EXPONENTIAL TENSOR MODULES

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Presented to the Faculty of the Graduate School of

The University of Texas at Arlington

in Partial fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

The University of Texas at Arlington

May 2022

ACKNOWLEDGEMENTS

I would like to thank my advisor Dr. Dimitar Grantcharov for his guidance and continual support. At times, we discussed mathematics, talked about many things outside of math, and even watched basketball together, for which I'm forever grateful. I appreciate your patience in guiding me through the ups and downs of this journey.

Also, it is my honor to have Dr. David Jorgensen, Dr. Michaela Vancliff, Dr. Ruth Gornet, and Dr. Gaik Ambartsoumian on my committee. I am very lucky to be in your classes in the past, and I have learned so much mathematics from you all.

I want to acknowledge the faculty and staff in the Department of Mathematics at the University of Texas at Arlington, which provided such a great working environment during my time here. I also want to thank my fellow students for their support and meaningful conversations.

I would like to thank my parents, Noi Nguyen and Thao Hoang, and my sister, Linh Nguyen, for their unconditional love and support.

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Abstract

Representation theory of Lie algebra of a finite dimensional reductive Lie algebra $\mathfrak{g}$ is a long-standing problem. The ultimate goal is to classify all representations of $\mathfrak{g}$. However. the only case only case when a complete classification is obtained is the case of $\mathfrak{g} = \mathfrak{sl}(2)$, [5]. Hence, it is natural to study certain categories of representations of $\mathfrak{g}$ for which some finiteness conditions on the action of certain elements of $\mathfrak{g}$ is enforced. In this thesis, we introduce a class of representations $T(g, V, S)$ of $\mathfrak{sl}(n+1)$ of mixed tensor type. By varying the polynomial g, the $\mathfrak{gl}(n)$-module V, and the set S, we obtain important classes of weight representations over the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(n+1)$, and representations that are free over $\mathfrak{h}$. Moreover, An isomorphism theorem and simplicity criterion for $T(g, V, S)$ is provided.

Contents

Chapter 1

Introduction

There are two important, but opposite in nature, categories of modules of finite-dimensional reductive Lie algebras $\mathfrak{a}$. The first one consists of weight modules, namely, those that decompose into direct sums of their weight spaces relative to a fixed Cartan subalgegra $\mathfrak{h}$. The second one is the category of $\mathfrak{h}$-free modules. The classification of the simple objects in these categories is far from reach unless one imposes an additional finiteness condition. In particular, simple weight $\mathfrak{a}$-modules with finite weight multiplicities have been classified by O. Mathieu, [13], following works of G. Benkart, D. Britten, S. Fernando, V. Futorny, A. Joseph, F. Lemire, and others. On the other hand, the classification of all simple $\mathfrak{h}$-free modules of finite-rank is still an open problem and the only known case is when the rank equals one, [15].

An important part of Mathieu's breakthrough paper [13] is the new notion of *coherent family* - a "big" weight module whose support coincides with the whole $\mathfrak{h}^*$. Coherent families have explicit geometric realizations via sections of vector bundles of algebraic varieties (called tensor coherent families), and also can be constructed purely algebraically through twisted localization of highest weight modules. The geometric realization is especially convenient in the case of $\mathfrak{sl}(n+1)$, when the coherent families are direct sums of tensor products $T(P,V) = P \otimes V$ of mixed type. More precisely, P is a module over the algebra $\mathcal{D}(n)$ of polynomial differential operators of $\mathcal{O} = \mathbb{C}[t_1, ..., t_n]$ and V is a module over the Lie algebra $\mathfrak{gl}(n)$. As a result, we have an explicit presentation of the root elements of $\mathfrak{sl}(n+1)$ in terms of differential operator presentation. The tensor modules of mixed type $T(P, V)$ were introduced by Shen, [18] and Rudakov, [17], and play important role in the representation theory of various Lie algebras of derivations and vector fields, see for example, [4], [10], [12], [19], [20].

Throughout the thesis we fix $\mathfrak{s} = \mathfrak{sl}(n+1)$. One of the tools used in the study of $\mathfrak{h}$-free $\mathfrak{s}$-modules is the weighting functor $\mathcal{W}$. This functor maps an $\mathfrak{h}$-free module M of finite rank to a coherent family $\mathcal{M}$ and raises a natural question about further connections between the categories of weight and $\mathfrak{h}$-free modules. A main purpose of the present thesis is to make such connection and in particular to combine both types of modules together. This is done thanks to applying two functors on the tensor modules $T(\mathcal{O}, V)$. The two functors are exponentiation $\exp_g$ by a polynomial g, and a Fourier transform ψ_S relative to a subset S of $\{1, 2, ..., n\}$. As a result we define the *exponential tensor modules* $T(g, V, S)$. The case of $g = 0$ and improper set S is closely related to the geometric realization of Mathieu's tensor coherent families. When $g = 0$, by varying S we obtain all injective partly-irreducible coherent families. In particular, every simple bounded $\mathfrak{s}$-module appears as a submodule of some coherent family of general type.

The case when g has degree 1 and V is 1-dimensional leads to the complete list of $\mathfrak{h}$-free modules of rank 1. Furthermore, we obtain interesting classes of Whittaker modules and weight modules relative to other Cartan subalgebras. Also, connections with the weighting functor and Witten deformation of the de Rham complex are discovered. Two of the main results in the present paper are a simplicity criterion and an isomorphism theorem for $T(g, V, S)$. In particular, we provide new families of simple non-weight modules over $\mathfrak{sl}(n+1)$ obtained through explicit presentation of the Lie algebra in terms of differential operators. The main tools used in the proofs of these two results are the twisted localization and translation functors.

Chapter 2

Preliminary Concepts

All vector spaces, algebras, and tensor products are assumed to be over $\mathbb{C}$ unless otherwise stated. We set $[\![k]\!] = \{1, 2, ..., k\}$ for a positive integer k. By $\mathbb{Z}_{\geq k}$ we denote the set of all integers i such that $i \geq k$. We define $\mathbb{Z}_{>k}$, $\mathbb{Z}_{\leq k}$, and $\mathbb{Z}_{<k}$ similarly. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.
By S_k we denote the symmetric group of k letters.

2.1 Lie algebras

Definition 2.1.1. *An $\mathbb{F}$-Lie algebra is a pair $(\mathfrak{g}, [\cdot, \cdot])$ consisting of a vector space $\mathfrak{g}$ over a field $\mathbb{F}$ and a (Lie) bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$, which is a bilinear map and satisfies the following conditions:*

1. $[x, y] = -[y, x]$ *(skew symmetry),*

2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *(Jacobi identity)*
   $\forall a, b \in \mathbb{F}$ *and* $\forall x, y, z \in \mathfrak{g}$.

Remark 1. *A Lie algebra can be defined over any arbitrary field $\mathbb{F}$. However, the ground field $\mathbb{F}$ will be $\mathbb{C}$ throughout the rest of this thesis.*

For a vector space V, we denote $\mathrm{End}(V)$ the algebra of endomorphisms of V. $\mathrm{End}(V)$ is a Lie algebra when paired with the commutator as the Lie bracket. We denote this Lie algebra by $\mathfrak{gl}(V)$.

A special case of $\mathfrak{gl}(V)$ is when V is a finite dimensional vector space of dimension n over $\mathbb{C}$. By fixing a basis of V, $\mathfrak{gl}(V)$ can be identified as the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$, where $\mathfrak{gl}(n,\mathbb{C}) = \{n \times n$ matrices with entries in $\mathbb{C}\}$. $\mathfrak{gl}(n,\mathbb{C})$ is called the *general linear Lie algebra*. The basis of $\mathfrak{gl}(n,\mathbb{C})$ is $\{E_{ij} : 1 \leq i,j \leq n\}$, where E_{ij} is the elementary $n \times n$ matrix with 1 in the ij position and 0 elsewhere. Moreover, the Lie bracket on its basis elements can be computed explicitly using the following formula:

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj},$$

where $\delta_{i,j}$ is the Kronecker delta function.

Remark 2. *In general, we can define a Lie algebra structure on A, where A is an associative algebra over a field. By letting $[x,y] = xy - yx$, where xy is the multiplication structure on A, $(A,[.,.])$ becomes a Lie algebra.*

Example 2.1.2 (Witt algebra and Virasoro algebra)**.** *Let $\mathcal{W}_n$ be vector space with basis $\{l_m | m \in \mathbb{Z}\}$ together with the Lie bracket defined as $[l_m, l_n] = (m-n)l_{m+n}$ for all $m,n \in \mathbb{Z}$. $\mathcal{W}_n$ is an infinite dimensional Lie algebra, which is called the Witt algebra.*

Let $\mathcal{V}$ be a vector space with basis $\{l_m | m \in \mathbb{Z}\} \cup \{c\}$, where the Lie bracket is defined as following:

$$[l_m, l_n] = (m-n)l_{m+n} + \delta_{m+n,0}\tfrac{c}{12}(m^3 - m),$$
$$[c, \mathcal{V}] = 0.$$

$\mathcal{V}$ is the central extension of $\mathcal{W}_n$, which is called the Virasoro algebra.

Witt algebra and Virasoro algebra have many applications in Physics.

Definition 2.1.3. *A vector subspace* $\mathfrak{h}$ *of* $\mathfrak{g}$ *is a* Lie subalgebra *if* $[x,y] \in \mathfrak{h}$ *for all* $x, y \in \mathfrak{h}$.

Since $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$ for every $x, y \in \mathfrak{gl}(n,\mathbb{C})$, $\mathfrak{sl}(n,\mathbb{C}) = \{n \times n$ traceless matrices with entries in $\mathbb{C}\}$ is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{C})$. The Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ is called the *special linear algebra.*

Remark 3. *For the sake of simplicity, we denote* $\mathfrak{gl}(n,\mathbb{C})$ *by* $\mathfrak{gl}(n)$ *,and* $\mathfrak{sl}(n,\mathbb{C})$ *by* $\mathfrak{sl}(n)$.

In particular, The Lie algebra $\mathfrak{sl}(2)$ will be used frequently throughout this thesis as examples. $\mathfrak{sl}(2)$ has the following standard basis:

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\ \text{and } e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Nonetheless, the Lie bracket with respect to the above basis is given by the following formulas:

$$[h,e] = 2e, [h,f] = -2f, [e,f] = h.$$

Definition 2.1.4. *An* ideal $\mathfrak{i}$ *of a Lie algebra* $\mathfrak{g}$ *is a subalgebra of* $\mathfrak{g}$ *such that* $[x,y] \in \mathfrak{i}$ *for all* $x \in \mathfrak{i}, y \in \mathfrak{g}$.

Example 2.1.5. $\mathfrak{sl}(n)$ *is an ideal of* $\mathfrak{gl}(n)$.

Let A be a subset of $\mathfrak{g}$. The set of elements in $\mathfrak{g}$ that commute with all elements a in A is called the *centralizer* of A in $\mathfrak{g}$. We denote this set by $C_{\mathfrak{g}}(A)$. Nonetheless, the centralizer of $\mathfrak{g}$ in $\mathfrak{g}$ is called the *center* of $\mathfrak{g}$. By definition, the *center* of a Lie algebra $\mathfrak{g}$ is an ideal of $\mathfrak{g}$.

Example 2.1.6. *The center of* $\mathfrak{gl}(n)$ *is the set of all scalar multiple of the identity matrix* $\mathcal{I}_n$.

Definition 2.1.7. *The derived series of a Lie algebra* $\mathfrak{g}$ *is the sequence of ideals* $\mathfrak{g}^{(0)} = \mathfrak{g}, \mathfrak{g}^{(1)} = [\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}], ..., \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}]$. *The* lower central series *of a Lie algebra* $\mathfrak{g}$ *is the sequence of ideals* $\mathfrak{g}^0 = \mathfrak{g}, \mathfrak{g}^1 = [\mathfrak{g}^0, \mathfrak{g}^0], ..., \mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}^{k-1}]$.

A lie algebra $\mathfrak{g}$ is *nilpotent* if there exists $k \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{g}^k = 0$. A lie algebra $\mathfrak{g}$ is *solvable* if there exists $k \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{g}^{(k)} = 0$. By definition, $\mathfrak{g}^{(k)} \subset \mathfrak{g}^k$. Thus, every nilpotent Lie algebra is sovable. We denote $\mathrm{Rad}(\mathfrak{g})$ the largest solvable ideal of $\mathfrak{g}$, which is called the *radical* of $\mathfrak{g}$.

One example of nilpotent lie algebra is the subalgebra of $\mathfrak{gl}(n)$ consisting of all strictly upper triangular matrices, which is denoted by $\mathfrak{n}(n)$. Note that $\mathfrak{n}(n)$ is also a solvable Lie algebra. However, The lie subalgebra consisting of all upper triangular matrices $\mathfrak{t}(n)$ is solvable only.

Definition 2.1.8. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras. A linear map $\phi : \mathfrak{g} \to \mathfrak{g}'$ is a* homomorphism *of Lie algebras if $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}$.*

One crucial example of Lie algebra homomorphisms is $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$. This map together with the vector space V is a representation of $\mathfrak{g}$, which will be studied throughout this thesis.

Definition 2.1.9. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. The* normalizer *of $\mathfrak{h}$ is the subalgebra $N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} | [x, \mathfrak{h}] \subset \mathfrak{h}\}$. A subalgebra $\mathfrak{h}$ is* self-normalizing *if $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.*

2.2 Semisimple Lie algebras, Root systems, and their properties

Definition 2.2.1. *A non-abelian Lie algebra $\mathfrak{g}$ is* simple *if the only ideals of $\mathfrak{g}$ are 0 and $\mathfrak{g}$.*

A lie algebra $\mathfrak{g}$ is *semisimple* if $\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{g}_i$, where $\mathfrak{g}_i$ is simple for all i. In fact, if $\mathrm{Rad}\, g = 0$, $\mathfrak{g}$ is semisimple.

The lie algebra $\mathfrak{gl}(n)$ is not semisimple since $\mathfrak{gl}(n) = \mathfrak{sl}(n) \oplus \mathbb{C}\mathcal{I}_n$. However, $\mathfrak{sl}(n)$ is simple and thus, it is also semisimple.

A lie algebra $\mathfrak{g}$ comes with a symmetric, bilinear form called the *Killing form*, which is defined as follow:

$$\mathbf{k}(x,y) := \mathrm{Tr}([x,[y,\_]]), \text{ for } x,y \in \mathfrak{g}.$$

Note that $\mathbf{k} : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ is also associative in the sense that $\mathbf{k}([x,y],z) = \mathbf{k}(x,[y,z]) \forall x,y,z \in \mathfrak{g}$.

The killing form $\mathbf{k}$ is non-degenerate if and only if $\mathfrak{g}$ is semisimple. This condition is known as a *Cartan criterion.*

Definition 2.2.2. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. $\mathfrak{h}$ is called a* Cartan subalgebra *if $\mathfrak{h}$ is a nilpotent self-normalizing subalgebra of $\mathfrak{g}$.*

In the case $\mathfrak{g} = \mathfrak{gl}(n)$, it is easy to see that the Cartan subalgebra $\mathfrak{h}$ is the one, which contains all diagonal matrices of $\mathfrak{g}$. Specifically, $\mathfrak{h} = \mathrm{Span}_{\mathbb{C}}\{E_{ii} : 1 \leq i \leq n\}$.

We denote $\mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h},\mathbb{C}) = \{f : \mathfrak{h} \to \mathbb{C} : f$ is an $\mathbb{C}$-linear map$\}$. For $\alpha \in \mathfrak{h}^*$, we denote $\mathfrak{g}_\alpha = \{g \in \mathfrak{g} : [h,g] = \alpha(h)g, \forall h \in \mathfrak{h}\}$.

Remark 4. *For $\mathfrak{g}$ is a semi-simple Lie algebra over $\mathbb{C}$, the Cartan subalgebra $\mathfrak{h}$ acts diagonally on $\mathfrak{g}$. Moreover, $\mathfrak{h}$ is abelian. Hence, we have the following decomposition*

$$\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{h}^*} \mathfrak{g}_\alpha.$$

Definition 2.2.3. *The* root system *of a Lie algebra $\mathfrak{g}$ is the set $\Phi = \{\alpha \in \mathfrak{h}^* : \mathfrak{g}_\alpha \neq 0\}$.*

For each $\alpha \in \Phi$, $\mathfrak{g}_\alpha$ is callled a *root space* corresponding to the root α. A nonzero element in $\mathfrak{g}_\alpha$ is called a *root vector.* We note that $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ since $\mathfrak{g}_0 = \mathfrak{h}$. For $\alpha \in \Phi$, we pick $e_\alpha \in \mathfrak{g}_\alpha, f_\alpha \in \mathfrak{g}_{-\alpha}$, and let $h_\alpha := [e_\alpha, f_\alpha]$ such that e_α, f_α, and h_α form a standard basis of $\mathfrak{sl}(2)$. Note that $\alpha(h_\alpha) = 2$.

Theorem 2.2.1 (Properties of Φ). *Let $\alpha, \beta \in \Phi$ such that $\beta \notin \{\pm\alpha\}$, we have the following:*

1. $\beta(h_\alpha) \in \mathbb{Z}$;

2. $\beta - \beta(h_\alpha)\alpha \in \Phi$;

3. $\mathbb{C}\alpha \cap \Phi = \{\pm\alpha\}$;

4. *if* $\alpha + \beta \in \Phi$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$;

5. Φ *spans* $\mathfrak{h}^*$.

Definition 2.2.4. *Define the operator $\langle .,. \rangle : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ by $\langle \alpha, \beta \rangle = 2\frac{(\alpha,\beta)}{(\beta,\beta)}$, where $(.,.)$ is the standard inner product on the vector space $\mathfrak{h}^*$.*

Definition 2.2.5. *Let σ_α be the reflection of $\mathfrak{h}^*$ across the hyperplane orthogonal the root α, defined by $\lambda \mapsto \lambda - \langle \lambda, \alpha \rangle \alpha$.*

Definition 2.2.6. *The set $W = \{\sigma_\alpha : \alpha \in \Phi\}$ is called the* Weyl group *of Φ.*

We denote $\rho = \frac{1}{2}\sum_{\alpha \in \phi^+} \alpha$. By definition, W is a subgroup of $GL(\mathfrak{h}^*)$. We define the *dot action* of W on $\mathfrak{h}^*$ as $w \cdot \lambda = w(\lambda + \rho) - \rho$. We denote the stabilizer of λ via the dot action by $W_\lambda = \{w \in W : w \cdot \lambda = \lambda\}$.

Definition 2.2.7. *Let $\pi \subset \Phi$. π is called a* base *of Φ if each root $\beta \in \Phi$ can be expressed uniquely as $\pm \sum_{\alpha \in \pi} c_\alpha \alpha$, where $c_\alpha \in \mathbb{Z}_{\geq 0}$.*

We denote $\Phi^+ := \{\alpha \in \Phi : \alpha \in \mathbb{Z}_{\geq 0}\Phi\}$ and $\Phi^- := \{\alpha \in \Phi : \alpha \in \mathbb{Z}_{\leq 0}\Phi\}$. Φ^+ and Φ^- are called the set of positive and negative roots, respectively. Note that the root system Φ can be written as the disjoint union $\Phi = \Phi^+ \sqcup \Phi^-$.

By letting $\mathfrak{n}_- := \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha$ and $\mathfrak{n}_+ := \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$, we obtain a triangular decomposition of our semi-simple complex Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

This triangular decomposition depends on π and $\mathfrak{h}$.

Definition 2.2.8. *Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$. $\mathfrak{b}$ is called the* Borel *subalgebra of $\mathfrak{g}$ with respect to π and $\mathfrak{h}$.*

As an example, we consider the following root system:

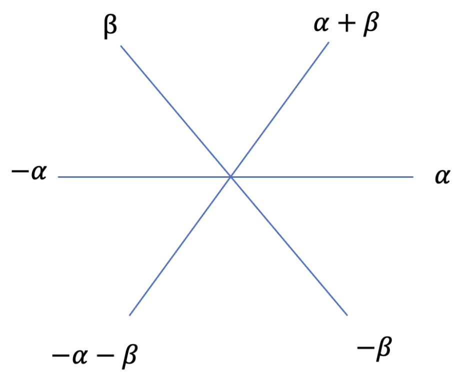


Figure 2.1: Root system of type A_2

The base of the above root system is $\pi = \{\alpha, \beta\}$. The set of positive roots is $\Phi^+ = \{\alpha, \beta, \alpha + \beta\}$ and the set of negative roots is $\Phi^- = \{-\alpha, -\beta, -\alpha - \beta\}$.

Given $\mathfrak{g} = \mathfrak{sl}(n)$, a common basis of $\mathfrak{sl}(n)$ is $\{E_{ij}, E_{kk} - E_{k+1,k+1} : 1 \leq i, j \leq n, i \neq j, 1 \leq k \leq n-1\}$. Note that a Cartan subalgebra of $\mathfrak{sl}(n)$ with respects to the above basis can be realized as $\mathrm{Span}_{\mathbb{C}}\{E_{kk} - E_{k+1,k+1} : 1 \leq k \leq n-1\}$. Let $\{\epsilon_i : 1 \leq i \leq n\}$ be dual basis of $\{E_{ii} : 1 \leq i \leq n\}$, where $\epsilon_i(E_{jj}) = \delta_{ij}$. We have the following information of the Lie algebra $\mathfrak{sl}(n)$:

- Root system: $\Phi = \{\varepsilon^i - \varepsilon^j : i \neq j\}$.
- $\Phi_+ = \{\varepsilon^i - \varepsilon^j : i < j\}$, $\Phi_- = \{\varepsilon^i - \varepsilon^j : i > j\}$.
- Basis $\pi = \{\varepsilon^i - \varepsilon^{i+1} : i = 1, ..., n-1\}$.

- Root subspaces:

$$(\mathfrak{sl}(n))_{\varepsilon^i - \varepsilon^j} \quad := \quad \begin{cases} \mathrm{Span}_{\mathbb{C}}\{e_{ij}\}, & i < j \\ \\ \mathrm{Span}_{\mathbb{C}}\{e_{ji}\}, & i > j \end{cases}$$

- Weyl group: $W = \mathcal{S}_n$ (permutations of the ε^i).

<u>$\mathfrak{sl}(2)$ **case**</u>

- $\mathfrak{sl}(2) = \mathrm{Span}_{\mathbb{C}}\{f, h, e\}$, where f, h, e are defined as above.

- $\mathfrak{n}_- = \mathrm{Span}_{\mathbb{C}}\{f\}, \mathfrak{h} = \mathrm{Span}_{\mathbb{C}}\{h\}, \mathfrak{n}_+ = \mathrm{Span}_{\mathbb{C}}\{e\}$.

- $\Phi = \{\pm\alpha\}$, where $\alpha(h) = 2$.

- $\pi = \alpha$.

Remark 5. *In the case that $\mathfrak{g} = \mathfrak{sl}(2)$, $\mathfrak{h}^* \cong \mathbb{C}$. It is convenient to identify $\mathfrak{h}^*$ as $\mathbb{C}$ and the simple root α as 2.*

2.3 Lie algebra Modules

Definition 2.3.1. *A representation of a Lie algebra $\mathfrak{g}$ is a pair (ρ, V), where V is a vector space and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra homomorphism.*

A $\mathfrak{g}$-representation (ρ, V) could be seen as $\mathfrak{g}$-module V with action $x \cdot v = \rho(x)(v)$. In particular, a vector space V together with the bilinear action $\mathfrak{g} \times M \to M$ defined by $(g, m) \mapsto g \cdot m$, is a $\mathfrak{g}$-module if

$$[x, y] \cdot m = x \cdot y \cdot m - y \cdot x \cdot m, \text{ for all } x, y \in \mathfrak{g} \text{ and } m \in M.$$

Remark 6. *Through out the rest of this thesis, module language will be used instead of representation.*

Example 2.3.2. *Let* $\mathfrak{g} = \mathfrak{gl}(n)$ *and* $V = \mathbb{C}^n$*, we define the* natural module *by letting each element in* $\mathfrak{gl}(n)$ *acts on* $\mathbb{C}^n$ *by the normal matrix multiplication. This module structure also holds for any Lie subalgebras of* $\mathfrak{gl}(n)$*.*

Example 2.3.3. *Let* $\mathfrak{g}$ *be any Lie algebra. The* Adjoint module *of* $\mathfrak{g}$ *is defined as* $x \cdot y = ad_x(y) = [x, y] \forall x, y \in \mathfrak{g}$.

A subspace W of a $\mathfrak{g}$-module V is a *submodule* if W is invariant under the action of $\mathfrak{g}$, namely $\mathfrak{g}W \subset W$. By default, $\{0\}$ and V are submodules of V. Moreover, If V contains no other submodules beside $\{0\}$ and V, then V is a *simple*, or *irreducible*. On one hand, A module V is said to be *semi-simple* or *completely reducible* if V can be written as dirrect sum of simple modules. On the other hand, V is an *indecomposible* module if it is not the direct sum of proper submodules.

Example 2.3.4. *Let* $\mathfrak{g} = \mathfrak{gl}(n)$*, and* $V = \mathbb{C}[x_1, x_2, ..., x_n]$*.* V *is a* $\mathfrak{g}$*-module, where the* $\mathfrak{g}$*-action can be defined as* $E_{ij} \cdot f(x_1, x_2, ..., x_n) = x_i \frac{\partial f}{\partial x_j}$*, for all* $f(x_1, x_2, ..., x_n) \in \mathbb{C}[x_1, x_2, ..., x_n]$*. Note that* V *is a completely reducible module since* $V = \bigoplus_{i=0}^{\infty} \mathbb{C}[x_1, x_2, ..., x_n]_i$*, where* $\mathbb{C}[x_1, x_2, ..., x_n]_i$ *is a space of homogeneous polynomials of degree i.*

Definition 2.3.5. *Let* V *and* W *be* $\mathfrak{g}$*-module. A* $\mathfrak{g}$-module homomorphism *is a linear map* $\phi : V \to W$ *such that* $\phi(g \cdot v) = g \cdot \phi(v)$*, for all* $g \in \mathfrak{g}, v \in V$*.*

Definition 2.3.6. *Let* W *be a submodule of* V *over a Lie algebra* $\mathfrak{g}$*. The* quotient vector space V/W *is a* $\mathfrak{g}$*-module where the* $\mathfrak{g}$*-action is defined as* $x \cdot (v + W) = x \cdot v + W$*.*

2.3.1 The universal enveloping algebra

Let (V, φ) be a representation of Lie algebra $\mathfrak{g}$ with $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra homomorphism. It is difficult to study representations of Lie algebra since $\mathrm{im}(\varphi)$ is not closed under composition of linear maps. Hence, it is natural to assign an object with associative-structure to each Lie algebra $\mathfrak{g}$ such that Lie bracket is encoded as commutator.

We recall that any associative algebra A is a Lie algebra with $[x,y] = xy - yx\ \forall x, y \in U$. We denote this Lie algebra $(A, [.,.])$.

Definition 2.3.7. *The universal enveloping algebra of a Lie algebra* $\mathfrak{g}$ *is defined as an associative algebra* U *together with a Lie algebra homomorphism* $\iota : \mathfrak{g} \to (U, [.,.])$ *such that for an associative algebra* A *and any Lie algebra homomorphism* $\alpha : \mathfrak{g} \to (A, [.,.])$*, there exist a unique homomorphism of associative algebra* $\gamma : U \to A$*, such that* $\alpha = \gamma \circ \iota$*. That is, the following diagram is a commutative diagram:*

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\iota"] \arrow[d, "\alpha"'] & U \arrow[dl, "\gamma"] \\
A &
\end{tikzcd}$$

For a $\mathbb{C}$-vector space V, we define the *tensor algebra* $T(V)$ of V as the direct sum $T(V) := \bigoplus_{k=0}^{\infty} V^{\otimes k}$ where $V^{\otimes 0} := F, V^{\otimes k} := V \otimes \ldots \otimes V$(k-copies of V). Note that $T(V)$ is an associative algebra over $\mathbb{C}$.

Construction of the universal enveloping algebra: for a Lie algebra $\mathfrak{g}$, the universal enveloping algebra $\mathcal{U}(\mathfrak{g}) := T(\mathfrak{g})/J$, where J is a 2-sided ideal generated by $\{x \otimes y - y \otimes x - [x,y] : x, y \in \mathfrak{g}\}$.

Remark 7. *Let* $\mathfrak{h}$ *be a Cartan subalgebra of* $\mathfrak{g}$ *with basis* $\{h_1, \ldots, h_k\}$*.* $\mathcal{U}(\mathfrak{h}) \cong \mathbb{C}[h_1, \ldots, h_k]$ *as associative algebras. This observation will be used frequently in this thesis.*

Theorem 2.3.1 (**Poincare-Birkhoff-Witt theorem**)**.** *Let* $(g_1, g_2, \ldots, g_n)$ *be an ordered basis of* $\mathfrak{g}$*. Then* $\{g_1^{k_1} g_2^{k_2} \ldots g_n^{k_n} : k_i \in \mathbb{Z}_{\geq 0}\}$ *forms a basis of* $\mathcal{U}(\mathfrak{g})$*.*

Example 2.3.8. *Let* $\mathfrak{g} = \mathfrak{sl}(2)$*, the basis of* $\mathcal{U}(\mathfrak{sl}(2))$ *is* $\{f^i h^j e^k : i, j, k \in \mathbb{Z}_{\geq 0}\}$.

We denote the centralizer of $\mathfrak{h}$ in $\mathcal{U}(\mathfrak{g})$ by $\mathcal{U}(\mathfrak{g})^0 := \{u \in \mathcal{U}(\mathfrak{g}) : [h, u] = 0 \forall h \in \mathfrak{h}\}$. By definition, $\mathcal{U}(\mathfrak{g})^0$ is an associative and unital subalgebra of $\mathcal{U}(\mathfrak{g})$. The *center* of $\mathcal{U}(\mathfrak{g})$ is denoted by $\mathcal{Z}(\mathfrak{g})$. By definition of $\mathcal{Z}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})^0$, $\mathcal{Z}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})^0$.

Example 2.3.9. *In the case* $\mathfrak{g} = \mathfrak{sl}(2)$*, we denote* $c := (h+1)^2 + 4fe \in \mathcal{U}(\mathfrak{sl}(2))$*. It can be checked by direct computation that* $c \in \mathcal{Z}(\mathfrak{sl}(2))$*. Nonetheless, this element* c *is called the* Casimir *element of the Lie algebra* $\mathfrak{sl}(2)$*. Moreover,* $\mathcal{Z}(\mathfrak{sl}(2)) = \mathbb{C}[c]$ *and* $\mathcal{U}(\mathfrak{sl}(2))^0 = \mathbb{C}[c, h]$.

2.3.2 Weight modules and highest weight modules

Definition 2.3.10. *An* $\mathfrak{g}$*-module* M *is a* weight module *if* $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda$*, where* $M^\lambda = \{m \in M \mid h \cdot m = \lambda(h)m, \text{ for every } h \in \mathfrak{h}\}$.

The *weight space* of M corresponds to λ is M^λ and $\dim M^\lambda$ is the multiplicity of the weight λ. For a weight module M, we denote the *support* of M as $\mathrm{Supp}(M) := \{\lambda \in \mathfrak{h}^* : M^\lambda \neq 0\}$. In fact, $M = \bigoplus_{\lambda \in \mathrm{Supp}(M)} M^\lambda$.

Lemma 2.3.2. *For any* $\alpha \in \Phi$*, and* $\lambda \in \mathfrak{h}^*$*,* $\mathfrak{g}_\alpha M^\lambda \subset M^{\lambda+\alpha}$.

Proof. Let $v \in M^\lambda$, $g \in \mathfrak{g}_\alpha$, and $h \in \mathfrak{h}$.
$h \cdot (g \cdot v) = g \cdot (h \cdot v) + [h, g] \cdot v = g \cdot \lambda(h)v + \alpha(h)g \cdot v = (\lambda + \alpha)(h)g \cdot v$
Hence, $g \cdot v \in M^{\lambda+\alpha}$. □

We denote the space $\mathbb{Z}\Phi$ of all $\mathbb{Z}$-linear combinations of roots, which is called the *root lattice* corresponds to the root system Φ. Note that $\mathbb{Z}\Phi$ forms an additive subgroup of $\mathfrak{h}^*$. From the above lemma, M is simple if $\mathrm{Supp}(M)$ is in a single coset of $\mathfrak{h}^*/\mathbb{Z}\Phi$.

In the case that $\mathfrak{g} = \mathfrak{sl}(2)$, the root lattice is $2\mathbb{Z}$.

Example 2.3.11. *Adjoint module of* $\mathfrak{g}$ *is a weight module if* $\mathfrak{g}$ *is a semi-simple Lie algebra over* $\mathbb{C}$ *since* $\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{h}^*} \mathfrak{g}_\alpha$*, where* $\mathfrak{g}_\alpha = \{g \in \mathfrak{g} : [h, g] = \alpha(h)g, \forall h \in \mathfrak{h}\}$. *In this example,* $\operatorname{Supp}\mathfrak{g} = \Phi \cup \{0\}$.

Moreover, The adjoint module $\mathfrak{g}$ on $\mathcal{U}(\mathfrak{g})$ is a weight module since the adjoint action of $\mathfrak{h}$ on $\mathcal{U}(\mathfrak{g})$ is diagonalizable. In the case of $\mathfrak{g} = \mathfrak{sl}(2)$, the adjoint action of $\mathfrak{h}$ on a basis element of $\mathcal{U}(\mathfrak{sl}(2))$ can be computed explicitly as follow:

$$[h, f^i h^j e^k] = 2(k-i) f^i h^j e^k.$$

However, the regular module of $\mathfrak{g}$ on $\mathcal{U}(\mathfrak{g})$ is *not* a weight module since $\mathfrak{h}$ acts freely on $\mathcal{U}(\mathfrak{g})$ by PBW theorem.

Example 2.3.12. *Let* $\xi \in \mathbb{C}/2\mathbb{Z}$ *and* $a \in \mathbb{C}$. *We consider the vector space* $V(\xi, a)$ *with basis* $\{v_\mu : \mu \in \xi\}$. $V(\xi, a)$ *is a weight module of* $\mathfrak{sl}(2)$ *under the following action:*

$$\begin{aligned} e \cdot v_\mu &= \frac{1}{4}(a - (\mu+1)^2) v_{\mu+2}, \\ f \cdot v_\mu &= v_{\mu-2}, \\ h \cdot v_\mu &= \mu v_\mu. \end{aligned}$$

Note that $\operatorname{Supp}(V(\xi, a)) = \xi$, where $\xi \in \mathbb{C}/2\mathbb{Z}$. This module is usually called a *dense module*. Nonetheless, any simple weight $\mathfrak{sl}(2)$-module is a subquotient of some $V(\xi, a)$, which is described in detail in [14].

Definition 2.3.13. *For* $\lambda, \mu \in \mathfrak{h}^*$, *we define the following partial order* $\lambda \le \mu$ *if* $\mu - \lambda \in \mathbb{Z}_{\ge 0}\Phi^+$.

Definition 2.3.14. *Let* M *be a weight module, and* $\lambda \in \operatorname{Supp}(M)$. *If* λ *is maximal with respects to the above partial order in* $\operatorname{Supp}(M)$, *then* λ *is called the* highest weight *of* M. *Nonetheless,* M *is called a* highest weight module *of the weight* λ *in this case.*

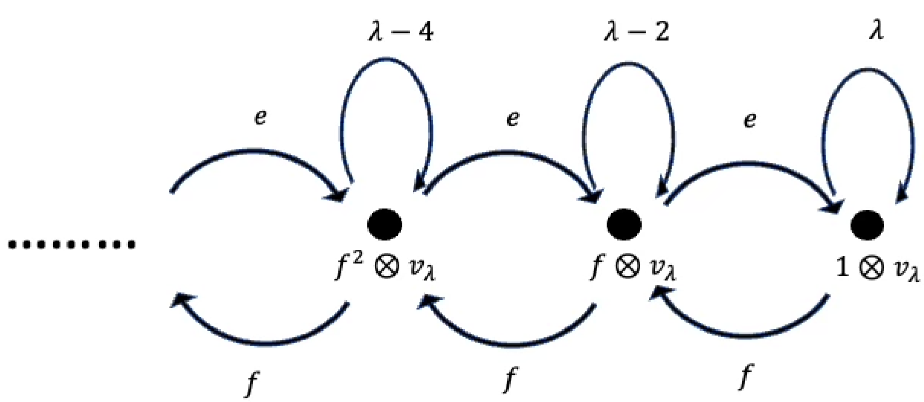
Highest weight modules were first studied by Verma in his thesis in 1966 [22]. Verma modules are typical examples of highest weight modules and they can be constructed as follow:

For $\lambda \in \mathfrak{h}^*$, let C_λ be the 1 dimensional $\mathfrak{h}$–module where $h \cdot v = \lambda(h)v$, for all $h \in \mathfrak{h}$. By setting $\mathfrak{n}_+ C_\lambda = 0$, C_λ becomes a $\mathfrak{h} \oplus \mathfrak{n}_+$–module ($\mathcal{U}(\mathfrak{b})$-module) .The Verma module corresponds to weight λ is a $\mathfrak{g}$–module, which is defined as $M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} C_\lambda$.

Lemma 2.3.3. *Let $M(\lambda)$ be a Verma module of weight λ of a Lie algebra $\mathfrak{g}$. Then:*

1. $M(\lambda) \cong \mathcal{U}(\mathfrak{n}_-)$ *as* $\mathcal{U}(\mathfrak{n}_-)$*-module;*
2. $\dim V^\lambda = 1$*;*
3. $\mathrm{Supp}(M(\lambda)) = \lambda - \mathbb{Z}_{\geq 0}\Phi_+$.

Construction of the Verma module $M(\lambda)$ of $\mathfrak{sl}(2)$:
Fix $\lambda \in \mathbb{C}$ and let $v_\lambda \in C_\lambda$ be a basis element. C_λ is a $\mathcal{U}(\mathfrak{b})$-module since $h \cdot v_\lambda = \lambda v_\lambda$ and $e \cdot v_\lambda = 0$. Through the above definition of Verma modules and PBW theorem, $M(\lambda) = \mathrm{Span}_{\mathbb{C}}\{f^i \otimes v_\lambda : i \in \mathbb{Z}_{\geq 0}\}$. Moreover, the action $\mathfrak{sl}(2)$ on $M(\lambda)$ can be visualized through the following diagram:



In this example, $\mathrm{Supp}(M(\lambda)) = \lambda - 2i$, for all $i \in \mathbb{Z}_{\geq 0}$. The weight multiplicity is 1 for every $\nu \in \mathrm{Supp}(M(\lambda))$.

By construction, $M(\lambda)$ has a unique maximal submodule. We denote $L(\lambda)$ the simple quotient of $M(\lambda)$ by its maximal submodule.

Definition 2.3.15. *Let $\lambda \in \mathfrak{h}^*$. λ is* integral *if $\lambda(h_\alpha) \in \mathbb{Z}$, for all $\alpha \in \pi$.*

It is a well-known result that $L(\lambda)$ is a finite dimensional $\mathfrak{g}$-module if and only if $\lambda(h_\alpha) \in \mathbb{Z}_{\geq 0}$, for all $\alpha \in \pi$. In this case, λ is called an *integral, dominant* weight. This result also gives a classification of all simple finite dimensional $\mathfrak{g}$ module. Nonetheless, this result will be used in the later chapters.

In the case the $\mathfrak{g} = \mathfrak{sl}(2)$, $\mathfrak{h}^* \cong \mathbb{C}$, we consider $\lambda = n \in \mathbb{Z}_{\geq 0}$. The Verma module $M(n)$ has a maximal submodule $M(-n-2)$, and $L(n) = M(n)/M(-n-2)$. Moreover, $L(n)$ is a finite dimensional of dimension n+1.

2.3.3 Coherent families

We recall that $U(\mathfrak{g})^0$ is the centralizer of $\mathfrak{h}$ in $U(\mathfrak{g})$.

Definition 2.3.16. *A* coherent $\mathfrak{g}$-family of degree d *is a weight $\mathfrak{g}$-module $\mathcal{M}$ such that:*

(i) $\dim \mathcal{M}^\lambda = d$ *for every* $\lambda \in \mathfrak{h}^*$

(ii) For any $u \in U(\mathfrak{g})^0$, the map $\lambda \mapsto \mathrm{Tr}\,(u|\mathcal{M}^\lambda)$ is polynomial in λ.

Coherent family was first introduced by Mathieu in 2000 in order to classify simple weight modules with finite weight multiplicities of reductive Lie algebras. It can be understood as a "big" weight module where its support is the whole $\mathfrak{h}^*$.

Note that since finitely generated bounded modules have finite length (Lemma 3.3 in [13]), we can define the semisimplification $\mathcal{M}^{ss}$ of a coherent family $\mathcal{M}$. Namely, $\mathcal{M}^{ss} = \bigoplus_{\lambda \in \mathfrak{h}^*/\mathbb{Z}\Phi} \mathcal{M}[\lambda]^{ss}$, where $\mathcal{M}[\lambda] = \bigoplus_{\alpha \in \mathbb{Z}\Phi} \mathcal{M}^{\lambda+\alpha}$.

Proposition 2.3.17. *Let $\mathcal{M}$ be a coherent family.*

1. *For any $\lambda \in \mathfrak{h}^*$, $\mathcal{M}[\lambda]$ has finite length.*

2. *The coherent family $\mathcal{M}$ and $\mathcal{M}[\lambda]^{ss}$ have the same simple subquotients.*

Example 2.3.18. *Fix $a \in \mathbb{C}$ and consider the the following module*

$$V(a) = \bigoplus_{\xi \in \mathbb{C}/2\mathbb{Z}} V(\xi, a),$$

where $V(\xi, a)$ is defined in 2.3.12.

Coherent family can be expressed in terms of differential operators as the following example.

Example 2.3.19. *Fix $a \in \mathbb{C}$. Let $\mathcal{M}(a)$ be the $\mathfrak{sl}(2)$-module with basis $\{t^\lambda : \lambda \in \mathbb{C}\}$, where the action is given by the following formulas:*

$$\begin{aligned} e &\mapsto at - t^2\partial_t, \\ f &\mapsto \partial_t, \\ h &\mapsto 2t\partial_t - aI. \end{aligned}$$

This is a coherent family of degree 1.

A coherent family can be obtained by applying *twisted localization functor* on a single Verma module. It is known that Coherent families exist for Lie algebras of type A and C ([13]).

2.3.4 $\mathcal{U}(\mathfrak{h})$– free modules

Remark 8. *If M is a weight module, then $\mathcal{U}(\mathfrak{h})$ acts locally finitely on M, i.e.* $\dim \mathcal{U}(\mathfrak{h}) \cdot m < \infty$, $\forall m \in M$.

In contrast with weight modules, a $\mathcal{U}(\mathfrak{h})$-free module is a module on which $\mathcal{U}(\mathfrak{h})$ acts freely. In particular, a $\mathcal{U}(\mathfrak{g})$-module M is a $\mathcal{U}(\mathfrak{h})$-free module of rank k if and only if $\operatorname{Res}_{\mathcal{U}(\mathfrak{h})}^{\mathcal{U}(\mathfrak{g})} M \cong \mathcal{U}(\mathfrak{h})^k$.

Example 2.3.20. *Let $N = \mathbb{C}[t]e^t$ be an $\mathfrak{sl}(2)$-module where the $\mathfrak{sl}(2)$-action is defined as follow:*

$$\begin{aligned} e &\mapsto -t^2\partial_t, \\ f &\mapsto \partial_t, \\ h &\mapsto 2t\partial_t. \end{aligned}$$

Note that the action of h on N is a free action from direct computation. In fact, N is a $\mathcal{U}(\mathfrak{h})$– free module of rank 1, which will be presented as a case of exponential tensor module in later chapter.

Example 2.3.21. *Let $M = \mathbb{C}[h]$ be a $\mathfrak{sl}(2)$-module where the $\mathfrak{sl}(2)$-action is defined as follow:*

$$\begin{aligned} h \cdot f(h) &= 2hf(h), \\ e \cdot f(h) &= hf(h-1), \\ f \cdot f(h) &= -hf(h+1). \end{aligned}$$

Note that the $\mathfrak{sl}(2)$-module M is free over $\mathcal{U}(\mathfrak{h})$ of rank 1. Moreover, simple $\mathfrak{sl}(n+1)$–modules that are free over $\mathcal{U}(\mathfrak{h})$ of rank 1 was classified by Nilsson in 2013 ([15]), which is summarized as follow:

Let $h_k := e_{k,k} - \frac{1}{n+1}\sum_{i=1}^{n+1} e_{i,i}$. Recall that $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{sl}(n+1)$ spanned by h_i, $i = 1, ..., n$. We identify $U(\mathfrak{h})$ with $\mathbb{C}[h_1, ..., h_n]$. Let $\sigma_i \in \operatorname{Aut}(\mathbb{C}[h])$ be defined by $\sigma_i(f(h_1, ..., h_n)) = f(h_1, ..., h_i - 1, ..., h_n)$, $i \in [\![n]\!]$. Then, following [15], for $S \subset [\![n]\!]$ and

$b \in \mathbb{C}$ we define the $\mathfrak{s}$-module M_b^S as follows. The underlying space of M_b^S is $\mathbb{C}[h_1, ..., h_n]$ and the $\mathfrak{sl}(n+1)$-action is defined by

$$h_k \cdot f \quad := \quad h_k f, \qquad k \in [\![n]\!];$$

$$e_{i,n+1} \cdot f \quad := \quad \begin{cases} (h_1 + ... + h_n + b)\sigma_i f, & i \in S, \\ (h_1 + ... + h_n + b)(h_i - b - 1)\sigma_i f, & i \notin S; \end{cases}$$

$$e_{n+1,j} \cdot f \quad := \quad \begin{cases} -(h_j - b)\sigma_j^{-1} f, & j \in S, \\ -\sigma_j^{-1} f, & j \notin S; \end{cases}$$

$$e_{i,j} \cdot f \quad := \quad \begin{cases} (h_j - b)\sigma_i \sigma_j^{-1} f, & i, j \in S, \\ \sigma_i \sigma_j^{-1} f, & i \in S, j \notin S, \\ (h_i - b - 1)(h_j - b)\sigma_i \sigma_j^{-1} f, & i \notin S, j \in S, \\ (h_i - b - 1)\sigma_i \sigma_j^{-1} f, & i, j \notin S. \end{cases}$$

One of the main results in [15] is that the modules $F_{\mathbf{a}}(M_b^S)$ and $F_\tau F_{\mathbf{a}}(M_b^S)$, for $S \subset [\![n]\!]$, $\mathbf{a} \in \mathbb{C}^n$, form a skeleton in the category of $\mathfrak{sl}(n+1)$-modules which are free of rank 1 when restricted to $U(\mathfrak{h})$. In this thesis, we will present the list of these modules as a particular case of exponential tensor modules.

2.3.5 Central character

Definition 2.3.22. *Let M be a $\mathfrak{g}$-module. The* central character *of M is a Lie algebra homomorphism $\chi : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$, where $z.m = \chi(z)m$, for all $z \in \mathcal{Z}(\mathfrak{g})$ and $m \in M$.*

Note that every simple modules have central characters. However, the converse is not always true. A well-known counter example is the class of Verma modules. In fact, Verma modules might not be simple but always have central character. Moreover, a Verma module $M(\lambda)$, its submodules, and subquotients, have the same central character corresponds to the weight λ. This central character is usually denoted as χ_λ.

Another important result is that every central character can be realized as a central character of a Verma module. In another words, if χ is the central character of a module M, $\chi = \chi_\lambda$ for some $\lambda \in \mathfrak{h}^*$.

A module M is said to have *generalized central character* if $M = \bigoplus_\chi M^\chi$, where $M^\chi := \{m \in M :$ for each $z \in \mathcal{Z}(\mathfrak{g})$ there is $k(z) \in \mathbb{Z}_{\geq 1}$ such that $(z - \chi(z))^{k(z)} = 0\}$. Since $z \in \mathcal{Z}(\mathfrak{g})$, M^χ is a submodule of M.

2.3.6 Harish-Chandra Homomorphism

We denote $I := \mathcal{U}(\mathfrak{g})^0 \bigcap \mathcal{U}(\mathfrak{g})\mathfrak{n}_+$. It is easy to see that I is a 2 sided ideal of $\mathcal{U}(\mathfrak{g})^0$. Furthermore, we also have $\mathcal{U}(\mathfrak{g})^0 = \mathcal{U}(\mathfrak{h}) \oplus I$ as vector spaces.

Definition 2.3.23. *The Harish-Chandra homomorphism is a homomorphism of associative algebras, which is defined as the projection $\varphi : \mathcal{U}(\mathfrak{g})^0 \to \mathcal{U}(\mathfrak{h})$.*

By $\mathcal{U}(\mathfrak{h})^{(W,\cdot)}$, we denote the algebra $\mathcal{U}(\mathfrak{h})$, which is invariant under the dot action of W.

Theorem 2.3.4. *The restriction of the Harish-Chandra homomorphism to $\mathcal{Z}(\mathfrak{g})$ induces an isomorphism between $\mathcal{Z}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{h})^{(W,\cdot)}$.*

Corollary 2.3.4.1. *Let $\lambda, \mu \in \mathfrak{h}^*$, $\chi_\lambda = \chi_\mu$ iff $\lambda \in W \cdot \mu$.*

2.4 Functors

2.4.1 Twisting functors

Let $\phi : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra automorphism. We define a the *endofunctor* F on $\mathfrak{g}$-module as follow:

$$F_\phi : \mathfrak{g}\text{-module} \to \mathfrak{g}\text{-module, where } M \mapsto M^\phi.$$

Here M is a $\mathfrak{g}$-module and M^ϕ is identified as M where the $\mathfrak{g}$-action is twisted by the automorphism ϕ . Explicitly, the action of $\mathfrak{g}$ on M^ϕ is given by $x \bullet m = \phi(x) \cdot m$, for all $m \in M$, $x \in \mathfrak{g}$.

2.4.2 Translation functors

Let V be a finite-dimensional $\mathfrak{g}$-module, and let $\eta, \lambda \in \mathfrak{h}^*$ be such that $\lambda - \eta \in \operatorname{Supp} V$. Let $\mathfrak{g}^\mu$-mod denote the category of $\mathfrak{s}$-modules which admit a generalized central character χ_μ.

The translation functor $T_V^{\eta,\lambda} : \mathfrak{g}^\eta\text{-mod} \to \mathfrak{g}^\lambda\text{-mod}$ is defined by $T_V^{\eta,\lambda}(M) = (M \otimes V)^{\chi_\lambda}$, where $(M \otimes V)^{\chi_\lambda}$ stands for the direct summand of $M \otimes V$ admitting generalized central character χ_λ. Assume in addition that $\lambda - \eta$ belongs to the W-orbit of the highest weight of V, the stabilizers of $\eta + \rho$ and $\lambda + \rho$ in the Weyl group coincide and $\eta + \rho$, $\lambda + \rho$ lie in the same Weyl facet. Then $T_V^{\eta,\lambda} : \mathfrak{g}^\eta\text{-mod} \to \mathfrak{g}^\lambda\text{-mod}$ defines an equivalence of categories (see [3]).

2.4.3 Weighting functor for $\mathfrak{sl}(n+1)$

The definition of weighting functor $\mathcal{W}$ appeared first in [16] attributing the idea to O. Mathieu. For any module M over $\mathfrak{sl}(n+1)$, the *weighting* $\mathcal{W}(M)$ of M is a coherent family defined as follows. Let M$ax\, U(\mathfrak{h})$ denote the set of maximal ideals of $U(\mathfrak{h})$. Also, for $\lambda \in \mathfrak{h}^*$ by $\overline{\lambda} : U(\mathfrak{h}) \to \mathbb{C}$ we denote the algebra homomorphism such that $\overline{\lambda}|\mathfrak{h} = \lambda$. Then

$$\mathcal{W}(M) \coloneqq \bigoplus_{\mathfrak{m} \in \mathrm{M}ax\, U(\mathfrak{h})} M/\mathfrak{m}M = \bigoplus_{\lambda \in \mathfrak{h}^*} M/\ker(\overline{\lambda})M$$

has an $\mathfrak{sl}(n+1)$-module structure via the action $x_\alpha \cdot (v + \ker(\overline{\lambda})M) \coloneqq (x_\alpha \cdot v) + \ker(\overline{\lambda + \alpha})M$, where x_α is in the α-root space of $\mathfrak{sl}(n+1)$.

Example 2.4.1. *Let* $h_1 = \frac{1}{2}(e_{11} - e_{22})$. *Let* $M = \mathbb{C}[h]$ *be a* $\mathfrak{sl}(2)$- *module where the* $\mathfrak{sl}(2)$-*action is defined as follow:*

$$\begin{aligned} h_1 \cdot f(h) &= hf(h), \\ e \cdot f(h) &= hf(h-1), \\ f \cdot f(h) &= -hf(h+1). \end{aligned}$$

We recall that $\Phi_{\mathfrak{sl}(2)} = \{\pm\alpha\}$, where $\alpha(h_1) = 1$. Let $v_\lambda = 1 + \ker(\overline{\lambda})M$. We apply weighting functor on M as follow:

- $e_{12} \cdot v_\lambda = e_{12} \cdot 1 + \ker(\overline{\lambda + \alpha})M = h + \ker(\overline{\lambda + \alpha})M = \lambda + \alpha(h) + \ker(\overline{\lambda + \alpha})M = \lambda + 1 + \ker(\overline{\lambda + \alpha})M = (\lambda + 1)v_{\lambda+1}$,

- $e_{21} \cdot v_\lambda = e_{21} \cdot 1 + \ker(\overline{\lambda - \alpha})M = -h + \ker(\overline{\lambda - \alpha})M = -\lambda + \alpha(h) + \ker(\overline{\lambda - \alpha})M = -\lambda + 1 + \ker(\overline{\lambda - \alpha})M = (-\lambda + 1)v_{\lambda-1}$,

- $h_1 \cdot v_\lambda = h_1 \cdot 1 + \ker(\overline{\lambda})M = h + \ker(\overline{\lambda})M = \lambda v_\lambda$.

Note that $\mathcal{W}(M)$ is a vector space with basis $\{v_\lambda : \lambda \in \mathbb{C}\}$, where the $\mathfrak{sl}(2)$-action is defined as above. Nonetheless, $\mathcal{W}(M)$ is a coherent family of degree 1. Moreover, This observation can be generalized as the following proposition.

Proposition 2.4.2 ([16])**.** *If* M *is a* $U(\mathfrak{h})$*-free module of rank* d*,* $\mathcal{W}(M)$ *is a coherent family of degree* d*.*

Chapter 3

The Lie algebras $\mathcal{D}(n)$ and $\mathfrak{sl}(n+1)$

Throughout the remainder of the thesis, $\mathcal{O} = \mathbb{C}[t_1, ..., t_n]$ and $\mathcal{O}_0$ will stand for the maximal ideal of $\mathcal{O}$ generated by $t_1, ..., t_n$. We use the multi-index notation. In particular, $\boldsymbol{t}^{\boldsymbol{\nu}} = t_1^{\nu_1}...t_n^{\nu_n}$, where $\boldsymbol{t} = (t_1, ..., t_n)$ and $\boldsymbol{\nu} = (\nu_1, ..., \nu_n)$ with $\nu_i \in \mathbb{C}$. If n is fixed, we set $\mathbb{C}[\boldsymbol{t}] = \mathbb{C}[t_1, ..., t_n] = \mathcal{O}$, $\mathbb{C}[\boldsymbol{t}^{\pm 1}] = \mathbb{C}[t_1^{\pm 1}, ..., t_n^{\pm 1}]$, and $\boldsymbol{t}^{\boldsymbol{\nu}}\mathbb{C}[\boldsymbol{t}^{\pm 1}] = t_1^{\nu_1}...t_n^{\nu_n}\mathbb{C}[t_1^{\pm 1}, ..., t_n^{\pm 1}]$, where the latter is the span of all (formal) monomials $t_1^{\nu_1+k_1}...t_n^{\nu_n+k_n}$, $k_i \in \mathbb{Z}$. We set $\partial_i := \frac{\partial}{\partial t_i}$ and use the notation t_i for the element in $\mathrm{End}(\mathcal{O})$ corresponding to multiplication by t_i. Let $e_{i,j}$ stand for the (i,j)th elementary matrix of $\mathfrak{gl}(n+1)$, while $E_{i,j}$ will stand for the (i,j)th elementary matrix of $\mathfrak{gl}(n)$. We fix the following basis of $\mathfrak{sl}(n+1)$: $\{h_k, e_{i,j} \mid 1 \le i,j \le n+1, i \ne j, k = 1, ..., n\}$, where $h_k := e_{k,k} - \frac{1}{n+1}\sum_{i=1}^{n+1} e_{i,i}$. In particular, $e_{i,i} - e_{j,j} = h_i - h_j$ if $1 \le i,j \le n$ and $e_{i,i} - e_{n+1,n+1} = h_i + \sum_{j=1}^{n} h_j$. Unless otherwise stated, whenever $e_{i,j}$ is used we assume that $i \ne j$. Moreover, the Lie bracket on the basis of $\mathfrak{sl}(n+1)$ can be computed explicitly using the following formula:

$$[e_{i,j}, e_{i',j'}] = \delta_{j,i'}e_{i,j'} - \delta_{i,j'}e_{i',j}, 1 \le i, i', j, j' \le n+1, i \ne j, i' \ne j';$$

$$[h_k, e_{i,j}] = (\delta_{k,i} - \delta_{k,j})e_{i,j};$$

$$[h_k, h_k'] = 0.$$

This choice of basis of the Lie algebra $\mathfrak{sl}(n+1)$ is purely for the convenience in terms of computations. For n is a fixed positive integer, $\mathfrak{s} = \mathfrak{sl}(n+1)$, and $\mathfrak{g} \simeq \mathfrak{gl}(n)$ is a fixed subalgebra of $\mathfrak{s}$ defined in Section 4.2.4.

3.1 The Lie algebra $\mathcal{D}(n)$

Definition 3.1.1. *$\mathcal{D}(n)$ is the associative subalgebra of* $\operatorname{End}(\mathcal{O})$ *generated by t_i, ∂_i, $i = 1, ..., n$, subject to*

$$t_i t_j - t_j t_i = \partial_i \partial_j - \partial_j \partial_i = 0;\ \partial_i t_j - t_j \partial_i = \delta_{ij}.$$

Remark 9. *$\mathcal{D}(n)$ is a Lie algebra with the Lie bracket defined as $[x, y] = xy - yx, \forall x, y \in \mathcal{D}(n)$.*

3.2 Automorphisms of $\mathcal{D}(n)$ and $\mathfrak{sl}(n+1)$

3.2.1 Some automorphisms of $\mathfrak{sl}(n+1)$

By τ we will denote the negative transpose on $\mathfrak{gl}(n+1)$, i.e. $\tau(e_{i,j}) = -e_{j,i}$ and we use the same letter for the restriction of τ on $\mathfrak{sl}(n+1)$. Then τ is an involutive automorphism. For $\mathbf{a} \in (\mathbb{C}^*)^{n+1}$, we set $\varphi_{\mathbf{a}}(e_{i,j}) = \frac{a_i}{a_j} e_{i,j}$. Then $\varphi_{\mathbf{a}}$ and τ are automorphisms of $\mathfrak{sl}(n+1)$ and $\varphi_{\mathbf{a}} = \varphi_{\mathbf{a}'}$ is and only if $\mathbf{a}' = c\mathbf{a}$ for some $c \in \mathbb{C}^*$. By F_τ and $F_{\mathbf{a}}$ we denote the endofunctors on $\mathfrak{g}$-mod corresponding to the twists by τ and $\varphi_{\mathbf{a}}$, respectively.

3.2.2 Fourier transform and Exponentiation on $\mathcal{D}(n)$

The Fourier transform on $\mathcal{D}(n)$ is an automorphism of $\mathcal{D}(n)$ defined by a subset S of $\{1, 2, ..., n\}$ as follows:

$$\begin{aligned} \psi_S(t_i) &= \partial_i,\ \psi_S(\partial_i) = -t_i,\ \text{if } i \in S, \\ \psi_S(t_j) &= t_j,\ \psi_S(\partial_j) = \partial_j,\ \text{if } j \notin S. \end{aligned}$$

For an arbitrary polynomial $g \in \mathcal{O}$, we define the automorphism θ_g of $\mathcal{D}(n)$ via $\theta_g(t_i) = t_i$, $\theta_g(\partial_i) = \partial_i + \frac{\partial g}{\partial t_i}$, for $i = 1, ..., n$. We will call θ_g, the *g-exponentiation* on $\mathcal{D}(n)$. Since $\theta_g = \theta_{g+c}$, *we will assume that $g \in \mathcal{O}_0$ whenever θ_g is considered.*

If M is a $\mathcal{D}(n)$-module, by M^{exp_g} we will denote the modules obtained from M after twisting by θ_g. Alternatively, $M^{\exp_g}$ can be thought as the space Me^g with the natural action of $\mathcal{D}(n)$. In the special case when g is a homogeneous linear polynomial $g = \sum_{i=1}^n b_i t_i$, we will denote θ_g and M^{exp_g} by θ_b and $M^{\exp_b}$, respectively, where $b = (b_1, ..., b_n)$ is in $\mathbb{C}^n$.

3.2.3 Twisted localization of $\mathcal{D}(n)$-modules and $\mathfrak{sl}(n+1)$-modules

We first recall some properties of the twisted localization functor in general. Let $\mathcal{U}$ be an associative unital algebra and $\mathcal{H}$ be a commutative subalgebra of $\mathcal{U}$. We assume in addition that $\mathcal{H} = \mathbb{C}[\mathfrak{h}]$ for some vector space $\mathfrak{h}$, and that

$$\mathcal{U} = \bigoplus_{\mu \in \mathfrak{h}^*} \mathcal{U}^\mu,$$

where

$$\mathcal{U}^\mu = \{x \in \mathcal{U} | [h, x] = \mu(h)x, \forall h \in \mathfrak{h}\}.$$

Definition 3.2.1. *An elelment $a \in \mathcal{U}$ is said to be* ad-nilpotent *if for any $u \in \mathcal{U}$, there exists an $n(u) > 0$ so that* $\operatorname{ad}(a)^{n(u)}(u) = 0$.

In order to localize a noncommutative associative algebra $\mathcal{U}$ with respects to a subset A, A need to be an Ore subset of $\mathcal{U}$. It is easy to check that set $\langle a\rangle = \{a^n \mid n \geq 0\}$ is an Ore subset of $\mathcal{U}$ for a be an ad-nilpotent element of $\mathcal{U}$. We define the $\langle a\rangle$-localization $D_{\langle a\rangle}\mathcal{U}$ of $\mathcal{U}$. For a $\mathcal{U}$-module M by $D_{\langle a\rangle}M = D_{\langle a\rangle}\mathcal{U} \otimes_{\mathcal{U}} M$ we denote the $\langle a\rangle$-localization of M. Note that if a is injective on M, then M is isomorphic to a submodule of $D_{\langle a\rangle}M$. In the latter case we will identify M with that submodule.

We next recall the definition of the generalized conjugation of $D_{\langle a\rangle}\mathcal{U}$ relative to $x \in \mathbb{C}$. This is the automorphism $\phi_x : D_{\langle a\rangle}\mathcal{U} \to D_{\langle a\rangle}\mathcal{U}$ given by

$$\phi_x(u) = \sum_{i\geq 0} \binom{x}{i} \operatorname{ad}(a)^i(u)a^{-i}.$$

If $x \in \mathbb{Z}$, then $\phi_x(u) = a^x u a^{-x}$. With the aid of ϕ_x we define the twisted module $\Phi_x(M) = M^{\phi_x}$ of any $D_{\langle a\rangle}\mathcal{U}$-module M. Finally, we set $D^x_{\langle a\rangle}M = \Phi_x D_{\langle a\rangle}M$ for any $\mathcal{U}$-module M and call it the *twisted localization* of M relative to a and x. We will use the notation $a^x \cdot m$ (or simply $a^x m$) for the element in $D^x_{\langle a\rangle}M$ corresponding to $m \in D_{\langle a\rangle}M$. In particular, the following formula holds in $D^x_{\langle a\rangle}M$:

$$u(a^x m) = a^x \left(\sum_{i\geq 0} \binom{-x}{i} \operatorname{ad}(a)^i(u)a^{-i}m \right)$$

for $u \in \mathcal{U}$, $m \in D_{\langle a\rangle}M$.

We will apply the twisted localization functor for $(\mathcal{U}, \mathcal{H})$ in the following three cases:

(i) $\mathcal{U} = \mathcal{D}(n)$, $\mathfrak{h} = \bigoplus_{i=1}^n (\mathbb{C}x_i\partial_i)$;

(ii) $\mathcal{U} = U(\mathfrak{sl}(n+1))$, $\mathfrak{h} = \bigoplus_{i=1}^n (\mathbb{C}h_i)$;

(iii) $\mathcal{U} = U(\mathfrak{gl}(n))$, $\mathfrak{h} = \bigoplus_{i=1}^n (\mathbb{C}E_{ii})$.

In case (i), for simplicity, we will use the following notation: $D_i^+ = D_{\langle t_i\rangle}$, $D_i^- = D_{\langle \partial_i\rangle}$. Also, for $\mathcal{U} = \mathcal{D}(n)$ and a $\mathcal{U}$-module M, we set $D^+_{(i)}M = (D_i^+\mathcal{U}/\mathcal{U}) \otimes_{\mathcal{U}} M$ and $D^-_{(i)}M = (D_i^-\mathcal{U}/\mathcal{U}) \otimes_{\mathcal{U}} M$. In the particular case, when t_i (respectively, ∂_i) acts injectively on M, then $D^+_{(i)}M \simeq D^+_i M/M$ (respectively, $D^-_{(i)}M \simeq D^-_i M/M$). Also, we set $D^+_S = \prod_{i\in S} D^+_i$ and $D^+_{(S)} = \prod_{i\in S} D^+_{(i)}$.

In case (ii), we will often consider the following setting. If Σ is a set of commuting roots (i.e. $\alpha,\beta \in \Sigma$ implies $\alpha+\beta \notin \Sigma$) and $f_\alpha \in \mathfrak{s}^{-\alpha}$ for $\alpha \in \Sigma$, then we consider $D_\Sigma = \prod_{\alpha\in\Sigma} D_{\langle f_\alpha\rangle}$. Also, if Σ is a linearly independent set, and $\mu = \sum_\alpha \mu_\alpha \alpha$, then we set $D^\mu_\Sigma = \prod_{\alpha\in\Sigma} D^{\mu_\alpha}_{\langle f_\alpha\rangle}$.

3.2.4 Families of differential operator presentations of $\mathfrak{sl}(n+1)$

Proposition 3.2.2. *Let V be a $\mathfrak{gl}(n)$-module and S be a subset of $\{1,2,...,n\}$. Then the correspondence*

$$
\begin{array}{rcl}
h_k & \mapsto & -t_k\partial_k \otimes 1 + 1\otimes E_{kk} - 1\otimes 1, \ \textit{for } k \notin S \\
h_k & \mapsto & t_k\partial_k \otimes 1 + 1\otimes E_{kk}, \ \textit{for } k \in S \\
e_{i,j} & \mapsto & 1\otimes E_{ij} - t_j\partial_i \otimes 1, \ \textit{for } i,j \notin S \\
e_{i,j} & \mapsto & 1\otimes E_{ij} + t_i\partial_j \otimes 1, \ \textit{for } i,j \in S \\
e_{i,j} & \mapsto & 1\otimes E_{ij} + t_it_j \otimes 1, \ \textit{for } i\in S, j \notin S \\
e_{i,j} & \mapsto & 1\otimes E_{ij} - \partial_i\partial_j \otimes 1, \ \textit{for } i\notin S, j \in S \\
e_{n+1,j} & \mapsto & -t_j \otimes 1, \ \textit{for } j \notin S \\
e_{n+1,j} & \mapsto & -\partial_j \otimes 1, \ \textit{for } j \in S \\
e_{i,n+1} & \mapsto & -\sum\limits_{j\notin S}\partial_j\otimes E_{ij} + \sum\limits_{l\in S} t_l\otimes E_{il} + \sum\limits_{j\notin S} t_j\partial_j\partial_i\otimes 1 - \sum\limits_{l\in S} t_l\partial_l\partial_i \otimes 1 - \sum\limits_{j=1}^{n}\partial_i\otimes E_{jj} \\
 & & +((n+1)-|S|)\partial_i\otimes 1, \ \textit{for } i\notin S \\
e_{i,n+1} & \mapsto & -\sum\limits_{j\notin S}\partial_j\otimes E_{ij} + \sum\limits_{l\in S} t_l\otimes E_{il} - \sum\limits_{j\notin S} t_it_j\partial_j\otimes 1 + \sum\limits_{l\in S} t_it_l\partial_l \otimes 1 + \sum\limits_{j=1}^{n} t_i\otimes E_{jj} \\
 & & -(n-|S|)t_i\otimes 1, \ \textit{for } i\in S.
\end{array}
$$

extends to a homomorphism $\omega_{V,S} : \mathfrak{sl}(n+1) \to \mathcal{D}(n)\otimes \mathrm{End}(V)$.

Proof. The case when $S = [\![n]\!]$ has been known for long time and usually is attributed to

Rudakov, [17], and Shen, [18]. The case of arbitrary S follows from $S = [\![n]\!]$ by applying the appropriate Fourier transform. Namely, $\omega_{V,S} = (\psi^3_{\widehat{S}} \otimes 1)\omega_{V,[\![n]\!]}$, where $\widehat{S} = [\![n]\!] \setminus S$. $\square$

The case $S = \varnothing$

We will denote $\omega_{V,\varnothing}$ by ω_V. In this case we have that

$$\begin{aligned}
h_k &\mapsto -t_k\partial_k \otimes 1 + 1 \otimes E_{kk} - 1 \otimes 1, \text{ for all } k,\\
e_{i,j} &\mapsto 1 \otimes E_{ij} - t_j\partial_i \otimes 1, \text{ for all } i \neq j,\\
e_{n+1,j} &\mapsto -t_j \otimes 1, \text{ for all } j,\\
e_{i,n+1} &\mapsto -\sum_{j=1}^{n} \partial_j \otimes E_{ij} + \sum_{j=1}^{n} t_j\partial_j\partial_i \otimes 1 - \sum_{j=1}^{n} \partial_i \otimes E_{jj} + (n+1)\partial_i \otimes 1, \text{ for all } i.
\end{aligned}$$

One easily checks that $\omega_{V,S} = (\psi_S \otimes 1)\omega_V$. Furthermore, the above correspondence define a homomorphism $\omega : U(\mathfrak{sl}(n+1)) \to \mathcal{D}(n) \otimes U(\mathfrak{gl}(n))$ that will play important role in Section 4.2.4.

The case $S = [\![n]\!]$

In this other "extreme" case we have the following presentation:

$$\begin{aligned}
h_k &\mapsto t_k\partial_k \otimes 1 + 1 \otimes E_{kk}, \text{ for all } k,\\
e_{ij} &\mapsto 1 \otimes E_{ij} + t_i\partial_j \otimes 1, \text{ for all } i \neq j,\\
e_{n+1,j} &\mapsto -\partial_j \otimes 1, \text{ for all } j,\\
e_{i,n+1} &\mapsto \sum_{l=1}^{n} t_l \otimes E_{il} + \sum_{l=1}^{n} t_i t_l \partial_l \otimes 1 + \sum_{j=1}^{n} t_i \otimes E_{jj}, \text{ for all } i.
\end{aligned}$$

Chapter 4

Exponential tensor modules

For a $\mathcal{D}(n)$-module P, a subset S of $[\![n]\!]$, and a $\mathfrak{gl}(n)$-module V, by $T(P,V)$ we denote the space $P \otimes V$ considered as a module over $\mathfrak{s} = \mathfrak{sl}(n+1)$ through the homomorphism $\omega_V = \omega_{V,\varnothing}$. In particular, the $\mathfrak{s}$-module with underlying space $P \otimes V$ obtained from the homomorphism $\omega_{V,S}$ is isomorphic to $T(P^{\psi_S}, V)$.

We will pay special attention at the case when $P = (\mathcal{O}^{\psi_S})^{\exp_g} = e^g\,(\mathbb{C}[\mathbf{t}])^{\psi_S}$, where g is a polynomial in $\mathbb{C}[\mathbf{t}]$. In this case, we will call

$$T(g,V,S) = T\left(\left((\mathbb{C}[\mathbf{t}])^{\psi_S}\right)^{\exp_g}, V, \varnothing\right) = T(e^g\,(\mathbb{C}[\mathbf{t}])^{\psi_S}, V).$$

exponential tensor module corresponding to g, V, and S. In the case when $g = \sum_{i=1}^n b_n t_n$, we set $T(b,V,S) = T(g,V,S)$ where $b \in \mathbb{C}^n$. In the case when $g = 0$ the modules $T(0,V,S)$ can be considered as Fourier transforms of the classical tensor modules studied originally by Rudakov, Shen, and others. The modules $T(0,V,S)$ play important role in the classification of simple torsion free $\mathfrak{s}$-modules of Mathieu, [13], as they are parts of coherent families defined in the next subsection. If $g \neq 0$, the modules $T(g,V,S)$ are not weight modules, as the following statement shows. The proof follows directly from the definition of ω_V, see §3.2.4.

Lemma 4.0.1. *The module $T(g,V,S)$ is a weight $\mathfrak{sl}(n+1)$-module if and only if $g=0$ and V is a weight $\mathfrak{gl}(n)$-module.*

4.1 Localization of exponential tensor modules

In this section we obtain some important results on localization of the exponential tensor modules $T(g,V,S)$. These results will help us to establish simplicity criteria for $T(g,V,S)$ in some particular cases of V.

Lemma 4.1.1. *If $P=\mathbb{C}[\boldsymbol{t}]$ is the defining representations of $\mathcal{D}(n)$, then*

$$e^g P^{\psi_S} \simeq D^+_{(S)}(e^g P) \simeq e^g D^+_{(S)}(P).$$

Proof. The second isomorphism is straightforward. It is enough to show $P^{\psi_S} \simeq D^+_{(S)}(P)$ as this implies $e^g P^{\psi_S} \simeq e^g D^+_{(S)}(P)$. Let $\widehat{S} = [\![n]\!] \setminus S$. Using the multi-index notation, we have that the space $D^+_{(S)}P$ has a basis consisting of all $t_S^{\mathbf{m}} t_{\widehat{S}}^{\boldsymbol{\ell}}$ with $\mathbf{m} \in (\mathbb{Z}_{<0})^{|S|}$ and $\boldsymbol{\ell} \in (\mathbb{Z}_{\geq 0})^{|\widehat{S}|}$. This follows from the standard fact that $t^{-1}, t^{-2}, \ldots$ form a basis of $D_{\langle t\rangle}\mathbb{C}[t]/\mathbb{C}[t]$.

For $\mathbf{k}=(k_1,\ldots,k_{|S|})$ with $k_j \in \mathbb{Z}_{\geq 0}$, and $p \in \mathbb{C}[\mathbf{t}]$ with $\partial_i p = 0$ for all $i \in S$, consider the map $t_S^{\mathbf{k}} p \mapsto \partial_S^{\mathbf{k}}(-t_S^{-1})p$. It is not difficult to check that this map extends to a homomorphism $P^{\psi_S} \to D^+_{(S)}(P)$. This is an isomorphism since it maps a basis element to a nonzero scalar multiple of the corresponding basis element. □

4.1.1 The case of one-dimensional V

We now focus on the case when V is one-dimensional representation of weight $a \in \mathbb{C}$. We denote this representation by V_a. In other words $V_a = a\,\mathrm{tr}$.

Proposition 4.1.1. *Given $g \in \mathcal{O}_0$, $a \in \mathbb{C}$, and $S \subset [\![n]\!]$, we have the following.*

(i) If $(n+1)(a-1) \notin \mathbb{Z}$, then $T(g,V_a,S)$ is simple.

(ii) *If* $(n+1)(a-1) \in \mathbb{Z}_{\leq -n-1}$*, then* $T(g, V_a, S)$ *is simple if and only if* $S = \varnothing$.

(iii) *If* $(n+1)(a-1) \in \{-n, -n+1, ..., -1\}$*, then* $T(g, V_a, S)$ *is simple if and only if* $S = \varnothing$ *or* $S = [\![n]\!]$.

(iv) *If* $(n+1)(a-1) \in \mathbb{Z}_{\geq 0}$*, then* $T(g, V_a, S)$ *is simple if and only if* $S = [\![n]\!]$.

Proof. By Lemma 4.1.1, $T(g, V_a, S) \simeq T(e^g D^+_{(S)}(\mathbb{C}[\mathbf{t}]), V_a)$. The module $T(e^g D^+_{(S)}(\mathbb{C}[\mathbf{t}]), V_a)$ has a basis $e^g t^{\mathbf{m}}$, where $m_i \in \mathbb{Z}_{<0}$ for $i \in S$ and $m_i \in \mathbb{Z}_{\geq 0}$ for $i \notin S$.

We next prove the "only if" part for all (i)-(iv). First assume that $(n+1)(a-1) \in \mathbb{Z}$. The coefficient $b_{\mathbf{m}}$ of $e^g \mathbf{t}^{\mathbf{m}} t_i^{-1}$ in the expansion of $e_{i,n+1}(e^g \mathbf{t}^{\mathbf{m}})$ is

$$b_{\mathbf{m}} = m_i \left(\sum_{j=1}^{n} m_j - 1 - (n+1)(a-1) \right). \tag{4.1.1}$$

From here we easily check that

$$T' = \mathrm{Span}\{e^g \mathbf{t}^{\mathbf{m}} \mid \sum_{j=1}^{n} m_j \geq (n+1)(a-1) + 1\}$$

is a submodule of $T(g, V_a, S)$. We have that $T' = T(g, V_a, S)$ if and only if $(n+1)(a-1) \in \mathbb{Z}_{\leq -1}$ and $S = \varnothing$; and $T' = 0$ if and only if $(n+1)(a-1) \in \mathbb{Z}_{\geq -n}$ and $S = [\![n]\!]$. This completes the proof of the "only if" statements.

It remains to show the "if" parts. Assume that the conditions for S in (i)-(iv) are satisfied and let M be a nontrivial submodule of $T(g, V_a, S)$. We first show *homogeneity of* M, namely, if $e^g f \in M$, $f = \sum_{\mathbf{m}} a_{\mathbf{m}} \mathbf{t}^{\mathbf{m}}$, then $e^g \mathbf{t}^{\mathbf{m}} \in M$, whenever $a_{\mathbf{m}} \neq 0$. Indeed, for $k = 1, ..., n$ we have that

$$h_k e^g f = -t_k \partial_k g e^g f - e^g t_k \partial_k f + (1-a) e^g f.$$

Thus $e^g t_k \partial_k f \in M$ which easily implies $e^g \mathbf{t}^{\mathbf{m}} \in M$ if $a_{\mathbf{m}} \neq 0$. Using the homogeneity of M

and applying multiple actions of $e_{i,n+1}$ and $e_{n+1,j}$ if necessary, we obtain that $e^g \mathbf{t}_S^{-1} \in M$. Then using again homogeneity of M and multiple actions of $e_{i,n+1}$ and $e_{n+1,j}$, we see that $e^g \mathbf{t}^{\mathbf{m}} \in M$ for all $\mathbf{m}$ such that $m_i \in \mathbb{Z}_{<0}$ for $i \in S$ and $m_i \in \mathbb{Z}_{\geq 0}$ for $i \notin S$. Hence, $M = T(g, V_a, S)$. □

4.1.2 The case of $V = \bigwedge^k \mathbb{C}^n$

Here we focus on the case when V is an exterior power of the natural module of $\mathfrak{gl}(n)$. We will use the notation $V = \bigwedge^k \mathbb{C}^n$ in this case. The next result gives simplicity criterion for $T(g, \bigwedge^k \mathbb{C}^n, S)$. The simplicity of $T(g, \bigwedge^k \mathbb{C}^n, S)$ as modules over the Lie algebra W_n was proven in [12].

Proposition 4.1.2. *Given $g \in \mathcal{O}_0$, $k \in \{0, 1, ..., n\}$, and $S \subset [\![n]\!]$, we have the following.*

(i) If $0 < k < n$, then $T(g, \bigwedge^k \mathbb{C}^n, S)$ is not simple.

(ii) If $k = 0$, then $T(g, \bigwedge^k \mathbb{C}^n, S)$ is simple if and only if $S = \varnothing$.

(iii) If $k = n$, then $T(g, \bigwedge^k \mathbb{C}^n, S)$ is simple if and only if $S = [\![n]\!]$.

Proof. The cases $k = 0$ and $k = n$ correspond to the cases $a = 0$ and $a = 1$ in §4.1.1. This proves parts (ii) and (iii). Part (i) follows from Lemma 3.2 in [12], but for reader's convenience we outline the important parts in the proof. The crucial part is that for any $\mathcal{D}(n)$-module P, there is a differential map:

$$d_P : T(P, \bigwedge \mathbb{C}^n) \to T(P, \bigwedge \mathbb{C}^n), \tag{4.1.2}$$

$d_P(f \otimes v) = \sum_{i=1}^n (t_i f) \otimes (e_i \wedge v)$, where $(e_1, ..., e_n)$ is the standard basis of $\mathbb{C}^n$. This map has the property that $d_P^2 = 0$ and that it is an $\mathfrak{sl}(n+1)$-homomorphism (in fact, it is a

W_n-homomorphism, too). This leads to the de Rham complex

$$0 \xrightarrow{d_P} T(P, \bigwedge^0 \mathbb{C}^n) \xrightarrow{d_P} T(P, \bigwedge^1 \mathbb{C}^n) \xrightarrow{d_P} \cdots \xrightarrow{d_P} T(P, \bigwedge^n \mathbb{C}^n) \xrightarrow{d_P} 0. \tag{4.1.3}$$

Thus $d_P\left[T(P, \wedge^{k-1}\mathbb{C}^n)\right]$ is a nontrivial proper submodule of $T(P, \wedge^k \mathbb{C}^n)$ for $k = 1, ..., n-1$. Letting $P = e^g\,(\mathbb{C}[\mathbf{t}])^{\psi_S}$ leads to the proof of the nonsimplicity of $T(g, \wedge^k \mathbb{C}^n, S)$ for $k = 1, ..., n-1$. □

Remark 10. *Consider the case $S = [\![n]\!]$. Then the map d_P defined in (4.1.2) is nothing but the standard de Rham differential. In particular, if $g = 0$, we have that $T(g, \wedge^k \mathbb{C}^n, [\![n]\!])$ is isomorphic to the module $\Omega^k_{\mathbb{C}^n}$ of k-forms on $\mathbb{C}^n$ and $d_P = d$ is the standard differential operator on the de Rham complex on $\mathbb{C}^n$. In the case of arbitrary g, $d_P = d_g$ is the Witten deformation of the standard de Rham differential by g defined in [21]. Namely, $d_P(\omega) = d(\omega) + dg \wedge \omega$.*

4.2 Exponential tensor modules in the cases $\deg g = 0, 1$

4.2.1 Tensor coherent families

Let V be a finite dimensional $\mathfrak{gl}(n)$-module and let $S \subset [\![n]\!]$. Note that the space $\mathbf{t}^\lambda \mathbb{C}[\mathbf{t}^{\pm 1}]$ has a natural structure of a $\mathcal{D}(n)$-module. Moreover, $\mathbf{t}^\lambda \mathbb{C}[\mathbf{t}^{\pm 1}]$ and $\mathbf{t}^\mu \mathbb{C}[\mathbf{t}^{\pm 1}]$ are isomorphic if and only if $\lambda - \mu \in \mathbb{Z}^n$. Hence, we may define $\mathbf{t}^\lambda \mathbb{C}[\mathbf{t}^{\pm 1}]$ for $\lambda \in \mathbb{C}^n/\mathbb{Z}^n$. Let $\mathcal{T}(V, S) = \bigoplus_{\lambda \in \mathbb{C}^n/\mathbb{Z}^n} T\left(\left(\mathbf{t}^\lambda \mathbb{C}[\mathbf{t}^{\pm 1}]\right)^{\psi_S}, V\right)$. Then one easily checks the following.

Proposition 4.2.1. *The $\mathfrak{s}$-module $\mathcal{T}(V, S)$ is a coherent family of degree* $\dim V$.

4.2.2 The case $g = 0$: injective coherent families

In this subsection we consider the case when $g = 0$ and V is a simple finite-dimensional $\mathfrak{gl}(n)$-module.

As mentioned earlier, coherent families are one of the main tools that O. Mathieu used in the classification of all simple torsion free modules of $\mathfrak{s}$ as the latter are submodules of partly-irreducible coherent families. We call a coherent family *partly-irreducible* (or just *irreducible*) if there is $\lambda \in \mathfrak{h}^*$ such that $\mathcal{M}[\lambda] = \bigoplus_{\alpha \in \mathbb{Z}\Phi} \mathcal{M}^{\lambda+\alpha}$ is an irreducible $\mathfrak{s}$-module. The coherent families $\mathcal{T}(V, S)$ are partly-irreducible except for the case when $V = L_{\mathfrak{g}}(\lambda)$ and $\lambda - \mathbf{1} \in \cup_{i=1}^{n} \mathcal{H}^i$. Below we provide more details about the non-irreducible case. For details we refer the reader to §11 in [13].

If $\mu \in \mathcal{H}^0$, we have a complex

$$0 \to \mathcal{T}(L_{\mathfrak{g}}(w_n \cdot \mu + \mathbf{1}), S) \to \mathcal{T}(L_{\mathfrak{g}}(w_{n-1} \cdot \mu + \mathbf{1}), S) \to \cdots \to \mathcal{T}(L_{\mathfrak{g}}(w_0 \cdot \mu + \mathbf{1}), S) \to 0$$

In the case $\mu = 0$, we have $L_{\mathfrak{g}}(w_{n-i} \cdot \mu + \mathbf{1}) = L_{\mathfrak{g}}(\omega_i) = \bigwedge^i(\mathbb{C}^n)$, and the above complex is the de Rham complex defined in (4.1.3). The complex for arbitrary μ is obtained from the complex for $\mu = 0$ after applying the translation functor from Proposition 4.2.6(iii).

Let now $\lambda - \mathbf{1} \in \mathcal{H}^i$ for $0 \leq i \leq n$, and let $\mu \in \mathcal{H}^0$ be such that $\lambda - \mathbf{1} = w_i \cdot \mu$. Then we can define the formal coherent family $\mathcal{T}'(L_{\mathfrak{g}}(\lambda), S) = \sum_{j=0}^{i} (-1)^j \mathcal{T}(L_{\mathfrak{g}}(\lambda[j]), S)$, where $\lambda[j] = w_{i-j} \cdot \mu + \mathbf{1}$. To understand how the formal coherent family, which is an element of a suitable Grothendieck K_0-group, can be considered as a genuine module, we refer the reader to Theorem 11.4 in [13]. Set for convenience $\mathcal{T}'(L_{\mathfrak{g}}(\lambda), S) = \mathcal{T}(L_{\mathfrak{g}}(\lambda), S)$ if $\lambda - \mathbf{1} \notin \mathcal{R}$.

Another important feature of the families $\mathcal{T}(V, S)$ and $\mathcal{T}'(V, S)$ is their injectivity. For a subset Σ of roots of $\mathfrak{s}$ and an $\mathfrak{s}$-module M, we say that M is Σ-*injective* if every nonzero α-root vector x_α of $\mathfrak{s}$ acts injectively on M for every $\alpha \in \Sigma$. Here is one important particular case of Σ. Let $\tilde{S}$ be a nonempty proper subset of $[\![n+1]\!]$, and let

$$\Sigma_{\tilde{S}} = \{\varepsilon_i - \varepsilon_j \mid i \in \tilde{S}, j \notin \tilde{S}\} \tag{4.2.1}$$

We say that an $\mathfrak{s}$-module M is *$\tilde{S}$-injective* if it is $\Sigma_{\tilde{S}}$-injective.

Proposition 4.2.2. *Let V be a simple finite-dimensional $\mathfrak{gl}(n)$-module and let S be a proper subset of $[\![n]\!]$. Then $\mathcal{T}'(V,S)$ is an $(S \cup \{n+1\})$-injective partly-irreducible coherent family, and $F_\tau \mathcal{T}'(V,S)$ is $([\![n]\!] \setminus S)$-injective partly-irreducible coherent family.*

Proof. The irreducibility of $\mathcal{T}'(V,S)$, and hence of $F_\tau \mathcal{T}'(V,S)$, follows from Theorem 11.4 in [13]. Although the latter theorem concerns the case $S = [\![n]\!]$, the case for arbitrary S follows from the fact that the semisimplification $\mathcal{T}'(V,S)^{ss}$ of $\mathcal{T}'(V,S)$ does not depend on the choice of S. The injectivity follows by the explicit formulas defining $\omega_{V,S}$ in Proposition 3.2.2. □

Coherent families may be constructed also via twisted localization of highest weight modules. We outline the construction of these families below and refer the reader to §4 in [13] for details.

If Σ is a set of commuting roots that is a basis of $\mathbb{Z}\Delta$ and M is a Σ-injective bounded simple module, then

$$\mathcal{E}_\Sigma(M) = \bigoplus_{\lambda \in \mathfrak{h}^*/\mathbb{Z}\Phi} D_\Sigma^\lambda M.$$

is a Σ-injective irreducible coherent family containing M as a submodule. We call $\mathcal{E}_\Sigma(M)$ *the Σ-injective coherent extension of M*. The injective coherent extensions are unique in the sense of the following proposition.

Proposition 4.2.3. *Let Σ be a set of commuting roots that is a basis of $\mathbb{Z}\Phi$ and M be a simple Σ-injective bounded $\mathfrak{s}$-module. Then $\mathcal{E}_\Sigma(M)$ is the unique, up to isomorphism, Σ-injective irreducible coherent family containing a submodule isomorphic to M.*

Proof. Let $\mathcal{M}$ be a Σ-injective irreducible coherent family containing M as a submodule and let $\lambda \in h^*$. Consider the modules $M_1 = \bigoplus_{\alpha \in \mathbb{Z}\Phi} \mathcal{M}^{\lambda+\alpha}$ and $M_2 = D_\Sigma^\lambda M$. These modules have the property $(M_1)^{ss} \simeq (M_2)^{ss}$ by the uniqueness of the semisimple coherent extension (Proposition 4.8 in [13]). Since M_1 and M_2 are both Σ-injective, there is a simple Σ-injective module L that is isomorphic to submodules of M_1 and M_2. On the other hand, both M_1 and M_2 are dense, i.e. $\dim M_1^{\lambda+\alpha} = \dim M_2^{\lambda+\alpha} = \deg M$ for all $\alpha \in \mathbb{Z}\Phi$. Hence, $M_1 \simeq D_\Sigma L \simeq M_1$, which implies the result. □

Remark 11. *Britten and Lemire introduced slightly different notion of Σ-injective coherent family in [6], see Definition 2.1. They imposed more restrictions on the action of the root elements on the family and as a result established a uniqueness result for the Σ-injective coherent families of degree d containing a Σ-injective (not necessarily simple) bounded module M of degree d. This uniqueness result was used to show that every simple torsion free module is a subquotient of $\mathcal{M} \otimes F$ where $\mathcal{M}$ is a coherent family of degree 1 and F is a finite-dimensional simple $\mathfrak{s}$-module.*

Propositions 4.2.2 and 4.2.3 imply the following.

Corollary 4.2.0.1. *If $\tilde{S}$ is a nonempty proper subset of $[\![n+1]\!]$, and $\mathcal{M}$ is a $\tilde{S}$-injective partly irreducible coherent family, then the following hold.*

(i) If $n+1 \in \tilde{S}$, then $\mathcal{M} \simeq \mathcal{T}'(V,S)$ for $S = \tilde{S} \setminus \{n+1\}$ and some V.

(ii) If $n+1 \notin \tilde{S}$, then $\mathcal{M} \simeq F_\tau \mathcal{T}'(V,S)$ for $S = [\![n]\!] \setminus \tilde{S}$ and some V.

Furthermore, every simple infinite-dimensional bounded $\mathfrak{s}$-module is a submodule of $\mathcal{T}'(V,S)$ or $F_\tau \mathcal{T}'(V,S)$ for some V and S.

Remark 12. *As mentioned in the proof of Proposition 4.2.2, $\mathcal{T}'(V)^{ss} = \mathcal{T}'(V,S)^{ss}$ is independent of the choice of S. In fact, if L is an infinite-dimensional simple submodule of $\mathcal{T}'(V)^{ss}$, then $\mathcal{T}'(V)^{ss}$ is the unique, up to isomorphism, semisimple partly-irreducible* *coherent family containing L as a submodule. In particular, if L is Σ-injective, then $\mathcal{T}'(V)^{ss} \simeq \mathcal{E}_{\Sigma}(L)^{ss}$. Since the $\tilde{S}$-injective partly-irreducible coherent families are classified, it is natural to attempt to classify all injective families $\mathcal{M}$ that are indecomposable, i.e. such that all $\mathcal{M}[\lambda]$ are indecomposable modules. All $\mathcal{T}'(V,S)$ and $F_\tau\mathcal{T}'(V,S)$ will be in this classification list except two of them, $\mathcal{T}(V,[\![n]\!])$ and $F_\tau\mathcal{T}(V,[\![n]\!])$, as they are not $\tilde{S}$-injective for any $\tilde{S}$.*

4.2.3 The case $\deg g = 1$ and $\mathfrak{h}$-free modules of $\mathfrak{sl}(n+1)$

In this subsection we fix $g(t) = \sum_{i=1}^{n} b_i t_i$ for $b_i \in \mathbb{C}^*$. If $\mathbf{b} = (b_1, b_2, ..., b_n)$, we will write $T(\mathbf{b}, V, S)$ for $T(g, V, S)$ and $\mathbf{b}t$ for $g(t) = \sum_{i=1}^{n} b_i t_i$.

Theorem 4.2.1. *Let $\boldsymbol{b} \in (\mathbb{C}^*)^n$. Then the following correspondence defines a homomorphism $U(\mathfrak{s}) \to \mathrm{End}(\mathbb{C}[\boldsymbol{h}] \otimes V)$:*

$$
\begin{aligned}
h_k &\mapsto h_k \otimes 1, \quad \textit{for all } k, \\
e_{ij} &\mapsto \frac{b_j}{b_i} h_i \sigma_i \sigma_j^{-1} \otimes 1 - \frac{b_j}{b_i} \sigma_i \sigma_j^{-1} \otimes E_{ii} + \sigma_i \sigma_j^{-1} \otimes E_{ij}, \quad \textit{for } i, j \in S, \\
e_{ij} &\mapsto \frac{b_i}{b_j} (h_j + 1) \sigma_i \sigma_j^{-1} \otimes 1 - \frac{b_i}{b_j} \sigma_i \sigma_j^{-1} \otimes E_{jj} + \sigma_i \sigma_j^{-1} \otimes E_{ij}, \quad \textit{for } i, j \notin S, \\
e_{ij} &\mapsto -b_i b_j \sigma_i \sigma_j^{-1} \otimes 1 + \sigma_i \sigma_j^{-1} \otimes E_{ij}, \quad \textit{for } i \notin S, j \in S, \\
e_{ij} &\mapsto \frac{-1}{b_i b_j} h_i (h_j + 1) \sigma_i \sigma_j^{-1} \otimes 1 + \frac{1}{b_i b_j} (h_j + 1) \sigma_i \sigma_j^{-1} \otimes E_{ii} + \frac{1}{b_i b_j} h_i \sigma_i \sigma_j^{-1} \otimes E_{jj} \\
&\quad - \frac{1}{b_i b_j} \sigma_i \sigma_j^{-1} \otimes E_{ii} E_{jj} + \sigma_i \sigma_j^{-1} \otimes E_{ij}, \quad \textit{for } i \in S, j \notin S, \\
e_{n+1,j} &\mapsto -b_j \sigma_j^{-1} \otimes 1, \ \textit{for } j \in S, \\
e_{n+1,j} &\mapsto \frac{1}{b_j} h_j \sigma_j^{-1} \otimes 1 - \frac{1}{b_j} \sigma_j^{-1} \otimes E_{jj} + \frac{1}{b_j} \sigma_j^{-1} \otimes 1, \ \textit{for } j \notin S, \\
e_{i,n+1} &\mapsto \frac{1}{b_i} (\sum_{j=1}^{n} h_j - 1) h_i \sigma_i \otimes 1 - \frac{1}{b_i} (\sum_{j=1}^{n} h_j) \sigma_i \otimes E_{ii} - \sum_{j \notin S} b_j \sigma_i \otimes E_{ij} + \sum_{p \in S} \frac{h_p}{b_p} \sigma_i \otimes E_{ip} \\
&\quad - \sum_{p \in S} \frac{1}{b_p} \sigma_i \otimes E_{ip} E_{pp} + \sum_{p \in S} \frac{1}{b_p} \sigma_i \otimes E_{ip}, \quad \textit{for } i \in S, \\
e_{i,n+1} &\mapsto -b_i (\sum_{j=1}^{n} h_j - 1) \sigma_i \otimes 1 - \sum_{j \notin S} b_j \sigma_i \otimes E_{ij} + \sum_{p \in S} \frac{h_p}{b_p} \sigma_i \otimes E_{ip} - \sum_{p \in S} \frac{1}{b_p} \sigma_i \otimes E_{ip} E_{pp} \\
&\quad + \sum_{p \in S} \frac{1}{b_p} \sigma_i \otimes E_{ip}, \ \textit{for } i \notin S.
\end{aligned}
$$

This homomorphism endows the space $\mathbb{C}[\boldsymbol{h}] \otimes V$ with an $\mathfrak{s}$-module structure. The resulting module is $\mathfrak{h}$-free of rank $\dim V$, and it is isomorphic to $T(\boldsymbol{b}, V, S)$.

Proof. The homomorphism in the theorem is the composition of $\omega_{V,S} : U(\mathfrak{s}) \to \mathcal{D}(n) \otimes \mathrm{End}(V)$ and the homomorphism $\mathcal{D}(n) \otimes \mathrm{End}(V) \to \mathrm{End}(\mathbb{C}[\mathbf{h}] \otimes V)$ where the latter is defined by the following maps:

$$
\begin{aligned}
1\otimes E_{ij} &\mapsto \sigma_i\sigma_j^{-1}\otimes E_{ij},\ \text{for all } i,j,\\
t_i\otimes 1 &\mapsto -\frac{1}{b_i}\left((h_i+1)\sigma_i^{-1}\otimes 1-\sigma_i^{-1}\otimes E_{ii}\right),\ \text{for } i\notin S,\\
\partial_i\otimes 1 &\mapsto b_i\sigma_i\otimes 1,\ \text{for } i\notin S\\
t_i\otimes 1 &\mapsto \frac{1}{b_i}\left(h_i\sigma_i\otimes 1-\sigma_i\otimes E_{ii}\right),\ \text{for } i\in S,\\
\partial_i\otimes 1 &\mapsto b_i\sigma_i^{-1}\otimes 1,\ \text{for } i\in S.
\end{aligned}
$$

To prove that the above define a homomorphism, we make a repeated use of the identity $h_i\sigma_i - \sigma_i h_i = \sigma_i$.

The isomorphism $T(\mathbf{b}, V, S)\to \mathbb{C}[\mathbf{h}]\otimes V$ is given explicitly by the formulas

$$
e^{\mathbf{b}t}\mathbf{t}^{\mathbf{k}}\otimes v\mapsto \prod_{i\in S}\binom{h_i-\mathrm{wt}(v)_i}{k_i}\frac{k_i!}{b_i^{k_i}}\prod_{j\notin S}\binom{-h_j+\mathrm{wt}(v)_j-1}{k_j}\frac{k_j!}{b_j^{k_j}}\otimes v,
$$

where v is a weight vector of V of weight $\mathrm{wt}(v)$. □

Proposition 4.2.4. *The following isomorphism of coherent $\mathfrak{s}$-families hold:*

$$
\mathcal{W}(T(\boldsymbol{b}, V, S))\simeq \mathcal{T}(V, [\![n]\!]\setminus S).
$$

Proof. Let $\dim V = N$ and let $(v_1, ..., v_N)$ be a basis of weight vectors of V. Denote by $\mathbf{1}_\ell\in\mathbb{C}[\mathbf{h}]^{\oplus N}$ the element $(0,...,1,...,0)$, where 1 is in the ℓth position. Let $v_{\lambda,\ell}=\mathbf{1}_\ell+\ker(\bar\lambda)T(\mathbf{b},V,S)$ for $\lambda\in\mathfrak{h}^*$ and $1\le\ell\le N$. Then the explicit isomorphism $\mathcal{W}(T(\mathbf{b},V,S))\to\mathcal{T}(V,[\![n]\!]\setminus S)$ is given by the formulas

$$
v_{\lambda,\ell}\mapsto\prod_{i\in S}(b_it_i)^{-\lambda_i+\mathrm{wt}(v_\ell)_i-1}\prod_{j\notin S}(b_jt_j)^{\lambda_j-\mathrm{wt}(v_\ell)_j}\otimes v_\ell.
$$

□

4.2.4 Central characters of the exponential tensor modules

For $\lambda \in \mathfrak{h}^*$, we set χ_λ to be the central character of the simple $\mathfrak{b}$-highest weight $\mathfrak{sl}(n+1)$-module $L_{\mathfrak{b}}(\lambda)$ with highest weight λ. The character of the corresponding simple $\mathfrak{b}_{\mathfrak{g}}$-highest weight $\mathfrak{g}_n$-module will be denoted by χ'_λ.

We identify $\mathfrak{h}^*$ with $\mathbb{C}^n$ and set $\rho_{\mathfrak{s}} = (n, n-1, ..., 1)$, $\rho_{\mathfrak{g}} = (n-1, n-2, ..., 0)$, and $\mathbf{1} = (1, 1, ..., 1)$. In particular, $\rho_{\mathfrak{s}} = \rho_{\mathfrak{g}} + \mathbf{1}$. The elements in $\lambda \in \mathfrak{h}^*$ naturally extend to algebra homomorphisms $\lambda : \mathbb{C}[\mathbf{h}] \to \mathbb{C}$. Then there exist isomorphisms $\xi_{\mathfrak{s}} : Z(\mathfrak{sl}(n+1)) \to \mathbb{C}[\mathbf{h}]$ and $\xi_{\mathfrak{g}} : Z(\mathfrak{g}_n) \to \mathbb{C}[\mathbf{h}]$ such that $\chi_\lambda(z) = (\lambda + \rho_{\mathfrak{s}})(\xi_{\mathfrak{s}}(z))$ and $\chi'_\lambda(z') = (\lambda + \rho_{\mathfrak{g}})(\xi_{\mathfrak{g}}(z'))$, respectively, for all $\lambda \in \mathfrak{h}^*$, $z \in Z(\mathfrak{sl}(n+1))$, and $z' \in Z(\mathfrak{g}_n)$. The maps $\xi_{\mathfrak{s}}$ and $\xi_{\mathfrak{g}}$ are certainly the restrictions of the Harish-Chandra homomorphisms $U(\mathfrak{sl}(n+1))^{\mathfrak{h}} \to U(\mathfrak{h})$ and $U(\mathfrak{g}_n)^{\mathfrak{h}} \to U(\mathfrak{h})$ to the corresponding centers, where $U(\mathfrak{a})^{\mathfrak{h}}$ stands for the centralizer of $\mathfrak{h}$ in $U(\mathfrak{a})$. Let $\xi : Z(\mathfrak{sl}(n+1)) \to Z(\mathfrak{g}_n)$ be the composition $\xi = \xi_{\mathfrak{g}}^{-1}\xi_{\mathfrak{s}}$.

Proposition 4.2.5. *With the notation as above, $\omega(z) = 1 \otimes (\xi(z))$ for all $z \in Z(\mathfrak{sl}(n+1))$.*

Proof. We will prove that the identity $\omega(z) = 1 \otimes \xi(z)$ holds when both sides are considered as endomorphisms on $T(\lambda) = T(0, \mathbb{C}[\mathbf{t}], L_{\mathfrak{b}_{\mathfrak{g}}}(\lambda))$ for $\lambda \in \mathbb{C}^n$. This is sufficient since $T(\lambda) = \mathbb{C}[\mathbf{t}] \otimes L_{\mathfrak{b}_{\mathfrak{g}}}(\lambda)$ is a faithful module over $\mathcal{D}(n) \otimes U(\mathfrak{gl}(n))$. Let v_0 be a $\mathfrak{b}_g$-highest weight vector of $L_{\mathfrak{b}_{\mathfrak{g}}}(\lambda)$. Using the formulas in §3.2.4, one easily checks that $1 \otimes v_0$ is a $\mathfrak{b}$-highest weight vector of the $\mathfrak{sl}(n+1)$-module $T(\lambda)$. Again by these formulas, the weight of $1 \otimes v_0$ is $\lambda - \mathbf{1}$. Therefore, $\omega(z) = \chi_{\lambda-\mathbf{1}}(z)\mathrm{I}d$ on $T(\lambda)$. On the other hand, if $z \in Z(\mathfrak{sl}(n+1))$, then

$$1 \otimes \xi(z) = \chi_\lambda(\xi(z))\mathrm{I}d = (\lambda + \rho_{\mathfrak{g}})(\xi_{\mathfrak{s}}(z)) = (\lambda - \mathbf{1} + \rho_{\mathfrak{s}})(\xi_{\mathfrak{s}}(z)) = \chi_{\lambda-\mathbf{1}}(z)\mathrm{I}d \tag{4.2.2}$$

on $T(\lambda)$. This completes the proof. □

Corollary 4.2.1.1. *If V is a $\mathfrak{gl}(n)$-module of central character χ_λ, and P is a $\mathcal{D}(n)$-module, then $T(P, V)$ has central character $\chi_{\lambda-\mathbf{1}}$.*

Proof. For brevity, write $T(V) = T(P, V)$. Then

$$\chi_{T(V)}(\omega(z))\mathrm{Id} = \omega(z) = 1 \otimes \xi(z) = \chi_V(\xi(z))\mathrm{Id} = \chi_\lambda(z')\mathrm{Id} = \chi_{\lambda-\mathbf{1}}(z)\mathrm{Id}.$$

The last identity follows from the identities (4.2.2) in the proof of the last proposition. □

We obtain the following isomorphism theorem of $T(g, V, S)$.

Theorem 4.2.2. *If V_1 and V_2 are simple finite-dimensional $\mathfrak{gl}(n)$-modules, $g_1, g_2 \in \mathcal{O}_0$ and $S_1, S_2 \subset [\![n]\!]$, then $T(g_1, V_1, S_1) \simeq T(g_2, V_2, S_2)$ if and only if $g_1 = g_2$, $V_1 \simeq V_2$, and $S_1 = S_2$.*

Proof. The isomorphism $V_1 \simeq V_2$ follows from Corollary 4.2.1.1. Then after a θ_{-g_2}-twist, we obtain $T(g_1 - g_2, V_1, S_1) \simeq T(0, V_1, S_2)$. From Lemma 4.0.1 we have $g_1 = g_2$. To conclude that $S_1 = S_2$, we note for example that $e_{n+1,j}$ acts locally nilpotently on $T(0, V, S)$ if and only if $j \in S$. □

We note that an isomorphism theorem for tensor modules over the Lie algebra W_n is established in [12] (Lemma 3.7).

4.2.5 Translation functors and exponential tensor modules

Recall that we have fixed $\mathfrak{h}$ to be the Cartan subalegbra of $\mathfrak{s} = \mathfrak{sl}(n+1)$ and $\mathfrak{g} \simeq \mathfrak{gl}(n)$ and $\mathfrak{b}_\mathfrak{s}$ and $\mathfrak{b}_\mathfrak{g}$ to be the corresponding Borel subalgebras. We will often write the weights of $\mathfrak{s}$ and $\mathfrak{g}$ as n-tuples.

For simplicity we set $L_\mathfrak{g}(\lambda) = L_{\mathfrak{b}_\mathfrak{g}}(\lambda)$ and $L_\mathfrak{s}(\lambda) = L_{\mathfrak{b}_\mathfrak{s}}(\lambda)$. Let $\Lambda^+_\mathfrak{g}$ (respectively, $\Lambda^+_\mathfrak{s}$) denote the sets of weights in $\mathfrak{h}^*$ such that $L_\mathfrak{g}(\lambda)$ (respectively, $L_\mathfrak{s}(\lambda)$) is finite dimensional. In other words, $(\lambda_1, \ldots, \lambda_n) \in \Lambda^+_\mathfrak{g}$ (respectively, $(\lambda_1, \ldots, \lambda_n) \in \Lambda^+_\mathfrak{s}$) if and only if $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$ for $i = 1, \ldots, n-1$ (respectively, $(\lambda_1, \ldots, \lambda_n) \in \Lambda^+_\mathfrak{g}$ and $\lambda_n + \sum_{i=1}^n \lambda_i \in \mathbb{Z}_{\geq 0}$).

In view of the Weyl group action, for $\lambda \in \mathfrak{h}^*$, it is convenient to define $\lambda_{n+1} = -\sum_{i=1}^n \lambda_i$ and $\widetilde{\lambda} = (\lambda_1, ..., \lambda_n, \lambda_{n+1})$. With this definition we have that the ith simple reflection of the Weyl group acts on $\widetilde{\lambda}$ via the transposition $s_i = (i, i+1)$ for $i = 1, ..., n$. Let $w_k = s_n s_{n-1} ... s_{n-k+1}$. In other words, we choose w_k to be the minimal S_n-coset representative of length k. By definition, $w_0 = \mathrm{Id}$. Let also, w_k' be the transposition $(n-k+1, n)$, i.e. $w_k' = s_n s_{n-1} ... s_{n-k+1} ... s_{n-1} s_n$. As usual, for $w \in S_{n+1}$ and $\lambda \in \mathbb{C}^n$, we set $w \cdot \widetilde{\lambda} = w(\widetilde{\lambda} + \widetilde{\rho}_{\mathfrak{s}}) - \widetilde{\rho}_{\mathfrak{s}}$. By definition, we have $w(\lambda) = w(\widetilde{\lambda})$ and $w \cdot \lambda = w \cdot \widetilde{\lambda}$. We let $\omega_k = \sum_{i=1}^k \varepsilon_i$ be the k-th fundamental weight. In particular, $\omega_n = \mathbf{1}$.

We represent $\Lambda_{\mathfrak{g}}^+$ as a disjoint union of three sets, $\Lambda_{\mathfrak{g}}^+ = \mathcal{N} \sqcup \mathcal{S} \sqcup \mathcal{R}$, where:

(i) $\lambda \in \mathcal{N}$, if $\lambda \in \Lambda_{\mathfrak{g}}^+$, $\lambda_n + \sum_{i=1}^n \lambda_i \notin \mathbb{Z}$,

(ii) $\lambda \in \mathcal{S}$, if $\lambda \in \Lambda_{\mathfrak{g}}^+$, $\lambda_n + \sum_{i=1}^n \lambda_i \in \mathbb{Z}$, and $w_k' \cdot \lambda = \lambda$ for some k,

(iii) $\lambda \in \mathcal{R}$, if $\lambda \in \Lambda_{\mathfrak{g}}^+$, $\lambda_n + \sum_{i=1}^n \lambda_i \in \mathbb{Z}$, and $w_k' \cdot \lambda \neq \lambda$ for all k.

The sets $\mathcal{N}$, $\mathcal{S}$, and $\mathcal{R}$, are nothing else, but the sets of λ such that $\widetilde{\lambda}$ is nonintegral, singular, and regular integral, respectively. We further decompose $\mathcal{S}$ and $\mathcal{R}$ as follows:

$$\mathcal{S} = \sqcup_{i=1}^n \mathcal{H}^{i-1,i}, \ \mathcal{R} = \sqcup_{i=0}^n \mathcal{H}^i,$$

where $\lambda \in \mathcal{H}^{i-1,i}$ if $w_i' \cdot \lambda = \lambda$; $\lambda \in \mathcal{H}^0$ if $\lambda_n + \sum_{i=1}^n \lambda_i \in \mathbb{Z}_{\geq 0}$; and $\lambda \in \mathcal{H}^i$, $i > 1$, if $\lambda = w_i \cdot \mu$ for some $\mu \in \mathcal{H}^0$. In particular, $\mathcal{H}^0 = \Lambda_{\mathfrak{s}}^+$. Also, if $\lambda = 0$, then $w_k \cdot 0 = \omega_{n-k} - \mathbf{1}$.

The following is easy to verify and will be helpful for the simplicity criterion, i.e. Theorem 4.2.4.

Lemma 4.2.3. *Let $\lambda \in \Lambda_{\mathfrak{g}}^+$ and $\lambda_0 = \infty$. Then the following hold.*

(i) $\lambda - \mathbf{1} \in \mathcal{N}$, if and only if $\lambda_n + \sum_{i=1}^n \lambda_i \notin \mathbb{Z}$;

(ii) $\lambda - \mathbf{1} \in \mathcal{H}^{k,k+1}$ if and only if $\lambda_{n-k} - (n-k) = -\sum_{i=1}^n \lambda_i$, for $k = 0, ..., n-1$;

(iii) $\lambda - \mathbf{1} \in \mathcal{H}^k$ *if and only if* $\lambda_{n-k} - (n-k) > -\sum_{i=1}^n \lambda_i > \lambda_{n-k+1} - (n-k+1)$, *for* $k = 0, ..., n$.

Proposition 4.2.6. *Let* $\lambda \in \Lambda_{\mathfrak{g}}^+$ *and* P *be a* $\mathcal{D}(n)$*-module. Then the following hold.*

(i) If $\lambda - \mathbf{1} \in \mathcal{N}$*, then there is* $a \in \mathbb{C}$ *such that* $(n+1)(a-1) \notin \mathbb{Z}$ *and* $T_V^{(a-1)\mathbf{1},\lambda-\mathbf{1}} T(P, V_a) \simeq T(P, L_{\mathfrak{g}}(\lambda))$*, where* $V = L_{\mathfrak{s}}(\mu)$ *with* $\mu = \lambda - a\mathbf{1}$*.*

(ii) If $\lambda - \mathbf{1} \in \mathcal{H}^{k-1,k}$ *and* $a_k = \frac{k}{n+1}$*, then* $T_V^{(a_k-1)\mathbf{1},\lambda-\mathbf{1}} T(P, V_{a_k}) \simeq T(P, L_{\mathfrak{g}}(\lambda))$*, where* $V = L_{\mathfrak{s}}(\mu)$ *with* $\widetilde{\mu} = (\lambda_1 - a_k, ..., \lambda_k - a_k, \lambda_k - a_k, ..., \lambda_n - a_k)$*.*

(iii) If $\lambda - \mathbf{1} \in \mathcal{H}^k$*, then* $T_V^{\omega_{n-k}-\mathbf{1},\lambda-\mathbf{1}} T(P, \Lambda^{n-k}\mathbb{C}^n) \simeq T(P, L_{\mathfrak{g}}(\lambda))$*, where* $V = L_{\mathfrak{s}}(\mu)$ *is such that* $\lambda - \mathbf{1} = w_k \cdot \mu$*.*

Moreover, the functors $T_V^{\eta-\mathbf{1},\lambda-\mathbf{1}}$ *used in (i)-(iii) are equivalences of categories.*

Proof. We first note that $T(P, W) \otimes V \simeq T(P, W \otimes V)$, and also that an exact sequence of $\mathfrak{g}$-modules

$$0 \to W_1 \to W \to W_2 \to 0$$

leads to an exact sequence

$$0 \to T(P, W_1) \to T(W, V) \to T(P, W_2) \to 0$$

of $\mathfrak{sl}(n+1)$-modules. These two statements are written as Remarks 2.1 and 2.2 in [9] in more restrictive setting, but the proofs are essentially the same. The idea is to observe that the modules $T(P, V)$ are $(\mathfrak{s}, \mathcal{O})$-modules (i.e. $\mathfrak{s}$-modules and $\mathcal{O}$-modules with compatible actions of $\mathfrak{s}$ and $\mathcal{O}$). More precisely, $T(P, V) = P \otimes_{\mathcal{O}} \widetilde{V}$, where $\widetilde{V} = \mathcal{O} \otimes V$. Note that $\widetilde{V}$ can be treated as the module of sections of the vector bundle $\mathcal{V}$ induced from V on the affine open subset of the projective space $\mathbb{P}^n$ consisting of $[t_0, t_1, ..., t_n]$ such that $t_i \neq 0$. For details on the geometric interpretation, see, for example, §11 in [13] or §2 in [9].

We next apply the tensor product invariance and the exactness of $T(P,V)$ together with Corollary 4.2.1.1. It remains to show that $L_{\mathfrak{g}}(\eta) \otimes L_{\mathfrak{s}}(\mu)$ has a direct summand isomorphic to $L_{\mathfrak{g}}(\lambda)$, and that there is no other direct summand $L_{\mathfrak{g}}(\mu)$ such that $\chi_{\mu-\mathbf{1}} = \chi_{\lambda-\mathbf{1}}$. Here $\eta = a\mathbf{1}, a_k\mathbf{1}, \omega_{n-k}$, for cases (i), (ii), (iii), respectively. This follows from the the Gelfand-Tsetlin decomposition of $L_{\mathfrak{s}}(\mu) = \bigoplus_p L_{\mathfrak{g}}(\mu^{(p)})$ into $\mathfrak{g}$-modules $L_{\mathfrak{g}}(\mu^{(p)})$ and looking at the supports of the resulting products $L_{\mathfrak{g}}(\eta) \otimes L_{\mathfrak{g}}(\mu^{(p)})$. □

Remark 13. *A Jordan-Hölder decomposition of $T(P, V_a) \otimes V$, where $a \in \mathbb{C}$ and V is a simple finite-dimensional $\mathfrak{sl}(n+1)$-module, is described explicitly in [6]. In particular, the authors prove their old conjecture that every simple torsion-free $\mathfrak{sl}(n+1)$-module is isomorphic to a submodule of $T(P, V_a) \otimes V$, for particular choices of P, a, and V. One should note that for some choices of a, the corresponding translation functors are not equivalences of categories, and that is why it is better to work with $T(P, \bigwedge^i \mathbb{C}^n)$.*

Combining Propositions 4.1.1, 4.1.2, and 4.2.6, we obtain one of our main results.

Theorem 4.2.4. *Let $\lambda \in \Lambda^+_{\mathfrak{g}}$, $g \in \mathcal{O}_0$, and $S \subset [\![n]\!]$. Then the following hold.*

(i) If $\lambda - \mathbf{1} \in \mathcal{N}$, then $T(g, L_{\mathfrak{g}}(\lambda), S)$ is simple.

(ii) If $\lambda - \mathbf{1} \in \mathcal{S}$, then $T(g, L_{\mathfrak{g}}(\lambda), S)$ is simple if and only if $S = \varnothing$ or $S = [\![n]\!]$.

(iii) If $\lambda - \mathbf{1} \in \mathcal{R}$, then:

- *(a) if $\lambda - \mathbf{1} \in \mathcal{H}^0$, $T(g, L_{\mathfrak{g}}(\lambda), S)$ is simple if and only if $S = [\![n]\!]$.*
- *(b) if $\lambda - \mathbf{1} \in \mathcal{H}^i$, $1 < i < n$, $T(g, L_{\mathfrak{g}}(\lambda), S)$ is not simple,*
- *(c) if $\lambda - \mathbf{1} \in \mathcal{H}^n$, $T(g, L_{\mathfrak{g}}(\lambda), S)$ is simple if and only if $S = \varnothing$.*

For the explicit conditions when $\lambda - \mathbf{1} \in \mathcal{N}$, etc. see Lemma 4.2.3.

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