Twisting Systems and some Quantum $\mathbb{P}^3$s with Point Scheme a Rank-2 Quadric

by

HUNG VIET TRAN

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Supervising Committee:

Michaela Vancliff, Ph.D., Supervising Professor
Gaik Ambartsoumian, Ph.D.
Ruth E. Gornet, Ph.D.
Dimitar Grantcharov, Ph.D.
David A. Jorgensen, Ph.D.

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ABSTRACT

TWISTING SYSTEMS AND SOME QUANTUM $\mathbb{P}^3$S WITH POINT SCHEME A RANK-2 QUADRIC

Hung Viet Tran, Ph.D.

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Supervising Professor: Michaela Vancliff

In 1996, J. J. Zhang introduced the concept of twisting a graded algebra by a twisting system, which generalizes the concept of twisting a graded algebra by an automorphism (the latter concept having been introduced in an article by M. Artin, J. Tate and M. Van den Bergh in 1991). Twisting using a twisting system is an equivalence relation and certain important algebraic properties of the original algebra are carried over to the twisted algebra. We call a twisting system nontrivial if it is not given by an automorphism. However, there are very few known examples of nontrivial twisting systems in the literature.

M. Vancliff and K. Van Rompay in 1997, and B. Shelton and M. Vancliff in 1999, were successful in finding one example each of a nontrivial twisting system. Their twisting systems were constructed on certain quadratic algebras A (on four generators) using two invertible linear maps t and τ that satisfy $t^2 =$ identity and $\tau^2 \in \text{Aut}(A)$. We extend their work on twisting systems using analogous maps that satisfy $t^n =$ identity and $\tau^n \in \text{Aut}(B)$, where $n \in \mathbb{N}$ and B is any finitely generated quadratic algebra.

We illustrate our new method for producing a nontrivial twisting system on an algebra that is a quadratic quantum $\mathbb{P}^3$ whose point scheme is given by a rank-2 quadric in $\mathbb{P}^3$. Such an algebra is a new addition to the literature.

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CHAPTER 1

Introduction

In 1996, J. J. Zhang introduced the concept of twisting a graded algebra by a twisting system, which generalizes the concept of twisting a graded algebra by an automorphism (the latter concept having been introduced in an article by M. Artin, J. Tate and M. Van den Bergh in 1991). Twisting using a twisting system is an equivalence relation and certain important algebraic properties of the original algebra are carried over to the twisted algebra. We call a twisting system nontrivial if it is not given by an automorphism. However, there are very few known examples of nontrivial twisting systems in the literature.

M. Vancliff and K. Van Rompay in 1997, and B. Shelton and M. Vancliff in 1999, were successful in finding one example each of a nontrivial twisting system. Their twisting systems were constructed on certain quadratic algebras A (on four generators) using two invertible linear maps t and τ that satisfy $t^2 =$ identity and $\tau^2 \in \mathrm{Aut}(A)$. Their methods entailed using geometric techniques specific to their algebras. We extend their work on twisting systems using analogous maps that satisfy $t^n =$ identity and $\tau^n \in \mathrm{Aut}(B)$, where $n \in \mathbb{N}$ and B is any finitely generated quadratic algebra.

More specifically, in Section 2.1, some basic definitions and concepts relevant to later chapters are described, such as the standard definitions of a graded algebra, a free algebra, and a quadratic algebra. Some more advanced standard concepts are also provided, such as those of Hilbert series, Koszul algebra, Ore extension of a ring, global homological dimension, Gelfand-Kirillov dimension, Artin-Schelter regularity, and Cohen-Macaulay property. In Section 2.2, we recall the concept of a twist of a graded algebra by an automorphism as given in [2, Section 8]. We note that twisting by an automorphism is reflexive and symmetric, but, in general, not transitive. In Section 2.3, we provide Zhang's concept of a twisting system [16]. As explained in [16], twisting by an automorphism is an example of twisting by a twisting system, and that twisting by a twisting system is reflexive, symmetric and transitive. Also, certain important properties of a graded algebra are preserved by twisting using a twisting system, as discussed in [16], and are recalled in Section 2.2. We conclude Section 2.2 with the example of a nontrivial twisting system that was constructed by M. Vancliff and K. Van Rompay in [15].

In Chapter 3, in our generalization of the twisting systems found in [13, 15], we consider a quadratic algebra $A = T(V)/\langle W \rangle$, where V is a finite-dimensional vector space, $T(V)$ is the tensor algebra on V, and W is a finite-dimensional subspace of $V \otimes V$. We consider two invertible linear maps t and τ that are defined on $T(V)$, and satisfy t^δ = identity and $\tau^\delta \in \mathrm{Aut}(A)$, where $\delta \in \mathbb{N}$. Linear maps $t_n : T(V) \to T(V)$ are defined, in terms of t and τ, in Definition 3.1. We prove that $\{t_n : n \in \mathbb{Z}\}$ is a twisting system of $T(V)$ in Lemma 3.3. Moreover, $\{t_n : n \in \mathbb{Z}\}$ is proved in Theorem 3.6 to be a twisting system of A if it satisfies the hypotheses of Proposition 3.5. In Definition 3.7, we call such a twisting system a twisting system of type $\mathbb{T}_\delta$. The main content of Chapter 3 is to prove that the twisting system of type $\mathbb{T}_\delta$ is well defined. In Corollary 3.9, we give some equivalent conditions for a twisting system of A to be of type $\mathbb{T}_\delta$. In particular, as shown in Corollary 3.9, if $\{t_n : n \in \mathbb{Z}\}$ is a twisting system of A of type $\mathbb{T}_\delta$, then it is a twisting system of A of type $\mathbb{T}_{k\delta}$ for all $k \in \mathbb{N}$. Unlike the methods in [13, 15], our method of constructing twisting systems does not entail the use of any geometric data.

In Chapter 4, we illustrate our method for producing a nontrivial twisting system of type $\mathbb{T}_\delta$ on an algebra that is proved in Chapter 5 to be a quadratic quantum $\mathbb{P}^3$, whose point scheme is given by a rank-2 quadric in $\mathbb{P}^3$. Specifically, we consider a family of quadratic algebras $A(\rho) = T(V)/\langle W \rangle$, where V is a four-dimensional vector space over a field $\Bbbk$ and the subspace W of $V \otimes V$ depends on a nonzero scalar $\rho \in \Bbbk$, and $\mathrm{char}(\Bbbk) \neq 2$. We construct a nontrivial twisting system of $A(\rho)$ of type $\mathbb{T}_2$ (and so also of type $\mathbb{T}_{2k}$ for all $k \in \mathbb{N}$, by Corollary 3.9). In conclusion of Chapter 4, the twist of $A(\rho)$ under the above twisting system $\{t_n : n \in \mathbb{Z}\}$ is given.

In the first section of Chapter 5, we study the family of algebras $A(\rho)$ that are used to produce the nontrivial twisting system example in Chapter 4. We show that each quadratic algebra $A(\rho)$ satisfies several desirable properties, as stated in Theorem 5.1.9, such as being noetherian, Artin-Schelter regular of global dimension four, Auslander-regular, satisfying the Cohen-Macaulay property, having Hilbert series $H(s) = (1 - s)^{-4}$ and being a domain. Consequently, these algebras are quadratic quantum $\mathbb{P}^3$s. In Section 5.2, we show that the point scheme of each quadratic algebra $A(\rho)$ is given by a rank-2 quadric in $\mathbb{P}^3$; that is, the point scheme is the union of two distinct planes in the projective space $\mathbb{P}^3$ (Theorem 5.2.1). Such algebras are a new addition to the literature. Moreover, the zero locus of the defining relations of $A(\rho)$ does not determine the defining relations of $A(\rho)$ (Theorem 5.2.3). We conclude the dissertation by stating some possible future research.

CHAPTER 2

Preliminaries

We begin by defining some relevant terms in abstract algebra, after which we will briefly discuss the concepts of twisting a graded algebra by an automorphism and twisting a graded algebra using a twisting system. As shown in [16], twisting by an automorphism is a special case of twisting using a twisting system. Also, certain important properties of twisting using a twisting system, as discussed in [16], will be mentioned.

Most of the definitions in Section 2.1 can be found in [3, 4, 11]. Also, most of the content of Sections 2.2 and 2.3 is based on [2, Section 8] and [16].

Throughout the dissertation, the notation $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0$ denotes the set $\mathbb{N} \cup \{0\}$, $\mathbb{Z}$ denotes the set of all integers and $\Bbbk$ denotes a field. If R is a ring, then $R^\times$ denotes the set of nonzero elements in R.

2.1 Some Abstract Algebra Definitions

Definition 2.1.1. (cf. [4, Pages 342 and 657])

Let R be a commutative ring with identity 1_R. If A is a ring with identity 1_A together with a ring homomorphism $f : R \longrightarrow A$ mapping 1_R to 1_A such that the subring $f(R)$ of A is contained in the center of A, then A is called an R-**algebra**.

In particular, let $\Bbbk$ be a field. A ring A is a **$\Bbbk$-algebra** if $\Bbbk$ is contained in the center of A, and the unity element of $\Bbbk$ is the unity element of A.

Definition 2.1.2. (cf. [11, Page 103])

Let R denote a nonzero ring. An element r of R is said to be **normal** in R if $rR = Rr$.

Definition 2.1.3. (cf. [11, Page 4])

A nonzero (possibly noncommutative) ring R is said to be a **domain** if the product of nonzero elements in R is always nonzero.

Definition 2.1.4. (cf. [3, Definition 1.1.1])

A $\Bbbk$-algebra A is called **$\mathbb{N}$-graded** if it has a decomposition $A = \bigoplus_{j=0}^{\infty} A_j$, where A_i is an abelian group for all i and $A_i A_j \subseteq A_{i+j}$ for all $i, j \geq 0$. Moreover, we view $A_i = \{0\}$ whenever $i < 0$.

We say that an $\mathbb{N}$-graded $\Bbbk$-algebra A is **connected** if $A_0 = \Bbbk$.

A nonzero element x in A is called **homogeneous of degree** n if $x \in A_n$ for some n.

A left, or right, ideal I of A is called **homogeneous** if it is generated by homogeneous elements, or equivalently if $I = \bigoplus_{j=0}^{\infty} (I \cap A_j)$.

Definition 2.1.5. (cf. [3, Definition 1.2.1])

Let A be an $\mathbb{N}$-graded $\Bbbk$-algebra. A left A-module M is **graded** if M has a decomposition $M = \bigoplus_{n\in\mathbb{Z}} M_n$, where M_n is an abelian group for all n and $A_i M_j \subseteq M_{i+j}$ for all $i \in \mathbb{N}_0$, $j \in \mathbb{Z}$.

Let $k \in \mathbb{Z}$. A homomorphism of graded A-modules $\phi : M \longrightarrow N$ is a **graded homomorphism of degree** k if $\phi(M_n) \subseteq N_{n+k}$ for all $n \in \mathbb{Z}$.

Definition 2.1.6. (cf. [3, Definition 1.1.3])

A $\Bbbk$-algebra A is **finitely generated** if there is a finite set of elements $x_1, \ldots, x_n \in A$ such that the set

$$S = \{x_{i_1} \cdots x_{i_m} \mid 1 \leq i_j \leq n, m \in \mathbb{N}\} \cup \{1\}$$

spans A as a vector space over $\Bbbk$. In this case, the elements $x_1, \ldots, x_n$ are called the **generators** of A, and we say A is **generated** by $x_1, \ldots, x_n$.

Definition 2.1.7. (cf. [3, Example 1.1.2])

If A is defined as in Definition 2.1.6, and if A has basis S as a vector space over $\Bbbk$, then we call A the **free algebra over** $\Bbbk$ **on** n **generators** $x_1, \ldots, x_n$, and write $A = \Bbbk\langle x_1, \ldots, x_n\rangle$. In this dissertation, the free algebra $A = \Bbbk\langle x_1, \ldots, x_n\rangle$ is connected and $\mathbb{N}$-graded, where A_d is the $\Bbbk$-span of all monomials $x_{i_1} \cdots x_{i_d}$ in S.

Remark 2.1.8. (cf. [4, Page 443])

Let $T(V) = \bigoplus_{j=0}^{\infty} V^{\otimes j}$ denote the tensor algebra of a finite-dimensional vector space $V = \Bbbk x_1 \oplus \cdots \oplus \Bbbk x_n$ where $n \in \mathbb{N}$. We note that $T(V) \cong \Bbbk\langle x_1, \ldots, x_n\rangle$.

Remark 2.1.9. (cf. [3, Page 15])

A connected $\mathbb{N}$-graded $\Bbbk$-algebra A (where $A \neq A_0$) is finitely generated if and only if there is a surjective graded ring homomorphism from some free algebra $\Bbbk\langle x_1, \ldots, x_n\rangle$ to A. It follows that $A \cong \Bbbk\langle x_1, \ldots, x_n\rangle/I$ for some homogeneous ideal I of $\Bbbk\langle x_1, \ldots, x_n\rangle$.

Definition 2.1.10. (cf. [12, Page 6])

If A is an $\mathbb{N}$-graded $\Bbbk$-algebra that is generated by homogeneous degree-one elements, where each defining relation of A is homogeneous of degree two, then A is called a **quadratic algebra**.

In particular, if A is a finitely generated quadratic $\Bbbk$-algebra, then $A \cong T(V)/\langle W\rangle$, where $T(V)$ is given in Remark 2.1.8 and W is a subspace of $V \otimes V$.

Definition 2.1.11. (cf. [7, Page 174])

Let $A = \bigoplus_{i=0}^{\infty} A_i$ be an $\mathbb{N}$-graded $\Bbbk$-algebra and let x denote a nonzero homogeneous element of A. We call x n-**regular** if both the right and left multiplication of elements of A_i by x is injective on A_i for all $i \leq n$.

Definition 2.1.12. (cf. [8, Section 5.5])

For an $\mathbb{N}$-graded finitely generated $\Bbbk$-algebra $A = \bigoplus_{i=0}^{\infty} A_i$, its **Hilbert series** $H_A(s)$ is defined to be

$$H_A(s) = \sum_{n\in\mathbb{Z}} \dim(A_n)s^n.$$

Definition 2.1.13. (cf. [7, Page 173])

For a quadratic $\Bbbk$-algebra $A = \bigoplus_{i=0}^{\infty} A_i$ with defining relations given by a subspace $W \subset A_1 \otimes A_1$, the **Koszul dual** of A is defined as $A^! = T(A_1^*)/\langle W^\perp \rangle$, where A_1^* is the dual of the vector space A_1 and $W^\perp = \{X \in A_1^* \otimes A_1^* \mid Xx = 0 \text{ for all } x \in W\}$.

Definition 2.1.14. (cf. [7, Page 173])

Let A be as given in Definition 2.1.13, where A_1 has $\Bbbk$-basis $\{x_1, \ldots, x_n\}$, and let $\{X_1, \ldots, X_n\}$ denote the dual basis in A_1^*. The **Koszul complex** for A is the complex of free left A-modules

$$\cdots \to A \otimes (A_m^!)^* \to \cdots \to A \otimes (A_1^!)^* \to A \otimes (A_0^!)^* \to {}_A\Bbbk \to 0,$$

where ${}_A\Bbbk$ is the trivial left A-module and the differential is right multiplication by $\sum_{i=1}^{n} x_i \otimes X_i \in A \otimes_{\Bbbk} A^!$. The algebra A is called a **Koszul algebra** if this complex is exact.

Definition 2.1.15. (cf. [5, Page 34], [11, Page 38])

Let R be a nonzero ring, σ a ring endomorphism of R, and δ a left σ-derivation of R (in particular, $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$ for all $r, s \in R$). A ring A is called an **Ore extension of** R if

1. A contains R as a subring,
2. there exists $x \in A$ such that A is a free left R-module with basis $\{1, x, x^2, \ldots\}$, and
3. $xr = \sigma(r)x + \delta(r)$ for all $r \in R$.

We write $A = R[x; \sigma, \delta]$ in this case.

Definition 2.1.16. (cf. [1, Page 41])

An algebra A is said to have **finite global homological dimension** d if

$$\begin{aligned} d &= \sup\{\text{projective dimension of } M \mid M = \text{left } A\text{-module}\} \\ &= \sup\{\text{projective dimension of } M \mid M = \text{right } A\text{-module}\} \end{aligned}$$

is finite. In this case we write $\operatorname{gldim}(A) = d$.

Definition 2.1.17. (cf. [6])

The **Gelfand-Kirillov dimension** (**GK-dimension**) of a $\Bbbk$-algebra A is defined as

$$\operatorname{GKdim}(A) = \sup_{V} \left\{ \limsup_{n\to\infty} \log_n(\dim_{\Bbbk}(\sum_{i=0}^{n} V^i)) \right\},$$

where the supremum is taken over all finite-dimensional subspaces V of A, $V^0 = \Bbbk$, and V^i denotes the subspace spanned by all monomials $v_1 \cdots v_i$ for all $i \in \mathbb{N}$, where $v_j \in V$ for all $j \in \mathbb{N}$.

If A is a $\Bbbk$-algebra and M is a left A-module, the **Gelfand-Kirillov dimension** (**GK-dimension**) of M is defined as

$$\operatorname{GKdim}(M) = \sup_{V,F} \left\{ \limsup_{n\to\infty} \log_n(\dim_{\Bbbk}(V^n F)) \right\},$$

where the supremum is taken over all finite-dimensional subspaces V of A containing unity and over all finite-dimensional subspaces F of M. One defines $\operatorname{GKdim}(M)$ for a right A-module M analogously.

Definition 2.1.18. (cf. [1, Pages 41-42])

Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a finitely generated, $\mathbb{N}$-graded, connected $\Bbbk$-algebra, generated by A_1. The algebra A is called **Artin-Schelter regular** (AS-regular) of dimension d if

1. A has finite global homological dimension d, and
2. A has finite Gelfand-Kirillov dimension, and
3. A is Gorenstein; that is, the projective modules appearing in a minimal projective resolution of the trivial left A-module ${}_A\Bbbk$ are finitely generated and $\mathrm{Ext}^q_A({}_A\Bbbk, A) \cong \delta^q_d \Bbbk_A$, where the grading of $\Bbbk_A$ is possibly shifted by some degree, and $\delta^q_d = 1$ if $q = d$ and 0 otherwise.

Definition 2.1.19. (cf. [8, Definition 2.1])

A noetherian ring A is **Auslander-regular** of dimension μ if

1. $\mathrm{gldim}(A) = \mu < \infty$, and
2. for all finitely generated A-modules M and for all $q \geq 0$, $j(N) \geq q$ for every A-submodule N of $\mathrm{Ext}^q_A(M, A)$, where

$$j(N) = \inf\{i \mid \mathrm{Ext}^i_A(N, A) \neq 0\} \in \mathbb{N} \cup \{0, \infty\}.$$

Definition 2.1.20. (cf. [8, Definition 5.8])

Let A be a noetherian $\Bbbk$-algebra with GKdim $(A) = \mu \in \mathbb{N}_0$. We say that A satisfies the **Cohen-Macaulay property** (CM property) if $\mathrm{GKdim}(M) + j(M) = \mu$ for all nonzero finitely generated A-modules M, where

$$j(M) = \inf\{i \mid \mathrm{Ext}^i_A(M, A) \neq 0\} \in \mathbb{N} \cup \{0, \infty\}.$$

Definition 2.1.20 is modeled on the commutative setting where, if A is a commutative noetherian local ring, then A is Cohen-Macaulay if and only if $\mathrm{Kdim}(N)$ + $j(N)$ = $\mathrm{Kdim}(A)$ for all nonzero finitely generated A-modules N, where Kdim denotes Krull dimension.

The following result is well known, but is included for completeness.

Lemma 2.1.21.

Let U be a vector space and W a finite-dimensional subspace of U. If $\phi : U \longrightarrow U$ is an invertible linear map such that $\phi(W) \subseteq W$, then $\phi(W) = W = \phi^{-1}(W)$.

Proof.

Consider $\phi|_W : W \to W$, which is injective. Since $\dim(W) < \infty$, we may apply the Rank-Nullity Theorem to $\phi|_W$, so the result follows. ■

2.2 Twist of a Graded Algebra by an Automorphism

The twist of a graded algebra by an automorphism was defined in [2, Section 8] for $\mathbb{Z}$-graded algebras. For our purposes, we will restrict our attention to mainly quadratic algebras.

Definition 2.2.1. [2, Section 8]

Let $A = \bigoplus_{j=0}^{\infty} A_j$ be a connected $\mathbb{N}$-graded $\Bbbk$-algebra and let $\phi \in \mathrm{Aut}(A)$ be a graded automorphism of degree zero. We form a new graded algebra B as follows:

1. as vector spaces, $B = A$, and
2. we define a multiplication $*$ on B by

$$a_1 * a_2 * \cdots * a_m = a_1\phi(a_2)\phi^2(a_3)\cdots\phi^{m-1}(a_m),$$

   for all $a_1, \ldots, a_m \in A_1$, $m \in \mathbb{N}$, and use distributivity to extend $*$ to the rest of B.

We call B **the twist of A by the automorphism ϕ.**

[2, Section 8] proved that the multiplication on B is well defined.

In particular, a twist of a quadratic algebra by a graded degree-0 automorphism is a quadratic algebra.

Example 2.2.2.

Consider the quadratic algebra

$$A = \frac{\Bbbk\langle x, y\rangle}{\langle yx - qxy\rangle},$$

where q is a nonzero element in $\Bbbk$, and $\deg(x) = \deg(y) = 1$. The map ϕ defined by $\phi(x) = x$, $\phi(y) = qy$ is an automorphism of A since

$$\phi(yx - qxy) = qyx - q^2xy = q(yx - qxy).$$

Applying Definition 2.2.1, we have

$$y * x = y\phi(x) = yx = qxy = x\phi(y) = x * y,$$

which implies that the twist B of A by ϕ is given by

$$B \cong \frac{\Bbbk\langle x, y\rangle}{\langle yx - xy\rangle} = \Bbbk[x, y];$$

that is, B is the (commutative) polynomial ring on two variables.

We note that twisting a graded algebra by an automorphism is reflexive and symmetric, but usually it is not transitive. To remedy this, twisting a graded algebra by a twisting system (see Definitions 2.3.1 and 2.3.4) was considered by Zhang in [16].

2.3 Twisting Systems

Definition 2.3.1. [16, Definition 2.1]

Let $A = \bigoplus_{j=0}^{\infty} A_j$ be an $\mathbb{N}$-graded $\Bbbk$-algebra, where $\Bbbk \subseteq A_0$. A set $\{t_n\}_{n\in\mathbb{Z}}$ of degree-0 graded $\Bbbk$-linear bijections of A is called **a twisting system of** A if

$$t_n(at_m(b)) = t_n(a)t_{n+m}(b)$$

or, equivalently,

$$t_n(ab) = t_n(a)t_{n+m}t_m^{-1}(b),$$

for all $a \in A_m$, $b \in A_i$, $m, i, n \in \mathbb{Z}$.

The following example shows that a graded degree-0 automorphism yields a twisting system.

Example 2.3.2. [16, Page 284]

Let A denote an $\mathbb{N}$-graded $\Bbbk$-algebra and ϕ a degree-0 graded automorphism of A. Writing $t_n = \phi^n$, for all $n \in \mathbb{Z}$, we have

$$\begin{aligned} t_n(at_m(b)) &= \phi^n(a\phi^m(b)) = \phi^n(a)\phi^n(\phi^m(b)) = \phi^n(a)\phi^{n+m}(b) \\ &= t_n(a)t_{n+m}(b), \end{aligned}$$

for all $a \in A_m$, $b \in A_i$, and for all $n, m, i \in \mathbb{Z}$. Thus, $\{t_n\}_{n\in\mathbb{Z}}$ is a twisting system of A.

In the following example, the map t_n is not, in general, a homomorphism for any $n \in \mathbb{Z}$.

Example 2.3.3. [16, Page 284]

Let A denote an $\mathbb{N}$-graded $\Bbbk$-algebra, and $y \in A_0$ an invertible element of A. Define $t_n(a) = ya$ for all $a \in A$ and $n \in \mathbb{Z}$. It follows that

$$t_n(at_m(b)) = yayb = t_n(a)t_{n+m}(b).$$

for all $a \in A_m$, $b \in A_i$, and for all $n, m, i \in \mathbb{Z}$. Thus, $\{t_n\}_{n\in\mathbb{Z}}$ is a twisting system of A.

Definition 2.3.4. [16, Definition/Proposition 2.3]

Let A and $\{t_n\}_{n\in\mathbb{Z}}$ be defined as in Definition 2.3.1. We form a new quadratic algebra B as follows:

1. as vector spaces, $B = A$, and
2. we define a multiplication $*$ on B by

$$a_1 * a_2 * \cdots * a_m = a_1 t_1(a_2) t_2(a_3) \cdots t_{m-1}(a_m),$$

   for all $a_1, \ldots, a_m \in A_1$, $m \in \mathbb{N}$, and use distributivity to extend $*$ to the rest of B.

We call B **the twist of A by the twisting system** $\{t_n\}_{n\in\mathbb{Z}}$.

In [16], Zhang defined the notion of twisting a graded algebra by a twisting system for more general graded algebras, but our focus is mainly on quadratic algebras. He also proved that the multiplication on B is well defined.

Definition 2.3.5.

A twist of a graded $\Bbbk$-algebra by a twisting system is called **nontrivial** if it is not a twist by an automorphism and we refer to such a twisting system as **nontrivial**.

By Example 2.3.2, twisting a graded algebra by an automorphism is a special case of twisting by a twisting system. Moreover, Example 2.3.3 is an example of a nontrivial twisting system.

The following result shows that twisting a graded algebra using a twisting system is an equivalence relation.

Proposition 2.3.6. [16, Proposition 2.5]

If A is a graded $\Bbbk$-algebra, then

1. *(reflexivity) A is a twist by a twisting system of itself;*
2. *(symmetry) if B is a twist by a twisting system of A, then A is a twist by a twisting system of B;*
3. *(transitivity) if B is a twist by a twisting system of A and if C is a twist by a twisting system of B, then C is a twist by a twisting system of A.*

Twisting systems are useful in part due to the following result.

Theorem 2.3.7. [16, Propositions 5.1 and 5.2 and Theorems 5.7 and 5.11]

If A is a connected, $\mathbb{N}$-graded, finitely generated $\Bbbk$-algebra, then the following properties are preserved under twisting by a twisting system:

1. *being a domain*
2. *Gelfand-Kirillov dimension*
3. *global dimension*

4. Krull dimension

5. being noetherian.

Moreover, for noetherian connected $\mathbb{N}$-graded $\Bbbk$-algebras, the following are preserved under a twist by a twisting system:

6. injective dimension

7. being Cohen-Macaulay

8. being Artin-Schelter Gorenstein (or Artin-Schelter regular)

9. being Auslander Gorenstein (or Auslander regular) and Cohen-Macaulay.

For example, in Example 2.2.2, where the algebra $\frac{\Bbbk\langle x,y\rangle}{\langle yx-qxy\rangle}$ twisted to the algebra $\frac{\Bbbk\langle x,y\rangle}{\langle yx-xy\rangle} = \Bbbk[x,y]$, we can apply Theorem 2.3.7 to the polynomial domain $\Bbbk[x,y]$ to infer some important properties of the algebra $\frac{\Bbbk\langle x,y\rangle}{\langle yx-qxy\rangle}$. In particular, since $\Bbbk[x,y]$ is a noetherian domain, we deduce immediately that $\frac{\Bbbk\langle x,y\rangle}{\langle yx-qxy\rangle}$ is also a noetherian domain.

We note that, by the transitivity of twisting systems (Proposition 2.3.6), twisting a graded $\Bbbk$-algebra A by an automorphism and twisting the resulting algebra by an automorphism yields a twist of A by a (potentially nontrivial) twisting system. Nevertheless, very few examples of nontrivial twisting systems are known in the literature. Among these few examples are those constructed in [13, 15] on certain quadratic algebras. In Example 2.3.8 below, we recall the example in [15]; the example in [13] utilized a similar construction to that of [15].

Example 2.3.8. [15, Section 4]

In this example, let A denote an algebra from [15, Section 4]. Such an algebra is quadratic and satisfies $A = T(V)/\langle W\rangle = \bigoplus_{i=0}^{\infty} A_i$, where V is a vector space of dimension four and W is a subspace of $V \otimes V$ of dimension six. In this setting, there exist invertible linear maps $\tau : A_1 \to A_1$ and $t : A_1 \to A_1$ that induce automorphisms of $T(V)$, but do not induce automorphisms of A. Nevertheless, $\tau^2 \in \mathrm{Aut}(A)$ and t^2 is the identity map. Let $t_1 : A \to A$ denote the linear map given by $t_1(\alpha) = \alpha$ for all $\alpha \in A_0$, and

$$t_1(a_1 \cdots a_m) = (\tau^{-1}t)(a_1)\tau^{-2}t\tau(a_2)\cdots\tau^{-m}t\tau^{m-1}(a_m),$$

where $m \in \mathbb{N}$ and $a_j \in A_1$ for all $j = 1, \ldots, m$. For all $n \in \mathbb{Z}$, let $t_n : A \to A$ denote the linear map $t_n = \tau^{r-n}t_r$, where $r \in \{0, 1\}$ satisfies $n \equiv r \pmod 2$, and t_0 is the identity map. By using certain geometric data associated to A, the authors of [15] prove that the set $\{t_n\}_{n\in\mathbb{Z}}$ is a twisting system of A.

We call this twisting system a **twisting system of type** $\mathbb{T}_2$ and we generalize this type of twisting system in Definition 3.7.

CHAPTER 3

Twisting Systems of Type $\mathbb{T}_\delta$

In this chapter, we generalize the twisting system construction discussed in Example 2.3.8 of Chapter 2. Explicitly, in Definition 3.7, we introduce the notion of a twisting system of type $\mathbb{T}_\delta$, where $\delta \in \mathbb{N}$, $\delta \geq 2$ (we recall that $\delta = 2$ in Example 2.3.8). Our method does not entail the use of any geometric data. The main objective of Chapter 3 is to prove that, under certain conditions, the twisting system of type $\mathbb{T}_\delta$ is well defined on any finitely generated quadratic algebra. We will prove in Corollary 3.9 that any twisting system of type $\mathbb{T}_\delta$ is also of type $\mathbb{T}_{k\delta}$ for any $k \in \mathbb{N}$.

Throughout this chapter, we use the same notation and definitions as given in Section 2.1. In particular, we recall that a twisting system is called nontrivial if it is not given by an automorphism. We write $\mathbb{Z}_\delta = \{0, 1, 2, \ldots, \delta - 1\}$.

Let V denote a finite-dimensional vector space, and $T(V)$ the tensor algebra on V. We take degree$(x) = 1$, for all nonzero elements $x \in V \subseteq T(V)$. Let $A = T(V)/\langle W \rangle = \bigoplus_{i=0}^{\infty} A_i$, where W is a finite-dimensional subspace of $V \otimes V$. We note that A is a quadratic algebra, where $V = A_1$. Let t and τ be two invertible linear graded functions that map $T(V)$ to $T(V)$.

Definition 3.1.

For $n \in \mathbb{Z}$, we define maps $t_n : T(V) \to T(V)$ by $t_n(\alpha) = \alpha$, for all $\alpha \in T(V)_0$ and

$$t_n(a_1 \cdots a_m) = \prod_{j=1}^{m} \tau^{1-j-n} t^n \tau^{j-1}(a_j),$$

where $m \in \mathbb{N}$, $a_j \in T(V)_1 = V$ for $j = 1, \ldots, m$, and extend t_n to $T(V)$ by linearity.

We note that t_0 is the identity function on $T(V)$ and that each t_n is a linear graded bijection from $T(V)$ to $T(V)$.

The main goal of this chapter is to prove that, under certain hypotheses, t_n is well defined on A for all $n \in \mathbb{N}$ and that $\{t_n : n \in \mathbb{Z}\}$ is a twisting system on A.

Remark 3.2.

Suppose that t_n is defined as in Definition 3.1 and that $\{t_n\}_{n \in \mathbb{Z}}$ yields a twisting system on A. If $t\tau = \tau t$ then, from Definition 3.1, we have

$$t_n(a_1 \cdots a_m) = \prod_{j=1}^{m} \tau^{1-j-n} t^n \tau^{j-1}(a_j) = \prod_{j=1}^{m} \tau^{1-j-n} \tau^{j-1} t^n(a_j) = \prod_{j=1}^{m} (\tau^{-1} t)^n(a_j),$$

and so $t_n = t_1^n \in \mathrm{Aut}(A)$ for all $n \in \mathbb{Z}$. It follows that the twisting system on A is reduced to being a twist by $t_1 = \tau^{-1} t \in \mathrm{Aut}(A)$. Conversely, if $t_1 \in \mathrm{Aut}(A)$, then $\tau^{-2} t \tau = \tau^{-1} t$ on V, which implies that $t\tau = \tau t$ on V.

Consequently, in seeking examples of nontrivial twisting systems on A in the setting of Definition 3.1, we require $t\tau \neq \tau t$ on V.

Lemma 3.3.

Let t_n be defined as in Definition 3.1 for all $n \in \mathbb{Z}$. If $a \in T(V)_m$ and $b \in T(V)_i$, where $m, i \in \mathbb{N}_0$, then

(a) $t_n(ab) = t_n(a)t_{n+m}t_m^{-1}(b)$, *for all $n \in \mathbb{Z}$, so $\{t_n : n \in \mathbb{Z}\}$ is a twisting system of $T(V)$, and*

(b) $t_n(T(V)_m b) \subseteq T(V)_m t_{n+m}t_m^{-1}(b)$, *for all $n \in \mathbb{Z}$.*

Proof.

It suffices to consider $a = a_1a_2\cdots a_m$ and $b = b_1b_2\cdots b_i$, where $a_j, b_j \in T(V)_1$, for all j. Define

$$c_j = \begin{cases} a_j & 1 \le j \le m \\ b_{j-m} & m+1 \le j \le m+i. \end{cases}$$

It follows that

$$\begin{aligned} t_n(a)t_{n+m}t_m^{-1}(b) &= t_n(a)\prod_{j=1}^{i} \tau^{1-j-(n+m)}t^{n+m}\tau^{j-1}\tau^{1-j}t^{-m}\tau^{j+m-1}(b_j) \\ &= t_n(a)\prod_{j=1}^{i} \tau^{1-j-(n+m)}t^{n}\tau^{j+m-1}(b_j) \\ &= \prod_{k=1}^{m} \tau^{1-k-n}t^{n}\tau^{k-1}(a_k)\prod_{j=1}^{i} \tau^{1-(m+j)-n}t^{n}\tau^{m+j-1}(b_j) \\ &= \prod_{k=1}^{m+i} \tau^{1-k-n}t^{n}\tau^{k-1}(c_k) \\ &= t_n(ab), \end{aligned}$$

for all $n \in \mathbb{Z}$. ∎

Lemma 3.4.

Let t_n be defined as in Definition 3.1 for all $n \in \mathbb{Z}$. Suppose t^δ = identity for some $\delta \in \mathbb{N}$. If $n \equiv r \pmod{\delta}$ for some $r \in \mathbb{Z}_\delta$, then $t_n = \tau^{r-n}t_r$ on $T(V)$.

Proof.

We write $n \equiv r \pmod{\delta}$ where $r \in \mathbb{Z}_\delta$. Since $t^\delta =$ identity, $t^n = t^r$ and

$$\begin{aligned}
t_n(a_1 \cdots a_m) &= \prod_{j=1}^{m} \tau^{1-j-n} t^n \tau^{j-1}(a_j) \\
&= \prod_{j=1}^{m} \tau^{r-n} \tau^{1-j-r} t^r \tau^{j-1}(a_j) \\
&= \tau^{r-n} \prod_{j=1}^{m} \tau^{1-j-r} t^r \tau^{j-1}(a_j) \\
&= \tau^{r-n} t_r(a_1 \cdots a_m),
\end{aligned}$$

where $m \in \mathbb{N}$, $a_j \in T(V)_1 = V$ for $j = 1, \ldots, m$. The result follows by linearity of t^n. ■

Proposition 3.5.

Let t_n be as in Lemma 3.4. Suppose $t^\delta =$ identity and $\tau^\delta \in \mathrm{Aut}(A)$ for some $\delta \in \mathbb{N}$. If $t_r(W) \subseteq W$ for all $r \in \mathbb{Z}_\delta$, then t_n is defined on A for all $n \in \mathbb{Z}$.

Proof.

We will prove that $t_n(\langle W \rangle) \subseteq \langle W \rangle$ for all $n \in \mathbb{Z}$. Let $m, i \in \mathbb{N}_0$. By Lemma 3.3(b), it follows that

$$t_n(T(V)_m W T(V)_i) \subseteq T(V)_m [t_{n+m} t_m^{-1}(W)] T(V)_i. \tag{3.1}$$

By Lemma 3.4, $t_n = \tau^{r-n} t_r$ where $n \equiv r \pmod{\delta}$ with $r \in \mathbb{Z}_\delta$. Writing $m \equiv M$ (mod δ), $m + n \equiv P \pmod{\delta}$, with $M, P \in \mathbb{Z}_\delta$, we have

$$t_{n+m} t_m^{-1} = \tau^{P-(n+m)} t_P t_M^{-1} \tau^{m-M}. \tag{3.2}$$

By assumption, $\tau^\delta(W) \subseteq W$ and $t_r(W) \subseteq W$ for all $r \in \mathbb{Z}_\delta$, so $t_M^{-1}(W) \subseteq W$ (by Lemma 2.1.21). Thus (3.2) implies $t_{n+m}t_m^{-1}(W) \subseteq W$. Hence (3.1) implies $t_n(\langle W \rangle) \subseteq \langle W \rangle$ for all $n \in \mathbb{Z}$. ■

Theorem 3.6.

Let t_n, t and τ be as in Proposition 3.5. If $t_r(W) \subseteq W$ for all $r \in \mathbb{Z}_\delta$, then $\{t_n : n \in \mathbb{Z}\}$ is a twisting system of A.

Proof.

By Lemma 3.3(a), $\{t_n : n \in \mathbb{Z}\}$ is a twisting system of $T(V)$. Also, by Proposition 3.5, t_n is defined on A for all $n \in \mathbb{Z}$. The result follows. ■

Definition 3.7.

We say the twisting system given by Theorem 3.6 is of type $\mathbb{T}_\delta$.

We present an example in Chapter 4 of a new example of a nontrivial twisting system of type $\mathbb{T}_2$.

Remark 3.8.

We note that in Definition 3.7, if $\{t_n : n \in \mathbb{Z}\}$ is a twisting system of A of type $\mathbb{T}_1$, then t is the identity map on A and $\tau \in \mathrm{Aut}(A)$. In particular, in this case, we have $t\tau = \tau t$, so by Remark 3.2, the twisting system $\{t_n : n \in \mathbb{Z}\}$ is reduced to being a twist by $t_1 = \tau^{-1}t \in \mathrm{Aut}(A)$.

Corollary 3.9.

Let t_n be defined as in Definition 3.1 for all $n \in \mathbb{Z}$ and let $\delta \in \mathbb{N}$. The following conditions are equivalent

(a) $\{t_n : n \in \mathbb{Z}\}$ *is a twisting system of A of type $\mathbb{T}_\delta$,*

(b) $\{t_n : n \in \mathbb{Z}\}$ *is a twisting system of A of type $\mathbb{T}_{k\delta}$ for all $k \in \mathbb{N}$,*

(c) $\{t_n : n \in \mathbb{Z}\}$ *is a twisting system of A of type $\mathbb{T}_{m\delta}$ for some $m \in \mathbb{N}$, $t^\delta =$ identity, and $\tau^\delta \in \mathrm{Aut}(A)$,*

(d) $\{t_n : n \in \mathbb{Z}\}$ *is a twisting system of A of type $\mathbb{T}_\alpha$ for some $\alpha \in \mathbb{N}$, $\alpha \geq \delta$, $t^\delta =$ identity, and $\tau^\delta \in \mathrm{Aut}(A)$,*

(e) $\{t_n : n \in \mathbb{Z}\}$ *is a twisting system of A simultaneously of types $\mathbb{T}_{\alpha_1}, \ldots, \mathbb{T}_{\alpha_m}$, where $\alpha_1, \ldots, \alpha_m \in \mathbb{N}$, $m \in \mathbb{N}$, and δ is the greatest common divisor of $\alpha_1, \ldots, \alpha_m$.*

Proof.

We first prove that if (a) holds then (b) holds. Assume that $\{t_n : n \in \mathbb{Z}\}$ is a twisting system of A of type $\mathbb{T}_\delta$. In particular, $t^\delta =$ identity, $\tau^\delta(W) \subseteq W$, and $t_s(W) \subseteq W$ for all $s \in \mathbb{Z}_\delta$. Let $\beta = k\delta$, for some $k \in \mathbb{N}$. We have $t^\beta = (t^\delta)^k =$ identity and $\tau^\beta(W) = (\tau^\delta)^k(W) \subseteq W$. If $r \in \mathbb{Z}_\beta$, then $r \equiv s \pmod{\delta}$, for some $s \in \mathbb{Z}_\delta$. By Lemma 3.4, it follows that $t_r = \tau^{s-r} t_s$, so $t_r(W) \subseteq W$, for all $r \in \mathbb{Z}_\beta$. By Theorem 3.6, $\{t_n : n \in \mathbb{Z}\}$ is a twisting system of A of type $\mathbb{T}_\beta$. It follows that (b) holds.

Clearly, if (b) holds, then for an arbitrary $m \in \mathbb{N}$, by taking $k = m$, $\{t_n : n \in \mathbb{Z}\}$ is a twisting system of A of type $\mathbb{T}_{m\delta}$. Also, by taking $k = 1$, we have $t^\delta =$ identity and $\tau^\delta \in \mathrm{Aut}(A)$ by Definition 3.7. Thus, (c) holds.

We now prove that if (c) holds then (d) holds. Assume (c) holds. By the hypothesis of part (c), $t^\delta =$ identity and $\tau^\delta \in \text{Aut}(A)$. By taking $\alpha = m\delta$, (d) holds.

We now prove that if (d) holds then (a) holds. Assume (d) holds. By the hypothesis of part (d), $t^\delta =$ identity and $\tau^\delta \in \text{Aut}(A)$. As sets, we may view $\mathbb{Z}_\delta \subseteq \mathbb{Z}_\alpha$, so the fact that $t_r(W) \subseteq W$ for all $r \in \mathbb{Z}_\alpha$ implies that $t_s(W) \subseteq W$ for all $s \in \mathbb{Z}_\delta$. By Theorem 3.6, $\{t_n : n \in \mathbb{Z}\}$ is a twisting system of A of type $\mathbb{T}_\delta$, so (a) holds.

We now prove that if (b) holds then (e) holds. By choosing $k_1, \ldots, k_m \in \mathbb{N}$, where $k_1, \ldots, k_m$ are relatively prime, and letting $\alpha_1 = k_1\delta, \ldots, \alpha_m = k_m\delta$, we see that if (b) holds then (e) holds.

We now prove that if (e) holds then (a) holds. Assume (e) holds. If $m = 1$, the result is immediate. Suppose $m \geq 2$. Since δ is the greatest common divisor of $\alpha_1, \ldots, \alpha_m$, there exist $n_1, \ldots, n_m \in \mathbb{Z}$ such that $\delta = n_1\alpha_1 + \cdots + n_m\alpha_m$. We have $t^\delta = (t^{\alpha_1})^{n_1} \cdots (t^{\alpha_m})^{n_m} =$ identity and $\tau^\delta = (\tau^{\alpha_1})^{n_1} \cdots (\tau^{\alpha_m})^{n_m} \in \text{Aut}(A)$. Since $\delta | \alpha_i$ for all i, we may view, as sets, $\mathbb{Z}_\delta \subseteq \mathbb{Z}_{\alpha_i}$ for all i, so the fact that $t_r(W) \subseteq W$ for all $r \in \mathbb{Z}_{\alpha_i}$ implies that $t_s(W) \subseteq W$ for all $s \in \mathbb{Z}_\delta$. By Theorem 3.6, $\{t_n : n \in \mathbb{Z}\}$ is a twisting system of A of type $\mathbb{T}_\delta$, so (a) holds. ■

Remark 3.10.

We note that in Corollary 3.9, if $\{t_n : n \in \mathbb{Z}\}$ is a twisting system of A of type $\mathbb{T}_\alpha$ and type $\mathbb{T}_\beta$ simultaneously, where $\alpha, \beta \in \mathbb{N}$ are relatively prime, then t is the identity map on A and $\tau \in \text{Aut}(A)$. In this case, by Definition 3.1, $t_n = \tau^{-n}$ for all $n \in \mathbb{Z}$, and, by Remark 3.8, the twist is a twist by an automorphism.

CHAPTER 4

An Example of a Nontrivial Twisting System of Type $\mathbb{T}_\delta$

Throughout this chapter, we use the same notation and definitions as provided in Section 2.1 and Chapter 3. In this chapter, we illustrate our method for producing a nontrivial twisting system on an algebra that is a quadratic quantum $\mathbb{P}^3$ whose point scheme is given by a rank-2 quadric in $\mathbb{P}^3$.

Suppose $\mathrm{char}(\Bbbk) \neq 2$. Consider the family of quadratic algebras

$$A(\rho) = \Bbbk\langle x_1, \ldots, x_4\rangle / \langle \omega_1, \ldots, \omega_6\rangle,$$

where $\rho \in \Bbbk^\times$ and

$$\begin{aligned}
&\omega_1 = \rho^2 x_1x_2 + x_2x_1, \quad \omega_2 = \rho x_1x_3 - x_3x_1, \quad \omega_3 = \rho x_1x_4 + x_4x_1,\\
&\omega_4 = x_2x_3 - \rho x_3x_2, \quad\ \ \omega_5 = x_2x_4 + \rho x_4x_2, \quad \omega_6 = x_3x_4 + x_4x_3 + x_1x_2.
\end{aligned}$$

We identify $\Bbbk\langle x_1, \ldots, x_4\rangle$ with $T(V)$, where $V = \Sigma_{i=1}^4 \Bbbk x_i$, and write $W = \Sigma_{i=1}^6 \Bbbk\omega_i$. In Chapter 5, we prove that $A(\rho)$ is a quantum $\mathbb{P}^3$ with point scheme given by a rank-2 quadric.

Consider the homomorphisms t and τ defined on $\Bbbk\langle x_1, \ldots, x_4\rangle$ given by the matrices:

$$[t] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad [\tau] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & -2 \end{bmatrix},$$

where we use row i to give the image of x_i, for all i (for example, $[t](x_4) = x_3 - x_4$). The matrices act in the usual manner on the dual, V^*, of V, which will be relevant in Chapter 5. We consider t_n, $n \in \mathbb{Z}$, as in Definition 3.1.

We now verify the conditions for $\{t_n : n \in \mathbb{Z}\}$ to be a nontrivial twisting system of $A(\rho)$. We notice that

1. the maps t and τ are automorphisms of $\Bbbk\langle x_1, \ldots, x_4\rangle$, since $[t]$ and $[\tau]$ are invertible (since $\mathrm{char}(\Bbbk) \neq 2$),
2. $t^2 =$ identity,
3. $t\tau \neq \tau t$, since the matrix entry $([\tau][t] - [t][\tau])_{43} = 2 \neq 0$ (since $\mathrm{char}(\Bbbk) \neq 2$),
4. $\tau(W) \not\subseteq W$, since $\tau(\omega_3) = 3\omega_2 - 2\omega_3 + 6x_3x_1 \notin W$ (since $\mathrm{char}(\Bbbk) \neq 2$),
5. $t(W) \not\subseteq W$, since $t(\omega_3) = \omega_2 - \omega_3 + 2x_3x_1 \notin W$ (since $\mathrm{char}(\Bbbk) \neq 2$),
6. $\tau^2(W) \subseteq W$ since
$$\begin{aligned} &\tau^2(\omega_1) = 16\omega_1, && \tau^2(\omega_2) = 4\omega_2, && \tau^2(\omega_3) = 4\omega_3, \\ &\tau^2(\omega_4) = 64\omega_4, && \tau^2(\omega_5) = 64\omega_5, && \tau^2(\omega_6) = 16\omega_6, \end{aligned}$$
and
7. $t_1(W) \subseteq W$ since we have $t_1(x_ix_j) = \tau^{-1}t(x_i)\tau^{-2}t\tau(x_j)$, $i, j = 1, \ldots, 4$, implies that
$$\begin{aligned} &t_1(\omega_1) = \tfrac{1}{4}\omega_1, && t_1(\omega_2) = \tfrac{1}{2}\omega_2, && t_1(\omega_3) = \tfrac{1}{4}(\omega_2 + 2\omega_3), \\ &t_1(\omega_4) = \tfrac{1}{8}\omega_4, && t_1(\omega_5) = \tfrac{1}{16}(\omega_4 + 2\omega_5), && t_1(\omega_6) = \tfrac{1}{4}\omega_6. \end{aligned}$$

Thus, by Theorem 3.6 and Remark 3.8, we have a nontrivial twisting system of $A(\rho)$ of type $\mathbb{T}_2$ (and so also of type $\mathbb{T}_{2k}$ for all $k \in \mathbb{N}$, by Corollary 3.9).

Let $A'(\rho)$ be the twist of $A(\rho)$ under the above twisting system $\{t_n : n \in \mathbb{Z}\}$. By Definition 2.3.4, if $a, b \in A(\rho)_1$, the multiplication $*$ on $A'(\rho)$ is defined as $a*b = at_1(b)$. Thus, writing $b = t_1^{-1}(c)$, for some $c \in A(\rho)_1$, we have

$$ac = a * t_1^{-1}(c) = a * (t^{-1}\tau)(c) = a * t\tau(c)$$

(as $t^2 =$ identity). Since

$$t\tau(x_1) = x_1, \qquad t\tau(x_2) = 4x_2, \qquad t\tau(x_3) = 2x_3, \qquad t\tau(x_4) = x_3 + 2x_4,$$

we thus obtain the new algebra

$$A'(\rho) = \Bbbk\langle x_1, \ldots, x_4\rangle / \langle \omega_1', \ldots, \omega_6'\rangle,$$

where, omitting the $*$, we have

$$\begin{aligned} \omega_1' &= 4\rho^2 x_1 x_2 + x_2 x_1, & \omega_2' &= 2\rho x_1 x_3 - x_3 x_1, \\ \omega_3' &= 2\rho x_1 x_4 + x_4 x_1 + \rho x_1 x_3, & \omega_4' &= 2x_2 x_3 - 4\rho x_3 x_2, \\ \omega_5' &= 2x_2 x_4 + 4\rho x_4 x_2 + x_2 x_3, & \omega_6' &= 2(x_3 x_4 + x_4 x_3) + x_3^2 + 4x_1 x_2. \end{aligned}$$

Although this method did not entail any geometric data, we show that, in Chapter 5, for all $\rho \in \Bbbk^\times$, $A(\rho)$ is a noetherian regular algebra of global dimension four (sometimes called a quantum $\mathbb{P}^3$) with point scheme given by the union of two distinct planes in $\mathbb{P}^3$.

CHAPTER 5

Some Quantum $\mathbb{P}^3$s with Point Scheme a Rank-2 Quadric

In this chapter, we introduce a family of AS-regular algebras that have point scheme given by a rank-2 quadric in the projective space $\mathbb{P}^3$.

In Section 5.1, we study the family of quadratic algebras $A(\rho)$, $\rho \in \Bbbk^\times$, that were used to produce the example in Chapter 4 of a nontrivial twisting system. We show that $A(\rho)$ satisfies several desirable properties, as stated in Theorem 5.1.9, and such algebras are often called quadratic quantum $\mathbb{P}^3$s. In Section 5.2, we show that the point scheme of $A(\rho)$ is given by a rank-2 quadric in $\mathbb{P}^3$; that is, it is the union of two distinct planes in the projective space $\mathbb{P}^3$ (Theorem 5.2.1). Moreover, we prove in Theorem 5.2.3 that the zero locus of the defining relations of $A(\rho)$ does not determine the defining relations of $A(\rho)$.

Throughout this chapter, we will use the same notation and definitions as given in Section 2.1 of Chapter 2. We continue to assume $\mathrm{char}(\Bbbk) \neq 2$.

5.1 Regularity of $A(\rho)$

We recall that, in Chapter 4, the family of quadratic algebras

$$A(\rho) = \Bbbk\langle x_1, \ldots, x_4\rangle / \langle \omega_1, \ldots, \omega_6\rangle$$

is given, where $\rho \in \Bbbk^\times$ and

$$\omega_1 = \rho^2 x_1 x_2 + x_2 x_1, \qquad \omega_2 = \rho x_1 x_3 - x_3 x_1, \qquad \omega_3 = \rho x_1 x_4 + x_4 x_1,$$
$$\omega_4 = x_2 x_3 - \rho x_3 x_2, \qquad \omega_5 = x_2 x_4 + \rho x_4 x_2, \qquad \omega_6 = x_3 x_4 + x_4 x_3 + x_1 x_2.$$

We will use the algebra A defined in Lemma 5.1.1 below to prove that $A(\rho)$ has many desirable properties.

Lemma 5.1.1. *For all $\rho \in \Bbbk^{\times}$, the algebra*

$$\begin{aligned} A &= \Bbbk\langle x_1, \ldots, x_4\rangle/\langle x_1x_2 - x_2x_1,\ x_1x_3 - x_3x_1,\ x_1x_4 - x_4x_1, \\ &\qquad x_2x_3 + x_3x_2,\ x_2x_4 + x_4x_2,\ x_3x_4 - x_4x_3 + x_1x_2\rangle \end{aligned}$$

is isomorphic to a twist by an automorphism of $A(\rho)$.

Proof. Let $\theta \in \mathrm{Aut}(A(\rho))$ be defined as $\theta(x_1) = \rho^2 x_1$, $\theta(x_2) = -x_2$, $\theta(x_3) = \rho x_3$, $\theta(x_4) = -\rho x_4$. For every $\rho \in \Bbbk^{\times}$, let $\mathcal{A}(\rho)$ denote the twist of $A(\rho)$ by θ; that is,

$$\begin{aligned} \mathcal{A}(\rho) &= \Bbbk\langle x_1, \ldots, x_4\rangle/\langle x_1x_2 - x_2x_1,\ x_1x_3 - x_3x_1,\ x_1x_4 - x_4x_1, \\ &\qquad x_2x_3 + x_3x_2,\ x_2x_4 + x_4x_2,\ x_3x_4 - x_4x_3 + (1/\rho)x_1x_2\rangle. \end{aligned}$$

Mapping $x_1 \mapsto \rho x_1$, $x_i \mapsto x_i$, $i \in \{2, 3, 4\}$, yields an isomorphism: $\mathcal{A}(\rho) \to A$. ■

We note that x_1 is central in A and x_2 is normal in A. Let $J = Ax_2$. From the normality of x_2 and the defining relations of A, it is immediate that A/J is isomorphic to the commutative polynomial ring $\Bbbk[x_1, x_3, x_4]$.

Corollary 5.1.2.

The algebra A/J is a noetherian domain, has Hilbert series $H_{A/J}(s) = (1-s)^{-3}$, is Koszul, AS-regular and Auslander-regular of global dimension 3, and satisfies the CM property.

Proof.

It is well known (e.g., [9]) that the commutative polynomial ring $B = \Bbbk[x_1, \dots, x_n]$ on n variables is a noetherian domain, has Hilbert series $H_B(s) = (1-s)^{-n}$, is Koszul, AS-regular and Auslander-regular of global dimension n, and satisfies the CM property.

Since A/J is isomorphic to the commutative polynomial ring $\Bbbk[x_1, x_3, x_4]$, the result follows. ■

We will show in Lemma 5.1.4 and Propositions 5.1.7 and 5.1.8 that the algebra A is a noetherian domain, has Hilbert series $H_A(s) = (1-s)^{-4}$, is Koszul, AS-regular and Auslander-regular of global dimension 4, and satisfies the CM property.

Lemma 5.1.3.

The algebra A is an Ore extension $R[x_4; \sigma, d]$ of

$$R = \Bbbk\langle x_1, x_2, x_3\rangle / \langle x_1x_2 - x_2x_1,\ x_1x_3 - x_3x_1,\ x_2x_3 + x_3x_2\rangle,$$

where

$$\sigma(x_1) = x_1, \qquad \sigma(x_2) = -x_2, \qquad \sigma(x_3) = x_3,$$
$$d(x_1) = 0, \qquad d(x_2) = 0, \qquad d(x_3) = x_1x_2.$$

Proof.

For the defining relations of A, let us write

$$\eta_1 = x_1x_2 - x_2x_1, \qquad \eta_2 = x_1x_3 - x_3x_1, \qquad \eta_3 = x_1x_4 - x_4x_1,$$
$$\eta_4 = x_2x_3 + x_3x_2, \qquad \eta_5 = x_2x_4 + x_4x_2, \qquad \eta_6 = x_3x_4 - x_4x_3 + x_1x_2.$$

Correspondingly, for the defining relations of R, we have $r_1 = \eta_1$, $r_2 = \eta_2$ and $r_3 = \eta_4$.

With a slight abuse of notation, we use the above definitions of σ and d on $\Bbbk\langle x_1, x_2, x_3\rangle$ and we next prove that these maps induce well-defined maps σ and d on R. It is immediate that $\sigma(r_1) = -r_1$, $\sigma(r_2) = r_2$ and $\sigma(r_3) = -r_3$, so $\sigma \in$

Aut(R). Using the equation $d(ab) = \sigma(a)d(b) + d(a)b$ for all $a, b \in R$, as in Definition 2.1.15, we obtain $d(x_1x_2) = 0$, $d(x_2x_1) = 0$, $d(x_1x_3) = x_1^2x_2$, $d(x_3x_1) = x_1x_2x_1$, $d(x_2x_3) = -x_2x_1x_2$, and $d(x_3x_2) = x_1x_2^2$, and hence

$$\begin{aligned} d(r_1) &= 0, \\ d(r_2) &= x_1^2x_2 - x_1x_2x_1 &&= x_1r_1, \\ d(r_3) &= -x_2x_1x_2 + x_1x_2^2 &&= r_1x_2, \end{aligned}$$

and so d is a well-defined left σ-derivation on R.

Furthermore, using the equation $x_4r = \sigma(r)x_4 + d(r)$ for all $r \in R$, as given in Definition 2.1.15, we obtain

$$\begin{aligned} x_4x_1 &= x_1x_4 + d(x_1) &&= x_1x_4, \\ x_4x_2 &= -x_2x_4 + d(x_2) &&= -x_2x_4, \\ x_4x_3 &= x_3x_4 + d(x_3) &&= x_3x_4 + x_1x_2, \end{aligned}$$

which give us the defining relations η_3, η_5 and η_6 of A.

Thus, $A = R[x_4; \sigma, d]$ is an Ore extension of R. ■

Lemma 5.1.4.

The algebra A is a noetherian domain, has Hilbert series $H_A(s) = (1-s)^{-4}$, is AS-regular, Auslander-regular and satisfies the CM property.

Proof.

The algebra R from Lemma 5.1.3 is a skew-polynomial ring, so $H_R(s) = (1-s)^{-3}$. Since $A = R[x_4; \sigma, d]$, where $\sigma \in \mathrm{Aut}(R)$, A is a free R-module with basis $\{1, x_4, x_4^2, \ldots\}$ and so $H_A(s) = (1-s)^{-4}$. By [11, Theorem 1.2.9] and Lemma 5.1.3, A is a noetherian domain. The fact that A is Auslander-regular and satisfies the CM-property follows from Lemma 5.1.3 and the Lemma in [10, Page 184].

Since $H_A(s) = (1-s)^{-4}$, the algebra A has polynomial growth. Combining this with A being Auslander-regular implies, by [8], that A is AS-regular. ■

We use the next few results (5.1.5 - 5.1.7) to prove in Proposition 5.1.8 that A has global homological dimension four.

Lemma 5.1.5.

The Koszul dual, $A^!$, of A has Hilbert series $H_{A^!}(s) = (1+s)^4$.

Proof.

The Koszul dual of A, as given in Definition 2.1.13, can be verified to be

$$A^! = \Bbbk\langle X_1, \ldots, X_4\rangle / \langle \mu_1, \ldots, \mu_{10}\rangle,$$

where

$$\begin{aligned}
&\mu_1 = X_1^2, && \mu_2 = X_2^2,\\
&\mu_3 = X_3^2, && \mu_4 = X_4^2,\\
&\mu_5 = X_1X_2 + X_2X_1 - (1/2)(X_3X_4 - X_4X_3), && \mu_6 = X_3X_4 + X_4X_3,\\
&\mu_7 = X_1X_3 + X_3X_1, && \mu_8 = X_1X_4 + X_4X_1,\\
&\mu_9 = X_2X_3 - X_3X_2, && \mu_{10} = X_2X_4 - X_4X_2.
\end{aligned}$$

From this data, we have the following equations in $A^!$:

$$X_1^2 = X_2^2 = X_3^2 = X_4^2 = 0,$$

$$
\begin{aligned}
X_2X_1 &= -X_1X_2 + X_3X_4,\\
X_3X_1 &= -X_1X_3,\\
X_3X_2 &= X_2X_3,\\
X_4X_1 &= -X_1X_4,\\
X_4X_2 &= X_2X_4,\\
X_4X_3 &= -X_3X_4.
\end{aligned}
$$

We will apply Bergman's Diamond Lemma. The ambiguities arising from the above relations are

$$
\begin{array}{llllllllllll}
\alpha_1 &=& X_1^3, & \alpha_2 &=& X_2X_1^2, & \alpha_3 &=& X_3X_1^2, & \alpha_4 &=& X_4X_1^2,\\
\alpha_5 &=& X_2^3, & \alpha_6 &=& X_2^2X_1, & \alpha_7 &=& X_3X_2^2, & \alpha_8 &=& X_4X_2^2,\\
\alpha_9 &=& X_3^3, & \alpha_{10} &=& X_3^2X_1, & \alpha_{11} &=& X_3^2X_2, & \alpha_{12} &=& X_4X_3^2,\\
\alpha_{13} &=& X_4^3, & \alpha_{14} &=& X_4^2X_1, & \alpha_{15} &=& X_4^2X_2, & \alpha_{16} &=& X_4^2X_3,\\
\alpha_{17} &=& X_3X_2X_1, & \alpha_{18} &=& X_4X_2X_1, & \alpha_{19} &=& X_4X_3X_1, & \alpha_{20} &=& X_4X_3X_2.
\end{array}
$$

Since α_1, α_5, α_9 and α_{13} have the form X_i^3, where $i = 1, \ldots, 4$, each one is resolvable to zero.

Since α_3, α_4, α_7, α_8, α_{10}, α_{11}, α_{12}, α_{14}, α_{15} and α_{16} involve only commutative relations or skew commutative relations and X_i^2, these ambiguities are resolvable to zero.

For α_2, by using the above set of equations, we have

$$
\begin{aligned}
X_2X_1^2 &= X_2(X_1^2) = 0,\\
X_2X_1^2 &= (X_2X_1)X_1\\
&= (-X_1X_2 + X_3X_4)X_1 = -X_1(-X_1X_2 + X_3X_4) + X_3X_4X_1\\
&= X_1^2X_2 - X_1X_3X_4 + (-1)^2X_1X_3X_4 = 0.
\end{aligned}
$$

Therefore, α_2 is resolvable to zero.

For α_6, by using the above set of equations, we have

$$\begin{aligned}
X_2^2X_1 &= (X_2^2)X_1 = 0, \\
X_2^2X_1 &= X_2(X_2X_1) \\
&= X_2(-X_1X_2 + X_3X_4) = -(-X_1X_2 + X_3X_4)X_2 + X_2X_3X_4 \\
&= X_1X_2^2 - (X_3X_4)X_2 + X_2X_3X_4 = 0.
\end{aligned}$$

Therefore, α_6 is resolvable to zero.

For α_{17}, by using the above set of equations, we have

$$\begin{aligned}
X_3X_2X_1 &= (X_3X_2)X_1 \\
&= (X_2X_3)X_1 = -(X_2X_1)X_3 \\
&= -(-X_1X_2 + X_3X_4)X_3 = X_1X_2X_3 + X_3^2X_4 \\
&= X_1X_2X_3, \\
X_3X_2X_1 &= X_3(X_2X_1) \\
&= X_3(-X_1X_2 + X_3X_4) = -(X_3X_1)X_2 + X_3^2X_4 \\
&= X_1(X_3X_2) \\
&= X_1X_2X_3.
\end{aligned}$$

Therefore, α_{17} is resolvable to $X_1X_2X_3$.

For α_{18}, by using the above set of equations, we have

$$\begin{aligned}
X_4X_2X_1 &= (X_4X_2)X_1 \\
&= (X_2X_4)X_1 = -(X_2X_1)X_4 \\
&= -(-X_1X_2 + X_3X_4)X_4 = X_1X_2X_4 - X_3X_4^2 \\
&= X_1X_2X_4, \\
X_4X_2X_1 &= X_4(X_2X_1) \\
&= X_4(-X_1X_2 + X_3X_4) = -(X_4X_1)X_2 + (X_4X_3)X_4 \\
&= (X_1X_4)X_2 - X_3X_4^2 \\
&= X_1X_2X_4.
\end{aligned}$$

Therefore, α_{18} is resolvable to $X_1X_2X_4$.

For α_{19}, by using the above set of equations, we have

$$\begin{aligned} X_4X_3X_1 &= (X_4X_3)X_1 \\ &= (-X_3X_4)X_1 = -(-1)^2X_1(X_3X_4) \\ &= -X_1X_3X_4, \\ X_4X_3X_1 &= X_4(X_3X_1) \\ &= (-1)^2X_1(X_4X_3) = X_1(-X_3X_4) \\ &= -X_1X_3X_4. \end{aligned}$$

Therefore, α_{19} is resolvable to $-X_1X_3X_4$.

For α_{20}, by using the above set of equations, we have

$$\begin{aligned} X_4X_3X_2 &= (X_4X_3)X_2 \\ &= (-X_3X_4)X_2 = -(X_3X_2)X_4 = -(X_2X_3)X_4 \\ &= -X_2X_3X_4, \\ X_4X_3X_2 &= X_4(X_3X_2) \\ &= X_4(X_2X_3) = X_2(X_4X_3) = X_2(-X_3X_4) \\ &= -X_2X_3X_4. \end{aligned}$$

Therefore, α_{20} is resolvable to $-X_2X_3X_4$.

Thus, the following monomials

$$\begin{array}{llllll} 1, & X_1, & X_2, & X_3, & X_4, & \\ X_1X_2, & X_1X_3, & X_1X_4, & X_2X_3, & X_2X_4, & X_3X_4, \\ X_1X_2X_3, & X_1X_2X_4, & X_1X_3X_4, & X_2X_3X_4, & & \\ X_1X_2X_3X_4, & & & & & \end{array}$$

constitute a $\Bbbk$-basis for $A^!$, which implies that $H_{A^!}(s) = (1+s)^4$.

We note that the Affine package in Maxima (using input code similar to that at [14]) also reveals that $H_{A^!}(s) = (1+s)^4$. ■

The following result from [7] is used to prove that A is Koszul.

Proposition 5.1.6. [7, Theorem 2.6]

Let B be a finitely generated quadratic algebra with 1-regular normal element $x \in B_1$. Suppose that $\phi \in \mathrm{Aut}(B)$ satisfies $xb = \phi(b)x$ for all $b \in B$ and that the algebra $C = B/xB$ is a Koszul algebra. We have $H_{B^!}(s) = (1+s)H_{C^!}(s)$ if and only if (i) B is a Koszul algebra and (ii) x is regular.

Proposition 5.1.7.

The algebra A is Koszul.

Proof.

Since A is a domain by Lemma 5.1.4, the normal element x_2 is regular (and hence 1-regular) in A. Moreover, A/Ax_2 is Koszul by Corollary 5.1.2. It follows that, in Proposition 5.1.6 we may take B to be A and x to be x_2. Since $H_{A^!}(s) = (1+s)^4 = (1+s)H_{(A/J)^!}(s)$ by Lemma 5.1.5 and Corollary 5.1.2, Proposition 5.1.6 implies that A is Koszul. ∎

Proposition 5.1.8.

The algebra A has global dimension four.

Proof.

The algebra A is Koszul by Proposition 5.1.7, and $H_{A^!}(s) = (1+s)^4$ by Lemma 5.1.5, so the trivial A-module has a minimal free resolution of length four. The result follows (cf. [1, Page 41]). ∎

Theorem 5.1.9.

The algebra $A(\rho)$ is noetherian, AS-regular of global dimension four, Auslander-regular, satisfies the CM property, has Hilbert series $H(s) = (1-s)^{-4}$ and is a domain.

Proof.

By Lemma 5.1.1, $A(\rho)$ is a twist of A by an automorphism and the stated properties hold for A, so they hold for $A(\rho)$ by Theorem 2.3.7. ■

5.2 The Point Scheme of $A(\rho)$

Let $V = \sum_{i=1}^{4} \Bbbk x_i$ and write V^* for the dual space of V. View A as $T(V)/\langle W' \rangle$, where W' is the subspace of $V \otimes V$ given by the defining relations of A. Let $\Gamma_A \subseteq \mathbb{P}(V^*) \times \mathbb{P}(V^*)$ denote the zero locus of W'. Let $\mathfrak{p}_A \subseteq \mathbb{P}(V^*)$ denote the image of Γ_A under the projection map $\pi : \mathbb{P}(V^*) \times \mathbb{P}(V^*) \to \mathbb{P}(V^*)$ onto the first component of $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$.

The defining relations of A can be written in the matrix form Mx, where x is the transpose of $[x_1, x_2, x_3, x_4]$ and M is a 6×4 matrix. By Lemma 5.1.4 and [1], the point scheme of A can be identified with $\mathfrak{p}_A$ and can be computed using the 4×4 minors of M.

Since A is isomorphic to a twist of $A(\rho)$ for all $\rho \in \Bbbk^\times$ (by Lemma 5.1.1), the point scheme of A is isomorphic to the point scheme of $A(\rho)$. We will prove that the point scheme of A is given by a rank-2 quadric in $\mathbb{P}^3$.

Theorem 5.2.1.

The point scheme of A is $\mathcal{V}(x_1x_2) \subseteq \mathbb{P}^3 = \mathbb{P}(V^*)$, and so is the union of two distinct planes. Similarly, for all $\rho \in \Bbbk^\times$, the point scheme of $A(\rho)$ is $\mathcal{V}(x_1x_2)$. In particular, the point scheme in each case is reduced.

Proof.

The defining relations of A can be expressed in the matrix form

$$Mx = \begin{bmatrix} -x_2 & x_1 & 0 & 0 \\ -x_3 & 0 & x_1 & 0 \\ -x_4 & 0 & 0 & x_1 \\ 0 & x_3 & x_2 & 0 \\ 0 & x_4 & 0 & x_2 \\ 0 & x_1 & -x_4 & x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1x_2 - x_2x_1 \\ x_1x_3 - x_3x_1 \\ x_1x_4 - x_4x_1 \\ x_2x_3 + x_3x_2 \\ x_2x_4 + x_4x_2 \\ x_3x_4 - x_4x_3 + x_1x_2 \end{bmatrix}.$$

The following polynomials arise from the determinants of the 4×4 minors of M:

$$\begin{aligned}
P_1 &= -2x_1^2x_2x_3, \\
P_2 &= -2x_1^2x_2x_4, \\
P_3 &= -x_1^3x_2, \\
P_4 &= 2x_1x_2^2x_3, \\
P_5 &= 2x_1x_2x_3^2, \\
P_6 &= x_1x_2(2x_3x_4 - x_1x_2), \\
P_7 &= 2x_1x_2^2x_4, \\
P_8 &= x_1x_2(2x_3x_4 + x_1x_2), \\
P_9 &= 2x_1x_2x_4^2, \\
P_{10} &= -x_1x_2^3, \\
P_{11} &= 0,
\end{aligned}$$

$$
\begin{aligned}
P_{12} &= x_1^2 x_2 x_3, \\
P_{13} &= x_1^2 x_2 x_4, \\
P_{14} &= -x_1 x_2^2 x_3, \\
P_{15} &= -x_1 x_2^2 x_4.
\end{aligned}
$$

Since $\mathrm{char}(\Bbbk) \neq 2$, in order to find the zero locus of these 15 polynomials, it suffices to consider the following ten distinct monomials:

$$
\begin{array}{lllll}
x_1^2 x_2 x_3, & x_1^2 x_2 x_4, & x_1^3 x_2, & x_1 x_2^2 x_3, & x_1 x_2 x_3^2, \\
x_1 x_2 x_3 x_4, & x_1^2 x_2^2, & x_1 x_2^2 x_4, & x_1 x_2 x_4^2, & x_1 x_2^3.
\end{array}
$$

Factoring $x_1 x_2$ from each of these ten monomials yields the following ten distinct monomials:

$$
x_1 x_3, \quad x_1 x_4, \quad x_1^2, \quad x_2 x_3, \quad x_3^2, \quad x_3 x_4, \quad x_1 x_2, \quad x_2 x_4, \quad x_4^2, \quad x_2^2.
$$

We note that these monomials generate $\langle x_1, x_2, x_3, x_4 \rangle^2$. Therefore, the 4×4 minors of M generate the ideal $I = \langle x_1 x_2 \rangle \langle x_1, x_2, x_3, x_4 \rangle^2$.

The point scheme of A is thus given by $\mathcal{V}(x_1 x_2)$ in the projective space $\mathbb{P}^3 = \mathbb{P}(V^*)$, since the saturation of I is $\langle x_1 x_2 \rangle$ (see Lemma 5.2.2). By Lemma 5.1.1, $A(\rho)$ is isomorphic to a twist of A by an automorphism, so the point scheme of $A(\rho)$ is also the union of two distinct planes. In the proof of Lemma 5.1.1, the maps are linear, so it follows that the point scheme of $A(\rho)$, for each $\rho \in \Bbbk^\times$, is also given by $\mathcal{V}(x_1 x_2)$. ■

Lemma 5.2.2.

Let $I = \langle x_1 x_2 \rangle \langle x_1, x_2, x_3, x_4 \rangle^2$ as in the proof of Theorem 5.2.1. The saturation of I is

$$
\mathrm{Sat}(I) = \langle x_1 x_2 \rangle.
$$

Proof.

Since $I = \langle x_1x_2\rangle\langle x_1, x_2, x_3, x_4\rangle^2$, so $I \subseteq \langle x_1x_2\rangle$. We have

$$\mathrm{Sat}(I) = \{s \in \Bbbk[x_1, x_2, x_3, x_4] \mid \forall k = 1, \ldots, 4, \exists n \in \mathbb{N}_0 \text{ such that } x_k^n s \in I\}.$$

If $s \in \mathrm{Sat}(I)$ then $x_1^n s \in I$ for some $n \in \mathbb{N}_0$. Since $\Bbbk[x_1, x_2, x_3, x_4]$ is a unique factorization domain, we have $x_2|s$ by definition of I. Similarly, $x_1|s$. Thus, $s \in \langle x_1x_2\rangle$. Therefore $\mathrm{Sat}(I) \subseteq \langle x_1x_2\rangle$.

Conversely, if $p \in \langle x_1x_2\rangle$, then $p = x_1x_2r$ for some $r \in \Bbbk[x_1, x_2, x_3, x_4]$. We note from P_3, P_5, P_9 and P_{10} in the proof of Theorem 5.2.1 that $x_1x_2x_k^2 \in I$, for all $k = 1, \ldots, 4$, so $x_k^2 p = x_k^2 x_1x_2r \in I$, for all $k = 1, \ldots, 4$. Hence $p \in \mathrm{Sat}(I)$, so $\langle x_1x_2\rangle \subseteq \mathrm{Sat}(I)$.

Therefore, the result follows. ■

Theorem 5.2.3.

The zero locus in $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$ of the defining relations of $A(\rho)$ does not determine the defining relations of $A(\rho)$.

Proof.

Let Γ be the zero locus in $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$ of the defining relations of $A(\rho)$. A point (α, β) in Γ, where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$, satisfies the six equations:

$$\rho^2\alpha_1\beta_2 + \alpha_2\beta_1 = 0, \tag{5.1}$$

$$\rho\alpha_1\beta_3 - \alpha_3\beta_1 = 0, \tag{5.2}$$

$$\rho\alpha_1\beta_4 + \alpha_4\beta_1 = 0, \tag{5.3}$$

$$\alpha_2\beta_3 - \rho\alpha_3\beta_2 = 0, \tag{5.4}$$

$$\alpha_2\beta_4 + \rho\alpha_4\beta_2 = 0, \tag{5.5}$$

$$\alpha_3\beta_4 + \alpha_4\beta_3 + \alpha_1\beta_2 = 0. \tag{5.6}$$

By Theorem 5.2.1, the point scheme is given by $\mathcal{V}(x_1x_2)$. Setting $\alpha_1 = 0$ and noting $\alpha_i \neq 0$ for some $i \in \{2,3,4\}$, equations (5.1)-(5.3) imply that $\beta_1 = 0$. It follows that equations (5.4)-(5.6) give us $(\alpha, \beta) = ((0, \alpha_2, \alpha_3, \alpha_4), (0, \alpha_2, \rho\alpha_3, -\rho\alpha_4))$. Similarly, setting $\alpha_2 = 0$, we obtain $(\alpha, \beta) = ((\alpha_1, 0, \alpha_3, \alpha_4), (\rho\alpha_1, 0, \alpha_3, -\alpha_4))$. Combining these two cases, we have

$$\Gamma = \{((\alpha_1, \alpha_2, \alpha_3, \alpha_4), (\rho^2\alpha_1, \alpha_2, \rho\alpha_3, -\rho\alpha_4)) \in \mathbb{P}^3 \times \mathbb{P}^3 \mid \alpha_1\alpha_2 = 0\},$$

where $\mathbb{P}^3 = \mathbb{P}(V^*)$.

The element $x_1 \otimes x_2 \in T(V)_2$ vanishes on Γ, but $x_1 \otimes x_2 \notin W$. Thus, the zero locus of the defining relations of $A(\rho)$ does not determine the defining relations of $A(\rho)$ (does not determine W). ■

From the results in this chapter, we can now conclude that the algebras $A(\rho)$ and A are quantum $\mathbb{P}^3$s whose respective point scheme is given by a rank-2 quadric in $\mathbb{P}^3$. Such algebras are a new addition to the literature.

5.3 Future Research

The results in [15] (respectively, [13]) classify the quantum $\mathbb{P}^3$s whose point scheme is given by a rank-4 (respectively, rank-3) quadric in $\mathbb{P}^3$. Analogous to that work, we propose classifying all quantum $\mathbb{P}^3$s whose point scheme is given by a rank-2 quadric in $\mathbb{P}^3$ (perhaps by using techniques listed in Section 3 of [13]).

We propose attempting a similar classification to that in the previous proposed research project with rank-1 quadrics in $\mathbb{P}^3$ instead of rank-2 quadrics (if such algebras exist), or prove that such algebras do not exist.

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BIOGRAPHICAL STATEMENT

Hung Tran was born in Vietnam. He grew up under the loving care of his parents and his sister. After high school in Vietnam, he attended the University of Texas at Austin, University of Florida, University of Texas at Dallas, University of Houston and currently University of Texas at Arlington. Legend says that Vietnamese people originated from the liaison of Lac Long Quan, a king of dragons, and Au Co, an angel from heaven, so that Vietnamese people should have the strength and courage of a dragon and, at the same time, the beauty and gentleness of an angel. Alas, Hung Tran has had none of these esteemed characteristics and instead possesses numerous shortcomings (and possibly this set of shortcomings is countably infinite). Realizing his shortcomings, he has tried to continue to learn and to humbly try to improve himself throughout his life.