Quantized enveloping superalgebra of type P

by

SABER MURAD AHMED

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ABSTRACT

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Saber Murad Ahmed, Ph.D.

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Supervising Professor: Dimitar Grantcharov

We introduce a new quantized enveloping superalgebra $\mathfrak{U}_q\mathfrak{p}_n$ attached to the Lie superalgebra $\mathfrak{p}_n$ of type P. The superalgebra $\mathfrak{U}_q\mathfrak{p}_n$ is a quantization of a Lie bisuperalgebra structure on $\mathfrak{p}_n$ and we study some of its basic properties. We determine representations of the superalgebra $\mathfrak{U}_q\mathfrak{p}_n$ and derive its Drinfeld-Jimbo relations. We prove the triangular decomposition of $\mathfrak{U}_q\mathfrak{p}_n$ and introduce some preliminary results concerning the highest weight representation of $\mathfrak{U}_q\mathfrak{p}_n$. We also introduce the periplectic q-Brauer algebra and prove that it is the centralizer of the $\mathfrak{U}_q\mathfrak{p}_n$-module structure on $\mathbb{C}(n|n)^{\otimes \ell}$. Finally, we propose a definition for a new periplectic q-Schur superalgebra.

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CHAPTER 1

INTRODUCTION

Many of the symbols and notation of the early chapters will be familiar to the casual mathematics reader of mathematical writing, and will be used consistently throughout. Any important notations will be mentioned when they are first introduced. A familiarity with the standard topics of a first-year algebra course and a first-year linear algebra course is assumed from the interested reader. Topics include vector spaces, rings, fields, modules, and algebras. A two-semester sequence of graduate algebra is recommended.

1.1 History of Lie Superalgebras

Lie algebras and superalgebras are important in many fields of mathematics and physics, such as quantum physics and particle physics. Studies of symmetries arise when studying particle states and the possibilities of such states, which are understood through symmetries. When relating particles of different statistics, such as fermions and bosons, then the concept of "supersymmetries" becomes important, and hence the rise of Lie superalgebras in the 1970s. This was of great interest to quantum chemists and quantum physicists, and thus was the original motivation for the study of Lie superalgebras. Studying the representation theory of Lie superalgebras is equivalent to studying the representation theory of their universal enveloping algebras, and this helped to drive the motivation for studying Lie superalgebras as well, among other related algebraic objects such as quantum groups.

Motivated by the usefulness in physics and other fields of mathematics, much research has been done to classify Lie superalgebras, determine various properties about these structures, and their representations. Kac classified the simple finite dimensional Lie superalgebras over algebraically closed fields of characteristic zero in [36] in 1977. Lie superalgebras of classical type are defined in [52]. Irreducible characters of classical Lie superalgebras for the finite-dimensional modules, and even for modules in the BGG category $\mathcal{O}$, have been worked out in [47], [7], [32], [11], [12], [5].

During this time quantum groups have gained much traction, yet Lie superalgebras have not been as well investigated and in some sense overlooked. Even more than thirty years after the work of Kac, the representation theory of Lie superalgebras is still not well understood. Despite this, there has been significant progress made in the study of Lie superalgebras.

Lie superalgebras can also be seen as a generalization of Lie algebras. Although the representation theory of Lie algebras and Lie superalgebras have some similarities, studying Lie superalgebra sand their representations proves to be much more difficult. For example, in the study of the category of finite-dimensional modules over a finite-dimensional simple Lie superalgebra, it is not true that the category is semisimple in general [37]. These categories are much more difficult to study compared to their classical counterparts.

One class of Lie superalgebras that have gained much traction in the past 30 years are referred to as the "strange" Lie superalgebras. One reason these are called "strange" are due to not having a direct analogue to Lie algebras. The representation theory of these strange Lie superalgebras is one of the more popular subjects of studies recently and there are many fundamental questions that remain open. There are two types of strange Lie superalgebras, P and Q, both of which are interesting due to the algebraic, geometric, and combinatorial properties of their representations. The study of the representations of type P Lie superalgebras, which are also called periplectic Lie superalgebras in the literature, has attracted considerable attention in the last five years. Interesting results on the category $\mathcal{O}$, the associated (affine) periplectic Brauer algebras, and related theories have been established in [3], [4], [10], [15], [16], [17], [18], [20], [27], [28], [39], [48], and more recently [1], [21], [22], among others. Despite these considerable advances in the understanding of the representation theory of type P Lie superalgebras, many aspects of it still remain mysterious.

1.2 History of Quantum Supergroups

Quantum groups emerged in the 1980's, with Drinfeld coining the term in 1985. In 1986 at the International Congress of Mathematicians, Drinfeld's results brought quantum groups to the attention of mathematicians internationally and set the foundation for its theory. The first discovered quantum group was the q-analogue of $SU(2)$, the special unitary group of rank 2. One of the problems of interest was understanding exactly solvable models in quantum mechanics, which involved integrable systems. Quantum groups also appeared in the work of physicists and mathematicians interested in the quantum inverse scattering method in statistical mechanics. At first, quantum groups were understood to be associative algebras whose defining relations are expressed in terms of a matrix of constants known as a quantum R-matrix. However, Drinfeld and Jimbo independently observed that these guantum groups are really Hopf algebras, and these particular Hopf algebras are deformations of universal enveloping algebras of Lie algebras. These deformations were originally intended to aid in the construction of solutions to the now famous Yang-Baxter equation. Very recently, quantum groups have been found to have far reaching connections in pure mathematics, such as category theory, representation theory, and topology. The theory of quantum groups has grown significantly, and continues to gather substantial attention ever since their introduction.

It was later realized that it was also possible to quantize the enveloping superalgebras of certain Lie superalgebras, with Lie superalgebras being pure algebraic objects that carried essential information for their corresponding Lie supergroup. The quantized Lie algebras and the quantized Lie superalgebras have been found to be useful in low dimensional topology, statistical physics, and noncommutative geometry. There are results of the quantization of the universal enveloping algebras of the strange Lie superalgebras. Olshanski constructed the quantized enveloping algebra of type Q, $\mathfrak{U}_q\mathfrak{q}_n$, in [44] through the FRT formalism as described in [29]. Nicolas Guay, Dimitar Grantcharov, and I constructed the quantized enveloping algebra of type P, $\mathfrak{U}_q\mathfrak{p}_n$, in [1] through the same formalism.

Quantum supergroups are the supersymmetric generalizations of quantum groups. Supersymmetric integrable lattice models and conformal field theories have a natural link with these quantum supergroups. Mathematically, they are one of the few known examples of noncommutative and noncocommutative graded Hopf algebras that have been studied [55]. Quantum supergroups and quantized enveloping superalgebras tend to be used interchangeably in the case of Lie superalgebras. Just like in the case of Lie superalgebras, the studies of their quantizations have not been as well investigated. The representation theory of quantized superalgebras is still not well understood. Therefore, any and all results of studying representation theory of these objects are significant in this field of study.

CHAPTER 2

Bialgebras

In this chapter, we will introduce the reader into some preliminary concepts about algebras that will be prevalent in future chapters. We will first remind the reader the definition of an algebra, although it may be presented in a way unfamiliar to those whose backgrounds in algebra are limited to a one or two semester graduate-level course. However, this presentation will prove to be useful for introducing coalgebras, and by extension Hopf algebras. All quantum groups or quantized enveloping algebras of Lie algebras are Hopf algebras, and also all quantum supergroups or quantized enveloping superalgebras of Lie superalgebras are Hopf superalgebras. As a result, it is important that the reader is at least aware of what a Hopf algebra is.

2.1 Coalgebras

The reader may be aware of the definition of an algebra given to be a ring A together with a ring map $\eta : \Bbbk \to A$, whose image is contained in the center of A. This map η equips A with the operation of scalar multiplication given by $(\alpha, a) \longmapsto \eta(\alpha)a$. In other words, an algebra A is a ring equipped with a map that defines scalar multiplication with the field and the ring A, giving the ring a vector space structure. We will paraphrase this definition below.

Definition 2.1.1. Let A be a vector space. A is an $\Bbbk$-*algebra* when equipped with the linear maps $\mu : A \otimes A \to A$ and $\eta : \Bbbk \to A$ that satisfies the following axiomatic commutative diagrams:

$$
\begin{tikzcd}
A \otimes \Bbbk \arrow[r, "id \otimes \iota"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes A \arrow[r, "\iota \otimes id"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] \\
& A
\end{tikzcd}
$$

Unit

$$
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu \otimes id"] \arrow[d, "id \otimes \mu"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"] & A
\end{tikzcd}
$$

Associativity

The map μ is called the product or the multiplication while η is called the unit of the algebra.

We say that A is *commutative* if, in addition to the above diagrams being satisfied, the following axiomatic commutative diagram is also satisfied:

$$
\begin{tikzcd}
A \otimes A \arrow[r, "\tau_{A,A}"] \arrow[dr, "\mu"'] & A \otimes A \arrow[d, "\mu"] \\
& A
\end{tikzcd}
$$

Commutatitivty

where $\tau_{A,A}$ is the map that switches the tensor factors: $\tau_{A,A}(x \otimes x') = x' \otimes x$ for all $x, x' \in A$.

Presenting the definition of an algebra in this way is useful as we can then systematically write the definition of what is called a coalgebra by changing the direction of every arrow in the diagrams above.

Definition 2.1.2. Let C be a vector space. C is an $\Bbbk$-*coalgebra* when equipped with the linear maps $\Delta : C \to C \otimes C$ and $\epsilon : C \to \Bbbk$ that satisfies the following axiomatic commutative diagrams:

$$\begin{tikzcd}
C \otimes \Bbbk & & C \otimes C \arrow[ll, "id \otimes \epsilon"'] \\
& C \arrow[ul, "\cong"] \arrow[ur, "\Delta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes C & & C \otimes C \arrow[ll, "\epsilon \otimes id"'] \\
& C \arrow[ul, "\cong"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

Counit

$$\begin{tikzcd}
C \otimes C \otimes C & C \otimes C \arrow[l, "\Delta \otimes id"'] \\
C \otimes C \arrow[u, "id \otimes \Delta"] & C \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}$$

Coassociativity

The map Δ is called the coproduct (or comultiplication) while ϵ is called the counit of the coalgebra.

We say that A is *cocommutative* if, in addition to the above diagrams being satisfied, the following axiomatic commutative diagram is also satisfied:

$$\begin{tikzcd}
A \otimes A & & A \otimes A \arrow[ll, "\tau_{A,A}"'] \\
& A \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

Cocommutatitivty

where $\tau_{A,A}$ is the map that switches the tensor factors: $\tau_{A,A}(x \otimes x') = x' \otimes x$ for all $x, x' \in A$.

Example 2.1.1. Let X be a set and let $C = \Bbbk[X]$ by the $\Bbbk$-vector space generated by the basis X. Define the coproduct Δ and counit ϵ on C by

$$\Delta(x) = x \otimes x \qquad \epsilon(x) = 1.$$

This gives a coalgebra structure onto C.

Definition 2.1.3. Let C and C' be coalgebras with coproducts Δ and Δ' and counits ϵ and ϵ', respectively. A linear map f from C to C' is a **coalgebra homomorphism** if

$$(f \otimes f) \circ \Delta = \Delta' \circ f \qquad \text{and} \qquad \epsilon = \epsilon' \circ f.$$

In other words, if f preserves the coproduct and counit of C and C'.

2.2 Hopf Algebras

Definition 2.2.1. A **bialgebra** B over a field $\Bbbk$ is a vector space over $\Bbbk$ with the linear maps $\mu : B \otimes B \to B$, the **multiplication**, $\Delta : B \to B \otimes B$, the **comultiplication**, $\iota : \Bbbk \to B$, the **unit**, and $\epsilon : B \to \Bbbk$, the **counit** satisfying the following conditions:

- B is an algebra over $\Bbbk$,
- B is a coalgebra over $\Bbbk$,
- multiplication and unit are coalgebra homomorphisms (or equivalently, comultiplication and the counit are algebra homomorphisms)

Example 2.2.1. For any Lie algebra $\mathfrak{g}$, $\mathfrak{U}(\mathfrak{g})$ equipped with comultiplication $\Delta(x) = x \otimes 1 + 1 \otimes x$ and counit $\epsilon(x) = 0$, for all $x \in \mathfrak{g}$, is a bialgebra.

Definition 2.2.2. A **Hopf algebra** H over a field $\Bbbk$ is a bialgebra over $\Bbbk$ with the linear maps $\mu : H \otimes H \to H$, the **multiplication**, $\Delta : H \to H \otimes H$, the **comultiplication**, $\iota : \Bbbk \to H$, the **unit**, $\epsilon : H \to \Bbbk$, the **counit**, and $S : H \to H$, the **antipode** that satisfies the following commutative diagrams

$$\begin{tikzcd}
H \otimes H & H \otimes H \arrow[l, "S \otimes id"'] \arrow[d, "\mu"] \\
H \arrow[u, "\Delta"] \arrow[r, "\iota \circ \epsilon"] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
H \otimes H & H \otimes H \arrow[l, "id \otimes S"'] \arrow[d, "\mu"] \\
H \arrow[u, "\Delta"] \arrow[r, "\iota \circ \epsilon"] & H
\end{tikzcd}$$

CHAPTER 3

Lie Superalgebras

In this chapter, we briefly discuss concepts and definitions from super linear algebra and develop a background in Lie superalgebras that will either be relevant to the main topic of study, or will be helpful in understanding other topics that could be relevant to the main topic. We first discuss super vector spaces; the most basic object of discussion. We will mention some parallels that super vector spaces have compared to vector spaces, like dimension and morphisms between spaces. We then describe superalgebras and various operations on superalgebras. We end the chapter with a discussion of Lie superalgebras and universal enveloping algebras of Lie superalgebras.

3.1 Super Vector Spaces

Definition 3.1.1. A *super vector space* V over a field $\Bbbk$, also known as a superspace, is a vector space over the field $\Bbbk$ that has a $\mathbb{Z}_2$-grading. Namely, there are vector subspaces V_0 and V_1 such that $V = V_0 \oplus V_1$.

Definition 3.1.2. Let $V = V_0 \oplus V_1$ be a vector super space over $\Bbbk$, and let W be a proper subset of V. W is a *super vector subspace*, or subsuperspace, if W is itself a super vector space over $\Bbbk$ with the same operations as V.

Definition 3.1.3. Let $V = V_0 \oplus V_1$ be a vector super space over k. The elements of V_0 are said to be *even*, and the elements of V_1 are said to be *odd*.

Definition 3.1.4. Let $V = V_0 \oplus V_1$ be a super vector space over a field $\Bbbk$. Let $dimV_0 = n$ and $\dim V_1 = m$. The *dimension* of V, denoted as $\dim V$, is the tuple

$(n|m)$. We say that V is *finite-dimensional* if V_0 and V_1 are finite dimensional vector spaces. The *superdimension* of V, denoted as sdim V, is defined to be $n - m$.

Example 3.1.1. The superspace $\Bbbk^{n|m} = \Bbbk^n \oplus \Bbbk^m$, where $\Bbbk_0^{n|m} = \Bbbk^n$ and $\Bbbk_1^{n|m} = \Bbbk^m$, is the space of points with $n + m$ coordinates, described with the standard basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+m}\}$, where the first n vectors of the basis are basis vectors of $\Bbbk^n$ and the next m vectors are basis vectors of $\Bbbk^m$

Example 3.1.2. Let V be any vector space over $\Bbbk$. We can endow V with a trivial $\mathbb{Z}_2$ grading to obtain the vector superspace $V = V \oplus \{\vec{0}\}$.

Something to note is that if V and W are vector spaces, then the vector superspaces $V' = V \oplus W$ and $V'' = W \oplus V$, where $V_0' = V_1'' = V$ and $V_1' = V_0'' = W$ are not the same. The definition of super space depends on a representation of the group $\mathbb{Z}_2$.

Definition 3.1.5. An element $v \in V$ for a vector superspace $V = V_0 \oplus V_1$ is called *homogeneous* if $v \in V_0$ or $v \in V_1$.

Definition 3.1.6. The *parity* of a homogeneous element $v \in V_i$ is denoted by $p(v) = i$, where p is the parity function on the set of homogoeneous elements of V to Z_2.

When looking at maps between superspaces, we want the grading to be preserved.

Definition 3.1.7. Let $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ be superspaces. A *morphism*, f from V to W is a linear map $f : V \to W$ that preserve the grading. In other words, $f(V_i) \subset W_i$ for all $i \in \mathbb{Z}_2$. A morphism f of superspaces is called an *isomorphism* if it is bijective.

Since, for a super vector space $V = V_0 \oplus V_1$, we can write each element in V as a sum of elements in V_0 and V_1, we can restrict our attention to the homogeneous elements of V and extend the relevant results by linearity. So any formula defining a linear object in which the parity function appear to elements in a superspace is assumed to be homogeneous, and we will adapt this throughout.

For superspaces V and W, we denote the set of morphisms from V to W as $\mathrm{Hom}(V, W)$. This vector space contains all parity preserving linear maps from V to W. This suggests existence of parity reversing linear maps from V to W.

Definition 3.1.8. Let $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ be superspaces. A linear map f from V to W is *parity-reversing* if $f(V_i) \subset W_{1-i}$, for $i \in \mathbb{Z}_2$.

An important result is that every linear map between superspaces can be written as a sum of parity-preserving and parity-reversing maps.

Proposition 3.1.1. *Let $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ be superspaces. Every linear map f from V to W can be written as $f = f_0 + f_1$, where f_0 is a parity-preserving map from V to W, and f_1 is a parity-reversing map from V to W.*

We denote the space of all linear maps from superspaces V to W as $\mathbf{Hom}(V, W)$, which is referred to as the *internal* Hom [50]. Due to Proposition 3.1.1, $\mathbf{Hom}(V, W)$ is a superspace, with $\mathbf{Hom}(V, W)_0$ being the space of all parity-preserving linear maps from V to W, and $\mathbf{Hom}(V, W)_1$ being the space of all parity-reversing linear maps from V to W.

Remark 3.1.2. Something to note that since $\mathbf{Hom}(V, W)$ is a superspace if V and W are superspaces, then $\mathrm{End}(V) = \mathbf{Hom}(V, V)$ is also a superspace. This plays a significant role in representation theory.

Should V and W be finite-dimensional superspaces, then we can fix a basis of V and W and represent linear maps from V to W using matrices. If $V = W$, $\dim(V) = (n|m)$, and $\Bbbk = \mathbb{C}$, then fixing a standard basis of V will give the vector superspace of all matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

denoted as $M_{n|m}(\mathbb{C})$. Here, A and D are $n \times n$ and $m \times m$ matrices, respectively.

3.2 Superalgebras

Now we extend the concept of superspaces to that of superalgebras.

Definition 3.2.1. A *superalgebra* is a superspace $A = A_0 \oplus A_1$ equipped with a bilinear multiplication that preserves the grading on A, i.e. $A_i A_j \subset A_{i+j}$ for $i, j \in \mathbb{Z}_2$.

An analogue of commutativity in superalgebras is as follows:

Definition 3.2.2. A superalgebra A is *supercommutative* if, for all $a, b \in A$,

$$ab = (-1)^{p(a)p(b)} ba$$

Example 3.2.1. Let V be a finite-dimensional vector superspace. Then we have that $\mathrm{End}(V)$ is a superalgebra, with multiplication defined as composition of functions.

Remark 3.2.1. If A is a supercommutative superalgebra, then every odd element $a \in A$ is nilpotent as $a^2 = -a^2 \implies a^2 = 0$.

Supercommutativity in superalgebras differs from commutativity in commutative algebras in that a sign factor appears. This sign factors appears whenever two (odd) elements are interchanged in a classical relation, otherwise known as the *rule of signs* [50].

We want to mention some algebraic constructions for superalgebras that will come up every now and again.

Definition 3.2.3. Let $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ be superalgebras. Their *direct sum* $V \oplus W$ is a superalgebra who space is the direct sum of the vector spaces V and W with the induced $\mathbb{Z}_2$-grading

$$(V \oplus W)_0 = V_0 \oplus W_0$$
$$(V \oplus W)_1 = V_1 \oplus W_1$$

Definition 3.2.4. Let V and W be superalgebras. Their *tensor product* $V \otimes W$ is a superalgebra who space is the tensor product of the vector spaces V and W with the induced $\mathbb{Z}_2$-grading

$$(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)$$
$$(V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0)$$

and operation defined by

$$(v_1 \otimes w_1)(v_2 \otimes w_2) = (-1)^{p(a_2)p(b_1)} v_1 v_2 \otimes w_1 w_2$$

Definition 3.2.5. Let V be a vector space. The m^{th} *tensor product* of V, $V^{\otimes m}$, is the tensor product of V with itself k times:

$$V^{\otimes m} = V \otimes V \otimes \ldots \otimes V$$

We denote $V^{\otimes 0} = \Bbbk$. The *tensor algebra* of V, $T(V)$, is constructed as the following:

$$T(V) = \bigoplus_{m=0}^{\infty} V^{\otimes m} = \Bbbk \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \ldots$$

with multiplication on homogeneous and elementary tensors determined by the isomorphism $V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes (m+n)}$:

$$(v_1 \otimes \ldots \otimes v_m) \cdot (w_1 \otimes \ldots \otimes w_n) = v_1 \otimes \ldots \otimes v_m \otimes w_1 \otimes \ldots \otimes w_n$$

Proposition 3.2.2. *Let V be a vector space. The tensor algebra $T(V)$ of V satisfies the following universal property: if A is associative algebra and $f : V \to A$ is a linear map, and $\iota : V \to T(V)$ is the inclusion map $\iota(v) = v$, then there exists a unique algebra homomorphism $\overline{f} : T(V) \to A$ such that $\iota \circ \overline{f} = f$*

Remark 3.2.3. If V is a superspace, using definitions 3.2.4 and 3.2.5, we can endow $T(V)$ with a $\mathbb{Z}_2$-grading to obtain a *tensor superalgebra* of V.

Two important algebras that can be obtained from the tensor algebra are the symmetric algebra and the exterior algebra.

Definition 3.2.6. Let V be a vector superspace. The *symmetric superalgebra* of a vector superspace V, denoted as $S(V)$, is the superalgebra obtained by taking the quotient of the tensor superalgebra $T(V)$ by the ideal $I(V)$ generated by all (homogeneous) elements of the form

$$u \otimes v - (-1)^{p(u)p(v)} v \otimes u$$

for all $u, v \in V$. The *kth symmetric power*, $S^k(V)$, of V is equal to $V^{\otimes k}$ modulo the submodule generated by all elements of the form

$$v_1 \otimes v_2 \otimes \ldots \otimes v_k - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \ldots \otimes v_{\sigma(k)}$$

for all $v_i \in V$ and for all σ in the symmetric group S_k.

Definition 3.2.7. Let V be a vector superspace. The *exterior superalgebra* of a vector superspace V, denoted as $\bigwedge(V)$, is the superalgebra obtained by taking the quotient of the tensor superalgebra $T(V)$ by the ideal $I(V)$ generated by all (homogeneous) elements of the form

$$v \otimes v$$

for all $v \in V$. The *kth exterior power*, $\bigwedge^k(V)$, of V is equal to $V^{\otimes k}$ modulo the submodule generated by all elements of the form

$$v_1 \otimes v_2 \otimes \ldots \otimes v_k$$

where $v_i = v_j$ for some $i \neq j$.

Remark 3.2.4. The definitions for the kth symmetric power and kth exterior power in the classical case are given as theorems in [26] (see Theorems 34 and 36 in section 11.5 in [26]). If V is a superspace, we can then use these facts and the grading on V to lift these theorems to the definitions above. Similarly, the symmetric and exterior superalgebras inherit the universal property from the tensor superalgebra in proposition 3.2.2.

Remark 3.2.5. We can rewrite the symmetric and exterior superalgebras as direct sums of their symmetric and exterior powers respectively. In other words, for a vector superspace V,

$$S(V) = \bigoplus_{m=0}^{\infty} S^m(V)$$
$$\bigwedge(V) = \bigoplus_{m=0}^{\infty} \bigwedge^m(V)$$

Another important superalgebra that should be adressed is a Hopf superalgebra, which is the "super" version of the Hopf algebra defined in Definition 2.2.2:

Definition 3.2.8. A **Hopf superalgebra** H over a field $\Bbbk$ is a bisuperalgebra over $\Bbbk$ with the linear maps $\mu : H \otimes H \to H$, the **multiplication**, $\Delta : H \to H \otimes H$, the **comultiplication**, $\iota : \Bbbk \to H$, the **unit**, $\epsilon : H \to \Bbbk$, the **counit**, and $S : H \to H$, the **antipode** that satisfies the following commutative diagrams

$$
\begin{tikzcd}
H \otimes H & H \otimes H \arrow[l, "S \otimes id"'] \arrow[d, "\mu"] \\
H \arrow[u, "\Delta"] \arrow[r, "\iota \circ \epsilon"] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
H \otimes H & H \otimes H \arrow[l, "id \otimes S"'] \arrow[d, "\mu"] \\
H \arrow[u, "\Delta"] \arrow[r, "\iota \circ \epsilon"] & H
\end{tikzcd}
$$

3.3 Lie Superalgebras

Throughout the rest of this chapter, assume that vector super spaces are over a field $\Bbbk$, unless otherwise stated. Now we define the main object of interest:

Definition 3.3.1. A *Lie superalgebra* is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ together with a bilinear map $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the superbracket, such that

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j} \text{ for } i, j \in \mathbb{Z}_2 \tag{3.1}$$

$$[a, b] = -(-1)^{p(a)p(b)}[b, a] \tag{3.2}$$

$$(-1)^{p(a)p(c)}[a, [b, c]] + (-1)^{p(a)p(b)}[b, [c, a]] + (-1)^{p(b)p(c)}[c, [a, b]] = 0 \tag{3.3}$$

In the definition above, note that equation 3.1 implies that the bilinear map $[\,,\,]$ preserves the grading, equation 3.2 defines a grading skew symmetry, and equation 3.3 is referred to as the super Jacobi identity.

Example 3.3.1. Given a vector superspace V, we can define a trivial Lie superbracket where $[X, Y] \equiv 0$ for all $X, Y \in V$, to obtain a trivial Lie superalgebra. Also, recall from Example 3.1.2 that any vector space V can be written trivially as $V = V \oplus \{0\}$ as a superspace. Therefore, we can also define a trivial Lie superalgebra of vector spaces by using the same superbracket defined on the trivial decomposition of V as a superspace.

Example 3.3.2. Let $V = V_0 \oplus V_1$ be a finite-dimensional vector superspace, and thus $\mathrm{End}(V)$ is a finite-dimensional superalgebra. We can endow $\mathrm{End}(V)$ with the superbracket $[\, ,\,] : \mathrm{End}(V) \times \mathrm{End}(V) \to \mathrm{End}(V)$ such that

$$[a, b] = ab - (-1)^{p(a)p(b)} ba$$

for $a, b \in \mathrm{End}(V)$. This Lie superalgebra is called the general Lie superalgebra, $\mathfrak{gl}(V)$. If $\dim V_0 = m$ and $\dim V_1 = n$, then by fixing a basis, we can write all elements $\mathfrak{gl}(V)$ as matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.4}$$

where A is a $m \times m$ matrix and D is a $n \times n$ matrix. The Lie superalgebra that contains all matrices of the form of (3.4) is denoted as $\mathfrak{gl}(m|n)$, which is isomorphic to $\mathfrak{gl}(V)$. The even elements are those where $B = C = 0$, and the odd elements are those where $A = D = 0$. Note that every element in $\mathfrak{gl}(m|n)$ can be written as a sum of the elementary matrices E_{ij} for $1 \leq i, j \leq m+n$. We have that $p(E_{ij}) = p(i)+p(j) \mod 2$, where $p(i) = 0$ for $1 \leq i \leq m$ and $p(i) = 1$ for $m + 1 \leq i \leq m + n$. The superbracket in terms of these elementary matrices is

$$[E_{ij}, E_{k\ell}] = \delta_{jk} E_{i\ell} - (-1)^{p(E_{ij})p(E_{k\ell})} \delta_{i\ell} E_{kj}$$

If $X \in \mathfrak{gl}(m|n)$ is in the form of (3.4), we can define some super-analogues of some properties of matrices in linear algebra. The only important properties for our purposes are the supertrace and supertranspose.

Definition 3.3.2. Let $X \in \mathfrak{gl}(m|n)$ be of the form in (3.4). The *supertrace* of X, denoted $\mathrm{Str}(X)$, is defined by

$$\mathrm{Str}(X) = \mathrm{tr}(A) - \mathrm{tr}(D)$$

Remark 3.3.1. Due to the properties of trace, we have that $\mathrm{Str}(X+Y) = \mathrm{Str}(X) + \mathrm{Str}(Y)$ and $\mathrm{Str}(cX) = c\,\mathrm{Str}(X)$, for $X, Y \in \mathfrak{gl}(m|n)$ and $c \in \Bbbk$.

Remark 3.3.2. The supertrace of all odd homogeneous elements of $\mathfrak{gl}(m|n)$ is 0.

Definition 3.3.3. Let $X \in \mathfrak{gl}(m|n)$ be of the form in (3.4). The *supertranspose* of X, denoted X^{st}, is defined by

$$X^{\mathrm{st}} = \begin{pmatrix} A^T & C^T \\ -B^T & D^T \end{pmatrix}$$

In terms of elementary matrices E_{ij}, the supertranspose is defined by

$$(E_{ij})^{st} = (-1)^{p(i)(p(j)+1)} E_{ji}$$

There is an analogue of subspaces for Lie superalgebras.

Definition 3.3.4. Let $\mathfrak{g}$ be a Lie superalgebra and let $\mathfrak{h}$ be a subsuperspace of $\mathfrak{g}$. $\mathfrak{h}$ is a *Lie subsuperalgebra* of $\mathfrak{g}$ if $[g_1, g_2]_{\mathfrak{g}} \in \mathfrak{h}$, for all $g_1, g_2 \in \mathfrak{h}$.

Example 3.3.3. The commutator subsuperalgebra of $\mathfrak{gl}(m|n)$ consists of all matrices of the form (3.4) with supertrace zero. This subsuperalgebra of $\mathfrak{gl}(m|n)$ is denoted by $\mathfrak{sl}(m|n)$.

Just like how there is a notion of isomorphisms with superspaces, we have the equivalent for Lie superalgebras.

Definition 3.3.5. Let $\mathfrak{g}$ and $\mathfrak{a}$ be Lie superalgebras with superbrackets $[\,,\,]_{\mathfrak{g}}$ and $[\,,\,]_{\mathfrak{a}}$ respectively. A *morphism*, f from $\mathfrak{g}$ to $\mathfrak{a}$ is a parity-preserving linear map $f : \mathfrak{g} \to \mathfrak{a}$ such that

$$f[g_1, g_2]_{\mathfrak{g}} = [f(g_1), f(g_2)]_{\mathfrak{a}}$$

for all $g_1, g_2 \in \mathfrak{g}$. A morphism f of Lie superalgebras is called an *isomorphism* if f is also bijective.

We bring up some other definitions that will be important for understanding what will be discussed in future chapters.

Definition 3.3.6. Let $\mathfrak{g}$ be a Lie superalgebra. If V and W are subspaces, then $[V, W]$, which is spanned by all $[v, w]$ for all $v \in V$ and $w \in W$, is a subspace of $\mathfrak{g}$.

Definition 3.3.7. A vector super subspace $\mathfrak{a}$ of $\mathfrak{g}$ is called an *ideal* if $[\mathfrak{a}, \mathfrak{g}] \subset \mathfrak{a}$.

Definition 3.3.8. A Lie superalgebra $\mathfrak{g}$ is called *solvable* if the derived series of $\mathfrak{g}$

$$\mathfrak{g} \leq [\mathfrak{g}, \mathfrak{g}] \leq [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \leq [[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]], [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]]] \leq \dots$$

terminates. If we let

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] \text{ for } i \geq 0$$

then the derived series of $\mathfrak{g}$ terminates if $\mathfrak{g}^{(n)} = 0$ for some large n.

Definition 3.3.9. A Lie superalgebra $\mathfrak{g}$ is called *nilpotent* if the lower central series of $\mathfrak{g}$

$$\mathfrak{g} \leq [\mathfrak{g}, \mathfrak{g}] \leq [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \leq [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]] \leq \dots$$

terminates. If we let

$$\mathfrak{g}^{[0]} = \mathfrak{g}, \quad \mathfrak{g}^{[i+1]} = [\mathfrak{g}, \mathfrak{g}^{[i]}] \text{ for } i \geq 0$$

then the lower central series of $\mathfrak{g}$ terminates if $\mathfrak{g}^{[n]} = 0$ for some large n.

Definition 3.3.10. A Lie superalgebra $\mathfrak{g}$ is called *simple* if the only $\mathbb{Z}_2$-graded ideals of $\mathfrak{g}$ are 0 and $\mathfrak{g}$. A Lie superalgebra $\mathfrak{g}$ is called *semisimple* if it contains no solvable ideals.

3.4 Bilinear Forms

We saw that the definition for Lie superalgebras (and Lie algebras) use a bilinear form. You may recall from linear algebra that (symmetric) bilinear forms play a role in determining orthogonal bases of a vector space. In studying representation theory of Lie superalgebras, bilinear forms prove to be an essential asset. This section will provide the more important details and definitions that will be used in future chapters.

Definition 3.4.1. A bilinear form B on a vector (super)space V, and thus on a Lie superalgebra, is a bilinear map $V \times V \to \Bbbk$. In other words, for all $u, v \in V$ and $\alpha, \beta \in \Bbbk$,

$$\mathsf{B}(\alpha u + \beta v, w) = \alpha \mathsf{B}(u, w) + \beta \mathsf{B}(v, w), \qquad \text{and}$$
$$\mathsf{B}(u, \alpha v + \beta w) = \alpha \mathsf{B}(u, v) + \beta \mathsf{B}(u, w)$$

Definition 3.4.2. Let $\mathfrak{g}$ be a Lie superalgebra. A bilinear form B on $\mathfrak{g}$ is

(i) *supersymmetric* if $\mathsf{B}(X, Y) = (-1)^{p(X)p(Y)} \mathsf{B}(Y, X)$ for all homogeneous $X, Y \in \mathfrak{g}$,

(ii) *skew-supersymmetric* if $\mathsf{B}(X, Y) = -(-1)^{p(X)p(Y)} \mathsf{B}(Y, X)$ for all homogeneous $X, Y \in \mathfrak{g}$,

(iii) *(ad)-invariant* if $\mathsf{B}([X, Y], Z) = \mathsf{B}(X, [Y, Z])$ for all homogeneous $X, Y, Z \in \mathfrak{g}$,

(iv) *non-degenerate* if for $X \in \mathfrak{g}$, $\mathsf{B}(X, Y) = 0$ for all homogeneous $Y \in \mathfrak{g}$, then $X = 0$.

(v) *even* if $\mathsf{B}(X, Y) = 0$ for all $X, Y \in \mathfrak{g}$ such that $p(X) = p(Y) + 1$, and

(vi) *odd* if $\mathsf{B}(X, Y) = 0$ for all $X, Y \in \mathfrak{g}$ such that $p(X) = p(Y)$.

Definition 3.4.3. Let B be a bilinear form on a Lie superalgebra $\mathfrak{g}$. A Lie subsuperalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is said to be B-*isotropic* if B restricted to $\mathfrak{a}$ vanishes. In other words, $\mathfrak{a}$ is a B-isotropic subspace of $\mathfrak{g}$ if $\mathsf{B}(X, Y) = 0$ for all $X, Y \in \mathfrak{a}$.

Definition 3.4.4. Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be Lie subsuperalgebras of $\mathfrak{g}$. We say that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are *transversal* if $\mathfrak{g} = \mathfrak{a}_1 + \mathfrak{a}_2$.

One bilinear form on $\mathfrak{gl}_{n|n}$ that we will use extensively is the form given by the supertrace

$$\mathsf{B}(X,Y) = \mathrm{Str}(XY)$$

$(X, Y \in \mathfrak{gl}_{n|n})$. An important fact is that this form is ad-invariant, supersymmetric and non-degenerate, which will prove necessary when discussing the Lie bisuperalgebra structure on $\mathfrak{p}_n$.

Proposition 3.4.1 ([36])**.** *The bilinear form* B *on* $\mathfrak{gl}(m|n)$ *given by the super-trace,* $\mathsf{B}(X,Y) = \mathrm{Str}(XY)$, *is ad-invariant, supersymmetric and non-degenerate.*

3.5 Lie bisuperalgebras

Definition 3.5.1. A *Lie superbialgebra* is a pair $(\mathfrak{g}, \delta)$, where $\mathfrak{g}$ is a Lie superalgebra, δ is a morphism of super vector spaces $\delta : \mathfrak{g} \to \Lambda^2(\mathfrak{g})$ which is a *cocycle* and satisfies the *co-super Jacobi identity.*

Definition 3.5.2. A *Manin supertriple* $(\mathfrak{a}, \mathfrak{a}_1, \mathfrak{a}_2)$ consists of a Lie superalgebra $\mathfrak{a}$ equipped with an ad-invariant supersymmetric non-degenerate bilinear form B, along with two Lie subsuperalgebras $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of $\mathfrak{a}$ which are B-isotropic transversal subsuperspaces of $\mathfrak{a}$.

A bilinear form B given in the previous definition defines a non-degenerate pairing between $\mathfrak{a}_1$ and $\mathfrak{a}_2$ and a supercobracket $\delta : \mathfrak{a}_1 \to \mathfrak{a}_1^{\otimes 2}$ by

$$\mathsf{B}^{\otimes 2}(\delta(X), Y \otimes Z) = \mathsf{B}(X, [Y, Z]), \tag{3.5}$$

for all $X \in \mathfrak{a}_1$ and $Y, Z \in \mathfrak{a}_2$, where $\mathsf{B}^{\otimes 2}$ is obtained by extending B to a non-degenerate pairing on $\mathfrak{gl}(m|n) \otimes_{\mathbb{C}} \mathfrak{gl}(m|n)$. This is a result from Definition 3.5.1 and the following proposition:

Proposition 3.5.1. *[Andruskiewitsch, 93] Let* $(\mathfrak{g}, \delta)$ *be a Lie bisuperalgebra and set* $\mathfrak{p}_1 = \mathfrak{g}$, $\mathfrak{p}_2 = \mathfrak{g}^*$, $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$. *Then* $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ *is a Manin supertriple. Conversely,* *any Manin supertriple with $\mathfrak{p}$ finite dimensional gives rise to a Lie superbialgebra structure on $\mathfrak{p}_1$.*

3.6 Universal Enveloping Superalgebras

Definition 3.6.1. Let $\mathfrak{g}$ be a Lie superalgebra. A universal enveloping algebra of $\mathfrak{g}$ is a pair (U,σ), where U is an associative $\Bbbk$-algebra with 1, and where $\sigma : \mathfrak{g} \to U$ is a linear map satisfying

$$\sigma([x, y]) = \sigma(x)\sigma(y) - (-1)^{p(x)p(y)}\sigma(y)\sigma(x) \tag{3.6}$$

such that the following universal property is satisfied: if U' is another associative $\Bbbk$-algebra with 1 and $\sigma' : \mathfrak{g} \to U'$ is a linear map satisfying (3.6), then there is a unique algebra homomorphism $\phi : U \to U'$ such that $\phi \circ \sigma = \sigma'$.

Theorem 3.6.1. *Let $\mathfrak{g}$ be a Lie superalgebra. Then there exists a unique universal enveloping superalgebra $U(\mathfrak{g})$ of $\mathfrak{g}$, up to isomorphism.*

Proof. We first show such a universal enveloping superalgebra exists. Let $T(\mathfrak{g})$ be the tensor algebra on $\mathfrak{g}$, and let I be the two sided ideal of $T(\mathfrak{g})$ generated by all elements of the form

$$x \otimes y - (-1)^{p(x)p(y)}y \otimes x - [x, y]$$

where x and y are homogeneous elements of $\mathfrak{g}$. Set $U(\mathfrak{g}) = T(\mathfrak{g})/I$. Let $\alpha : T(\mathfrak{g}) \to U(\mathfrak{g})$ be the natural map from $T(\mathfrak{g})$ to $U(\mathfrak{g})$, with $\ker \alpha = I$ and let $\iota : \mathfrak{g} \to T(\mathfrak{g})$ be the inclusion map. Let $\sigma = \alpha \circ \iota$. Note that for all homogeneous $x, y \in \mathfrak{g}$:

$$\begin{aligned}\sigma([x, y]) = \alpha \circ \iota([x, y]) = \alpha([x, y]) &= [x, y] \\ &= x \otimes y - (-1)^{p(x)p(y)}y \otimes x \\ &= \iota(x)\iota(y) - (-1)^{p(x)p(y)}\iota(y)\iota(x)\end{aligned}$$

$$= \alpha(\iota(x)\iota(y) - (-1)^{p(x)p(y)}\iota(y)\iota(x))$$
$$= \sigma(x)\sigma(y) - (-1)^{p(x)p(y)}\sigma(y)\sigma(x))$$

So we have that σ satisfies equation (3.6).

Now let U' be another associative $\Bbbk$-algebra with 1 and let $\sigma' : \mathfrak{g} \to U'$ be a linear map satisfying equation (3.6). Proposition 3.2.2 yields an algebra homomorphism $\phi' : T(\mathfrak{g}) \to U'$ such that $\phi' \circ \iota = \sigma'$. Since σ satisfies the property in equation (3.6), we have that I lies in $\ker \phi'$. ϕ' therefore can be extended to a algebra homomorphism $\phi : U(\mathfrak{g}) \to U'$ such that $\phi \circ \sigma' = \phi'$. Thus, we have that $\phi \circ \sigma = \sigma'$. Now suppose that there exists another map $\psi : U(\mathfrak{g}) \to U'$ such that $\psi \circ \sigma = \sigma'$. Then, for every $x \in \mathfrak{g}$, we have that $\phi \circ \sigma(x) = \sigma'(x) = \psi \circ \sigma(x)$, which implies that $\phi = \psi$ as $\sigma(\mathfrak{g})$ generates $U(\mathfrak{g})$ by construction. Therefore, we have that the pair $(U(\mathfrak{g}), \sigma)$ is a universal enveloping algebra of $\mathfrak{g}$.

To prove uniqueness, let $(U(\mathfrak{g}), \sigma)$ and (U', σ') be universal enveloping algebras of $\mathfrak{g}$. Using the fact that σ satisfies the universal property, there is a unique algebra homomorphism $\phi : U(\mathfrak{g}) \to U'$ such that $\phi \circ \sigma = \sigma'$. Switching roles of the algebras gives another unique algebra homomorphism $\psi : U' \to U(\mathfrak{g})$ such that $\psi \circ \sigma' = \sigma$. Combining these gives that $\phi \circ \psi = 1_{U'}$ and $\psi \circ \phi = 1_{U(\mathfrak{g})}$. Therefore, we get that $U(\mathfrak{g}) \cong U'$.

Lastly, since $\mathfrak{g}$ has a $\mathbb{Z}_2$ grading, $T(\mathfrak{g})$ also has a $\mathbb{Z}_2$ grading, and by extension $U(\mathfrak{g})$ has a $\mathbb{Z}_2$ grading. $\square$

We will use the notation xy as opposed to $x \otimes y$ to denote multiplication in $U(\mathfrak{g})$. Also, note that for any odd element x in a Lie superalgebra $\mathfrak{g}$, we have that

$$0 = [x, x] = xx + xx = 2x^2 \implies x^2 = 0$$

The following is a very important theorem for describing the generators of $\mathfrak{U}(\mathfrak{g})$ that is a (super) generalization for Lie algebras:

Theorem 3.6.2 (Poincare-Birkhoff-Witt Theorem). *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra. Let $\{x_1, x_2, \ldots, x_m\}$ be a basis for $\mathfrak{g}_0$ and let $\{y_1, y_2, \ldots, y_n\}$ be a basis for $\mathfrak{g}_1$. Then the set*

$$\{x_1^{r_1} x_2^{r_2} \ldots x_m^{r_m} y_1^{s_1} y_2^{s_2} \ldots y_m^{s_m} \mid r_1, r_2, \ldots, r_m \in \mathbb{Z} \in \mathbb{Z}^{\geq 0}, s_1, s_2, \ldots, s_n \in \{0, 1\}\}$$

is a basis for $\mathfrak{U}(\mathfrak{g})$.

3.7 Representations of Lie superalgebras

Definition 3.7.1. Let $\mathfrak{g}$ be a Lie superalgebra. A representation ρ of $\mathfrak{g}$ is a Lie superalgebra homomorphism

$$\rho : \mathfrak{g} \to \mathrm{End}(V)$$

where V is a vector superspace.

Representations allows us to treat elements of a Lie superalgebra as linear maps in $\mathrm{End}(V)$. Therefore, we can endow V with a $\mathfrak{g}$-module structure using the multiplication

$$g \cdot V \coloneqq \rho(g)(v)$$

for $g \in \mathfrak{g}$. We will write gV to indicate the action of an element $g \in \mathfrak{g}$ on V.

Definition 3.7.2. Let $\mathfrak{g}$ be a Lie superalgebra. A representation ρ of $\mathfrak{g}$ is *faithful* if it is injective (i.e. one-to-one).

Faithful representations allows us to view elements of a Lie superalgebra as linear maps, whose properties are much more studied. More specifically, if V and $\mathfrak{g}$ are finite-dimensional as superspaces, then by fixing a basis, we can view elements of a Lie superalgebra as matrices. This is because the Lie superalgebra $\{\rho(g) \in \mathrm{End}(V) \mid g \in \mathfrak{g}\}$, where ρ is a faithful representation of $\mathfrak{g}$, is isomorphic to $\mathfrak{g}$.

Example 3.7.1. Let $\mathfrak{g}$ be a Lie superalgebra. The *adjoint representation* $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is a representation that maps g to $\mathrm{ad}(g)$ for each $g \in \mathfrak{g}$, where

$$\mathrm{ad}(g)(x) = [g, x]$$

for $x \in \mathfrak{g}$.

3.8 Weights and Roots of Lie superalgebras

Definition 3.8.1. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal nilpotent subalgebra of $\mathfrak{g}_0$ such that it is equal to its own normalizer, i.e.

$$N(\mathfrak{h}) = \{X \in \mathfrak{g}_0 \mid [X, \mathfrak{h}] \subset \mathfrak{h}\} = \mathfrak{h}.$$

Example 3.8.1. Consider the Lie superalgebra $\mathfrak{gl}(m|n)$. Consider the subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(m|n)$ that consists of all diagonal matrices E_{ii} for $1 \leq i \leq m+n$. Note that $[E_{ii}, E_{jj}] = 0$ for all $1 \leq i, j \leq m+n$, and thus $\mathfrak{h}$ is nilpotent by definition. Also, $N(\mathfrak{h}) = \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{gl}(m|n)$.

Definition 3.8.2. Let $\mathfrak{g}$ be a Lie superalgebra, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let V be a $\mathfrak{g}$-module. For $\mu \in \mathfrak{h}^*$, we define the *weight space* V_μ of V attached to μ to be

$$V_\mu = \{v \in V \mid hv = \mu(h)v, \text{ for all } h \in \mathfrak{h}\}.$$

We call $\mu \in \mathfrak{h}^*$ a *weight* of V if $V_\mu \neq 0$.

Definition 3.8.3. Let $\mathfrak{g}$ be a Lie superalgebra and let V be a $\mathfrak{g}$-module. V is called a weight module if

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$$

and $\dim V_\mu < \infty$ for every $\mu \in \mathfrak{h}^*$.

Note that in definition 3.8.2, weight space involves looking at the eigenvalues of the action of $\mathfrak{g}$ onto a superspace V. If we have a representation of $\mathfrak{g}$ onto V, then we can use the definitions of weight spaces above. The weights of adjoint representations of a Lie superalgebra $\mathfrak{g}$ (which are the eigenvalues of the action of $\mathfrak{g}$ with itself), are instead called *roots*.

Definition 3.8.4. Let $\mathfrak{g}$ be a Lie superalgebra and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. For $\mu \in \mathfrak{h}^*$, we define the *root space* $\mathfrak{g}_\mu$ of $\mathfrak{g}$ attached to μ to be

$$\mathfrak{g}_\mu = \{g \in \mathfrak{g} \mid [h, g] = \mu(h)g, \text{ for all } h \in \mathfrak{h}\}.$$

We call $\mu \in \mathfrak{h}^*$ a *root* of V if $V_\mu \neq 0$.

Definition 3.8.5. The set of roots Δ of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a Cartan subalgebra $\mathfrak{h}$ is

$$\Delta = \{\mu \in \mathfrak{h}^* \mid \mu \neq 0, \mathfrak{g}_\mu \neq 0\}.$$

We call a root $\alpha \in \Delta$ *even* if $\mathfrak{g}_\alpha \cap \mathfrak{g}_0 \neq 0$ or *odd* if $\mathfrak{g}_\alpha \cap \mathfrak{g}_1 \neq 0$. We will denote the set of even and odd roots of $\mathfrak{g}$ as Δ_0 and δ_1 respectively.

CHAPTER 4

Lie Superalgebra of type P

It is well known that a semisimple Lie algebra is a direct sum of simple Lie algebras, but this is by no means true in the case of Lie superalgebras. However, Kac gave a construction in [36] that allows us to describe finite-dimensional semisimple Lie superalgebras in terms of finite-dimensional simple Lie superalgebras. Kac then went on to classify all simple Lie superalgebras over an algebraically closed field of characteristic zero. One class, the Lie superalgebras of type P, otherwise called the periplectic Lie superalgebras, $\mathfrak{p}_n$, will be the main focus.

We want to define a quantized enveloping superalgebra associated with $\mathfrak{p}_n$, which will first require us to determine a Lie bisuperalgebra structure on $\mathfrak{p}_n$. In this chapter, we will first define $\mathfrak{p}_n$, then show what the Lie bisuperalgebra structure is on $\mathfrak{p}_n$. We then finish the discussion with other essentials for when we discuss the quantized enveloping superalgebra in the next chapter.

From this chapter and on, we will be studying these Lie superalgebras over the complex numbers $\mathbb{C}$.

4.1 Definition of $\mathfrak{p}_n$

Let $\mathbb{C}(n|n)$ is the vector superspace $\mathbb{C}^n \oplus \mathbb{C}^n$ spanned by the odd standard basis vectors $e_{-n}, \ldots, e_{-1}$ and the even standard basis vectors $e_1, \ldots, e_n$. Let $M_{n|n}(\mathbb{C})$ be the vector superspace consisting of matrices $A = (a_{ij})$ with $a_{ij} \in \mathbb{C}$, with rows and columns labeled using the integers $-n, \ldots, -1$ and $1, \ldots, n$ (which means $i, j \in \{\pm 1, \pm 2, \ldots, \pm n\}$. For each elementary matrix E_{ij} in $M_{n|n}(\mathbb{C})$, the parity is determined by $p(E_{ij}) = p(i) + p(j) \mod 2$, where $p(i) = 0 \in \mathbb{Z}_2$ if $1 \le i \le n$ and $p(i) = 1 \in \mathbb{Z}_2$ if $-n \le i \le -1$.

We will use $\mathfrak{gl}_{n|n}$ to denote the Lie superalgebra $\mathfrak{gl}(n|n)$ over $\mathbb{C}$, with underlying vector superspace $M_{n|n}(\mathbb{C})$ described above. Recall that $\mathfrak{gl}(n|n)$ is equipped with the Lie superbracket

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - (-1)^{(p(i)+p(j))(p(k)+p(l))}\delta_{il}E_{kj}$$

Let ι be the involution on $\mathfrak{gl}(n|n)$ given by $\iota(X) = -\pi(X^{\mathrm{st}})$, where $\pi : \mathfrak{gl}(n|n) \to \mathfrak{gl}(n|n)$ is the linear map given by $\pi(E_{ij}) = E_{-i,-j}$.

Definition 4.1.1. The Lie superalgebra $\mathfrak{p}_n$ of type P, which is also called the periplectic Lie superalgebra, is the subspace of fixed points of $\mathfrak{gl}(n|n)$ under the involution ι, that is

$$\mathfrak{p}_n = \{X \in \mathfrak{gl}(n|n) \mid \iota(X) = X\}$$

Using the definition, we see that if $X \in \mathfrak{p}_n$, then as a matrix X has the form

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \tag{4.1}$$

where B is symmetric and C is skew-symmetric.

Let

$$\mathsf{E}_{ij} = E_{ij} + \iota(E_{ij}) = E_{ij} - (-1)^{p(i)(p(j)+1)}E_{-j,-i}$$

We can write the superbracket on $\mathfrak{p}_n$ in terms of the E_{ij}'s as follows:

$$\begin{aligned}[\mathsf{E}_{ji}, \mathsf{E}_{lk}] = &\delta_{il}\mathsf{E}_{jk} - (-1)^{(p(i)+p(j))(p(k)+p(l))}\delta_{jk}\mathsf{E}_{li} \\ &- \delta_{i,-k}(-1)^{p(l)(p(k)+1)}\mathsf{E}_{j,-l} - \delta_{-j,l}(-1)^{p(j)(p(i)+1)}\mathsf{E}_{-i,k}\end{aligned} \tag{4.2}$$

A basis of $\mathfrak{p}_n$ is provided by all the matrices E_{ij} with $1 \leq |j| < |i| \leq n$, $1 \leq i = j \leq n$, and $-n \leq i = -j \leq -1$. Note that $\mathsf{E}_{ij} = -(-1)^{p(i)(p(j)+1)}\mathsf{E}_{-j,-i}$ for all $i, j \in \{\pm 1, \ldots, \pm n\}$. This implies that $\mathsf{E}_{i,-i} = 0$ when $1 \leq i \leq n$.

There is an alternative definition for this Lie superalgebra: Let $V_n = V_0 \oplus V_1$ be a vector superspace of dimension $(n|n)$, and let B be an odd, non-degenerate, supersymmetric bilinear form given on the basis of V_n by

$$(v_i, v_j) = -\delta_{i,-j} \tag{4.3}$$

Then we can define the Lie superalgebra $\mathfrak{p}(V_n)$ to be the Lie subsuperalgebra of $\mathfrak{gl}(V_n)$ whose homogeneous maps preserve the bilinear form in (4.3):

$$\mathfrak{p}(V_n) = \{\varphi \in \mathfrak{gl}(V_n) \mid (\varphi(v), w) + (-1)^{p(a)p(\varphi)}(v, \varphi(w)) = 0 \text{ for all homogeneous } v, w \in V_n\}$$

Fixing a basis for V_n will identify $\mathfrak{p}(V_n)$ with a Lie superalgebra isomorphic to $\mathfrak{p}_n$. Different choices of the bilinear form will give isomorphic periplectic Lie superalgebras.

4.2 Lie bisuperalgebra structure of $\mathfrak{p}_n$

In order to construct a Lie bisuperalgebra structure on $\mathfrak{p}_n$, Proposition 3.5.1 allows us to use a Manin supertriple. First we need to determine a bilinear form on $\mathfrak{gl}_{n|n}$ to use. For the rest of this chapter, we will use the bilinear form B given by the supertrace

$$\mathsf{B}(X, Y) = \mathrm{Str}(XY).$$

Recall that B is ad-invariant, supersymmetric and non-degenrate by Proposition 3.4.1, which is necessary by definition of a Manin supertriple.

Now we want to find a Lie subsuperalgebra of $\mathfrak{gl}_{n|n}$ that is what we can consider an "orthogonal“ to $\mathfrak{p}_n$ with respect to the supertrace. The following Lie subsuperalgebra of $\mathfrak{gl}_{n|n}$ is what we use:

Definition 4.2.1. The "butterfly" Lie superalgebra $\mathfrak{b}_n$ is the subspace of $\mathfrak{gl}_{n|n}$ spanned by E_{ij} with $1 \leq |i| < |j| \leq n$ and by $E_{ii} + E_{-i,-i}, E_{i,-i}$ for $1 \leq i \leq n$.

Proposition 4.2.1. *($\mathfrak{gl}_{n|n}, \mathfrak{p}_n, \mathfrak{b}_n$) is a Manin supertriple.*

Proof. Through straightforward computations of the basis elements, we have that $\mathsf{B}(X, Y) = 0$ for $X, Y \in \mathfrak{p}_n$ or for $X, Y \in \mathfrak{b}_n$. So, $\mathfrak{p}_n$ and $\mathfrak{b}_n$ are B-isotropic. Also, note that every elementary matrix is obtained by some sum of basis elements in $\mathfrak{p}_n$ and $\mathfrak{b}_n$.

$$\begin{aligned}
E_{ii} &= (-1)^{p(i)}\mathsf{E}_{ii} + (E_{ii} + E_{-i,-i}) && \text{for } i \in \{\pm 1, \dots, \pm n\} \\
E_{i,-i} &= \frac{1}{2}\mathsf{E}_{i,-i} + \delta_{i>0}E_{i,-i} && \text{for } i \in \{\pm 1, \dots, \pm n\} \\
E_{ij} &= \begin{cases} 0 + E_{ij} & \text{if } 1 \leq |i| < |j| \leq n \\ \mathsf{E}_{ij} + (-1)^{p(i)(p(j)+1)}E_{-j,-i} & \text{if } 1 \leq |j| \leq |i| \leq n \end{cases}
\end{aligned}$$

So we have that every element in $\mathfrak{gl}_{n|n}$ can be written as a sum of elements in $\mathfrak{p}_n$ and $\mathfrak{b}_n$. Therefore $\mathfrak{gl}_{n|n} = \mathfrak{p}_n + \mathfrak{b}_n$. In fact, $\mathfrak{gl}_{n|n} = \mathfrak{p}_n \oplus \mathfrak{b}_n$ as $\mathfrak{p}_n \cap \mathfrak{b}_n = \{0\}$ due to the elementary matrices being linear independent in $\mathfrak{gl}_{n|n}$. Thus, $\mathfrak{p}_n$ and $\mathfrak{b}_n$ are transversal subspaces of $\mathfrak{gl}_{n|n}$. □

Remark 4.2.2. A similar Manin supertriple is given in [40].

To use equation (3.5) to determine the cobracket on $\mathfrak{p}_n$ explictly, we need to first extend the bilinear map B. We extend the form B to a non-degenerate pairing $\mathsf{B}^{\otimes 2}$ on $\mathfrak{gl}_{n|n} \otimes_{\mathbb{C}} \mathfrak{gl}_{n|n}$ by setting

$$\mathsf{B}^{\otimes 2}(X_1 \otimes X_2, Y_1 \otimes Y_2) = (-1)^{p(X_2)p(Y_1)}\mathsf{B}(X_1, Y_1)\mathsf{B}(X_2, Y_2) \tag{4.4}$$

for all homogeneous $X_1, X_2, Y_1, Y_2 \in \mathfrak{p}_n$. The sign $(-1)^{p(X_2)p(Y_1)}$ is necessary for the form to be ad-invariant.

Using the identity (3.5) as a result of the Manin supertriple $(\mathfrak{gl}_{n|n}, \mathfrak{p}_n . \mathfrak{p}_n)$, we can now compute the supercobracket δ for $\mathfrak{p}_n$. The formula for δ (assuming, without loss of generality, that $|j| \leq |i|$) is as follows:

$$\begin{aligned}\delta(\mathsf{E}_{ij}) = & \sum_{\substack{k=-n \\ |j|<|k|<|i|}}^{n} (-1)^{p(k)+1}\big(\mathsf{E}_{ik} \otimes \mathsf{E}_{kj} - (-1)^{(p(i)+p(k))(p(j)+p(k))}\mathsf{E}_{kj} \otimes \mathsf{E}_{ik}\big) \\ & -\frac{1}{2}((-1)^{p(i)}\mathsf{E}_{ii} - (-1)^{p(j)}\mathsf{E}_{jj}) \otimes \mathsf{E}_{ij} + \frac{1}{2}\mathsf{E}_{ij} \otimes ((-1)^{p(i)}\mathsf{E}_{ii} - (-1)^{p(j)}\mathsf{E}_{jj}) \\ & -\frac{\delta(i<0)}{2}\big(\mathsf{E}_{i,-i} \otimes \mathsf{E}_{-i,j} - (-1)^{p(j)}\mathsf{E}_{-i,j} \otimes \mathsf{E}_{i,-i}\big) \\ & +\frac{\delta(j>0)}{2}\big((-1)^{p(i)}\mathsf{E}_{-j,j} \otimes \mathsf{E}_{i,-j} + \mathsf{E}_{i,-j} \otimes \mathsf{E}_{-j,j}\big)\end{aligned} \tag{4.5}$$

Now we have a Lie bisuperalgebra structure on $\mathfrak{p}_n$. Following the ideas in [44], to obtain a quantized enveloping superalgebra of type P, we will effectively quantized the Lie bisuperalgebra structure on $\mathfrak{p}_n$. To accomplish this, we will need a particular element s that satisfies the classical Yang-Baxter equation:

$$[\mathsf{s}_{12}, \mathsf{s}_{13}] + [\mathsf{s}_{12}, \mathsf{s}_{23}] + [\mathsf{s}_{13}, \mathsf{s}_{23}] = 0.$$

We can construct such an element due to the following lemma from [2].

Lemma 4.2.3. *Let $\mathfrak{p}$ be a finite dimensional Lie superalgebra and suppose that $(\mathfrak{a}, \mathfrak{a}_1, \mathfrak{a}_2)$ is a Manin triple with respect to a certain supersymmetric, invariant, bilinear form $\mathsf{B}(\cdot,\cdot)$. Let $\{X_i\}_{i\in I}, \{X_i'\}_{i\in I}$ be dual bases of $\mathfrak{a}_1$ and $\mathfrak{a}_2$, respectively, in the sense that $\mathsf{B}(X_i', X_j) = \delta_{ij}$. Set $s = \sum_{i\in I} X_i \otimes X_i'$. Then s is a solution of the classical Yang-Baxter equation.*

Let

$$\mathsf{s} = \sum_{1\leq|j|<|i|\leq n} (-1)^{p(j)}\mathsf{E}_{ij} \otimes E_{ji} + \frac{1}{2}\sum_{1\leq i\leq n} \mathsf{E}_{ii} \otimes (E_{ii} + E_{-i,-i}) + \frac{1}{2}\sum_{1\leq i\leq n} \mathsf{E}_{-i,i} \otimes E_{i,-i} \tag{4.6}$$

This is obtained from the previous lemma, using the Manin supertriple from Proposition 4.2.1.

Remark 4.2.4. The fake Casimir used in [3] is also defined using the sum of tensor product of basis vectors in $\mathfrak{p}_n$ and their duals in $\mathfrak{p}_n^\perp$, but the fake Casimir differs from the element s.

Using Lemma 4.2.3, we obtain the following result:

Proposition 4.2.5. s *is a solution of the classical Yang-Baxter equation:* $[\mathsf{s}_{12}, \mathsf{s}_{13}] + [\mathsf{s}_{12}, \mathsf{s}_{23}] + [\mathsf{s}_{13}, \mathsf{s}_{23}] = 0$.

Something to note is that now we can explicitly write the cobracket on $\mathfrak{p}_n$ in a more compact form, in terms of s.

Lemma 4.2.6. *The super cobracket can also be expressed as*

$$\delta(X) = [X \otimes 1 + 1 \otimes X, \mathsf{s}]. \tag{4.7}$$

Proof. First note that, since s is even, $[X \otimes 1 + 1 \otimes X, \mathsf{s}] = -[\mathsf{s}, X \otimes 1 + 1 \otimes X]$. We have to check that $\mathsf{B}^{\otimes 2}(\delta(X), Y_1 \otimes Y_2) = \mathsf{B}^{\otimes 2}([X \otimes 1 + 1 \otimes X, \mathsf{s}], Y_1 \otimes Y_2)$ for all $Y_1, Y_2 \in \mathfrak{b}_n$ and all $X \in \mathfrak{p}_n$.

Since it is not obvious that $[X \otimes 1 + 1 \otimes X, \mathsf{s}] \in \mathfrak{p}_n \otimes \mathfrak{p}_n$, we first verify that $\mathsf{B}^{\otimes 2}([X \otimes 1 + 1 \otimes X, \mathsf{s}], Y_1 \otimes Y_2) = 0$ if $(Y_1, Y_2) \in \mathfrak{p}_n \oplus \mathfrak{p}_n, \mathfrak{p}_n \oplus \mathfrak{b}_n$ or $\mathfrak{b}_n \oplus \mathfrak{b}_n$. Since $\mathsf{B}(\mathfrak{b}_n, \mathfrak{b}_n) = 0 = \mathsf{B}(\mathfrak{p}_n, \mathfrak{p}_n)$, this is clear except perhaps if $Y_1 \in \mathfrak{b}_n$ and $Y_2 \in \mathfrak{p}_n$. Let us assume that Y_1 and Y_2 belong to the natural bases of $\mathfrak{b}_n$ and $\mathfrak{p}_n$ given above. Let $Y_1^\vee \in \mathfrak{p}_n$ and $Y_2^\vee \in \mathfrak{b}_n$ be the elements in the same bases (up to a scalar multiple) and dual to Y_1 and Y_2 (so the scalar in question is such that $\mathsf{B}(Y_1^\vee, Y_1) = 1 = \mathsf{B}(Y_2, Y_2^\vee)$.) Using that $(\cdot,\cdot)$ is supersymmetric and invariant, we obtain:

$$\begin{aligned}
\mathsf{B}^{\otimes 2}([X \otimes 1 + 1 \otimes X, \mathsf{s}]\ ,\ Y_1 \otimes Y_2) &= -\mathsf{B}^{\otimes 2}([\mathsf{s}, X \otimes 1 + 1 \otimes X], Y_1 \otimes Y_2) \\
&= \mathsf{B}^{\otimes 2}(\mathsf{s}, [X \otimes 1 + 1 \otimes X, Y_1 \otimes Y_2]) \\
&= -\mathsf{B}^{\otimes 2}(\mathsf{s}, [X, Y_1] \otimes Y_2 + (-1)^{p(X)p(Y_1)} Y_1 \otimes [X, Y_2]) \\
&= -\mathsf{B}^{\otimes 2}(Y_2 \otimes Y_2^\vee, [X, Y_1] \otimes Y_2) - (-1)^{p(X)p(Y_1)} \mathsf{B}^{\otimes 2}(Y_1^\vee \otimes Y_1, Y_1 \otimes [X, Y_2])
\end{aligned}$$

$$= -(-1)^{p(Y_2)(p(X)+p(Y_1))}\mathsf{B}(Y_2,[X,Y_1]) - (-1)^{p(X)p(Y_1)}\mathsf{B}(Y_1,[X,Y_2])$$
$$= -\mathsf{B}([X,Y_1],Y_2) - (-1)^{p(X)p(Y_1)}\mathsf{B}(Y_1,[X,Y_2])$$
$$= 0.$$

To verify (4.7), we pick $Y_1, Y_2 \in \mathfrak{b}_n$ and $X \in \mathfrak{p}_n$. We have:

$$\begin{aligned}
\mathsf{B}^{\otimes 2}([X\otimes 1 + 1\otimes X, \mathsf{s}], Y_1\otimes Y_2) &= -\mathsf{B}^{\otimes 2}([\mathsf{s}, X\otimes 1 + 1\otimes X], Y_1\otimes Y_2)\\
&= -(-1)^{p(X)p(Y_1)}\mathsf{B}^{\otimes 2}(Y_1^{\vee}\otimes Y_1, Y_1\otimes [X,Y_2])\\
&= -(-1)^{p(X)p(Y_1)}\mathsf{B}(Y_1,[X,Y_2])\\
&= -(-1)^{p(Y_1)p(Y_2)}\mathsf{B}([X,Y_2],Y_1)\\
&= -(-1)^{p(Y_1)p(Y_2)}\mathsf{B}(X,[Y_2,Y_1])\\
&= \mathsf{B}(X,[Y_1,Y_2])\\
&= \mathsf{B}^{\otimes 2}(\delta(X), Y_1\otimes Y_2).
\end{aligned}$$

□

4.3 Root System of $\mathfrak{p}_n$

Let $I \coloneqq \{1,\dots,n-1\}$ and $J \coloneqq \{1,\dots,n\}$. Let $\mathfrak{h}$ be the Cartan subsuperalgebra of $\mathfrak{p}_n$ with basis $\{k_1,\dots,k_n\}$, where $k_i \coloneqq \mathsf{E}_{ii}$ for $1 \le i \le n$. Let $\{\epsilon_1,\dots,\epsilon_n\}$ be the basis of $\mathfrak{h}^*$ dual to $\{k_1,\dots,k_n\}$.

The root system Δ of $\mathfrak{p}_n$ consists of the roots

$$\epsilon_i - \epsilon_j \text{ for } i \neq j$$
$$\epsilon_i + \epsilon_j \text{ for } i < j$$
$$-\epsilon_i - \epsilon_j \text{ for } i \le j.$$

Let $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $\beta_i = 2\epsilon_i$, and $\gamma_i = \epsilon_i + \epsilon_{i+1}$. Then, for $i \in I$ and $j \in J$, let

$$e_i \coloneqq \mathsf{E}_{-i-1,-i} \qquad e_{\bar{i}} \coloneqq \mathsf{E}_{i+1,-i} \qquad F_{\bar{j}} \coloneqq \mathsf{E}_{-j,j}$$

$$f_i \coloneqq \mathsf{E}_{i+1,i} \qquad f_i \coloneqq \mathsf{E}_{i+1,i}$$

The following shows what the relations of $\mathfrak{p}_n$ are in terms of the generating elements defined above.

Proposition 4.3.1 ([21]). *The Lie superalgebra $\mathfrak{p}_n$ is generated by the elements e_i, $e_{\bar{i}}$, f_i, $f_{\bar{i}}$ ($i \in I$), $\mathfrak{h}$ and $F_{\bar{j}}$ ($j \in J$) subject to following defining relations (for $h \in \mathfrak{h}$):*

$$[\mathfrak{h},\mathfrak{h}] = 0 \qquad [f_{\overline{i+1}}, f_i] = [e_{i+1}, f_{\bar{i}}]$$

$$[h, e_i] = \alpha_i(h)e_i \qquad [f_{\bar{i}}, f_i] = F_{\bar{i}}$$

$$[h, f_i] = -\alpha_i(h)f_i \qquad [e_i, f_{\bar{i}}] = F_{\overline{i+1}}$$

$$[h, e_{\bar{i}}] = \gamma_i(h)e_{\bar{i}} \qquad [e_{\bar{i}}, e_{\bar{j}}] = [f_{\bar{i}}, f_{\bar{j}}] = 0 \textit{ for } i,j \in I$$

$$[h, f_{\bar{i}}] = -\gamma_i(h)f_{\bar{i}} \qquad [f_i, e_{\bar{j}}] = 0 \textit{ if } i \neq j+1$$

$$[h, F_{\bar{i}}] = -\beta_i(h)F_{\bar{i}} \qquad [e_{\bar{i}}, e_j] = 0 \textit{ if } i \neq j+1$$

$$[e_i, e_j] = [f_i, f_j] = 0 \textit{ for } |i-j| \neq 1 \qquad [e_i, f_{\bar{j}}] = 0 \textit{ if } i \neq j, j+1$$

$$[e_i, f_j] = -\delta_{ij}(k_i - k_{i+1}) \qquad [f_{\bar{i}}, f_j] = 0 \textit{ if } i \neq j, j+1$$

$$[e_{\bar{i}}, f_{\bar{i}}] = -(k_i - k_{i+1}) \qquad [F_{\bar{j}}, e_i] = -\beta_i(k_j)f_{\bar{i}}$$

$$[f_{\bar{i}}, e_{\bar{j}}] = 0 \textit{ if } |i-j| > 1 \qquad [F_{\bar{j}}, f_i] = \beta_{i+1}(k_j)f_{\bar{i}}$$

$$[f_{\overline{i+1}}, e_{\bar{i}}] = [e_{i+1}, e_i] \qquad [e_i, [e_i, e_{i\pm1}]] = 0$$

$$[f_{\bar{i}}, e_{\overline{i+1}}] = [f_{i+1}, f_i] \qquad [f_i, [f_i, f_{i\pm1}]] = 0$$

$$[e_{\overline{i+1}}, e_i] = [f_{i+1}, e_{\bar{i}}] \qquad [e_{\overline{i+1}}, [e_{i+1}, e_i]] = e_{\bar{i}}$$

Note that the relations of $\mathfrak{p}_n$ in Proposition 4.3.1 gives that the elements e_i, f_i, $e_{\bar{i}}$, $f_{\bar{i}}$, and $F_{\bar{i}}$ are all root vectors of $\mathfrak{p}_n$ under the adjoint representation. More specifically, the root spaces of α_i, $-\alpha_i$, γ, $-\gamma_i$, and $-\beta_i$ are spanned, respectively, by $e_i, f_i, e_{\bar{i}}, f_{\bar{i}}$, and $F_{\bar{i}}$.

Some additional relations of $\mathfrak{p}_n$ are shown in the following lemma:

Lemma 4.3.2. *The following relations hold in $\mathfrak{p}_n$:*

(a) $[F_{\bar{j}}, e_{\bar{i}}] = \begin{cases} 2f_i & \text{if } j = i \\ 2e_i & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$

(b) $[F_{\bar{j}}, f_{\bar{i}}] = 0$

(c) $[e_i, [e_i, e_{\overline{i\pm 1}}]] = 0$

(d) $[f_i, [f_i, f_{\overline{i\pm 1}}]] = 0$

(e) $[F_{\bar{i}}, F_{\bar{j}}] = 0$ *for* $i, j \in J$

Proof. We will prove (a) and (c). The remaining parts can be deduced similarly.

First, we prove (a) for $j = n$. For every i we have

$$[F_{\bar{n}}, e_{\bar{i}}] = [[e_{n-1}, f_{\overline{n-1}}], e_{\bar{i}}] = [e_{\bar{i}}, [e_{n-1}, f_{\overline{n-1}}]] = [e_{n-1}, [f_{\overline{n-1}}, e_{\bar{i}}]] + [f_{\overline{n-1}}, [e_{\bar{i}}, e_{n-1}]] = [e_{n-1}, [f_{\overline{n-1}}, e_{\bar{i}}]].$$

If $i = n-2$, then we have that $[e_{n-1}, [f_{\overline{n-1}}, e_{\overline{n-2}}]] = [e_{n-1}, [e_{n-1}, e_{n-2}]] = 0$. If $i = n-1$, then we have that

$$[e_{n-1}, [f_{\overline{n-1}}, e_{\overline{n-1}}]] = [e_{n-1}, -k_{n-1} + k_n] = -[e_{n-1}, k_{n-1}] + [e_{n-1}, k_n] = 2e_{n-1}.$$

Otherwise, we have that $[F_{\bar{n}}, e_{\bar{i}}] = 0$.

Next we prove (a) for $j < n$. Using the relations in Proposition **??**, we have that:

$$[F_{\bar{j}}, e_{\bar{i}}] = [[f_{\bar{j}}, f_j], e_{\bar{i}}]$$

$$= [e_{\bar{i}}, [f_{\bar{j}}, f_j]]$$

$$= [f_{\bar{j}}, [f_j, e_{\bar{i}}]] - [f_j, [e_{\bar{i}}, f_{\bar{j}}]].$$

Note that $[F_{\bar{j}}, e_{\bar{i}}] = 0$ from above, unless $|i - j| \leq 2$. So, we need to check the three subcases $i - j = 0, 1, -1$.

If $j = i$, then

$$[F_{\bar{i}}, e_{\bar{i}}] = [f_{\bar{i}}, [f_i, e_{\bar{i}}]] - [f_i, [e_{\bar{i}}, f_{\bar{i}}]]$$

$$= -[f_i, -k_i + k_{i+1}]$$

$$= 2f_i.$$

If $i = j + 1$, then

$$[F_{\bar{j}}, e_{\overline{j+1}}] = [f_{\bar{j}}, [f_j, e_{\overline{j+1}}]] - [f_j, [e_{\overline{j+1}}, f_{\bar{j}}]]$$

$$= -[f_j, [f_{j+1}, f_j]]$$

$$= [f_j, [f_j, f_{j+1}]] + [f_j, [f_{j+1}, f_j]]$$

$$= 0.$$

If $j = i + 1$, then

$$[F_{\overline{i+1}}, e_{\bar{i}}] = [f_{\overline{i+1}}, [f_{i+1}, e_{\bar{i}}]] - [f_{i+1}, [e_{\bar{i}}, f_{\overline{i+1}}]]$$

$$= [f_{\overline{i+1}}, [e_{\overline{i+1}}, e_i]] - [f_{i+1}, [e_{i+1}, e_i]]$$

$$= [e_{\overline{i+1}}, [e_i, f_{\overline{i+1}}]] - [e_i, [f_{\overline{i+1}}, e_{\overline{i+1}}]] + [e_{i+1}, [e_i, f_{i+1}]] + [e_i, [f_{i+1}, e_{i+1}]]$$

$$= -[e_i, -k_{i+1} + k_{i+2}] - [e_i, -k_{i+1} + k_{i+2}]$$

$$= 2e_i.$$

Now, we prove (c). Note that $[e_i, e_{\overline{i-1}}] = 0$ for all $2 \leq i \leq n$, so $[e_i, [e_i, e_{\overline{i-1}}]] = 0$. Also,

$$[e_i, [e_i, e_{\overline{i+1}}]] = [e_i, [e_{\bar{i}}, f_{i+1}]]$$

$$= [e_{\bar{i}}, [f_{i+1}, e_i]] + [f_{i+1}, [e_i, e_{\bar{i}}]]$$
$$= 0.$$

□

Using the root space decomposition $\mathfrak{p}_n = \mathfrak{h} \oplus \left(\bigoplus_{\mu \in \Delta} (\mathfrak{p}_n)_\mu \right)$ we define the triangular decomposition

$$\mathfrak{p}_n = \mathfrak{p}_n^- \oplus \mathfrak{h} \oplus \mathfrak{p}_n^+,$$

where $\mathfrak{p}_n^-$ is spanned by $\{f_i, f_{\bar{i}}, F_{\bar{j}} \mid i \in I, j \in J\}$ and $\mathfrak{p}_n^+$ is spanned by $\{e_i, e_{\bar{i}} \mid i \in I\}$. Equivalently, $\Delta = \Delta_+ \sqcup \Delta_-$, where

$$\Delta_+ = \Delta(\mathfrak{p}_n^+) = \{\alpha_i, \gamma_i \mid i \in I\},\ \Delta_- = \Delta(\mathfrak{p}_n^-) = \{-\alpha_i, -\gamma_i, -\beta_j \mid i \in I, j \in J\}.$$

The cone of positive roots will be denoted by $\mathcal{Q}_+ := \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0}\alpha_i + \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0}\gamma_i$ and $\mathcal{Q}_- := -\sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0}\alpha_i - \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0}\gamma_i - \sum_{i=1}^{n} \mathbb{Z}_{\geq 0}\beta_i$ denotes the cone of negative roots. By $\mathcal{Q} = \mathcal{Q}_+ + \mathcal{Q}_-$ we denote the cone of roots.

We will also denote $\mathcal{P} := \bigoplus_{i=1}^{n} \mathbb{Z}\epsilon_i$ to be the weight lattice of $\mathfrak{p}_n$, and denote $\mathcal{P}^\vee := \bigoplus_{i=1}^{n} \mathbb{Z}k_i$ to be the coweight lattice.

CHAPTER 5

Quantized Enveloping Superalgebra of Type P

In this chapter, we define the quantized enveloping superalgebra of type P following the approach used in [29] and [44]. We will also prove that it is indeed a quantization of a Lie bisuperalgebra structure on $\mathfrak{p}_n$ discussed in the previous chapter. Some of its basic properties and relations will be discussed, and a PBW-type basis for the superalgebra will be given. For this and subsequent chapters, we will denote by $\mathbb{C}_q$ the field $\mathbb{C}(q)$ of rational functions in the variable q, and we will set $\mathbb{C}_q(n|n) = \mathbb{C}_q \otimes_{\mathbb{C}} \mathbb{C}(n|n)$

5.1 Quantized enveloping superalgebra

Definition 5.1.1. Let $S \in \mathrm{End}_{\mathbb{C}_q}(\mathbb{C}_q(n|n)^{\otimes 2})$ be given by the formula:

$$S = \sum_{i,j=-n}^{n} s_{ij} \otimes E_{ij} \tag{5.1}$$

where

$$s_{ij} = (q - q^{-1})(-1)^{p(i)} \mathsf{E}_{ji} \quad \text{if } |i| < |j|$$
$$s_{i,-i} = (q - q^{-1})\delta_{i>0} E_{-i,i}$$
$$s_{ii} = (q - 1)(E_{ii} - q^{-1} E_{-i,-i}) + 1$$

Remark 5.1.1. If we define S instead as an element of $\mathrm{End}_{\mathbb{C}[[\hbar]]}(\mathbb{C}_\hbar(n|n)^{\otimes 2})$ by the same formula as in definition 5.1.1 but with q, q^{-1} replaced by $e^{\hbar/2}, e^{-\hbar/2}$ and $\mathbb{C}_q(n|n)^{\otimes 2}$ replaced by $\mathbb{C}_\hbar(n|n)^{\otimes 2} = \mathbb{C}(n|n)^{\otimes 2}[[\hbar]]$, then $S = 1 + \hbar \mathsf{s} + o(\hbar)$. This allows us to think of S as a quantized version of s.

Theorem 5.1.2. *S is a solution of the quantum Yang-Baxter equation:*

$$S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}.$$

Proof. We will give a sketch of the proof here. A more thorough proof is given in [1]. Set $f(q) = S_{12}S_{13}S_{23} - S_{23}S_{13}S_{12}$. Then, consider $f(q)$ as a Laurent polynomial $\sum_{i=-3}^{3} f_i q^i$ with coefficients f_i in $\mathrm{End}_{\mathbb{C}}\left(\mathbb{C}_{n|n}^{\otimes 3}\right)$. Then one shows the eight relations $f(a) = 0,\ f'(b) = 0,\ f''(c) = 0$ for $a, b, c = \pm 1$ and $b = \pm\sqrt{-1}$. We can then deduce that $f(q)$ is a scalar multiple of $(q - q^{-1})^3$ and we show that the coefficient of q^3 in $f(q)$ is zero. $\square$

Using S, we can now define the main object of interest:

Definition 5.1.2. The *quantized enveloping superalgebra of* $\mathfrak{p}_n$ is the $\mathbb{Z}_2$-graded $\mathbb{C}_q-$algebra $\mathfrak{U}_q\mathfrak{p}_n$ generated by elements t_{ij}, t_{ii}^{-1} with $1 \le |i| \le |j| \le n$ and $i, j \in \{\pm 1, \ldots, \pm n\}$ which satisfy the following relations:

$$t_{ii} = t_{-i,-i},\ t_{-i,i} = 0 \text{ if } i > 0,\ t_{ij} = 0 \text{ if } |i| > |j|; \tag{5.2}$$

$$T_{12}T_{13}S_{23} = S_{23}T_{13}T_{12} \tag{5.3}$$

where $T = \sum_{|i| \le |j|} t_{ij} \otimes_{\mathbb{C}} E_{ij}$ and the last equality holds in $\mathfrak{U}_q\mathfrak{p}_n \otimes_{\mathbb{C}(q)} \mathrm{End}_{\mathbb{C}(q)}(\mathbb{C}_q(n|n))^{\otimes 2}$. The $\mathbb{Z}_2$-degree of t_{ij} is $p(i) + p(j)$.

$\mathfrak{U}_q\mathfrak{p}_n$ is a Hopf algebra with antipode given by $T \mapsto T^{-1}$ and with coproduct given by

$$\Delta(t_{ij}) = \sum_{k=-n}^{n} (-1)^{(p(i)+p(k))(p(k)+p(j))} t_{ik} \otimes t_{kj}.$$

Remark 5.1.3. Since, by Theorem 5.1.2, S is a solution to the quantum Yang-Baxter equation, relation (5.3) gives that $\mathbb{C}_q(n|n)$ is a representation of $\mathfrak{U}_q\mathfrak{p}_n$ via the mapping $T \mapsto S$. More specifically, $t_{ij} \mapsto s_{ij}$, where $S = \sum_{i,j=-n}^{n} s_{ij} \otimes E_{ij}$ as in Definition 5.1.1.

By comparing coefficients of $E_{ij} \otimes E_{k\ell}$ on both sides on relation (5.3), the defining relation (5.3) in Definition 5.1.2 can be written explicitly as

$$
\begin{aligned}
&(-1)^{(p(i)+p(j))(p(k)+p(l))} t_{ij}t_{kl} - t_{kl}t_{ij} + \theta(i,j,k)\big(\delta_{|j|<|l|} - \delta_{|k|<|i|}\big)\epsilon t_{il}t_{kj} \\
&\quad + (-1)^{(p(i)+p(j))(p(k)+p(l))}\big(\delta_{j>0}(q-1) + \delta_{j<0}(q^{-1}-1)\big)\big(\delta_{jl} + \delta_{j,-l}\big)t_{ij}t_{kl} \\
&\quad - \big(\delta_{i>0}(q-1) + \delta_{i<0}(q^{-1}-1)\big)\big(\delta_{ik} + \delta_{i,-k}\big)t_{kl}t_{ij} + \theta(i,j,k)\delta_{j>0}\delta_{j,-l}\epsilon t_{i,-j}t_{k,-l} \\
&\quad - (-1)^{p(j)}\delta_{i<0}\delta_{i,-k}\epsilon t_{-k,l}t_{-i,j} + (-1)^{p(j)(p(i)+1)}\epsilon \sum_{-n\le a\le n} \big((-1)^{p(i)p(a)}\theta(i,j,k)\delta_{j,-l}\delta_{|a|<|l|}t_{i,-a}t_{ka} \\
&\quad + (-1)^{p(-j)p(a)}\delta_{i,-k}\delta_{|k|<|a|}t_{al}t_{-a,j}\big) = 0
\end{aligned}
\tag{5.4}
$$

where

$$\theta(i,j,k) = \mathrm{sgn}(\mathrm{sgn}(i) + \mathrm{sgn}(j) + \mathrm{sgn}(k)) \text{ and } \epsilon = q - q^{-1}.$$

Remark 5.1.4. If t_{ij} is odd, then $t_{ij}^2 = 0$. This follows for example after taking $i = k$ and $j = l$ in (5.4).

We have that a PBW-type theorem holds for $\mathfrak{U}_q\mathfrak{p}_n$. Let $\prec$ be a total order on the set of generators t_{ij}, $1 \le |i| \le |j| \le n$, of $\mathfrak{U}_q\mathfrak{p}_n$, such that $t_{ij} \prec t_{kl}$ if

(i) $|i| > |k|$, or

(ii) $|i| = |k|$ and $|j| > |l|$, or

(iii) $i = k$ and $j = -l > 0$, or

(iv) $i = -k > 0$ and $|j| = |l|$.

Note that this order leads to a total lexicographic order on the set of words formed by the generators t_{ij}. Namely, if $A = A_1 \cdots A_r$ and $B = B_1 \cdots B_s$ are two such words in the sense that each A_k for $1 \le k \le r$ and each B_l for $1 \le l \le s$ is equal to some generator t_{ij}, then $A \prec B$ if $r < s$ or if $r = s$ and there is a p such that $A_k = B_k$ for $1 \le k \le p-1$ and $A_p \prec B_p$. Note that, in this order, the generators t_{ij} with $i = j$ or $i = -j$ are not grouped together. We call a generator of the from t_{ii} *diagonal*.

Definition 5.1.3. A word $A_1^{k_1}...A_r^{k_r}$ in the generators t_{ij} is called a *reduced monomial* if $A_1 \prec \cdots \prec A_r$, and $k_i \in \mathbb{Z}_{>0}$ if A_i is not diagonal, $k_i \in \mathbb{Z} \setminus \{0\}$ if A_i is diagonal, and $k_i = 1$ if A_i is odd.

Theorem 5.1.5. *The reduced monomials form a basis of* $\mathfrak{U}_q\mathfrak{p}_n$ *over* $\mathbb{C}_q$.

Proof. This proof is also found in [1]. We first show that the set of reduced monomials spans $\mathfrak{U}_q\mathfrak{p}_n$. Note that it is enough to show that all quadratic monomials are in the span of this set. Let $t_{ij}t_{kl}$ be a quadratic monomial which is not reduced. We have that either $t_{kl} \neq t_{ij}$, or $i = k$, $j = l$ and t_{ij} is odd. In the latter case, as explained in Remark 5.1.4, t_{ij}^2=0. In the former case, we proceed with a case-by-case reasoning considering seven mutually exclusive subcases:

(a) $|i| < |k|$ and $|j| \neq |l|$.

(b) $|i| < |k|$ and $j = l$.

(c) $|i| < |k|$ and $j = -l$.

(d) $|i| = |k|$ and $|j| < |l|$.

(e) $i = k$ and $j = -l < 0$.

(f) $i = -k < 0$ and $j = l$.

(g) $i = -k < 0$ and $j = -l$.

Let's consider in some details subcase (c). The remaining subcases are handled in a similar manner. In subcase (c), (5.4) simplifies to:

$$\begin{aligned}(-1)^{(p(i)+p(j))(p(k)+p(-j))}\left(\delta_{j>0}q+\delta_{j<0}q^{-1}\right)t_{ij}t_{k,-j} - t_{k,-j}t_{ij} + \theta(i,j,k)\delta_{j>0}\epsilon t_{i,-j}t_{kj} \\ + (-1)^{p(j)(p(i)+1)}\epsilon \sum_{-n\leq a\leq n}(-1)^{p(i)p(a)}\theta(i,j,k)\delta_{|a|<|j|}t_{i,-a}t_{ka} = 0\end{aligned} \tag{5.5}$$

Let us assume that $|l| = |j| = 1$. Then the previous equation reduces to

$$(-1)^{(p(i)+p(j))(p(k)+p(-j))}\left(\delta_{j>0}q+\delta_{j<0}q^{-1}\right)t_{ij}t_{k,-j} + \theta(i,j,k)\delta_{j>0}\epsilon t_{i,-j}t_{kj} = t_{k,-j}t_{ij}$$

Replacing j by $-j$ leads to the equation

$$(-1)^{(p(i)+p(-j))(p(k)+p(j))}\left(\delta_{j<0}q+\delta_{j>0}q^{-1}\right)t_{i,-j}t_{kj}+\theta(i,-j,k)\delta_{j<0}\epsilon t_{ij}t_{k,-j}=t_{kj}t_{i,-j}$$

The monomials $t_{k,-j}t_{ij}$ and $t_{kj}t_{i,-j}$ are properly ordered and the previous two equations can be solved to express $t_{ij}t_{k,-j}$ and $t_{i,-j}t_{kj}$ in terms of the former.

We then proceed by descending induction on $|j|$ and show that $t_{ij}t_{k,-j}$ can be expressed as a linear combination of properly ordered monomials. The base case $|j|=1$ was completed above. We use again (5.5) and the corresponding equation obtained after switching j and $-j$. In these two equations, by induction, the monomials $t_{i,-a}t_{ka}$ with $|a|<|j|$ can be expressed as linear combinations of properly ordered monomials. Moreover, $t_{k,-j}t_{ij}$ and $t_{kj}t_{i,-j}$ are already correctly ordered. As in the case $|l|=|j|=1$, we can then solve those two equations to express $t_{ij}t_{k,-j}$ and $t_{i,-j}t_{kj}$ in terms of properly ordered monomials.

It remains to show that the reduced monomials form a linearly independent set. We follow the approach in [44]. Let $M_1,\ldots,M_r$ be pairwise distinct reduced monomials in the generators τ_{ij} such that $a_1M_1+\ldots+a_rM_r=0$ for some $a_1,\ldots,a_r\in\mathbb{C}_q$. Without loss of generality, we can assume that $a_i\in\mathcal{A}$. It is sufficient to prove that $a_1,...,a_r\in\mathcal{A}$ implies $a_1,...,a_r\in(q-1)\mathcal{A}$.

Recall that there is a surjective homomorphism $\theta:\mathfrak{U}_{\mathcal{A}}\mathfrak{p}_n\to\mathfrak{U}\mathfrak{p}_n$ More precisely, θ is the composite of ψ^{-1} from Theorem 5.2.1 and the projection $\mathfrak{U}_{\mathcal{A}}\mathfrak{p}_n\to\mathfrak{U}_{\mathcal{A}}\mathfrak{p}_n/(q-1)\mathfrak{U}_{\mathcal{A}}\mathfrak{p}_n$ from Theorem 5.2.1. Let $\overline{M}_i=\theta(M_i)$ and denote by $\bar{a}_i$ the image of a_i in $\mathcal{A}/(q-1)\mathcal{A}$. Since $M_1,\ldots,M_r$ are pairwise distinct reduced monomials, $\overline{M}_1,\ldots,\overline{M}_r$ are pairwise distinct monomials in $\mathfrak{U}\mathfrak{p}_n$. Then using that

$$\bar{a}_1\overline{M}_1+\ldots+\bar{a}_r\overline{M}_r=\theta(a_1M_1+\ldots+a_rM_r)=0$$

and the (classical) PBW Theorem for $\mathfrak{U}\mathfrak{p}_n$, we obtain $\bar{a}_1=\ldots=\bar{a}_r=0$. Hence $a_1,\ldots,a_r\in(q-1)\mathcal{A}$ as needed. $\square$

5.2 Limit when $q \to 1$ and quantization

We want to explain how we can view $\mathfrak{U}\mathfrak{p}_n$ as the limit when $q \to 1$ of $\mathfrak{U}_q\mathfrak{p}_n$. We also want to explain how the co-Poisson Hopf algebra structure on $\mathfrak{U}\mathfrak{p}_n$ that is inherited from the cobracket δ on $\mathfrak{p}_n$ can be recovered from the coproduct on $\mathfrak{U}_q\mathfrak{p}_n$. This will be important to know when relating the representation theory of $\mathfrak{U}_q\mathfrak{p}_n$ to that of $\mathfrak{U}\mathfrak{p}_n$.

Set $\tau_{ij} = \frac{t_{ij}}{q-q^{-1}}$ if $i \neq j$ and set $\tau_{ii} = \frac{t_{ii}-1}{q-1}$. Let $\mathcal{A}$ be the localization of $\mathbb{C}[q, q^{-1}]$ at the ideal generated by $q-1$. Let $\mathfrak{U}_{\mathcal{A}}\mathfrak{p}_n$ be the $\mathcal{A}$-subalgebra of $\mathfrak{U}_q\mathfrak{p}_n$ generated by τ_{ij} when $1 < |i| < |j| < n$.

Theorem 5.2.1. *The map $\psi : \mathfrak{U}\mathfrak{p}_n \longrightarrow \mathfrak{U}_{\mathcal{A}}\mathfrak{p}_n/(q-1)\mathfrak{U}_{\mathcal{A}}\mathfrak{p}_n$ given by $\psi(\mathsf{E}_{ji}) = (-1)^{p(j)}\overline{\tau}_{ij}$ for $|i| < |j|$, $1 \leq i = j \leq n$, and $\psi(\mathsf{E}_{-i,i}) = -2\overline{\tau}_{i,-i}$ for $1 \leq i \leq n$, is an associative $\mathbb{C}$-superalgebra isomorphism.*

Proof. In order to check that $\psi([\mathsf{E}_{ji}, \mathsf{E}_{kl}]) = [\psi(\mathsf{E}_{ji}), \psi(\mathsf{E}_{kl})]$, we proceed as follows. We apply ψ on both sides of (4.2). To show that the resulting right hand side coincides with $[\psi(\mathsf{E}_{ji}), \psi(\mathsf{E}_{kl})]$, we use (5.4) and pass to the quotient $\mathfrak{U}_{\mathcal{A}}\mathfrak{p}_n/(q-1)\mathfrak{U}_{\mathcal{A}}\mathfrak{p}_n$. This is done via a long case-by-case verification for i, j, k, l in each of the following 26 cases:

1. $|i| = |j| < |k| < |l|$
2. $|k| < |i| = |j| < |l|$
3. $|k| < |l| < |i| = |j|$
4. $|k| = |l| < |i| < |j|$
5. $|i| < |k| = |l| < |j|$
6. $|i| < |j| < |k| = |l|$
7. $|i| = |k| < |j| < |l|$
8. $|i| = |k| < |l| < |j|$
9. $|i| < |k| < |l| = |j|$
10. $|k| < |i| < |l| = |j|$
11. $|k| < |l| = |i| < |j|$
12. $|i| < |j| = |k| < |l|$
13. $|i| < |j| < |k| < |l|$
14. $|i| < |k| < |j| < |l|$
15. $|i| < |k| < |l| < |j|$
16. $|k| < |i| < |j| < |l|$
17. $|k| < |i| < |l| < |j|$
18. $|k| < |l| < |i| < |j|$
19. $|k| = |l| < |i| = |j|$
20. $|i| = |j| < |k| = |l|$
21. $|i| = |k| < |j| = |l|$
22. $|i| = |j| = |k| < |l|$
23. $|k| < |i| = |j| = |l|$
24. $|i| = |k| = |l| < |j|$
25. $|i| < |j| = |k| = |l|$
26. $|i| = |j| = |k| = |l|$

We will show case (10), and all other cases follow similarly. This case splits into two subcases: when $j = l$ and when $j = -l$.

When $j = l$, (5.4) gives us that

$$\begin{aligned}
0 &= (-1)^{(p(i)+p(j))(p(k)+p(j))}(q-q^{-1})^2\tau_{ij}\tau_{kj} - (q-q^{-1})^2\tau_{kj}\tau_{ij} \\
&- (-1)^{(p(i)+p(j))(p(i)+p(k))+p(i)}(q-q^{-1})^3\tau_{ij}\tau_{kj} \\
&+ (-1)^{(p(i)+p(j))(p(k)+p(j))}(q^{\operatorname{sgn}(j)}-1)(q-q^{-1})^2\tau_{ij}\tau_{kj} - \delta_{j>0}(-1)^{p(i)(p(k)+1)}(q-q^{-1})^3\tau_{i,-j}\tau_{k,-j}
\end{aligned}$$

This simplifies to

$$\begin{aligned}
0 &= (-1)^{(p(i)+p(j))(p(k)+p(j))}\tau_{ij}\tau_{kj} - \tau_{kj}\tau_{ij} - (-1)^{(p(i)+p(j))(p(i)+p(k))+p(i)}(q-q^{-1})\tau_{ij}\tau_{kj} \\
&+ (-1)^{(p(i)+p(j))(p(k)+p(j))}(q^{\operatorname{sgn}(j)}-1)\tau_{ij}\tau_{kj} - \delta_{j>0}(-1)^{p(i)(p(k)+1)}(q-q^{-1})\tau_{i,-j}\tau_{k,-j}.
\end{aligned}$$

Hence, in $\mathfrak{U}_{\mathcal{A}}\mathfrak{g}/(q-1)\mathfrak{U}_{\mathcal{A}}\mathfrak{g}$ we have that:

$$0 = (-1)^{(p(i)+p(j))(p(k)+p(j))}\overline{\tau}_{ij}\overline{\tau}_{kj} - \overline{\tau}_{kj}\overline{\tau}_{ij}.$$

The above together with $[\mathsf{E}_{ji}, \mathsf{E}_{jk}] = 0$ implies that:

$$\begin{aligned}
[\psi(\mathsf{E}_{ji}), \psi(\mathsf{E}_{jk})] &= [(-1)^{p(j)}\overline{\tau}_{ij}, (-1)^{p(j)}\overline{\tau}_{kj}] \\
&= (-1)^{(p(i)+p(j))(p(k)+p(j))}\left((-1)^{(p(i)+p(j))(p(k)+p(j))}\overline{\tau}_{ij}\,\overline{\tau}_{kj} - \overline{\tau}_{kj}\,\overline{\tau}_{ij}\right) \\
&= 0 = \psi\left([\mathsf{E}_{ji}, \mathsf{E}_{jk}]\right).
\end{aligned}$$

Now, in the case that $j = -l$, (5.4) gives us that

$$\begin{aligned}
E^{ij,kl}(q) &= (-1)^{(p(i)+p(j))(p(k)+p(j)+1)}(q-q^{-1})^2\tau_{ij}\tau_{k,-j} - (q-q^{-1})^2\tau_{k,-j}\tau_{ij} \\
&- (-1)^{(p(i)+p(j))(p(i)+p(k))+p(i)}(q-q^{-1})^3\tau_{i,-j}\tau_{kj} \\
&+ (-1)^{(p(i)+p(j))(p(k)+p(j)+1)}(q^{\operatorname{sgn}(j)}-1)(q-q^{-1})^2\tau_{ij}\tau_{k,-j} \\
&- \delta_{j>0}(-1)^{(p(i)+p(j))(p(i)+p(k))}(q-q^{-1})^3\tau_{i,-j}\tau_{kj} \\
&+ (-1)^{p(j)(p(k)+1)}(q-q^{-1})^2((q-1)\tau_{i,-k}\tau_{kk} + \tau_{i,-k} + (-1)^{p(i)}(q-q^{-1})\tau_{ik}\tau_{k,-k})
\end{aligned}$$

$$+ (-1)^{p(j)(p(k)+1)+p(k)p(i)}(q-q^{-1})^3 \sum_{|k|<|a|<|j|} (-1)^{p(i)p(a)} \delta_{|a|<|j|} \tau_{i,-a}\tau_{ka}$$

This simplifies to

$$\begin{aligned}
0 &= (-1)^{(p(i)+p(j))(p(k)+p(j)+1)}\tau_{ij}\tau_{k,-j} - \tau_{k,-j}\tau_{ij} \\
&\quad - (-1)^{(p(i)+p(j))(p(i)+p(k))+p(i)}(q-q^{-1})\tau_{i,-j}\tau_{kj} + (-1)^{(p(i)+p(j))(p(k)+p(j)+1)}(q^{\mathrm{sgn}(j)}-1)\tau_{ij}\tau_{k,-j} \\
&\quad - \delta_{j>0}(-1)^{(p(i)+p(j))(p(i)+p(k))}(q-q^{-1})\tau_{i,-j}\tau_{kj} \\
&\quad + (-1)^{p(j)(p(k)+1)}((q-1)\tau_{i,-k}\tau_{kk} + \tau_{i,-k} + (-1)^{p(i)}(q-q^{-1})\tau_{i,k}\tau_{k,-k}) \\
&\quad + (-1)^{p(j)(p(k)+1)+p(k)p(i)}(q-q^{-1}) \sum_{|k|<|a|<|j|} (-1)^{p(i)p(a)}\delta_{|a|<|j|}\tau_{i,-a}\tau_{ka}.
\end{aligned}$$

Hence, in $\mathfrak{U}_{\mathcal{A}}\mathfrak{g}/(q-1)\mathfrak{U}_{\mathcal{A}}\mathfrak{g}$, we have:

$$0 = (-1)^{(p(i)+p(j))(p(k)+p(j)+1)}\overline{\tau}_{ij}\overline{\tau}_{k,-j} - \overline{\tau}_{k,-j}\overline{\tau}_{ij} + (-1)^{p(j)(p(k)+1)}\overline{\tau}_{i,-k}$$

By combining the above with $[\mathsf{E}_{ji}, \mathsf{E}_{-j,k}] = (-1)^{p(i)(p(j)+1)+1}\mathsf{E}_{-i,k}$, we obtain:

$$\begin{aligned}
[\psi(\mathsf{E}_{ji}), \psi(\mathsf{E}_{-j,k})] &= [(-1)^{p(j)}\overline{\tau}_{ij}, (-1)^{p(j)+1}\overline{\tau}_{k,-j}] \\
&= (-1)^{(p(i)+p(j))(p(k)+p(j)+1)+1}\left((-1)^{(p(i)+p(j))(p(k)+p(j)+1)}\overline{\tau}_{ij}\,\overline{\tau}_{k,-j} - \overline{\tau}_{k,-j}\,\overline{\tau}_{ij}\right) \\
&= (-1)^{(p(i)+p(j))(p(k)+p(j)+1)+1}\left((-1)^{p(j)(p(k)+1)+1}\overline{\tau}_{i,-k}\right) \\
&= (-1)^{p(j)p(i)+p(i)p(k)+p(i)}\psi\left(\mathsf{E}_{-k,i}\right) \\
&= (-1)^{p(j)(p(i)+1)+1}\psi\left(\mathsf{E}_{-i,k}\right) \\
&= \psi\left([\mathsf{E}_{ji}, \mathsf{E}_{-j,k}]\right).
\end{aligned}$$

Note that ψ is surjective due to how $\mathfrak{U}_{\mathcal{A}}\mathfrak{p}_n$ is defined. So, it remains to prove that it is injective. (The rest of this proof can be found in [1].) Recall by Remark 5.1.3 that the space $\mathbb{C}_q(n|n)$ is a representation of $\mathfrak{U}_q\mathfrak{p}_n$ via the assignment $t_{ij} \mapsto s_{ij}$ (where $S = \sum_{i,j=-n}^{n} s_{ij} \otimes E_{ij}$). Therefore, by restriction, $\mathbb{C}_q(n|n)$ is also a representation of $\mathcal{U}_{\mathcal{A}}\mathfrak{p}_n$ by restriction. More explicitly,

$$\tau_{ij} \mapsto (-1)^{p(i)}\mathsf{E}_{ji} \text{ if } |i| < |j|,$$

$$\tau_{i,-i} \mapsto E_{-i,i},$$
$$\tau_{ii} \mapsto (E_{ii} - q^{-1}E_{-i,-i}) \text{ if } 1 \leq i \leq n.$$

Set $\mathbb{C}_{\mathcal{A}}(n|n) = \mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}(n|n)$. The space $\mathbb{C}_{\mathcal{A}}(n|n)$ is a $\mathfrak{U}_{\mathcal{A}}\mathfrak{p}_n$-submodule and so are all the tensor powers $\mathbb{C}_{\mathcal{A}}(n|n)^{\otimes \ell}$. We thus have a superalgebra homomorphism $\phi_\ell : \mathcal{U}_{\mathcal{A}}\mathfrak{p}_n \longrightarrow \mathrm{End}_{\mathcal{A}}(\mathbb{C}_{\mathcal{A}}(n|n)^{\otimes \ell})$ for each $\ell \geq 1$.

Let π_ℓ be the quotient homomorphism

$$\mathrm{End}_{\mathcal{A}}(\mathbb{C}_{\mathcal{A}}(n|n)^{\otimes \ell}) \longrightarrow \mathrm{End}_{\mathcal{A}}(\mathbb{C}_{\mathcal{A}}(n|n)^{\otimes \ell})/(q-1)\mathrm{End}_{\mathcal{A}}(\mathbb{C}_{\mathcal{A}}(n|n)^{\otimes \ell}) \cong \mathrm{End}_{\mathbb{C}}(\mathbb{C}(n|n)^{\otimes \ell}).$$

The composite $\pi_\ell \circ \phi_\ell$ descends to a homomorphism $\overline{\pi_\ell \circ \phi_\ell}$ from $\mathfrak{U}_{\mathcal{A}}\mathfrak{p}_n/(q-1)\mathfrak{U}_{\mathcal{A}}\mathfrak{p}_n$ to $\mathrm{End}_{\mathbb{C}}(\mathbb{C}(n|n)^{\otimes \ell})$. The composite $\overline{\pi_\ell \circ \phi_\ell} \circ \psi$ is the superalgebra homomorphism $\mathfrak{U}\mathfrak{p}_n \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}(n|n)^{\otimes \ell})$ induced by the natural $\mathfrak{p}_n$-module structure on $\mathbb{C}(n|n)^{\otimes \ell}$ twisted by the automorphism of $\mathfrak{p}_n$ given by $\mathsf{E}_{ij} \mapsto (-1)^{p(i)+p(j)}\mathsf{E}_{ij}$.

We can combine the homomorphisms $\overline{\pi_\ell \circ \phi_\ell} \circ \psi$ for all $\ell \geq 1$ to obtain a homomorphism

$$\mathcal{U}\mathfrak{p}_n \longrightarrow \prod_{\ell=1}^{\infty} \mathrm{End}_{\mathbb{C}}(\mathbb{C}(n|n)^{\otimes \ell}).$$

This map is injective since $\mathbb{C}(n|n)$ is a faithful representation of $\mathfrak{p}_n$. It follows that ψ is injective as well. □

Remark 5.2.2. If we replace $\mathbb{C}(q)$ by $\mathbb{C}((\hbar))$, q by $e^{\hbar/2}$, and $\mathcal{A}$ by $\mathbb{C}[[\hbar]]$, similar to remark 5.1.3, then we obtain an analogue to Theorem 5.2.1 which would also hold true.

Recall that the Lie bisuperalgebra structure of $\mathfrak{p}_n$ is induced by the cobracket δ, with an explicit formula from equation (4.5). This cobracket δ then extends to a Poisson co-bracket on $\mathfrak{U}\mathfrak{p}_n$, which we will also denote as δ. A very important result that we want to show is that $\mathfrak{U}_{\mathbb{C}[[\hbar]]}\mathfrak{p}_n$ is a quantization of the co-Poisson Hopf superalgebra structure on $\mathfrak{U}\mathfrak{p}_n$, which is Theorem 5.2.3.

For convenience, for $A \in \mathfrak{U}_{\mathbb{C}[[\hbar]]}\mathfrak{p}_n$, we denote by $\overline{A}$ both the image of A in $\mathfrak{U}_{\mathbb{C}[[\hbar]]}\mathfrak{p}_n/h\mathfrak{U}_{\mathbb{C}[[\hbar]]}\mathfrak{p}_n$ and the corresponding element in $\mathfrak{U}\mathfrak{p}_n$ via the isomorphism of the $\hbar$-analogue of Theorem 5.2.1. We similarly identify the corresponding elements in $\left(\mathfrak{U}_{\mathbb{C}[[\hbar]]}\mathfrak{p}_n/h\mathfrak{U}_{\mathbb{C}[[\hbar]]}\mathfrak{p}_n\right) \otimes \left(\mathfrak{U}_{\mathbb{C}[[\hbar]]}\mathfrak{p}_n/h\mathfrak{U}_{\mathbb{C}[[\hbar]]}\mathfrak{p}_n\right)$ and $\mathfrak{U}\mathfrak{p}_n \otimes \mathfrak{U}\mathfrak{p}_n$.

Theorem 5.2.3. *If $A \in \mathfrak{U}_{\mathbb{C}[[\hbar]]}\mathfrak{p}_n$, we have $\overline{\hbar^{-1}(\Delta(A) - \Delta(A)^\circ)} = \delta(\overline{A})$. Hence, $\mathfrak{U}_{\mathbb{C}[[\hbar]]}\mathfrak{p}_n$ is a quantization of the co-Poisson Hopf superalgebra structure on $\mathfrak{U}\mathfrak{p}_n$.*

Proof. It is enough to show that the identity above holds for the generators τ_{ij} of $\mathfrak{U}_{\mathbb{C}[[\hbar]]}\mathfrak{g}$. Let $A = \tau_{ij}$. First note that the identity is trivially satisfied for $i = j$, as both sides are zero. Assume now that $i \neq j$. Then:

$$\begin{aligned}
\hbar^{-1}\left(\Delta(\tau_{ij}) - \Delta(\tau_{ij})^\circ\right) = {} & \left(\frac{e^{\hbar/2} - e^{-\hbar/2}}{\hbar}\right) \sum_{\substack{k=-n \\ |i|<|k|<|j|}}^{n} \left((-1)^{(p(i)+p(k))(p(j)+p(k))}\tau_{ik} \otimes \tau_{kj} - \tau_{kj} \otimes \tau_{ik}\right) \\
& + \left(\frac{e^{\hbar/2} - 1}{\hbar}\right)\left(\tau_{ii} \otimes \tau_{ij} - \tau_{ij} \otimes \tau_{ii} + \tau_{ij} \otimes \tau_{jj} - \tau_{jj} \otimes \tau_{ij}\right) \\
& - \left(\frac{e^{\hbar/2} - e^{-\hbar/2}}{\hbar}\right) \delta_{i>0}\left((-1)^{p(j)}\tau_{i,-i} \otimes \tau_{-i,j} + \tau_{-i,j} \otimes \tau_{i,-i}\right) \\
& + \left(\frac{e^{\hbar/2} - e^{-\hbar/2}}{\hbar}\right) \delta_{j<0}\left((-1)^{p(i)}\tau_{i,-j} \otimes \tau_{-j,j} - \tau_{-j,j} \otimes \tau_{i,-j}\right)
\end{aligned}$$

Thus, in $\mathfrak{U}_{\mathbb{C}[[\hbar]]}\mathfrak{g}/\hbar\mathfrak{U}_{\mathbb{C}[[\hbar]]}\mathfrak{g}$, we have:

$$\begin{aligned}
\overline{\hbar^{-1}\left(\Delta(\tau_{ij}) - \Delta(\tau_{ij})^\circ\right)} = {} & \sum_{\substack{k=-n \\ |i|<|k|<|j|}}^{n} \left((-1)^{(p(i)+p(k))(p(j)+p(k))}\overline{\tau}_{ik} \otimes \overline{\tau}_{kj} - \overline{\tau}_{kj} \otimes \overline{\tau}_{ik}\right) \\
& + \frac{1}{2}\left(\overline{\tau}_{ii} \otimes \overline{\tau}_{ij} - \overline{\tau}_{ij} \otimes \overline{\tau}_{ii} + \overline{\tau}_{ij} \otimes \overline{\tau}_{jj} - \overline{\tau}_{jj} \otimes \overline{\tau}_{ij}\right) \\
& - \delta_{i>0}\left(\overline{\tau}_{-i,j} \otimes \overline{\tau}_{i,-i} + (-1)^{p(j)}\overline{\tau}_{i,-i} \otimes \overline{\tau}_{-i,j}\right) \\
& + \delta_{j<0}\left((-1)^{p(i)}\overline{\tau}_{i,-j} \otimes \overline{\tau}_{-j,j} - \overline{\tau}_{-j,j} \otimes \overline{\tau}_{i,-j}\right)
\end{aligned}$$

We next compute $\delta(\overline{\tau}_{ij})$ using the isomorphism of Theorem 5.2.1 and equation (4.5).

$$\delta(\overline{\tau}_{ij}) = (-1)^{p(j)}\delta(\mathsf{E}_{ji})$$

$$
\begin{aligned}
&= \sum_{\substack{k=-n\\ |i|<|k|<|j|}}^{n} (-1)^{p(j)+p(k)} \left((-1)^{(p(i)+p(k))(p(j)+p(k))}\mathsf{E}_{ki}\otimes \mathsf{E}_{jk} - \mathsf{E}_{jk}\otimes \mathsf{E}_{ki}\right)\\
&\quad - \frac{1}{2}(-1)^{p(j)}\left((-1)^{p(j)}\mathsf{E}_{jj} - (-1)^{p(i)}\mathsf{E}_{ii}\right)\otimes \mathsf{E}_{ji} + \frac{1}{2}(-1)^{p(j)}\mathsf{E}_{ji}\otimes\left((-1)^{p(j)}\mathsf{E}_{jj} - (-1)^{p(i)}\mathsf{E}_{ii}\right)\\
&\quad - \frac{\delta_{j<0}}{2}(-1)^{p(j)}\mathsf{E}_{j,-j}\otimes \mathsf{E}_{-j,i} + \frac{\delta_{i>0}}{2}\mathsf{E}_{-i,i}\otimes \mathsf{E}_{j,-i}\\
&\quad + \frac{\delta_{j<0}}{2}(-1)^{p(i)+p(j)}\mathsf{E}_{-j,i}\otimes \mathsf{E}_{j,-j} + \frac{\delta_{i>0}}{2}(-1)^{p(j)}\mathsf{E}_{j,-i}\otimes \mathsf{E}_{-i,i}\\
&= \overline{\hbar^{-1}\left(\Delta(\tau_{ij}) - \Delta(\tau_{ij})^{\circ}\right)}
\end{aligned}
$$

as needed. □

5.3 Drinfeld-Jimbo Relations of $\mathfrak{U}_q\mathfrak{p}_n$

Let

$$
\begin{gathered}
q^{k_i} := t_{ii}, \qquad e_i := -\tau_{-i,-i-1} = \frac{-1}{q-q^{-1}}t_{-i,-i-1}, \qquad f_{\bar{i}} := -\tau_{i,-i-1} = \frac{-1}{q-q^{-1}}t_{i,-i-1},\\
f_i := \tau_{i,i+1} = \frac{1}{q-q^{-1}}t_{i,i+1} \qquad e_{\bar{i}} := \tau_{-i,i+1} = \frac{1}{q-q^{-1}}t_{-i,i+1} \qquad F_{\bar{i}} := -2\tau_{i,-i} = \frac{-2}{q-q^{-1}}t_{i,-i}
\end{gathered}
\tag{5.6}
$$

From relation (5.6), we can write the generators of $\mathfrak{U}_q\mathfrak{p}_n$ in terms of the generators above:

$$
\begin{aligned}
t_{-i,-i-j} &= -(q-q^{-1})q^{-\sum_{h=1}^{j-1}k_{i+h}}\prod_{h=1}^{j-1}\text{ad } e_{i+h}(e_i)\\
t_{-i,i+j} &= (q-q^{-1})q^{-\sum_{h=1}^{j-1}k_{i+h}}\prod_{h=1}^{j-1}\text{ad } f_{i+h}(e_{\bar{i}})\\
t_{i,-i-j} &= -(q-q^{-1})q^{-\sum_{h=1}^{j-1}k_{i+h}}\prod_{h=1}^{j-1}\text{ad } e_{i+h}(f_{\bar{i}})\\
t_{i,i+j} &= (q-q^{-1})q^{-\sum_{h=1}^{j-1}k_{i+h}}\prod_{h=1}^{j-1}\text{ad } f_{i+h}(f_i)
\end{aligned}
\tag{5.7}
$$

where $\text{ad } a_i(a_j) := [a_i, a_j]$, $\prod_{h=1}^{j}\text{ad } a_{i+h}(a_i) := \text{ad } a_{i+j}\text{ad } a_{i+j-1}\ldots\text{ad } a_{i+1}(a_i)$, and

$\prod_{h=1}^{0} \text{ad } a_{i+h}(a_i) := a_i$, for $a_i = e_i, e_{\bar{i}}, f_i, f_{\bar{i}}$. From these we obtain the following relations

$$t_{i,i+j} = q^{-k_{i+j-1}}(f_{i+j-1}t_{i,i+j-1} - t_{i,i+j-1}f_{i+j-1})$$

$$t_{-i,i+j} = q^{-k_{i+j-1}}(f_{i+j-1}t_{-i,i+j-1} - t_{-i,i+j-1}f_{i+j-1})$$

$$t_{i,-i-j} = q^{-k_{i+j-1}}(e_{i+j-1}t_{i,-i-j+1} - t_{i,-i-j+1}e_{i+j-1})$$

$$t_{-i,-i-j} = q^{-k_{i+j-1}}(e_{i+j-1}t_{-i,-i-j+1} - t_{-i,-i-j+1}e_{i+j-1})$$

Alternatively, for $i > 0$, we can write the generators of $\mathfrak{U}_q\mathfrak{p}_n$ inductively as follows

$$t_{ij} = -q^{-k_{i+1}}(f_i t_{i+1,j} - t_{i+1,j}f_i)$$

$$t_{-i,j} = q^{-k_{i+1}}(e_i t_{i+1,j} - t_{i+1,j}e_i)$$

Our first main result is the following presentation of $\mathfrak{U}_q\mathfrak{p}_n$.

Proposition 5.3.1. *The quantum superalgebra $\mathfrak{U}_q\mathfrak{p}_n$ is isomorphic to the unital associative algebra over $\mathbb{C}(q)$ generated by the elements $e_i, f_i, e_{\bar{i}}, f_{\bar{i}}$ for $i \in I$, $F_{\bar{i}}$ for $i \in J$, and q^h for $h \in \mathcal{P}^\vee$, satisfying the following relations*

$$q^0 = 1, \quad q^{h_1+h_2} = q^{h_1}q^{h_2} \quad \textit{for } h_1, h_2 \in \mathcal{P}^\vee$$

$$q^h e_i = q^{\alpha_i(h)} e_i q^h, \quad q^h f_i = q^{-\alpha_i(h)} f_i q^h \quad \textit{for } h \in \mathcal{P}^\vee$$

$$q^h e_{\bar{i}} = q^{\gamma_i(h)} e_{\bar{i}} q^h, \quad q^h f_{\bar{i}} = q^{-\gamma_i(h)} f_{\bar{i}} q^h, \quad q^h F_{\bar{i}} = q^{-\beta_i(h)} F_{\bar{i}} q^h \quad \textit{for } h \in \mathcal{P}^\vee$$

$$e_i e_j - e_j e_i = 0, \quad f_i f_j - f_j f_i = 0, \quad f_{\bar{i}} f_{\bar{j}} + f_{\bar{j}} f_{\bar{i}} = 0 \quad \textit{if } |i-j| > 1$$

$$e_{\bar{i}} e_{\bar{j}} + e_{\bar{j}} e_{\bar{i}} = 0, \quad F_{\bar{i}} F_{\bar{j}} + F_{\bar{j}} F_{\bar{i}} = 0 \quad \textit{if } |i-j| > 0$$

$$e_i f_j - f_j e_i = 0 \quad \textit{if } j \neq i, i+1$$

$$e_i f_{\bar{j}} - f_{\bar{j}} e_i = 0, \quad f_i f_{\bar{j}} - f_{\bar{j}} f_i = 0, \quad e_{\bar{i}} f_{\bar{j}} + f_{\bar{j}} e_{\bar{i}} = 0 \quad \textit{if } |i-j| > 1$$

$$e_i e_{\bar{j}} - e_{\bar{j}} e_i = 0, \quad f_j e_{\bar{i}} - e_{\bar{i}} f_j = 0 \quad \textit{if } j \neq i+1$$

$$F_{\bar{i}} e_j - e_j F_{\bar{i}} = 0, \quad F_{\bar{i}} f_j - f_j F_{\bar{i}} = 0 \quad \textit{if } i \neq j, j+1$$

$$F_{\bar{i}}e_{\bar{j}} - e_{\bar{j}}F_{\bar{i}} = 0, \quad F_{\bar{i}}f_{\bar{j}} - f_{\bar{j}}F_{\bar{i}} = 0 \quad \textit{if } i \neq j, j+1$$

$$e_{\bar{i}}^2 = 0, \quad f_{\bar{i}}^2 = 0, \quad F_{\bar{i}}^2 = 0$$

$$e_{i+1}e_i - e_ie_{i+1} = e_{\bar{i}}f_{\overline{i+1}} + f_{\overline{i+1}}e_{\bar{i}}, \quad f_{i+1}f_i - f_if_{i+1} = f_{\bar{i}}e_{\overline{i+1}} + e_{\overline{i+1}}f_{\bar{i}}$$

$$e_{\overline{i+1}}e_i - e_ie_{\overline{i+1}} = f_{i+1}e_{\bar{i}} - e_{\bar{i}}f_{i+1}, \quad f_{\overline{i+1}}f_i - f_if_{\overline{i+1}} = e_{i+1}f_{\bar{i}} - f_{\bar{i}}e_{i+1}$$

$$e_if_i - f_ie_i = -\frac{q^{2k_i} - q^{2k_{i+1}}}{q^2-1} + \frac{q^2-1}{q^2}f_{\bar{i}}e_{\bar{i}}$$

$$f_{\bar{i}}e_{\bar{i}} + q^2e_{\bar{i}}f_{\bar{i}} = -\frac{q^2}{q^2-1}(q^{2k_i} - q^{2k_{i+1}})$$

$$qe_if_{\bar{i}} - q^{-1}f_{\bar{i}}e_i = \frac{(1+q^2)}{2}q^{k_{i+1}}F_{\overline{i+1}} = q^{-1}f_{\overline{i+1}}f_{i+1} - qf_{i+1}f_{\overline{i+1}}$$

$$qF_{\overline{i+1}}e_i - e_iF_{\overline{i+1}} = 0, \quad qF_{\bar{i}}f_i - f_iF_{\bar{i}} = 0$$

$$F_{\bar{i}}e_i - qe_iF_{\bar{i}} = -2f_{\bar{i}}q^{k_i}, \quad q^{-1}F_{\overline{i+1}}f_i - f_iF_{\overline{i+1}} = 2q^{k_{i+1}}f_{\bar{i}}$$

$$F_{\bar{i}}e_{\bar{i}} + qe_{\bar{i}}F_{\bar{i}} = 2f_iq^{k_i}, \quad F_{\bar{i}}f_{\bar{i}} + q^{-1}f_{\bar{i}}F_{\bar{i}} = 0$$

$$F_{\overline{i+1}}e_{\bar{i}} + qe_{\bar{i}}F_{\overline{i+1}} = 2e_iq^{k_{i+1}}, \quad F_{\overline{i+1}}f_{\bar{i}} + q^{-1}f_{\bar{i}}F_{\overline{i+1}} = 0$$

$$q^{-1}e_i^2e_{i+1} - (q+q^{-1})e_ie_{i+1}e_i + qe_{i+1}e_i^2 = 0$$

$$qe_{i+2}^2e_i - (q+q^{-1})e_{i+1}e_ie_{i+1} + q^{-1}e_ie_{i+1}^2 = 0$$

$$qf_i^2f_{i+1} - (q+q^{-1})f_if_{i+1}f_i + q^{-1}f_{i+1}f_i^2 = 0$$

$$q^{-1}f_{i+1}^2f_i - (q+q^{-1})f_{i+1}f_if_{i+1} + qf_if_{i+1}^2 = 0$$

$$q^{-1}e_i^2e_{\overline{i+1}} - (q+q^{-1})e_ie_{\overline{i+1}}e_i + qe_{\overline{i+1}}e_i^2 = 0$$

$$qf_i^2f_{\overline{i+1}} - (q+q^{-1})f_if_{\overline{i+1}}f_i + q^{-1}f_{\overline{i+1}}f_i^2 = 0$$

$$e_{i+1}e_ie_{\overline{i+1}} - e_ie_{i+1}e_{\overline{i+1}} - q^2e_{\overline{i+1}}e_{i+1}e_i + q^2e_{\overline{i+1}}e_ie_{i+1} = q^{2k_{i+1}}e_{\bar{i}}$$

$$2qq^{k_{i+1}}(f_{i+1}f_{\bar{i}} - f_{\bar{i}}f_{i+1}) = (1-q^{-2})F_{\overline{i+1}}(f_{i+1}f_i - f_if_{i+1})$$

$$-2qq^{k_{i+1}}(f_{\overline{i+1}}e_i - e_if_{\overline{i+1}}) = (1-q^{-2})F_{\overline{i+1}}(e_{i+1}e_i - e_ie_{i+1})$$

$$-2qq^{k_{i+1}}(f_{\overline{i+1}}f_{\bar{i}} + f_{\bar{i}}f_{\overline{i+1}}) = (1-q^{-2})F_{\overline{i+1}}(f_{\overline{i+1}}f_i - f_if_{\overline{i+1}})$$

$$2qq^{k_{i+1}}(f_{i+1}e_i - e_if_{i+1}) = (1-q^{-2})F_{\overline{i+1}}(e_{\overline{i+1}}e_i - e_ie_{\overline{i+1}})$$

Proof. Let U be the unital associative algebra over $\mathbb{C}(q)$ generated by the elements $e_i, f_i, e_{\bar{i}}, f_{\bar{i}}$ for $i \in I$, $F_{\bar{i}}$ for $i \in J$, and q^h for $h \in \mathcal{P}^\vee$ with defining relations given in the proposition. We can obtain these relations using (5.6) and (5.4). This gives a well-defined algebra isomorphism $\psi : U \to \mathfrak{U}_q\mathfrak{p}_n$. The relations in (5.7) immediately shows that ψ is surjective.

It remains to show that ψ is injective, which is enough to show that the relation in (5.4) is obtained by the relations in the statement of the proposition above. This is done on a case-by-case basis. We simplify the expression in (5.4) in each of the same 26 cases as in Proposition 5.2.1:

1. $|i| = |j| < |k| < |l|$
2. $|k| < |i| = |j| < |l|$
3. $|k| < |l| < |i| = |j|$
4. $|k| = |l| < |i| < |j|$
5. $|i| < |k| = |l| < |j|$
6. $|i| < |j| < |k| = |l|$
7. $|i| = |k| < |j| < |l|$
8. $|i| = |k| < |l| < |j|$
9. $|i| < |k| < |l| = |j|$
10. $|k| < |i| < |l| = |j|$
11. $|k| < |l| = |i| < |j|$
12. $|i| < |j| = |k| < |l|$
13. $|i| < |j| < |k| < |l|$
14. $|i| < |k| < |j| < |l|$
15. $|i| < |k| < |l| < |j|$
16. $|k| < |i| < |j| < |l|$
17. $|k| < |i| < |l| < |j|$
18. $|k| < |l| < |i| < |j|$
19. $|k| = |l| < |i| = |j|$
20. $|i| = |j| < |k| = |l|$
21. $|i| = |k| < |j| = |l|$
22. $|i| = |j| = |k| < |l|$
23. $|k| < |i| = |j| = |l|$
24. $|i| = |k| = |l| < |j|$
25. $|i| < |j| = |k| = |l|$
26. $|i| = |j| = |k| = |l|$

We will prove cases 2 and 25 here, as the rest will follow using similar ideas with varying levels of complexity.

Suppose that $|k| < |i| = |j| < |l|$ in (5.4). We want to prove that the relation

$$0 = (-1)^{p(i)+p(j))(p(k)+p(\ell))} t_{ij} t_{k\ell} - t_{k\ell} t_{ij}$$

are obtained from the relations in the proposition. We do this through induction on $|i| - |k|$ first, then on $|\ell| - |i|$. We will first consider the case of when $i = -j$ $(i > 0)$

and $k, \ell > 0$. The other cases will follow similarly. Suppose that $\ell = i+1$. For the base case $i = k+1$, we have the following:

$$
\begin{aligned}
&t_{k+1,-k-1}t_{k,k+2} - t_{k,k+2}t_{k+1,-k-1} \\
&\qquad = -\frac{q-q^{-1}}{2}[F_{\overline{k+1}}t_{k,k+2} - t_{k,k+2}F_{\overline{k+1}}] \\
&\qquad = -\frac{(q-q^{-1})^2}{2}[F_{\overline{k+1}}q^{-k_{k+1}}(f_{k+1}f_k - f_kf_{k+1}) - q^{-k_{k+1}}(f_{k+1}f_k - f_kf_{k+1})F_{\overline{k+1}}] \\
&\qquad = -\frac{(q-q^{-1})^2}{2}[q^{-2}q^{-k_{k+1}}F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1}) - q^{-k_{k+1}}(f_{k+1}f_k - f_kf_{k+1})F_{\overline{k+1}}] \\
&\qquad = -\frac{(q-q^{-1})^2}{2}q^{-k_{k+1}}[q^{-2}F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1}) - (f_{k+1}f_k - f_kf_{k+1})F_{\overline{k+1}}] \\
&\qquad = -\frac{(q-q^{-1})^2}{2}q^{-k_{k+1}}[F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1}) - (f_{k+1}f_k - f_kf_{k+1})F_{\overline{k+1}} \\
&\qquad + (q^{-2}-1)F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1})] \\
&\qquad = -\frac{(q-q^{-1})^2}{2}q^{-k_{k+1}}[F_{\overline{k+1}}f_{k+1}f_k - F_{\overline{k+1}}f_kf_{k+1} - f_{k+1}f_kF_{\overline{k+1}} + f_kf_{k+1}F_{\overline{k+1}} \\
&\qquad + (q^{-2}-1)F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1})] \\
&\qquad = -\frac{(q-q^{-1})^2}{2}q^{-k_{k+1}}[q^{-1}f_{k+1}F_{\overline{k+1}}f_k - F_{\overline{k+1}}f_kf_{k+1} - f_{k+1}f_kF_{\overline{k+1}} + qf_kF_{\overline{k+1}}f_{k+1} \\
&\qquad + (q^{-2}-1)F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1})] \\
&\qquad = -\frac{(q-q^{-1})^2}{2}q^{-k_{k+1}}[f_{k+1}(q^{-1}F_{\overline{k+1}}f_k - f_kF_{\overline{k+1}}) - q(q^{-1}F_{\overline{k+1}}f_k - f_kF_{\overline{k+1}})f_{k+1} \\
&\qquad + (q^{-2}-1)F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1})] \\
&\qquad = -\frac{(q-q^{-1})^2}{2}q^{-k_{k+1}}[2f_{k+1}q^{k_{k+1}}f_{\bar{i}} - 2qq^{k_{k+1}}f_{\bar{i}}f_{k+1} + (q^{-2}-1)F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1})] \\
&\qquad = -\frac{(q-q^{-1})^2}{2}q^{-k_{k+1}}[2qq^{k_{k+1}}(f_{k+1}f_{\bar{i}} - f_{\bar{i}}f_{k+1}) + (q^{-2}-1)F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1})] \\
&\qquad = -\frac{(q-q^{-1})^2}{2}q^{-k_{k+1}}[(1-q^{-2})F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1}) + (q^{-2}-1)F_{\overline{k+1}}(f_{k+1}f_k - f_kf_{k+1})] \\
&\qquad = 0
\end{aligned}
$$

The induction step for $i - k \geq 2$ is as follows:

$$t_{i,-i}t_{k,i+1} - t_{k,i+1}t_{i,-i} = -\frac{q-q^{-1}}{2}[F_{\bar{i}}t_{k,i+1} - t_{k,i+1}F_{\bar{i}}]$$

$$
\begin{aligned}
&= -\frac{q-q^{-1}}{2}[F_{\bar{i}}q^{-k_{k+1}}(f_k t_{k+1,i+1} - t_{k+1,i+1}f_k) - t_{k,i+1}F_{\bar{i}}] \\
&= -\frac{q-q^{-1}}{2}[q^{-k_{k+1}}F_{\bar{i}}(f_k t_{k+1,i+1} - t_{k+1,i+1}f_k) - t_{k,i+1}F_{\bar{i}}] \\
&= -\frac{q-q^{-1}}{2}[q^{-k_{k+1}}(f_k t_{k+1,i+1} - t_{k+1,i+1}f_k)F_{\bar{i}} - t_{k,i+1}F_{\bar{i}}] \\
&= -\frac{q-q^{-1}}{2}[t_{k,i+1}F_{\bar{i}} - t_{k,i+1}F_{\bar{i}}] \\
&= 0
\end{aligned}
$$

Now we let ℓ vary, and we will prove this relation through induction on $\ell - i$. The base case $\ell = i+1$ was treated above. The induction step for $\ell - i \geq 2$ is as follows:

$$
\begin{aligned}
t_{i,-i}t_{k\ell} - t_{k\ell}t_{i,-i} &= -\frac{q-q^{-1}}{2}[F_{\bar{i}}q^{-k_{\ell-1}}(f_{\ell-1}t_{k,\ell-1} - t_{k,\ell-1}f_{\ell-1}) - q^{-k_{\ell-1}}(f_{\ell-1}t_{k,\ell-1} - t_{k,\ell-1}f_{\ell-1})F_{\bar{i}}] \\
&= -\frac{q-q^{-1}}{2}[q^{-k_{\ell-1}}F_{\bar{i}}(f_{\ell-1}t_{k,\ell-1} - t_{k,\ell-1}f_{\ell-1}) - q^{-k_{\ell-1}}(f_{\ell-1}t_{k,\ell-1} - t_{k,\ell-1}f_{\ell-1})F_{\bar{i}}] \\
&= -\frac{q-q^{-1}}{2}[q^{-k_{\ell-1}}(f_{\ell-1}t_{k,\ell-1} - t_{k,\ell-1}f_{\ell-1})F_{\bar{i}} - q^{-k_{\ell-1}}(f_{\ell-1}t_{k,\ell-1} - t_{k,\ell-1}f_{\ell-1})F_{\bar{i}}] \\
&= 0
\end{aligned}
$$

Now, suppose that $|i| < |j| = |k| = |l|$ in (5.4). We want to show that the relation

$$0 = (-1)^{(p(i)+p(j))(p(j)+p(k))}q^{\mathrm{sgn}(j)}t_{ij}t_{kj} - t_{kj}t_{ij}$$

when $j = \ell$, and

$$0 = (-1)^{(p(i)+p(j))(p(j)+p(k))}q^{\mathrm{sgn}(j)}t_{ij}t_{k,-j} - t_{k,-j}t_{ij} + (-1)^{p(i)}\delta_{j>0}(q-q^{-1})t_{i,-j}t_{kj}$$

when $j = -\ell$, are obtained from the relations in the proposition. We prove this through induction on $|j| - |i|$. We will consider the case of when $i > 0$ and $j = k > 0$. The other cases all follow similarly. For the base case $j = i + 1$, the relations

$$qt_{i,i+1}t_{i+1,i+1} = t_{i+1,i+1}t_{i,i+1}$$

when $j=\ell$, and

$$qt_{i,i+1}t_{i+1,-i-1}-t_{i+1,-i-1}t_{i,i+1}=-(q-q^{-1})t_{i,-i-1}t_{i+1,i+1}$$

when $j=-\ell$, are obtained from the proposition as $q^{k_{i+1}}f_i=qf_iq^{k_{i+1}}$ and $f_iF_{\overline{i+1}}-q^{-1}F_{\overline{i+1}}f_i=-2q^{k_{i+1}}f_{\bar{i}}$. The induction step for $j-i\geq 2$ is as follows:

$$\begin{aligned}
qt_{ij}t_{jj} &= qt_{ij}q^{k_j}\\
&= qq^{-k_{i+1}}(f_it_{i+1,j}-t_{i+1,j}f_i)q^{k_j}\\
&= q^{k_j}q^{-k_{i+1}}(f_it_{i+1,j}-t_{i+1,j}f_i)\\
&= q^{k_j}t_{ij}\\
&= t_{jj}t_{ij}
\end{aligned}$$

for $j=\ell$, and

$$\begin{aligned}
qt_{ij}t_{j,-j} &= -q\frac{q-q^{-1}}{2}q^{-k_{i+1}}(f_it_{i+1,j}-t_{i+1,j}f_i)F_{\bar{j}}\\
&= -q\frac{q-q^{-1}}{2}q^{-k_{i+1}}f_it_{i+1,j}F_{\bar{j}}+q\frac{q-q^{-1}}{2}q^{-k_{i+1}}t_{i+1,j}f_iF_{\bar{j}}\\
&= -q\frac{q-q^{-1}}{2}q^{-k_{i+1}}f_it_{i+1,j}F_{\bar{j}}+q\frac{q-q^{-1}}{2}q^{-k_{i+1}}t_{i+1,j}f_iF_{\bar{j}}\\
&= -\frac{q-q^{-1}}{2}q^{-k_{i+1}}f_i(F_{\bar{j}}t_{i+1,j}+2t_{i+1,-j}q^{k_j})+\frac{q-q^{-1}}{2}q^{-k_{i+1}}(F_{\bar{j}}t_{i+1,j}+2t_{i+1,-j}q^{k_j})f_i\\
&= -\frac{q-q^{-1}}{2}q^{-k_{i+1}}F_{\bar{j}}f_it_{i+1,j}-(q-q^{-1})q^{-k_{i+1}}f_it_{i+1,-j}q^{k_j}\\
&+\frac{q-q^{-1}}{2}q^{-k_{i+1}}F_{\bar{j}}t_{i+1,j}f_i+(q-q^{-1})q^{-k_{i+1}}t_{i+1,-j}q^{k_j}f_i\\
&= \frac{q-q^{-1}}{2}q^{-k_{i+1}}F_{\bar{j}}(f_it_{i+1,j}-t_{i+1,j}f_i)-(q-q^{-1})q^{-k_{i+1}}(f_it_{i+1,-j}-t_{i+1,-j}f_i)q^{k_j}\\
&= \frac{q-q^{-1}}{2}F_{\bar{j}}q^{-k_{i+1}}(f_it_{i+1,j}-t_{i+1,j}f_i)-(q-q^{-1})q^{-k_{i+1}}(f_it_{i+1,-j}-t_{i+1,-j}f_i)q^{k_j}\\
&= \frac{q-q^{-1}}{2}F_{\bar{j}}t_{ij}-(q-q^{-1})t_{i,-j}q^{k_j}\\
&= t_{j,-j}t_{ij}-(q-q^{-1})t_{i,-j}t_{jj}
\end{aligned}$$

for $j=-\ell$. $\square$

The following are some relations of $\mathfrak{U}_q\mathfrak{p}_n$ that resulted from the relations in Proposition 5.3.1:

Lemma 5.3.2. *The following relations holds in $\mathfrak{U}_q\mathfrak{p}_n$:*

a) $e_i f_i - f_i e_i = e_{\bar{i}} f_{\bar{i}} + f_{\bar{i}} e_{\bar{i}}$

b) $\frac{2}{1+q^2} f_{\overline{i+1}} f_{i+1} f_i - f_{\overline{i+1}} f_i f_{i+1} - f_{i+1} f_i f_{\overline{i+1}} + q^2 f_i f_{i+1} f_{\overline{i+1}} = q^2 q^{2k_{i+1}} f_{\bar{i}} - \frac{1-q^2}{1+q^2} f_{i+1} f_{\overline{i+1}} f_i$

c) $f_i e_i = e_i f_i + q^2 \frac{q^{2k_i} - q^{k_{i+1}}}{q^2 - 1} + (q^2-1) e_{\bar{i}} f_{\bar{i}}$

Set $\deg e_i = \alpha_i$, $\deg f_i = -\alpha_i$, $\deg q^h = 0$, $\deg e_{\bar{i}} = \gamma_i$, $\deg f_{\bar{i}} = -\gamma_i$, and $\deg F_{\bar{i}} = -\beta_i$. Noting that all the defining relations of the quantum superalgebra $\mathfrak{U}_q\mathfrak{p}_n$ are homogeneous, we see immediately that $\mathfrak{U}\mathfrak{p}_n$ has a root-space decomposition,

$$\mathfrak{U}_q\mathfrak{p}_n = \bigoplus_{\alpha \in \mathcal{Q}} (\mathfrak{U}_q)_\alpha$$

where $(\mathfrak{U}_q)_\alpha = \left\{ v \in \mathfrak{U}_q\mathfrak{p}_n \mid q^h v q^{-h} = q^{\alpha(h)} v \text{ for all } h \in \mathcal{P}^\vee \right\}$.

Through direct computations, we can express the coproduct Δ in terms of the new generators in Proposition 5.3.1:

Lemma 5.3.3. *In terms of the generators $e_i, f_i, e_{\bar{i}}, f_{\bar{i}}$ for $i \in I$, $F_{\bar{i}}$ for $i \in J$, and q^h for $h \in \mathcal{P}^\vee$ in $\mathfrak{U}_q\mathfrak{p}_n$,*

$$\Delta(q^h) = q^h \otimes q^h$$

$$\Delta(e_i) = q^{k_i} \otimes e_i + e_i \otimes q^{k_{i+1}} - \frac{q-q^{-1}}{2} e_{\bar{i}} \otimes F_{\overline{i+1}}$$

$$\Delta(f_i) = q^{k_i} \otimes f_i + f_i \otimes q^{k_{i+1}} + \frac{q-q^{-1}}{2} F_{\bar{i}} \otimes e_{\overline{i+1}}$$

$$\Delta(e_{\bar{i}}) = q^{k_i} \otimes e_{\bar{i}} + e_{\bar{i}} \otimes q^{k_{i+1}}$$

$$\Delta(f_{\bar{i}}) = q^{k_i} \otimes f_{\bar{i}} + f_{\bar{i}} \otimes q^{k_{i+1}} - \frac{q-q^{-1}}{2} F_{\bar{i}} \otimes e_i + \frac{q-q^{-1}}{2} f_i \otimes F_{\overline{i+1}}$$

$$\Delta(F_{\bar{i}}) = q^{k_i} \otimes F_{\bar{i}} + F_{\bar{i}} \otimes q^{k_i}$$

Remark 5.3.4. This Lemma is crucial as the tensor representations of $\mathfrak{U}_q\mathfrak{p}_n$ onto $\mathbb{C}_q(n|n)$ uses the coproduct in terms of the new generators.

5.4 Triangular decomposition of $\mathfrak{U}_q\mathfrak{p}_n$

Let $\mathfrak{U}_q^+$ (respectively $\mathfrak{U}_q^-$) be the sub-superalgebra of $\mathfrak{U}_q\mathfrak{p}_n$ generated by the elements e_i and $e_{\bar{i}}$ (respectively f_i, $f_{\bar{i}}$, and $F_{\bar{j}}$) for $i = 1, \ldots, n-1$ (and $j = 1, \ldots, n$). Also, let $\mathfrak{U}_q^0$ be generated by q^h for $h \in \mathcal{P}^\vee$. We want to show that $\mathfrak{U}_q\mathfrak{p}_n$ has a triangular decomposition, which will require the following lemma:

Lemma 5.4.1. *Let $\mathfrak{U}_q^{\geq 0}$ (respectively $\mathfrak{U}_q^{\leq 0}$) be generated by generated by $\mathfrak{U}_q^0$ and $\mathfrak{U}_q^+$ ($\mathfrak{U}_q^0$ and $\mathfrak{U}_q^+$ respectively). Then the following isomorphisms holds:*

$$\mathfrak{U}_q^{\geq 0} \cong \mathfrak{U}_q^0 \otimes \mathfrak{U}_q^+ \qquad \mathfrak{U}_q^{\leq 0} \cong \mathfrak{U}_q^- \otimes \mathfrak{U}_q^0$$

Proof. We will prove the second part. Let $\{f_\zeta \mid \zeta \in \Omega\}$ be a basis of $\mathfrak{U}_q^-$ consisting of monomials in f_i's, $f_{\bar{i}}$'s, and $F_{\bar{j}}$'s ($1 \leq i \leq n-1$, $1 \leq j \leq n$), with Ω being an index set. Consider the map $\varphi : \mathfrak{U}_q^- \otimes \mathfrak{U}_q^0 \to \mathfrak{U}_q^{\leq 0}$ by $\varphi(f_\zeta \otimes q^h) = f_\zeta q^h$. The defining relations of $\mathfrak{U}_q\mathfrak{p}_n$ implies that $f_\zeta q^h$ span $\mathfrak{U}_q^{\leq 0}$, so φ is surjective. Thus it is enough to show that the set $\{f_\zeta q^h \mid \zeta \in \Omega, h \in \mathcal{P}^\vee\}$ is linearly independent over $\mathbb{C}(q)$.

Suppose

$$\sum_{\substack{\zeta \in \Omega \\ h \in \mathcal{P}^\vee}} C_{\zeta,h} f_\zeta q^h = 0$$

where $C_{\zeta,h}$ is some constant in $\mathbb{C}(q)$. We may write

$$\sum_{\alpha \in \mathcal{Q}_-} \left(\sum_{\substack{\deg f_\zeta = \alpha \\ h \in \mathcal{P}^\vee}} C_{\zeta,h} f_\zeta q^h \right) = 0$$

Write $\alpha = -\sum_{i=1}^{n-1}(m_i\alpha_i + n_i\gamma_i) - \sum_{i=1}^{n} r_i\beta_i$, for $m_i, n_i, r_i \in \mathbb{Z}_{\geq 0}$, and let $h_\alpha = \sum_{i=1}^{n-1}(m_i + n_i)k_{i+1} + \sum_{i=1}^{n} r_ik_i$ and $h'_\alpha = \sum_{i=1}^{n-1}(m_i + n_i)k_i + r_ik_i$. Since $\mathfrak{U}_q\mathfrak{p}_n = \bigoplus_{\alpha\in\mathcal{Q}}(\mathfrak{U}_q)_\alpha$, we have, for each $\alpha \in \mathcal{Q}_-$:

$$\sum_{\substack{\deg f_\zeta=\alpha\\ h\in\mathcal{P}^\vee}} C_{\zeta,h}f_\zeta q^h = 0 \tag{5.8}$$

Since f_ζ is a monomial in f_i's, $f_{\bar{i}}$'s, and $F_{\bar{i}}$'s, we have

$$\Delta(f_\zeta) = f_\zeta \otimes q^{h_\alpha} + (\text{ intermediate terms }) + q^{h'_\alpha} \otimes f_\zeta$$

which shows the terms of degree $(\alpha, 0)$ in $\Delta(f_\zeta)$. Applying the comultiplication to (5.8) gives

$$\sum_{\substack{\deg f_\zeta=\alpha\\ h\in\mathcal{P}^\vee}} C_{\zeta,h}(f_\zeta q^h \otimes q^{h+h_\alpha} + \ldots + q^{h+h'_\alpha} \otimes f_\zeta q^h) = 0$$

Collecting the terms of degree $(\alpha, 0)$ gives that

$$\sum_{\substack{\deg f_\zeta=\alpha\\ h\in P^\vee}} C_{\zeta,h}(f_\zeta q^h \otimes q^{h+h_\alpha}) = 0$$

Since the set $\{q^h \mid h \in \mathcal{P}^\vee\}$ is a linearly independent set, we have that, for all $h \in \mathcal{P}^\vee$:

$$\sum_{\deg f_\zeta=\alpha} C_{\zeta,h}f_\zeta q^h = 0$$
$$\implies \sum_{\deg f_\zeta=\alpha} C_{\zeta,h}f_\zeta = 0$$

Due to the linear independence of f_ζ, we conclude that all $C_{\zeta,h} = 0$, as desired.

□

Theorem 5.4.2. *There is a $\mathbb{C}(q)$-linear isomorphism*

$$\mathfrak{U}_q(\mathfrak{p}_n) \cong \mathfrak{U}_q^- \otimes \mathfrak{U}_q^0 \otimes \mathfrak{U}_q^+$$

Proof. Let $\{f_\zeta \mid \zeta \in \Omega\}$, $\{q^h \mid h \in \mathcal{P}^\vee\}$, and $\{e_\tau \mid \tau \in \Omega'\}$, be monomial bases of $\mathfrak{U}_q^-$, $\mathfrak{U}_q^0$, and $\mathfrak{U}_q^+$ respectively, where Ω is the index set as in the proof of Lemma 5.4.1, and Ω' is a different indexing set. From the defining relations of $\mathfrak{U}_q\mathfrak{p}_n$ in proposition 5.3.1, we can always move e_i and $e_{\bar{i}}$ $(1 \leq i \leq n-1)$ to the far right in each monomial. Thus, we have that $f_\zeta q^h e_\tau$ spans $\mathfrak{U}_q\mathfrak{p}_n$, and so it suffices to show that the elements $f_\zeta q^h e_\tau$ are linearly independent over $\mathbb{C}(q)$.

Suppose

$$\sum_{\substack{\zeta\in\Omega,\tau\in\Omega' \\ h\in\mathcal{P}^\vee}} C_{\zeta,h,\tau} f_\zeta q^h e_\tau = 0$$

where $C_{\zeta,h,\tau}$ is some constant in $\mathbb{C}(q)$. Due to the root space decomposition of $\mathfrak{U}_q\mathfrak{p}_n$, we have that, for all $\alpha \in \mathcal{Q}$:

$$\sum_{\substack{deg f_\zeta+\deg e_\tau=\alpha \\ h\in\mathcal{P}^\vee}} C_{\zeta,h,\tau} f_\zeta q^h e_\tau = 0 \tag{5.9}$$

Using the partial ordering on $\mathfrak{h}^*$, where $\lambda \leq \mu$ if and only if $\lambda - \mu \in \mathcal{Q}_+$ for $\lambda, \mu \in \mathfrak{h}^*$, we can choose $\gamma = \deg f_\zeta$ and $\beta = \deg e_\tau$, which are minimal and maximal respectively, among those for which $\gamma + \beta = \alpha$ and $C_{\zeta,h,\tau} \neq 0$. If $\gamma = -\sum_{i=1}^{n-1}(m_i\alpha_i + n_i\gamma_i) - \sum_{i=1}^{n} r_i\beta_i$, set $h_\gamma = \sum_{i=1}^{n-1}(m_i+n_i)k_{i+1} + \sum_{i=1}^{n} r_ik_i$, and if $\beta = \sum_{i=1}^{n-1}(m_i'\alpha_i + n_i'\gamma_i)$, set $h_\beta = \sum_{i=1}^{n-1}(m_i'k_i + n_i'k_i)$, for $m_i, m_i', n_i, n_i', r_i \in \mathbb{Z}^{\geq 0}$.

The term of degree $(0,\beta)$ in $\Delta(e_\tau)$ is $q^{h_\beta} \otimes e_\tau$ and the term of degree $(\gamma, 0)$ in $\Delta(f_\zeta)$ is $f_\zeta \otimes q^{h_\gamma}$. Applying the comultiplication to the sum in (5.9), since the terms of degree (γ, β) in (5.9) must sum to zero, we have that

$$\sum_{\substack{\deg f_\zeta = \gamma \\ \deg e_\tau = \beta \\ h \in \mathcal{P}^\vee}} C_{\zeta,h,\tau}(f_\zeta q^{h+h_\beta} \otimes q^{h+h_\gamma} e_\tau) = 0$$

By lemma 5.4.1, the elements $f_\zeta q^h$ are linearly independent for $\zeta \in \Omega, h \in \mathcal{P}^\vee$. Thus, for all $h \in \mathcal{P}^\vee$, we have that

$$\sum_{\deg e_\tau = \beta} C_{\zeta,h,\tau} q^{h+h_\gamma} e_\tau = 0$$
$$\implies \sum_{\deg e_\tau = \beta} C_{\zeta,h,\tau} e_\tau = 0$$

Due to the linear independence of e_τ, we conclude that all $C_{\zeta,h,\tau} = 0$, as desired.

□

CHAPTER 6

Periplectic q-Brauer algebra

In [45] and [46], Schur proved that the action of the symmetric group S_k on $V^{\otimes k}$ generates the centralizer algebra of $\mathfrak{gl}_n$. We know of this result today as the Schur-Weyl duality.

In [42], D. Moon identified the centralizer of the action of $\mathfrak{p}_n$ on the tensor space $\mathbb{C}_{n|n}^{\otimes k}$. This centralizer is called the periplectic Brauer algebra in the literature: see [15, 10, 17, 18].

Recall that we have a representation of $\mathfrak{U}_q\mathfrak{p}_n$ on $\mathbb{C}_q(n|n)$ via the assignment $t_{ij} \mapsto s_{ij}$ as per Remark 5.1.3. We can then extend this to a representation of $\mathfrak{U}_q\mathfrak{p}_n$ on $\mathbb{C}_q(n|n)^{\otimes k}$, for each $k \geq 1$, through the coproduct on $\mathfrak{U}_q\mathfrak{p}_n$. More explicitly, the action on $\mathbb{C}_q(n|n)^{\otimes k}$ by $\mathfrak{U}_q\mathfrak{p}_n$ is given by

$$t_{ij}(e_a \otimes e_b) = \sum_{k=-n}^{n} (-1)^{(p(i)+p(k))(p(k)+p(j))+(p(k)+p(j))p(a)} t_{ik}(e_a) \otimes t_{kj}(e_b) \tag{6.1}$$

where

$$t_{ii}(e_a) = \sum_{b=-n}^{n} q^{\delta_{bi}(1-2p(i))+\delta_{b,-i}(2p(i)-1)} E_{bb}(e_a);$$

$$t_{i,-i}(e_a) = (q-q^{-1})\delta_{i>0} E_{-i,i}(e_a);$$

$$t_{ij}(e_a) = (q-q^{-1})(-1)^{p(i)} \mathsf{E}_{ji}(e_a), \text{ if } |i| \neq |j|.$$

In this chapter, we determine the centralizer of the action of $\mathfrak{U}_q\mathfrak{p}_n$ on $\mathbb{C}_q(n|n)^{\otimes k}$. This centralizer is referred to as the periplectic q-Brauer algebra. Quantum analogs of the Brauer algebra were studied in [41] where they appear as centralizers of the action of twisted quantized enveloping algebras $\mathfrak{U}_q^{tw}\mathfrak{o}_n$ and $\mathfrak{U}_q^{tw}\mathfrak{sp}_n$ on tensor representations (here, $\mathfrak{sp}_n$ is the symplectic Lie algebra).

6.1 Definition

We start with the definition of the algebra of interest:

Definition 6.1.1. The periplectic q-Brauer algebra $\mathfrak{B}_{q,k}$ is the associative $\mathbb{C}(q)$-algebra generated by elements t_i and c_i for $1 \le i \le k-1$ satisfying the following relations:

$$(\mathsf{t}_i - q)(\mathsf{t}_i + q^{-1}) = 0, \;\; \mathsf{c}_i^2 = 0, \;\; \mathsf{c}_i\mathsf{t}_i = -q^{-1}\mathsf{c}_i, \;\; \mathsf{t}_i\mathsf{c}_i = q\mathsf{c}_i \quad \text{for } 1 \le i \le k-1; \tag{6.2}$$

$$\mathsf{t}_i\mathsf{t}_j = \mathsf{t}_j\mathsf{t}_i, \;\; \mathsf{t}_i\mathsf{c}_j = \mathsf{c}_j\mathsf{t}_i, \;\; \mathsf{c}_i\mathsf{c}_j = \mathsf{c}_j\mathsf{c}_i \quad \text{if } |i-j| \ge 2; \tag{6.3}$$

$$\mathsf{t}_i\mathsf{t}_j\mathsf{t}_i = \mathsf{t}_j\mathsf{t}_i\mathsf{t}_j, \;\; \mathsf{c}_{i+1}\mathsf{c}_i\mathsf{c}_{i+1} = -\mathsf{c}_{i+1}, \;\; \mathsf{c}_i\mathsf{c}_{i+1}\mathsf{c}_i = -\mathsf{c}_i \quad \text{for } 1 \le i \le k-2; \tag{6.4}$$

$$\mathsf{t}_i\mathsf{c}_{i+1}\mathsf{c}_i = -\mathsf{t}_{i+1}\mathsf{c}_i + (q-q^{-1})\mathsf{c}_{i+1}\mathsf{c}_i, \;\; \mathsf{c}_{i+1}\mathsf{c}_i\mathsf{t}_{i+1} = -\mathsf{c}_{i+1}\mathsf{t}_i + (q-q^{-1})\mathsf{c}_{i+1}\mathsf{c}_i \tag{6.5}$$

Remark 6.1.1. Setting $q = 1$ in this definition yields the algebra A_k from Definition 2.2 in [42].

Remark 6.1.2. Let S_k denote the symmetric group on k elements. The Hecke algebra H_k, generated by $\{h(s_i) = \mathsf{t}_i\}$, is a subalgebra of $\mathfrak{B}_{q,k}$. The generators of H_k satisfies the relations:

$$h(\sigma)h(\sigma') = h(\sigma\sigma') \text{ if } \ell(\sigma\sigma') = \ell(\sigma) + \ell(\sigma') \tag{6.6}$$

$$(h(s_i) - q)(h(s_i) + q^{-1}) = 0 \tag{6.7}$$

$$h(s_i)h(s_{i+1})h(s_i) = h(s_{i+1})h(s_i)h(s_{i+1}) \tag{6.8}$$

where $\sigma, \sigma' \in S_k$ and $\ell(\sigma)$ is the length of the permutation σ, which are the same relations as that in Definition 6.1.1.

Lemma 6.1.3. *Consider $\mathbb{C}(q)$ as purely odd $\mathfrak{U}_q\mathfrak{p}_n$-module. We have $\mathfrak{U}_q\mathfrak{p}_n$-module homomorphisms $\vartheta : \mathbb{C}_q(n|n) \otimes \mathbb{C}_q(n|n) \to \mathbb{C}(q)$ and $\epsilon : \mathbb{C}(q) \to \mathbb{C}_q(n|n) \otimes \mathbb{C}_q(n|n)$ given by $\vartheta(e_a \otimes e_b) = \delta_{a,-b}(-1)^{p(a)}$ and $\epsilon(1) = \sum_{a=-n}^{n} e_a \otimes e_{-a}$.*

Proof. It is enough to check that, for all the generators t_{ij} of $\mathfrak{U}_q\mathfrak{p}_n$ and any tensor $\mathbf{v} \in \mathbb{C}_q(n|n) \otimes \mathbb{C}_q(n|n)$,

$$\vartheta(t_{ij}(\mathbf{v})) = t_{ij}(\vartheta(\mathbf{v})) \text{ and } \epsilon(t_{ij}(1)) = t_{ij}(\epsilon(1)). \tag{6.9}$$

Using the formula for the coproduct, we have:

$$t_{ij}(e_a \otimes e_{-a}) = \sum_{k=-n}^{n} (-1)^{(p(i)+p(k))(p(k)+p(j))+(p(k)+p(j))p(a)} t_{ik}(e_a) \otimes t_{kj}(e_{-a}) \tag{6.10}$$

This can be made more explicit using

$$\begin{aligned} t_{ii}(e_a) &= \sum_{b=-n}^{n} q^{\delta_{bi}(1-2p(i))+\delta_{b,-i}(2p(i)-1)} E_{bb}(e_a); \\ t_{i,-i}(e_a) &= (q-q^{-1})\delta_{i>0} E_{-i,i}(e_a); \\ t_{ij}(e_a) &= (q-q^{-1})(-1)^{p(i)} \mathsf{E}_{ji}(e_a), \text{ if } |i| \neq |j|. \end{aligned} \tag{6.11}$$

We directly prove that (6.9) is satisfied through computations of $t_{ij}(e_{a_1} \otimes e_{a_2})$. We split this proof into three cases depending on the value of i and j. We will only show the case for $j = -i$.

Using (6.11) with (6.10) gives the following:

$$\begin{aligned} t_{i,-i}(e_{a_1} \otimes e_{a_2}) &= (-1)^{p(a_1)} \delta_{i>0} t_{ii}(e_{a_1}) \otimes t_{i,-i}(e_{a_2}) + \delta_{i>0} t_{i,-i}(e_{a_1}) \otimes t_{-i,-i}(e_{a_2}) \\ &= (-1)^{p(a_1)} \delta_{i>0} q^{\delta_{a_1 i}(1-2p(i))+\delta_{a_1,-i}(2p(i)-1)} (q-q^{-1})(e_{a_1} \otimes E_{-i,i}(e_{a_2})) \\ &+ \delta_{i>0} q^{\delta_{a_2 i}(1-2p(i))+\delta_{a_2,-i}(2p(i)-1)} (q-q^{-1})(E_{-i,i}(e_{a_1}) \otimes e_{a_2}) \\ &= (-1)^{p(a_1)} \delta_{i>0} \delta_{a_2,i} q^{-\delta_{a_1 i}+\delta_{a_1,-i}} (q-q^{-1})(e_{a_1} \otimes e_{-i}) \\ &+ \delta_{i>0} \delta_{a_1,i} q^{-\delta_{a_2 i}+\delta_{a_2,-i}} (q-q^{-1})(e_{-i} \otimes e_{a_2}) \end{aligned}$$

If $a_1 = -a_2 = i$, we obtain

$$t_{i,-i}(e_{a_1} \otimes e_{a_2}) = q^{-1}(q-q^{-1})(e_{-i} \otimes e_{-i}).$$

If $a_1 = -a_2 = -i$, we get

$$t_{i,-i}(e_{a_1} \otimes e_{a_2}) = -q^{-1}(q - q^{-1})(e_{-i} \otimes e_{-i}).$$

It follows that $t_{i,-i}\left(\sum_{a=-n}^{n} e_a \otimes e_{-a}\right) = 0$, and thus (6.9) holds for ϵ.

Now observe the following:

$$\begin{aligned}\vartheta(t_{i,-i}(e_{a_1} \otimes e_{a_2})) &= \delta_{i>0}\delta_{a_2,i}\delta_{a_1,i}q^{-\delta_{a_1 i}+\delta_{a_1,-i}}(q - q^{-1}) - \delta_{i>0}\delta_{a_1,i}\delta_{a_2,i}q^{-\delta_{a_2 i}+\delta_{a_2,-i}}(q - q^{-1}) \\ &= \delta_{i>0}\delta_{a_2,i}\delta_{a_1,i}q^{-1}(q - q^{-1}) - \delta_{i>0}\delta_{a_1,i}\delta_{a_2,i}q^{-1}(q - q^{-1}) = 0\end{aligned}$$

This shows that (6.9) holds for ϑ.

The case for when $i = j$ is similar, and the case of when $|i| \neq |j|$ is split further into several cases, depending on the relationship of i and j with a_1 and a_2, each approached in a similar manner. $\square$

By composing ϑ and ϵ, we obtain a $\mathfrak{U}_q\mathfrak{p}_n$-module homomorphism $\epsilon \circ \vartheta : \mathbb{C}_q(n|n)^{\otimes 2} \to \mathbb{C}_q(n|n)^{\otimes 2}$. In terms of elementary matrices, this linear map, which we abbreviate by $\mathfrak{c}$, is given by

$$\sum_{a,b=-n}^{n} (-1)^{p(a)p(b)} E_{ab} \otimes E_{-a,-b}.$$

The super-permutation operator P on $\mathbb{C}_q(n|n)^2$ is given by

$$P = \sum_{a,b=-n}^{n} (-1)^{p(b)} E_{ab} \otimes E_{ba}.$$

Note that this means that $\mathfrak{c} = P^{(\pi\circ \mathrm{st})_2}$ where $(\pi \circ \mathrm{st})_2$ stands for the map $\pi \circ \mathrm{st}$ applied to the second tensor in the previous formula for P.

We can extend $\mathfrak{c}$ to a $\mathfrak{U}_q\mathfrak{p}_n$-module homomorphism $\mathfrak{c}_i : \mathbb{C}_q(n|n)^{\otimes l} \to \mathbb{C}_q(n|n)^{\otimes l}$ for $1 \leq i \leq l - 1$ by applying $\mathfrak{c}$ to the i^{th} and $(i + 1)^{th}$ tensors.

The linear map $\mathbb{C}_q(n|n)^{\otimes l} \to \mathbb{C}_q(n|n)^{\otimes l}$ given by $P_i S_{i,i+1}$ where P_i is the super-permutation operator acting on the i^{th} and $(i + 1)^{th}$ tensors is also a $\mathfrak{U}_q\mathfrak{p}_n$-module homomorphism: this is a consequence of the fact that S is a solution of the quantum Yang-Baxter relation.

Proposition 6.1.4. *The tensor superspace $\mathbb{C}_q(n|n)^{\otimes l}$ is a module over $\mathfrak{B}_{q,l}$ if we let t_i act as $P_iS_{i,i+1}$ and c_i act as $\mathfrak{c}_i$.*

Proof. The linear operators $P_iS_{i,i+1}$ satisfy the braid relation (the first relation in (6.4)), which is a consequence of the fact that S is a solution of the quantum Yang-Baxter relation. The relations (6.3) for the operators $P_iS_{i,i+1}$ and $\mathfrak{c}_i$ can be easily verified. The other relations can be checked via direct computations. Therefore, it is enough to check the relations (6.2) on $\mathbb{C}_q(n|n)^{\otimes 2}$ and the relations (6.5) on $\mathbb{C}_q(n|n)^{\otimes 3}$. We briefly sketch some of those computations below.

First, note that $\mathfrak{c}P = -\mathfrak{c}$ and $P\mathfrak{c} = \mathfrak{c}$. Also, we easily obtain the following:

$$\mathfrak{c}\left((q-1)\sum_{i=1}^{n} E_{ii}\otimes E_{ii}\right) = \mathfrak{c}\left((q^{-1}-1)\sum_{i=1}^{n} E_{-i,-i}\otimes E_{-i,-i}\right) = 0,$$

$$\mathfrak{c}\left((q-1)\sum_{i=1}^{n} E_{ii}\otimes E_{-i,-i}\right) = (q-1)\sum_{a=-n}^{n}\sum_{b=1}^{n} E_{ab}\otimes E_{-a,-b},$$

$$\mathfrak{c}\left((q^{-1}-1)\sum_{i=1}^{n} E_{-i,-i}\otimes E_{ii}\right) = (q^{-1}-1)\sum_{a=-n}^{n}\sum_{b=-n}^{-1} (-1)^{p(a)} E_{ab}\otimes E_{-a,-b},$$

$$\mathfrak{c}\left(\sum_{i=-n}^{-1} E_{i,-i}\otimes E_{-i,i}\right) = -\sum_{a=-n}^{n}\sum_{b=1}^{n} E_{ab}\otimes E_{-a,-b},$$

$$\mathfrak{c}\left(\sum_{1\le|j|<|i|\le n} (-1)^{p(j)}\mathsf{E}_{ij}\otimes E_{ji}\right) = \sum_{a=-n}^{n}\sum_{1\le|j|<|i|\le n} (-1)^{p(a)(p(i)+1)+p(j)} E_{a,-i}\otimes E_{-a,i} = 0.$$

Therefore, we have that $\mathfrak{c}(S-1) = (q^{-1}-1)\mathfrak{c}$, hence $\mathfrak{c}S = q^{-1}\mathfrak{c}$. Now using that $\mathfrak{c} = -\mathfrak{c}P$, we obtain the third relation in (6.2). Similarly, we prove $(S-1)\mathfrak{c} = (q-1)\mathfrak{c}$, and then using $P\mathfrak{c} = \mathfrak{c}$, we obtain the fourth relation in (6.2).

For the remaining relations we use the following formula:

$$PS = \sum_{i,j=-n}^{n} (-1)^{p(j)} E_{ij}\otimes E_{ji} + (q-1)\sum_{i=1}^{n}(E_{-i,i}\otimes E_{i,-i})$$

$$+ (q-1)\sum_{i=1}^{n}(E_{ii}\otimes E_{ii}) - (q^{-1}-1)\sum_{i=1}^{n}(E_{i,-i}\otimes E_{-i,i})$$
$$- (q^{-1}-1)\sum_{i=1}^{n}(E_{-i,-i}\otimes E_{-i,-i}) + (q-q^{-1})\sum_{i=-n}^{-1}(E_{-i,-i}\otimes E_{ii})$$
$$+ (q-q^{-1})\sum_{|j|<|i|}(E_{jj}\otimes E_{ii}) + (q-q^{-1})\sum_{|j|<|i|}\big((-1)^{p(i)p(j)}E_{ji}\otimes E_{-j,-i}\big)$$

□

6.2 Centralizer of $\mathfrak{U}_q(\mathfrak{p})_n$-action on $\mathbb{C}_q(n|n)^{\otimes k}$

As mentioned after the definition of $\mathfrak{B}_{q,l}$, the module structure given in Proposition 6.1.4 commutes with the action of $\mathfrak{U}_q(\mathfrak{p}_n)$ on $\mathbb{C}_q(n|n)^{\otimes l}$. This, as a result, gives us the algebra homomorphisms

$$\mathfrak{B}_{q,l} \longrightarrow \mathrm{End}_{\mathfrak{U}_q(\mathfrak{p}_n)}(\mathbb{C}_q(n|n)^{\otimes l}) \text{ and } \mathfrak{U}_q(\mathfrak{p}_n) \longrightarrow \mathrm{End}_{\mathfrak{B}_{q,l}}(\mathbb{C}_q(n|n)^{\otimes l}). \tag{6.12}$$

The following main theorem states that $\mathfrak{B}_{q,l}$ is the full centralizer of the action of $\mathfrak{U}_q(\mathfrak{p}_n)$ on $\mathbb{C}_q(n|n)^{\otimes l}$ when $n \geq l$.

Theorem 6.2.1. *The map $\mathfrak{B}_{q,l} \longrightarrow \mathrm{End}_{\mathfrak{U}_q(\mathfrak{p}_n)}(\mathbb{C}_q(n|n)^{\otimes l})$ is surjective and it is injective when $n \geq l$.*

Proof. This is a q-analogue of Theorem 4.5 in [42]. The proof follows the lines of the proof of Theorem 3.28 in [6], using Proposition 6.2.2 and Theorem 4.5 in [42] along with Lemma 3.27 in [6], which can be applied in this instance. □

Proposition 6.2.2. *The quotient algebra $\mathfrak{B}_{q,l}(\mathcal{A})/(q-1)\mathfrak{B}_{q,l}(\mathcal{A})$ is isomorphic to the algebra A_l given in Definition 2.2 in [42].*

Proof. It follows immediately from the definitions of both A_l and $\mathfrak{B}_{q,l}(\mathcal{A})$ that we have a surjective algebra homomorphism $A_l \twoheadrightarrow \mathfrak{B}_{q,l}(\mathcal{A})/(q-1)\mathfrak{B}_{q,l}(\mathcal{A})$. The fact that this homomorphism is injective can be proved as in the proof of Proposition 3.21 in [6] using Theorem 4.1 in [42]. □

The q-Schur superalgebras of type Q were introduced in [6] and [24, 25]. Considering *loc. cit.* and the earlier work on q-Schur algebras for $\mathfrak{gl}_n$ (see for instance [23]), the following definition is natural.

Definition 6.2.1. The q-Schur superalgebra $S_q(\mathfrak{p}_n, l)$ of type P is the centralizer of the action of $\mathfrak{B}_{q,l}$ on $\mathbb{C}_q(n|n)^{\otimes l}$, that is, $S_q(\mathfrak{p}_n, l) = \mathrm{End}_{\mathfrak{B}_{q,l}}(\mathbb{C}_q(n|n)^{\otimes l})$.

By definition, (6.12) gives an algebra homomorphism $\mathfrak{U}_q(\mathfrak{p}_n) \longrightarrow S_q(\mathfrak{p}_n, l)$. It is an open question whether or not this map is surjective. We also have an algebra homomorphism $\mathfrak{B}_{q,l} \longrightarrow \mathrm{End}_{S_q(\mathfrak{p}_n,l)}(\mathbb{C}_q(n|n)^{\otimes l})$ from (6.12). It is natural to expect that it should be an isomorphism, perhaps under certain conditions on n and l, which is also open.

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BIOGRAPHICAL STATEMENT

Saber Ahmed was born in Mogadishu, Somalia, in 1995. He received his B.S. degrees in both Mathematics and Chemistry from the University of Texas at Austin in 2017. He has been a PhD student in mathematics at the University of Texas at Arlington since obtaining B.S. degrees in 2017, up until 2022. His research interests focuses on the representation theory of Lie superalgebras and quantized Lie superalgebras.