OPTION PRICING WITH INVESTMENT STRATEGY UNDER STOCHASTIC INTEREST RATES

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NILOOFAR GHORBANI

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The members of the Committee approve the doctoral dissertation of Niloofar Ghorbani

Prof. Andrzej Korzeniowski
Supervising Professor ______________________________

Prof. Shan Sun-Mitchell ______________________________

Dr. Suvra Pal ______________________________

Dr. Li Wang ______________________________

Dean of the Graduate School ______________________________

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ABSTRACT

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Niloofar Ghorbani, Ph.D.

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Supervising Professor: Prof. Andrzej Korzeniowski

Equity options are the most common types of financial derivatives that give an investor the right but not the obligation to buy or sell shares of stock at a given price in the future for a premium (option price) paid at present. The Classical Black-Scholes Formula solved a longstanding mathematical problem of finding no arbitrage option price by means of stochastic Ito calculus based on Geometric Brownian Motion dynamics of the stock price and a fixed interest rate over the option time horizon. We extend the Black-Scholes Model by adding a component of investor's buying and selling strategies for Call and Put Option, in addition to relaxing the interest rate from fixed to evolving randomly, whereby reflecting the actual market environment. We first present a solution to an open problem regarding Call Option price under linear investment hedging for stochastic interest rate modeled by Cox-Ingersol-Ross process, via a Monte-Carlo simulation method. Next, we extend Put and Call Option pricing under linear investment strategy from the Black-Scholes setting to Hull-White interest rate model. Finally, based on our findings, we derive suitable modifications for practical implementation which inherently reflect the discrete nature of market transactions.

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CHAPTER 1

Introduction

This thesis is concerned with the effect of investment strategy on reducing the loss sustained by Option writer when building the Call or Put Option models, to lower the Option price of the classical Black-Scholes models.

In recent years, thanks to steady growth of financial derivatives market, various generalizations of the classical Option pricing model were developed. Namely, a combination of stochastic interest rates along with dynamic investing strategies in the underlying security prior to Option expiration has been proposed for the purpose of hedging the investment risks. It turned out that selling a security proportionally to its dropping price for Put Option and buying the security proportionally to its rising price for Call Option (both under European Black -Scholes Model) resulted in lower Option price as shown by Wang and Wang [24] [25]. Zhang *et al.* [28] extended the result for Call Option to stochastic interest rates following the Vasicek model and asked whether Call Option price can be established for stochastic interest rates under the Cox-Ingersoll-Ross (*CIR*) model [4]. An extensive background and the literature on the subject can be found in [28].

The main obstacle in solving the problem for *CIR* is the fact that the closed form solution to the stochastic differential equation (*SDE*) is no longer available (in general), unlike in Vasicek interest rate given explicitly by the gaussian process.

In chapter 2 we present an effective way for calculating Call Option price in the case of randomly evolving interest rates for the Cox-Ingersoll-Ross model. The method uses Monte Carlo simulation of interest rates path integrals, which is readily carried out thanks to the Ornstein-Uhlenbeck OU process representation [6].

In chapter 3 we obtained the closed form of the Put and Call Option price for the linear investment strategy under the Hull-White stochastic interest rates [11]. In particular, a protective put option can serve as an insurance policy against losses for the stock holder. Since the option price associated with trading of the underlying security is based on continuous stock trading (impossible to implement!), a feasible discrete variant is in order. Recently Li *et al.* [12] proposed a discretized method for the Call Option under the classical Black-Scholes with linear investment strategy.

Chapter 4 is concerned with a feasible market implementation for our Hull-White pricing model which in the special case covers the Vasicek model.

In chapter 5 we outline research that include potential extensions to Jump-diffusion processes for Options under investment strategies, monograph [10].

In this thesis we study the Option pricing models in financial derivatives market. The random assets are stochastic processes, i.e., families $(X_t)_{t\in I}$ of random variables indexed by a time interval I. For fundamental background, Brownian motion, Ito stochastic calculus, classical Black-Scholes Option pricing models we refer to the monograph [1].

1.1 Brownian Motion

The standard Brownian motion is a stochastic process $(B_t)_{t\in R^+}$ such that

- $B_0 = 0$ almost surely,
- The sample trajectories $t \to B_t$ are continuous, with probability 1

- For any finite sequence of times $t_0 < t_1 < \cdots < t_n$, the increments

$$B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}} \tag{1.1}$$

are mutually independent random variables.

- For any given times $0 \leq s < t$, $B_t - B_s$ has the Gaussian distribution $\mathcal{N}(0, t-s)$ with mean zero and variance $t - s$.

Martingale Property Standard Brownian motion is martingale wit respect to the filtration $\mathcal{F}_t$, i.e.,

$$E[B_t \mid \mathcal{F}_s] = B_s, \quad 0 \leq s < t. \tag{1.2}$$

1.2 Stochastic Integral

[1]-[5] The stochastic integral with respect to Brownian motion $(B_t)_{t\in R^+}$ of any simple predictable process $(u_t)_{t\in R^+}$ of the form

$$u_t := \sum_{i=1}^{n} F_i \mathbb{1}_{(t_{i-1},t_i]}(t), \qquad t \in R^+, \tag{1.3}$$

where F_i is an $\mathcal{F}_{t_{i-1}}$-measurable random variable for $i = 1, 2, \cdots, n$, and $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$.

The next proposition extends the construction of the stochastic integral from simple predictable processes to square-integrable $(\mathcal{F}_t)_{t\in[0,T]}$-adapted processes $(u_t)_{t\in R^+}$ for which the value of u_t at time t can only depend on information contained in the Brownian path up to time t.

Proposition 1.2.1. The stochastic integral with respect to Brownian motion $(B_t)_{t\in R^+}$ extends to all adapted processes $(u_t)_{t\in R^+}$ such that

$$\|u\|^2_{L^2(\Omega\times[0,T])} := E\left[\int_0^T \mid u_t \mid^2 dt\right] < \infty, \tag{1.4}$$

with the Ito isometry

$$\| \int_0^T u_t dB_t \|^2_{L^2(\Omega)} := E\left[\left(\int_0^T u_t dB_t\right)^2\right] = E\left[\int_0^T | u_t |^2 \, dt\right]. \tag{1.5}$$

1.3 Ito Formula

For any Ito process $(X_t)_{t\in R^+}$ of the form

$$X_t = X_0 + \int_0^t \nu_s ds + \int_0^t u_s dB_s, \qquad t \in R^+ \tag{1.6}$$

or in differential notation

$$dX_t = \nu_t dt + u_t dB_t, \tag{1.7}$$

where $(u_t)_{t\in R+}$ and $(\nu_t)_{t\in R+}$ are square-integrable adapted processes, we have

$$\begin{aligned} f(t, X_t) =& f(0, X_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s)\, ds + \int_0^t \nu_s \frac{\partial f}{\partial x}(s, X_s)\, ds + \int_0^t u_s \frac{\partial f}{\partial x}(s, X_s)\, dB_s \\ &+ \frac{1}{2}\int_0^t | u_s |^2 \frac{\partial^2 f}{\partial x^2}(s, X_s)\, ds. \end{aligned} \tag{1.8}$$

or in differential notation,

$$\begin{aligned} df(t, X_t) =& \frac{\partial f}{\partial t}(t, X_t)\, dt + \nu_t \frac{\partial f}{\partial x}(t, X_t)\, dt + u_t \frac{\partial f}{\partial x}(t, X_t)\, dB_t \\ &+ \frac{1}{2} | u_t |^2 \frac{\partial^2 f}{\partial x^2}(t, X_t)\, dt. \end{aligned} \tag{1.9}$$

1.4 European Call Options

A Call option with strike price K and the expiration time T is the right (but not an obligation) to buy 1 share of stock at the price K, called the strike price, at the future time T. Buying an option comes at a price C and the problem is to determine what C should be in the absence of arbitrage. Options return profit $h(S_T)$ only when the stock price S_T at time T exceeds the strike price K, in which case the option holder will gain

$$h(S_T) = max[S_T - K, 0] \geqslant 0 \tag{1.10}$$

1.5 European Put Options

A Put Option with strike price K and the expiration time T is the right (but not an obligation) to sell 1 share of stock at the price K, called the strike price, at the future time T. Options return profit $h(S_T)$ only when the strike price K exceeds the stock price S_T at time T, in which case the option holder will gain

$$h(S_T) = max[K - S_T, 0] \geqslant 0 \tag{1.11}$$

1.6 Black-Scholes Option Pricing Formula

The seminal work of Black and Scholes [1] found an Option pricing formula for the period prior to the Option expiration time. A basis for their model is the generally accepted absence of arbitrage. An arbitrage is a risk-free profit making scheme which takes advantage of the inefficiencies in financial markets resulted from securities valuation based upon Present-Future frame of reference. We include a sample of extensive literature on the subject of classical option pricing such as monographs, textbooks, and research articles ([1]-[4], [7], [16]-[22]) in the References. Turning to recently emerging new field of option pricing with investment strategies there are only a few published research articles ([6], [12]-[15], [24]- [26], [28]).

Given the stock price process

$$S_u = S_t e^{\left(\mu - \frac{\sigma^2}{2}\right)(u-t)+\sigma(X_u - X_t)}, \qquad 0 \leqslant t \leqslant u \leqslant T, \tag{1.12}$$

Black-Scholes Call option price $C(t)$ at time t, reads

$$\begin{aligned} C_t &= E\left[e^{-r(T-t)} max\left[Z_T - K, 0\right]\right] \\ &= S_t N\left(\frac{\left(r+\frac{\sigma^2}{2}\right)(T-t)+ln\left(\frac{S_t}{K}\right)}{\sigma\sqrt{T-t}}\right) \\ &\quad -Ke^{-r(T-t)} N\left(\frac{\left(r-\frac{\sigma^2}{2}\right)(T-t)+ln\left(\frac{S_t}{K}\right)}{\sigma\sqrt{T-t}}\right) \end{aligned} \tag{1.13}$$

where

$$Z_u = S_t e^{\left(r - \frac{\sigma^2}{2}\right)(u-t) + \sigma(W_u - W_t)} \tag{1.14}$$

with $EZ_u = S_t e^{r(u-t)}$, $W(\cdot)$ = standard Brownian motion, K = strike price, $T - t$ = expiration time in years, r = risk free return rate and $N(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ = cumulative standard normal distribution $\sim \mathcal{N}(0,1)$.

CHAPTER 2

Adaptive Risk Hedging for Call Options under Cox-Ingersoll-Ross Interest Rates

2.1 Introduction

This chapter is concerned with the derivation of the Call Option price for the linear investment under *CIR* interest rate. A key benefit of *CIR* process is that in some economies the interest rates always stay positive, and consequently the Vasicek model is not applicable due to allowing the interest rates to become negative. In what follows we adopt the model setup and notation from [28]. European Call Option under the linear investment strategy triggers stock buying whenever the stock price exceeds the strike price. The investment fraction is defined by:

$$Q(S) = \begin{cases} 0 & S \leq K \\ \dfrac{\beta}{\alpha K}(S-K) & K \leq S \leq (1+\alpha)K \\ \beta & S \geq (1+\alpha)K \end{cases} \tag{2.1}$$

where

S is stock price.

$Q(S)$ is the stock investment proportion, which is equal to the value of the stock investment divided by A, where A is the entire investment amount.

K is strike price of the Option.

α is the investment strategy index, indicating the stock investment occurs during the period in which the stock price increases from K to $(1+\alpha)K$.

β is the maximum value of the stock investment proportion.

It was found in [28] that the Call Option value V_T based on the linear investment with parameters α, β, strike price K, and the terminal stock price S_T reads as follows:

$$V_T = \begin{cases} 0 & S_T \leq K \\ \left(1+\frac{\beta}{\alpha}\right)(S_T-K) - \frac{\beta S_T}{\alpha} ln\left(\frac{S_T}{K}\right) & K \leq S_T \leq (1+\alpha)K \\ S_T - K - \frac{\beta S_T}{\alpha} ln(1+\alpha) + K\beta & S_T \geq (1+\alpha)K \end{cases} \tag{2.2}$$

We will use the above formula for the stock price that satisfies *SDE* with drift depending on the random interest rate, whose *SDE* follows *CIR*.

2.2 The Market Model

Consider the stock price S_t dynamics

$$dS_t = r_t S_t dt + \sigma_1 S_t dW_{1,t}, \qquad S(0) = S_0 > 0, \quad 0 \leq t \leq T. \tag{2.3}$$

By Ito lemma, the stock price at time T can be expressed as

$$S_T = S_0 e^{\int_0^T \left(r_0 - \frac{\sigma_1^2}{2}\right) ds + \int_0^T \sigma_1 dW_{1,t}}. \tag{2.4}$$

Furthermore, consider the interest rate r_t dynamics known as Cox-Ingersoll-Ross model

$$dr_t = a\left(b - r_t\right) dt + \sigma_2 \sqrt{r_t} dW_{2,t}, \qquad r(0) = r_0 > 0, \quad 0 \leq t \leq T \tag{2.5}$$

where $W_{1,t}$ and $W_{2,t}$ are independent Brownian motions on the probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geqslant 0}, P\right)$ adapted to the filtration $\mathcal{F}_t$.

The definition and lemma below are standard.

Definition 2.1.1. [2] [20] A numeraire is any strictly positive $\mathcal{F}_t$-adapted stochastic process N_t that can be taken as a unit of reference when pricing an asset X_t as follows

$$\hat{X}_t = \frac{X_t}{N_t}. \tag{2.6}$$

Lemma 2.1.1. [2] [20] Assume there exists a numeraire N and the corresponding probability measure Q_N. Then the price of any traded asset (without intermediate payments) X relative to N is a martingale under Q_N

$$E^{Q_N}\left[\frac{X_T}{N_T}|\mathcal{F}_t\right] = \frac{X_t}{N_t}, \qquad 0 \leqslant t \leqslant T. \tag{2.7}$$

In this section we consider the money market account $B_t = e^{\int_0^t r_s ds}$ with the stochastic interest rate r_t as numeraire. The measure associated with this numeraire is a risk-neutral measure denoted by Q and by the lemma reads

$$E^{Q}\left[\frac{X_T}{B_T}|\mathcal{F}_t\right] = \frac{X_t}{B_t}, \qquad 0 \leqslant t \leqslant T. \tag{2.8}$$

The derivative price is then obtained by calculating the conditional expectation of its terminal payoff

$$H_0 = E_Q\left[e^{-\int_0^T r_s ds} H_T|\mathcal{F}_0\right] \tag{2.9}$$

where H_T is the derivative's payoff at time T. The filtration $\mathcal{F}_0$ does not have an effect on calculation of the expectation and Formula (2.9) can be written as

$$H_0 = E_Q\left[e^{-\int_0^T r_s ds} H_T\right]. \tag{2.10}$$

Indeed, the option price C at initial time discounted by the money market account numeraire under the risk-neutral measure is represented by

$$C = E_Q\left[e^{-\int_0^T r_s ds} V_T\right] \tag{2.11}$$

where V_T is the Call Option payoff at maturity time

$$V_T = h\left(S_T\right) = h\left(S_0 e^{\int_0^T \left(r_s - \frac{\sigma_1^2}{2}\right) ds + \int_0^T \sigma_1 dW_{1,t}}\right) \tag{2.12}$$

with

$$h(x) = max\left[x - K, 0\right] \geqslant 0, \tag{2.13}$$

and V_T previously defined by (2.2).

2.3 *CIR* Model via Ornstein-Uhlenbeck Process

Cox-Ingersoll-Ross (1985) introduced a square-root term in the diffusion coefficient of the Vasicek model which brings a solution to the positivity problem encountered in Vasicek model. It is well-known that in general there is no closed-form solution to the CIR model Equation (2.5). However, it turns out that in some cases one can obtain the closed form solution in terms of the Ornstein- Uhlenbeck (OU) process. For the sake of completeness, we state and verify this fact in the following lemma.

Lemma 2.2.1. [9] Consider the n-dimensional OU process

$$dX_t^i = -\alpha X_t^i dt + \sigma dW_t^i \tag{2.14}$$

where W_t^i are n independent Brownian motions, $i = 1, ..., n$. Let

$$Y_t = \sum_{i=1}^{n} \left(X_t^i\right)^2. \tag{2.15}$$

Note that

$$\begin{aligned} d\left(X_t^i\right)^2 &= 2X_t^i dX_t^i + 2d\left\langle X^i\right\rangle_t \\ &= \left(-2\alpha\left(X_t^i\right)^2 + \sigma^2\right) dt + 2\sigma X_t^i dW_t^i \end{aligned}$$

Thus

$$dY_t = d\left(\sum_{i=1}^{n}\left(X_t^i\right)^2\right) = \sum_{i=1}^{n} d\left(X_t^i\right)^2$$
$$= \left(-2\alpha Y_t + n\sigma^2\right)dt + 2\sigma\sum_{i=1}^{n} X_t^i dW_t^i,$$

where the second step follows from the independence of the Brownian motions. Next note that the process

$$Z_t = \int_0^t \sum_{i=1}^{n} X_u^i dW_u^i$$

is a martingale with quadratic variation

$$\langle Z\rangle_t = \int_0^t \sum_{i=1}^{n}\left(X_u^i\right)^2 du = \int_0^t Y_u du.$$

Consequently, by Levy's characterization theorem, the process

$$\tilde{W}_t = \int_0^t \frac{1}{\sqrt{Y_u}} \sum_{i=1}^{n} X_u^i dW_u^i$$

is a Brownian motion. Therefore

$$dY_t = \left(-2\alpha Y_t + n\sigma^2\right)dt + 2\sigma\sqrt{Y_t}d\tilde{W}_t$$

whereas

$$dr_t = a\left(b - r_t\right)dt + \sigma_2\sqrt{r_t}dW_{2,t}.$$

Direct comparison ($Y_t \equiv r_t$) yields

$$a = 2\alpha, \quad b = \frac{n\sigma^2}{a} = \frac{n\sigma^2}{2\alpha} \quad \text{and } \sigma_2 = 2\sigma.$$

To solve (3.14) multiply by $X_t e^{\alpha t}$ to get

$$d\left(X_t^i e^{\alpha t}\right) = e^{\alpha t} dX_t^i + \alpha e^{\alpha t} X_t^i dt = \sigma e^{\alpha t} dW_t^i$$

which upon integration from 0 to t gives

$$X_t^i = e^{-\alpha t} X_t^i(0) + \int_0^t \sigma e^{-\alpha(t-s)} dW_s^i. \tag{2.16}$$

Notice that (2.15)-(2.16) imply r_t has non-central chi square distribution.

The parameter a corresponds to the speed of adjustment to the mean b, and σ_2 is the short rate volatility. The drift $a(b-r)$ is exactly the same as in Vasicek model, however, the volatility in *CIR* model is $\sqrt{r_t}\sigma_2$ as opposed to σ_2 for Vasicek. The drift ensures mean reversion of the interest rate towards the long run value b, with the speed of adjustment governed by the strictly positive parameter a. To ensure that interest rate r_t stays positive for all t we must assume $2ab > \sigma_2^2$ in equation (2.5) which in turn requires $n \geqslant 3$ in (2.15).

It is worth noting that in the Vasicek model Zhang *et al.* [28] utilized a zero-coupon bond as numeraire, which lead to the Option price under the forward measure. This approach entails to drift change in the *SDE* for the interest rate. This method, when applied to the *CIR* model, would require extension of our Lemma to *OU* process with variable drift, and ultimately would have introduced more complexity to the closed-form representation of the interest rate. As a result, our representation was derived under the risk-neutral measure, which is more suitable for Monte Carlo simulation.

2.4 Call Option Price under *CIR* Model

Stock price S_T under *CIR* model reads (2.4). We represent the S_T in terms of R, C and Z

$$S_T = S_0 e^{\int_0^T r_s ds - \int_0^T \frac{\sigma_1^2}{2} ds + \int_0^T \sigma_1 dW_{1,t}} \tag{2.17}$$

$$S_T = S_0 e^{R+C+Z} \tag{2.18}$$

where

$$R = \int_0^T r_s ds$$
$$C = -\int_0^T \frac{\sigma_1^2}{2} ds$$
$$Z = \int_0^T \sigma_1 dW_{1,t} \sim N\left(0, \sigma_1^2 T\right)$$

with independent random variables R, Z.

Remark 2.3.1. Even though r_t has known non-central chi-square distribution (by (2.15), $Y \equiv r$), the distribution of its integral $\int_0^t r_s ds$ is unknown (unlike gaussian with known mean and variance in the Vasicek case) and thus not suitable for direct calculations. Nevertheless, the path integral $\int_0^t r_s ds$ leads to straightforward Monte Carlo simulation, thanks to squared OU process representation of r_t.

Theorem 2.3.1. Based on the notation established in (2.17)-(2.18), the explicit form of V_T reads

$$V_T = \begin{cases} 0, \\ \qquad\qquad\qquad\qquad R+Z \le ln\left(\frac{K}{S_0}\right) + \frac{\sigma_1^2}{2}T \\ \left(1+\frac{\beta}{\alpha}\right)\left(S_0 e^{-\frac{\sigma_1^2}{2}T+R+Z} - K\right) - \frac{\beta}{\alpha} S_0 e^{-\frac{\sigma_1^2}{2}T+R+Z}\left(lnS_0 - \frac{\sigma_1^2}{2}T + R + Z - lnK\right), \\ \qquad\qquad\qquad\qquad ln\frac{K}{S_0} + \frac{\sigma_1^2}{2}T \le R+Z \le ln\frac{K(1+\alpha)}{S_0} + \frac{\sigma_1^2}{2}T \\ S_0 e^{-\frac{\sigma_1^2}{2}T+R+Z} - K - \frac{\beta}{\alpha}\left(S_0 e^{-\frac{\sigma_1^2}{2}T+R+Z}\right)(ln(1+\alpha)) + K\beta, \\ \qquad\qquad\qquad\qquad R+Z > ln\frac{K(1+\alpha)}{S_0} + \frac{\sigma_1^2}{2}T \end{cases} \tag{2.19}$$

Even though the distribution of S_T is unknown, (2.12) can still be calculated by Monte Carlo simulation thanks to the path integral representation of S_T.

2.5 Monte Carlo Simulation

2.5.1 Discretization

In order to do simulation and use the theorem (2.3.1) we need to simulate the stock price S_T as expressed by (2.17)-(2.18). The only part that requires attention is the path integral $R = \int_0^t dt$. To calculate R we implement Riemann approximation with the discretization on $[0, T]$ as follows:

$$r_t = \sum_{i=1}^{n} (X_t^i)^2 = \sum_{i=1}^{n} \left[e^{-\alpha t} X_t^i(0) + \int_0^t \sigma e^{-\alpha(t-s)} dW_s^i\right]^2.$$

We have

$$\int_0^T r_t dt = \int_0^T \sum_{i=1}^n \left(X_t^i\right)^2 dt = \sum_{i=1}^n \int_0^T \left(X_t^i\right)^2 dt$$
$$\approx \sum_{i=1}^n \left[\sum_{i=1}^m h \left[e^{-\alpha t} X_t^i(0) + \sum_{k=1}^l \sigma e^{-\alpha hl - \alpha hk} \left(W_{hk}^i - W_{h(k-1)}^i \right) \right]^2 \right]$$

with $m = \frac{T}{h}$ for the time step size h.

2.6 Example

We illustrate our method by simulating the Black-Scholes European Call under *CIR* interest rates for six months, one year and a two year Leap ($T = 0.5, 1, 2$). We chose $h = 0.01$ and number of trials $N = 10,000$, for accuracy of the Brownian Motion approximation and simulation respectively. The results are listed in the Table 2.1.

Simulation Parameters:

- Investment Indexes $\alpha = 0.2$ and $\beta = 0.5$.
- Stock volatility $\sigma_1 = 0.5$.
- Interest rate volatility $\sigma_2 = 0.2$.
- *CIR* model parameters $a = 1, b = 0.02$.
- The initial stock price $S_0 = 40$.
- The strike price $K = 45$.

Table 2.1. Estimated value of Call Option price.

Terminal time T	0.5	1	2
Call Option price C_T			
CIR with Investment Strategy	2.25	2.26	2.22
CIR without Investment Strategy	3.42	3.32	3.27

As expected, by Table 2.1, the Option price with investment under *CIR* is smaller than the Option price without investment and shows that the Linear Investment hedging lowers the investment risk for the Call Option holder.

CHAPTER 3

Put Options with Linear Investment for Hull-White Interest Rates

3.1 Introduction

The gist of considering dynamic investment strategies, such as presented in this chapter, is two-fold. Firstly, unlike in the classical Black-Scholes model where the investor buys options and has no position in the underlying stock throughout the option time horizon, the dynamic investment strategy requires the investor to continuously trade the stock, whereby lowering the investor risk which is manifested by the lower option price. Secondly, the interest rates are no longer constant and are assumed stochastic.

This chapter is concerned with Put Option hedging by linear investment strategy under the Hull-White stochastic interest rates model. European Put Option with the linear investment strategy triggers stock selling whenever the stock price falls below the strike price and stays in the range $[(1-\alpha)K, K]$. Following [26] we state the relevant facts regarding the hedging strategy. The investment fraction is defined by:

$$Q(S) = \begin{cases} \beta & S \leq (1-\alpha)K \\ \frac{(1-\beta)}{\alpha K}\left[S-(1-\alpha)K\right]+\beta & (1-\alpha)K \leq S \leq K \\ 1 & S \geq K \end{cases} \tag{3.1}$$

where

S is stock price.

Q(S) is the stock investment proportion, which is equal to the value of the stock investment divided by A, where A is the entire investment amount.

K is strike price of the option.

α is the investment strategy index, indicating the stock investment occurs during the period in which the stock price drops from K to $(1-\alpha)K$.

β is the minimum value of the stock investment proportion.

It was found in [26] that the Put Option value V_T based on the linear investment with parameters α, β, strike price K reads as follows:

$$V_T = \begin{cases} 0 & S_T \geq K \\ \frac{1-\beta}{\alpha}\left(\frac{2\alpha-1}{2}K + (1-\alpha)S_T - \frac{S_T^2}{2K}\right) & (1-\alpha)K \leq S_T \leq K \\ (K-\beta S_T) - \frac{(1-\beta)K(2-\alpha)}{2} & S_T \leq (1-\alpha)K \end{cases} \tag{3.2}$$

where S_T is the terminal stock price.

We will use the above formula for the stock price that satisfies a stochastic differential equation *(SDE)* with drift depending on the random interest rate, whose *SDE* follows the Hull-White model.

3.2 The Market Model

The evolution of the stock price S_t satisfies the following *SDE*

$$dS_t = \mu_t S_t dt + \sigma_1 S_t dW_{1,t} \tag{3.3}$$

with mean return μ_t, constant volatility σ_1 and a standard Brownian motion $W_{1,t}$. The stock price dynamic under the risk-neutral measure is then as follows

$$dS_t = r_t S_t dt + \sigma_1 S_t dW_{1,t} \tag{3.4}$$

where r_t is the interest rate.

By Ito formula the stock price at time T can be expressed as

$$S_T = S_0 e^{\int_0^T \left(r_s - \frac{\sigma_1^2}{2} \right) ds + \int_0^T \sigma_1 dW_{1,t}} \tag{3.5}$$

where S_0 is the initial stock price.

Wang et al. [26] proposed a put Option model based on a dynamic investment strategy for the Black-Scholes Option pricing. In this chapter we extend their result to the stochastic Hull-White interest rate model r_t which satisfies the following *SDE*

$$dr_t = (\theta(t) - ar_t)\, dt + \sigma_2 dW_{2,t} \tag{3.6}$$

with a and σ_2 constants and $W_{2,t}$ standard Brownian motion independent from $W_{1,t}$.

Remark 3.2.1. Special case of Hull-White model, $\theta(t) = ab$, becomes the Vasicek model.

In general, Calin [3], $\theta(t)$ satisfies the following equation

$$\theta(t) = \partial_t f(0,t) + af(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \tag{3.7}$$

where $f(0,t)$ is the yield curve determined by the bond price.

The solution to (3.4) reads

$$r_t = r_0 e^{-at} + e^{-at} \int_0^t \theta(s) e^{as} ds + \sigma_2 e^{-at} \int_0^t e^{as} dW_{2,s}. \tag{3.8}$$

Note that the first two terms are deterministic and the last is a Wiener integral, thus the process r_t is normally distributed, with mean and variance

$$\begin{aligned} E[r_t] &= r_0 e^{-at} + e^{-at} \int_0^t \theta(s) e^{as} ds \\ Var[r_t] &= \frac{\sigma_2^2}{2a}(1 - e^{-2at}). \end{aligned} \tag{3.9}$$

Integrating (3.6) yields

$$\int_0^t r_s ds = \frac{r_0(1 - e^{-at})}{a} + \int_0^t e^{-as} \int_0^s \theta(u) e^{au} du ds + \sigma_2 \int_0^t e^{-as} \int_0^s e^{au} dW_{2,u} \tag{3.10}$$

when the interest rates are stochastic, the bond price is calculated by conditional expectation

$$P(t,T) = E\left[e^{-\int_t^T r_s ds}|\mathcal{F}_t\right] \tag{3.11}$$

where $\mathcal{F}_t$ denotes the information available in the market at time t.

Lemma 3.2.1. The Hull-White zero-coupon bond price is as follows

$$P(0,T) = e^{\frac{r_0(e^{-aT}-1)}{a} - \int_0^T e^{-as}\int_0^s \theta(u)e^{au}duds + \frac{\sigma_2^2}{2a^2}\left[T + \frac{1-e^{-2aT}}{2a} - \frac{2}{a}\left(1-e^{-aT}\right)\right]}. \tag{3.12}$$

Proof: By (3.10)

$$\begin{aligned} P(0,T) &= E\left[e^{-\int_0^T r_s ds}|\mathcal{F}_0\right] \\ &= e^{\frac{r_0\left(1-e^{-aT}\right)}{a}} e^{-\int_0^T e^{-as}\int_0^s \theta(u)e^{au}duds} E\left[e^{-\sigma_2\int_0^T e^{-as}\int_0^s e^{au}dW_{2,u}ds}|\mathcal{F}_0\right] \end{aligned} \tag{3.13}$$

The proof will be carried out in several steps.

Step 1: Set $X_T = \int_0^T e^{-as}\int_0^s e^{au}dW_{2,u}ds$, then

$$E\left[X_T\right] = \int_0^T e^{-as}E\left[\int_0^s e^{au}dW_{2,u}\right]ds = 0 \tag{3.14}$$

since $\int_0^s e^{au}dW_{2,u}$ is gaussian with mean 0 and variance $\frac{e^{2as}-1}{2a}$.

Step 2: By product rule

$$\begin{aligned} d\left(X_s\int_0^s e^{au}dW_{2,u}\right) &= \int_0^s e^{au}dW_{2,u}dX_s + X_s d\int_0^s e^{au}dW_{2,u} + \underbrace{dX_s d\int_0^s e^{au}dW_{2,u}}_{0} \\ &= e^{-as}\left(\int_0^s e^{au}dW_{2,u}\right)^2 ds + X_s e^{as}dW_{2,s} \end{aligned} \tag{3.15}$$

Integrating the above gives

$$X_T\int_0^T e^{as}dW_{2,s} = \int_0^T e^{-as}\left(\int_0^s e^{au}dW_{2,u}\right)^2 ds + \int_0^T X_s e^{as}dW_{2,s}. \tag{3.16}$$

By taking the expectation and using the fact that the Wiener integral has zero mean, we obtain

$$E\left[X_T\int_0^T e^{as}dW_{2,s}\right] = \int_0^T e^{-as}E\left[\left(\int_0^T e^{as}dW_{2,s}\right)^2\right]ds$$
$$= \int_0^T e^{-as}\left(\frac{e^{2as}-1}{2a}\right) \tag{3.17}$$
$$= \frac{1}{a^2}\left(\frac{e^{aT}+e^{-aT}}{2}-1\right)$$

Step 3: Applying Ito Lemma:

$$d\left(X_T^2\right) = 2X_T dX_T + \left(dX_T\right)^2 = 2X_T e^{-aT}\int_0^T e^{as}dW_{2,s}dt \tag{3.18}$$

then integrating and applying step 2, yields

$$E\left[X_T^2\right] = 2\int_0^T e^{-as}E\left[X_s\int_0^s e^{au}dW_{2,u}\right]ds = \frac{2}{a^2}\int_0^T\left(\frac{1+e^{-2as}}{2}-e^{-as}\right)ds$$
$$= \frac{1}{a^2}\left[T+\frac{1}{2a}\left(1-e^{-2aT}\right)+\frac{2}{a}\left(e^{-aT}-1\right)\right] \tag{3.19}$$

Step 4: Using a stochastic variant of Fubini's theorem we interchange the Riemann and the Wiener integrals as follows

$$X_T = \int_0^T e^{-as}\int_0^s e^{au}dW_{2,u}ds = \int_0^s e^{au}\int_0^T e^{-as}dsdW_{2,u}$$
$$= \int_0^T e^{-as}ds\int_0^s e^{au}dW_{2,u} = \frac{1}{a}\left(1-e^{-aT}\right)\int_0^s e^{au}dW_{2,u} \tag{3.20}$$

which implies that X_T is normally distributed with mean 0 and variance $E[X_T^2]$ computed in step 3. Therefore

$$E\left[e^{-\sigma_2\int_0^T e^{-as}\int_0^s e^{au}dW_{2,u}ds}\right] = E\left[e^{\sigma_2 X_T}\right] = e^{\left(\frac{\sigma_2^2}{2}\right)Var(X_T)}$$
$$= e^{\frac{\sigma_2^2}{2a^2}\left[T+\frac{1-e^{-2aT}}{2a}-\frac{2}{a}\left(1-e^{-aT}\right)\right]} \tag{3.21}$$

which gives rise to the formula (zero coupon bond price).

3.3 Hull-White under T-forward Measure

The stochastic model for the bond price under Hull-White model is as follows [13].

$$\begin{aligned} dP(t,T) &= r_t P(t,T)dt + \nu(t,T)P(t,T)dW_t \\ &= r_t P(t,T)dt - \frac{\sigma_2}{a}\left(1 - e^{-a(T-t)}\right) P(t,T)dW_t \end{aligned} \tag{3.22}$$

In order to simplify the calculation of option value under the stochastic interest rate, we use the technique of changing the measure and numeraire. Following general considerations in Brigo and Mercurio [2] the dynamic of Hull-White model under the zero-coupon bond as numeraire can be obtained using the following.

Proposition 3.3.1. [2] Assume two numeraires B and P evolve under a probability measure Q

$$\begin{aligned} dB_t &= (...)\, dt + \sigma_t^B dW_t \\ dP_t &= (...)\, dt + \sigma_t^P dW_t \end{aligned} \tag{3.23}$$

Then the drift of process X under numeraire P is

$$\mu_t^P(x_t) = \mu_t^B(X_t) - \sigma_t(X_t)\left(\frac{\sigma_t^B}{B_t} - \frac{\sigma_t^P}{P_t}\right) \tag{3.24}$$

and

$$dW_t^P = dW_t + \left(\frac{\sigma_t^B}{B_t} - \frac{\sigma_t^P}{P_t}\right) dt. \tag{3.25}$$

Note. By the Proposition for money market account $dB_t = r_t B_t dt$ and zero-coupon bond $dP(t,T) = r_t P(t,T)dt - \frac{\sigma}{a}\left(1 - e^{-a(T-t)}\right) P(t,T)dW_{2,t}$, r_t for Hull-White model under T-forward measure Q^T satisfies the following SDE

$$dr_t = \left(\theta(t) - ar_t - \frac{\sigma_2^2}{a}\left(1 - e^{-a(T-t)}\right)\right) dt + \sigma dW_t^T \tag{3.26}$$

where

$$dW_t^T = dW_t + \frac{\sigma}{a}\left(1 - e^{-a(T-t)}\right) dt. \tag{3.27}$$

Now that we obtained the evolution of Hull-White under T-forward measure, we solve (3.24) via multiplying by the integrating factor e^{at} to get

$$d\left(r_t e^{at}\right) = e^{at}\theta(t)dt - e^{at}\frac{\sigma_2^2}{a}\left(1 - e^{-a(T-t)}\right)dt + e^{at}\sigma_2 dW_{2,t}^T. \tag{3.28}$$

Integrating from 0 to t yields

$$\begin{aligned} r_t = r_0 e^{-at} - \frac{\sigma_2^2}{a}\left[\frac{\left(e^{at}-1\right)e^{-aT}\left(2e^{aT} - e^{at} - 1\right)}{2a}\right]e^{-at} \\ + e^{-at}\int_0^t \theta(s)e^{as}ds + \sigma_2\int_0^t e^{-a(t-s)}dW_{2,s}^T \end{aligned} \tag{3.29}$$

By integrating over $[0,T]$ we obtain

$$\begin{aligned} \int_0^T r_t dt = r_0\frac{1-e^{-aT}}{a} - \frac{\sigma_2^2}{a}\left[\frac{e^{-2aT}\left(\left(2aT-3\right)e^{2aT} + 4e^{aT} - 1\right)}{2a^2}\right] \\ + \int_0^T e^{-at}\int_0^t \theta(s)e^{as}dsdt + \sigma_2\int_0^T\int_0^t e^{-a(t-s)}dW_{2,u}^T dt. \end{aligned} \tag{3.30}$$

3.4 Put Option Price

The Put Option price is expressed as a product of the expectation of V_T under the T-forward measure and the price of zero-coupon bond. Notice that by (3.3) stock price S_T is lognormally distributed and we denote its probability density by $f(s)$ for $S_T = s$.

Theorem 3.4.1. (Put Option Price) The Put Option price at time 0 under the Hull-White interest rate is given by

$$\begin{aligned} P_T = P(0,T)\left(\left(K - \frac{(1-\beta)K(2-\alpha)}{2}\right)N\left[d_1\right] - \beta e^{\mu_T + \frac{1}{2}\sigma_T^2}N\left[d_2\right]\right) \\ - P(0,T)\left(\frac{1-\beta}{\alpha}\frac{2\alpha-1}{2}K\left[N\left(d_3\right) - N\left(d_1\right)\right]\right) \\ - P(0,T)\left(\frac{1-\beta}{\alpha}(1-\alpha)e^{\mu_T + \frac{1}{2}\sigma_T^2}\left[N\left(d_4\right) - N\left(d_2\right)\right]\right) \\ + P(0,T)\left(\frac{\beta-1}{2\alpha K}e^{\mu_T + \frac{1}{2}\sigma_T^2}\left[N\left(d_4\right) - N\left(d_2\right)\right]\right) \end{aligned} \tag{3.31}$$

where

$$P(0,T) = e^{\frac{r_0(e^{-aT}-1)}{a} - \int_0^T e^{-as} \int_0^s \theta(u)e^{au}duds + \frac{\sigma_2^2}{2a^2}\left[T + \frac{1-e^{-2aT}}{2a} - \frac{2}{a}\left(1-e^{-aT}\right)\right]}$$

$$d_1 = \frac{ln(1-\alpha)K-\mu_T}{\sigma_T} \qquad d_2 = \frac{ln(1-\alpha)K-\mu_T-\sigma_T^2}{\sigma_T}$$

$$d_3 = \frac{lnK-\mu_T}{\sigma_T} \qquad d_4 = \frac{lnk-\mu_T-\sigma_T^2}{\sigma_T}$$

$$\mu_T = lnS_0 - \frac{\sigma_1^2}{2}T + r_0\frac{1-e^{-aT}}{a} - \frac{\sigma_2^2}{a}\left[\frac{e^{-2aT}\left((2aT-3)e^{2aT}+4e^{aT}-1\right)}{2a^2}\right] + \int_0^T e^{-at}\int_0^t \theta(s)e^{as}dsdt$$

$$\sigma_T^2 = \sigma_1^2 T + \frac{\sigma_2^2}{a^2}\left[T - 2\frac{1-e^{-aT}}{a} + \frac{1-e^{-2aT}}{2a}\right]$$

Proof.

$$\begin{aligned} E^T\left[V_T\right] &= \int_0^{(1-\alpha)K}\left[(K-\beta S_T) - \frac{(1-\beta)K(2-\alpha)}{2}\right] f(S_T)dS_T \\ &\quad + \int_{(1-\alpha)K}^{K} \frac{1-\beta}{\alpha}\left[\frac{2\alpha-1}{2}K + (1-\alpha)S_T - \frac{S_T^2}{2K}\right] f(S_T)dS_T \\ &= \int_0^{(1-\alpha)K}\left[(K-\beta S_T) - \frac{(1-\beta)K(2-\alpha)}{2}\right] f(S_T)d(S_T) \\ &\quad + \int_{(1-\alpha)K}^{K} \frac{1-\beta}{\alpha}\left[\frac{2\alpha-1}{2}K + (1-\alpha)S_T\right] f(S_T)dS_T \\ &\quad - \int_{(1-\alpha)K}^{K}\left[\frac{1-\beta}{\alpha}\frac{S_T^2}{2K}\right] f(S_T)dS_T \end{aligned} \tag{3.32}$$

We split evaluating $E^T[V_T]$ into integrals I_1, I_2 and I_3 as follows

$$\begin{aligned} I_1 &= \int_0^{(1-\alpha)K}\left[(K-\beta S_T) - \frac{(1-\beta)K(2-\alpha)}{2}\right] f(S_T)d(S_T) \\ I_2 &= \int_{(1-\alpha)K}^{K} \frac{1-\beta}{\alpha}\left[\frac{2\alpha-1}{2}K + (1-\alpha)S_T\right] f(S_T)dS_T \\ I_3 &= -\int_{(1-\alpha)K}^{K}\left[\frac{1-\beta}{\alpha}\frac{S_T^2}{2K}\right] f(S_T)dS_T \end{aligned} \tag{3.33}$$

Set $y = lnS_T$, then $f(e^y)e^y$ is the probability density function of lnS_T, and the mean and variance under the T-forward measure can be expressed form (3.3) as follows

$$\begin{aligned}
\mu_T = E^T\left[lnS_T\right] &= E^T\left[lnS_0 + \int_0^T \left(r_t - \frac{\sigma_1^2}{2}\right)dt + \int_0^T \sigma_1 dW_{1,t}^T\right] \\
&= lnS_0 - \frac{\sigma_1^2}{2}T + E^T\left[\int_0^T r_t dt\right] + \underbrace{E^T\left[\int_0^T \sigma_1 dW_{1,t}^T\right]}_{0} \\
&= lnS_0 - \frac{\sigma_1^2}{2}T + r_0\frac{1-e^{-aT}}{a} - \frac{\sigma_2^2}{a}\left[\frac{e^{-2aT}\left((2aT-3)\,e^{2aT} + 4e^{aT} - 1\right)}{2a^2}\right] \\
&\quad + \int_0^T e^{-at}\int_0^t \theta(s)e^{as}dsdt
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
\sigma_T^2 = Var^T\left[lnS_T\right] &= Var^T\left[lnS_0 + \int_0^T \left(r_t - \frac{\sigma_1^2}{2}\right)dt + \int_0^T \sigma_1 dW_{1,t}^T\right] \\
&= Var^T\left[lnS_0\right] + Var^T\left[\int_0^T \left(r_t - \frac{\sigma_1^2}{2}\right)dt\right] + Var^T\left[\int_0^T \sigma_1 dW_{1,t}^T\right] \\
&= Var^T\left[\int \sigma_1 dW_{1,t}^T\right] + Var^T\left[\int_0^T \sigma_2 e^{-at}\int_0^t e^{as}dW_{2,s}^T dt\right] \\
&= \int_0^T \sigma_1^2 dt + Var^T\left[\sigma_2\int_0^T\int_0^t e^{-a(t-s)}dW_{2,s}^T dt\right] \\
&= \sigma_1^2 T + \frac{\sigma^2}{a^2}\left[T - 2\frac{1-e^{-aT}}{a} + \frac{1-e^{-2aT}}{2a}\right].
\end{aligned} \tag{3.35}$$

Moreover, we have

$$f\left(e^y\right)e^y = \frac{1}{\sqrt{2\pi}\sigma_T}e^{-\frac{1}{2}\frac{(y-\mu_T)^2}{\sigma_T^2}} \tag{3.36}$$

and therefore

$$
\begin{aligned}
I_1 &= \int_0^{(1-\alpha)K} \left((K - \beta S_T) - \frac{(1-\beta)K(2-\alpha)}{2} \right) f(S_T) dS_T \\
&= \int_0^{(1-\alpha)K} \left(K - \frac{(1-\beta)K(2-\alpha)}{2} \right) f(S_T) dS_T - \beta \int_0^{(1-\alpha)K} S_T f(S_T) dS_T \\
&= \int_{-\infty}^{ln(1-\alpha)K} \left(K - \frac{(1-\beta)K(2-\alpha)}{2} \right) f\left(e^y\right) e^y dy - \beta \int_{-\infty}^{ln(1-\alpha)} e^y f\left(e^y\right) e^y dy \\
&= \left(K - \frac{(1-\beta)K(2-\alpha)}{2} \right) \frac{1}{\sqrt{2\pi}\sigma_T} \int_{-\infty}^{ln(1-\alpha)K} e^{-\frac{1}{2}\frac{(y-\mu_T)^2}{\sigma_T^2}} dy \\
&\quad - \beta \frac{1}{\sqrt{2\pi}\sigma_T} \int_{-\infty}^{ln(1-\alpha)K} e^y e^{-\frac{1}{2}\frac{(y-\mu_T)^2}{\sigma_T^2}} dy
\end{aligned}
\tag{3.37}
$$

Setting $z = \frac{y-\mu_T}{\sigma_T}$ gives

$$
\begin{aligned}
I_1 &= \left(K - \frac{(1-\beta)K(2-\alpha)}{2} \right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{ln(1-\alpha)K-\mu_T}{\sigma_T}} e^{-\frac{1}{2}z^2} dz \\
&\quad - \beta \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{ln(1-\alpha)K-\mu_T}{\sigma_T}} e^{\mu_T + z\sigma_T} e^{-\frac{1}{2}z^2} dz \\
&= \left(K - \frac{(1-\beta)K(2-\alpha)}{2} \right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{ln(1-\alpha)K-\mu_T}{\sigma_T}} e^{-\frac{1}{2}z^2} dz \\
&\quad - \beta \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{ln(1-\alpha)K-\mu_T}{\sigma_T}} e^{\mu_T + \frac{1}{2}\sigma_T^2} e^{-\frac{1}{2}(z-\sigma_T^2)} dz \\
&= \left(K - \frac{(1-\beta)K(2-\alpha)}{2} \right) N\left[\frac{ln(1-\alpha)K - \mu_T}{\sigma_T} \right] \\
&\quad - \beta e^{\mu_T + \frac{1}{2}\sigma_T^2} N\left[\frac{ln(1-\alpha)K - \mu_T - \sigma_T^2}{\sigma_T} \right]
\end{aligned}
\tag{3.38}
$$

where $N(x)$ is the standard normal cumulative distribution function.

$$
\begin{aligned}
I_2 &= \int_{(1-\alpha)K}^{K} \frac{1-\beta}{\alpha}\left(\frac{2\alpha-1}{2}K+(1-\alpha)S_T\right)f(S_T)dS_T \\
&= \int_{(1-\alpha)K}^{K}\left(\frac{1-\beta}{\alpha}\frac{2\alpha-1}{2}K\right)f(S_T)dS_T+\frac{1-\beta}{\alpha}(1-\alpha)\int_{(1-\alpha)K}^{K}S_Tf(S_T)dS_T \\
&= \frac{1-\beta}{\alpha}\frac{2\alpha-1}{2}K\int_{ln(1-\alpha)K}^{lnK}f\left(e^y\right)e^ydy+\frac{1-\beta}{\alpha}(1-\alpha)\int_{ln(1-\alpha)K}^{lnK}e^yf\left(e^y\right)e^ydy \\
&= \frac{1-\beta}{\alpha}\frac{2\alpha-1}{2}K\frac{1}{\sqrt{2\pi}\sigma_T}\int_{ln(1-\alpha)K}^{lnK}e^{-\frac{1}{2}\frac{(y-\mu_T)^2}{\sigma_T^2}}dy \\
&\quad+\frac{1-\beta}{\alpha}(1-\alpha)\frac{1}{\sqrt{2\pi}\sigma_T}\int_{ln(1-\alpha)K}^{lnK}e^ye^{-\frac{1}{2}\frac{(y-\mu_T)^2}{\sigma_T^2}}dy \\
&= \frac{1-\beta}{\alpha}\frac{2\alpha-1}{2}K\frac{1}{\sqrt{2\pi}}\int_{\frac{ln(1-\alpha)K-\mu_T}{\sigma_T}}^{\frac{lnK-\mu_T}{\sigma_T}}e^{-\frac{1}{2}z^2}dz \\
&\quad+\frac{1-\beta}{\alpha}(1-\alpha)\frac{1}{\sqrt{2\pi}}\int_{\frac{ln(1-\alpha)K-\mu_T}{\sigma_T}}^{\frac{lnK-\mu_T}{\sigma_T}}e^{\mu_T+z\sigma_T}e^{-\frac{1}{2}z^2}dz \\
&= \frac{1-\beta}{\alpha}\frac{2\alpha-1}{2}K\left[N\left(\frac{lnK-\mu_T}{\sigma_T}\right)-N\left(\frac{ln(1-\alpha)K-\mu_T}{\sigma_T}\right)\right] \\
&\quad+\frac{1-\beta}{\alpha}(1-\alpha)e^{\mu_t+\frac{1}{2}\sigma_T^2}\left[N\left(\frac{lnK-\mu_T-\sigma_T^2}{\sigma_T}\right)-N\left(\frac{ln(1-\alpha)K-\mu_T-\sigma_T^2}{\sigma_T}\right)\right]
\end{aligned}
\tag{3.39}
$$

and

$$
\begin{aligned}
I_3 &= -\int_{(1-\alpha)K}^{K}\left[\frac{1-\beta}{\alpha}\frac{S_T^2}{2K}\right]f(S_T)dS_T=\frac{\beta-1}{2\alpha K}\int_{(1-\alpha)K}^{K}S_T^2f(S_T)dS_T \\
&= \frac{\beta-1}{2\alpha K}\int_{ln(1-\alpha)K}^{lnK}e^{2y}f\left(e^y\right)dy=\frac{\beta-1}{2\alpha K}\int_{ln(1-\alpha)K}^{lnK}e^ye^yf\left(e^y\right)dy \\
&= \frac{\beta-1}{2\alpha K}\frac{1}{\sqrt{2\pi}\sigma_T}\int_{ln(1-\alpha)K}^{lnK}e^ye^{-\frac{1}{2}\frac{(y-\mu_T)^2}{\sigma_T^2}}dy \\
&= \frac{\beta-1}{2\alpha K}\frac{1}{\sqrt{2\pi}}\int_{\frac{ln(1-\alpha)K-\mu_T}{\sigma_T}}^{\frac{lnK-\mu_T}{\sigma_T}}e^{\mu_T+z\sigma_T}e^{-\frac{1}{2}z^2}dz \\
&= \frac{\beta-1}{2\alpha K}e^{\mu_T+\frac{1}{2}\sigma_T^2}\left[N\left(\frac{lnK-\mu_T-\sigma_T^2}{\sigma_T}\right)-N\left(\frac{ln(1-\alpha)K-\mu_T-\sigma_T^2}{\sigma_T}\right)\right]
\end{aligned}
\tag{3.40}
$$

Corollary. 3.4.1. (Put Option Price for Vasicek Model) Observe that in the special case we recover the Put Option price for the Vasicek model when $\theta(t) = ab$.

3.5 Applications to Call Option

European Call Option under the linear investment strategy triggers stock buying whenever the stock price exceeds the strike price. The investment fraction is defined by:

$$Q(S) = \begin{cases} 0 & S \leq K \\ \frac{\beta}{\alpha K}(S - K) & K \leq S \leq (1+\alpha)K \\ \beta & S \geq (1+\alpha)K \end{cases}$$

where

S is stock price.

$Q(S)$ is the stock investment proportion, which is equal to the value of the stock investment divided by A, where A is the entire investment amount.

K is strike price of the option.

α is the investment strategy index, indicating the stock investment occurs during the period in which the stock price increases from K to $(1+\alpha)K$.

β is the maximum value of the stock investment proportion.

Zhang *et al.* [28] derived the Call Option price $C \equiv C_T$ based on the linear investment for the Vasicek interest rate model and we extend their result to the Hull-White model.

Theorem 3.5.1. The Call Option price with the linear investment strategy at time 0 for the Hull-White model is given by

$$
\begin{aligned}
C_T =& P(0,T)\left(1+\frac{\beta}{\alpha}\left(1-\mu_T+lnK-\sigma_T^2\right)\right)e^{\mu_T+\frac{1}{2}\sigma_T^2}\left[N\left(d_1\right)-N\left(d_2\right)\right]\\
&-P(0,T)\left(1+\frac{\beta}{\alpha}\right)K\left[N(d_3)-N(d_4)\right]\\
&-P(0,T)\frac{\beta}{\alpha}\frac{\sigma_T}{\sqrt{2\pi}}e^{\mu_T+\frac{1}{2}\sigma_T^2}\left(e^{-\frac{d_2^2}{2}}-e^{-\frac{d_1^2}{2}}\right)\\
&+P(0,T)\left(1-\frac{\beta}{\alpha}ln(1+\alpha)\right)e^{\mu_T+\frac{1}{2}\sigma_T^2}N(-d_1)\\
&+P(0,T)K(\beta-1)N(-d_3)
\end{aligned}
$$

with $P(0,T)$, d_1, d_2, d_3, d_4, μ_T and σ_T^2 defined below

$$
P(0,T)=e^{\frac{r_0(e^{-aT}-1)}{a}-\int_0^T e^{-as}\int_0^s\theta(u)e^{au}duds+\frac{\sigma_2^2}{2a^2}\left[T+\frac{1-e^{-2aT}}{2a}-\frac{2}{a}\left(1-e^{-aT}\right)\right]}
$$

$$
\begin{aligned}
d_1 &= \tfrac{ln(1+\alpha)K-\mu_T-\sigma_T^2}{\sigma_T} & d_2 &= \tfrac{lnK-\mu_T-\sigma_T^2}{\sigma_T}\\
d_3 &= \tfrac{ln(1+\alpha)K-\mu_T}{\sigma_T} & d_4 &= \tfrac{lnK-\mu_T}{\sigma_T}
\end{aligned}
$$

$$
\begin{aligned}
\mu_T &= lnS_0-\frac{\sigma_1^2}{2}T+r_0\frac{1-e^{-aT}}{a}-\frac{\sigma_2^2}{a}\left[\frac{e^{-2aT}\left(\left(2aT-3\right)e^{2aT}+4e^{aT}-1\right)}{2a^2}\right]\\
&+\int_0^T e^{-at}\int_0^t\theta(s)e^{as}dsdt\\
\sigma_T^2 &= \frac{\sigma_2^2}{a^2}\left[T-2\frac{1-e^{-aT}}{a}+\frac{1-e^{-2aT}}{2a}\right]+\sigma_1^2T
\end{aligned}
$$

Proof: The formula for C_T has been derived in Zhang *et al.* [28] for the Vasicek model with explicit dependence on the bond price $P(0,T)$, μ_T and σ_T^2. Since in the Hull-White model the respective bond price $P(0,T)$, μ_T and σ_T^2 have been found in (3.10), (3.32) and (3.33) respectively, and the derivation of the Call Option price C_T in Hull-White model is analogous to that of Vasicek model we omit the proof of the formula C_T.

Corollary 3.5.1. (Call Option price for Vasicek Model). Observe that in the special case we recover the Call Option price for the Vasicek model when $\theta(t) = ab$.

CHAPTER 4

Market Implementation

To generalize the adaptive hedging European Option for practical application, a discrete trading strategy for Call Option is derived by Meng Li *et al.* [12], and the Option pricing formula is deducted based on the non-random interest rates. We extended their strategy by considering a European Put Option, in the cases of both non-random and random interest rates models. The Put Option price with discrete investment strategy under the Vasicek model and the extended Vasicek model (Hull-White) is derived.

4.1 Call Options

4.1.1 Discrete Trading Strategy

In order to make the linear investment strategy for Option pricing more adaptable to the real market, we work on discrete linear trading strategy. Adaptive risk hedging European Call Option with discrete trading position was first introduced by Li *et al.* [12].

4.1.2 Strategy Assumptions

The adaptive risk hedging for European Call Option is proposed based on the following assumptions:

- The Call Option holder holds one Option contract and an initial Capital of amount $A = Q \times K$ at the beginning of the Option period, where Q is the number of shares of stock for Option contract and K is the Call Option strike price.

- The Call Option holder should buy the underlying stock according to the price changes subject to the discrete trading strategy.
- The potential loss sustained by the Option holder through adaptive trading is not the responsibility of the Option writer.
- There are no transaction costs for buying the underlying stock.

4.1.3 Description of Discrete Trading Strategy

The Call Option contains a capital amount A initially. When the price of the underlying stock goes up to $(K+\delta)$, $\delta \geqslant 0$ the Option holder spends $\beta_0 A$ to buy stocks. The parameter β_0 is a constant between 0 and 1, called the initial capital utilization coefficient. The Option holder linearly adjusts the capital utilization to increase the holding while the price continues to increase and hits a series of equally spaced points S_n where $\{S_n : S_n = K + n\Delta, n = 1, 2, ..., N\}$, with Δ, a positive constant, the price distance for two consecutive trading actions and N the total number of trades within the option valid period.

The strategy parameters α (strategy index) and β (maximum capital utilization), both positive numbers, illustrate that the maximum amount of capital tradable is βA when the price reaches $(1+\alpha)(K+\sigma)$. The strategy assumes the option holder will evenly $(\frac{\beta A}{N})$ distribute the capital over each of the potential trades corresponding to S_n, $n = 1, 2, ..., N$, by purchasing $\frac{\beta A}{N S_n}$ shares of stocks.

4.1.4 The Value Function *V(S)*

It was found in [11] that the Call Option value V_T based on the discrete trading strategy reads

$$V_T = \begin{cases} 0 & S_T < K \\ S_T - K & K \leq S_T < S_1 \\ S_T - \frac{\beta S_T K}{N} \sum_{n=1}^{m} \frac{1}{K+n\Delta} + \left(\frac{\beta m}{N} - 1\right) K & S_m \leq S_T < S_{m+1} \leq S_N \\ S_T - \frac{\beta S_T K}{N} \sum_{n=1}^{N} \frac{1}{K+n\Delta} + (\beta - 1)K & S_N \leqslant S_T \end{cases} \tag{4.1}$$

with the investment parameters α, β, strike price K, and the terminal stock price S_T. The Call Option pricing formula is derived by [12] for the special case of interest rate to be fixed based on the geometric fractional brownian motion stock price behavior, and is comparable with classical B-S model. In what follows, we compute the Call Option price under the discrete position strategy for the case of non-random, Vasicek and Hull-White interest rate models.

4.1.5 Option Price under Fixed Interest Rates r

Suppose S_T, the stock price dynamic under the risk-neutral measure be as

$$dS_t = rS_t dt + \sigma_1 S_t dW_{1,t}, \quad S(0) = S_0 > 0 \ \ 0 \leq t \leq T \tag{4.2}$$

By Ito formula, the stock price at the maturity time T can be obtained and reads

$$S_T = S_0 e^{\left[(r - \frac{1}{2}\sigma_1^2)t + \sigma_1 W_{1,T}\right]} \tag{4.3}$$

Based on the value function V_T in (4.1), the Call Option price under risk-neutral measure can be evaluated as

$$C = e^{-rT} E[V_T] \tag{4.4}$$

where,

$$E[V_T] = \int_K^{S_1} (S_T - K)\, f(S_T) dS_T$$
$$+ \int_{S_m}^{S_{m+1}} \left[\left(1 - \frac{\beta K}{N} \sum_{n=1}^{m} \frac{1}{K + n\Delta} \right) S_T + \left(\frac{\beta m}{N} - 1 \right) K \right] f(S_T) dS_T$$
$$+ \int_{S_N}^{+\infty} \left[\left(1 - \frac{\beta K}{N} \sum_{n=1}^{N} \frac{1}{K + n\Delta} \right) S_T + (\beta - 1) K \right] f(S_T) dS_T \tag{4.5}$$

with

$$I_1 = \int_K^{S_1} (S_T - K)\, f(S_T) dS_T$$
$$I_2 = \int_{S_m}^{S_{m+1}} \left[\left(1 - \frac{\beta K}{N} \sum_{n=1}^{m} \frac{1}{K + n\Delta} \right) S_T + \left(\frac{\beta m}{N} - 1 \right) K \right] f(S_T) dS_T$$
$$I_3 = \int_{S_N}^{+\infty} \left[\left(1 - \frac{\beta K}{N} \sum_{n=1}^{N} \frac{1}{K + n\Delta} \right) S_T + (\beta - 1) K \right] f(S_T) dS_T \tag{4.6}$$

calculating the three integrals I_1, I_2 and I_3 with a similar method to the previous chapters and using the probability density function of lnS_T, whose mean and variance are as follows

$$\begin{aligned} \mu = E\left[lnS_T\right] &= E\left[lnS_0 + \left(r - \frac{1}{2}\sigma_1^2 \right) T + \sigma_1 W_{1,T} \right] \\ &= lnS_0 + \left(r - \frac{1}{2}\sigma_1^2 \right) T \end{aligned} \tag{4.7}$$

$$\begin{aligned} \sigma_1^2 &= Var\left[lnS_0 + \left(r - \frac{1}{2}\sigma_1^2 \right) T + \sigma_1 W_{1,T} \right] \\ &= \sigma_1^2 T \end{aligned} \tag{4.8}$$

gives the Call Option price.

$$
\begin{aligned}
I_1 &= \int_K^{S_1} (S_T - K)\, f(S_T) dS_T \\
&= \int_{lnK}^{lnS_1} (e^y - K)\, f\left(e^y\right) e^y dy \\
&= \int_{lnK}^{lnS_1} (e^y - K)\, \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}} dy \\
&= \int_{lnK}^{lnS_1} \frac{1}{\sqrt{2\pi}\sigma} e^y e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}} dy - K \int_{lnK}^{lnS_1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}} dy \\
&= \int_{\frac{lnK-\mu}{\sigma}}^{\frac{lnS_1-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{\mu+\sigma z} e^{-\frac{1}{2}z^2} dz - K \int_{\frac{lnK-\mu}{\sigma}}^{\frac{lnS_1-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{\frac{lnK-\mu}{\sigma}}^{\frac{lnS_1-\mu}{\sigma}} e^{\mu+\frac{1}{2}\sigma_2} e^{-\frac{1}{2}(z-\sigma)^2} dz - K \frac{1}{\sqrt{2\pi}} \int_{\frac{lnK-\mu}{\sigma}}^{\frac{lnS_1-\mu}{\sigma}} e^{-\frac{1}{2}z^2} dz \\
&= e^{\mu+\frac{1}{2}\sigma_2} \left[N\left(\frac{lnS_1-\mu-\sigma^2}{\sigma}\right) - N\left(\frac{lnK-\mu-\sigma^2}{\sigma}\right)\right] \\
&\quad - K\left[N\left(\frac{lnS_1-\mu}{\sigma}\right) - N\left(\frac{lnK-\mu}{\sigma}\right)\right].
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
I_2 &= \int_{S_m}^{S_{m+1}} \left[\left(1 - \frac{\beta K}{N}\sum_{n=1}^{m}\frac{1}{K+n\Delta}\right) S_T + \left(\frac{\beta m}{N} - 1\right) K\right] f(S_T) dS_T \\
&= \int_{S_m}^{S_{m+1}} \left(1 - \frac{\beta K}{N}\sum_{n=1}^{m}\frac{1}{K+n\Delta}\right) S_T f(S_T) dS_T + \int_{S_m}^{S_{m+1}} \left(\frac{\beta m}{N} - 1\right) K f(S_T) dS_T \\
&= \left(1 - \frac{\beta K}{N}\sum_{n=1}^{m}\frac{1}{K+n\Delta}\right) \int_{S_m}^{S_{m+1}} e^y f\left(e^y\right) e^y dy + \left(\frac{\beta m}{N} - 1\right) K \int_{S_m}^{S_{m+1}} f(e^y) e^y dy \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left(1 - \frac{\beta K}{N}\sum_{n=1}^{m}\frac{1}{K+n\Delta}\right) \int_{S_m}^{S_{m+1}} e^y e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}} dy \\
&\qquad + \frac{1}{\sqrt{2\pi}\sigma}\left(\frac{\beta m}{N} - 1\right) K \int_{S_m}^{S_{m+1}} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}} dy \\
&= \frac{1}{\sqrt{2\pi}} \left(1 - \frac{\beta K}{N}\sum_{n=1}^{m}\frac{1}{K+n\Delta}\right) \int_{\frac{S_m-\mu}{\sigma}}^{\frac{S_{m+1}-\mu}{\sigma}} e^{\mu+z\sigma} e^{-\frac{1}{2}z^2} dz + \frac{1}{\sqrt{2\pi}} \left(\frac{\beta m}{N} - 1\right) K \int_{\frac{S_m-\mu}{\sigma}}^{\frac{S_{m+1}-\mu}{\sigma}} e^{-\frac{1}{2}z^2} dz \\
&= e^{\mu+\frac{1}{2}\sigma^2} \left(1 - \frac{\beta K}{N}\sum_{n=1}^{m}\frac{1}{K+n\Delta}\right) \left[N\left(\frac{lnS_{m+1}-\mu-\sigma^1}{\sigma}\right) - N\left(\frac{lnS_m-\mu-\sigma^2}{\sigma}\right)\right] \\
&\qquad + \left(\frac{\beta m}{N} - 1\right) K \left[N\left(\frac{lnS_{m+1}-\mu}{\sigma}\right) - N\left(\frac{lnS_m-\mu}{\sigma}\right)\right].
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
I_3 &= \int_{S_N}^{+\infty} \left[\left(1 - \frac{\beta K}{N} \sum_{n=1}^{N} \frac{1}{K + n\Delta}\right) S_T + (\beta - 1) K \right) f(S_T) dS_T \\
&= \left(1 - \frac{\beta K}{N} \sum_{n=1}^{N} \frac{1}{K + n\Delta}\right) \int_{lnS_N}^{+\infty} e^y f\left(e^y\right) e^y dy + (\beta - 1) K \int_{lnS_N}^{+\infty} f\left(e^y\right) e^y dy \\
&= \left(1 - \frac{\beta K}{N} \sum_{n=1}^{N} \frac{1}{K + n\Delta}\right) \frac{1}{\sqrt{2\pi}\sigma} \int_{lnS_N}^{+\infty} e^y e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}} dy \\
&\qquad + (\beta - 1) K \frac{1}{\sqrt{2\pi}\sigma} \int_{lnS_N}^{+\infty} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}} dy \\
&= e^{\mu + \frac{1}{2}\sigma^2} \left(1 - \frac{\beta K}{N} \sum_{n=1}^{N} \frac{1}{K + n\Delta}\right) \left[N\left(\frac{\mu + \sigma^2 - lnS_N}{\sigma}\right) \right] \\
&\qquad + (\beta - 1) K N \left(\frac{\mu - lnS_N}{\sigma}\right).
\end{aligned} \tag{4.11}
$$

Thus, from formulas (4.4) to (4.11), the Option price under fixed interest rate can be given by

$$
\begin{aligned}
C &= e^{-rT} E\left[V_T\right] \\
&= e^{-rT} e^{\mu + \frac{1}{2}\sigma^2} \left[N\left(d_1\right) - \left(d_2\right)\right] \\
&\quad - K e^{-rT} \left[N\left(d_3\right) - N\left(d_4\right)\right] \\
&\quad + e^{-rT} e^{\mu + \frac{1}{2}\sigma^2} \left(1 - \frac{\beta K}{N} \sum_{n=1}^{m} \frac{1}{K + n\Delta}\right) \left[N\left(d_5\right) - N\left(d_6\right)\right] \\
&\quad + e^{-rT} \left(\frac{\beta m}{N} - 1\right) \left[N\left(d_7\right) - N\left(d_8\right)\right] \\
&\quad + e^{-rT} e^{\mu + \frac{1}{2}\sigma^2} \left(1 - \frac{\beta K}{N} \sum_{n=1}^{N} \frac{1}{K + n\Delta}\right) \left[N\left(d_9\right)\right] \\
&\quad + e^{-rT} K(\beta - 1) N\left(d_{10}\right)
\end{aligned} \tag{4.12}
$$

where,

$$d_1 = \frac{lnS_1 - \mu - \sigma^2}{\sigma} \qquad d_6 = \frac{lnS_m - \mu - \sigma^2}{\sigma}$$
$$d_2 = \frac{lnK - \mu - \sigma^2}{\sigma} \qquad d_7 = \frac{lnS_{m+1} - \mu}{\sigma}$$
$$d_3 = \frac{lnS_1 - \mu - \sigma^2}{\sigma} \qquad d_8 = \frac{lnS_m - \mu}{\sigma}$$
$$d_4 = \frac{lnK - \mu}{\sigma} \qquad d_9 = \frac{\mu + \sigma^2 - lnS_N}{\sigma}$$
$$d_5 = \frac{lnS_{m+1} - \mu - \sigma^2}{\sigma} \qquad d_{10} = \frac{\mu - lnS_N}{\sigma}$$

and

$$\mu = lnS_0 + \left(\mu - \frac{1}{2}\sigma_1^2\right) T$$
$$\sigma^2 = \sigma_1^2 T$$

4.1.6 Option Price under Stochastic Interest Rates

4.1.6.1 Vasicek Interest Rates Model

Solution to the Vasicek interest rate model, is used to obtain the Call Option price C according to the following formula

$$C = P(0,T)E^T\left[V_T\right]$$

where

$$S_T = S_0 e^{\int_0^T \left(r_s - \frac{1}{2}\sigma_1^2\right)ds + \int_0^T \sigma_1 dW_{1,t}}$$

and $P(0,T)$ is the price of the zero-coupon bond. Also the mean μ_T and Variance σ_2^2 under the T-forward measure in the case of the Vasicek model has been obtained by Zhang [28]. Thus the expectation of V_T under the T-forward measure based on the discrete dynamic strategy can be obtained similarly to (4.12) and is

$$
\begin{aligned}
C = {} & P(0,T)e^{\mu_T+\frac{1}{2}\sigma_T^2}\left[N\left(d_1\right)-\left(d_2\right)\right] \\
& - KP(0,T)\left[N\left(d_3\right)-N\left(d_4\right)\right] \\
& + P(0,T)e^{\mu_T+\frac{1}{2}\sigma_T^2}\left(1-\frac{\beta K}{N}\sum_{n=1}^{m}\frac{1}{K+n\Delta}\right)\left[N\left(d_5\right)-N\left(d_6\right)\right] \\
& + P(0,T)\left(\frac{\beta m}{N}-1\right)\left[N\left(d_7\right)-N\left(d_8\right)\right] \\
& + P(0,T)e^{\mu_T+\frac{1}{2}\sigma_T^2}\left(1-\frac{\beta K}{N}\sum_{n=1}^{N}\frac{1}{K+n\Delta}\right)N\left(d_9\right) \\
& + P(0,T)K(\beta-1)N\left(d_{10}\right)
\end{aligned}
\tag{4.13}
$$

where,

$$
\begin{aligned}
d_1 &= \frac{lnS_1-\mu_T-\sigma_T^2}{\sigma_T} & d_6 &= \frac{lnS_m-\mu_T-\sigma_T^2}{\sigma_T} \\
d_2 &= \frac{lnK-\mu_T-\sigma_T^2}{\sigma_T} & d_7 &= \frac{lnS_{m+1}-\mu_T}{\sigma_T} \\
d_3 &= \frac{lnS_1-\mu_T-\sigma_T^2}{\sigma_T} & d_8 &= \frac{lnS_m-\mu_T}{\sigma_T} \\
d_4 &= \frac{lnK-\mu_T}{\sigma_T} & d_9 &= \frac{\mu_T+\sigma_T^2-lnS_N}{\sigma_T} \\
d_5 &= \frac{lnS_{m+1}-\mu_T-\sigma_T^2}{\sigma_T} & d_{10} &= \frac{\mu_T-lnS_N}{\sigma_T}
\end{aligned}
$$

and

$$
\begin{aligned}
\mu_T &= lnS_0-\frac{\sigma_1^2}{2}T-r_0\frac{1-e^{-aT}}{a}+\left(\frac{\theta}{a}-\frac{\sigma_2^2}{a^2}\right)\left[T-\frac{1-e^{-aT}}{a}\right]+\frac{\sigma_2^2}{2a}\frac{1-e^{-aT}}{a} \\
\sigma_T^2 &= \frac{\sigma_2^2}{a^2}\left[T-2\frac{1-e^{-aT}}{a}+\frac{1-e^{-2aT}}{2a}\right]+\sigma_1^2T
\end{aligned}
$$

with the zero-Coupon bond price as

$$
P(0,T)=exp\left\{\left[\frac{1-e^{-aT}}{a}-T\right]\left(\frac{\theta}{a}-\frac{\sigma_2^2}{2a^2}\right)-\frac{\sigma_2^2\frac{1-e^{-aT}}{a}}{4a}\right\}e^{-r_0\frac{1-e^{-aT}}{a}}. \tag{4.14}
$$

4.1.6.2 Hull-White Interest Rates Model

We derived the Option price under Hull-White interest rates model using the solution to the Hull-White *SDE* obtained in chapter 3, (3.8). Since our derivation is under T-forward measure, we need to consider the zero-coupon bond price $P(0,T)$ and the mean μ_T and Variance σ_T^2 for this model under the T-forward measure. The Option price in this case reads (4.13), where

$$\mu_T = lnS_0 - \frac{\sigma_1^2}{2}T + r_0\frac{1-e^{-aT}}{a} - \frac{\sigma_2^2}{a}\left[\frac{e^{-2aT}\left(\left(2aT-3\right)e^{2aT}+4e^{aT}-1\right)}{2a^2}\right]$$
$$+\int_0^T e^{-at}\int_0^t \theta(s)e^{as}dsdt$$
$$\sigma_T^2 = \frac{\sigma_2^2}{a^2}\left[T - 2\frac{1-e^{-aT}}{a} + \frac{1-e^{-2aT}}{2a}\right] + \sigma_1^2 T$$

and the zero-coupon bond price is

$$P(0,T) = e^{\frac{r_0(e^{-aT}-1)}{a} - \int_0^T e^{-as}\int_0^s \theta(u)e^{au}duds + \frac{\sigma_2^2}{2a^2}\left[T + \frac{1-e^{-2aT}}{2a} - \frac{2}{a}\left(1-e^{-aT}\right)\right]}$$

4.2 Put Options

European Put Option with linear investment strategy triggers stock selling whenever the stock price S_t falls below the strike price K and stays in the range $[(1-\alpha)K, K]$.

4.2.1 Description of Discrete Trading Strategy

The Put Option contains a specified capital amount A and holding of Q initially. When the stock price drops from K to $(K-\delta)$, $\delta \geqslant 0$ the Option holder sells $\beta_0.A$ proportion of stocks. Parameter β_0 is called the initial capital utilization coefficient which is a constant between 0 and 1. The Option holder linearly adjusts the capital utilization to decrease the holding if the price continues to fall until it reaches $(1-\alpha)(K-\delta)$ and the total capital spending reaches $\beta.A$ with $\beta < ... < \beta_1 < \beta_0$.

The Option holder will only sell when the stock price hits a series of equally spaced points S_n, where $\{S_n : S_n = K - n\Delta, n = 1, ..., N\}$, with Δ a positive constant, the price distance for two consecutive trading actions and N the totall number of trades within the Option valid period. The strategy parameters α (investment index) and β (minimum capital utilization), both positive numbers, illustrate the maximum amount of capital tradable is $\beta_0.A$ when the price reaches $(1-\alpha)(K-\delta)$. The strategy assumes the Option holder will evenly $\left(\frac{\beta A}{N}\right)$ distribute the capital over each of the potential trades corresponding to S_n, $n = 1, 2, ..., N$ by selling $\frac{\beta A}{N S_n}$ shares of stocks.

4.2.2 The Value Function $V(S)$

In the classical European Options, the Option writer will sustain a loss of the amount L as the stock price falls below K

$$L = Q(K-S) = \frac{A}{K}(K-S). \tag{4.15}$$

For the adaptive hedging Option purpose, the investor is required to sell a proportion of underlying stock in order to hedge the risk. In a discrete position strategy, the Option holder's income R based on such transactions throughout the Option valid period can be calculated as,

$$R = \frac{\beta A}{N S_n}(S_n - S) \tag{4.16}$$

when the stock price falls from S_n to S, with $S_N \leqslant S_{m+1} < S < S_n$.

The Option holder makes a cumulative income $R(S)$ as

$$R(S) = \sum_{n=1}^{m} \frac{\beta A}{N S_n} (S_n - S) \tag{4.17}$$

Thus, the total loss taken by the Option writer can be obtained as

$$\begin{aligned} L(S) =& \frac{A}{K} (K - S) - \sum_{n=1}^{m} \frac{\beta A}{N S_n} (S_n - S) \\ =& \frac{A}{K} (K - S) - \frac{\beta A}{N} \sum_{n=1}^{m} \frac{(S_n - S)}{S_n} \\ =& \frac{A}{K} (K - S) - \frac{\beta A}{N} \sum_{n=1}^{m} \frac{S_n}{S_n} + \frac{\beta A S}{N} \sum_{n=1}^{m} \frac{1}{S_n} \\ =& A - \frac{AS}{K} - \frac{\beta A m}{N} + \frac{\beta A S}{N} \sum_{n=1}^{m} \frac{1}{K - n\Delta} \\ =& \left(1 - \frac{\beta m}{N}\right) A - \frac{AS}{K} + \frac{\beta A S}{N} \sum_{n=1}^{m} \frac{1}{K - n\Delta} \end{aligned} \tag{4.18}$$

however, for $S \leqslant S_N$, the Option writer's loss L is

$$L(S) = (1 - \beta)A - \frac{AS}{K} + \frac{\beta A S}{N} \sum_{n=1}^{m} \frac{1}{K - n\Delta}. \tag{4.19}$$

Therefore, the following function represents the Option writer's loss based on the stock price S

$$L(S) = \begin{cases} (1-\beta)A - \frac{AS}{K} + \frac{\beta AS}{N} \sum\limits_{n=1}^{m} \frac{1}{K-n\Delta} & S \leqslant S_N \\ \left(1 - \frac{\beta m}{N}\right) A - \frac{AS}{K} + \frac{\beta AS}{N} \sum\limits_{n=1}^{m} \frac{1}{K-n\Delta} & S_N \leq S_{m+1} < S \leq S_m \\ \frac{A}{K}(K - S) & S_1 < S \leqslant K \\ 0 & S > K \end{cases} \tag{4.20}$$

Consequently the intrinsic value function $V(S)$ is derived by dividing the loss function by the number of shares of stock within the Option depending on the stock price finishing price S_T

$$V(S_T) = \begin{cases} (1-\beta)K - S_T + \frac{\beta S_T K}{N} \sum\limits_{n=1}^{m} \frac{1}{K-n\Delta} & S_T \leqslant S_N \\ \left(1 - \frac{\beta m}{N}\right) K - S_T + \frac{\beta S_T K}{N} \sum\limits_{n=1}^{m} \frac{1}{K-n\Delta} & S_N \leq S_{m+1} < S_T \leq S_m \\ (K - S_T) & S_1 < S_T \leqslant K \\ 0 & S_T > K \end{cases} \tag{4.21}$$

4.2.3 Option price under Fixed Interest Rates r

Considering the assumptions of section (4.5) and the intrinsic value function $V(S_T)$ in (4.21), the Put Option price under risk neutral measure is evaluated as

$$P = e^{-rT} E\left[V(S_T)\right] \tag{4.22}$$

where,

$$\begin{aligned} E\left[V\left(S_T\right)\right] = & \int_0^{S_N} \left[(1-\beta)K - S_T + \frac{\beta S_T K}{N} \sum_{n=1}^{m} \frac{1}{K - n\Delta}\right] f\left(S_T\right) dS_T \\ & + \int_{s_{m+1}}^{S_m} \left[\left(1 - \frac{\beta m}{N}\right) K - S_T + \frac{\beta S_T K}{N} \sum_{n=1}^{m} \frac{1}{K - n\Delta}\right] f\left(S_T\right) dS_T \\ & + \int_{S_1}^{K} (K - S_T) f\left(S_T\right) dS_T \end{aligned} \tag{4.23}$$

calculating the three integrals I_1, I_2 and I_3 similar to the previous chapters and using the probability density function of lnS_T, whose mean and variance has been obtained in (4.7) and (4.8) gives the Put Option price.

$$
\begin{aligned}
I_1 =& \int_0^{S_N} \left[(1-\beta)K - S_T + \frac{\beta S_T K}{N} \sum_{n=1}^{m} \frac{1}{K-n\Delta} \right] f\left(S_T\right) dS_T \\
=& \int_0^{S_N} (1-\beta)K f\left(S_T\right) dS_T - \int_0^{S_N} \left(1 - \frac{\beta K}{N} \sum_{n=1}^{m} \frac{1}{K-n\Delta}\right) S_T f\left(S_T\right) dS_T \\
=&(1-\beta)K \int_{-\infty}^{lnS_N} f\left(e^y\right) e^y dy - \left(1 - \frac{\beta K}{N} \sum_{n=1}^{m} \frac{1}{K-n\Delta}\right) \int_{-\infty}^{lnS_N} e^y f\left(e^y\right) e^y dy \\
=&(1-\beta)K \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{lnS_N} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}} dy - \left(1 - \frac{\beta K}{N} \sum_{n=1}^{m} \frac{1}{K-n\Delta}\right) \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{lnS_N} e^y e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}} dy \\
=&(1-\beta)K \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{lnS_N-\mu}{\sigma}} e^{-\frac{1}{2}z^2} dz - \left(1 - \frac{\beta K}{N} \sum_{n=1}^{m} \frac{1}{K-n\Delta}\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{lnS_N-\mu}{\sigma}} e^{\mu+z\sigma} e^{-\frac{1}{2}z^2} dz \\
=&(1-\beta)K \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{lnS_N-\mu}{\sigma}} e^{-\frac{1}{2}z^2} dz - \left(1 - \frac{\beta K}{N} \sum_{n=1}^{m} \frac{1}{K-n\Delta}\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{lnS_N-\mu}{\sigma}} e^{\mu+\frac{1}{2}\sigma^2} e^{-\frac{1}{2}(z-\sigma)^2} dz \\
=&(1-\beta)K N\left(\frac{lnS_N-\mu}{\sigma}\right) - \left(1 - \frac{\beta K}{N} \sum_{n=1}^{m} \frac{1}{K-n\Delta}\right) N\left(\frac{lnS_N-\mu-\sigma^2}{\sigma}\right)
\end{aligned}
\tag{4.24}
$$

and

$$
\begin{aligned}
I_2 =& \int_{s_{m+1}}^{S_m} \left[\left(1-\frac{\beta m}{N}\right) K - S_T + \frac{\beta S_T K}{N} \sum_{n=1}^{m} \frac{1}{K-n\Delta} \right] f\left(S_T\right) dS_T \\
=& \left(1-\frac{\beta m}{N}\right) K \int_{s_{m+1}}^{S_m} f\left(e^y\right) e^y dy - \left(1 - \frac{\beta K}{N} \sum_{n=1}^{m} \frac{1}{K-n\Delta}\right) \int_{s_{m+1}}^{S_m} e^y f\left(e^y\right) e^y dy \\
=& \left(1-\frac{\beta m}{N}\right) K \frac{1}{\sqrt{2\pi}\sigma} \int_{s_{m+1}}^{S_m} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}} dy - \left(1 - \frac{\beta K}{N} \sum_{n=1}^{m} \frac{1}{K-n\Delta}\right) \frac{1}{\sqrt{2\pi}\sigma} \int_{s_{m+1}}^{S_m} e^y e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}} dy \\
=& \left(1-\frac{\beta m}{N}\right) K \frac{1}{\sqrt{2\pi}} \int_{\frac{S_{m+1}-\mu}{\sigma}}^{\frac{S_m-\mu}{\sigma}} e^{-\frac{1}{2}z^2} dz - \left(1 - \frac{\beta K}{N} \sum_{n=1}^{m} \frac{1}{K-n\Delta}\right) \frac{1}{\sqrt{2\pi}} \int_{\frac{S_{m+1}-\mu}{\sigma}}^{\frac{S_m-\mu}{\sigma}} e^{\mu+\frac{1}{2}\sigma^2} e^{-\frac{1}{2}(z-\sigma)^2} dz \\
=& \left(1-\frac{\beta m}{N}\right) K \left[N\left(\frac{lnS_m-\mu}{\sigma}\right) - N\left(\frac{lnS_{m+1}-\mu}{\sigma}\right) \right] \\
&- \left(1 - \frac{\beta K}{N} \sum_{n=1}^{m} \frac{1}{K-n\Delta}\right) \left[N\left(\frac{lnS_m-\mu-\sigma^2}{\sigma}\right) - N\left(\frac{lnS_{m+1}-\mu-\sigma^2}{\sigma}\right) \right]
\end{aligned}
\tag{4.25}
$$

similarly,

$$
\begin{aligned}
I_3 =& \int_{S_1}^{K} (K - S_T) f\left(S_T\right) dS_T \\
=& K \int_{lnS_1}^{lnK} f\left(e^y\right) e^y dy - \int_{lnS_1}^{lnK} e^y f\left(e^y\right) e^y dy \\
=& K \frac{1}{\sqrt{2\pi}\sigma} \int_{lnS_1}^{lnK} e^{\frac{(y-\mu)^2}{\sigma^2}} dy - \frac{1}{\sqrt{2\pi}\sigma} \int_{lnS_1}^{lnK} e^y e^{\frac{(y-\mu)^2}{\sigma^2}} dy \\
=& K \left[N\left(\frac{lnK - \mu}{\sigma}\right) - N\left(\frac{lnS_1 - \mu}{\sigma}\right) \right] - e^{\mu + \frac{1}{2}\sigma^2} \left[N\left(\frac{lnK - \mu - \sigma^2}{\sigma}\right) - N\left(\frac{lnS_1 - \mu - \sigma^2}{\sigma}\right) \right]
\end{aligned}
\tag{4.26}
$$

4.2.4 Option price under Stochastic Interest Rates

4.2.4.1 Vasicek Interest Rates Model

The price of Put Option P with the underlying stock price dynamic and interest rate following the Vasicek model reads

$$
P = P(0,T) E^T \left[V_T\right] \tag{4.27}
$$

where $P(0,T)$ is the zero-coupon bond price in (4.14) and μ_T, σ_T^2 the mean and variance under T-forward measure. Thus,

$$
\begin{aligned}
P =& P(0,T) E^T \left[V_T\right] \\
=& P(0,T) K (1-\beta) N\left(d_1\right) \\
& - P(0,T) \left(1 - \frac{\beta K}{N} \sum_{n=1}^{m} \frac{1}{K - n\Delta}\right) N\left(d_2\right) \\
& + P(0,T) \left(1 - \frac{\beta m}{N}\right) K \left[N\left(d_3\right) - N\left(d_4\right)\right] \\
& - P(0,T) \left(1 - \frac{\beta K}{N} \sum_{n=1}^{m} \frac{1}{K - n\Delta}\right) \left[N\left(d_5\right) - N\left(d_6\right)\right] \\
& + P(0,T) K \left[N\left(d_7\right) - N\left(d_8\right)\right] \\
& - P(0,T) e^{\mu + \frac{1}{2}\sigma^2} \left[N\left(d_9\right) - N\left(d_{10}\right)\right]
\end{aligned}
\tag{4.28}
$$

where,

$$d_1 = \frac{lnS_N - \mu_T}{\sigma_T} \qquad d_6 = \frac{lnS_{m+1} - \mu_T - \sigma_T^2}{\sigma_T}$$

$$d_2 = \frac{lnS_N - \mu_T - \sigma_T^2}{\sigma_T} \qquad d_7 = \frac{lnK - \mu_T}{\sigma_T}$$

$$d_3 = \frac{lnS_m - \mu_T}{\sigma_T} \qquad d_8 = \frac{lnS_1 - \mu_T}{\sigma_T}$$

$$d_4 = \frac{lnS_{m+1} - \mu_T}{\sigma_T} \qquad d_9 = \frac{lnK - \mu_T + \sigma_T^2}{\sigma_T}$$

$$d_5 = \frac{lnS_m - \mu_T - \sigma_T^2}{\sigma_T} \qquad d_{10} = \frac{\ln S_1 - mu_T - \sigma_T^2}{\sigma_T}$$

and

$$\mu_T = lnS_0 - \frac{\sigma_1^2}{2}T - r_0\frac{1 - e^{-aT}}{a} + \left(\frac{\theta}{a} - \frac{\sigma_2^2}{a^2}\right)\left[T - \frac{1 - e^{-aT}}{a}\right] + \frac{\sigma_2^2}{2a}\frac{1 - e^{-aT}}{a}$$

$$\sigma_T^2 = \frac{\sigma_2^2}{a^2}\left[T - 2\frac{1 - e^{-aT}}{a} + \frac{1 - e^{-2aT}}{2a}\right] + \sigma_1^2 T$$

4.2.4.2 Hull-White Interest Rates Model

The Put Option price under the extended Vasicek, Hull-White model reads (4.28) where the mean μ_T and σ_T^2 for this model under T-forward measure are as

$$\mu_T = lnS_0 - \frac{\sigma_1^2}{2}T + r_0\frac{1 - e^{-aT}}{a} - \frac{\sigma_2^2}{a}\left[\frac{e^{-2aT}\left((2aT - 3)\, e^{2aT} + 4e^{aT} - 1\right)}{2a^2}\right]$$
$$+ \int_0^T e^{-at}\int_0^t \theta(s)e^{as}dsdt$$
$$\sigma_T^2 = \frac{\sigma_2^2}{a^2}\left[T - 2\frac{1 - e^{-aT}}{a} + \frac{1 - e^{-2aT}}{2a}\right] + \sigma_1^2 T$$

where the zero-coupon bond price is

$$P(0,T) = e^{\frac{r_0(e^{-aT}-1)}{a} - \int_0^T e^{-as}\int_0^s \theta(u)e^{au}duds + \frac{\sigma_2^2}{2a^2}\left[T + \frac{1-e^{-2aT}}{2a} - \frac{2}{a}\left(1 - e^{-aT}\right)\right]}.$$

CHAPTER 5

Conclusions and Future Research

In this study we first presented an effective way for calculating Call Option price in the case of randomly evolving interest rates for the Cox-Ingersoll-Ross model. The method uses Monte Carlo simulation of interest rates path integrals, which is readily carried out thanks to OU process representation.

Furthermore, we obtained the closed form of the Put and Call Option price for the linear investment strategy under the Hull-White stochastic interest rates. In particular, a protective put option can serve as an insurance policy against losses for the stock holder.

Since the option price associated with trading of the underlying security is based on continuous stock trading (impossible to implement!), a feasible discrete variant is in order. Recently Li et al. [12] proposed a discretized method for the Call Option under the classical Black-Scholes with linear investment strategy. A feasible market implementation for our Hull-White pricing model was presented in chapter 4.

Our approach can be extended to any other stochastic interest rate model with suitable solution representation (e.g. some transformation of Brownian Motion) of its underlying *SDE*.

A natural extension of our work is Option pricing based on dynamic investment strategy under Jump-diffusion processes, which typically fits market data better than the simple diffusion processes. The classical Black-Scholes model [1] is a log-normal diffusion process, i.e., the log-return is normally distributed. However, in the real market due to the presence of jumps on stock prices, perfect hedging by diffusion processes is impossible. Merton in [17] introduced the Jump-diffusion model for Option pricing when the underlying asset returns are discontinuous. In the way that the basic building block of diffusion models is Brownian motion, the jump processes starting point is Poisson process.

Poisson Processes share some important properties with Brownian motions, namely, Martingale property. Ito formulas and Lemmas has been derived for Jump processes. Also, analogous to obtaining a Brownian motion without drift from a Brownian motion with drift using Girsanov's theorem, we can change the measure for Possion and compound Poisson processes. For the fundamental definitions and theorems on the subject of Jump diffusion models, see [22].

As far as future research is concerned, we intend to focus on extending our results from chapters 2-3 to Jump-diffusions as those processes provide a better fit to market data than continuous diffusions. Another aspect worth exploring amounts to developing non-linear investment strategies which could incorporate investor's risk aversion via relevant utility functions, e.g., concave for risk avoiding or convex for risk seeking. Unlike the linear investment strategy, derivation of closed form solutions will present a clear challenge due to evaluation of expected values for non-linear functional of Jump-diffusions.

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BIOGRAPHICAL STATEMENT

Niloofar Ghorbani was born in Tehran, Iran, in 1990. She received her B.S. from Khajeh Nasir University of Technology and M.S. degree from Tarbiat Modares University, Iran, in 2013 and 2017, respectively, in Mathematics. She has been a PhD student in the University of Texas at Arlington in Mathematics since 2017. Her research interests focus on Probability and Statistics, Mathematical Finance, Option Pricing with Investment Strategy, Random Processes and Stochastic Analysis.