phi_1(x,y) = a[1,1] + a[2,1]x + a[3,1]y + a[4,1]\ x^2 + \\ + a[5,1]xy + a[6,1]y^2 + a[7,1]x^3 + a[8,1]x^2y + \ldots\ldots\ldots \\ + a[14,1]xy^3 + a[15,1]y^4 + a[16,1]x^5 + a[17,1]x^4y + \ldots\ldots \\ + a[25,1]x^3y^3 + a[26,1]x^2y^4 + a[27,1]xy^5 + a[28,1]y^6.\end{aligned} \tag{3.78}$$

The coefficients a [i,j] which are obtained by solving the equations of boundary conditions are as follows :

Table 3.3 – Coefficients of 6th order polynomial for isotropic plate with mixed boundary conditions.

a[5,1]	a[9,1]	a[12,1]	a[17,1]	a[18,1]
-1.00	1.00	2.00	-1.00	-2.00

Once again, following the same procedure as in simply supported boundary condition, the eigenvalue problem is solved using the generalized method, the entire procedure is carried out by calculating the eigenvalues and corresponding eigenvectors using the Cholesky decomposition. Wherein, the Cholesky decomposition is applied to matrix [B].

As before, the result obtained using the Cholesky decomposition was pretty accurate and compatible with the direct eigenvalue method. In order to improve on the percent error and get good convergence, a higher order approximating polynomial must be considered. A seventh order polynomial is formed by adding terms to Eq. (3.78).

$$\phi_1(x,y) = a[1,1] + a[2,1]x + a[3,1]y + a[4,1]x^2 + a[5,1]xy + \ldots\ldots$$

$$\begin{aligned} &+ a[11,1]x^4 + a[12,1]x^3y + a[13,1]x^2y^2 + \ldots\ldots\ldots \\ &+ a[18,1]x^3y^2 + a[19,1]x^2y^3 + a[20,1]xy^4 + \ldots \\ &+ a[25,1]x^3y^3 + a[26,1]x^2y^4 + a[27,1]xy^5 + \ldots\ldots \\ &+ a[34,1]x^2y^5 + a[35,1]xy^6 + a[36,1]y^7. \end{aligned} \quad (3.79)$$

In order to evaluate the result for a seventh order polynomial, it is necessary to change the sixth order polynomial to a seventh order polynomial. By applying the boundary conditions to this new approximating function and solving for all the coefficients of the seventh order using the *Mathematica* program, we get a system of three trial functions $\phi_1, \phi_2 and \phi_3$.

The corresponding coefficients to these three trial functions are:

Table 3.4 – Coefficients of 7th order polynomial for isotropic plate with mixed boundary conditions.

a[5,1]	a[9,1]	a[12,1]	a[17,1]	a[18,1]
-1.00	1.00	2.00	-1.00	-2.00

a[5,1]	a[9,1]	a[12,1]	a[18,1]	a[23,1]
-2.33	2.33	3.33	-3.33	-1.00

a[5,1]	a[12,1]	a[14,1]	a[17,1]	a[25,1]
-1.00	2.00	1.00	-1.00	-2.00

Substituting these values into Eq. (3.79) we get the following trial functions:

$$\phi_1(x,y) = \{x(1-2x^2+x^3)(-1+y)y\} \tag{3.80}$$

$$\phi_2(x,y) = \{\frac{x}{3}(7-10x^2+3x^4)(-1+y)y\} \tag{3.81}$$

$$\phi_3(x,y) = \{x(1-2x^2+x^3)y(-1+y^2)\} \tag{3.82}$$

As incase of simply supported boundary condition, While the first order eigenvalue system was solved by simply multiplying the single element matrix [A] by the single element inverse matrix [B], this third order system is solved using the *Mathematica* eigenvalue function. The elements of matrices [A] and [B] found by the definite integral command in *Mathematica* are then operated upon using the eigenvalue function in *Mathematica.*

This convergence of the critical buckling load from an over estimated value toward the exact value is a property of the Galerkin method. Such a property occurs when the order of the approximating functions is increased.

CHAPTER 4

NUMERICAL RESULTS

This chapter presents the numerical results achieved for the problems discussed in chapter 3. All of the computation was carried out with the help of a symbolic algebra software system*, Mathematica* [11].

4.1 Simply supported boundary condition

In our example, we analyze the buckling load of a square plate for the given simply supported boundary condition. In this type of boundary condition wherein all sides are simply supported, the plate is prevented from deflecting but free to rotate about a line along the boundary edge, such as a hinge.

The conditions on a simply supported edge paralled to the y-axis at $x = a$, are.

$$w|_{X=a} = 0 \tag{4.1}$$

$$M_x|_{X=a} = -D\left(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial y^2}\right)_{X=a} = 0 \tag{4.2}$$

On a simply supported edge parallel to the x-axis at $y = a$, the change of w with respect to the x-coordinate vanishes; thus, the condition along this boundary are

$$w|_{Y=a} = 0 \tag{4.3}$$

$$M_y|_{Y=a} = -D\left(\frac{\partial^2 w}{\partial y^2} + \upsilon \frac{\partial^2 w}{\partial x^2}\right)_{Y=a} = 0 \tag{4.4}$$

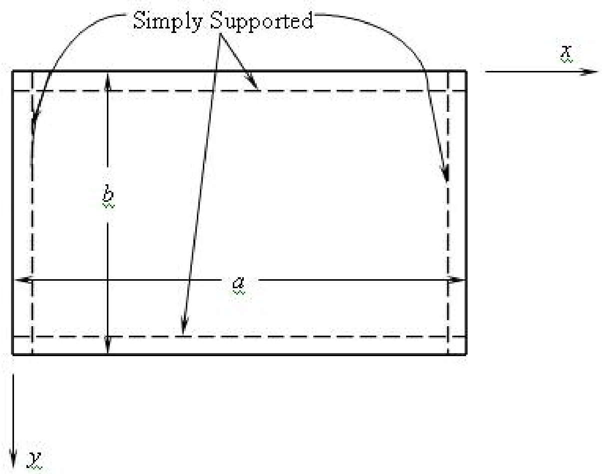

Fig 4.1 Simply supported plate

Once the boundary conditions have been applied, the next step is to choose the suitable trial functions which satisfy the boundary conditions completely. For the current boundary conditions, the trial function can be chosen as

$$\phi_i(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{N} a[i,j]u_j(x,y) \quad (4.5)$$

where , $$u_j(x,y) = x^{L_j}.y^{M_j} \quad (4.6)$$

In the simply supported boundary condition example, it is found that an eight order polynomial is the lowest order possible to satisfy the boundary conditions.

$$\phi_i = a[1,1] + a[2,1]x + a[3,1]y + a[4,1]\ xy$$

$$\begin{aligned} &+ a[5,1]x^2 + a[6,1]y^2 + a[7,1]xy^2 + \ldots.. \\ &+ a[42,1]x^3y^5 + a[43,1]x^2y^6 + a[44,1]xy^7 + a[45,1]y^8 \end{aligned} \quad (4.7)$$

Where $a[1,1]$ through $a[45,1]$ are unknown coefficients of ϕ_1.

Table 4.1 Comparison of eigenvalues for simply supported plate

Order	Direct Approach	Cholesky's Decomposition
8	389.969	389.969
9	389.69, 2463.33	389.969,401401.4
10	389.636, 2463.33, 6304.32	389.969, 401401.4, 6.34*10^7
11	389.637, 2435.33, 6304.32	389.969, 401401.4
12	389.63, 6234.5	389.96, 20443.4, 401401.4
13	383.26, 2435.07, 6234.56	389.969, 401401.39
14	831.41, 2373.52, 6388.73, complex	389.969, 401401.39
15	670.524,1147.44, 4740.78, complex	389.969, 401401.39

Once the eigenvalue problem is solved using the generalized method, the entire procedure is carried out by calculating the eigenvalues and corresponding eigenvectors using the Cholesky decomposition. Wherein the Cholesky decomposition is applied to matrix [B] as

$$B = U^T U, \tag{4.8}$$

Where, U is the upper triangular matix and U^T is the transpose of U.

The result obtained using the Cholesky decomposition was very accurate and compatible with the direct eigenvalue method.

It was seen that the values obtained for the eigenvalue for the least order of the polynomial matched with that of a simple one-dimensional problem. The published values for higher orders were not to be found and so it was not possible for any comparison to be done for them.

4.2 Mixed boundary condition

In example 2, we are going to find the buckling load of a square plate with mixed boundary conditions. we consider two opposite edges to be simply supported and the other two edges of the plate to be fixed. If a plate is simply supported along two opposite sides and clamped on the other two sides at y=*0*, and y=*b*. The boundary conditions are

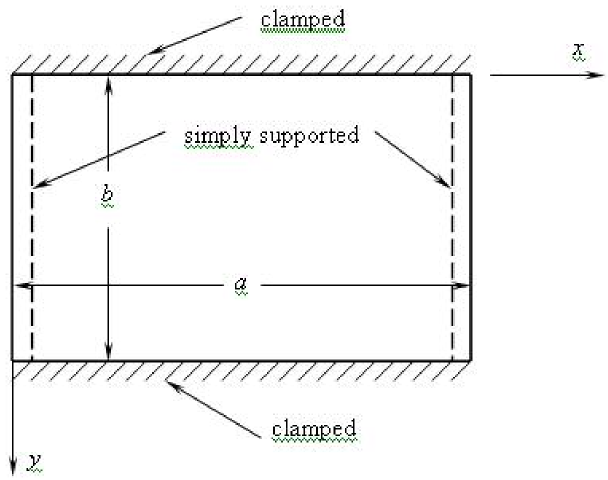

Fig 4.2 Plate with mixed boundary conditions

$$w = 0, \tag{4.9}$$

$$\frac{\partial w}{\partial y} = 0 \qquad \text{at } y=0,\ b \tag{4.10}$$

Also, on the simply supported edge parallel to the y-axis, the boundary conditions at *x=0,* and *x=a* are

$$w=0, \tag{4.11}$$

and $$M\,|_{x=a} = -D\left(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial y^2}\right)_{x=a} = 0. \tag{4.12}$$

For the Mixed boundary condition example, it is found that a sixth order polynomial is the lowest order possible to satisfy the boundary conditions.

$$\phi_i = a[1,1] + a[2,1]x + a[3,1]y + a[4,1]\ x^2$$

$$\begin{aligned} &+ a[5,1]xy + a[6,1]y^2 + a[7,1]x^3 + \ldots.. \\ &+ a[14,1]xy^3 + a[15,1]y^4 + a[16,1]x^5 + \ldots\ldots. \\ &+ a[25,1]x^3y^3 + a[26,1]x^2y^4 + a[27,1]xy^5 + a[28,1]y^6. \end{aligned} \quad (4.13)$$

Where $a[1,1]$ to a [28, 1] are unknown coefficients of ϕ_i.

Table 4.2 Comparison of eigenvalues for plate with mixed boundary condition

Order	Direct Approach	Cholesky's Decomposition
6	294.96	294.96
7	294.96,926.71,2376.34	294.96,19509.7
8	926.71,2376.34,4910.4,cmplx	294.96,19509.7,304128.4
9	2646.6,4910.4,complex	294.96,19509.7,304128.4
10	236.23,364.25,2464.6,complex	294.96,19509.7,304128.4
11	134.067,236.48,364.43,complex	294.96,19509.7
12	139.19,1233.01,2441.74,complex	294.96,19509.7,304128.4
13	5317.25,10442.9,complex	294.96,19509.7
14	6615.23, 11432.9,complex	294.96,19509.7,304128.4
15	88.491,440.52,1192.64,complex	294.96,19509.7,304128.4

Once a general Nth order polynomial is defined as mentioned in equation (4.13), the next step is to apply the boundary conditions and solve for the unknown coefficients. Once again following on the lines of example 1 and using both eigenvalue function and Cholesky decomposition we see that the results obtained were accurate and compatible with each other.

4.3 Weinstein's Theory

Weinstein's theory is related to the stability of a square plate with four clamped edges under uniform thrust in all directions in its plane. Both Weinstein and Trefftz [19] have obtained approximate solutions by the Rayleigh method and have shown how lower limits to the buckling load may be obtained by considering the cases in which the edge conditions are less stringent. Using the exact method, they considered the stability of the rectangular plate loaded by compressive forces distributed over two opposite edges and varying linearly from zero at one corner.

According to Weinstein,

Consider the class of all even functions w(x, y) which have continuous derivatives of the fourth order in the square $S : |x| \leq \frac{\pi}{2}, |y| \leq \frac{\pi}{2}$,

and which satisfy the conditions

$$w = 0 \tag{4.14}$$

$$\frac{\partial w}{\partial x}|_{x=0} = 0. \tag{4.15}$$

On the boundary, $C(|x| = \frac{\pi}{2}, |y| = \frac{\pi}{2})$ of the square plate, it is required to find the least value λ of the expression

$$\frac{I(w)}{D(w)},$$

where *I(w)* and *D(w)* denote the following integrals taken over the square plate S:

$$I(w) = \iint_S (w_{xx} + w_{yy})^2 \, dxdy; \qquad D(w) = \iint_S (w_x{}^2 + w_y{}^2) dxdy.$$

also, $\Delta w = w_{xx} + w_{yy}$; $(\text{grad } w)^2 = w_x^{\,2} + w_y^{\,2}$

The function *w* which renders *(I / D)* a minimum satisfies the differential equation

$$\Delta\Delta w + \lambda \Delta w = 0, \tag{4.16}$$

with (4.14) and (4.15) as boundary condtions.

Solving the equation (4.16) they obtained the inequality $\lambda <$ 5.33….., this number being the value of (I/ D) for the function

$$w^* = \cos^2 x \cos^2 y \tag{4.17}$$

Taking a set of functions which, like w^*, satisfy the boundary conditions (4.14) and (4.15), Weinstein could obtain in a similar way by the Ritz method a non-increasing sequence of upper limits for the true value of λ. According to Weinstein, such a sequence would always converge but not necessarily to λ, except when the chosen set is complete. For instance, Weinstein computed the value of (I / D) for

$$w^{**} = a \cos^2 x \cos^2 y + b \cos^3 x \cos^3 y, \tag{4.18}$$

it was found that, with a suitable choice of *a:b*, the inequality λ **< 5.31173.**

Weinstein has obtained solutions both for the case when the two edges parallel to the load are simply supported and for the case when they are clamped. In both the cases, the loaded edges were always simply supported.

When the Mathematica program was modified according to the parameters in Weinstein's theory it was seen that the results were very accurate and matched with the results obtained by Weinstein's theory. The minimum order of polynomial for the given set of boundary conditions came out to be eighth.

$$\phi_i = a[1,1] + a[2,1]x + a[3,1]y + a[4,1]\ xy$$

$$\begin{aligned} &+ a[5,1]x^2 + a[6,1]y^2 + a[7,1]xy^2 + ... \\ &+ a[42,1]x^3y^5 + a[43,1]x^2y^6 + a[44,1]xy^7 + a[45,1]y^8 \end{aligned} \qquad (4.19)$$

Where $a[1,1]$ through $a[45,1]$ are unknown coefficients of ϕ_1.

Applying the set of boundary conditions wherein slope and deflection on each side is zero.

$$\begin{aligned} &w|_{x=0,a} = 0 && w|_{y=0,a} = 0 \\ &\frac{\partial w}{\partial x}|_{x=0,a} = 0 && \frac{\partial w}{\partial y}|_{y=0,a} = 0 \end{aligned} \qquad (4.20)$$

Using the set of equations for the boundary condition, we can evaluate the coefficients of the polynomial. Solving these equations we get a single trial function for the eighth order polynomial. Applying the Laplacian operator *L* over the entire area of the plate, we evaluate the elements of matrices [A] and [B]. Then applying the Cholesky decomposition on matrix [B], the eigenvalues that are calculated come out to be

$\lambda =$ **5.4713,** which is very close to the desired value of $\lambda =$ **5.33.**

To get better convergence of the results, a higher order polynomial was considered and Cholesky decomposition was applied again to achieve more stable results.

4.4 Rectangular plate axially compressed in one direction

Consider a simply supported rectangular plate subjected to a uniform axially compressive force N_x per unit length along the edges $x = 0, a.$ putting N_y and N_{xy} equal to zero, we get the governing equation as

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2} = 0. \qquad (4.21)$$

In terms of the non-dimensional parameters ξ and η, defined as

$$x = a\xi, \quad y = b\eta, \qquad p = a/b \qquad (4.22)$$

rewriting Eq. (4.21) as

$$\frac{\partial^4 w}{\partial \xi^4} + 2p^2\frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + p^4\frac{\partial^4 w}{\partial \eta^4} + \lambda^2\frac{\partial^2 w}{\partial \xi^2} = 0. \qquad (4.23)$$

Since all the edges are simply supported, the boundary conditions are given by

The conditions on a simply supported edge paralled to the y-axis at $x = a$, are.

$$w|_{X=a} = 0 \qquad (4.24)$$

$$M_x|_{X=a} = -D\left(\frac{\partial^2 w}{\partial x^2} + \upsilon\frac{\partial^2 w}{\partial y^2}\right)_{X=a} = 0 \qquad (4.25)$$

On a simply supported edge parallel to the x-axis at $y = a$,

$$w|_{Y=a} = 0 \qquad (4.26)$$

$$M_y|_{Y=a} = -D\left(\frac{\partial^2 w}{\partial y^2} + \upsilon\frac{\partial^2 w}{\partial x^2}\right)_{Y=a} = 0 \qquad (4.27)$$

In a non-dimensional form, these conditions can be written as

$$w = \frac{\partial^2 w}{\partial \xi^2} + \upsilon p^2\frac{\partial^2 w}{\partial \eta^2} = 0 \qquad (\xi = 0, 1), \qquad (4.28)$$

$$w = \upsilon \frac{\partial^2 w}{\partial \xi^2} + p^2 \frac{\partial^2 w}{\partial \eta^2} = 0 \qquad (\eta = 0, 1) \tag{4.29}$$

Since $w = 0$ along $\xi = 0, 1$ and $\eta = 0, 1$, we have

$$\frac{\partial^2 w}{\partial \xi^2} = 0 \qquad (\eta = 0, 1),$$

$$\frac{\partial^2 w}{\partial \eta^2} = 0 \qquad (\xi = 0, 1)$$

Making use of these relations, we find the boundary conditions from Eqs. (4.28) and (4.29) reduce to

$$w = \frac{\partial^2 w}{\partial \xi^2} = 0 \quad (\xi = 0, 1), \tag{4.30}$$

$$w = \frac{\partial^2 w}{\partial \eta^2} = 0 \quad (\eta = 0, 1). \tag{4.31}$$

The differential equation (4.23) and the boundary conditions (4.30) and (4.31) contain terms which are of even order in ξ and η; hence, a variable separable type of solution is sought. Thus, for Eq. (4.23), assume a solution of the form

$$w(\xi,\eta) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \sin m\pi\xi \;\; \sin n\pi\eta \quad (m = 1,2,\ldots, n = 1,2,\ldots\ldots), \tag{4.32}$$

Where m and n define the number of half waves that the plate buckles in the *X*- and *Y*- direction, respectively, and A_{mn} represents the amplitudes of the shape function or the mode shapes. Eq (4.32) satisfies the boundary conditions exactly.

Since we need to calculate the minimum value for N_x, minimizing the equation to get least value of N_x. we get,

$$(N_x)_{cr} = \frac{4D\pi^2}{b^2} \tag{4.33}$$

D – Flexural rigidity

b – Plate width

Thus, a simply supported plate buckles with one half wave in the Y – direction and p half waves in the X – direction, i.e.; p must be an integer. This implies that the plate buckles into square plates.

Assuming D and b as unity, then from Eq. (4.33) the minimum critical load N_x is calculated to be **39.478**.

The minimum order of polynomial for the given set of boundary conditions came out to be of eighth order. This eighth order polynomial has 45 coefficients to be evaluated to calculate the buckling load.

$$\begin{aligned}\phi_i &= a[1] + xa[2] + ya[3] + x^2 a[4] + xy\,a[5] + y^2a[6] + x^3a[7] + x^2ya[8] + xy^2a[9] + y^3a[10] + \\ & x^4a[11] + x^3ya[12] + x^2y^2a[13] + xy^3a[14] + y^4a[15] + x^5a[16] + x^4ya[17] + x^3y^2a[18] + x^2y^3a[19] + \\ & xy^4a[20] + y^5a[21] + x^6a[22] + x^5ya[23] + x^4y^2a[24] + x^3y^3a[25] + x^2y^4a[26] + xy^5a[27] + y^6a[28] + x^7a[29] \\ & + x^6ya[30] + x^5y^2a[31] + x^4y^3a[32] + x^3y^4a[33] + x^2y^5a[34] + xy^6a[35] + y^7a[36] + x^8a[37] + x^7ya[38] + \\ & x^6y^2a[39] + x^5y^3a[40] + x^4y^4a[41] + x^3y^5a[42] + x^2y^6a[43] + xy^7a[44] + y^8a[45]\end{aligned} \tag{4.34}$$

Using the set of equations for the boundary condition, we can evaluate the coefficients of the polynomial.

$$\begin{aligned} & w|_{x=0,a} = 0 && w|_{y=0,a} = 0 \\ & \frac{\partial^2 w}{\partial x^2}|_{x=0,a} = 0 && \frac{\partial^2 w}{\partial y^2}|_{y=0,a} = 0 \end{aligned} \tag{4.35}$$

Once the boundary conditions are applied, the next step is to solve all the equations involving the coefficients of the N^{th} order polynomial. The procedure is carried out using the Galerkin method to arrive at the eigenvalues for each order of the polynomial.

Table 4.3 - Comparison of Eigenvalues for a simply supported plate loaded axially

Order	Direct Approach	Cholesky's Decomposition
8	39.506	39.506
9	39.5066,62.205,249.55	39.5066,10136.4
10	**39.478**,62.205,113.41,159.2	39.5066, 10136.4, 40664.8
11	**39.478**,61.687,113.41,159.2	39.5066, 10136.4, 40664.8

Similarly evaluating for 10^{th} order polynomial, we end up with 66 coefficients that are to be determined in order to arrive at the minimum buckling load value. Applying the given boundary conditions and solving the problem using the Galerkin method.

$$\begin{aligned}\phi_i = {} & a[1] + xa[2] + ya[3] + x^2 a[4] + xy\, a[5] + y^2 a[6] + x^3 a[7] + x^2 y a[8] + xy^2 a[9] + y^3 a[10] + \\ & x^4 a[11] + x^3 y a[12] + x^2 y^2 a[13] + xy^3 a[14] + y^4 a[15] + x^5 a[16] + x^4 y a[17] + x^3 y^2 a[18] + x^2 y^3 a[19] + \\ & xy^4 a[20] + y^5 a[21] + x^6 a[22] + x^5 y a[23] + x^4 y^2 a[24] + x^3 y^3 a[25] + \ldots\ldots + \\ & x^5 y^3 a[40] + x^4 y^4 a[41] + x^3 y^5 a[42] + x^2 y^6 a[43] + xy^7 a[44] + y^8 a[45] + \ldots\ldots + y^9\, a[55] + x^{10} a[56] + \ldots\ldots \\ & x^5 y^5 a[61] + x^4 y^6 a[62] + x^3 y^7 a[63] + x^2 y^8 a[64] + x\, y^9 a[65] + y^{10} a[66]\end{aligned} \tag{4.36}$$

Solving for the coefficients of the trial functions, we get a single trial function for the eighth order polynomial. Galerkin method is used to evaluate the eigenvalues. The minimum eigenvalue calculated using the *Mathematica* program is $\lambda =$ **39.506.**

On further increasing the order of the polynomial to 10^{th} order, and following the procedure of applying the boundary conditions and following the Galerkin method, we get the value of $\lambda =$ **39.478** which is exactly the same as theoretical value.

Hence we can conclude that the lowest eigenvalue calculated using the *Mathematica* program is very close to the value obtained from the theoretical solution of N_x. To achieve better convergence and accuracy of the results, a higher order polynomial is required.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

The Galerkin method was used to solve the governing differential equation of the plate for different boundary conditions. The procedure used for solving for the coefficients of the approximating polynomials and the Galerkin method would be inconceivable if it were not for the symbolic software. By successfully manipulating the given expressions and symbolically retaining the variables, the usage of symbolic algebra software is suited for parametric study.

One particular advantage symbolic software has over present numerical methods is the ability to retain parameters throughout the calculations and to input them at appropriate times determined by the user. The method of selecting polynomials that satisfy all boundary conditions, geometric and natural, provides for better accuracy and faster convergence. This allows for lower order polynomials to be used as approximating function.

Examples of the *Mathematica* programs used in this method are included in the Appendices. The procedures explained in chapter 3 along with the examples provided in the Appendix are sufficient to show the steps necessary to reach a solution. The results obtained throughout this thesis were close to existing data.

Weinstein's theory was one of the first works done on plate analysis and it really formed a basis of understanding the plate behavior under different conditions. It formed the yardstick for future work that was carried out in this field.

The method of solving for the buckling load presented here takes advantage of the speed and convenience of symbolic software. With the availability of routines such as *Mathematica* and *Matlab,* classical analytical procedures can be considered when analyzing engineering problems. These routines provide for a way of solving a problem other than numerical techniques and finite element methods. Future applications of symbolic software are encouraged. These applications will hopefully lead to a better understanding of classical analytical procedures and also make ways for faster and more accurate methods in solving such types of problems.

APPENDIX A

MATHEMATICA PROGRAMS TO SOLVE THE BUCKLING LOAD EQUATION OF A PLATE FOR DIFFERENT CASES

Program 1

This program calculates the buckling load for a simply supported plate using the Galerkin method. The region considered for the plate in this case is,

x=0, x=1, and y=0, y=1

(* Definition of trial function which satisfies the given boundary conditions. The least order defined for this case.*)

```
poly[n_]:= Sum [a(i + j)(i + j + 1) / 2 + i + 1] x^iy^ j,{i, 0,n},{j, 0,n - i}]
order = 8;
```

(*Generating the trial function using the above function definition*)

```
poly[order]
```

$a[1] + xa[2] + ya[3] + x^2 a[4] + xy\,a[5] + y^2a[6] + x^3a[7] + x^2ya[8] + xy^2a[9] + y^3a[10] + x^4a[11] + x^3ya[12] + x^2y^2a[13] + xy^3a[14] + y^4a[15] + x^5a[16] + x^4ya[17] + x^3y^2a[18] + x^2y^3a[19] + xy^4a[20] + y^5a[21] + x^6a[22] + x^5ya[23] + x^4y^2a[24] + x^3y^3a[25] + x^2y^4a[26] + xy^5a[27] + y^6a[28] + x^7a[29] + x^6ya[30] + x^5y^2a[31] + x^4y^3a[32] + x^3y^4a[33] + x^2y^5a[34] + xy^6a[35] + y^7a[36] + x^8a[37] + x^7ya[38] + x^6y^2a[39] + x^5y^3a[40] + x^4y^4a[41] + x^3y^5a[42] + x^2y^6a[43] + xy^7a[44] + y^8a[45]$

(*Applying the boundary conditions *)

```
eq1 = CoefficientList[poly[order] / . x → 0 , y]
eq2 = CoefficientList[poly[order] / . x → 1 , y]
eq3 = CoefficientList[poly[order] / . y → 0 , x]
eq4 = CoefficientList[poly[order] / . y → 1 , x]
```

```
eq5 = CoefficientList[D[poly[order] , {x, 2}] + υ ( D[poly[order], {y, 2}] ) / . x → 0 , y]

eq6 = CoefficientList[D[poly[order] , {x, 2}] + υ ( D[poly[order], {y, 2}] ) / . x → 1 , y]

eq7 = CoefficientList[D[poly[order] , {y, 2}] + υ ( D[poly[order], {x, 2}] ) / . y → 0 , x]

eq8 = CoefficientList[D[poly[order] , {y, 2}] + υ ( D[poly[order], {x, 2}] ) / . y → 1 , x]

(*Flatten command used for grouping all the eight equations together *)

Eq = Flatten [ { eq1, eq2, eq3, eq4, eq5, eq6, eq7, eq8} ]

(* Tabulating all the eight equations using the Table command *)

Equn = Table[ Eq[ [ i ] ] == 0,  {i , 1, Length[Eq] } ]

(* Solving all the eight equations and generating the coefficients *)

sol = Solve[ Equn, Table[a[i],  {i, 1, 60} ] ] [ [ 1] ]

(* Substituting the values of coefficients back into the equations *)

poly1 = poly[order] / . sol

(* Simplifying the equation by taking out common terms *)

m = Table[Coefficient[ poly1, a[i]  ] ,  {i, 1, Length[poly[order] ] } ]  // Simplify

(* Deleting all the integers to separate out all the derived base functions *)

  e = DeleteCases[ m, _Integer]

  (* Calculation of Matrix A*)

A = Table[ Integrate[( D[e[[i]], {x, 4}] + D[e[[i]], {y, 4}] + (2 * D[D[e[[i]], {y, 2}], {x, 2}])) *
e[[j]], {x, 0, 1} , {y, 0, 1}], {i, 1, 1}, {j, 1, 1}] // N; MatrixForm[A]

(* Calculation of Matrix B*)

B = Table[Inte grate[ e[[i]]  * e[[j]], {x,  0,  1}, {y,  0,  1}], {i, 1, 1}, {j, 1, 1}] // N; MatrixForm [B]
```

(* Calculating inverse of Matrix B*)

B1 = Inverse [B]

(* Calculation of the product of Matrices A and B1*)

A2 = %.A; MatrixForm[A2]

(* Calculation of Eigenvalues for the Matrix A2*)

E1 = Eigenvalues [A2]

When the Cholesky decomposition is used, the entire procedure is carried out till evaluating the matrices [A] and [B]. After calculating [A] and [B] the following steps are followed for Cholesky decomposition.

(* To load the Cholesky decomposition routine *)

<< LinearAlgebra ' Cholesky'

(* Calculating the Cholesky decomposition for matrix [B]*)

(u = CholeskyDecomposition[B]) // MatrixForm

(* Calculation of transpose for matrix u*)

ut = Transpose[u]

(* Calculation of inverse of ut*)

uti = Inverse[ut]

(* Calculating the product of uti , A and ui*)

A2 = (uti) * A * (ui)

(* Calculating the Eigenvalues for A2*)

E1 = Eigenvalues [A2]

Program 2

This program calculates the buckling load for mixed boundary conditions using the Galerkin method. The region considered for the plate in this case is

$x = 0, x = 1$ and $y = 0, y = 1$

(* Definition of trial function which satisfies the given boundary conditions. The least order defined for this case.*)

```
poly[n_] := Sum [a(i + j) (i + j + 1) / 2 + i + 1] x^iy^ j, {i, 0, n}, {j, 0, n - i}]
order = 6;
```

(*Generating the trial function using the above function definition*)

```
poly[order]
```

$a[1] + xa[2] + ya[3] + x^2 a[4] + xy a[5] + y^2a[6] + x^3a[7] + x^2ya[8] + xy^2a[9] + y^3a[10] + x^4a[11] + x^3ya[12] + x^2y^2a[13] + xy^3a[14] + y^4a[15] + x^5a[16] + x^4ya[17] + x^3y^2a[18] + x^2y^3a[19] + xy^4a[20] + y^5a[21] + x^6a[22] + x^5ya[23] + x^4y^2a[24] + x^3y^3a[25] + x^2y^4a[26] + xy^5a[27] + y^6a[28]$

(*Applying the boundary conditions *)

```
eq1 = CoefficientList[poly[order] / . x → 0 , y]

eq2 = CoefficientList[poly[order] / . x → 1 , y]

eq3 = CoefficientList[poly[order] / . y → 0 , x]

eq4 = CoefficientList[poly[order] / . y → 1 , x]

eq5 = CoefficientList[D[poly[order] , {x, 2}] + υ ( D[poly[order], {y, 2}] ) / . x → 0 , y]

eq6 = CoefficientList[D[poly[order] , {x, 2}] + υ ( D[poly[order], {y, 2}] ) / . x → 1 , y]
```

```
eq7 = CoefficientList[D[poly[order], {x,1}] /. y → 0, x]

eq8 = CoefficientList[D[poly[order], {x,1}] /. y → 1, x]

(*Flatten command used for grouping all the eight equations together *)

Eq = Flatten[{eq1, eq2, eq3, eq4, eq5, eq6, eq7, eq8}]

(* Tabulating all the eight equations using the Table command *)

Equn = Table[Eq[[i]] == 0, {i, 1, Length[Eq]}]

(* Solving all the eight equations and generating the coefficients *)

sol = Solve[Equn, Table[a[i], {i, 1, 60}]][[1]]

(* Substituting the values of coefficients back into the equations *)

poly1 = poly[order] /. sol

(* Simplifying the equation by taking out common terms *)

m = Table[Coefficient[poly1, a[i]], {i, 1, Length[poly[order]]}] // Simplify

(* Deleting all the integers to separate out all the derived base functions *)

e = DeleteCases[m, _Integer]

(* Calculation of Matrix A*)

A = Table[Integrate[(D[e[[i]], {x, 4}] + D[e[[i]], {y, 4}] + (2 * D[D[e[[i]], {y, 2}], {x, 2}])) *
e[[j]], {x, 0, 1}, {y, 0, 1}], {i, 1, 1}, {j, 1, 1}] // N; MatrixForm[A]

(* Calculation of Matrix B*)

B = Table[Integrate[e[[i]] * e[[j]], {x, 0, 1}, {y, 0, 1}], {i, 1, 1}, {j, 1, 1}] // N; MatrixForm[B]

(* To load the Cholesky decomposition routine *)

<< LinearAlgebra'Cholesky'
```

(* Calculating the Cholesky decomposition for matrix [B]*)

(u = CholeskyDecomposition[B]) // MatrixForm

(* Calculation of transpose for matrix u*)

ut = Transpose[u]

(* Calculation of inverse of ut*)

uti = Inverse[ut]

(* Calculating the product of uti , A and ui*)

A2 = (uti) * A * (ui)

(* Calculating the Eigenvalues for A2*)

E1 = Eigenvalues [A2]

Program 3

This program calculates the buckling load using Weinstein's theory. The region considered for the square plate according to Weinstein's theory is

$$x = 0, x = \pi, y = 0, y = \pi$$

(* Definition of trial function which satisfies the given boundary conditions. The least order defined for this case.*)

```
poly[n_] := Sum [a(i + j) (i + j + 1) / 2 + i + 1] x^iy^ j, {i, 0, n}, {j, 0, n - i}]
order = 8;
```

(*Generating the trial function using the above function definition*)

```
poly[order]
```

$a[1] + x a[2] + y a[3] + x^2 a[4] + x y a[5] + y^2 a[6] + x^3 a[7] + x^2 y a[8] + x y^2 a[9] + y^3 a[10] + x^4 a[11] + x^3 y a[12] + x^2 y^2 a[13] + x y^3 a[14] + y^4 a[15] + x^5 a[16] + x^4 y a[17] + x^3 y^2 a[18] + x^2 y^3 a[19] + x y^4 a[20] + y^5 a[21] + x^6 a[22] + x^5 y a[23] + x^4 y^2 a[24] + x^3 y^3 a[25] + x^2 y^4 a[26] + x y^5 a[27] + y^6 a[28] + x^7 a[29] + x^6 y a[30] + x^5 y^2 a[31] + x^4 y^3 a[32] + x^3 y^4 a[33] + x^2 y^5 a[34] + x y^6 a[35] + y^7 a[36] + x^8 a[37] + x^7 y a[38] + x^6 y^2 a[39] + x^5 y^3 a[40] + x^4 y^4 a[41] + x^3 y^5 a[42] + x^2 y^6 a[43] + x y^7 a[44] + y^8 a[45]$

(*Applying the boundary conditions *)

```
eq1 = CoefficientList[poly[order] / . x → 0 , y]

eq2 = CoefficientList[poly[order] / . x → Pi , y]

eq3 = CoefficientList[poly[order] / .  y → 0 , x]

eq4 = CoefficientList[poly[order] / . y → Pi , x]
```

```
eq5 = CoefficientList[D[poly[order], {y, 1}]  / . y → 0 , x]

eq6 = CoefficientList[D[poly[order], {y, 1}]  / . y → Pi , x]

eq7 = CoefficientList[D[poly[order], {x, 1}]  / . x → 0 , y]

eq8 = CoefficientList[D[poly[order], {x, 1}]  / . x → Pi  , y]

(*Flatten command used for grouping all the eight equations together *)

Eq = Flatten [ { eq1, eq2, eq3, eq4, eq5, eq6, eq7, eq8} ]

(* Tabulating all the eight equations using the Table command *)

Equn = Table[ Eq[ [ i ] ] == 0,  {i , 1, Length[Eq] } ]

(* Solving all the eight equations and generating the coefficients *)

sol = Solve[ Equn, Table[a[i],  {i, 1, 60} ] ] [ [ 1] ]

(* Substituting the values of coefficients back into the equations *)

poly1 = poly[order] / . sol

(* Simplifying the equation by taking out common terms *)

m = Table[Coefficient[ poly1, a[i]  ],  {i, 1, Length[poly[order] ] } ]  // Simplify

(* Deleting all the integers to separate out all the derived base functions *)

  e = DeleteCases[ m, _Integer]

  (* Calculation of Matrix A*)

A = Table[ Integrate[( D[e[[i]], {x, 4}] + D[e[[i]], {y, 4}] + (2 * D[D[e[[i]], {y, 2}], {x, 2}])) *
e[[j]], {x, 0, Pi} , {y, 0, Pi}], {i, 1, 1}, {j, 1, 1}] // N; MatrixForm[A]
```

(* Calculation of Matrix B*)

B = Table[Inte grate[(D[e[[i]] , {x, 2}] + (D[e[[i]], {y, 2}])) * e[[j]],

{x, 0, Pi}, {y, 0, Pi}, {i, 1, 1}, {j, 1, 1}] // N; MatrixForm [B]

When the Cholesky decomposition is used, the entire procedure is carried out till evaluating the matrices [A] and [B]. After calculating [A] and [B] the following steps are followed for Cholesky decomposition.

(* To load the Cholesky decomposition routine *)

<< LinearAlgebra ' Cholesky'

(* Calculating the Cholesky decomposition for matrix [B]*)

(u = CholeskyDecomposition[B]) // MatrixForm

(* Calculation of transpose for matrix u*)

ut = Transpose[u]

(* Calculation of inverse of ut*)

uti = Inverse[ut]

(* Calculating the product of uti , A and ui*)

A2 = (uti) * A * (ui)

(* Calculating the Eigenvalues for A2*)

E1 = Eigenvalues [A2]

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BIOGRAPHICAL INFORMATION

Yattender Rishi Dubey received his Bachelor in Engineering in Mechanical Engineering from Visveswaraiah Technological University, Belgaum, India in 2002. Immediately after his graduation, he worked as a Graduate Trainee at Indian Institute of Chemical Technology, Hyderabad, India from August 2002 to June 2003. His responsibilities involved process equipment design and process control for the pilot plant used for the manufacturing of benzaldehyde from the oxidation of toluene.

Later in the fall of 2003, he joined The University of Texas at Arlington to pursue his Masters of Science in Mechanical Engineering. His master's thesis dealt with the study of buckling load for an isotropic plate using the Galerkin method. His research interests include finite element methods, engineering design and optimization using numerical techniques and extending the Galerkin theory to more complex structural analysis.