# ON A CUBIC NONLINEAR EQUATION MODEL ARISING IN SHALLOW WATER THEORY

by

OSAMA SALAMEH ALKHAZALEH

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# ON A CUBIC NONLINEAR EQUATION MODEL ARISING IN SHALLOW WATER THEORY

OSAMA SALAMEH ALKHAZALEH

The University of Texas at Arlington

Supervising Professor: Dr. Yue Liu

## Abstract

The shallow water waves theory produces numerous integrable equations with cubic nonlinearity as asymptotic models. We began our work by formally deriving a model equation for the free surface elevation $\eta$ with higher-order terms from shallow water in the Euler equation for an incompressible fluid with the simplest bottom and surface conditions. This model equation is truncated at the order $O(\varepsilon^3, \varepsilon\mu)$ and contains higher-order terms, which are useful for deriving a class of unidirectional wave equations including cubic nonlinear terms.

Next, we derived an equation with cubic nonlinearity as the asymptotic method from the classical shallow-water theory by employing suitable scalings, appropriate asymptotic expansions truncating, and a particular Kodama transformation to expand $\eta$ in terms of $u$ and its derivatives. This equation is relates to several different crucial shallow water

equations, including the CH, mCH, and Novikov types.

Last, we analyzed a special case of our approximate equation called mCH-Novikov equation by applying the method of characteristics by using conserved quantities to arrive at a Riccati-type differential inequality. This proved that the wave-breaking phenomenon of this equation is the curvature blow-up.

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# Chapter 1

## INTRODUCTION

### 1.1 Developments, Motivation, and Goals

In addition to the critical behavior of boundaries, the Euler equations of fluid dynamics are embodied in the theory of water waves. Simplified mathematical models have been presented as suitable estimates in many specialized physical contexts given the complexity and problems inherent in the analytical and experimental investigation of the entire system. This dissertation follows the same research path. We examine the shallow-water (or long-wave) approximations of the oscillatory gravitation water wave system currently in place. Double asymptotic expansions are commonly used to approximate the governing equations formally in the following two fundamental dimensionless positive parameters:

$$\text{the amplitude parameter } \varepsilon := \frac{a}{h_0}, \qquad \text{and the shallowness parameter } \mu := \frac{h_0^2}{\lambda^2}, \tag{1.1}$$

where $a, h_0$ and $\lambda$ are the typical amplitude of the wave, the depth of the water, and the wavelength, respectively.

The shallow water (or long-wave) regime corresponds to the following assumptions: $\mu \ll 1$ ($\mu$ to be small). Further connecting $\varepsilon$ with $\mu$ allows for the derivation of model equations in specific asymptotic regimes.

The Korteweg–de Vries (KdV) equation [1] is considered one of the most well-known and basic long-wave asymptotic models that permit actual nonlinear behavior. The wave amplitude is presumed to be small but finite: $\varepsilon = O(\mu)$ in the KdV modeling, reflecting the nonlinear effect. This scaling is then used to construct the Benjamin-Bona-Mahony (BBM) type equations [2], which are a set of asymptotically equivalent equations for general initial data. Like all physically relevant waves, both the KdV equation and the BBM class have smooth soliton and global solutions; consider, for instance, [3, 4]. However, due to the significant dispersive impact that regularizes the progressively nonlinear steepening, some other basic nonlinear phenomena such as wave-breaking and surface singularities, are excluded from the KdV model. This emphasizes the necessity for stronger nonlinear effects in model equations to properly characterize single wave occurrences for larger amplitude waves.

Regimes that include relatively high nonlinearities are defined by larger values of $\varepsilon$, such as the so-called Camassa–Holm (CH) scaling for shallow water waves of moderate amplitude

$$\mu \ll 1, \qquad \varepsilon = O(\mu^{\frac{1}{2}}). \tag{1.2}$$

Considering this, a two-parameter family of approximation equations [5] is developed using this optimization. This includes the well-known Camassa–Holm equations

$$m_t + u m_x + 2u_x m = 0, \quad m = u - u_{xx},$$

and the Degasperis–Procesi (DP) equation

$$m_t + um_x + 3u_x m = 0, \quad m = u - u_{xx},$$

where $u$ is the horizontal component of the velocity field at some specific depth, and $m$ is the so-called momentum density. The CH equation was initially conceived of as a bi-Hamiltonian equation in [6], and the DP equation was first developed in the analysis of integral equations in [7]. Later, the CH equation was introduced [8] in the context of water waves. Like the KdV equation, the CH and DP equations are completely integrable. In contrast to KdV, both CH and DP can allow solutions with some degree of singularity, such as peaking waves [9, 10, 11, 12, 13] and breaking waves [7, 8, 14, 15].

Discovery of the CH and DP equations has prompted researchers to seek additional generalization models with fascinating traits and applications. Because these two equations are both quadratic nonlinear, it makes sense to ask what form of singularity can be generated when non-linearity grows stronger and the hyperbolic attribute thus becomes more prevalent. This entails accounting for greater amplitude waves in the context of asymptotic modeling. Following ideas similar to those presented in [5], Quirchmayr [16] analyzed a shallow-water regime for waves of *large amplitude* by replacing the CH scaling (1.2) by $\varepsilon = O(\mu^{1/4})$. Truncating at the order of $O(\varepsilon^6, \mu^{3/2})$, such a scaling produces a model equation for the free surface elevation $\eta$ with nonlinearity up to the seventh order. Relating the horizontal velocity $u$ with $\eta$, taking advantage of the freedom of evaluating $u$ at a certain depth of the water, and omitting several power nonlinear terms allows a cubic nonlinear equation for $u$ to be obtained. For such a model, a blow-up criterion can be formulated, which remains in terms of the gradient catastrophe.

Recent works examining a new type of singularity formation for cubic nonlinear models are one of the motives for this study. Namely, this refers to the curvature blow-up, in which the second derivative $u_{xx}$ of solution becomes unbounded in finite time while the solution u and its slope $u_x$ stay bounded, is one of the motives for this study. The modified Camassa–Holm (mCH) equation [17, 18], and the generalized modified Camassa–Holm (gmCH) equation [9, 19], provide examples. Nevertheless, these equations have properties of conservation of energy and momentum persistence, allowing $u$ and $u_x$ to be controlled. However, the presence of higher-order nonlocal nonlinearity causes the larger derivative to blow up. To that end, we would like to perform a modeling with Quirchmayr's scaling [16] to derive cubic nonlinear equations that explain the curvature blow-up phenomenon. It is noteworthy that the mCH, gmCH, Novikov [20], and other higher-order nonlinear descendants of the CH equation are derived in the context of integrable systems. Another purpose of this research is to suggest a hydrodynamic approach for obtaining some of those cubic nonlinear models, including the mCH and Novikov equations. Regarding our modeling, we also highlight that, in contrast to [16], we prefer our model equations to include all nonlinear elements at the correct truncation order, which might make a rigorous argument for their hydrodynamic relevance to the entire water wave problem. In general, we expect cubic nonlinearity to arise at the order of $O(\varepsilon^2\mu)$, leaving the $O(\mu^2)$ terms as higher order terms and result in a scaling requirement of $\varepsilon = O(\mu^{1/2})$. As a result, rather of choosing $\varepsilon = O(\mu^{1/4})$ as in [16], we impose the following

$$\mu \ll 1, \qquad \varepsilon = O(\mu^{\frac{2}{5}}). \tag{1.3}$$

For waves of moderately large amplitude, this corresponds to a shallow-water regime. We develop an equation for the scaled surface elevation $\eta$ in the same way as was done for the CH equation.

$$
\begin{aligned}
2(\eta_x+\eta_t)+\frac{1}{3}\mu\eta_{xxx}+3\varepsilon\eta\eta_x-\frac{3}{4}\varepsilon^2\eta^2\eta_x+\frac{3}{8}\varepsilon^3\eta^3\eta_x+\varepsilon\mu\left(\frac{23}{12}\eta_x\eta_{xx}+\frac{5}{6}\eta\eta_{xxx}\right)\\
+\frac{115}{192}\varepsilon^4\eta^4\eta_x+\varepsilon^2\mu\left(-\frac{5}{16}\eta\eta_x\eta_{xx}-\frac{3}{4}\eta^2\eta_{xxx}+\frac{21}{16}\eta_x^3\right)=0+O(\varepsilon^5,\mu^2).
\end{aligned}
\tag{1.4}
$$

A comparable surface equation was found in [16] with a greater amplitude scaling of $\varepsilon = O(\mu^{1/4})$. A cubic nonlinear equation for $u$ is then obtained by connecting the horizontal velocity $u$ with elevation $\eta$.

Unlike [16], we use the so-called Kodama transformation [21] to adapt the idea of [22] to expand $\eta$ in terms of $u$ and its derivatives. The expansion has the following structure in particular:

$$\eta \sim u+\varepsilon A+\mu B+\varepsilon\mu C+\mu^2 D+\varepsilon^2 E+\varepsilon^3 K+\varepsilon^2\mu G+\varepsilon\mu^2 H,$$

where

$$
\begin{aligned}
&A:=\lambda_1 u^2,\quad B:=\lambda_2 u_{xx},\quad E:=\lambda_3 u^3,\quad K=\lambda_0 u^4,\quad C:=\lambda_4 u_x^2+\lambda_5 u u_{xx},\\
&D:=\lambda_6 u_{xxxx},\quad G:=\lambda_7 u u_x^2+\lambda_8 u^2 u_{xx},\quad H:=\lambda_9 u_x u_{xxx}+\lambda_{10} u u_{xxxx}+\lambda_{11} u_{xx}^2.
\end{aligned}
$$

Dullin et al. [23] employed this type of transformation to derive shallow wave model under the impact of surface tension, which was previously described by Kodama in [21]. A further splitting of $u_{xxt}$, combined with an equation for $u_t$, yields one more degree of freedom, $\nu$, as (3.4) and (3.5) show. The equations are intended to produce a specific form that imposes the same number of constraints on these parameters, leading to exact parameter values in the resulting model equations. Crucially, this allows us to obtain the CH-mCH-Novikov equation

$$m_t + 3uu_x - + \frac{k_1\varepsilon^2}{4}((u^2 - \beta\mu u_x^2)m)_x + \frac{k_2\varepsilon^2}{4}(u^2 m_x + 3uu_x m) = 0 + O(\varepsilon^5, \mu^2), \tag{1.5}$$

where $m = u - \beta\mu u_{xx}, \beta = -\dfrac{5}{6(\sigma-3)}, k_2 = -\dfrac{3\left(10k_1\sigma - 10k_1 - 13\sigma + 39\right)}{10(4\sigma-3)}$, and $k_1$ satisfies

$$800k_1^2\left(86\sigma^2 - 398\sigma + 645\right) + 30k_1\left(10768\sigma^3 - 62120\sigma^2 + 188109\sigma - 233883\right)$$
$$+4000k_1^3(\sigma-3) + 3\left(90496\sigma^4 - 557360\sigma^3 + 2242656\sigma^2 - 8124759\sigma + 9843417\right) = 0$$

Mathematically, under suitable scaling the quadratic terms in (1.5) can be dropped in a formal scaling limit, leaving (1.5) as the mCH-Novikov equation

$$m_t + k_1((u^2 - u_x^2)m)_x + k_2(u^2 m_x + 3uu_x m) = 0, \tag{1.6}$$

where $k_1$ and $k_2$ satisfy conditions given above.

As previously stated, our second goal is to study the formations of singularities due to spatial nonlinear effects and to build initial data that result in the finite time curvature blow-up using cubic nonlinear models. To accomplish this, we concentrate our efforts on equation (1.6), which contains only cubic nonlinearities, and consider the following Cauchy problem

$$\begin{cases} m_t + k_1\left[(u^2 - u_x^2)m\right]_x + k_2\left(u^2 m_x + 3uu_x m\right) = 0, \\ u(0,x) = u_0(x), \end{cases} \quad t > 0,\ x \in \mathbb{S}. \tag{1.7}$$

Furthermore, we allow ourselves to consider a general range of model parameters for $k_1$ and $k_2$ than that given before. In the blow-up analysis, the two groups of cubic nonlinearities in (4.3) appear to play quite different roles. When $k_1 = 0$, (4.3) becomes the Novikov equation

$$m_t + k_2(u^2 m_x + 3uu_x m) = 0, \tag{1.8}$$

Then, when the initial momentum density $m_0$ may not change sign, the solution exists globally for all time [24]. When the Novikov nonlinearity is absent ($k_2 = 0$ i.e., the mCH equation), it has been demonstrated [17, 18, 25] that the curvature can still blow up in finite time even if $m_0$ does not change sign. This raises the obvious of how the interaction of these two groups of cubic nonlinearities affects the singularity formation mechanism. As Brandolese et al. [26, 27] demonstrated, it is also worth noting that many quadratic nonlinear CH-type equations have such a strong non-diffusive character that excessively localized information about the data is enough to cause solutions to blow up in finite time. The nonlinear nonlocal effects of the equations are over-dominated by the local nonlinearities of the equations, which leads to this phenomenon. As studied in [17, 9], this hyperbolic feature appears to be partially counter balanced by the stronger nonlocal impacts due to the higher nonlinearity of the equations. Consequently, it would be fascinating to investigate how the original data's local structures may affect the evolution of solutions to equation (4.3), particularly the formation of singularity. Because the equation contains both mCH and Novikov forms of nonlinearity, a relaxed local-in-space blow-up criterion in the spirit of [17, 9] is a reasonable to expect for this. However, as previously mentioned, the two types of nonlinearities do not appear to work well together for causing blow-ups, making the analysis somewhat subtle.

The proper blow-up quantity to examine at is identified using an improved Beale–Kato –Majda type blow-up criterion (cf. Lemma (4.2.4)). Following the dynamics of quantity along the features reveals precise local and nonlocal interactions between the solution and its slope, as Lemma (4.2.6) showed. The nonlocal convolution is controlled by combining the two conservation laws. This enables deduction of the key monotonicity property of $u$, $u_x$ and $m$ along with the characteristics, which leads to a Ricatti dynamics for $m$. This finding applies to a wide variety of $k_1$ and $k_2$ parameter values.

We also offer an alternative technique that does not rely on the application of conservation laws. Instead, using the sign preservation of the momentum density m, the nonlocal terms can be proved to have good signs as long as the initial momentum density does not change sign. The local terms, then, must be examined. It has been determined that as long as the local oscillation $|u_x/u|$ is moderate, a Ricatti type inequality can be obtained. Fast oscillations are already ruled out by the sign condition on $m$. With carefully chosen data, a further refined analysis of the evolution of $u_x/u$ can be performed, demonstrating that minor oscillations will continue along with the features, thus closing the argument.

## 1.2 Background

### 1.2.1 Camassa-Holm Equation (CH)

Camassa and Holm [8] proposed a new equation (CH) for shallow water waves in 1993:

$$m_t + um_x + 2u_x m = 0, \tag{1.9}$$

where $m = u - u_{xx}$, $u$ is the horizontal component of the velocity field at a specific depth, and $m$ is the so-called momentum density.

The CH scaling

$$\mu \ll 1, \qquad \varepsilon = O(\mu^{\frac{1}{2}}), \tag{1.10}$$

in [5] is used to derive this equation. The CH equation describes the motion of the unidirectional waves traveling over an underlying shear flow [28] and shallow water waves over a flat bottom [5, 29, 8, 30, 6]. It additionally appears in the models of propagation of axially symmetric waves in cylindrical hyperelastic rods [31, 32].

For a large class of initial data, the CH equation is an integrable and may be solved using the inverse scattering method [33, 34]. ]. Both the CH and KdV equation have share many properties, such as that they possess bi-Hamiltonian structures, admit soliton solutions and a Lax-pair, and are completely integrable. Unlike the KdV equation, however, the CH equation has received a considerable amount of attention from researchers due to its remarkable mathematical characteristics.

One of the most remarkable wave phenomena in nature is wave-breaking. The weakly nonlinear KdV equation is a simple model that does not possess wave-breaking phenomena [4, 35] because it demands a complete nonlinear transition [36]. However, the CH equation (quadratic nonlinear) can be used to represent the phenomenon of wave breaking in which the solution remains bound while the slope grows infinite in finite time [37, 38, 39, 40, 41].

Following wave breaking, the solutions can be continued as global weak solutions [42].

Another interesting mathematical feature of the CH equation is its peakons or peaked solitary waves. They are a type of global weak solution that is smooth except at the crest and is preserved under collisions with other wave-like solitons [11, 43, 44, 45]. The stability of these peakons has been demonstrated [46, 43, 32]. The CH equation contains both single-peakon and multi-peakon solutions [47].

Moreover, The CH equation has other remarkable properties: a variety of interesting geometric formulations, well-posedness, and infinity of conservation laws. Since the CH model exhibits all these properties and more, it has been given special attention in shallow water wave theory.

### 1.2.2 Modified Camassa-Holm Equation (mCH)

The discovery of the quadratic nonlinear CH equations in turn motivates the search for generalization equations with cubic or higher-order nonlinearities. The modified Camassa-Holm (mCH) equation is one of the most significant of these generalizations. The mCH equation (also called FORQ equation [48, 49]) was proposed by several researchers, including Fokas [50], Fuchssteiner [51], Olver and Rosenau [52], and Qiao [53] and is typically written as

$$m_t + ((u^2 - u_x^2)m)_x = 0, \tag{1.11}$$

where $u$ and $m$ are the velocity and the potential density, respectively [53]. This equation was derived by applying the general method of tri-Hamiltonian duality to the bi Hamiltonian representation of the modified Korteweg-deVries (mKdV) equation [51, 52]. The equation is formally integrable and can be rewritten as the bi-Hamiltonian form and the Lax pair [52]. Consequently, the inverse scattering transform approach can be used to solve it [53].

In addition, the mCH equation has new aspects such as blow up and wave breaking, that are not present in the CH equation. Another important distinction from the CH equation is in addition to peakons; the mCH equation admits weak kink solutions and cusp solitons [54, 55]. The authors [18] showed that when the scaling limit equation of the mCH equation is paired with the first-order term $u_x$, the short-pulse equation is satisfied. The stability of these peakons has been demonstrated [56].

### 1.2.3 Novikov Equation

The Novikov equation is considered a cubic generalization of the CH equation, which was derived by Vladimir Novikov [20, 57] using the symmetry classification of integrable equations. It takes the following form

$$m_t + k_2(u^2 m_x + 3uu_x m) = 0, \tag{1.12}$$

where $u$ is the horizontal velocity and $m$ is the so-called momentum density.

The Novikov equation is known to be integrable because it possesses the Lax pair [20]. Novikov was able to isolate equation (1.12) and deduce its symmetries using the perturbative symmetry approach [57]. This gives the conditions for a PDE to reveal an infinite number of symmetries. A matrix Lax pair representation to the Novikov equation was discovered by Hone and Wang [58]. Using the Liouville transformation, (1.12) is identical to the first equation in the Sawada-Kotera hierarchy's negative flow [59].

Using the scattering theory, authors in [60] explicitly obtained multi-peakons of the Novikov equation. The Cauchy problem for the Novikov equation proved to be locally well-posedness in the Sobolev space $H^s(\mathbb{R})$ with $s > \frac{3}{2}$ [61]. For the periodic case, it is local well-posedness in Sobolev space $H^s(\mathbb{R})$ with $s > \frac{5}{2}$ [62].

The Novikov equation as CH and mCH equations enjoys many interesting mathematical features such as admitting a bi-Hamiltonian structure [58], its infinite conserved quantities, peakon solutions, and its wave breaking phenomenon [7, 9, 20].

# Chapter 2

## DERIVATION OF THE FREE SURFACE EQUATION

### 2.1 Introduction

In this section, we briefly discuss fundamental and essential concepts in water wave propagation, including non-dimensionalization, scaling, and asymptotic expansions.

It is convenient and typical for the applied mathematical to introduce nondimensionalized quantities to analyze the asymptotic behavior of the solutions to a differential equation particularly a water waves equations. The nondimensionalization is the partial or full removal of physical dimensions from an equation involving physical quantities by a suitable substitution of variables [63]. After a suitable nondimensionalization, this method simplifies and parameterizes problems based on the typical scales anticipated on the physical grounds of flow, such as a typical depth of the water $h_0$, the typical wavelength $\lambda$, and the typical amplitude of the wave $a$. These scales are used to form fundamental and independent positive dimensionless parameters like the amplitude parameter $\varepsilon = \frac{a}{h_0}$, and the shallowness parameter $\mu := \frac{h_0^2}{\lambda^2}$.

It is additionally very useful to scale the variables for the small parameters thrown up by the non-dimensionalization. The role that the wave's amplitude plays in the formulation of the water wave drives the necessity of scaling.

To obtain valuable and related solutions in these cases, we are required to apply asymptotic techniques.

**Definition 1.** *[64] A sequence of functions $\phi_n$, $n \in \mathbb{N}$ is an asymptotic sequence as $x \to 0$ if for each $n$, we have:*

$$\phi_{n+1} = o(\phi_n), \quad as \quad x \to 0.$$

*We call $\phi_n$ asymptotic sequence. If $\phi_n$ is an asymptotic sequence and $f$ is a function, we write:*

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x), \quad as \quad x \to 0, \tag{2.1}$$

*if for each $N$ we have:*

$$f(x) - \sum_{n=0}^{N} a_n \phi_n(x) \quad as \quad x \to 0.$$

*We call (2.1) the asymptotic expansion of $f$ with respect to $\phi_n$ as $x \to 0$*

**Remark 1.** *The notation $\phi_{n+1} = o(\phi_n)$, as $x \to 0$ in definition (1) means:*

$$\lim_{x \to 0} \left| \frac{\phi_{n+1}(x)}{\phi_n(x)} \right| = 0.$$

We can perform asymptotic expansion to achieve an approximate solution to an equation with a small parameter. This method is attractive for nonlinear equations, particularly so if there are no clear strategies to solve them.

## 2.2 The Derivation

The main goal of this section is to formally derive of model equation (1.4) from the Euler equations for the free surface. Compared with the model equation derived in [29], which is truncated at the order $O(\varepsilon^3, \varepsilon\mu)$, the new model (1.4) contains additional higher order terms that are useful for deriving a class of unidirectional wave equations including cubic nonlinear terms.

Consider the two-dimensional incompressibility flows in the domain $\{(x,z) : 0 < z < h(x,t)\}$ with a parametrization of the shape of the free surface $h = h(x,t)$. In this scenario, the horizontal and vertical directions are represented by $x$ and $z$, respectively. The system under study is the Euler equations with the irrotational condition in the form of

$$\begin{cases} u_t + uu_x + wu_z = -\frac{1}{\rho}P_x, \\ w_t + uw_x + ww_z = -\frac{1}{\rho}P_z - g, \\ u_x + w_z = 0, \\ u_z - w_x = 0, \end{cases}$$

where the pressure is written as $P(t,x,z) = p_a + \rho g(h_0 - z) + p(t,x,z)$, where $p_a$ is the constant atmospheric pressure, and $p$ is a pressure variable measuring the hydrostatic pressure distribution. In addition, we pose the "no-flow" condition on the flat bed, i.e., $w|_{z=0} = 0$. On the surface $z = h_0 + \eta$, the dynamic condition $P = p_a$ and the kinematic condition yield

$$p = \rho g \eta \quad \text{and} \quad w = \eta_t + u\eta_x.$$

Next we nondimensionalize the system using the following scaling,

$$x \to \lambda x,\ z \to h_0 z,\ \eta \to a\eta,\ t \to \frac{\lambda}{\sqrt{gh_0}}t,\ u \to \sqrt{gh_0}u,\ w \to \sqrt{\mu g h_0}w,\ p \to \rho g h_0 p.$$

Recalling (1.1), we further assume that $u, w$, and $p$ are proportional to the wave amplitude, that is, $u \to \varepsilon u, \quad w \to \varepsilon w, \quad p \to \varepsilon p$. To examine the problem in an appropriate far field, we follow the approach employing the far field variable with the right-going wave:

$$\xi = \varepsilon^{1/2}(x - t), \quad \tau = \varepsilon^{3/2}t. \tag{2.2}$$

We also transform $w \to \sqrt{\varepsilon}$ to keep mass conservation. Therefore, the governing equations are seen as

$$\begin{cases} -u_\xi + \varepsilon(u_\tau + uu_\xi + wu_z) = -p_\xi & \text{in} \quad 0 < z < 1 + \varepsilon\eta, \\ \varepsilon\mu\{-w_\xi + \varepsilon(w_\tau + uw_\xi + ww_z)\} = -p_z & \text{in} \quad 0 < z < 1 + \varepsilon\eta, \\ u_\xi + w_z = 0 & \text{in} \quad 0 < z < 1 + \varepsilon\eta, \\ u_z - \varepsilon\mu w_\xi = 0 & \text{in} \quad 0 < z < 1 + \varepsilon\eta, \\ p = \eta & \text{on} \quad z = 1 + \varepsilon\eta, \\ w = -\eta_\xi + \varepsilon(\eta_\tau + u\eta_\xi) & \text{on} \quad z = 1 + \varepsilon\eta, \\ w = 0 & \text{on} \quad z = 0. \end{cases} \tag{2.3}$$

On the boundary, we use Taylor expansion: $f(1 + \varepsilon\eta) = \sum_{n=0}^{\infty} \frac{(\varepsilon\eta)^n}{n!} f^{(n)}(1)$. It turns out the equation on the fixed domain:

$$\begin{cases} -u_\xi + \varepsilon(u_\tau + uu_\xi + wu_z) = -p_\xi & \text{in} \quad 0 < z < 1, \\ \varepsilon\mu\{-w_\xi + \varepsilon(w_\tau + uw_\xi + ww_z)\} = -p_z & \text{in} \quad 0 < z < 1, \\ u_\xi + w_z = 0 & \text{in} \quad 0 < z < 1, \\ u_z - \varepsilon\mu w_\xi = 0 & \text{in} \quad 0 < z < 1, \\ p + \varepsilon\eta p_z + \frac{\varepsilon^2\eta^2}{2} p_{zz} + \frac{\varepsilon^3\eta^3}{6} p_{zzz} = \eta & \text{on} \quad z = 1, \\ w + \varepsilon\eta w_z + \frac{\varepsilon^2\eta^2}{2} w_{zz} + \frac{\varepsilon^3\eta^3}{6} w_{zzz} = -\eta_\xi + \varepsilon\eta_\tau + \varepsilon\eta_\xi\left(u + \varepsilon\eta u_z + \frac{\varepsilon^2\eta^2}{2} u_{zz} + \frac{\varepsilon^3\eta^3}{6} u_{zzz}\right) & \text{on} \quad z = 1, \\ w = 0 & \text{on} \quad z = 0. \end{cases} \tag{2.4}$$

A double asymptotic expansion is then introduced to seek a solution of the system formally,

$$q \sim \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varepsilon^n \mu^m q_{nm} \quad \text{as} \quad \varepsilon \to 0,\ \mu \to 0,$$

where $q$ will be taken to be the functions $u$, $w$, $p$ and $\eta$, and all functions $q_{nm}$ satisfy the far field conditions $q_{nm} \to 0$ as $|\xi| \to \infty$ for every $n$, $m = 0, 1, 2, 3, \ldots$ Substituting the asymptotic expansions of $u$, $w$, $p$, $\eta$ into (2.4), we check all the coefficients at every order $O(\varepsilon^i \mu^j)$ $(i,\ j = 0, 1, 2, 3, \ldots)$.

For example at $O(1)$ we obtain

$$\begin{cases} -u_{00,\xi} = -p_{00,\xi} & \text{in} \quad 0 < z < 1, \\ 0 = p_{00,z} & \text{in} \quad 0 < z < 1, \\ u_{00,\xi} + w_{00,z} = 0 & \text{in} \quad 0 < z < 1, \\ u_{00,z} = 0 & \text{in} \quad 0 < z < 1, \\ p_{00} = \eta_{00}, \quad w_{00} = -\eta_{00,\xi} & \text{on} \quad z = 1, \\ w_{00} = 0 & \text{on} \quad z = 0. \end{cases} \tag{2.5}$$

From the fourth equation in (2.5) it follows that $u_{00}$ is independent of $z$. Thanks to the third equation in (2.5) and the boundary condition of $w$ on $z = 0$, we get

$$w_{00} = w_{00}|_{z=0} + \int_0^z w_{00,z'} dz' = -z u_{00,\xi},$$

which along with the boundary condition on $z = 1$ implies $u_{00,\xi}(\tau, \xi) = \eta_{00,\xi}(\tau, \xi)$. Therefore

$$u_{00}(\tau, \xi) = \eta_{00}(\tau, \xi), \quad w_{00} = -z\eta_{00,\xi},$$

Therefore, this has been made of the far field conditions $u_{00}$, $\eta_{00} \to 0$ as $|\xi| \to \infty$. On the other hand, from the second equation in (2.5), it follows that

$$p_{00} = p_{00}|_{z=1} + \int_1^z p_{00,z'}\, dz' = \eta_{00}.$$

At $O(\varepsilon^1\mu^0) = O(\varepsilon)$ we obtain

$$\begin{cases} -u_{10,\xi} + u_{00,\tau} + u_{00}u_{00,\xi} = -p_{10,\xi} & \text{in} \quad 0 < z < 1, \\ 0 = p_{10,z} & \text{in} \quad 0 < z < 1, \\ u_{10,\xi} + w_{10,z} = 0 & \text{in} \quad 0 < z < 1, \\ u_{10,z} = 0 & \text{in} \quad 0 < z < 1, \\ p_{10} + p_{00,z}\eta_{00} = \eta_{10} & \text{on} \quad z = 1, \\ w_{10} + \eta_{00}w_{00,z} = -\eta_{10,\xi} + \eta_{00,\tau} + u_{00}\eta_{00,\xi} & \text{on} \quad z = 1, \\ w_{10} = 0 & \text{on} \quad z = 0. \end{cases} \tag{2.6}$$

From the fourth equation in (2.6), we know that $u_{10}$ is independent to $z$, that is, $u_{10} = u_{10}(\tau, \xi)$. Thanks to the third equation in (2.6) and the boundary conditions of $w$ on $z = 0$, we get

$$w_{10} = w_{10}|_{z=0} + \int_0^z w_{10,z'} dz' = -z u_{10,\xi}. \tag{2.7}$$

Hence, from the third equation in (2.6) and (2.7) and the boundary conditions of $w$ on $z = 1$, we obtain that

$$u_{10,\xi} = \eta_{10,\xi} - \eta_{00,\tau} - (u_{00}\eta_{00})_\xi \quad \text{and} \quad w_{10} = z(\eta_{00,\tau} + 2\eta_{00}\eta_{00,\xi} - \eta_{10,\xi}). \tag{2.8}$$

Thanks to the second equation in (2.6) , we deduce that

$$p_{10,\xi} = \eta_{10,\xi} = u_{10,\xi} + \eta_{00,\tau} + (u_{00}\eta_{00})_\xi. \tag{2.9}$$

Taking account of the first equation in (2.6) and (2.8), it must be

$$-p_{10,\xi} = -u_{10,\xi} + \eta_{00,\tau} + \eta_{00}\eta_{00,\xi},$$

which along with (2.9) and (2.8) implies

$$2\eta_{00,\tau} + 3\eta_{00}\eta_{00,\xi} = 0.$$

Similarly, at the orders $O(\varepsilon^0\mu^1)$, $O(\varepsilon^2\mu^0)$, $O(\varepsilon^1\mu^1)$, $O(\varepsilon^3\mu^0)$, $O(\varepsilon^4\mu^0)$ and $O(\varepsilon^2\mu^1)$, the relation between $p_{ij}$, $\eta_{ij}$, $u_{ij}$, $w_{ij}$ and their $\tau$-derivatives in these orders can be obtained; see [65]. Here we focus on the $O(\varepsilon^3\mu^1)$-order.

For the $O(\varepsilon^3\mu^1)$-order approximation when $0 < z < 1$, the following system is obtained

$$\begin{cases} -p_{31,\xi} = -u_{31,\xi} + u_{21,\tau} + \big(u_{00}u_{21} + u_{10}u_{11} + u_{20}u_{01}\big)_\xi \\ \qquad\qquad +w_{00}u_{21,z} + w_{10}u_{11,z}, \\ -p_{31,z} = -w_{20,\xi} + w_{10,\tau} + u_{00}w_{10,\xi} + u_{10}w_{00,\xi} + \big(w_{00}w_{10}\big)_z, \\ u_{31,\xi} + w_{31,z} = 0, \\ u_{31,z} - w_{20,\xi} = 0. \end{cases} \tag{2.10}$$

The boundary condition on $z = 0$ is $w_{31} = 0$, and on $z = 1$, the conditions read

$$\begin{cases} \eta_{31} = p_{31} + \eta_{21}p_{00,z} + \eta_{00}p_{21,z} + \eta_{11}p_{10,z} + \eta_{10}p_{11,z} + \eta_{20}p_{01,z} + \eta_{01}p_{20,z}, \\ w_{31} + \eta_{21}w_{00,z} + \eta_{00}w_{21,z} + \eta_{11}w_{10,z} + \eta_{10}w_{11,z} + \eta_{20}w_{01,z} + \eta_{01}w_{20,z} - \eta_{21,\tau} + \frac{\eta_{00}^2}{2}w_{11,zz} \\ \qquad = -\eta_{31,\xi} + u_{21}\eta_{00,\xi} + u_{00}\eta_{21,\xi} + u_{20}\eta_{01,\xi} + u_{01}\eta_{20,\xi} + u_{10}\eta_{11,\xi} + u_{11}\eta_{10,\xi} + \eta_{00}\eta_{00,\xi}u_{11,z}. \end{cases}$$

Next, we plug $w_{20,\xi}$, $w_{00,\xi}$ and $w_{10}$, which can easily be obtained from [65], into the second equation in (2.10). It takes the form of

$$p_{31,z} = -zu_{20,\xi\xi} + zu_{10,\xi\tau} + zu_{00}u_{10,\xi\xi} + zu_{10}\eta_{00,\xi\xi} - (w_{00}w_{10})_z.$$

Taking the $\xi$ derivative of the above and integrating in $z$ on $[1, z]$, we know

$$\begin{aligned} p_{31,\xi} =& \int_1^z p_{31,z'\xi}dz' + p_{31,\xi}|_{z=1} \\ =& \frac{z^2-1}{2}\Big( -u_{20,\xi\xi\xi} + u_{10,\xi\xi\tau} + (u_{00}u_{10,\xi\xi})_\xi + (u_{10}\eta_{00,\xi\xi})_\xi\Big) + (w_{00}w_{10})_\xi|_{z=1} \\ &- (w_{00}w_{10})_\xi + \Big(\eta_{00}\big( -w_{10,\xi} + w_{00,\tau} + u_{00}w_{00,\xi} + w_{00}w_{00,z}\big)\Big)_\xi|_{z=1} \\ &- \big(\eta_{10}w_{00,\xi}\big)_\xi|_{z=1} + \eta_{31,\xi}. \end{aligned} \tag{2.11}$$

On the other hand, we have $u_{21,z} = w_{10,\xi}$ and $u_{11,z} = w_{00,\xi}$ from [65]. Then the first equation in (2.10) becomes

$$-p_{31,\xi} = -u_{31,\xi} + u_{21,\tau} + (u_{00}u_{21} + u_{10}u_{11} + u_{20}u_{01})_\xi + (w_{00}w_{10})_\xi. \tag{2.12}$$

Combining (2.11) with (2.12), it leads to

$$\begin{aligned} 0 =& -u_{31,\xi} + u_{21,\tau} + (u_{00}u_{21} + u_{10}u_{11} + u_{20}u_{01})_\xi + \eta_{31,\xi} + \big(\eta_{10}\eta_{00,\xi\xi}\big)_\xi \\ &+ (w_{00}w_{10})_\xi|_{z=1} + \Big(u_{10,\xi\xi}\eta_{00} - \eta_{00,\xi\tau}\eta_{00} - \eta_{00}^2\eta_{00,\xi\xi} + \eta_{00}\eta_{00,\xi}^2\Big)_\xi \\ &+ \frac{z^2-1}{2}\Big( -u_{20,\xi\xi\xi} + u_{10,\xi\xi\tau} + (u_{00}u_{10,\xi\xi})_\xi + (u_{10}\eta_{00,\xi\xi})_\xi\Big). \end{aligned} \tag{2.13}$$

Now we will simplify equation (2.13). Because the fourth equation in (2.10) gives that

$$u_{31,\xi} = -\frac{z^2}{2}u_{20,\xi\xi\xi} + \partial_\xi\Phi_{31}(\tau,\xi).$$

for some $\Phi_{31}(\tau,\xi)$ independent of $z$, the third equation in (2.10) and the boundary condition on $\{z=0\}$ for $w_{31}$ yield that

$$w_{31} = w_{31}|_{z=0} + \int_0^z w_{31,z'}\,dz' = -\int_0^z u_{31,\xi}\,dz' = \frac{z^3}{6}u_{20,\xi\xi\xi} - z\partial_\xi\Phi_{31}(\tau,\xi).$$

Hence, combining with the boundary condition for $w_{31}$ on $z=1$, we have

$$\frac{1}{6}u_{20,\xi\xi\xi} - \partial_\xi\Phi_{31}(\tau,\xi) = -\eta_{31,\xi} + \eta_{21,\tau} + H_{4,\xi}|_{z=1} - \frac{\eta_{00}^2}{2}\eta_{00,\xi\xi\xi} - \eta_{00}\eta_{00,\xi}\eta_{00,\xi\xi},$$

where $H_4 := u_{00}\eta_{21} + u_{21}\eta_{00} + u_{20}\eta_{01} + u_{01}\eta_{20} + u_{11}\eta_{10} + u_{10}\eta_{11}$. Therefore $\Phi_{31}(\tau,\xi)$ satisfies

$$\partial_\xi\Phi_{31}(\tau,\xi) = \eta_{31,\xi} - \eta_{21,\tau} + \frac{1}{6}u_{20,\xi\xi\xi} - H_{4,\xi}|_{z=1} + \frac{\eta_{00}^2}{2}\eta_{00,\xi\xi\xi} + \eta_{00}\eta_{00,\xi}\eta_{00,\xi\xi}.$$

This in turn implies that

$$u_{31,\xi} = \eta_{31,\xi} - \eta_{21,\tau} - \left(\frac{z^2}{2} - \frac{1}{6}\right)u_{20,\xi\xi\xi} - H_{4,\xi}|_{z=1} + \frac{\eta_{00}^2}{2}\eta_{00,\xi\xi\xi} + \eta_{00}\eta_{00,\xi}\eta_{00,\xi\xi}.$$

It then follows from (2.13) that

$$\begin{aligned} 0 = {} & 2\eta_{21,\tau} + \frac{1}{3}u_{20,\xi\xi\xi} + (\eta_{00}\eta_{21})_\xi - (\eta_{00,\xi\tau}\eta_{00})_\xi + (\eta_{00}\eta_{00,\xi}^2)_\xi + (w_{00}w_{10})_\xi|_{z=1} \\ & + H_{4,\xi}|_{z=1} + \frac{2}{3}(\eta_{00}u_{10,\xi\xi})_\xi + \frac{1}{2}(\eta_{10}\eta_{00,\xi\xi})_\xi - \frac{7}{8}(\eta_{00}^2\eta_{00,\xi\xi})_\xi - \frac{1}{3}u_{10,\xi\xi\tau} \\ & - \left(u_{00}\int \eta_{11,\tau}\,d\xi\right)_\xi - (u_{00}H_2|_{z=1})_\xi + (u_{10}\phi_{11})_\xi - \int \eta_{11,\tau\tau}\,d\xi - H_{2,\tau}|_{z=1} \\ & + (u_{20}\eta_{01})_\xi - \frac{\eta_{00}^2}{2}\eta_{00,\xi\xi\xi} - \eta_{00}\eta_{00,\xi}\eta_{00,\xi\xi}, \end{aligned} \tag{2.14}$$

where $H_2 := u_{00}\eta_{11} + u_{11}\eta_{00} + u_{10}\eta_{01} + u_{01}\eta_{10}$ and $\phi_{11} := \frac{1}{3}\eta_{00,\xi\xi} - \frac{1}{2}\eta_{00}\eta_{01} + \eta_{11}$.

To obtain an equation for $\eta$ only, we substitute $u_{10}, u_{11}, u_{20}$, and $u_{21}$ into (2.14) to get that

$$
\begin{aligned}
H_{4,\xi}|_{z=1} &+ (u_{20}\eta_{01})_\xi - \left(u_{00}\int \eta_{11,\tau}\, d\xi\right)_\xi - (u_{00}H_2|_{z=1})_\xi + \frac{2}{3}(\eta_{00}u_{10,\xi\xi})_\xi \\
&= (2\eta_{00}\eta_{21} + 3\eta_{01}\eta_{20} + \eta_{10}\eta_{11})_\xi - (\eta_{00}\eta_{10}\eta_{01})_\xi + \frac{1}{4}(\eta_{00}^3\eta_{01})_\xi - \frac{1}{2}(\eta_{00,\xi\xi}\eta_{10})_\xi \\
&\quad + (\phi_{11}\eta_{10})_\xi - \frac{1}{4}(\eta_{00}^2\eta_{11})_\xi - 2\left(\eta_{00}\int \eta_{11,\tau}\, d\xi\right)_\xi + \frac{1}{3}(u_{10,\xi\xi}\eta_{00})_\xi - 2(H_2|_{z=1}\eta_{00})_\xi.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&2(\eta_{10}\phi_{11})_\xi - \frac{1}{4}(\eta_{00}^2\phi_{11})_\xi \\
&= \frac{2}{3}(\eta_{00,\xi\xi}\eta_{10})_\xi - (\eta_{00}\eta_{01}\eta_{10})_\xi + 2(\eta_{11}\eta_{10})_\xi - \frac{1}{12}(\eta_{00}^2\eta_{00,\xi\xi})_\xi + \frac{1}{8}(\eta_{01}\eta_{00}^3)_\xi - \frac{1}{4}(\eta_{11}\eta_{00}^2)_\xi.
\end{aligned}
$$

From [65], it is easy to see that

$$
\begin{aligned}
&-2\left(\eta_{00}\int \eta_{11,\tau}\, d\xi\right)_\xi \\
&\quad = 3(\eta_{00}^2\eta_{11} + \eta_{00}\eta_{10}\eta_{01})_\xi - \frac{3}{4}(\eta_{00}^3\eta_{01})_\xi + \frac{1}{3}(\eta_{00}\eta_{10,\xi\xi})_\xi + \frac{13}{24}(\eta_{00}\eta_{00,\xi}^2)_\xi + \frac{5}{6}(\eta_{00}^2\eta_{00,\xi\xi})_\xi.
\end{aligned}
$$

and

$$
\begin{aligned}
\int \eta_{11,\tau\tau}\, d\xi = &-\frac{3}{2}(\eta_{00}\eta_{11,\tau} + \eta_{00,\tau}\eta_{11} + (\eta_{10}\eta_{01})_\tau) + \frac{3}{8}(\eta_{00}^2\eta_{01})_\tau - \frac{1}{6}\eta_{10,\xi\xi\tau} \\
&- \frac{13}{24}\eta_{00,\xi}\eta_{00,\xi\tau} - \frac{5}{12}(\eta_{00}\eta_{00,\xi\xi})_\tau.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\int \eta_{11,\tau\tau}\, d\xi + H_{2,\tau}|_{z=1} \\
=&\frac{3}{4}(\eta_{00}^3\eta_{01})_\xi - \frac{1}{12}\eta_{00}\eta_{10,\xi\xi\xi} + \frac{13}{16}(\eta_{00,\xi}^2\eta_{00})_\xi - \frac{3}{2}(\eta_{00}\eta_{10}\eta_{01})_\xi + \frac{39}{32}(\eta_{00}^2\eta_{00,\xi\xi})_\xi \\
&- \frac{11}{96}(\eta_{00,\xi\xi}\eta_{00}^2)_\xi - \frac{3}{8}\eta_{00}^2\eta_{00,\xi\xi\xi} - \frac{1}{6}\eta_{10,\xi\xi\tau} - \frac{1}{12}\eta_{10}\eta_{00,\xi\xi\xi} - \frac{3}{4}\left(\eta_{00}^2\eta_{11}\right)_\xi.
\end{aligned}
$$

Then, (2.14) becomes

$$
\begin{aligned}
0 = \ & 2\eta_{21,\tau} + \frac{1}{3}\eta_{20,\xi\xi\xi} + 3(\eta_{00}\eta_{21} + \eta_{01}\eta_{20} + \eta_{10}\eta_{11})_\xi - \frac{3}{4}(\eta_{00}^2\eta_{11} + 2\eta_{01}\eta_{10}\eta_{00})_\xi \\
& + \frac{3}{8}(\eta_{00}^3\eta_{01})_\xi + \frac{5}{6}(\eta_{10}\eta_{00,\xi\xi\xi} + \eta_{00}\eta_{10,\xi\xi\xi}) + \frac{23}{12}(\eta_{00,\xi}\eta_{10,\xi\xi} + \eta_{00,\xi\xi}\eta_{10,\xi}) \\
& + \frac{21}{16}(\eta_{00,\xi}^3) - \frac{5}{16}(\eta_{00}\eta_{00,\xi}\eta_{00,\xi\xi}) - \frac{3}{4}(\eta_{00}^2\eta_{00,\xi\xi\xi}).
\end{aligned} \tag{2.15}
$$

The asymptotic expansion introduced earlier shows

$$
\eta := \eta_{00} + \varepsilon\eta_{10} + \varepsilon^2\eta_{20} + \varepsilon^3\eta_{30} + \mu\eta_{01} + \varepsilon\mu\eta_{11} + \varepsilon^2\mu\eta_{21} + O(\varepsilon^4, \mu^2).
$$

Recall the $\eta_{ij}$ equations obtained from [65], which are given by

$$
\begin{aligned}
& 2\eta_{00,\tau} + 3\eta_{00}\eta_{00,\xi} = 0, \\
& 2\eta_{01,\tau} + 3(\eta_{00}\eta_{01})_\xi + \frac{1}{3}\eta_{00,\xi\xi\xi} = 0, \\
& 2\eta_{10,\tau} + 3(\eta_{00}\eta_{10})_\xi - \frac{1}{4}(\eta_{00}^3)_\xi = 0, \\
& 2\eta_{20,\tau} + 3(\eta_{00}\eta_{20})_\xi + \frac{3}{2}(\eta_{10}^2)_\xi - \frac{3}{4}(\eta_{00}^2\eta_{10})_\xi - \frac{3}{8}(\eta_{00}^4)_\xi = 0, \\
& 2\eta_{30,\tau} + 3(\eta_{00}\eta_{30} + \eta_{10}\eta_{20})_\xi - \frac{3}{4}(\eta_{00}^2\eta_{20} + \eta_{00}\eta_{10}^2)_\xi - \frac{3}{8}(\eta_{00}^3\eta_{10})_\xi + \frac{115}{192}(\eta_{00}^5)_\xi = 0, \\
& 2\eta_{11,\tau} + 3(\eta_{00}\eta_{11} + \eta_{10}\eta_{01})_\xi - \frac{3}{4}(\eta_{00}^2\eta_{01})_\xi + \frac{1}{3}\eta_{10,\xi\xi\xi} - \frac{23}{24}(\eta_{00,\xi}^2)_\xi - \frac{5}{6}(\eta_{00}\eta_{00,\xi\xi})_\xi = 0.
\end{aligned}
$$

From (2.15) we obtain

$$
\begin{aligned}
& 2\eta_\tau + 3\eta\eta_\xi + \frac{1}{3}\mu\eta_{\xi\xi\xi} - \frac{3}{4}\varepsilon\eta^2\eta_\xi + \frac{3}{8}\varepsilon^2\eta^3\eta_\xi + \frac{115}{192}\varepsilon^3\eta^4\eta_\xi + \alpha\eta^5\eta_\xi + \varepsilon\mu\left(\frac{23}{12}\eta_\xi\eta_{\xi\xi} + \frac{5}{6}\eta\eta_{\xi\xi\xi}\right) \\
& \quad + \varepsilon^2\mu\left(-\frac{5}{16}\eta\eta_\xi\eta_{\xi\xi} - \frac{3}{4}\eta^2\eta_{\xi\xi\xi} + \frac{21}{16}\eta_\xi^3\right) = 0 + O(\varepsilon^5, \varepsilon^3\mu, \mu^2),
\end{aligned} \tag{2.16}
$$

where $\alpha$ is some constant we do not specify here.

Recall the original transformation $x = \varepsilon^{-\frac{1}{2}}\xi + \varepsilon^{-\frac{3}{2}}\tau,\ t = \varepsilon^{-\frac{3}{2}}\tau$, namely,

$$\frac{\partial}{\partial \xi} = \varepsilon^{-\frac{1}{2}}\partial_x, \quad \frac{\partial}{\partial \tau} = \varepsilon^{-\frac{3}{2}}(\partial_x + \partial_t). \tag{2.17}$$

The equation (2.16) transforms to

$$\begin{aligned} 2(\eta_x + \eta_t) + \frac{1}{3}\mu\eta_{xxx} + 3\varepsilon\eta\eta_x + \varepsilon^2 A_1\eta^2\eta_x + \varepsilon^3 A_2\eta^3\eta_x + \varepsilon\mu\big(A_3\eta_x\eta_{xx} + A_4\eta\eta_{xxx}\big) \\ + A_8\varepsilon^4\eta^4\eta_x + \varepsilon^2\mu\big(A_5\eta\eta_x\eta_{xx} + A_6\eta^2\eta_{xxx} + A_7\eta_x^3\big) = 0 + O(\varepsilon^5, \varepsilon^3\mu, \mu^2). \end{aligned} \tag{2.18}$$

where $A_1 = -\frac{3}{4}$, $A_2 = \frac{3}{8}$, $A_3 = \frac{23}{12}$, $A_4 = \frac{5}{6}$, $A_5 = -\frac{5}{16}$, $A_6 = -\frac{3}{4}$, $A_7 = \frac{21}{16}$, $A_8 = \frac{115}{192}$.

**Remark 2.** *It is noted that the high-order terms $O(\varepsilon^5, \mu^2)$ in (2.16) only depend on the function $\eta$ and its $\xi$ derivative. By the scaling invariance in (2.18), $O(\varepsilon^5, \mu^2)$ would not generate any lower order terms in (2.18) under the transformation in (2.17).*

# Chapter 3

## DERIVATION OF MODEL EQUATIONS WITH CUBIC NONLINEAR TERMS

### 3.1 Introduction

In this chapter, we derive the model equations to incorporate cubic nonlinearities of various types, including the CH, mCH, and Novikov types, as demonstrated in (1.5).

In comparison to [16], we intend for our model equations to include all of the nonlinear terms in the appropriate truncation order, reasonably allowing for a precise explanation of their hydrodynamic relevance to the entire water wave problem. Since the cubic nonlinearity is expected to arise on the order of $O(\varepsilon^2\mu)$, leaving the $O(\mu^2)$ terms as higher order terms, a scaling requirement $\varepsilon = O(\mu^{1/2})$ is naturally imposed. Therefore, we choose

$$\mu \ll 1, \qquad \varepsilon = O(\mu^{\frac{2}{5}}), \tag{3.1}$$

which corresponds to a shallow-water regime for waves of *mildly large amplitude*. Then, we derive an equation for the scaled surface elevation $\eta$ (1.4).

Next, we expand $\eta$ in terms of $u$ along with its derivatives using the so-called Kodama transformation [21]. In particular, the expansion takes the following form

$$\eta \sim u + \varepsilon A + \mu B + \varepsilon\mu C + \mu^2 D + \varepsilon^2 E + \varepsilon^3 K + \varepsilon^2\mu G + \varepsilon\mu^2 H.$$

The Kodama transformation takes a given shallow water elevation equation and transforms it into an equation of the same form but with different coefficients. These equations are asymptotically identical, meaning that, when the small parameters approach 0, their solutions converge to the same form. In other words, it is a transformation commonly employed to transform higher-order nonlinear differential equations to their asymptotically equal forms [66]. Indeed, many authors [23, 67, 68] have used and generalized this transformation.

A cubic nonlinear equation for $u$ is constructed by connecting the horizontal velocity $u$ with $\eta$ using the freedom of evaluating $u$ at a specific water depth and omitting some nonlinear power terms. In particular, this allows us to obtain CH-mCH-Novikov equation.

## 3.2 The Derivation

Having derived the equation of the free surface $\eta$ in Chapter (2), this section focuses on the derivation of the model equations that incorporate cubic nonlinearities of multiple kinds, including the CH, mCH, and Novikov types, as given in (1.5).

As the introduction indicates, we assume $\mu \ll 1$ and work in the regime where $\varepsilon = O(\mu^{\frac{2}{5}})$, which we refer to as the shallow-water regime for waves of mildly large amplitude. Since we expect our final model equations to be cubic nonlinear, a higher-order approximation (in $\varepsilon$ and $\mu$) is necessary. Thus, it is natural to post the Kodama transformation of the form

$$\eta = u + \varepsilon A + \mu B + \varepsilon\mu C + \mu^2 D + \varepsilon^2 E + \varepsilon^3 K + \varepsilon^2\mu G + \varepsilon\mu^2 H,$$

where $A, B, C, D, E, H, K,$ and $G$ are parameters related to $u$ and its derivatives but independent of $\varepsilon$ and $\mu$. This creates a certain degree of freedom in the expansion that may later be optimized. For example, to obtain the CH-type terms, as described in [29], it is possible to choose $A = \lambda_1 u^2$, $B = \lambda_2 u_{xx}$, and $K = \lambda_0 u^4$, where $\lambda_0,\ \lambda_1$ and $\lambda_2$ are some constants to be determined later. As a result, (3.2) becomes

$$\eta = u + \lambda_1 \varepsilon u^2 + \lambda_2 \mu u_{xx} + \varepsilon\mu C + \mu^2 D + \varepsilon^2 E + \lambda_0 \varepsilon^4 u^4 + \varepsilon^2 \mu G + \varepsilon \mu^2 H.$$

To proceed, we plug the Kodama transformation (3.2) into (2.18). The resulting equation will consists of $u$-terms. Collecting at each order, we have

$$\begin{aligned}
O_0(1) &:= 2(u_x + u_t),\\
O_0(\varepsilon) &:= 4\lambda_1\varepsilon(uu_x + uu_t) + 3\varepsilon uu_x,\\
O_0(\varepsilon^2) &:= 2\varepsilon^2(E_x + E_t) + 9\lambda_1\varepsilon^2u^2u_x + A_1\varepsilon^2u^2u_x,\\
O_0(\varepsilon^3) &:= 3\varepsilon^3(uE)_x + 6\lambda_1^2\varepsilon^3u^3u_x + A_1\lambda_1\varepsilon^3(u^4)_x + A_2\varepsilon^3u^3u_x + 2\lambda_0\varepsilon^3((u^4)_x + (u^4)_t),\\
O_0(\varepsilon^4) &:= \varepsilon^4\left(\lambda_0 + \lambda_1 + A_1\lambda_1^2 + A_2\lambda_1 + \frac{A_8}{5}\right)(u^5)_x + A_1\varepsilon^4(u^2E)_x,\\
O_0(\mu) &:= 2\lambda_2\mu(u_{xxx} + u_{xxt}) + \frac{1}{3}\mu u_{xxx},\\
O_0(\mu^2) &:= 2\mu^2(D_x + D_t) + \frac{\lambda_2}{3}\mu^2u_{xxxxx},\\
O_0(\varepsilon\mu) &:= 2\varepsilon\mu(C_x + C_t) + (2\lambda_1 + 3\lambda_2 + A_3)\varepsilon\mu u_xu_{xx} + \left(\frac{2}{3}\lambda_1 + 3\lambda_2 + A_4\right)\varepsilon\mu uu_{xxx},\\
O_0(\varepsilon^2\mu) &:= \frac{1}{3}\varepsilon^2\mu E_{xxx} + 2\varepsilon^2\mu(G_x + G_t) + 3\varepsilon^2\mu(uC)_x + 3\lambda_2\lambda_1\varepsilon^2\mu(u^2u_{xx})_x + \lambda_2A_1\varepsilon^2\mu(u^2u_{xx})_x\\
&\quad + 2\lambda_1A_3\varepsilon^2\mu(uu_x^2)_x + A_4\lambda_1\varepsilon^2\mu u^2u_{xxx} + \lambda_1A_4\varepsilon^2\mu u(u^2)_{xxx}\\
&\quad + A_5\varepsilon^2\mu uu_xu_{xx} + A_6\varepsilon^2\mu u^2u_{xxx} + A_7\varepsilon^2\mu u_x^3,\\
O_0(\varepsilon\mu^2) &:= \frac{1}{3}\varepsilon\mu^2C_{xxx} + 2\varepsilon\mu^2(H_x + H_t) + 3\varepsilon\mu^2(uD)_x + 3\lambda_2^2\varepsilon\mu^2u_{xx}u_{xxx}\\
&\quad + A_3\lambda_2\varepsilon\mu^2(u_xu_{xxx})_x + A_4\lambda_2\varepsilon\mu^2u_{xx}u_{xxx} + A_4\lambda_2\varepsilon\mu^2uu_{xxxxx},\\
O_0(\varepsilon^2\mu^2) &:= \frac{1}{3}\varepsilon^2\mu^2G_{xxx} + 3\lambda_1\varepsilon^2\mu^2(u^2D)_x + 3\lambda_2\varepsilon^2\mu^2(u_{xx}C)_x + 3\varepsilon^2\mu^2(Hu)_x\\
&\quad + \lambda_2^2A_1\varepsilon^2\mu^2(uu_{xx}^2)_x + A_1\varepsilon^2\mu^2(u^2D)_x + A_3\varepsilon^2\mu^2(u_xC_x)_x + 2A_3\lambda_1\lambda_2\varepsilon^2\mu^2(uu_xu_{xxx})_x
\end{aligned}$$

$$+ A_4\varepsilon^2\mu^2 C u_{xxx} + A_4\lambda_1\lambda_2\varepsilon^2\mu^2 u^2 u_{xxxxx} + A_4\lambda_1\lambda_2\varepsilon^2\mu^2 u_{xx}(u^2)_{xxx} + A_4\varepsilon^2\mu^2 u C_{xxx}$$
$$+ A_5\lambda_2\varepsilon^2\mu^2 u u_x u_{xxxx} + A_5\lambda_2\varepsilon^2\mu^2 u_x u_{xx}^2 + A_5\lambda_2\varepsilon^2\mu^2 u u_{xx} u_{xxx}$$
$$+ A_6\lambda_2\varepsilon^2\mu^2 u^2 u_{xxxxx} + 2A_6\lambda_2\varepsilon^2\mu^2 u u_{xx} u_{xxx} + 3A_7\lambda_2\varepsilon^2\mu^2 u_x^2 u_{xxx}.$$

This yields the following equation

$$u_t + u_x + \frac{1}{2}\Big[O_0(\varepsilon) + O_0(\varepsilon^2) + O_0(\varepsilon^3) + O_0(\mu^2) + O_0(\mu) + O_0(\varepsilon\mu) + O_0(\varepsilon\mu^2)\Big] = 0 + O(\varepsilon^3\mu, \varepsilon^2\mu^2, \mu^3).$$

Here, the subscript in $O_0$ is included solely to emphasize that the terms may change at each step.

The next step is to eliminate the $t$ derivative using the equation itself. As before, we expand the time derivatives, namely

$$\begin{aligned} u_t =& - u_x - 2\lambda_1\varepsilon(uu_x + uu_t) - \frac{3}{2}\varepsilon u u_x \\ &- \frac{1}{2}\Big[O_0(\varepsilon^2) + O_0(\varepsilon^3) + O_0(\mu^2) + O_0(\mu) + O_0(\varepsilon\mu) + O_0(\varepsilon\mu^2)\Big] + O(\varepsilon^3\mu, \varepsilon^2\mu^2, \mu^3). \end{aligned} \tag{3.2}$$

To have the whole $\varepsilon^2\mu^2$-order terms, we must take $\mu^2$ and $\varepsilon\mu^2$-order terms back even though, at the end, they are ignored as high-order. Next, we present this procedure in greater detail.

***Step 1***. At order $\varepsilon$, we substitute (3.2) into $\frac{1}{2}O_0(\varepsilon)$, we get

$$\begin{aligned} 2\lambda_1\varepsilon(uu_x + uu_t) + \frac{3}{2}\varepsilon u u_x =& \frac{3}{2}\varepsilon u u_x - \frac{4}{3}\lambda_1^2\varepsilon^2((u^3)_x + (u^3)_t) - \lambda_1\varepsilon^2(u^3)_x \\ &- \lambda_1\varepsilon u\Big[O_0(\varepsilon^2) + O_0(\varepsilon^3) + O_0(\mu) + O_0(\varepsilon\mu) + O_0(\mu^2) + O_0(\varepsilon\mu^2)\Big]. \end{aligned} \tag{3.3}$$

This expansion generates higher order terms. It leads to the following terms in asymptotic order

$$
\begin{aligned}
O_1(\varepsilon) &:= \frac{3}{2}\varepsilon u u_x, & O_1(\varepsilon^2) &:= \frac{1}{2}O_0(\varepsilon^2) - \frac{4}{3}\lambda_1^2\varepsilon^2((u^3)_x + (u^3)_t) - \lambda_1\varepsilon^2(u^3)_x,\\
O_1(\varepsilon^3) &:= \frac{1}{2}O_0(\varepsilon^3) - \lambda_1\varepsilon u O_0(\varepsilon^2), & O_1(\varepsilon^4) &:= \frac{1}{2}O_0(\varepsilon^4) - \lambda_1\varepsilon u O_0(\varepsilon^3),\\
O_1(\varepsilon\mu) &:= \frac{1}{2}O_0(\varepsilon\mu) - \lambda_1\varepsilon u O_0(\mu), & O_1(\mu) &:= \frac{1}{2}O_0(\mu),\\
O_1(\mu^2) &:= \frac{1}{2}O_0(\mu^2), & O_1(\varepsilon^2\mu) &:= \frac{1}{2}O_0(\varepsilon^2\mu) - \lambda_1\varepsilon u O_0(\varepsilon\mu),
\end{aligned}
$$

$$
O_1(\varepsilon\mu^2) := \tfrac{1}{2}O_0(\varepsilon\mu^2) - \lambda_1\varepsilon u O_0(\mu^2), \quad O_1(\varepsilon^2\mu^2) := \tfrac{1}{2}O_0(\varepsilon^2\mu^2) - \lambda_1\varepsilon u O_0(\varepsilon\mu^2).
$$

***Step 2***. For $O_1(\varepsilon^2)$ term, we can choose $E = \lambda_3 u^3$. Then we expand the time derivatives as

$$
u_t = -u_x - 2\lambda_1\varepsilon(uu_x + uu_t) - \frac{3}{2}\varepsilon u u_x - \frac{1}{2}\big[O_0(\mu) + O_0(\mu^2) + O_0(\varepsilon^2)\big] + O(\varepsilon^2, \varepsilon\mu).
$$

Hence the $O_1(\varepsilon^2)$-order term takes the following form,

$$
\begin{aligned}
O_1(\varepsilon^2) = &\left(\frac{1}{2}\lambda_1 + \frac{A_1}{6}\right)\varepsilon^2(u^3)_x - (6\lambda_3 - 8\lambda_1^2)\lambda_1\varepsilon^3u^2(uu_x + uu_t) - \left(\frac{9}{2}\lambda_3 - 6\lambda_1^2\right)\varepsilon^3u^3u_x\\
&- \left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\varepsilon^2u^2\big[O_0(\varepsilon^2) + O_0(\mu) + O_0(\mu^2)\big].
\end{aligned}
$$

We now denote $I_{f(u)}$ to be the coefficient of $f(u)$ in the equation. Then coefficient of $u^2u_x$ is given by $I_{u^2u_x} := \dfrac{3}{2}\lambda_1 + \dfrac{A_1}{2}$. The following terms in asymptotic order take the form

$$
\begin{aligned}
&O_2(\mu) := O_1(\mu), \qquad O_2(\varepsilon^2) := \left(\frac{3}{2}\lambda_1 + \frac{A_1}{2}\right)\varepsilon^2u^2u_x, \quad O_2(\varepsilon\mu) := O_1(\varepsilon\mu),\\
&O_2(\varepsilon^3) := O_1(\varepsilon^3) - (6\lambda_3 - 8\lambda_1^2)\lambda_1\varepsilon^3u^2(uu_x + uu_t) - \left(\frac{9}{2}\lambda_3 - 6\lambda_1^2\right)\varepsilon^3u^3u_x,\\
&O_2(\varepsilon^4) := O_1(\varepsilon^4) - \left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\varepsilon^2u^2O_0(\varepsilon^2), \qquad O_2(\mu^2) := O_1(\mu^2),\\
&O_2(\varepsilon^2\mu) := O_1(\varepsilon^2\mu) - \left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\varepsilon^2u^2O_0(\mu),
\end{aligned}
$$

$$O_2(\varepsilon\mu^2) := O_1(\varepsilon\mu^2), \quad O_2(\varepsilon^2\mu^2) := O_1(\varepsilon^2\mu^2) - \left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\varepsilon^2 u^2 O_0(\mu^2).$$

Now the equation has the form of

$$u_t + u_x + O_1(\varepsilon) + O_2(\varepsilon^2) + O_2(\mu) + O_2(\varepsilon\mu) + O_2(\varepsilon^2\mu) + O_2(\varepsilon^4) = 0 + O(\varepsilon^3\mu, \mu^3),$$

and the expression for $u_t$ is given by

$$\begin{aligned} u_t = &- u_x - \frac{3}{2}\varepsilon u u_x - (\frac{1}{2}\lambda_1 + \frac{A_1}{6})\varepsilon^2(u^3)_x - \frac{1}{2}O_0(\mu) - \frac{1}{2}O_0(\varepsilon\mu) + \lambda_1\varepsilon u O_0(\mu) \\ &- \frac{1}{2}O_0(\varepsilon^2\mu) + \lambda_1\varepsilon u O_0(\varepsilon\mu) + (\frac{3}{2}\lambda_3 - 2\lambda_1^2)\varepsilon^2 u^2 O_0(\mu) + O(\varepsilon^2, \mu^2). \end{aligned} \tag{3.4}$$

***Step 3***. We now consider $O_2(\mu)$ term. Here, another parameter is required. To this end, splitting the time derivative $\lambda_2\mu u_{xxt}$, it appears that

$$\lambda_2\mu u_{xxt} = \lambda_2(1-\nu)\mu u_{xxt} + \lambda_2\nu\mu u_{xxt}, \tag{3.5}$$

where $\nu$ is the new parameter which will be determined later. We remove the $u_{xxt}$ term by eliminating the $t$ derivatives using (3.4). Thereby, we have

$$\lambda_2\nu\mu u_{xxt} = -\lambda_2\nu\mu u_{xxx} - \frac{3}{2}\lambda_2\nu\varepsilon\mu(uu_x)_{xx} + \lambda_2\nu\mu(F_{\varepsilon^2} + F_\mu + F_{\varepsilon\mu} + F_{\varepsilon^2\mu})_{xx} + O(\varepsilon^3\mu, \varepsilon\mu^3),$$

where we define

$$F_{\varepsilon^2} := -\left(\frac{1}{2}\lambda_1 + \frac{A_1}{6}\right)\varepsilon^2(u^3)_x, \qquad F_\mu := -\lambda_2\mu(u_{xxx} + u_{xxt}) - \frac{1}{6}\mu u_{xxx},$$

$$F_{\varepsilon\mu} := -\frac{1}{2}O_0(\varepsilon\mu) + \lambda_1 u\varepsilon O_0(\mu), \qquad F_{\varepsilon^2\mu} := -\frac{1}{2}O_0(\varepsilon^2\mu) + \lambda_1 u\varepsilon O_0(\varepsilon\mu) + \left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\varepsilon^2 u^2 O_0(\mu).$$

This way $O_2(\mu)$ takes the form

$$\lambda_2\mu(u_{xxx}+u_{xxt})+\frac{1}{6}\mu u_{xxx}=\left(\lambda_2(1-\nu)+\frac{1}{6}\right)\mu u_{xxx}+\lambda_2(1-\nu)\mu u_{xxt}-\frac{3}{2}\nu\lambda_2\varepsilon\mu(3u_xu_{xx}+uu_{xxx})$$
$$+\lambda_2\nu\mu(F_{\varepsilon^2}+F_\mu+F_{\varepsilon\mu}+F_{\varepsilon^2\mu})_{xx}.$$

The coefficient of $u_{xxt}$ can be written as $I_{u_{xxt}}:=\lambda_2(1-\nu)$.

This procedure leads to the following terms in asymptotic order:

$$O_3(\mu):=(\lambda_2(1-\nu)+\frac{1}{6})\mu u_{xxx}+\lambda_2(1-\nu)\mu u_{xxt},\quad O_3(\varepsilon^3):=O_2(\varepsilon^3),$$
$$O_3(\varepsilon\mu):=O_2(\varepsilon\mu)-\frac{3}{2}\nu\lambda_2\varepsilon\mu(3u_xu_{xx}+uu_{xxx}),\qquad O_3(\varepsilon^4):=O_2(\varepsilon^4),$$
$$O_3(\varepsilon^2\mu):=O_2(\varepsilon^2\mu)+\lambda_2\nu\mu(F_{\varepsilon^2})_{xx},\qquad O_3(\mu^2):=O_2(\mu^2)+\lambda_2\nu\mu(F_\mu)_{xx},$$
$$O_3(\varepsilon\mu^2):=O_2(\varepsilon\mu^2)+\lambda_2\nu\mu(F_{\varepsilon\mu})_{xx},\qquad O_3(\varepsilon^2\mu^2):=O_2(\varepsilon^2\mu^2)+\lambda_2\nu\mu(F_{\varepsilon^2\mu})_{xx}.$$

***Step 4.*** We now consider $O_3(\varepsilon\mu)$ term. Choose $C=\lambda_4u_x^2+\lambda_5uu_{xx}$. From (3.4), the expression for $u_t$ is given by

$$u_t=-u_x-\frac{3}{2}\varepsilon uu_x-\frac{1}{2}(O_0(\mu)+O_0(\varepsilon\mu))+\lambda_1u\varepsilon O_0(\mu)+O(\varepsilon^2\mu,\mu^2,\varepsilon^2).$$

This operation produces $O_3(\varepsilon\mu)$ of the form

$$\begin{aligned}\varepsilon\mu(C_x+C_t)-2\lambda_1\lambda_2\varepsilon\mu u(u_{xxx}+u_{xxt})&=-3\lambda_4\varepsilon^2\mu u_x(uu_x)_x-\frac{3}{2}\lambda_5\varepsilon^2\mu uu_xu_{xx}-\left(\frac{3}{2}\lambda_5-3\lambda_1\lambda_2\right)\varepsilon^2\mu u(uu_x)_{xx}\\&-\lambda_4\varepsilon\mu u_x(O_0(\mu))_x-\lambda_5\varepsilon\mu u_{xx}\frac{1}{2}O_0(\mu)-\left(\frac{1}{2}\lambda_5-\lambda_1\lambda_2\right)\varepsilon\mu uO_0(\mu)_{xx}\\&-\lambda_4\varepsilon\mu u_x\Big(O_0(\varepsilon\mu)+2\lambda_1u\varepsilon O_0(\mu)\Big)_x-\frac{\lambda_5}{2}\varepsilon\mu u_{xx}O_0(\varepsilon\mu)\\&+\lambda_5\lambda_1uu_{xx}\varepsilon^2\mu O_0(\mu)-\left(\frac{1}{2}\lambda_5-\lambda_1\lambda_2\right)\varepsilon\mu uO_0(\varepsilon\mu)_{xx}\\&+(\lambda_5-2\lambda_1\lambda_2)\lambda_1\varepsilon^2\mu u(uO_0(\mu))_{xx}.\end{aligned}$$

The $\varepsilon\mu$-order term turns out to be

$$\varepsilon\mu\left[\left(\frac{3}{2}\lambda_2+\frac{1}{2}A_4-\frac{3}{2}\nu\lambda_2\right)uu_{xxx}+\left(\lambda_1+\frac{3}{2}\lambda_2+\frac{1}{2}A_3-\frac{9}{2}\nu\lambda_2\right)u_xu_{xx}\right].$$

Denote the coefficients of $uu_{xxx}$ and $u_xu_{xx}$ by

$$\begin{cases} I_{uu_{xxx}} := \frac{3}{2}\lambda_2+\frac{1}{2}A_4-\frac{3}{2}\nu\lambda_2, \\ I_{u_xu_{xx}} := \lambda_1+\frac{3}{2}\lambda_2+\frac{1}{2}A_3-\frac{9}{2}\nu\lambda_2. \end{cases}$$

The terms in asymptotic order are

$$\begin{aligned}
O_4(\mu^2) :=& O_3(\mu^2), \qquad O_4(\varepsilon^3) := O_3(\varepsilon^3), \qquad O_4(\varepsilon^4) := O_3(\varepsilon^4),\\
O_4(\varepsilon^2\mu) :=& O_3(\varepsilon^2\mu) - 3\lambda_4\varepsilon^2\mu u_x(uu_x)_x - \frac{3}{2}\lambda_5\varepsilon^2\mu uu_xu_{xx} - \left(\frac{3}{2}\lambda_5-3\lambda_1\lambda_2\right)\varepsilon^2\mu u(uu_x)_{xx},\\
O_4(\varepsilon\mu^2) :=& O_3(\varepsilon\mu^2) - \lambda_4\varepsilon\mu u_x(O_0(\mu))_x - \lambda_5\varepsilon\mu u_{xx}\frac{1}{2}O_0(\mu) - \left(\frac{1}{2}\lambda_5-\lambda_1\lambda_2\right)\varepsilon\mu u(O_0(\mu))_{xx},\\
O_4(\varepsilon^2\mu^2) :=& O_3(\varepsilon^2\mu^2) - \lambda_4\varepsilon\mu u_x(O_0(\varepsilon\mu)+2\lambda_1\varepsilon uO_0(\mu))_x - \frac{\lambda_5}{2}\varepsilon\mu u_{xx}O_0(\varepsilon\mu) + \lambda_5\lambda_1\varepsilon^2\mu uu_{xx}O_0(\mu)\\
&- \left(\frac{1}{2}\lambda_5-\lambda_1\lambda_2\right)\varepsilon\mu u(O_0(\varepsilon\mu))_{xx} + (\lambda_5-2\lambda_1\lambda_2)\lambda_1\varepsilon^2\mu u(uO_0(\mu))_{xx}.
\end{aligned}$$

***Step 5***. Next we consider $\varepsilon^3$-order which has the form

$$\begin{aligned}
O_4(\varepsilon^3) =& \frac{1}{2}O_0(\varepsilon^3) - \lambda_1\varepsilon uO_0(\varepsilon^2) - (6\lambda_3-8\lambda_1^2)\lambda_1\varepsilon^3u^2(uu_x+uu_t) - \left(\frac{9}{2}\lambda_3-6\lambda_1^2\right)\varepsilon^3u^3u_x\\
=& \left(\frac{3}{8}\lambda_3+\frac{1}{8}A_2+\frac{1}{4}\lambda_1A_1\right)\varepsilon^3(u^4)_x + 2\lambda_0\varepsilon^3\big((u^4)_x+(u^4)_t\big),
\end{aligned}$$

where we have replaced $u_t$ by $-u_x-\frac{3}{2}\varepsilon uu_x$. The coefficient is denoted by

$$I_{(u^4)_x} := \frac{3}{8}\lambda_3+\frac{1}{8}A_2+\frac{1}{4}\lambda_1A_1.$$

Also, at $\varepsilon^4$-order we have

$$O_5(\varepsilon^4) := \frac{1}{2}O_0(\varepsilon^4) - \lambda_1\varepsilon u O_0(\varepsilon^3) - \left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\varepsilon^2 u^2 O_0(\varepsilon^2) - 12\lambda_0\varepsilon^4 u^4 u_x.$$

Since

$$O_0(\varepsilon^4) = \varepsilon^4\left(\lambda_0 + \lambda_1 + A_1\lambda_1^2 + A_2\lambda_1 + \frac{A_8}{5} + A_1\lambda_3\right)(u^5)_x,$$

we can simplify $O_5$ as

$$\begin{aligned} O_5(\varepsilon^4) &= \frac{1}{2}O_0(\varepsilon^4) - \lambda_1\varepsilon u O_0(\varepsilon^3) - \left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\varepsilon^2 u^2 O_0(\varepsilon^2) - 12\lambda_0\varepsilon^4 u^4 u_x \\ &= \frac{1}{2}\big(-19\lambda_0 + 5\lambda_1 + 5A_1\lambda_1^2 + 5A_2\lambda_1 + A_8 + 5A_1\lambda_3\big)\varepsilon^4 u^4 u_x \\ &\quad - \lambda_1\varepsilon^4\big(12\lambda_3 + 6\lambda_1^2 + 4A_1\lambda_1 + A_2\big)u^4 u_x - \left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\big(9\lambda_1 + A_1\big)u^4 u_x \\ &= \varepsilon^4\frac{1}{2}\Big(-19\lambda_0 - \frac{3}{4}\lambda_1^2 + \frac{49}{8}\lambda_1 + \frac{A_8}{5} - \frac{3}{2}\lambda_3 + 24\lambda_1^3 - 51\lambda_1\lambda_3\Big)u^4 u_x. \end{aligned}$$

Then $I_{u^4u_x} = -19\lambda_0 - \frac{3}{4}\lambda_1^2 + \frac{49}{8}\lambda_1 + \frac{A_8}{5} - \frac{3}{2}\lambda_3 + 24\lambda_1^3 - 51\lambda_1\lambda_3$ and the terms which involve $\mu$ keep the same.

***Step 6***. Finally, we consider the $\varepsilon^2\mu$-order which has the form

$$\begin{aligned} O_4(\varepsilon^2\mu) = \frac{1}{2}O_0(\varepsilon^2\mu) - \lambda_1\varepsilon u O_0(\varepsilon\mu) - u^2\left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\varepsilon^2 O_0(\mu) + \lambda_2\nu\mu(F_{\varepsilon^2})_{xx} \\ - 3\lambda_4\varepsilon^2\mu u_x(uu_x)_x - \frac{3}{2}\lambda_5\varepsilon^2\mu u u_x u_{xx} - \left(\frac{3}{2}\lambda_5 - 3\lambda_1\lambda_2\right)\varepsilon^2\mu u(uu_x)_{xx}. \end{aligned}$$

We choose $G = \lambda_7 u u_x^2 + \lambda_8 u^2 u_{xx}$ to keep the scaling in the equation. From (3.4), the expression for $u_t$ is given by $u_t = -u_x - \lambda_2\mu(u_{xxx} + u_{xxt}) - \frac{1}{6}\mu u_{xxx}$. We eliminate $u_t$ by (3.4) itself, namely

$$u_t = -u_x - \frac{1}{6}\mu u_{xxx} + O(\varepsilon\mu). \tag{3.6}$$

Thereby, there appears the relation

$$\varepsilon^2\mu(G_x+G_t) = -\frac{1}{6}\lambda_7\varepsilon^2\mu^2u_x^2u_{xxx} - \frac{1}{3}\lambda_7\varepsilon^2\mu^2uu_xu_{xxxx} - \frac{1}{6}\lambda_8\varepsilon^2\mu^2u^2u_{xxxxx} - \frac{1}{3}\lambda_8\varepsilon^2\mu^2uu_{xx}u_{xxx}.$$

Hence, $\frac{1}{2}O_0(\varepsilon^2\mu)$ takes the form

$$\begin{aligned}\frac{1}{2}O_0(\varepsilon^2\mu) = & \frac{1}{6}\varepsilon^2\mu\lambda_3(u^3)_{xxx} + \frac{3}{2}\varepsilon^2\mu\lambda_4(uu_x^2)_x + \frac{3}{2}\varepsilon^2\mu\lambda_5(u^2u_{xx})_x + \frac{3}{2}\lambda_2\lambda_1\varepsilon^2\mu(u^2u_{xx})_x \\ & + \frac{1}{2}\varepsilon^2\mu\lambda_2A_1(u^2u_{xx})_x + \varepsilon^2\mu\lambda_1A_3(uu_x^2)_x + \frac{1}{2}A_4\lambda_1\varepsilon^2\mu u^2u_{xxx} + \frac{1}{2}\varepsilon^2\mu\lambda_1A_4u(u^2)_{xxx} \\ & + \frac{1}{2}A_5\varepsilon^2\mu uu_xu_{xx} + \frac{1}{2}A_6\varepsilon^2\mu u^2u_{xxx} + \frac{1}{2}A_7\varepsilon^2\mu u_x^3 - \frac{1}{6}\lambda_7\varepsilon^2\mu^2u_x^2u_{xxx} \\ & - \frac{1}{3}\lambda_7\varepsilon^2\mu^2uu_xu_{xxxx} - \frac{1}{6}\lambda_8\varepsilon^2\mu^2u^2u_{xxxxx} - \frac{1}{3}\lambda_8\varepsilon^2\mu^2uu_{xx}u_{xxx}.\end{aligned}$$

We now deal with $-\lambda_1u\varepsilon O_0(\varepsilon\mu)$. By definition $C = \lambda_4u_x^2 + \lambda_5uu_{xx}$ and (3.6), it follows that

$$-2\lambda_1\varepsilon^2\mu u(C_x+C_t) = -\varepsilon^2\mu^2\left(-\frac{1}{3}\lambda_1\lambda_4uu_xu_{xxxx} - \frac{1}{6}\lambda_1\lambda_5uu_{xx}u_{xxx} - \frac{1}{6}\lambda_1\lambda_5u^2u_{xxxxx}\right).$$

Then we know

$$\begin{aligned}-\lambda_1u\varepsilon O_0(\varepsilon\mu) = & -\lambda_1\varepsilon^2\mu\left[\left(\frac{6}{3}\lambda_1 + 3\lambda_2 + A_3\right)uu_xu_{xx} + \left(\frac{2}{3}\lambda_1 + 3\lambda_2 + A_4\right)u^2u_{xxx}\right] \\ & + \varepsilon^2\mu^2\left(\frac{1}{3}\lambda_1\lambda_4uu_xu_{xxxx} + \frac{1}{6}\lambda_1\lambda_5uu_{xx}u_{xxx} + \frac{1}{6}\lambda_1\lambda_5u^2u_{xxxxx}\right).\end{aligned}$$

Similarly, we have

$$-\left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\varepsilon^2u^2O_0(\mu) = -\left(\frac{1}{2}\lambda_3 - \frac{2}{3}\lambda_1^2\right)\varepsilon^2\mu u^2u_{xxx} - \lambda_2\left(\frac{1}{2}\lambda_3 - \frac{2}{3}\lambda_1^2\right)\varepsilon^2\mu^2u^2u_{xxxxx},$$

and $\lambda_2\nu\mu(F_{\varepsilon^2})_{xx} = -\lambda_2\nu\varepsilon^2\mu(\frac{1}{2}\lambda_1 + \frac{A_1}{6})(u^3)_{xxx}$.

Thus, we have

$$O_5(\varepsilon^2\mu) := \frac{1}{6}\varepsilon^2\mu\lambda_3(u^3)_{xxx} + \frac{3}{2}\varepsilon^2\mu\lambda_4(uu_x^2)_x + \frac{3}{2}\varepsilon^2\mu\lambda_5(u^2u_{xx})_x + \frac{3}{2}\lambda_2\lambda_1\varepsilon^2\mu(u^2u_{xx})_x$$
$$+ \frac{1}{2}\varepsilon^2\mu\lambda_2A_1(u^2u_{xx})_x + \varepsilon^2\mu\lambda_1A_3(uu_x^2)_x + \frac{1}{2}A_4\lambda_1\varepsilon^2\mu u^2u_{xxx} + \frac{1}{2}\varepsilon^2\mu\lambda_1A_4u(u^2)_{xxx}$$
$$+ \frac{1}{2}A_5\varepsilon^2\mu uu_xu_{xx} + \frac{1}{2}A_6\varepsilon^2\mu u^2u_{xxx} + \frac{1}{2}A_7\varepsilon^2\mu u_x^3 - \lambda_1\varepsilon^2\mu(\frac{6}{3}\lambda_1 + 3\lambda_2 + A_3)uu_xu_{xx}$$
$$- \lambda_1\varepsilon^2\mu\left(\frac{2}{3}\lambda_1 + 3\lambda_2 + A_4\right)u^2u_{xxx} - \left(\frac{1}{2}\lambda_3 - \frac{2}{3}\lambda_1^2\right)\varepsilon^2\mu u^2u_{xxx} - \lambda_2\nu\varepsilon^2\mu\left(\frac{1}{2}\lambda_1 + \frac{A_1}{6}\right)(u^3)_{xxx}$$
$$- 3\lambda_4\varepsilon^2\mu u_x(uu_x)_x - \frac{3}{2}\lambda_5\varepsilon^2\mu uu_xu_{xx} - \left(\frac{3}{2}\lambda_5 - 3\lambda_1\lambda_2\right)\varepsilon^2\mu u(uu_x)_{xx}.$$

More precisely, we have

$$I_{u^2u_{xxx}} := \frac{3}{2}(1-\nu)\lambda_1\lambda_2 + \frac{A_1}{2}(1-\nu)\lambda_2 + \frac{1}{2}A_4\lambda_1 + \frac{1}{2}A_6,$$
$$I_{uu_xu_{xx}} := 3\lambda_3 - 3\lambda_5 - 9(1-\nu)\lambda_2\lambda_1 + A_1(1-3\nu)\lambda_2 + (A_3+3A_4)\lambda_1 + \frac{1}{2}A_5 - 2\lambda_1^2,$$
$$I_{u_x^3} := \frac{1}{2}A_7 + \lambda_3 - \lambda_2\nu(3\lambda_1 + A_1) - \frac{3}{2}\lambda_4 + A_3\lambda_1.$$

In the asymptotic order, we have

$$O_5(\mu^2) := \frac{1}{2}O_0(\mu^2) + \lambda_2\nu\mu(F_\mu)_{xx},$$
$$O_5(\varepsilon\mu^2) := \frac{1}{2}O_0(\varepsilon\mu^2) - \lambda_1\varepsilon uO_0(\mu^2) + \lambda_2\nu\mu(F_{\varepsilon\mu})_{xx}$$
$$- \lambda_4\varepsilon\mu u_x(O_0(\mu))_x - \lambda_5\varepsilon\mu u_{xx}\frac{1}{2}O_0(\mu) - \left(\frac{1}{2}\lambda_5 - \lambda_1\lambda_2\right)\varepsilon\mu uO_0(\mu)_{xx},$$
$$O_5(\varepsilon^2\mu^2) := \frac{1}{2}O_0(\varepsilon^2\mu^2) - \lambda_1\varepsilon uO_0(\varepsilon\mu^2) - \left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\varepsilon^2u^2(O_0(\mu^2) + \lambda_2\nu\mu(F_{\varepsilon^2\mu})_{xx}$$
$$- \lambda_4\varepsilon\mu u_x(O_0(\varepsilon\mu) + 2\lambda_1u\varepsilon O_0(\mu))_x - \frac{\lambda_5}{2}\varepsilon\mu u_{xx}O_0(\varepsilon\mu) + \lambda_5\lambda_1uu_{xx}\varepsilon^2\mu O_0(\mu)$$
$$- \left(\frac{1}{2}\lambda_5 - \lambda_1\lambda_2\right)\varepsilon\mu u(O_0(\varepsilon\mu))_{xx} + (\lambda_5 - 2\lambda_1\lambda_2)\lambda_1\varepsilon^2\mu u(uO_0(\mu))_{xx}$$
$$+ \varepsilon^2\mu^2\lambda_1\left(\frac{1}{3}\lambda_4uu_xu_{xxxx} + \frac{1}{6}\lambda_5uu_{xx}u_{xxx} + \frac{1}{6}\lambda_5u^2u_{xxxxx}\right) - \lambda_2\left(\frac{1}{2}\lambda_3 - \frac{2}{3}\lambda_1^2\right)\varepsilon^2\mu^2u^2u_{xxxxx}$$
$$- \frac{1}{6}\lambda_7\varepsilon^2\mu^2u_x^2u_{xxx} - \frac{1}{3}\lambda_7\varepsilon^2\mu^2uu_xu_{xxxx} - \frac{1}{6}\lambda_8\varepsilon^2\mu^2u^2u_{xxxxx} - \frac{1}{3}\varepsilon^2\mu^2\lambda_8uu_{xx}u_{xxx}.$$

This procedure can be continued successively, and finally the coefficient of the terms at the order of $\varepsilon^2\mu^2$-order are obtained as

$$\begin{pmatrix} -6 & 0 & -4A_1 \\ 0 & 0 & -(30\lambda_1 + 10A_1) \\ 0 & -\frac{15}{2} & -(45\lambda_1 + 15A_1) \end{pmatrix}\begin{pmatrix} \lambda_{10} \\ \lambda_{11} \\ \lambda_6 \end{pmatrix} + \begin{pmatrix} C_2 \\ C_4 \\ C_5 \end{pmatrix} = \begin{pmatrix} I_{uu_xu_{xxxx}} - \lambda_8 \\ I_{u_x^2u_{xxx}} + 6\lambda_9 - \lambda_8 - \lambda_7 \\ I_{u_xu_{xx}^2} + \frac{9}{2}\lambda_9 - \lambda_8 - 2\lambda_7 \end{pmatrix}. \tag{3.7}$$

## The special form of the CH-mCH-Novikov equation

The equation (1.5) requires specific values of the parameters in the Kodama transformation. These can be determined through the following procedure.

Note that the CH-type equation requires

$$I_{u_{xxt}} = -\beta, \qquad I_{uu_{xxx}} = -\frac{\sigma\beta}{2}, \qquad I_{u_xu_{xx}} = -\sigma\beta,$$

for some parameter $\beta$. It is determined that $\beta = -\dfrac{5}{6(\sigma - 3)}$ and $\lambda_1, \lambda_2, \nu$ are given by

$$\begin{cases} \lambda_2(1-\nu) = -\beta, \\ \lambda_1 + \frac{3}{2}(1-3\nu)\lambda_2 = -\dfrac{\sigma\beta}{2} - \dfrac{A_3}{2}. \end{cases} \tag{3.8}$$

On the other hand, equation (1.5) requires that

$$I_{u^2u_x} = \frac{1}{4}(3k_1 + 4k_2), \quad I_{u^2u_{xxx}} = -\frac{1}{4}\beta(k_1 + k_2).$$

Therefore

$$\begin{cases} \frac{3}{2}\lambda_1 + \frac{A_1}{2} = \frac{1}{4}(3k_1 + 4k_2), \\ \frac{3}{2}(1-\nu)\lambda_1\lambda_2 + \frac{A_1}{2}(1-\nu)\lambda_2 + \frac{1}{2}A_4\lambda_1 + \frac{1}{2}A_6 = -\frac{1}{4}\beta(k_1+k_2), \end{cases} \tag{3.9}$$

where $A_1 = -3/4$, $A_3 = 23/12$, $A_4 = 5/6$,$A_6 = -3/4$. Combining this with (3.8) we have

$$\lambda_1 = \frac{-10k_1 - 72\sigma + 171}{60 - 80\sigma}, \quad \lambda_2 = \frac{-20k_1\sigma + 60k_1 - 164\sigma^2 + 649\sigma - 921}{-480\sigma^2 + 1800\sigma - 1080},$$

$$k_2 = -\frac{3\left(10k_1\sigma - 10k_1 - 13\sigma + 39\right)}{10(4\sigma - 3)}, \quad \nu = \frac{20k_1\sigma - 60k_1 + 164\sigma^2 - 1049\sigma + 1221}{20k_1\sigma - 60k_1 + 164\sigma^2 - 649\sigma + 921}, \tag{3.10}$$

where $k_1 \in \mathbb{R}$ is arbitrary. The coefficients of $(u^4)_x$ and $(u^5)_x$ must vanish for equation (1.5) to emerge, and hence

$$I_{u^3u_x} = \frac{3}{8}\lambda_3 + \frac{1}{8}A_2 + \frac{1}{4}\lambda_1 A_1 = 0,$$
$$I_{u^4u_x} = -19\lambda_0 - \frac{3}{4}\lambda_1^2 + \frac{49}{8}\lambda_1 + \frac{A_8}{5} - \frac{3}{2}\lambda_3 + 24\lambda_1^3 - 51\lambda_1\lambda_3 = 0,$$

where $A_2 = 3/8$ and $A_8 = 115/192$.

Then, we have

$$\lambda_3 = \frac{5k_1 + 26\sigma - 78}{20(4\sigma - 3)}, \tag{3.11}$$

and

$$\lambda_0 = \frac{1}{114000(4\sigma-3)^3}\Big(15k_1\left(73024\sigma^2 - 470364\sigma + 804501\right) + 225k_1^2(1028\sigma - 3579) + 18000k_1^3 + 2732464\sigma^3 - 18963756\sigma^2 + 55497258\sigma - 60937623\Big). \tag{3.12}$$

Also, for other terms, we require that

$$I_{u_x^3} = -\frac{1}{4}\beta k_1, \qquad I_{uu_xu_{xx}} = -\frac{1}{4}\beta(4k_1 + 3k_2).$$

With this choice, the implication is

$$\begin{cases} \lambda_3 - \frac{3}{2}\lambda_4 + \frac{1}{2}A_7 - \lambda_2\nu(3\lambda_1 + A_1) + A_3\lambda_1 = -\frac{1}{4}\beta k_1, \\ 3\lambda_3 - 3\lambda_5 - 9(1-\nu)\lambda_2\lambda_1 + A_1(1-3\nu)\lambda_2 + (A_3 + 3A_4)\lambda_1 + \frac{1}{2}A_5 - 2\lambda_1^2 = -\frac{1}{4}\beta(4k_1 + 3k_2), \end{cases}$$

where $A_5 = 11/16$, $A_7 = 21/16$. Then we obtain

$$\lambda_4 = \frac{-\frac{18(117-10k_1)^2}{(3-4\sigma)^2} + \frac{7839 - 670k_1}{3-4\sigma} - \frac{1000k_1}{\sigma - 3} + 14688}{10800}, \tag{3.13}$$

and

$$\lambda_5 = -\frac{-400k_1^2(\sigma-3) - 40k_1(\sigma+87)(4\sigma+3) + \sigma(8\sigma(5898\sigma - 46907) + 705093) - 214443}{2400(3-4\sigma)^2(\sigma-3)}. \tag{3.14}$$

This way $\nu, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$, and $k_2$ are obtained in terms of $k_1$.

Lastly, the coefficients of $\varepsilon^2\mu^2$-order terms should satisfy that:

$$I_{u^2u_{xxxxxx}} = I_{uu_xu_{xxxxx}} = I_{uu_{xx}u_{xxx}} = 0, \quad I_{u_x^2u_{xxx}} = \frac{k_1}{4}\beta^2, \quad I_{u_xu_{xx}^2} = k_1\frac{1}{2}\beta^2.$$

Since the coefficient of the term $u^2u_{xxxxxx}$ needs to be zero, it requires that

$$-\frac{A_4}{2}\lambda_1\beta - \frac{A_6}{2}\beta - \lambda_2\lambda_3 + \frac{4}{3}\lambda_1^2\lambda_2 - \frac{1}{6}\lambda_1\lambda_5 = 0,$$

where $\beta = -\dfrac{5}{6(\sigma-3)}$ and $\lambda_i (i = 1, 2, 3)$ only depend on $k_1$. This way the parameter $k_1$

should be a real root of the following equation

$$\begin{aligned} 800k_1^2\left(86\sigma^2 - 398\sigma + 645\right) + 30k_1\left(10768\sigma^3 - 62120\sigma^2 + 188109\sigma - 233883\right) \\ +4000k_1^3(\sigma - 3) + 3\left(90496\sigma^4 - 557360\sigma^3 + 2242656\sigma^2 - 8124759\sigma + 9843417\right) = 0. \end{aligned} \tag{3.15}$$

Notice that since the determinant of the matrix in (3.7) is nonzero, we can obtain $\lambda_6$, $\lambda_{10}$ and $\lambda_{11}$ for any parameters $\lambda_7$, $\lambda_8$ and $\lambda_9$.

In summary, if we take the Kodama transformation to be

$$\begin{aligned} \eta = u + \lambda_1\varepsilon u^2 + \lambda_2\mu u_{xx} + \varepsilon\mu(\lambda_4 u_x^2 + \lambda_5 u u_{xx}) + \varepsilon^2\lambda_3 u^3 + \varepsilon^3\lambda_0 u^4 + \mu^2(\lambda_6 u_{xxxx}) \\ + \varepsilon^2\mu(\lambda_7 u u_x^2 + \lambda_8 u^2 u_{xx}) + \varepsilon\mu^2(\lambda_9 u_x u_{xxx} + \lambda_{10} u u_{xxxx} + \lambda_{11} u_{xx}^2), \end{aligned}$$

where the parameters satisfy conditions (3.10)-(3.15) and $\lambda_7$, $\lambda_8$,and $\lambda_9$ can be any real numbers, then we arrive at:

$$\begin{aligned} m_t + (1 + \frac{3\varepsilon}{2}u)u_x - \frac{1}{4}\mu u_{xxx} - \varepsilon\mu\sigma\beta(u_x u_{xx} + \frac{1}{2}u u_{xxx}) + \frac{k_1\varepsilon^2}{4}((u^2 - \beta\mu u_x^2)m)_x \\ + \frac{k_2\varepsilon^2}{4}(u^2 m_x + 3u u_x m) = 0. \end{aligned} \tag{3.16}$$

# Chapter 4

## BLOW-UP ANALYSIS

### 4.1 Introduction

Breaking waves phenomena are commonly observed in the ocean and near the shore. They are essential for many reasons: they move sediment in shallow water, provide a source of turbulent energy for mixing the ocean's upper layers, and they improve the gas and particle matter exchange between the air and the sea [69].

Given a partial differential equation and initial data (in other words, a Cauchy problem), an essential and crucial question is proposed: is the equation well-posed? Although the issue above is essential and fundamental, it is not necessarily simple or easy to answer. In particular, the solution $u$ of the mCH-Novikov equation belongs to $C([0,T];H^s)$. This means that $u(.,x) \in C([0,T))$ and $u(t,.) \in H^s$, where $H^s$ is a Sobolov space whereas $T > 0$ denotes the lifespan or what is sometimes called the solution's maximal time of existence. Generally, this depends on the initial data and the space.

The maximal time of existence adds a new element to our problem. If the equation is well-posed and $T = \infty$, then we have a global well-posed equation. In turn, this means that the solution exists for any $t$. However, if the equation is well-posed and $T < \infty$ , We have a local well-posed equation, meaning that the solution exists as $t < T$. For the last case, we say the problem's solution $u$ develops a finite time blow-up.

This is the case because the solution cannot exist for all t-values. Therefore, another basic question in the theory of nonlinear partial differential equations is when and how the wave breaking phenomena can exist (solution form a singularity) and what its nature is. Depending on the problem, the blow-up phenomenon might manifest differently. For instance, it can occur when the problem's solution becomes unbounded as $t$ approaches $T$. In greater detail, a blow-up in finite time occurs if $T < \infty$ and

$$\limsup_{t \to T} \|u(t, )\| = \infty.$$

A blow-up may also occur in the following circumstances: assume that $T < \infty$, and we will obtain another type of blow-up if

$$\sup |u(t,x)| < \infty, \qquad \text{and} \qquad \limsup_{t\to T} (\sup_{x\in\mathbb{R}} |u_x(t,x)|) = \infty.$$

This type of blow-up is known as wave breaking, where $u_x$ becomes unbounded in finite time while $u$ stays bounded. From a geometrical standpoint, this indicates that, as $t$ approaches $T$, the tangent line to the curve $x \to (x, u(t,x))$ tends to become the perpendicular line to the $x$-direction [70]. Finally, higher nonlinearity can even cause curvature blow-up, which occurs when the second derivative of the solution becomes unbounded in finite time while the solution and its gradient remain bounded [71].

## 4.2 Blow-Up

Using the method from Chapter (3), many shallow-water models can be derived when we choose suitable parameters in the Kodama transformation and perform certain rescaling. In particular, we can obtain the mCH-Novikov equation.

Consider the same form of Kodama transformation as before; choose $\sigma = 1$. Then, applying scaling transformation

$$u \to \frac{\varepsilon}{2} u, \quad t \to (\beta\mu)^{\frac{1}{2}} t, \quad x \to (\beta\mu)^{\frac{1}{2}} x,$$

to equation (1.5) leads to

$$m_t + u_x - \frac{3}{5} u_{xxx} + 2u_x m + u m_x + k_1((u^2 - u_x^2)m)_x + k_2(u^2 m_x + 3u u_x m) = 0. \tag{4.1}$$

If we further scale $t \to \delta^2 t$ and $u \to \delta^{-1} u$, then (4.1) takes the form of

$$\delta^{-2} m_t + u_x - \frac{3}{5} u_{xxx} + \delta^{-1}(2u_x m + u m_x) + k_1 \delta^{-2}((u^2 - u_x^2)m)_x + k_2 \delta^{-2}(u^2 m_x + 3u u_x m) = 0.$$

Rewriting it as

$$m_t + \delta^2 u_x - \delta^2 \frac{3}{5} u_{xxx} + \delta(2u_x m + u m_x) + k_1((u^2 - u_x^2)m)_x + k_2(u^2 m_x + 3u u_x m) = 0, \tag{4.2}$$

and taking $\delta \to 0$, we procure the mCH-Novikov equation

$$m_t + k_1[(u^2 - u_x^2)m]_x + k_2(u^2 m_x + 3u u_x m) = 0.$$

In this chapter, we consider the periodic mCH-Novikov equation with cubic nonlinearity, which is derived with the asymptotic method from the classical shallow water theory. The approximate model equation is obtained by introducing suitable scaling and by truncating the asymptotic expansions of the unknowns to appropriate order along with the Kodama transformation. When the parameters take different values in mCH-Novikov equation, we get several different important shallow water equations, such as mCH equation and Novikov equation. Our analysis applies the method of characteristics and uses conserved quantities to arrive at a Riccati-type differential inequality. It is proven that the wave-breaking phenomenon of the mCH-Novikov equation is the curvature blow-up.

Having derived the model equations in Chapter (2) and Chapter (3), we now turn our attention to the blow-up analysis. In particular, as the introduction explains, we consider the Cauchy problem for the periodic mCH-Novikov equation

$$\begin{cases} m_t + k_1\left[(u^2 - u_x^2)m\right]_x + k_2\left(u^2 m_x + 3uu_x m\right) = 0, & t > 0,\ x \in \mathbb{S}, \\ u(t,0) = u(t,1), u(0,x) = u_0(x), \end{cases} \tag{4.3}$$

where $k_1, k_2 \in \mathbb{R}$.

**Lemma 4.2.1.** *Suppose that $u_0 \in H^s(\mathbb{S})$ with $s > \frac{5}{2}$. Assume $u$ is the corresponding solution to (4.3) with the initial data $u_0$. Then*

$$H_1[u] = \int_{\mathbb{S}} (u^2 + u_x^2) dx = \int_{\mathbb{S}} (u_0^2 + u_{0,x}^2) dx, \tag{4.4}$$

*and*

$$H_2[u] = \int_{\mathbb{S}} \left(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4\right) dx = \int_{\mathbb{S}} \left(u_0^4 + 2u_0^2u_{0,x}^2 - \frac{1}{3}u_{0,x}^4\right) dx. \tag{4.5}$$

*Proof.* We rewrite (4.3) as the following equation

$$u_t - u_{txx} + k_1((u^2 - u_x^2)(u - u_{xx}))_x + k_2(u^2(u_x - u_{xxx}) + 3uu_x(u - u_{xx})) = 0. \tag{4.6}$$

Multiplying equation (4.6) by $u$ and integrating by parts, we have

$$\int_{\mathbb{S}} uu_t dx = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} u^2 dx,$$
$$\int_{\mathbb{S}} uu_{txx} dx = -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} u_x^2 dx,$$

$$k_1 \int_{\mathbb{S}} u((u^2 - u_x^2)(u - u_{xx}))_x dx = -k_1 \int_{\mathbb{S}} (u^3 u_x - u^2 u_x u_{xx} - u u_x^3 + u_x^3 u_{xxx}) dx = 0,$$

$$k_2 \int_{\mathbb{S}} u(u^2(u_x - u_{xxx}) + 3uu_x(u - u_{xx}))) = k_2 \int_{\mathbb{S}} (4u^3 u_x - u^3 u_{xxx} - 3u^2 u_x u_{xx}) dx = 0.$$

Then we obtain

$$\frac{d}{dt} \int_{\mathbb{S}} (u^2 + u_x^2) dx = 0. \tag{4.7}$$

Similarly, we multiply equation (4.5) by $u_x$ and integrate by parts, then we get

$$\int_{\mathbb{S}} \left( u^4 + 2u^2 u_x^2 - \frac{1}{3} u_x^4 \right) = 0. \tag{4.8}$$

This completes the poof of lemmea (4.2.1). □

Lemma (4.2.1) shows that the following two functionals are conserved quantities for (4.3)

$$H_1[u] = \int_{\mathbb{R}} \left( u^2 + u_x^2 \right) dx, \quad \text{and} \qquad H_2[u] = \int_{\mathbb{S}} \left( u^4 + 2u^2 u_x^2 - \frac{1}{3} u_x^4 \right) dx. \tag{4.9}$$

The local well-posedness theory can be obtained following the standard argument of [72] with a slight modification.

**Theorem 4.2.2.** *Let $u_0 \in H^s$ with $s > \frac{5}{2}$. Then there exists a time $T > 0$ such that the Cauchy problem (4.3) has a unique strong solution $u \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$.*

### 4.2.1 Blow-Up Criterion

Similar to the other CH-type equations, (4.3) can be reformulated into a nonlocal transport form. Therefore, from standard transport theory, a Beale–Kato–Majda type of blow-up criterion is obtainable. A further refined analysis leads to the following lemma. The proof of this result follows an idea similar to one proposed in [18], so we therefore omit it.

**Lemma 4.2.3.** *Let $u_0 \in H^s$ with $s > \frac{5}{2}$ and $u$ be the corresponding solution to (4.3).*

*Assume that $T^*_{u_0} > 0$ is the maximum time of existence. Then*

$$T^*_{u_0} < \infty \;\Rightarrow\; \int_0^{T^*_{u_0}} \|k_1 m u_x(\tau) + 2k_2 u u_x(\tau)\|_{L^\infty}\, d\tau = \infty. \tag{4.10}$$

**Remark 3.** *The blow-up criterion (4.10) implies that the lifespan $T^*_{u_0}$ does not depend on the regularity index $s$ of the initial data $u_0$.*

Furthermore, we prove the following wave-breaking criteria.

**Lemma 4.2.4.** *Suppose that $u_0 \in H^s(\mathbb{S})$ with $s > \frac{5}{2}$. Then the corresponding solution $u$ to the Cauchy problem (4.3) blows up in finite time $T^* > 0$ if and only if*

$$\liminf_{t\to T^*} \inf_{x\in\mathbb{S}} \left\{k_1 m(t,x) u_x(t,x) + 2k_2 u(t,x) u_x(t,x)\right\} = -\infty. \tag{4.11}$$

*Proof.* In view of Remark (3), it suffices to consider the case $s = 3$. Suppose that if $k_1 m u_x + k_2 u u_x$ is bounded from below on $[0, T^*_{u_0}) \times \mathbb{S}$. In other words, there exists a constant $M > 0$ such that

$$\left(k_1 m u_x + 2k_2 u u_x\right)(t,x) \geq -M \quad \text{on} \quad [0, T^*_{u_0}) \times \mathbb{S}. \tag{4.12}$$

Multiplying (4.3) by $m$ and integrating over $\mathbb{S}$. Then integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} m^2\, dx + \int_{\mathbb{S}} \left(k_1 u_x m + 2k_2 u u_x\right) m^2\, dx = 0. \tag{4.13}$$

The initial condition implies that $m_0 \in H^{s-2} \subset L^q$ for any $2 \leq q \leq \infty$. Similarly, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} m_x^2\, dx + k_1 \int_{\mathbb{S}} \left((u^2 - u_x^2)m\right)_{xx} m_x\, dx + k_2 \int_{\mathbb{S}} \left(u^2 m_x + 3u u_x m\right)_x m_x\, dx = 0.$$

Integrating by parts, the second term yields

$$k_1\int_{\mathbb{S}}\left[(u^2-u_x^2)m\right]_{xx}m_x\ dx=\int_{\mathbb{S}}(5k_1u_xm)\,m_x^2\ dx-\int_{\mathbb{S}}\left(\frac{2}{3}k_1u_xm\right)m^2dx.$$

Integrating by parts, the third term can be computed as

$$k_2\int_{\mathbb{S}}\left(u^2m_x+3uu_xm\right)_x m_x=\int_{\mathbb{S}}(4k_2uu_x)\,m_x^2\ dx-\int_{\mathbb{S}}(6k_2uu_x)\,m^2dx-\int_{\mathbb{S}}12k_2um_xm^2dx.$$

In this way, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}}m_x^2\ dx+\int_{\mathbb{S}}(5k_1u_xm+4k_2uu_x)\,m_x^2\ dx-\int_{\mathbb{S}}\left(\frac{2}{3}k_1u_xm+6k_2uu_x\right)m^2dx-\int_{\mathbb{S}}12k_2um_xm^2dx=0.$$

So together with (4.13), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}}(m^2+m_x^2)\ dx=&-\int_{\mathbb{S}}(k_1u_xm+2k_2uu_x)\,m^2\ dx-\int_{\mathbb{S}}(5k_1u_xm+4k_2uu_x)\,m_x^2\ dx\\
&+\int_{\mathbb{S}}\left(\frac{2}{3}k_1u_xm+6k_2uu_x\right)m^2dx+\int_{\mathbb{S}}12k_2um_xm^2dx\\
=&\int_{\mathbb{S}}\left(4k_2uu_x-\frac{1}{3}k_1u_xm\right)m^2\ dx-\int_{\mathbb{S}}(5k_1u_xm+4k_2uu_x)\,m_x^2\ dx\\
&-\int_{\mathbb{S}}4k_2u_xm^3dx\\
\le&\ 5\int_{\mathbb{S}}M(m^2+m_x^2)\ dx+\int_{\mathbb{S}}\ k_2uu_x\left(\frac{14}{3}m^2+6m_x^2\right)\ dx-\int_{\mathbb{S}}4k_2u_xm^3dx\\
\le&\ 5\int_{\mathbb{S}}M(m^2+m_x^2)\ dx+\int_{\mathbb{S}}\ k_2uu_x\left(\frac{14}{3}m^2+6m_x^2\right)\ dx\\
&+\int_{\mathbb{S}}4k_2(u^2-u_x^2)mm_xdx\\
\le&\ 5\int_{\mathbb{R}}M(m^2+m_x^2)\ dx+6|k_2|\|u\|_{L^\infty}\|u_x\|_{L^\infty}\|m\|_{H^1}^2+4|k_2|\|u\|_{H^1}\|m\|_{L^2}\|m\|_{H^1}^2.
\end{aligned}$$

Therefore,

$$\frac{d}{dt}\|m\|_{H^1}^2\le C\|m\|_{H^1}^2,$$

where $C = C(u_0, m_0)$. Applying Gronwall's inequality, it follows that

$$\|m(t)\|_{H^1}^2 \leq e^{Ct}\|m_0\|_{H^1}^2,$$

for $t \in [0, T_{u_0}^*)$. From Lemma (4.2.3) this ensures that the solution does not blow up in finite time.

On the other hand, if

$$\liminf_{t\uparrow T_{u_0}^*} \left[\inf_{x\in\mathbb{S}} \left(k_1 m(t,x)u_x(t,x) + 2k_2 u(t,x)u_x(t,x)\right)\right] = -\infty,$$

then either $u_x$ or $m$ blows up in finite time. The proof of Lemma (4.2.4) is hence completed.

□

### 4.2.2 Dynamics along the Characteristics

We perform our blow-up analysis along the characteristics of equation (4.3). So, let us define the characteristics associated with the mCH-Novikov equation (4.3) as

$$\begin{cases} q_t(t,x) = \left[k_1\left(u^2 - u_x^2\right) + k_2 u^2\right](t, q(t,x)), \\ q(0,x) = x, \end{cases} \qquad x \in \mathbb{S}, \quad t \in [0,T). \tag{4.14}$$

**Proposition 1.** *Suppose $u_0 \in H^s(\mathbb{S})$ with $s > \frac{5}{2}$, and let $T > 0$ be the maximal existence time of the strong solution $u$ to the corresponding initial value problem (4.3). Then (4.14) has a unique solution $q \in C^1([0,T)\times\mathbb{S},\mathbb{S})$ such that $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{S}$ with*

$$q_x(t,x) = \exp\left(2\int_0^t \left(k_1 m u_x + k_2 u u_x\right)(s, q(s,x))\, ds\right) > 0, \quad \forall\ (t,x) \in [0,T)\times\mathbb{S}. \tag{4.15}$$

*Moreover, for all* $(t,x) \in [0,T) \times \mathbb{S}$ *it holds that*

$$m(t,q(t,x)) = m_0(x) \exp\left( - \int_0^t (2k_1 m u_x + 3k_2 u u_x)(s,q(s,x))\, ds \right), \tag{4.16}$$

*where* $m_0(x) = m(0,x)$.

A direct consequence of Proposition (1) is that the momentum density satisfies the sign-persistence property.

**Corollary 4.2.4.1.** *Suppose* $u_0 \in H^s(\mathbb{S})$ *with* $s > \frac{5}{2}$. *Let* $T > 0$ *be the maximal existence time of the strong solution* $u$ *to the corresponding initial value problem* (4.3). *If* $m_0(x) > 0$ *for all* $x \in \mathbb{S}$, *then* $m(t,x) > 0$ *for all* $(t,x) \in [0,T) \times \mathbb{S}$.

The following lemma play important roles in the blow-up phenomena. The proof follows an idea similar to [73].

**Lemma 4.2.5.** *Assume* $m_0 \in H^s(\mathbb{S})$ *with* $s > \frac{5}{2}, m_0 \geq 0$ *for all* $x \in \mathbb{S}$. *Let* $T > 0$ *be the maximal existence time of the solution* $m(t,x)$ *to the periodic problem* (4.3) *with the initial data* $m_0$. *Then*

$$|u_x(t,x)| \leq u(t,x).$$

Denote $p(x) = \dfrac{\cosh((x - [x] - \frac{1}{2})}{2\sinh(\frac{1}{2})}$, here $[x]$ represents the largest integer part of $x$, wich is the fundamental solution of $1 - \partial_x^2$ on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$, that is $(1-\partial_x^2)^{-1} f = p * f$, where $*$ denotes the convolution product on $\mathbb{S}$, defined by:

$$\begin{aligned} p * f(t,x) &= \int_0^1 p(x-y) f(t,y) dy \\ &= \int_0^1 \frac{\cosh((x-y) - [x-y] - \frac{1}{2})}{2\sinh(\frac{1}{2})} f(t,y) dy \\ &= \int_0^x \frac{\cosh(x-y-\frac{1}{2})}{2\sinh(\frac{1}{2})} f(t,y) dy + \int_x^1 \frac{\cosh((x-y+\frac{1}{2})}{2\sinh(\frac{1}{2})} f(t,y) dy. \end{aligned}$$

Define $p_+(x) = \dfrac{e^{x-[x]-\frac{1}{2}}}{4\sinh(\frac{1}{2})}$, $\qquad p_-(x) = \dfrac{e^{-x+[x]+\frac{1}{2}}}{4\sinh(\frac{1}{2})}$.

Then we have the relation

$$p = p_+ + p_-, \qquad p_x = p_- - p_+.$$

Now we compute the dynamics of a few important quantities along the characteristics $q(t, x_0)$. Denote $'$ the derivative $\partial_t + (k_1(u^2 - u_x^2) + k_2u^2)\partial_x$ along the characteristics, and

$$\widehat{u}(t) := u(t, q(t, x_0)), \quad \widehat{u_x}(t) := u_x(t, q(t, x_0)), \quad \widehat{m}(t) := m(t, q(t, x_0)), \quad \widehat{M}(t) := (mu_x)(t, q(t, x_0)).$$

**Lemma 4.2.6.** *Let $u_0 \in H^s(\mathbb{S})$, $s \geq 3$. Then $u(t, x)$, $u_x(t, x)$, $m(t, x)$ and $(mu_x)(t, x)$ satisfy the following integro-differential equations*

$$\widehat{u}'(t) = -\frac{2}{3}k_1\widehat{u_x}^3 + \left(\frac{k_1}{3} + \frac{k_2}{2}\right)\left[p_+ * (u - u_x)^3 - p_- * (u + u_x)^3\right](t, q(t, x_0)), \tag{4.17}$$

$$\begin{aligned}\widehat{u_x}'(t) = {} & k_1\left(\frac{1}{3}\widehat{u}^3 - \widehat{u}\widehat{u_x}^2\right) + \frac{k_2\widehat{u}}{2}(\widehat{u}^2 - \widehat{u_x}^2) \\ & - \left(\frac{k_1}{3} + \frac{k_2}{2}\right)\left[p_+ * (u - u_x)^3 + p_- * (u + u_x)^3\right](t, q(t, x_0)),\end{aligned} \tag{4.18}$$

$$\widehat{m}'(t) = -(2k_1\widehat{m}\widehat{u_x} + 3k_2\widehat{u}\widehat{u_x})\widehat{m}, \tag{4.19}$$

$$\begin{aligned}\widehat{M}'(t) = {} & -2k_1\widehat{M}^2 + \frac{\widehat{m}\widehat{u}}{6}\left[(2k_1 + 3k_2)\widehat{u}^2 - (6k_1 + 21k_2)\widehat{u_x}^2\right] \\ & - \left(\frac{k_1}{3} + \frac{k_2}{2}\right)\widehat{m}\left[p_+ * (u - u_x)^3 + p_- * (u + u_x)^3\right](t, q(t, x_0)).\end{aligned} \tag{4.20}$$

*Proof.* The proof of (4.19) is immediately obtaineble from the equation (4.3).

$$u_t = -k_1 p * \left[(u^2 - u_x^2)m\right]_x + k_2 p * \left(u^2 m_x + 3uu_x m\right). \tag{4.21}$$

The structure of the right-hand side of the above equation suggests that we may recall the results from [17] and [9]. First, from [17, (3.1)] we know that

$$p * \left[(u^2 - u_x^2)m\right]_x = (u^2 - u_x^2)u_x + \frac{2}{3}u_x^3 - \frac{1}{3}\left[p_+ * (u - u_x)^3 - p_- * (u + u_x)^3\right].$$

From [9, (3.7)] we have

$$p * \left(u^2 m_x + 3uu_x m\right) = u^2 u_x - \frac{1}{2}\left[p_+ * (u - u_x)^3 - p_- * (u + u_x)^3\right].$$

Plugging the above two into (4.21), we obtain (4.17).

The proof of (4.18) proceeds in the same way. Differentiating (4.21) we obtain

$$u_{xt} = -k_1 p * \left[(u^2 - u_x^2)m\right]_{xx} - k_2 p * \left(u^2 m_x + 3uu_x m\right)_x. \tag{4.22}$$

From [17, (3.2)], it follows that

$$p * \left[(u^2 - u_x^2)m\right]_{xx} = (u^2 - u_x^2)u_{xx} + \left(\frac{1}{3}u^3 - uu_x^2\right) - \frac{1}{3}\left[p_+ * (u - u_x)^3 + p_- * (u + u_x)^3\right].$$

From [9, (3.8)], we know

$$p * \left(u^2 m_x + 3uu_x m\right)_x = u^2 u_{xx} - \frac{u}{2}(u^2 - u_x^2) - \frac{1}{2}\left[p_+ * (u - u_x)^3 + p_- * (u + u_x)^3\right].$$

Therefore (4.18) is obtained by combining the above two equations.

Finally (4.20) can be derived from (4.18) and (4.19). □

### 4.2.3 Choice of Data and Blow-Up: $2k_1 + 3k_2 \neq 0$

The blow-up criterion (4.11) together with the conservation law $H_1[u]$ indicates two possible scenarios for the formation of singularity, namely the wave-breaking ($|u_x| \to \infty$) or curvature blow-up ($|m| \to \infty$). In this section, we seek data which leads to the second scenario.

### Non-Sign-Changing Data

For this, we first utilize the sign-persistence property Corollary (4.2.4.1) to consider data with positive momentum $m_0 \geq 0$, so that from the identities

$$u(t,x) = p * m(t,x), \qquad u_x(t,x) = p_x * m(t,x),$$

we have

$$u(t,x) \geq 0, \qquad u \pm u_x = 2p_\mp * m \geq 0. \tag{4.23}$$

This allows us to control the convolution terms in Lemma (4.2.6).

The main result of this subsection is the following.

**Theorem 4.2.7.** *Suppose that $k_1 < 0$, and $2k_1/3 < k_2 \leq -2k_1/9$. Let $m_0 \in H^s(\mathbb{S})$ for $s > 5/2$ and $m_0 \geq 0$. Assume that there exists some point $x_0 \in \mathbb{S}$ such that $m_0(x_0) > 0$ and*

$$u_{0,x}(x_0) \geq \max\left\{\sqrt{\frac{2k_1 + 3k_2}{4k_1}}, \sqrt{\frac{2k_1 + 3k_2}{6k_1 + 21k_2}}\right\} u_0(x_0). \tag{4.24}$$

*Then the corresponding solution $u(t,x)$ blows up in finite time with an estimate of the blow-up time $T^*$ as*

$$T^* \leq -\frac{1}{2k_1 m_0(x_0) u_{0,x}(x_0)}.$$

*Proof.* From Corollary (4.2.4.1) we know that $m(t,x) \geq 0$ and $\widehat{m} > 0$. It then follows from (4.2.5) that

$$u(t,x) \geq |u_x(t,x)| \geq 0, \qquad \widehat{u}(t) > 0. \tag{4.25}$$

Therefore $u_x$ does not blow up, and then Lemma (4.2.4) indicates that it suffices to consider the quantity $M(t,x) = (mu_x)(t,x)$.

From the condition of the theorem, (4.25), and (4.20) it holds that

$$\begin{aligned}\widehat{M}' &= -2k_1\widehat{M}^2 + \frac{\widehat{m}\widehat{u}}{6}\left[(2k_1+3k_2)\widehat{u}^2 - (6k_1+21k_2)\widehat{u_x}^2\right] \\ &\quad - \left(\frac{k_1}{3}+\frac{k_2}{2}\right)\hat{m}\left[p_+ * (u-u_x)^3 + p_- * (u+u_x)^3\right](t,q(t,x_0)) \\ &\geq -2k_1\widehat{M}^2 + \frac{\widehat{m}\hat{u}}{6}\left[(2k_1+3k_2) - (6k_1+21k_2)\frac{\widehat{u_x}^2}{\widehat{u}^2}\right].\end{aligned} \tag{4.26}$$

Since $\widehat{u}, \widehat{m} > 0$, it is now clear that in order to arrive at a Riccati-type inequality $\widehat{M}' \gtrsim \widehat{M}^2$, one would like to have $(2k_1+3k_2) - (6k_1+21k_2)\widehat{u_x}^2/\widehat{u}^2 \geq 0$, that is,

$$\frac{\widehat{u_x}^2}{\widehat{u}^2} \geq \frac{2k_1+3k_2}{6k_1+21k_2}, \tag{4.27}$$

which involves the competition between $u$ and its derivative $u_x$ along the characteristics. In particular, a finite-time blow-up of $\widehat{M}$ can be realized if the ration $|u_x/u|$ stays reasonable big along the characteristics. A quick computation shows that

$$\begin{aligned}\left(\frac{\widehat{u_x}}{\widehat{u}}\right)' &= \frac{\widehat{u}^2 - \widehat{u_x}^2}{\widehat{u}^2}\left[\left(\frac{k_1}{3}+\frac{k_2}{2}\right)\widehat{u}^2 - \frac{2k_1}{3}\widehat{u_x}^2\right] \\ &\quad - \frac{2k_1+3k_2}{6\widehat{u}^2}\left[(\widehat{u}+\widehat{u_x})p_+ * (u-u_x)^3 + (\widehat{u}-\widehat{u_x})p_- * (u+u_x)^3\right] \\ &\geq \widehat{u}^2\left[\left(\frac{k_1}{3}+\frac{k_2}{2}\right) - \left(k_1+\frac{k_2}{2}\right)\left(\frac{\widehat{u_x}}{\widehat{u}}\right)^2 + \frac{2k_1}{3}\left(\frac{\widehat{u_x}}{\widehat{u}}\right)^4\right] \\ &= \frac{2k_1}{3}\widehat{u}^2\left[\left(\frac{\widehat{u_x}}{\widehat{u}}\right)^2 - 1\right]\left[\left(\frac{\widehat{u_x}}{\widehat{u}}\right)^2 - \frac{2k_1+3k_2}{4k_1}\right].\end{aligned} \tag{4.28}$$

From (4.24), we have chosen the initial data so that

$$\left(\frac{\widehat{u_x}}{\widehat{u}}\right)(0) \geq \max\left\{\sqrt{\frac{2k_1+3k_2}{4k_1}}, \sqrt{\frac{2k_1+3k_2}{6k_1+21k_2}}\right\}.$$

Recall from (4.25) that $|\frac{\widehat{u_x}}{\widehat{u}}| \leq 1$. The assumptions on $k_1$ and $k_2$ ensure that the right-hand side of the above is less than 1. Therefore, $\frac{\widehat{u_x}}{\widehat{u}}$ increases initially, and a continuity argument implies that it increase for later time, and hence

$$\left(\frac{\widehat{u_x}}{\widehat{u}}\right)(t) \geq \left(\frac{\widehat{u_x}}{\widehat{u}}\right)(0) \geq \max\left\{\sqrt{\frac{2k_1+3k_2}{4k_1}}, \sqrt{\frac{2k_1+3k_2}{6k_1+21k_2}}\right\}.$$

In particular, we have:

$$\frac{\widehat{u_x}^2}{\widehat{u}^2} \geq \frac{2k_1+3k_2}{6k_1+21k_2}. \tag{4.29}$$

Plugging this into (4.26) it yields that $\widehat{M}'(t) \geq -2k_1\widehat{M}^2$, and thus $\widehat{M}(t)$ blows up in finite time with an estimate of the blow-up time $T^*$ as

$$T^* \leq -\frac{1}{2k_1\widehat{M}(0)} = -\frac{1}{2k_1 m_0(x_0)u_{0,x}(x_0)},$$

completing the proof of the theorem. □

**Remark 4.** *Note that in the conditions of Theorem (4.2.7) we require that* $\frac{2k_1}{3} < k_2 \leq -\frac{2k_1}{9}$ *The second inequality is needed in (4.29). The first inequality is also required since from the sign condition on* $m$ *we know that* $|u_x| \leq u$*, and therefore in (4.24) we need* $\frac{2k_1+3k_2}{4k_1} < 1$.

**Remark 5.** *Using a similar argument one can prove the finite time blow-up for data such that* $m_0 \leq 0$*,* $m_0(x_0) < 0$ *and*

$$u_{0,x}(x_0) \leq \max\left\{\sqrt{\frac{2k_1+3k_2}{4k_1}}, \sqrt{\frac{2k_1+3k_2}{6k_1+21k_2}}\right\} u_0(x_0).$$

Recall from Lemma (4.2.4) that when $m$ does not change sign, the true blow-up quantity is $k_1 m u_x$. In the setting of Theorem (4.2.7) and Remark (5) where $k_1 < 0$, we seek data that leads to $mu_x \to +\infty$. Thus using a similar argument we can handle the case when $k_1 > 0$, as indicated in the following corollary.

**Corollary 4.2.7.1.** *Suppose that $k_1 > 0$, and $-2k_1/9 \le k_2 < 2k_1/3$. Let $m_0 \in H^s(\mathbb{S})$ for $s > 5/2$ and $m_0 \ge 0$. Assume that there exists some point $x_0 \in \mathbb{S}$ such that $m_0(x_0) > 0$ and*

$$u_{0,x}(x_0) \le -\max\left\{\sqrt{\frac{2k_1+3k_2}{4k_1}}, \sqrt{\frac{2k_1+3k_2}{6k_1+21k_2}}\right\} u_0(x_0). \tag{4.30}$$

*Then the corresponding solution $u(t,x)$ blows up in finite time with an estimate of the blow-up time $T^*$ as*

$$T^* \le -\frac{1}{2k_1 m_0(x_0) u_{0,x}(x_0)}.$$

*Proof.* We still consider the dynamics of $\widehat{M}$ and look to have $\widehat{M} \to -\infty$ in finite time.

$$\begin{aligned}\widehat{M}' &= -2k_1\widehat{M}^2 + \frac{\widehat{m}\widehat{u}}{6}\left[(2k_1+3k_2)\widehat{u}^2 - (6k_1+21k_2)\widehat{u_x}^2\right] \\ &\quad -\left(\frac{k_1}{3}+\frac{k_2}{2}\right)\widehat{m}\left[p_+ * (u-u_x)^3 + p_- * (u+u_x)^3\right](t, q(t,x_0)) \\ &\le -2k_1\widehat{M}^2 + \frac{\widehat{m}\widehat{u}}{6}\left[(2k_1+3k_2)\widehat{u}^2 - (6k_1+21k_2)\widehat{u_x}^2\right].\end{aligned} \tag{4.31}$$

Now the goal is to have $(2k_1+3k_2)\widehat{u}^2 - (6k_1+21k_2)\widehat{u_x}^2 \le 0$, that is,

$$\frac{\widehat{u_x}^2}{\widehat{u}^2} \ge \frac{2k_1+3k_2}{6k_1+21k_2}, \tag{4.32}$$

and this again leads to considering $\widehat{u_x}/\widehat{u}$. From (4.28) we have

$$\begin{aligned}\left(\frac{\widehat{u_x}}{\widehat{u}}\right)' &= \frac{\widehat{u}^2 - \widehat{u_x}^2}{\widehat{u}^2}\left[\left(\frac{k_1}{3}+\frac{k_2}{2}\right)\widehat{u}^2 - \frac{2k_1}{3}\widehat{u_x}^2\right] \\ &\quad - \frac{2k_1+3k_2}{6\widehat{u}^2}\left[(\widehat{u}+\widehat{u_x})p_+ * (u-u_x)^3 + (\widehat{u}-\widehat{u_x})p_- * (u+u_x)^3\right] \\ &\le \frac{\widehat{u}^2 - \widehat{u_x}^2}{\widehat{u}^2}\left[\left(\frac{k_1}{3}+\frac{k_2}{2}\right)\widehat{u}^2 - \frac{2k_1}{3}\widehat{u_x}^2\right].\end{aligned} \tag{4.33}$$

Therefore we know that when (4.30) is satisfied, $\widehat{u_x}/\widehat{u}$ decreases, and thus

$$\frac{\widehat{u_x}^2}{\widehat{u}^2} \geq \max\left\{\frac{2k_1 + 3k_2}{4k_1}, \frac{2k_1 + 3k_2}{6k_1 + 21k_2}\right\}.$$

This way we obtain the desired Riccati inequality for $\widehat{M}$

$$\widehat{M}'(t) \leq -2k_1 \widehat{M}^2,$$

which implies that $\widehat{M}(t) \to -\infty$ as $t \to T^*$ where $T^* \leq -\dfrac{1}{2k_1 m_0(x_0) u_{0,x}(x_0)}$. □

**General Data**

Next we consider a general momentum density $m_0$ and look for the blow-up data. In this case we follow the standard procedure of utilizing the conservation laws $H_1[u]$. This will be the key to obtain the convolution estimate.

$$\frac{1}{3}\|u_x\|_{L^4}^4 = \int_{\mathbb{S}} \left(u^4 + 2u^2 u_x^2\right) dx - H_2[u_0] \leq 2\|u\|_{L^\infty}^2 H_1[u_0] - H_2[u_0] \leq \frac{e+1}{e-1} H_1^2[u_0] - H_2[u_0].$$

Therefore, the convolution estimates follow as

$$\begin{aligned} \left|p_\pm * (u \mp u_x)^3\right| &\leq \|p_\pm\|_{L^\infty} \left\|(u \mp u_x)^3\right\|_{L^1} \leq \frac{e^{\frac{1}{2}}}{2\sinh\frac{1}{2}} \left(\|u\|_{L^3}^3 + \|u_x\|_{L^3}^3\right) \\ &\leq \frac{e}{(e-1)} \left(\left(\frac{e+1}{2(e-1)}\right)^{\frac{1}{2}} H_1^{\frac{3}{2}}[u_0] + \left(\frac{3(e+1)}{e-1} H_1^2[u_0] - H_2[u_0]\right)^{\frac{1}{2}} H_1[u_0]\right) =: A. \end{aligned} \tag{4.34}$$

The blow-up result in this section is the following.

**Theorem 4.2.8.** *Suppose $k_1,\ k_2 < 0$. Let $m_0 \in H^s(\mathbb{S})$ with $s > 5/2$. Assume that there* *exists an* $x_0 \in \mathbb{S}$ *such that*

$$\begin{aligned} &m_0(x_0) > 0, \quad u_0(x_0) > 0, \quad u_{0,x}(x_0) \geq \sqrt[3]{A_1}, \\ &u_0(x_0)u_{0,x}^2(x_0) \geq \frac{2k_1+3k_2}{3(2k_1+k_2)}A_2, \end{aligned} \tag{4.35}$$

*where*

$$A_1 = \frac{2k_1+3k_2}{2k_1}A, \qquad A_2 = 2A + \left(\frac{e+1}{2(e-1)}H_1[u_0]\right)^{3/2},$$

*and* $A$ *is given in* (4.34). *Then the solution* $u(t,x)$ *blows up in finite time with an estimate of the blow-up time* $T^*$ *as*

$$T^* \leq -\frac{1}{2k_1 m_0(x_0)u_{0,x}(x_0)}.$$

*Proof.* Plugging (4.34) in (4.17) and (4.18) we obtain that

$$\begin{aligned} &\widehat{u}' \geq -\frac{2}{3}k_1\widehat{u_x}^3 + \frac{2k_1+3k_2}{3}A, \\ &\widehat{u_x}' \geq -\left(k_1+\frac{k_2}{2}\right)\widehat{u}\widehat{u_x}^2 + \frac{2k_1+3k_2}{6}\left[2A + \left(\frac{e+1}{2(e-1)}H_1[u_0]\right)^{3/2}\right]. \end{aligned}$$

Hence we know that $\widehat{u}$ is increasing when $\widehat{u_x}^3 \geq A_1$, and $\widehat{u_x}$ is increasing when

$$-\left(k_1+\frac{k_2}{2}\right)\widehat{u}\widehat{u_x}^2 \geq -\frac{2k_1+3k_2}{6}A_2.$$

From the assumption (4.35) we know that the above two conditions are satisfied initially. Hence a continuity argument yields that over the time of existence of solutions, $\widehat{u}$ and $\widehat{u_x}$ are both increasing. In particular,

$$\widehat{u}(t) \geq u_0(x_0) > 0, \qquad \widehat{u_x}(t) \geq u_{0,x}(x_0) > 0. \tag{4.36}$$

Recall that $\widehat{m}$ satisfies $\widehat{m}' = \widehat{u_x}(-2k_1\widehat{m}^2 - 3k_2\widehat{m}\widehat{u})$. From (4.35) and (4.36) we see that $\widehat{m}'$ increases initially. Then a continuity argument ensures that $\widehat{m}$ increases (and hence is positive) over the time of existence. Therefore

$$\widehat{m}' = \widehat{u_x}(-2k_1\widehat{m}^2 - 3k_2\widehat{m}\widehat{u}) \geq -2k_1\widehat{u_x}\widehat{m}^2 \geq -2k_1 u_{0,x}(x_0)\widehat{m}^2.$$

Hence $\widehat{m}(t)$ (and thus $\widehat{M}$ since $\widehat{u_x}(t) \geq u_{0,x}(x_0) > 0$) blows up to $+\infty$ in finite time with an estimate on the blow-up time $T^*$ as

$$T^* \leq -\frac{1}{2k_1 m_0(x_0) u_{0,x}(x_0)},$$

which completes the proof of the theorem. □

Similarly for positive $k_1$ and $k_2$ we have

**Corollary 4.2.8.1.** *Suppose $k_1,\ k_2 > 0$. Let $m_0 \in H^s(\mathbb{S})$ with $s > 5/2$. Assume that there exists an $x_0 \in \mathbb{S}$ such that*

$$\begin{aligned} &m_0(x_0) > 0, \quad u_0(x_0) > 0, \quad u_{0,x}(x_0) \leq -\sqrt[3]{A_1}, \\ &u_0(x_0)u_{0,x}^2(x_0) \geq \frac{2k_1 + 3k_2}{3(2k_1 + k_2)} A_2. \end{aligned} \tag{4.37}$$

*Then the solution $u(t,x)$ blows up in finite time with an estimate of the blow-up time $T^*$ as*

$$T^* \leq -\frac{1}{2k_1 m_0(x_0) u_{0,x}(x_0)}.$$

### 4.2.4 Choice of Data and Blow-Up: $2k_1 + 3k_2 = 0$

In the previous section, we require that $2k_1 + 3k_2 \neq 0$. In fact when $2k_1 + 3k_2 = 0$, the dynamics in Lemma (4.2.6) can be simplified as

$$\begin{aligned}
\widehat{u}' &= -\frac{2}{3}k_1 \widehat{u_x}^3, \\
\widehat{u_x}' &= -\left(k_1 + \frac{k_2}{2}\right)\widehat{u}\widehat{u_x}^2 = -\frac{2}{3}k_1 \widehat{u}\widehat{u_x}^2, \\
\widehat{m}' &= -(2k_1 \widehat{m}\widehat{u_x} + 3k_2 \widehat{u}\widehat{u_x})\widehat{m} = -2k_1 \widehat{m}\widehat{u_x}(\widehat{m} - \widehat{u}), \\
\widehat{M}' &= -2k_1 \widehat{M}^2 + \frac{4}{3}k_1 \widehat{u}\widehat{u_x}\widehat{M} = -2k_1 \widehat{u_x}\widehat{M}\left(\widehat{m} - \frac{2}{3}\widehat{u}\right).
\end{aligned} \tag{4.38}$$

In particular, the convolution terms all vanish and the dynamics is completely local. However, the dynamics of $\widehat{M}$ does not immediately lead to a Riccati type inequality. Instead, it involves the competition between $\widehat{u}$ and $\widehat{m}$.

#### The case when $k_1 < 0$

Note from (4.38) that when $k_1 < 0$,

$$\operatorname{sign}(\widehat{u}') = \operatorname{sign}(\widehat{u_x}), \quad \operatorname{sign}(\widehat{u_x}') = \operatorname{sign}(\widehat{u}). \tag{4.39}$$

Using this we first derive the following theorem which requires $m$ to be non-sign-changing.

**Theorem 4.2.9.** *Suppose that $k_1 < 0$, $2k_1 + 3k_2 = 0$. Let $m_0 \in H^s(\mathbb{S})$ for $s > 5/2$. Assume that*

*(a) $m_0 \geq 0$ and there exists some point $x_0 \in \mathbb{S}$ such that*

$$m_0(x_0) > 0, \quad u_{0,x}(x_0) > 0, \quad m_0(x_0) \geq \frac{4}{3}u_0(x_0), \quad \text{or} \tag{4.40}$$

*(b)* $m_0 \le 0$ *and there exists some point* $x_0 \in \mathbb{S}$ *such that*

$$m_0(x_0) < 0, \quad u_{0,x}(x_0) < 0, \quad m_0(x_0) \ge \frac{4}{3} u_0(x_0). \tag{4.41}$$

*Then the corresponding solution* $u(t,x)$ *blows up in finite time with an estimate of the blow-up time* $T^*$ *as*

$$T^* \le -\frac{1}{k_1 m_0(x_0) u_{0,x}(x_0)}.$$

*Proof.* Because $k_1 < 0$, the goal is to show that $\widehat{M} \to +\infty$ in finite time.

(a) Since now $m \ge 0$, $\widehat{m} > 0$ and $k_1 < 0$, we know from (4.39) that $\widehat{u} > 0$ and hence $\widehat{u_x}' > 0$. So $\widehat{u_x}(t) > 0$ if $\widehat{u_x}(0) > 0$. Then the last equation in (4.38) suggests that in order to derive a Riccati type inequality for $\widehat{M}$, one would like to have $\widehat{m} - \frac{2}{3}\widehat{u} \ge \varepsilon \widehat{m}$, for some $\varepsilon > 0$, that is,

$$\frac{\widehat{m}}{\widehat{u}} \ge \frac{2}{3(1-\varepsilon)}. \tag{4.42}$$

Now we can check the dynamics of $\widehat{m}/\widehat{u}$.

$$\left(\frac{\widehat{m}}{\widehat{u}}\right)' = -\frac{2k_1 \widehat{m}\widehat{u_x}}{\widehat{u}^2}\left(\widehat{m}\widehat{u} - \widehat{u}^2 - \frac{1}{3}\widehat{u_x}^2\right) \ge -\frac{2k_1 \widehat{m}\widehat{u_x}}{\widehat{u}^2}\left(\widehat{m}\widehat{u} - \frac{4}{3}\widehat{u}^2\right), \tag{4.43}$$

where we have used $|u_x| \le u$ to obtain the last inequality.

Therefore $\widehat{m}/\widehat{u}$ increases when $\widehat{m} \ge \frac{4}{3}\widehat{u}$. So when $\widehat{m}(0) \ge \frac{4}{3}\widehat{u}(0)$ we have

$$\frac{\widehat{m}}{\widehat{u}}(t) \ge \frac{\widehat{m}}{\widehat{u}}(0) \ge \frac{4}{3},$$

indicating that we may take $\varepsilon = \frac{1}{2}$ in (4.42). Thus from the last equation in (4.38) we have

$$\widehat{M}' \ge -k_1 \widehat{M}^2,$$

leading to $\widehat{M}(t) \to +\infty$ as $t \to T^*$ where $T^*$ satisfies

$$T^* \leq -\frac{1}{k_1 m_0(x_0) u_{0,x}(x_0)},$$

proving part (a).

(b) Similarly as in (a), we can deduce from (4.41) that

$$\widehat{m}(t) < 0, \quad \widehat{u}(t) \leq \widehat{u}(0) < 0, \quad \widehat{u_x}(t) \leq \widehat{u_x}(0) < 0. \tag{4.44}$$

To obtain a Riccati type inequality for $\widehat{M}$, it suffices to ask that $\widehat{m} - \frac{2}{3}\widehat{u} \leq \varepsilon \widehat{m}$, for some $\varepsilon > 0$, which leads to (4.42) again.

Following the dynamics of $\widehat{m}/\widehat{u}$ and keeping track of the signs as in (4.44) it follows that (4.43) still holds. Hence the rest of the argument goes the same way as in (a). □

**The case when $k_1 > 0$**

In this case it follows from (4.38) that

$$\text{sign}(\widehat{u}') = -\text{sign}(\widehat{u_x}), \quad \text{sign}(\widehat{u_x}') = -\text{sign}(\widehat{u}). \tag{4.45}$$

The corresponding blow-up results are as follows.

**Theorem 4.2.10.** *Suppose that $k_1 > 0$, $2k_1 + 3k_2 = 0$. Let $m_0 \in H^s(\mathbb{S})$ for $s > 1/2$. Assume that*

*(a) $m_0 \geq 0$ and there exists some point $x_0 \in \mathbb{S}$ such that*

$$m_0(x_0) > 0, \quad u_{0,x}(x_0) < 0, \quad m_0(x_0) \geq \frac{4}{3} u_0(x_0), \tag{4.46}$$

*or*

*(b)* $m_0 \leq 0$ *and there exists some point* $x_0 \in \mathbb{S}$ *such that*

$$m_0(x_0) < 0, \quad u_{0,x}(x_0) > 0, \quad m_0(x_0) \leq \frac{4}{3} u_0(x_0), \tag{4.47}$$

*Then the corresponding solution* $u(t,x)$ *blows up in finite time with an estimate of the blow-up time* $T^*$ *as*

$$T^* \leq -\frac{1}{k_1 m_0(x_0) u_{0,x}(x_0)}. \tag{4.48}$$

*Proof.* Tracking the dynamics of $\widehat{M}$ and using (4.45) we see that to obtain a Riccati type inequality for $\widehat{M}$ it suffices to have (4.42) for some $\varepsilon > 0$, for both cases (a) and (b). Thus computing $(\widehat{m}/\widehat{u})'$ and using that $|u_x| \leq u$ we get

$$\left(\frac{\widehat{m}}{\widehat{u}}\right)' = -\frac{2k_1 \widehat{m}\widehat{u_x}}{\widehat{u}^2}\left(\widehat{m}\widehat{u} - \widehat{u}^2 - \frac{1}{3}\widehat{u_x}^2\right) \geq -\frac{2k_1 \widehat{m}\widehat{u_x}}{\widehat{u}^2}\left(\widehat{m}\widehat{u} - \frac{4}{3}\widehat{u}^2\right),$$

which implies that

$$\frac{\widehat{m}}{\widehat{u}} \ \text{ increases if } \ \frac{\widehat{m}}{\widehat{u}} \geq \frac{4}{3}. \tag{4.49}$$

This in turn leads to $\widehat{M}' \leq -k_1 \widehat{M}^2$ and hence the blow-up of $\widehat{M}$ with an estimate of the blow-up time as (4.48).

Finally, the theorem is proved by realizing that (4.49) is satisfied if either (4.46) or (4.47) holds. □

# Appendix A

## The Coefficient of $\varepsilon^2\mu^2$ for the Derivation of the Free Surface Equation

The expansion of derivative of $t$ on $\mu^2, \varepsilon\mu^2$-order will generate $\varepsilon^2\mu^2$-order terms. While $\mu^2, \varepsilon\mu^2$ are high order terms under Camassa–Holm regime, we will eliminate the $t$ derivatives in $\mu^2$ and $\varepsilon\mu^2$-order to produce the full terms for $\varepsilon^2\mu^2$-order.

For $\mu^2$-order, it takes the form of

$$O_6(\mu^2) \coloneqq \mu^2(D_x + D_t) + \frac{1}{6}\lambda_2\mu^2 u_{xxxxxx} - \lambda_2^2\nu\mu^2(u_{xxx} + u_{xxt})_{xx} - \frac{1}{6}\lambda_2\nu\mu^2 u_{xxxxxx}.$$

By the scaling in the equation, $D$ should have the form $D = \lambda_6 u_{xxxx}$ for a parameter $\lambda_6$. Thus, we have

$$\mu^2(D_x + D_t) = \mu^2\lambda_6(u_{xxxxx} + u_{xxxxt}). \tag{A.1}$$

From (3.4), we observe that $u_t = -u_x - \frac{3}{2}\varepsilon u u_x - (\frac{1}{2}\lambda_1 + \frac{A_1}{6})\varepsilon^2(u^3)_x$. Therefore,

$$
\begin{aligned}
\mu^2\lambda_6(u_{xxxxx}+u_{xxxxt}) - \lambda_2^2\nu\mu^2(u_{xxx}+u_{xxt})_{xx}
&= \frac{3}{2}\lambda_2^2\nu\varepsilon\mu^2(uu_x)_{xxxx} + \lambda_2^2\nu(\frac{1}{2}\lambda_1 + \frac{A_1}{6})\varepsilon^2\mu^2(u^3)_{xxxxx} \\
&\quad - \frac{3}{2}\lambda_6\varepsilon\mu^2(uu_x)_{xxxx} - \lambda_6(\frac{1}{2}\lambda_1 + \frac{A_1}{6})\varepsilon^2\mu^2(u^3)_{xxxxx}.
\end{aligned}
$$

It generates higher order terms, viz.

$$
\begin{aligned}
O_6(\varepsilon\mu^2) :& \frac{1}{2}O_0(\varepsilon\mu^2) - \lambda_1 u\varepsilon O_0(\mu^2) + \lambda_2\nu\mu(F_{\varepsilon\mu})_{xx} - \lambda_4\varepsilon\mu u_x(O_0(\mu))_x \\
& - \frac{\lambda_5}{2}\varepsilon\mu u_{xx}O_0(\mu) - (\frac{1}{2}\lambda_5 - \lambda_1\lambda_2)\varepsilon\mu u(O_0(\mu))_{xx} + \frac{3}{2}(\lambda_2^2\nu - \lambda_6)\varepsilon\mu^2(uu_x)_{xx}, \\
O_6(\varepsilon^2\mu^2) :& \frac{1}{2}O_0(\varepsilon^2\mu^2) - \lambda_1 u\varepsilon O_0(\varepsilon\mu^2) - u^2(\frac{3}{2}\lambda_3 - 2\lambda_1^2)\varepsilon^2 O_0(\mu^2) + \lambda_2\nu\mu(F_{\varepsilon^2\mu})_{xx} \\
& - \lambda_4\varepsilon\mu u_x(O_0(\varepsilon\mu) + 2\lambda_1 u\varepsilon O_0(\mu))_x - \lambda_5\varepsilon\mu u_{xx}\frac{1}{2}(O_0(\varepsilon\mu)) + \lambda_5\lambda_1\varepsilon^2\mu u u_{xx}O_0(\mu) \\
& - (\frac{1}{2}\lambda_5 - \lambda_1\lambda_2)\varepsilon\mu u(O_0(\varepsilon\mu))_{xx} + (\lambda_5 - 2\lambda_1\lambda_2)\lambda_1\varepsilon^2\mu u(uO_0(\mu))_{xx} \\
& + \varepsilon^2\mu^2\lambda_1(\frac{1}{3}\lambda_4 uu_x u_{xxxx} + \frac{1}{6}\lambda_5 uu_{xx}u_{xxx} + \frac{1}{6}\lambda_5 u^2 u_{xxxxx}) - \lambda_2(\frac{1}{2}\lambda_3 - \frac{2}{3}\lambda_1^2)\varepsilon^2\mu^2 u^2 u_{xxxxx} \\
& - \frac{1}{6}\lambda_7\varepsilon^2\mu^2 u_x^2 u_{xxx} - \frac{1}{3}\lambda_7\varepsilon^2\mu^2 uu_x u_{xxxx} - \frac{1}{6}\lambda_8\varepsilon^2\mu^2 u^2 u_{xxxxx} - \frac{1}{3}\varepsilon^2\mu^2\lambda_8 uu_{xx}u_{xxx} \\
& + (\lambda_2^2\nu - \lambda_6)(\frac{1}{2}\lambda_1 + \frac{A_1}{6})\varepsilon^2\mu^2(u^3)_{xxxxx}.
\end{aligned}
$$

Next, we consider $\varepsilon\mu^2$-order i.e. $O_6(\varepsilon\mu^2)$. Here, the term $\frac{1}{2}O_0(\varepsilon\mu^2)$ is the only part which will product the $\varepsilon^2\mu^2$-order is $\varepsilon\mu^2(H_x + H_t)$. Hence, we choose $H = \lambda_9 u_x u_{xxx} + \lambda_{10}uu_{xxxx} + \lambda_{11}u_{xx}^2$ and expand $u_t$ up to $\varepsilon$-order, i.e., $u_t = -u_x - \frac{3}{2}\varepsilon uu_x$. It yields that

$$
\begin{aligned}
\varepsilon\mu^2(H_x + H_t) =& -\frac{3\lambda_9}{2}\varepsilon^2\mu^2(uu_x)_x u_{xxx} - \frac{3\lambda_9}{2}\varepsilon^2\mu^2 u_x(uu_x)_{xxx} - \frac{3\lambda_{10}}{2}\varepsilon^2\mu^2 uu_x u_{xxxxx} \\
& - \frac{3\lambda_{10}}{2}\varepsilon^2\mu^2 u(uu_x)_{xxxx} - 3\lambda_{11}\varepsilon^2\mu^2 u_{xx}(uu_x)_{xx}.
\end{aligned}
$$

Recalling the definition $D = \lambda_6 u_{xxxx}$ and $O_0(\mu) = 2\lambda_2\mu(u_{xxx} + u_{xxt}) + \frac{1}{3}\mu u_{xxx}$, we have

$$\begin{aligned}
-\lambda_1 u\varepsilon O_0(\mu^2) - \lambda_4\varepsilon\mu u_x(O_0(\mu))_x - \lambda_5\varepsilon\mu u_{xx}\frac{1}{2}O_0(\mu) - (\frac{1}{2}\lambda_5 - \lambda_1\lambda_2)\varepsilon\mu u(O_0(\mu))_{xx} \\
=& 3\lambda_6\lambda_1\varepsilon^2\mu^2 u(uu_x)_{xxxx} - \frac{\lambda_1}{3}\varepsilon\mu^2\lambda_2 uu_{xxxxxx} - \frac{\lambda_4}{3}\varepsilon\mu^2 u_x u_{xxxx} \\
&+ 3\lambda_2\lambda_4\varepsilon^2\mu^2 u_x(uu_x)_{xxx} - \frac{\lambda_5}{6}\varepsilon\mu^2 u_{xx}u_{xxx} + \frac{3}{2}\lambda_5\varepsilon^2\mu^2\lambda_2 u_{xx}(uu_x)_{xx}, \\
&- (\frac{1}{6}\lambda_5 - \frac{1}{3}\lambda_1\lambda_2)\varepsilon\mu^2 uu_{xxxxx} + \frac{3}{2}(\lambda_5 - 2\lambda_1\lambda_2)\varepsilon^2\mu^2 u\lambda_2(uu_x)_{xxxx}.
\end{aligned}$$

Also, there is

$$\begin{aligned}
\lambda_2\nu\mu(F_{\varepsilon\mu})_{xx} =& 3\lambda_2\nu\lambda_4\varepsilon^2\mu^2(u_x(uu_x)_x)_{xx} + \frac{3}{2}\lambda_2\nu\lambda_5\varepsilon^2\mu^2(uu_xu_{xx})_{xx} + \frac{1}{3}\nu\lambda_1\lambda_2\varepsilon\mu^2 uu_{xxxxx} \\
&+ \frac{3}{2}\lambda_2\nu\lambda_5\varepsilon^2\mu^2(u(uu_x)_{xx})_{xx} - \frac{1}{2}\lambda_2\nu(2\lambda_1 + 3\lambda_2 + A_3)\varepsilon\mu^2(u_xu_{xx})_{xx} \\
&- 3\lambda_1\lambda_2^2\nu\varepsilon^2\mu^2 u(uu_x)_{xxxx} - \frac{1}{2}\lambda_2\nu(\frac{2}{3}\lambda_1 + 3\lambda_2 + A_4)\varepsilon\mu^2(uu_{xxx})_{xx}.
\end{aligned} \tag{A.2}$$

After expanding of derivative of $t$, $\varepsilon^2\mu^2$-order terms takes the form:

$$\begin{aligned}
O_7(\varepsilon^2\mu^2) : &\frac{1}{2}O_0(\varepsilon^2\mu^2) - \lambda_1 u\varepsilon O_0(\varepsilon\mu^2) - u^2(\frac{3}{2}\lambda_3 - 2\lambda_1^2)\varepsilon^2 O_0(\mu^2) + \lambda_2\nu\mu(F_{\varepsilon^2\mu})_{xx} \\
&- \lambda_4\varepsilon\mu u_x(O_0(\varepsilon\mu) + 2\lambda_1 u\varepsilon O_0(\mu))_x - \frac{\lambda_5}{2}\varepsilon\mu u_{xx}O_0(\varepsilon\mu) + \lambda_5\lambda_1 uu_{xx}\varepsilon^2\mu O_0(\mu) \\
&- (\frac{1}{2}\lambda_5 - \lambda_1\lambda_2)\varepsilon\mu u(O_0(\varepsilon\mu))_{xx} + (\lambda_5 - 2\lambda_1\lambda_2)\lambda_1\varepsilon^2\mu u(uO_0(\mu))_{xx} \\
&+ \varepsilon^2\mu^2\lambda_1(\frac{1}{3}\lambda_4 uu_xu_{xxxx} + \frac{1}{6}\lambda_5 uu_{xx}u_{xxx} + \frac{1}{6}\lambda_5 u^2 u_{xxxxx}) - \lambda_2(\frac{1}{2}\lambda_3 - \frac{2}{3}\lambda_1^2)\varepsilon^2\mu^2 u^2 u_{xxxxx} \\
&- \frac{1}{6}\lambda_7\varepsilon^2\mu^2 u_x^2 u_{xxx} - \frac{1}{3}\lambda_7\varepsilon^2\mu^2 uu_xu_{xxxx} - \frac{1}{6}\lambda_8\varepsilon^2\mu^2 u^2 u_{xxxxx} - \frac{1}{3}\varepsilon^2\mu^2\lambda_8 uu_{xx}u_{xxx} \\
&+ (\lambda_2^2\nu - \lambda_6)(\frac{1}{2}\lambda_1 + \frac{A_1}{6})\varepsilon^2\mu^2(u^3)_{xxxxx} - \frac{3\lambda_9}{2}\varepsilon^2\mu^2(uu_x)_x u_{xxx} - \frac{3\lambda_9}{2}\varepsilon^2\mu^2 u_x(uu_x)_{xxx} \\
&- \frac{3\lambda_{10}}{2}\varepsilon^2\mu^2 uu_xu_{xxxx} - \frac{3\lambda_{10}}{2}\varepsilon^2\mu^2 u(uu_x)_{xxxx} - 3\lambda_{11}\varepsilon^2\mu^2 u_{xx}(uu_x)_{xx} + 3\lambda_6\lambda_1\varepsilon^2\mu^2 u(uu_x)_{xxxx}
\end{aligned}$$

$$+\,3\lambda_2\nu\lambda_4\varepsilon^2\mu^2(u_x(uu_x)_x)_{xx}+\frac{3}{2}\lambda_2\nu\lambda_5\varepsilon^2\mu^2(uu_xu_{xx})_{xx}+\frac{3}{2}\lambda_2\nu\lambda_5\varepsilon^2\mu^2(u(uu_x)_{xx})_{xx}$$
$$-\,3\lambda_1\lambda_2^2\nu\varepsilon^2\mu^2u(uu_x)_{xxxx}+3\lambda_2\lambda_4\varepsilon^2\mu^2u_x(uu_x)_{xxx}+\frac{3}{2}\lambda_5\varepsilon^2\mu^2\lambda_2u_{xx}(uu_x)_{xx}$$
$$+\,\frac{3\lambda_2}{2}(\lambda_5-2\lambda_1\lambda_2)\varepsilon^2\mu^2u(uu_x)_{xxxx}.$$

Now we try to write the specific form of $O_7(\varepsilon^2\mu^2)$. We substitute the definition $C=\lambda_4u_x^2+\lambda_5uu_{xx},\ D=\lambda_6u_{xxxx},\ G=\lambda_7uu_x^2+\lambda_8u^2u_{xx},\ H=\lambda_9u_xu_{xxx}+\lambda_{10}uu_{xxxx}+\lambda_{11}u_{xx}^2$ into the first term of $O_7(\varepsilon^2\mu^2)$. Then it takes the form of

$$
\begin{aligned}
&\frac{1}{2}O_0(\varepsilon^2\mu^2)\\
&=\frac{1}{6}\varepsilon^2\mu^2(\lambda_7uu_x^2+\lambda_8u^2u_{xx})_{xxx}+\frac{3}{2}\lambda_1\lambda_6\varepsilon^2\mu^2(u^2u_{xxxx})_x+\frac{3}{2}\lambda_2\varepsilon^2\mu^2(\lambda_4u_x^2u_{xx}+\lambda_5uu_{xx}^2)_x\\
&\quad+\frac{3}{2}\varepsilon^2\mu^2(\lambda_9uu_xu_{xxx}+\lambda_{10}u^2u_{xxxx}+\lambda_{11}uu_{xx}^2)_x+\frac{A_1}{2}\varepsilon^2\mu^2\lambda_2^2(uu_{xx}^2)_x\\
&\quad+\frac{A_4}{2}\varepsilon^2\mu^2u_{xxx}(\lambda_4u_x^2+\lambda_5uu_{xx})+\frac{A_1}{2}\lambda_6\varepsilon^2\mu^2(u^2u_{xxxx})_x+\frac{A_3}{2}\varepsilon^2\mu^2(u_x(\lambda_4u_x^2+\lambda_5uu_{xx})_x)_x\\
&\quad+A_3\lambda_1\lambda_2\varepsilon^2\mu^2(uu_xu_{xxx})_x+\frac{A_4}{2}\lambda_1\lambda_2\varepsilon^2\mu^2u^2u_{xxxxx}+\frac{A_4}{2}\lambda_1\lambda_2\varepsilon^2\mu^2u_{xx}(u^2)_{xxx}\\
&\quad+\frac{A_4}{2}\varepsilon^2\mu^2u(\lambda_4u_x^2+\lambda_5uu_{xx})_{xxx}+\frac{A_5}{2}\lambda_2\varepsilon^2\mu^2uu_xu_{xxxx}+\frac{A_5}{2}\lambda_2\varepsilon^2\mu^2u_xu_{xx}^2\\
&\quad+\frac{A_5}{2}\lambda_2\varepsilon^2\mu^2uu_{xx}u_{xxx}+\frac{A_6}{2}\lambda_2\varepsilon^2\mu^2u^2u_{xxxxx}+A_6\lambda_2\varepsilon^2\mu^2uu_{xx}u_{xxx}+\frac{3A_7}{2}\lambda_2\varepsilon^2\mu^2u_x^2u_{xxx}.
\end{aligned}
$$

Also, the others can be rewritten as

$$
\begin{aligned}
&-\lambda_1\varepsilon uO_0(\varepsilon\mu^2)-u^2\left(\frac{3}{2}\lambda_3-2\lambda_1^2\right)\varepsilon^2O_0(\mu^2)\\
&\quad=-\frac{1}{3}\lambda_1\varepsilon^2\mu^2u(\lambda_4u_x^2+\lambda_5uu_{xx})_{xxx}-3\lambda_1\lambda_6\varepsilon^2\mu^2u(uu_{xxxx})_x-3\lambda_1\lambda_2^2\varepsilon^2\mu^2uu_{xx}u_{xxx}\\
&\qquad-A_3\lambda_2\lambda_1\varepsilon^2\mu^2u(u_xu_{xxx})_x-A_4\lambda_1\lambda_2\varepsilon^2\mu^2uu_{xx}u_{xxx}-A_4\lambda_1\lambda_2\varepsilon^2\mu^2u^2u_{xxxxx}\\
&\qquad-\left(\frac{1}{2}\lambda_3-\frac{2}{3}\lambda_1^2\right)\lambda_2\varepsilon^2\mu^2u^2u_{xxxxx}.
\end{aligned}
$$

Hence,

$$I_1 := \lambda_2\nu\mu(F_{\varepsilon^2\mu})_{xx} - \lambda_4\varepsilon\mu u_x(O_0(\varepsilon\mu) + 2\lambda_1\varepsilon u O_0(\mu))_x - \lambda_5\varepsilon\mu\frac{1}{2}u_{xx}O_0(\varepsilon\mu)$$
$$+ \lambda_5\lambda_1\varepsilon^2\mu u u_{xx}O_0(\mu) - (\frac{1}{2}\lambda_5 - \lambda_1\lambda_2)\varepsilon\mu u(O_0(\varepsilon\mu))_{xx}$$
$$+ (\lambda_5 - 2\lambda_1\lambda_2)\lambda_1\varepsilon^2\mu u(uO_0(\mu))_{xx}.$$

Then,

$$I_1 = -\frac{1}{2}\lambda_2\nu\mu O_0(\varepsilon^2\mu)_{xx} + (\lambda_1\lambda_2\nu - \frac{\lambda_5}{2})\varepsilon\mu u_{xx}O_0(\varepsilon\mu) + (2\lambda_1\lambda_2\nu - \lambda_4)\varepsilon\mu u_x O_0(\varepsilon\mu)_x$$
$$+ (\lambda_1\lambda_2(1+\nu) - \frac{\lambda_5}{2})\varepsilon\mu u O_0(\varepsilon\mu)_{xx} + (\lambda_2\nu(3\lambda_3 - 4\lambda_1^2) + 2\lambda_5\lambda_1 - 2\lambda_1^2\lambda_2)\varepsilon^2\mu u u_{xx}O_0(\mu)$$
$$+ (\lambda_2\nu(3\lambda_3 - 4\lambda_1^2) - 2\lambda_4\lambda_1)\varepsilon^2\mu u_x^2 O_0(\mu) + (\lambda_2\nu(\frac{3}{2}\lambda_3 - 2\lambda_1^2) + (\lambda_5 - 2\lambda_1\lambda_2)\lambda_1)\varepsilon^2\mu u^2 O_0(\mu)_{xx}$$
$$+ (2\lambda_2\nu(3\lambda_3 - 4\lambda_1^2) - 2\lambda_4\lambda_1 + 2(\lambda_5 - 2\lambda_1\lambda_2)\lambda_1)\varepsilon^2\mu u u_x(O_0(\mu))_x.$$

These operations produce the $\varepsilon^2\mu^2$-order of the form

$$O_8(\varepsilon^2\mu^2) := \frac{1}{6}\varepsilon^2\mu^2(\lambda_7 u u_x^2 + \lambda_8 u^2 u_{xx})_{xxx} + \frac{3}{2}\lambda_1\lambda_6\varepsilon^2\mu^2(u^2 u_{xxxx})_x + \frac{3}{2}\lambda_2\varepsilon^2\mu^2(\lambda_4 u_x^2 u_{xx} + \lambda_5 u u_{xx}^2)_x$$
$$+ \frac{3}{2}\varepsilon^2\mu^2(\lambda_9 u u_x u_{xxx} + \lambda_{10}u^2 u_{xxxx} + \lambda_{11}u u_{xx}^2)_x + \frac{A_1}{2}\varepsilon^2\mu^2\lambda_2^2(u u_{xx}^2)_x$$
$$+ \frac{A_4}{2}\varepsilon^2\mu^2 u_{xxx}(\lambda_4 u_x^2 + \lambda_5 u u_{xx}) + \frac{A_1}{2}\lambda_6\varepsilon^2\mu^2(u^2 u_{xxxx})_x + \frac{A_3}{2}\varepsilon^2\mu^2((2\lambda_4 + \lambda_5)u_x^2 u_{xx} + \lambda_5 u u_x u_{xxx}))_x$$
$$+ A_3\lambda_1\lambda_2\varepsilon^2\mu^2(u u_x u_{xxx})_x + \frac{A_4}{2}\lambda_1\lambda_2\varepsilon^2\mu^2 u^2 u_{xxxxx} + \frac{A_4}{2}\lambda_1\lambda_2\varepsilon^2\mu^2 u_{xx}(u^2)_{xxx}$$
$$+ \frac{A_4}{2}\varepsilon^2\mu^2 u(\lambda_4 u_x^2 + \lambda_5 u u_{xx})_{xxx} + \frac{A_5}{2}\lambda_2\varepsilon^2\mu^2 u u_x u_{xxxx} + \frac{A_5}{2}\lambda_2\varepsilon^2\mu^2 u_x u_{xx}^2 + \frac{A_5}{2}\lambda_2\varepsilon^2\mu^2 u u_{xx}u_{xxx}$$
$$+ \frac{A_6}{2}\lambda_2\varepsilon^2\mu^2 u^2 u_{xxxxx} + A_6\lambda_2\varepsilon^2\mu^2 u u_{xx}u_{xxx} + \frac{3A_7}{2}\lambda_2\varepsilon^2\mu^2 u_x^2 u_{xxx} - \frac{1}{3}\lambda_1\varepsilon^2\mu^2 u(\lambda_4 u_x^2 + \lambda_5 u u_{xx})_{xxx}$$
$$- 3\lambda_1\lambda_6\varepsilon^2\mu^2 u(u u_{xxxx})_x - 3\lambda_1\lambda_2^2\varepsilon^2\mu^2 u u_{xx}u_{xxx} - A_3\lambda_1\lambda_2\varepsilon^2\mu^2 u(u_x u_{xxx})_x$$
$$- A_4\lambda_1\lambda_2\varepsilon^2\mu^2 u u_{xx}u_{xxx} - A_4\lambda_1\lambda_2\varepsilon^2\mu^2 u^2 u_{xxxxx} - \lambda_2(\frac{3}{6}\lambda_3 - \frac{2}{3}\lambda_1^2)\varepsilon^2\mu^2 u^2 u_{xxxxx}$$
$$- \frac{1}{6}\lambda_2\nu\lambda_3\varepsilon^2\mu^2(u^3)_{xxxxx} - \frac{3}{2}\lambda_2\nu\varepsilon^2\mu^2(\lambda_4 u u_x^2 + \lambda_5 u^2 u_{xx})_{xxx} - \frac{3}{2}\lambda_2^2\nu\lambda_1\varepsilon^2\mu^2(u^2 u_{xx})_{xxx}$$

$$
\begin{aligned}
&- \frac{A_1}{2}\lambda_2^2\nu\varepsilon^2\mu^2(u^2u_{xx})_{xxx} - A_3\lambda_1\lambda_2\nu\varepsilon^2\mu^2(uu_x^2)_{xxx} - \frac{A_4}{2}\lambda_1\lambda_2\nu\varepsilon^2\mu^2(u^2u_{xxx})_{xx} \\
&- \frac{A_4}{2}\lambda_1\lambda_2\nu\varepsilon^2\mu^2(u(u^2)_{xxx})_{xx} - \frac{A_5}{2}\lambda_2\nu\varepsilon^2\mu^2(uu_xu_{xx})_{xx} - \frac{A_6}{2}\lambda_2\nu\varepsilon^2\mu^2(u^2u_{xxx})_{xx} - \frac{A_7}{2}\lambda_2\nu\varepsilon^2\mu^2(u_x^3)_{xx} \\
&+ (\lambda_1\lambda_2\nu - \frac{\lambda_5}{2})\varepsilon^2\mu^2((2\lambda_1 + 3\lambda_2 + A_3)u_xu_{xx}^2 + (\frac{2}{3}\lambda_1 + 3\lambda_2 + A_4)uu_{xx}u_{xxx})) \\
&+ (2\lambda_1\lambda_2\nu - \lambda_4)\varepsilon^2\mu^2((2\lambda_1 + 3\lambda_2 + A_3)u_xu_{xx}^2 + (\frac{8}{3}\lambda_1 + 6\lambda_2 + A_3 + A_4)u_x^2u_{xxx}) \\
&+ (2\lambda_1\lambda_2\nu - \lambda_4)\varepsilon^2\mu^2(\frac{2}{3}\lambda_1 + 3\lambda_2 + A_4)uu_xu_{xxxx} \\
&+ (\lambda_1\lambda_2\nu - (\frac{1}{2}\lambda_5 - \lambda_1\lambda_2))\varepsilon^2\mu^2((\frac{10}{3}\lambda_1 + 9\lambda_2 + 2A_4 + A_3)uu_xu_{xxxx} + (\frac{2}{3}\lambda_1 + 3\lambda_2 + A_4)u^2u_{xxxxx}) \\
&+ (\lambda_1\lambda_2\nu - (\frac{1}{2}\lambda_5 - \lambda_1\lambda_2))\varepsilon^2\mu^2(\frac{20}{3}\lambda_1 + 12\lambda_2 + A_4 + 3A_3)uu_{xx}u_{xxx}) \\
&+ \frac{1}{3}(\lambda_2\nu(3\lambda_3 - 4\lambda_1^2) + 2\lambda_5\lambda_1 - 2\lambda_1^2\lambda_2)\varepsilon^2\mu^2uu_{xx}u_{xxx} + \frac{1}{3}(\lambda_2\nu(3\lambda_3 - 4\lambda_1^2) - 2\lambda_4\lambda_1)\varepsilon^2\mu^2u_x^2u_{xxx} \\
&+ \frac{1}{3}(2\lambda_2\nu(3\lambda_3 - 4\lambda_1^2) - 2\lambda_4\lambda_1 + \frac{2}{3}(\lambda_5 - 2\lambda_1\lambda_2)\lambda_1)\varepsilon^2\mu^2uu_xu_{xxxx} \\
&+ \frac{1}{3}(\lambda_2\nu(\frac{3}{2}\lambda_3 - 2\lambda_1^2) + (\lambda_5 - 2\lambda_1\lambda_2)\lambda_1)\varepsilon^2\mu^2u^2u_{xxxxx} - \lambda_2(\frac{1}{2}\lambda_3 - \frac{2}{3}\lambda_1^2)\varepsilon^2\mu^2u^2u_{xxxxx} \\
&+ \varepsilon^2\mu^2\lambda_1(\frac{1}{3}\lambda_4uu_xu_{xxxx} + \frac{1}{6}\lambda_5uu_{xx}u_{xxx} + \frac{1}{6}\lambda_5u^2u_{xxxxx}) + \frac{3}{2}\lambda_2(\lambda_5 - 2\lambda_1\lambda_2)\varepsilon^2\mu^2u(uu_x)_{xxxx} \\
&- \frac{1}{6}\lambda_7\varepsilon^2\mu^2u_x^2u_{xxx} - \frac{1}{3}\lambda_7\varepsilon^2\mu^2uu_xu_{xxxx} - \frac{1}{6}\lambda_8\varepsilon^2\mu^2u^2u_{xxxxx} - \frac{1}{3}\lambda_8\varepsilon^2\mu^2uu_{xx}u_{xxx} \\
&+ (\lambda_2^2\nu - \lambda_6)(\frac{1}{2}\lambda_1 + \frac{A_1}{6})\varepsilon^2\mu^2(u^3)_{xxxxx} - \frac{3\lambda_9}{2}\varepsilon^2\mu^2(uu_x)_xu_{xxx} - \frac{3\lambda_9}{2}\varepsilon^2\mu^2u_x(uu_x)_{xxx} \\
&- \frac{3\lambda_{10}}{2}\varepsilon^2\mu^2uu_xu_{xxxx} - \frac{3\lambda_{10}}{2}\varepsilon^2\mu^2u(uu_x)_{xxxx} - 3\lambda_{11}\varepsilon^2\mu^2u_{xx}(uu_x)_{xx} + 3\lambda_6\lambda_1\varepsilon^2\mu^2u(uu_x)_{xxxx} \\
&+ 3\lambda_2\nu\lambda_4\varepsilon^2\mu^2(u_x(uu_x)_x)_{xx} + \frac{3}{2}\lambda_2\nu\lambda_5\varepsilon^2\mu^2(uu_xu_{xx})_{xx} + \frac{3}{2}\lambda_2\nu\lambda_5\varepsilon^2\mu^2(u(uu_x)_{xx})_{xx} \\
&- 3\lambda_1\lambda_2^2\nu\varepsilon^2\mu^2u(uu_x)_{xxxx} + 3\lambda_2\lambda_4\varepsilon^2\mu^2u_x(uu_x)_{xxx} + \frac{3}{2}\lambda_2\lambda_5\varepsilon^2\mu^2u_{xx}(uu_x)_{xx}.
\end{aligned}
$$

Here, the coefficient of $u^2u_{xxxxx}$ could be denoted by

$$
C_1 := \frac{A_4}{2}\lambda_1\lambda_2(1-\nu) + \frac{A_6}{2}\lambda_2(1-\nu) - \lambda_2\lambda_3 + \frac{4}{3}\lambda_1^2\lambda_2 - \frac{1}{6}\lambda_1\lambda_5.
$$

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## BIOGRAPHICAL STATEMENT

Osama Salameh Alkhazaleh was born and raised in Almafraq, Jordan. He earned his Bachelor of Science degree in mathematics from Al Albayt University in 2009. In August 2012, he received his Master of Science in mathematics at the Jordan University of Science and Technology. During his Master's program, he worked as a teaching assistant in the mathematics department for two years. After receiving his master's degree, he worked as a lecturer at Umm-Alqura University in Mecca, K.S.A from 2012 to 2015. In the fall of 2017, Osama continued his education by beginning the Ph.D. program at the University of Texas at Arlington under the supervision of Dr. Yue Liu. During his doctoral program, he was awarded GTA Fellowship between 2018 and 2021.