HIGHER-ORDER NONSTANDARD FINITE DIFFERENCE METHODS FOR AUTONOMOUS DIFFERENTIAL EQUATIONS WITH APPLICATIONS IN MATHEMATICAL ECOLOGY

by

FAWAZ KARHAN ALALHARETH

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ABSTRACT

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Fawaz Karhan Alalhareth, Ph.D.

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Supervising Professor: Hristo V. Kojouharov

Nonstandard finite difference (NSFD) methods have been widely used to numerically solve various problems in biology. In recent years, NSFD methods have been proposed that preserve essential properties of the solutions of general autonomous differential equations, such as positivity and elementary stability, among others. However, those methods are only of first-order accuracy. In the first part of this dissertation, we construct and analyze two second-order modified positive and elementary stable nonstandard (PESN) numerical methods for n-dimensional autonomous differential equations. The new PESN methods are generalized versions of the explicit Euler's method and second-order accurate, thereby improving the order of accuracy of the underlying numerical method. In the second part of this dissertation, we analyze several chemostat models with a constant input of one species. Chemostat models have been extensively used to represent microbial growth and competition in homogeneous environments. First, we consider a simple growth chemostat model for a donor bacteria and one limiting resource. Since competition is crucial in nature,

we next propose a model when there is competition between a resident bacteria and the donor bacteria for a single limiting substrate in the presence of a lethal toxin. Resident bacteria can become donor bacteria by changing the genetics of the resident bacteria. This change can occur by a plasmid. Therefore, we propose a model when there is competition in the presence of a constant homogeneous plasmid. The proposed chemostat models are non-linear ordinary differential equations, and their exact solutions cannot be obtained analytically. Moreover, their solutions remain positive for all time. Therefore, using accurate and efficient numerical methods that also preserve the solutions' positivity property is essential when working with chemostat models. In the last part of the dissertation, the new second-order PESN (SOPESN) methods are used to approximate the solutions of the earlier presented chemostat models. In addition, the SOPESN methods are compared with several standard and nonstandard finite difference methods to numerically demonstrate their advantages when solving models in mathematical ecology.

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CHAPTER 1

Introduction

Ordinary differential equations (ODEs) are often used to model various complex processes in economics, engineering, biology, chemistry, and many other scientific fields. Most models are nonlinear differential equations that cannot be solved analytically. Moreover, the models are dynamical systems with essential properties such as the positivity, boundedness, and monotonicity of solutions and the linear stability properties of the equilibria, among others. Consequently, it becomes vital to construct numerical methods that accurately approximate the solutions of the ODEs and preserve their essential properties. Standard finite-difference numerical methods, especially explicit ones, such as the Euler and the Runge–Kutta methods, usually require severe restrictions on the discrete time-step size to mimic the ODE solution correctly.

About three decades ago, nonstandard finite difference (NSFD) methods were introduced by Mickens to numerically solve ODEs by preserving the essential properties of their exact solutions (see [31] and references therein). Since then, the NSFD methods have been extensively developed to approximate the solutions of ordinary differential equations. The NSFD schemes have been constructed to be qualitatively stable [4] with respect to the desired property of the solution of the ODE for any time-step size. Dynamically consistent [29, 5] NSFD methods have been developed to preserve boundedness, conservation laws, monotonicity of solutions, and the equilibria's linear stability, among other properties (See [35, 36] and references therein).

Biological models require that solutions (concentrations or populations) remain positive for any time. In this context, several NSFD methods have also been developed to preserve both the positivity and elementary stability properties of solutions for specific autonomous dynamical systems [12, 9, 11, 13, 2, 1]. A methodology for constructing positive and elementary stable nonstandard (PESN) numerical methods has also been proposed for general productive-destructive systems [14, 48] as well as for solving general autonomous systems with positive solutions [49]. However, most NSFD methods that preserve both the positivity and elementary stability properties of solutions are only first-order accuracy. Second-order NSFD methods have been proposed. For instance, in [15, 25, 16], we proposed a methodology for constructing NSFD methods that are second-order accuracy and are elementary stable. However, they do not preserve the positivity property. The positivity property of the solutions is a very important property for the solutions when the ODEs are used to model any biological system.

One crucial biological system is the gut microbiome, because it plays an essential role in human health. For instance, the gut microbiome could help weight loss or prevent obesity in individuals [8]. Moreover, the gut microbiota has crucial effects on mental illness, maintenance of mental health, and heart failure [27, 46, 7]. The human gut contains many bacteria; some are good for our health, but others are bad [47]. The harmful bacteria can become a good bacteria by changing the genetics of the harmful bacteria and vice versa. The change of the genetics in a bacteria can occur by the plasmid. Moreover, the plasmid plays fundamental roles in antibiotic resistance and bioremediation [6]. Therefore, understanding the dynamic plasmid bearing (donor) or non-plasmid bearing (resident) is crucial.

A laboratory device that can be used to represent the human gut is the chemostat. The chemostat is an experimental device that was invented simultaneously by Monod

[32] and Novick-Szilard [34]. It plays a vital role in ecology. The chemostat can be used as a model of a simple lake, the wastewater treatment process, or the growth of microorganisms for industrial or experimental purposes. Moreover, it can be used as a model of organisms' competition for a nutrient. The simple chemostat model of one organism consuming the nutrient appeared in the fifteen, and it is described with two non-linear ordinary differential equations [39]. Since then, there have been many modifications to the simple chemostat model to represent significant biological problems such as competition, competition in the presence of an inhibitor, competition in the presence of plasmid, and many others [44, 22, 40, 21]. However, most of the chemostat models are non-linear ordinary differential equations. They can not be solved analytically. Thus, it becomes essential to construct numerical methods that are efficient and preserve some important properties of the chemostat models, such as positivity and the local stability of the equilibria.

This dissertation is organised as follows. In Chapter 2, definitions and preliminary work needed for the rest of the dissertation are given. In Chapter 3, we construct and analyze second-order PESN methods for general classes of ODEs. In Chapter 4, we construct and an analyze several chemostat models with microbial input. In Chapter 5, several examples and applications are given which demonstrate the effectiveness of the proposed NSFD methods developed in Chapter 3, and then use the SOPESN methods to numerically solve the chemostat models and validate the theoretical result in Chapter 4. Finally, in Chapter 6, concluding remarks are given.

CHAPTER 2

Definitions and preliminaries

We consider autonomous differential equations of the form

$$\vec{x}' = \vec{f}(\vec{x}); \quad \vec{x}(t_0) = \vec{x}_0 \in \mathbb{R}^n_+, \tag{2.0.1}$$

where $\mathbb{R}^n_+ = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_i > 0, \forall i = 1, ..., n\}$, the vector-function $\vec{x} = [x_1, \ldots, x_n]$ is unknown, and the vector-function $\vec{f} = [f_1, \ldots, f_n] \in C^2(\mathbb{R}^n_+; \mathbb{R}^n)$ such that $\mathbb{R}^n_+$ is positively invariant [44]. It is also assumed that equation (2.0.1) has a finite number of equilibria and each of them is hyperbolic. In addition, we also consider the following general sub-class of autonomous differential equations, termed productive-destructive equations [14]:

$$\vec{x}' = \vec{P}(\vec{x}) - \vec{D}(\vec{x}); \quad \vec{x}(t_0) = \vec{x}_0 \in \mathbb{R}^n_+, \tag{2.0.2}$$

where $\vec{P} = [P_1, \ldots, P_n]$, $\vec{D} = [D_1, \ldots, D_n]$, and $P_i, D_i \in C^2(\mathbb{R}^n_+; \mathbb{R}_+)$. Productive-destructive equations (2.0.2) are widely used for models in the sciences and engineering, with the functions $\vec{P}$ and $\vec{D}$ representing the production and destruction processes, respectively.

A general finite difference method to approximate System (2.0.1) for all $i = 1, \ldots, n$ can be written as

$$D_{i,h}(x_i^k) = F_{i,h}(f_i; \vec{x}^k), \tag{2.0.3}$$

where $D_{i,h}(x_i^k) \approx x_i'(t_k)$, $x_i^k \approx x_i(t_k)$, $F_{i,h}(f_i; \vec{x}^k)$ approximates $f_i(\vec{x}(t_k))$ in Equation (2.0.1) and $t_k = t_0 + kh$, where $h > 0$ and $k \geq 0$.

Definition 2.0.1. ([4, 10, 50]) The finite-difference method (2.0.3) for solving Equation (2.0.1) is a NSFD method if at least one of the following conditions is satisfied for all $i = 1, \ldots, n$:

- $D_{i,h}(\vec{x}^k) = \dfrac{x_i^{k+1} - x_i^k}{\varphi_i(h)}$, where $\varphi_i(h) = h + \mathcal{O}(h^2)$ is a non-negative function;
- $F_{i,h}(f_i; \vec{x}^k) = g_i(\vec{x}^k, \vec{x}^{k+1}, h)$, where $g_i(\vec{x}^k, \vec{x}^{k+1}, h)$ is a non-local approximation of the i-th component of the right-hand side of System (2.0.1).

Definition 2.0.2 ([4, 10, 50])**.** The finite difference method (2.0.3) is called elementary stable if, for any value of the step size h, its only fixed points $\vec{x}^*$ are those of the equation (2.0.1) and the linear stability properties of each $\vec{x}^*$ being the same for both the differential equation and the numerical method.

Definition 2.0.3 ([14, 48])**.** The finite difference method (2.0.3) is called positive, if, for any value of the step size h, and $\vec{x}_0 \in \mathbb{R}^n_+$ its solution remains positive, i.e., $\vec{x}_k \in \mathbb{R}^n_+$ for all $k \in \mathbb{N}$.

Definition 2.0.4 ([44])**.** Let $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a function of two variables. The function π is said to be a continuous dynamical system if π is continuous and has the following properties:

1. $\pi(\vec{x}_0, 0) = \vec{x}_0$;
2. $\pi(\vec{x}_0, t + s) = \pi(\pi(\vec{x}_0, t), s)$.

One example of a dynamical system is an ordinary differential equation of the form

$$\vec{x}' = \vec{f}(\vec{x}), \tag{2.0.4}$$

with $\vec{x} \in \mathbb{R}^n$ and $\vec{f} \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ by defining

$$\pi(\vec{x}_0, t) = \vec{x}(t),$$

where $\vec{x}(t)$ is the solution of Equation (2.0.4) satisfying the initial condition $\vec{x}(0) = \vec{x}_0$.

Definition 2.0.5 ([44])**.** The dynamical system is said to be dissipative if all positive trajectories eventually lie in a bounded set.

Positive and elementary stable nonstandard numerical methods can be designed using the following theorem [48, 49]:

Theorem 2.0.1. *The numerical discretization of Equation* (2.0.1)*:*

$$\frac{x_i^{k+1} - x_i^k}{\varphi(h)} = w_i^k f_i(\vec{x}^k), \tag{2.0.5}$$

where

$$w_i^k = \begin{cases} 1, & \text{if } f_i(\vec{x}^k) \geq 0 \\ \dfrac{x_i^{k+1}}{x_i^k}, & \text{if } f_i(\vec{x}^k) < 0 \end{cases},$$

for $i = 1, \ldots, n$ *and the nonstandard denominator function* $\varphi(h)$ *satisfies the properties listed in [49, Equation (8)], represents a positive and elementary stable nonstandard (PESN1) numerical method for Equation* (2.0.1)*.*

The numerical discretization of Equation (2.0.2)*:*

$$\frac{x_i^{k+1} - x_i^k}{\varphi(h)} = P_i(\vec{x}^k) - D_i(\vec{x}^k)\frac{x_i^{k+1}}{x_i^k}, \tag{2.0.6}$$

for $i = 1, \ldots, n$ *and the nonstandard denominator function* $\varphi(h)$ *satisfies the properties listed in [48, Theorem 5.1], represents a positive and elementary stable nonstandard (PESN2) numerical method for Equation* (2.0.2)*.*

However, the PESN methods (2.0.5) and (2.0.6) are only of first-order accuracy. In the following chapter, we present a novel approach that modifies the denominator functions in the PESN1 and PESN2 methods to achieve a second-order accuracy of the corresponding methods.

CHAPTER 3

Second-order modified PESN numerical methods

3.1 Introduction

Modeling natural dynamical systems with ODEs requires that the physically relevant solutions remain positive for all times. In this context, NSFD methods have been developed to preserve both the positivity and elementary stability properties of solutions for specific autonomous dynamical systems [12, 9, 11, 13, 2, 1]. A methodology for constructing positive and elementary stable nonstandard (PESN) numerical methods has been proposed for general productive-destructive systems [14, 48] as well as for solving general autonomous dynamical systems with positive solutions [49]. However, most NSFD methods that preserve both the positivity and elementary stability properties of solutions are only first-order accurate.

In this chapter, two nonstandard finite difference methods, which are not only positive and elementary stable but also of second-order accuracy (SOPESN), are developed and analyzed to approximate the solutions of general autonomous differential equations, including productive-destructive equations. The proposed new SOPESN methods are generalizations of the PESN methods [14, 48, 49] with novel nonstandard denominator functions [30] that also guarantee a second-order accuracy, thereby improving the order of accuracy of the underlying explicit Euler's method.

Before proving the main results, we state the following lemma which is used to show that the new SOPESN1 and SOPESN2 methods are elementary stable:

Lemma 3.1.1. *Suppose* $\{x^n\}$ *and* $\{y^n\}$ *are sequences of positive real numbers such that* $x^n \to 0$ *as* $n \to \infty$ *while* $y^n \to b \in \mathbb{R}_+ \cup \{\infty\}$ *as* $n \to \infty$*. Then, there is* $N \in \mathbb{N}$ *such that* $n \geq N$ *implies*

$$x^n < y^n.$$

Proof. First, let us suppose $b \in \mathbb{R}^+$, then there is a positive integer N_1 such that if $n \geq N_1$ implies

$$\left| y^n - b \right| < \frac{b}{2}.$$

Therefore, using the absolute value definition implies

$$\frac{b}{2} < y^n < \frac{3b}{2}, \quad \text{for all } \ n \geq N_1.$$

But, since $x^n \to 0$ as $n \to \infty$, then there is a positive integer N_2 such that if $n \geq N_2$ we have

$$x^n < \frac{b}{2}.$$

Now taking $N = \max\{N_1, N_2\}$ implies

$$x^n < y^n, \quad \text{for all } \ n \geq N.$$

If $y^n \to \infty$, then there is a positive integer N_3 such that

$$y^n > 1, \quad \text{for all } \ n \geq N_3.$$

Also, since $x^n \to 0$ as $n \to \infty$, then there is a positive integer N_4 such that

$$x^n < 1, \quad \text{for all } \ n \geq N_4.$$

Therefore, taking $N = \max\{N_3, N_4\}$ we have

$$x^n < y^n, \quad \text{for all } \ n \geq N.$$

□

3.2 Second order modified PESN (SOPESN1) method for a general class of autonomous differential equations

The following result holds for the new SOPESN1 method for Equation (2.0.1):

Theorem 3.2.1. *Let $\vec{f} \in C^2(\mathbb{R}^n_+;\mathbb{R}^n)$ and let $\varphi_i : \mathbb{R}_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ satisfy the following:*

$$\varphi_i(h,\vec{x}) = h - q_i(\vec{x})\frac{h^2}{2} + \mathcal{O}(h^3), \tag{3.2.1}$$

where

$$q_i(\vec{x}) = \begin{cases} -\sum_{j=1}^{n} \dfrac{\partial f_i(\vec{x})}{\partial x_j}\dfrac{f_j(\vec{x})}{f_i(\vec{x})}, & f_i(\vec{x}) > 0 \\ \dfrac{2f_i(\vec{x})}{x_i} - \sum_{j=1}^{n} \dfrac{\partial f_i(\vec{x})}{\partial x_j}\dfrac{f_j(\vec{x})}{f_i(\vec{x})}, & f_i(\vec{x}) < 0 \end{cases}. \tag{3.2.2}$$

Then, the PESN1 method (2.0.5) *with deniminator function* (3.2.1)-(3.2.2)*:*

$$\frac{x_i^{k+1} - x_i^k}{\varphi_i(h,\vec{x}^k)} = w_i^k f_i(\vec{x}^k), \tag{3.2.3}$$

where

$$w_i^k = \begin{cases} 1, & \text{if } f_i(\vec{x}^k) \geq 0 \\ \dfrac{x_i^{k+1}}{x_i^k}, & \text{if } f_i(\vec{x}^k) < 0 \end{cases},$$

preserves the positivity of solutions of Equation (2.0.1)*, is elementary stable, and second-order accurate (SOPESN1 method).*

Proof. The proof is organized as follows. First, we prove that the numerical method is positivity preserving, then, that it is second order accurate, and finally, that it is elementary stable.

1. *Positivity*: The SOPESN1 method (3.2.3) can be written in the following explicit form:

$$x_i^{k+1} = \begin{cases} x_i^k + \varphi_i(h,\vec{x}_k) f_i(\vec{x}^k), & \text{if } f_i(\vec{x}^k) \geq 0 \\ \dfrac{(x_i^k)^2}{x_i^k - \varphi_i(h,\vec{x}^k) f_i(\vec{x}^k)}, & \text{if } f_i(\vec{x}^k) < 0 \end{cases}. \tag{3.2.4}$$

Note that if $f_i(\vec{x}^k) \geq 0$, then $\varphi_i(h, \vec{x}^k) f_i(\vec{x}^k) \geq 0$ and if $f_i(\vec{x}^k) < 0$, then $-\varphi_i(h, \vec{x}^k) f_i(\vec{x}^k) > 0$, since $\varphi_i(h, \vec{x}^k) > 0$. Therefore, the positivity of the SOPESN1 method follows from Equation (3.2.4), since $x_i^k > 0$ implies that $x_i^{k+1} > 0$, for all $i = 1, \ldots, n$.

2. *Second order of accuracy*: To prove the SOPESN1 method (3.2.3) is of second-order accuracy, we separately consider the following three cases:

   - If $f_i(\vec{x}) < 0$, then the SOPESN1 method (3.2.3) can be written in the explicit form
   $$x_i^{k+1} = \frac{(x_i^k)^2}{x_i^k - \varphi_i(h, \vec{x}^k) f_i(\vec{x}^k)}.$$
   Note that when $\varphi_i(h, \vec{x}) = h + \mathcal{O}(h^2)$ and $x_i \neq 0$, then
   $$\frac{\varphi_i(h, \vec{x}) f_i(\vec{x})}{x_i} \to 0 \quad \text{as} \quad h \to 0.$$
   Therefore, there exists $\delta_i > 0$ such that
   $$\left| \frac{\varphi_i(h, \vec{x}) f_i(\vec{x})}{x_i} \right| < 1,$$
   and consequently
   $$\frac{1}{1 - \dfrac{f_i(\vec{x}) \varphi_i(h, \vec{x})}{x_i}} = 1 + \frac{f_i(\vec{x}) \varphi_i(h, \vec{x})}{x_i} + \frac{f_i^2(\vec{x}) \varphi_i^2(h, \vec{x})}{x_i^2} + \mathcal{O}(h^3), \quad (3.2.5)$$
   whenever $0 < h < \delta_i$, for all $i = 1, \ldots, n$. Using Equation (3.2.5) and a Taylor series expansion about t_k yields the following:
   $$\begin{aligned}
   & x_i(t_{k+1}) - \frac{(x_i(t_k))^2}{x_i(t_k) - \varphi_i(h, \vec{x}(t_k)) f_i(\vec{x}(t_k))} \\
   & = x_i(t_{k+1}) - \frac{x_i(t_k)}{1 - \dfrac{\varphi_i(h, \vec{x}(t_k)) f_i(\vec{x}(t_k))}{x_i(t_k)}} \\
   & = x_i(t_{k+1}) - x_i(t_k) \left[1 + \frac{\varphi_i(h, \vec{x}(t_k)) f_i(\vec{x}(t_k))}{x_i(t_k)} + \frac{\varphi_i^2(h, \vec{x}(t_k)) (f_i(\vec{x}(t_k)))^2}{(x_i(t_k))^2} \right]
   \end{aligned}$$

$$
\begin{aligned}
&+ \mathcal{O}(h^3) \\
&= x_i'(t_k)h + x_i''(t_k)\frac{h^2}{2} - \varphi_i(h, \vec{x}(t_k))f_i(\vec{x}(t_k)) - \frac{\varphi_i^2(h, \vec{x}(t_k))f_i^2(\vec{x}(t_k))}{x_i(t_k)} \\
&+ \mathcal{O}(h^3) \\
&= f_i(\vec{x}(t_k))h + \frac{h^2}{2}\sum_{j=1}^{n}\frac{\partial f_i(\vec{x}(t_k))}{\partial x_j}f_j(\vec{x}(t_k)) \\
&\quad - \varphi_i(h, \vec{x}(t_k))f_i(\vec{x}(t_k)) - \frac{\varphi_i^2(h, \vec{x}(t_k))f_i^2(\vec{x}(t_k))}{x_i(t_k)} + \mathcal{O}(h^3).
\end{aligned}
$$

Now using $\varphi_i(h, \vec{x}) = h + \Big(- \dfrac{2f_i(\vec{x})}{x_i} + \displaystyle\sum_{j=1}^{n}\frac{\partial f_i(\vec{x})}{\partial x_j}\frac{f_j(\vec{x})}{f_i(\vec{x})}\Big)\frac{h^2}{2} + \mathcal{O}(h^3)$.

$$
x_i(t_{k+1}) - \frac{x_i^2(t_k)}{x_i(t_k) - \varphi_i(h, \vec{x}(t_k))f_i(\vec{x}(t_k))} = \mathcal{O}(h^3). \tag{3.2.6}
$$

- If $f_i(\vec{x}) > 0$, then the SOPESN1 method (3.2.3) can be written as

$$
x_i^{k+1} = x_i^k + \varphi_i(h, \vec{x}^k)f_i(\vec{x}^k).
$$

Using Taylor series expansion about t_k yields the following:

$$
\begin{aligned}
&x_i(t_{k+1}) - \Big[x_i(t_k) + \varphi_i(h, \vec{x}(t_k))f_i(\vec{x}(t_k))\Big] \\
&= x_i(t_k) + (x_i(t_k))'h + \Big(\sum_{j=1}^{n}\frac{\partial f_i(\vec{x}(t_k))}{\partial x_j}f_j(\vec{x}(t_k))\Big)\frac{h^2}{2} - x_i(t_k) \\
&- \varphi_i(h, \vec{x}(t_k))f_i(\vec{x}(t_k)) + \mathcal{O}(h^3) \\
&= f_i(\vec{x}(t_k))h + \Big(\sum_{j=1}^{n}\frac{\partial f_i(\vec{x}(t_k))}{\partial x_j}f_j(\vec{x}(t_k))\Big)\frac{h^2}{2} \\
&- \varphi_i(h, \vec{x}(t_k))f_i(\vec{x}(t_k)) + \mathcal{O}(h^3).
\end{aligned}
$$

Now using $\varphi_i(h, \vec{x}) = h + \Big(\displaystyle\sum_{j=1}^{n}\frac{\partial f_i(\vec{x}(t_k))}{\partial x_j}\frac{f_j(\vec{x}(t_k))}{f_i(\vec{x}(t_k))}\Big)\frac{h^2}{2} + \mathcal{O}(h^3)$ implies

$$
x_i(t_{k+1}) - \Big[x_i(t_k) + \varphi_i(h, \vec{x}(t_k))f_i(\vec{x}(t_k))\Big] = \mathcal{O}(h^3). \tag{3.2.7}
$$

- If $f_i(\vec{x}^k) = 0$, then the SOPESN1 method (3.2.3) reduces to $x_i^{k+1} = x_i^k$, which represents an exact scheme for solving Equation (2.0.1).

Equations (3.2.6) and (3.2.7) prove the second order accuracy of the SOPESN1 method (3.2.3).

3. *Elementary stability*: It is easy to see from the formulation (3.2.3) that all fixed points $\vec{x}^*$ of the SOPESN1 method are equilibria of Equation (2.0.1) and vice versa. To prove the numerical scheme (3.2.3) is elementary stable, let $\vec{x}^*$ be an equilibrium of system (2.0.1) and $J = J(\vec{x}^*)$ be the Jacobian evaluated at $\vec{x}^*$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. We assumed the Jacobian matrices to be diagonalizable, which is frequently considered in applications [3]. Setting $\vec{\epsilon} = \vec{x} - \vec{x}^*$, then linearized version of System (2.0.1) can be written as

$$\vec{\epsilon}' = J\vec{\epsilon}, \tag{3.2.8}$$

We consider the following three cases:

(a) **All eigenvalues of the matrix J are real:** Since J is diagonalizable, then there is nonsingular matrix P_r whose columns are the eigenvectors of J satisfying [19]

$$\Lambda_r = P_r^{-1} J P_r,$$

where $\Lambda_r = diag(\lambda_1, \ldots, \lambda_n)$. Now, let $\vec{\eta} = P_r^{-1}\vec{\epsilon}$ and multiplying $\vec{\epsilon}' = J\vec{\epsilon}$ by P_r^{-1} on both sides yields:

$$P_r^{-1}\vec{\epsilon}' = P_r^{-1} J\vec{\epsilon},$$

which is equivalent to

$$P_r^{-1}\vec{\epsilon}' = P_r^{-1}(P_r\Lambda_r P_r^{-1})\vec{\epsilon},$$

i.e,

$$\vec{\eta}' = \Lambda_r \vec{\eta}. \tag{3.2.9}$$

Following the approach in [3], we apply the numerical scheme (3.2.3) to (3.2.9).

First, let us assume $\vec{x}^*$ is a locally asymptotically stable equilibrium for (2.0.1). Then, $\lambda_i < 0$ for all $i = 1, \ldots, n$, and

$$\eta_i^{k+1} = \Big(\frac{\eta_i^k}{\eta_i^k - \varphi_i(h, \vec{\eta}^k)\lambda_i \eta_i^k}\Big)\eta_i^k,$$

since $\varphi_i(h, \vec{\eta}^k) > 0$ we can conclude

$$\Big|\frac{\eta_i^k}{\eta_i^k - \varphi_i(h, \vec{\eta}^k)\lambda_i \eta_i^k}\Big| < 1.$$

which shows that $\vec{x}^*$ is locally asymptotically stable for Scheme (3.2.3). Next, if $\vec{x}^*$ is unstable, then $\lambda_{j_0} > 0$ for some $j_0 \in \{1, \ldots, n\}$. After applying the numerical scheme to (3.2.9) yields:

$$\eta_{j_0}^{k+1} = \Big(1 + \varphi_{j_0}(h, \vec{\eta}^k)\lambda_{j_0}\Big)\eta_{j_0}^k,$$

since $\varphi_{j_0}(h, \vec{\eta}^k) > 0$, then clearly we have

$$\Big|1 + \varphi_{j_0}(h, \vec{\eta}^k)\lambda_{j_0}\Big| > 1.$$

Therefore, $\vec{x}^*$ is unstable for the numerical scheme (3.2.3).

(b) **All eigenvalues of J are complex with $\mathrm{Im}(\lambda_i) \neq 0$ for all $i = 1, \ldots, n$:** Since J is a real matrix, we note if λ_i is an eigenvalue of J, then $\bar{\lambda_i}$ (complex conjugate) is also an eigenvalue of J. Therefore, n must be an even positive integer. We recall there is a invertible matrix P_c satisfying

$$\Lambda_c = P_c^{-1} J P_c.$$

The Jordan normal form satisfies [19]

$$\Lambda_c = diag(B_{\lambda_1}, \ldots, B_{\lambda_n}),$$

where $B_{\lambda_i} = \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}$ with $\mathrm{R}(\lambda_i) = a_i$ and $\mathrm{Im}(\lambda_i) = b_i$. Similar to the argument in the Case (a), we apply the numerical scheme (3.2.3) to

$$\vec{\eta}' = \Lambda_c \vec{\eta}. \tag{3.2.10}$$

First, if $\vec{x}^*$ is a locally asymptotically stable equilibrium, then $a_i < 0$ for all $i = 1, \ldots, n$. Let $i \in \{1, \ldots, n\}$, then $a_i < 0$ and b_i is either positive or negative.

- If $b_i < 0$, then

  $$\eta_i^{k+1} = \Big(\frac{\eta_i^k}{\eta_i^k - \varphi_i(h, \vec{\eta}^k)(a_i\eta_i^k + b_i\eta_{i+1}^k)}\Big)\eta_i^k,$$

  since $a_i\eta_i^k + b_i\eta_{i+1}^k < 0$, and η_i^k, η_{i+1}^k are positive, then

  $$\eta_i^k - \varphi_i(h, \vec{\eta}^k)(a_i\eta_i^k + b_i\eta_{i+1}^k) > \eta_i^k.$$

  Hence,

  $$0 < \frac{\eta_i^k}{\eta_i^k - \varphi_i(h, \vec{\eta}^k)(a_i\eta_i^k + b_i\eta_{i+1}^k)} < 1.$$

  i.e.

  $$\Big|\frac{\eta_i^k}{\eta_i^k - \varphi_i(h, \vec{\eta}^k)(a_i\eta_i^k + b_i\eta_{i+1}^k)}\Big| < 1.$$

  Hence, $\eta_i^k \to 0$ as $k \to \infty$. We claim $\eta_{i+1}^k \to 0$ as $k \to \infty$. Let $k+1 \in \mathbb{N}$. We consider two cases:

  Case (b.I): $-b_i\eta_i^k + a_i\eta_{i+1}^k < 0$, then after applying the numerical scheme to (3.2.10) we have:

  $$\eta_{i+1}^{k+1} = \Big(\frac{\eta_{i+1}^k}{\eta_{i+1}^k - \varphi_{i+1}(h, \vec{\eta}^k)(-b_i\eta_i^k + a_i\eta_{i+1}^k)}\Big)\eta_{i+1}^k,$$

and $\left|\frac{\eta_{i+1}^k}{\eta_{i+1}^k-\varphi_{i+1}(h,\vec{\eta}^k)(-b_i\eta_i^k+a_i\eta_{i+1}^k)}\right|<1$ which implies $\eta_{i+1}^{k+1}\to 0$ as $k\to\infty$.
Case (b.II): $-b_i\eta_i^k+a_i\eta_{i+1}^k\geq 0$:

$$\begin{aligned}\eta_{i+1}^{k+1}&=\eta_{i+1}^k+\varphi_{i+1}(h,\vec{\eta}^k)\Big(-b_i\eta_i^k+a_i\eta_{i+1}^k\Big)\\&\geq\eta_{i+1}^k.\end{aligned}$$

Therefore, $\left\{\eta_{i+1}^k\right\}$ is a monotonically increasing sequence and hence it either converges to a positive number or tends to infinity as $k\to\infty$. Therefore, by Lemma (3.1.1) there is $N_1\in\mathbb{N}$ such that $m\geq N_1$ implies

$$-b_i\eta_i^m<-a_i\eta_{i+1}^m,$$

after applying the numerical scheme to Equation (3.2.10), we have:

$$\eta_{i+1}^{m+1}=\Big(\frac{\eta_{i+1}^m}{\eta_{i+1}^m-\varphi_{i+1}(h,\vec{\eta}^m)(-b_i\eta_i^m+a_i\eta_{i+1}^m)}\Big)\eta_{i+1}^m,$$

and since

$$\Big|\frac{\eta_{i+1}^m}{\eta_{i+1}^m-\varphi_{i+1}(h,\vec{\eta}^m)(-b_i\eta_i^m+a_i\eta_{i+1}^m)}\Big|<1,$$

then that implies $\eta_{i+1}^{m+1}\to 0$ as $m\to\infty$. This contradicts the limit is either positive or ∞. Therefore, the case $-b_i\eta_i^k+a_i\eta_{i+1}^k\geq 0$ is not possible for a large enough k.

- If $b_i>0$, then

$$\eta_{i+1}^{k+1}=\Big(\frac{\eta_{i+1}^k}{\eta_{i+1}^k-\varphi_{i+1}(h,\vec{\eta}^k)(-b_i\eta_i^k+a_i\eta_{i+1}^k)}\Big)\eta_{i+1}^k,$$

since $-b_i\eta_i^k+a_i\eta_{i+1}^k<0$, and η_i^k, η_{i+1}^k are positive, then

$$\eta_{i+1}^k-\varphi_{i+1}(h,\vec{\eta}^k)(-b_i\eta_i^k+a_i\eta_{i+1}^k)>\eta_{i+1}^k.$$

Hence,

$$0<\frac{\eta_{i+1}^k}{\eta_{i+1}^k-\varphi_{i+1}(h,\vec{\eta}^k)(-b_i\eta_i^k+a_i\eta_{i+1}^k)}<1.$$

i.e.

$$\Big|\frac{\eta_{i+1}^k}{\eta_{i+1}^k - \varphi_{i+1}(h,\vec{\eta}^k)(-b_i\eta_i^k + a_i\eta_{i+1}^k)}\Big| < 1.$$

Therefore, $\eta_{i+1}^k \to 0$ as $k \to \infty$ for all $i = 1, \ldots, n$. We claim $\eta_i^k \to 0$ as $k \to \infty$. Suppose for the sake of a contradiction not. Recall $\{\eta_i^k\}$ is a positive sequence, similar argument as above η_i^k converges to a positive number or tends to infinity. Using Lemma (3.1.1) we can conclude for large enough k we have

$$b_i\eta_{i+1}^k < -a\eta_i^k,$$

applying the numerical scheme to (3.2.10) yields:

$$\eta_i^{k+1} = \Big(\frac{\eta_i^k}{\eta_i^k - \varphi_i(h,\vec{\eta}^k)(a_i\eta_i^k + b_i\eta_{i+1}^k)}\Big)\eta_i^k,$$

but since $a_i\eta_i^k + b_i\eta_{i+1}^k < 0$ and η_i^k, η_{i+1}^k are all positive, then

$$\eta_i^k - \varphi_i(h,\vec{\eta}^k)(a_i\eta_i^k + b_i\eta_{i+1}^k) > \eta_i^k.$$

Hence,

$$0 < \frac{\eta_i^k}{\eta_i^k - \varphi_i(h,\vec{\eta}^k)(a_i\eta_i^k + b_i\eta_{i+1}^k)} < 1,$$

which implies

$$\Big|\frac{\eta_i^k}{\eta_i^k - \varphi_i(h,\vec{\eta}^k)(a_i\eta_i^k + b_i\eta_{i+1}^k)}\Big| < 1.$$

Therefore, $\eta_i^k \to 0$ as $k \to \infty$ contradicts the limit is either positive or infinity. Therefore, $\eta_i^k \to 0$ as $k \to \infty$. Since i was an arbitrary, we can conclude $\eta_i^k \to 0$ as $k \to \infty$ for all $i = 1, \ldots, n$. Therefore, $\vec{x}^*$ is locally asymptotically stable for Scheme (3.2.3).

Next, let us assume $\vec{x}^*$ is unstable. Then, there is an eigenvalues λ_{j_0} of J with $a_{j_0} > 0$. Now after applying the numerical scheme to Equation (3.2.10) we consider two cases:

- If $b_{j_0} < 0$,

$$\begin{aligned}\eta_{j_0+1}^{k+1} &= \eta_{j_0+1}^{k} + \varphi_{j_0+1}(h, \vec{\eta}^{k})(-b_{j_0}\eta_{j_0}^{k} + a_{j_0}\eta_{j_0+1}^{k}) \\ &= \Big(1 + a_{j_0}\varphi_{j_0+1}(h, \vec{\eta}^{k})\Big)\eta_{j_0+1}^{k} - \varphi_{j_0+1}(h, \vec{\eta}^{k})b_{j_0}\eta_{j_0}^{k} \\ &> \Big(1 + a_{j_0}\varphi_{j_0+1}(h, \vec{\eta}^{k})\Big)\eta_{j_0+1}^{k}.\end{aligned}$$

  Since $\Big|1 + a_{j_0}\varphi_{j_0+1}(h, \vec{\eta}^{k})\Big| > 1$, then $\Big(1 + a_{j_0}\varphi_{j_0+1}(h, \vec{\eta}^{k})\Big)\eta_{j_0+1}^{k} \to \infty$ as $k \to \infty$. Therefore, $\eta_{j_0+1}^{k+1} \to \infty$ as $k \to \infty$.

- $b_{j_0} > 0$, similar argument as above will show $\eta_{j_0}^{k} \to \infty$ as $k \to \infty$.

Therefore, $\vec{x}^*$ is unstable fixed point of the numerical scheme (3.2.3).

(c) J **has both real and complex eigenvalues:** Suppose $\lambda_1, \ldots, \lambda_k$ are the real eigenvalues of J and $\lambda_j = a_j + ib_j$ and $\bar{\lambda}_j = a_j - ib_j$ for $j = k+1, \ldots, \frac{n+k}{2}$. Then, there is a nonsingular matrix P_{rc} satisfying [19]

$$\Lambda_{rc} = P_{rc}^{-1} J P_{rc},$$

Similar to the argument in Case (a), we apply the numerical scheme to

$$\vec{\eta}' = \Lambda_{rc}\vec{\eta}, \tag{3.2.11}$$

where $J_{rc} = diag(B_{\lambda_1}, \ldots, B_{\lambda_n})$ and

$$B_{\lambda_i} = \begin{cases} \lambda_i, \ \mathrm{Im}(\lambda_i) = 0 \\ \begin{pmatrix} \mathrm{R}(\lambda_i) & \mathrm{Im}(\lambda_i) \\ -\mathrm{Im}(\lambda_i) & \mathrm{R}(\lambda_i) \end{pmatrix}, \ \mathrm{Im}(\lambda_i) \neq 0 \end{cases}.$$

Now let $\vec{x}^*$ is a stable equilibrium point of System (2.0.1). Then, $\mathrm{R}(\lambda_i) < 0$ for all i. To prove $\vec{x}^*$ is a stable fixed point. We note that if $\mathrm{Im}(\lambda_i) = 0$, we use same argument as in Case (a) to show $\eta_i^{k+1} \to 0$ as $k \to \infty$. If $\mathrm{Im}(\lambda_i) \neq 0$, we use same argument used in Case (b) to show $\eta_i^{k+1} \to 0$

as $k \to \infty$. Hence, $\vec{x}^*$ is locally asymptotically stable for Scheme (3.2.3). Finally, if $\vec{x}^*$ is unstable for (2.0.1), then, there is λ_{j_0} with $\mathrm{R}(\lambda_{j_0}) > 0$ for some $j_0 \in \{1, \ldots, n\}$. Also, $\mathrm{Im}(\lambda_{j_0}) = 0$ or $\mathrm{Im}(\lambda_i) \neq 0$ using similar argument as in Case (a) or Case (b), respectively, will show that $\vec{x}^*$ is an unstable fixed point for Scheme (3.2.3).

□

The following corollary is for the SOPESN1 method when $n = 1$.

Corollary 3.2.1. *Let $f \in C^2(\mathbb{R}_+; \mathbb{R})$ and let $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the following specific form:*

$$\varphi(h, x) = h - q(x)\frac{h^2}{2} + \mathcal{O}(h^3), \tag{3.2.12}$$

where

$$q(x) = \begin{cases} -f_x(x), & f(x) > 0 \\ \dfrac{2f(x)}{x} - f_x(x), & f(x) < 0 \end{cases}. \tag{3.2.13}$$

Then, the PESN1 method (2.0.5) *with deniminator function* (3.2.12)-(3.2.13)*:*

$$\frac{x^{k+1} - x^k}{\varphi(h, x^k)} = w^k f(x^k), \tag{3.2.14}$$

where

$$w^k = \begin{cases} 1, & \text{if } f(x^k) \geq 0 \\ \dfrac{x^{k+1}}{x^k}, & \text{if } f(x^k) < 0 \end{cases},$$

preserves the positivity of solutions of Equation (2.0.1) *when $n = 1$, is elementary stable, and second-order accurate (SOPESN1 method).*

3.3 Second order modified PESN (SOPESN2) method for n-dimensional productive-destructive systems

The following result holds for the new SOPESN2 method for Equation (2.0.2):

Theorem 3.3.1. *Let $\vec{P}, \vec{D} \in C^2(\mathbb{R}^n_+; \mathbb{R}^n_+)$ and $\varphi_i : \mathbb{R}_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ satisfies the following condition:*

$$\varphi_i(h, \vec{x}) = h - q_i(\vec{x})\frac{h^2}{2} + \mathcal{O}(h^3), \tag{3.3.1}$$

where

$$q_i(\vec{x}) = -\sum_{j=1}^{n} \Big(\frac{\partial P_i(\vec{x})}{\partial x_j} - \frac{\partial D_i(\vec{x})}{\partial x_j}\Big)\Big(\frac{P_j(\vec{x}) - D_j(\vec{x})}{P_i(\vec{x}) - D_i(\vec{x})}\Big) - \frac{2D_i(\vec{x})}{x_i}, \tag{3.3.2}$$

whenever $D_i(\vec{x}) \neq P_i(\vec{x})$. Then, the PESN2 method (2.0.6) *with deniminator function* (3.3.1)-(3.3.2)*:*

$$\frac{x_i^{k+1} - x_i^k}{\varphi_i(h, \vec{x}^k)} = P_i(\vec{x}^k) - D_i(\vec{x}^k)\frac{x_i^{k+1}}{x_i^k}, \tag{3.3.3}$$

preserves the positivity of solutions of Equation (2.0.2)*, is elementary stable, and second-order accurate (SOPESN2 method).*

Proof. The proof is organized similarly to the proof of Theorem 3.2.1.

1. *Positivity*: The positivity of the SOPESN2 method follows from the fact that when $\vec{x}^k \in \mathbb{R}^n_+$, then $P_i(\vec{x}^k) > 0$ and $D_i(\vec{x}^k) > 0$ for all $i = 1, \ldots, n$, and therefore,
$$x_i^{k+1} = \frac{x_i^k\Big(x_i^k + \varphi_i(h, \vec{x}^k)P_i(\vec{x}^k)\Big)}{x_i^k + \varphi_i(h, \vec{x}^k)D_i(\vec{x}^k)} > 0,$$
since $\varphi_i(h, \vec{x}^k) > 0$.
2. *Second order of accuracy*: To prove the SOPESN2 method is of second-order accuracy, we consider the following two cases:
   - If $D_i(\vec{x}^k) \neq P_i(\vec{x}^k)$, then the SOPESN2 method (3.3.3) can be written in the following explicit form:
$$x_i^{k+1} = \frac{x_i^k\Big(x_i^k + \varphi_i(h, \vec{x}^k)P_i(\vec{x}^k)\Big)}{x_i^k + \varphi_i(h, \vec{x}^k)D_i(\vec{x}^k)}.$$

Note that when $\varphi_i(h,\vec{x}) = h + \mathcal{O}(h^2)$ and $x_i \neq 0$, then there exists $\delta_i > 0$ such that

$$\frac{1}{1+\dfrac{D_i(\vec{x})\varphi_i(h,\vec{x})}{x_i}} = 1 - \frac{D_i(\vec{x})\varphi_i(h,\vec{x})}{x_i} + \frac{D_i^2(\vec{x})\varphi_i^2(h,\vec{x})}{x_i^2} + \mathcal{O}(h^3), \quad (3.3.4)$$

whenever $0 < h < \delta_i$. Using Equation (3.3.4) and a Taylor series expansion about t_k yields the following

$$\begin{aligned}
&x_i(t_{k+1}) - \frac{x_i(t_k) + \varphi_i(h,\vec{x}(t_k))P_i(\vec{x}(t_k))}{1+\frac{\varphi_i(h,\vec{x}(t_k))D_i(\vec{x}(t_k))}{x_i(t_k)}} \\
&= x_i(t_{k+1}) - [x_i(t_k) + P_i(\vec{x}(t_k))\varphi_i(h,\vec{x}(t_k))]\left[1 - \frac{D_i(\vec{x}(t_k))\varphi_i(h,\vec{x}(t_k))}{x_i(t_k)}\right. \\
&\quad \left. + \left(\frac{D_i(\vec{x}(t_k))\varphi_i(h,\vec{x}(t_k))}{x_i(t_k)}\right)^2\right] + \mathcal{O}(h^3) \\
&= x_i(t_{k+1}) - x_i(t_k) + D_i(\vec{x}(t_k))\varphi_i(h,\vec{x}(t_k)) - \frac{D_i^2(\vec{x}(t_k))\varphi_i^2(h,\vec{x}(t_k))}{x_i(t_k)} \\
&\quad - P_i(x(t_k))\varphi_i(h,\vec{x}(t_k)) + \frac{P_i(\vec{x}(t_k))D_i(\vec{x}(t_k))\varphi_i^2(h,\vec{x}(t_k))}{x_i(t_k)} + \mathcal{O}(h^3) \\
&= x_i'(t_k)h + x_i''(t_k)\frac{h^2}{2} + D_i(\vec{x}(t_k))\varphi_i(h,\vec{x}(t_k)) - \frac{D_i^2(\vec{x}(t_k))\varphi_i^2(h,\vec{x}(t_k))}{x_i(t_k)} \\
&\quad - P_i(\vec{x}(t_k))\varphi_i(h,\vec{x}(t_k)) + \frac{P_i(\vec{x}(t_k))D_i(\vec{x}(t_k))\varphi_i^2(h,\vec{x}(t_k))}{x_i(t_k)} + \mathcal{O}(h^3) \\
&= \Big(P_i(\vec{x}(t_k)) - D_i(\vec{x}(t_k))\Big)h - \varphi_i(h,\vec{x}(t_k))\Big(P_i(\vec{x}(t_k)) - D_i(\vec{x}(t_k))\Big) \\
&\quad + \sum_{j=1}^{n}\Big(\frac{\partial P_i(\vec{x}(t_k))}{\partial x_j} - \frac{\partial D_i(\vec{x}(t_k))}{\partial x_j}\Big)\Big(P_j(\vec{x}(t_k)) - D_j(\vec{x}(t_k))\Big)\frac{h^2}{2} \\
&\quad + \varphi_i^2(h,\vec{x}(t_k))\frac{D_i(\vec{x}(t_k))}{x_i(t_k)}\Big(P_i(\vec{x}(t_k)) - D_i(\vec{x}(t_k))\Big) + \mathcal{O}(h^3).
\end{aligned}$$

Now using Equations (3.3.1)-(3.3.2) for $\varphi_i(h, x(t_n))$ and substituting in the above expression results in

$$x_i(t_{k+1}) - \frac{x_i(t_k) + \varphi_i(h,\vec{x}(t_k))P_i(\vec{x}(t_k))}{1+\dfrac{\varphi_i(h,\vec{x}(t_k))D_i(\vec{x}(t_k))}{x_i(t_k)}} = \mathcal{O}(h^3).$$

- If $D_i(\vec{x}^k) = P_i(\vec{x}^k)$, then the SOPESN2 method (3.3.3) reduces to $x_i^{k+1} = x_i^k$, which represents an exact scheme for solving Equation (2.0.2) at $t = t_k$.

The above two cases prove the second-order accuracy of the SOPESN2 method (3.3.3).

3. *Elementary stability*: It is easy to see from the formulation (3.3.3) that all fixed points x^* of the SOPESN2 method are equilibria of Equation (2.0.2) and vice versa. To prove the numerical scheme (3.3.3) is elementary stable, let $\vec{x}^*$ be an equilibrium of Equation (2.0.2) and $J = J(x^*)$ be the Jacobian evaluated at $\vec{x}^*$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. We again assumed the Jacobian matrices to be diagonalizable. Setting $\vec{\epsilon} = \vec{x} - \vec{x}^*$, then the linearized version of System (2.0.2) can be written as

$$\vec{\epsilon}' = J\vec{\epsilon}. \tag{3.3.5}$$

We consider the following three cases:

(a) **All eigenvalues of the matrix J are real:** Since J is diagonalizable, then there is nonsingular matrix P_r whose columns are the eigenvectors of J satisfying [19]

$$\Lambda_r = P_r^{-1} J P_r,$$

where $\Lambda_r = diag(\lambda_1, \ldots, \lambda_n)$. Now let $\vec{\eta} = P_r^{-1}\vec{\epsilon}$. Multiplying Equation (3.3.5) by P_r^{-1} on both sides yields:

$$P_r^{-1}\vec{\epsilon}' = P_r^{-1} J\vec{\epsilon},$$

which is equivalent to

$$P_r^{-1}\vec{\epsilon}' = P_r^{-1}(P_r\Lambda_r P_r^{-1})\vec{\epsilon},$$

i.e,

$$\vec{\eta}' = \Lambda_r\vec{\eta}. \tag{3.3.6}$$

Following the approach in [3], we apply the numerical scheme to Equation (3.3.6). First, let us assume $\vec{x}^*$ be a locally asymptotically stable equilibrium for (2.0.2). Then, $\lambda_i < 0$ for all $i = 1, \ldots, n$, and

$$\eta_i^{k+1} = \left(\frac{1}{1 - \varphi_i(h, \vec{\eta}^k)\lambda_i} \right) \eta_i^k.$$

To show $\vec{x}^*$ is a locally asymptotically stable fixed point for Method (3.3.3) is equivalent to show:

$$\left| \frac{1}{1 - \varphi_i(h, \vec{\eta}^k)\lambda_i} \right| < 1.$$

Since $\lambda_i < 0$ and $\varphi_i(h, \vec{\eta}^k) > 0$, we have $1 - \varphi_i(h, \vec{\eta}^k)\lambda_i > 1$ which implies

$$0 < \frac{1}{1 - \varphi_i(h, \vec{\eta}^k)\lambda_i} < 1. \tag{3.3.7}$$

Therefore, by the inequality (3.3.7) and since i is arbitrary we can conclude that $\vec{x}^*$ is a stable fixed point of the numerical scheme (3.3.3).

Next, if $\vec{x}^*$ is an unstable equilibrium for (2.0.2), we need to show $\vec{x}^*$ is an unstable fixed point of the numerical scheme (3.3.3). But, since $\vec{x}^*$ is unstable, then there is an eigenvalue λ_{j_0} of J such that $\lambda_{j_0} > 0$, after applying the numerical scheme (3.3.3) to Equation (3.3.6) yields:

$$\eta_{j_0}^{k+1} = \Big(1 + \varphi_{j_0}(h, \vec{\eta}^k)\lambda_{j_0}\Big)\eta_{j_0}^k.$$

Since $\lambda_{j_0} > 0$, then $\Big|1 + \varphi_{j_0}(h, \vec{\eta}^k)\lambda_{j_0}\Big| > 1$. Therefore, $\vec{x}^*$ is an unstable fixed point of the numerical method (3.3.3).

(b) **All eigenvalues of J are complex with $\mathrm{Im}(\lambda_i) \neq 0$ for all $i = 1, \ldots, n$:** Since J is a real matrix, we note if λ_i is an eigenvalue of J, then $\bar{\lambda_i}$ (complex conjugate) is also an eigenvalue of J. Therefore, n must be an even positive integer. We recall there is a invertible matrix P_c satisfying [19]

$$\Lambda_c = P_c^{-1} J P_c.$$

The Jordan normal form satisfies

$$\Lambda_c = diag(B_{\lambda_1}, \ldots, B_{\lambda_n}),$$

where $B_{\lambda_i} = \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}$ with $\mathrm{R}(\lambda_i) = a_i$ and $\mathrm{Im}(\lambda_i) = b_i$. Similar to the argument in Case (a) we apply the numerical scheme to

$$\vec{\eta}' = J_c \vec{\eta}. \tag{3.3.8}$$

First, if $\vec{x}^*$ is a locally asymptotically stable equilibrium for (3.3.3), then $a_i < 0$ for all $i = 1, \ldots, n$. Let $i \in \{1, \ldots, n\}$, then $a_i < 0$ and b_i is either positive or negative.

- If $b_i < 0$, then

$$\eta_i' = a_i \eta_i + b_i \eta_{i+1},$$
$$\eta_{i+1}' = -b_i \eta_i + a_i \eta_{i+1}.$$

  Since $P_i(\vec{\eta}^k) = 0$ and $D_i(\vec{\eta}^k) = -(a_i \eta_i^k + b_i \eta_{i+1}^k)$, after applying the numerical scheme to (3.3.8) yields:

$$\eta_i^{k+1} = \Big(\frac{\eta_i^k}{\eta_i^k - \varphi_i(h, \vec{\eta}^k)(a_i \eta_i^k + b_i \eta_{i+1}^k)} \Big) \eta_i^k.$$

  Using that $a_i \eta_i^k + b_i \eta_{i+1}^k < 0$ and $\varphi_i(h, \vec{\eta}^k) > 0$ we can conclude that

$$\Big| \frac{\eta_i^k}{\eta_i^k - \varphi_i(h, \vec{\eta}^k)(a_i \eta_i^k + b_i \eta_{i+1}^k)} \Big| < 1.$$

  That implies $\eta_i^{k+1} \to 0$ as $k \to \infty$. Now we claim $\eta_{i+1}^{k+1} \to 0$ as $k \to \infty$. Now, suppose for the sake of a contradiction that this is not true. Since

$$\eta_{i+1}' = -b_i \eta_i + a_i \eta_{i+1},$$

then $P_{i+1}(\vec{\eta}^k) = -b_i\eta_i^k$ and $D_{i+1}(\vec{\eta}^k) = -a_i\eta_{i+1}^k$. Therefore,

$$\eta_{i+1}^{k+1} = \Big(\frac{\eta_{i+1}^k - \varphi_{i+1}(h,\vec{\eta}^k)b_i\eta_i^k}{\eta_{i+1}^k - \varphi_{i+1}(h,\vec{\eta}^k)a_i\eta_{i+1}^k}\Big)\eta_{i+1}^k,$$

Now since $\eta_{i+1}^{k+1} \nrightarrow 0$ as $k \to \infty$, then

$$\frac{\eta_{i+1}^k - \varphi_{i+1}(h,\vec{\eta}^k)b_i\eta_i^k}{\eta_{i+1}^k - \varphi_{i+1}(h,\vec{\eta}^k)a_i\eta_{i+1}^k} \geq 1,$$

thus $\eta_{i+1}^{k+1} \geq \eta_{i+1}^k$. That implies $\left\{\eta_{i+1}^{k+1}\right\}$ is a monotonically increasing positive sequence. Therefore, $\eta_{i+1}^{k+1} \to b \in \mathbb{R}_+ \cup \{\infty\}$ as $k \to \infty$. Using Lemma (3.1.1) for large enough k we have:

$$-b_i\eta_i^k < -a_i\eta_{i+1}^k,$$

hence, $P_i(\vec{\eta}^k) = 0$ and $D_i(\vec{\eta}^k) = -(-b_i\eta_i^k + a_i\eta_{i+1}^k)$.

Now applying the numerical method to Equation (3.3.8) yields:

$$\eta_{i+1}^{k+1} = \Big(\frac{\eta_{i+1}^k}{\eta_{i+1}^k - \varphi_i(h,\vec{\eta}^k)(-b_i\eta_i^k + a_i\eta_{i+1}^k)}\Big)\eta_{i+1}^k,$$

and $\left|\frac{\eta_{i+1}^k}{\eta_{i+1}^k - \varphi_i(h,\vec{\eta}^k)(-b_i\eta_i + a_i\eta_{i+1})}\right| < 1$. That implies $\eta_{i+1}^k \to 0$ as $k \to \infty$ contradicts $\eta_{i+1}^k \to b \in \mathbb{R}_+ \cup \{\infty\}$. Therefore, $\eta_{i+1}^k \to 0$ as $k \to \infty$.

- $b_i > 0$, similar argument as above shows that $\eta_i^k \to 0$ and $\eta_{i+1}^k \to 0$ as $k \to \infty$. Therefore, $\vec{x}^*$ is a locally asymptotically stable fixed point of the numerical scheme (3.3.3).

Next, if $\vec{x}^*$ is unstable equilibrium for System (2.0.2), then there is an eigenvalue λ_{j_0} such that $\mathrm{R}(\lambda_{j_0}) > 0$. After applying the numerical scheme to Equation (3.3.8) we consider the following two cases:

- If $b_{j_0} < 0$, then

$$\begin{aligned}\eta_{j_0+1}^{k+1} &= \eta_{j_0+1}^k + \varphi_{j_0+1}(h,\vec{\eta}^k)P_{j_0+1}(\vec{\eta}^k) \\ &= \eta_{j_0+1}^k + \varphi_{j_0+1}(h,\vec{\eta}^k)(-b_{j_0}\eta_{j_0}^k + a_{j_0}\eta_{j_0+1}^k)\end{aligned}$$

$$
\begin{aligned}
&> \eta_{j_0+1}^k + \varphi_{j_0+1}(h, \vec{\eta}^k) a_{j_0} \eta_{j_0+1}^k \\
&= \Big(1 + \varphi_{j_0+1}(h, \vec{\eta}^k) a_{j_0}\Big) \eta_{j_0+1}^k,
\end{aligned}
$$

But since $\Big|1 + \varphi_{j_0+1}(h, \vec{\eta}^k) a_{j_0}\Big| > 1$, then clearly $\eta_{j_0+1}^k \to \infty$ as $k \to \infty$.

- If $b_{j_0} > 0$, then

$$
\begin{aligned}
\eta_{j_0}^{k+1} &= \eta_{j_0}^k + \varphi_{j_0}(h, \vec{\eta}^k) P_{j_0}(\vec{\eta}^k) \\
&= \eta_{j_0}^k + \varphi_{j_0}(h, \vec{\eta}^k)(a_{j_0} \eta_{j_0}^k + b_{j_0} \eta_{j_0+1}^k) \\
&> \eta_{j_0}^k + \varphi_{j_0}(h, \vec{\eta}^k) a_{j_0} \eta_{j_0}^k \\
&= \Big(1 + \varphi_{j_0}(h, \vec{\eta}^k) a_{j_0}\Big) \eta_{j_0}^k.
\end{aligned}
$$

Therefore, since $\Big|1 + \varphi_{j_0}(h, \vec{\eta}^k) a_{j_0}\Big| > 1$, then $\eta_{j_0}^k \to \infty$ as $k \to \infty$.

This implies that $\vec{x}^*$ is an unstable fixed point of the numerical method (3.3.3).

(c) J **has both real and complex eigenvalues:** $\lambda_1, \ldots, \lambda_k$ are the real eigenvalues of J and $\lambda_j = a_j + ib_j$ and $\bar{\lambda}_j = a_j - ib_j$ for $j = k+1, \ldots, \frac{n+k}{2}$. Then, there is a nonsingular matrix P_{rc} satisfying [19]

$$
\Lambda_{rc} = P_{rc} A P_{rc}^{-1},
$$

where $\Lambda_{rc} = diag(B_{\lambda_1}, \ldots, B_{\lambda_n})$ and

$$
B_{\lambda_i} = \begin{cases} \lambda_i, \ \mathrm{Im}(\lambda_i) = 0 \\ \begin{pmatrix} \mathrm{R}(\lambda_i) & \mathrm{Im}(\lambda_i) \\ -\mathrm{Im}(\lambda_i) & \mathrm{R}(\lambda_i) \end{pmatrix}, \ \mathrm{Im}(\lambda_i) \neq 0 \end{cases}.
$$

Similar to the argument in Case (a) we apply the numerical scheme to

$$
\vec{\eta}' = J_{rc} \vec{\eta}. \tag{3.3.9}
$$

Now let $\vec{x}^*$ be a locally asymptotically stable equilibrium point of (2.0.2), then $\mathrm{R}(\lambda_i) < 0$ for all i. To prove $\vec{x}^*$ is a locally asymptotically stable fixed point for Scheme (3.3.3). We note that if $\mathrm{Im}(\lambda_i) = 0$, then we use the same argument as in Case (a) to show $\eta_i^{k+1} \to 0$ as $k \to \infty$. If $\mathrm{Im}(\lambda_i) \neq 0$, we use the same argument as in Case (b). That shows $\vec{x}^*$ is a locally asymptotically stable fixed point for Scheme (3.3.3). Finally, if $\vec{x}^*$ is an unstable equilibrium for System (2.0.2), then there is λ_{j_0} with $\mathrm{R}(\lambda_{j_0}) > 0$ for some $j_0 \in \{1, \ldots, n\}$. Also, either $\mathrm{Im}(\lambda_{j_0}) = 0$ or $\mathrm{Im}(\lambda_{j_0}) \neq 0$ using a similar argument as in Case (a) or Case (b), respectively, shows that $\vec{x}^*$ is an unstable fixed point for Scheme (3.3.3). Therefore, the numerical method (3.3.3) is elementary stable.

□

The following corollary is for SOPESN2 method when $n = 1$.

Corollary 3.3.1. *Let P, $D \in C^2(\mathbb{R}_+; \mathbb{R}_+)$ and $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following condition:*

$$\varphi(h, x) = h - q(x)\frac{h^2}{2} + \mathcal{O}(h^3), \tag{3.3.10}$$

where

$$q(x) = D_x(x) - P_x(x) - \frac{2D(x)}{x}. \tag{3.3.11}$$

Then, the PESN2 method (2.0.6) *with deniminator function* (3.3.10)-(3.3.11)*:*

$$\frac{x^{k+1} - x^k}{\varphi(h, x^k)} = P(x^k) - D(x^k)\frac{x^{k+1}}{x^k}, \tag{3.3.12}$$

preserves the positivity of solutions of Equation (2.0.2) *when $n = 1$, is elementary stable, and second-order accurate (SOPESN2 method).*

Remark 3.3.1. There exists many functions that satisfy the hypothesis of Theorems (3.2.1) and (3.3.1), e.g., one can choose

$$\varphi_i(h, \vec{x}^k) = \begin{cases} \frac{1-e^{-q_i(\vec{x}^k)h}}{q_i(\vec{x}^k)}, & \text{if } q_i(\vec{x}^k) \neq 0 \\ \\ h, & \text{if } q_i(\vec{x}^k) = 0 \end{cases} . \tag{3.3.13}$$

$\varphi_i(h, \vec{x}^k) = \dfrac{1 - e^{-q_i(\vec{x}^k)h}}{q_i(\vec{x}^k)}$, with $q_i(\vec{x}^k)$ as in Equation (3.2.2) and as in Equation (3.3.2), respectively.

CHAPTER 4

Chemostat Models with microbial input

4.1 Introduction

For many years, chemostat models have been used to mathematically model many important biological problems. For instance, the time dynamic of one bacteria B and the substrate S in the chemostat can be modeled by the following two differential equations [44, 39]:

$$\begin{aligned} \frac{dS}{dt} &= \underbrace{DS_{in}}_{\text{input}} - \underbrace{DS}_{\text{dilution}} - \underbrace{q\mu(S)B}_{\text{consumption by } B}, \\ \frac{dB}{dt} &= \underbrace{\mu(S)B}_{\text{growth}} - \underbrace{DB}_{\text{dilution}}. \end{aligned} \tag{4.1.1}$$

The asymptotic behavior of Model (4.1.1) can be found in details in [44, 39]. Model (4.1.1) has two equilibria, which are the wash-out equilibrium, $E_0 = (S_{in}, 0)$, and the positive equilibrium, $E_1 = (\mu^{-1}(D), \frac{S_{in}-\mu^{-1}(D)}{q})$. Moreover, the positive equilibrium is stable whenever it exists. However, when there is a constant toxin affecting the death rate of bacteria B, the above model doesn't represent the dynamic of this problem. Introducing a toxin in the system is important to answer crucial questions in commercial production with biological reactors, and to study detoxification problem [21]. The simple chemostat model with different removal rates has been studied before (see [20, 44]). The simple chemostat model of one bacteria including the bacteria's mortality rate is

$$\begin{aligned} \frac{dS}{dt} &= \underbrace{DS_{in}}_{\text{input}} - \underbrace{DS}_{\text{dilution}} - \underbrace{q\mu(S)B}_{\text{consumption by } B}, \\ \frac{dB}{dt} &= \underbrace{\mu(S)B}_{\text{growth}} - \underbrace{DB}_{\text{dilution}} - \underbrace{mB}_{\text{death of B}}. \end{aligned} \tag{4.1.2}$$

Model (4.1.2) has two equilibria, a boundary equilibrium $E_0 = (\frac{S_{in}}{D}, 0)$, and an interior equilibrium $E_1 = (\mu^{-1}(D+m), \frac{D}{q(D+m)}(S_{in} - \mu^{-1}(D+m)))$. Moreover, the positive equilibrium is locally asymptotically stable whenever it exists.

When there are more than one bacteria in the chemostat, they compete for the common limited resource. Since competition is crucial nature, the competition of organisms in the chemostat has been widely studied. For instance, the simple competition model in the chemostat can be modeled by the following nonlinear differential equations [44, 28, 17]:

$$\begin{aligned}
\frac{dS}{dt} &= \underbrace{DS_{in}}_{\text{input}} - \underbrace{DS}_{\text{dilution}} - \underbrace{q_1\mu_1(S)B_1}_{\text{consumption by } B_1} - \underbrace{q_2\mu_2(S)B_2}_{\text{consumption by } B_2}, \\
\frac{dB_1}{dt} &= \underbrace{\mu_1(S)B_1}_{\text{growth}} - \underbrace{DB_1}_{\text{dilution}}, \\
\frac{dB_2}{dt} &= \underbrace{\mu_2(S)B_2}_{\text{growth}} - \underbrace{DB_2}_{\text{dilution}}.
\end{aligned} \tag{4.1.3}$$

The asymptotic behavior of Model (4.1.3) is described by the Competitive exclusion principle [18, 44]. Which basically states that only one competitor survives. To tackle this problem, Robledo et all. [40] modified the above model by introducing an input concentration B_{in} to the inferior competitor.

$$\begin{aligned}
\frac{dS}{dt} &= \underbrace{DS_{in}}_{\text{input}} - \underbrace{DS}_{\text{dilution}} - \underbrace{q_1\mu_1(S)B_1}_{\text{consumption by } B_1} - \underbrace{q_2\mu_2(S)B_2}_{\text{consumption by } B_2}, \\
\frac{dB_1}{dt} &= \underbrace{DB_{in}}_{\text{input}} + \underbrace{\mu_1(S)B_1}_{\text{growth}} - \underbrace{DB_1}_{\text{dilution}}, \\
\frac{dB_2}{dt} &= \underbrace{\mu_2(S)B_2}_{\text{growth}} - \underbrace{DB_2}_{\text{dilution}}.
\end{aligned} \tag{4.1.4}$$

The authors proved the model has a positive equilibrium that is asymptotically stable when it exists. However, their model does not account for the death rate due to a constant toxin.

Antibiotic resistance is one of the top threats to the public's health. It occurs when a microorganism is not affected by one or more than one antibiotics. Plasmids can change bacteria's genetic material. This change can cause the bacteria to become antibiotic-resistant or remove an antibiotic resistance gene from the bacteria [41, 23]. Therefore, bacterial plasmids play an essential role in antibiotic resistance. There have been many mathematical models to understand the effect of the plasmid on the bacteria [41, 23, 45, 26]. The authors in [42] proposed a model of competition between plasmid-bearing and plasmid-free organisms in a chemostat based on the mass balances of the organisms.

$$\begin{aligned}
\frac{dS}{dt} &= \underbrace{DS_{in}}_{\text{input}} - \underbrace{DS}_{\text{dilution}} - \underbrace{q_1\mu_1(S)B_1}_{\text{consumption by } B_1} - \underbrace{q_2\mu_2(S)B_2}_{\text{consumption by } B_2}, \\
\frac{dB_1}{dt} &= \underbrace{\mu_1(S)B_1}_{\text{growth}} - \underbrace{DB_1}_{\text{dilution}} - \underbrace{q\mu_1(S)B_1}_{\text{loss of plasmid}}, \\
\frac{dB_2}{dt} &= \underbrace{\mu_2(S)B_2}_{\text{growth}} - \underbrace{DB_2}_{\text{dilution}} + \underbrace{q\mu_1(S)B_1}_{\text{loss of plasmid}}.
\end{aligned} \tag{4.1.5}$$

The parameter $q \in (0, 1)$, represents the probability that a plasmid is lost in reproduction. The mathematical analysis of the above model can be found in detail in [22]. The authors show that the positive equilibrium is asymptotically stable whenever it exists.

This chapter is organized as follows. We first start with a simple growth chemostat model, which represents the growth of the donor bacteria in the presence of a lethal toxin. Next, we consider a competition model representing the competition between the donor bacteria and the resident bacteria. Finally, we consider a competition model of the donor bacteria and the resident bacteria in the presence of a plasmid.

4.2 Simple chemostat model with microbial input and a constant death rate

The model for the growth of a single microbial population in a chemostat with supplying a microbial into the chemostat and a constant death rate due to a constant homogeneous toxin is based on the following assumptions:

1. The micro-organisms introduced in the vessel are of a single species.
2. The substrate (of concentration S) is the single limiting resource for growth.
3. The vessel is perfectly mixed.

$$\begin{aligned} \frac{dS}{dt} &= \underbrace{DS_{in}}_{\text{input}} - \underbrace{DS}_{\text{dilution}} - \underbrace{q\mu(S)B}_{\text{consumption by } B}, \\ \frac{dB}{dt} &= \underbrace{DB_{in}}_{input} - \underbrace{DB}_{\text{dilution}} + \underbrace{\mu(S)B}_{\text{growth}} - \underbrace{mB}_{\text{death of } B}, \end{aligned} \tag{4.2.1}$$

where D is a constant dilution rate, S_{in} is the concentration of the limiting nutrient, m is the bacteria's death rate, and B_{in} is the input concentration for B. Here, B is the population density of the microorganism, and S is the nutrient concentration. The function $\mu(S)$ is bounded and increasing $\mu(0) = 0$, $0 \leq \mu(S) \leq \mu^{max}$. For growth rate, the well-known Monod function is used $\mu(S) = \frac{\mu^{max} S}{K+S}$ where μ^{max} is the maximal growth rate as $S \to \infty$, and K is the half-saturation constant. Both μ^{max} and K can be measured experimentally. The parameter q is assumed to be constant and represents the efficiency at which the organism harvest nutrient for population growth.

4.2.1 Basic properties

4.2.1.1 Model well-posedness

Since Model (4.2.1) represents a biological system. All solutions should remain non-negative and bounded. The Cauchy problem associated to System (4.2.1) is

$$\vec{x}' = \vec{f}(\vec{x}); \quad \vec{x}(t_0) = \vec{x}_0, \tag{4.2.2}$$

where $\vec{x} = (S, B)$, $\vec{x}_0 = (S_0, B_0)$, and $\vec{f}(\vec{x}) \in \mathbb{R}^2$. Here, $\vec{f}(\vec{x})$ is the right-hand side in Equation (4.2.1). Clearly, the function $\vec{f}(\vec{x}(t))$ is continuously differentiable in $\mathbb{R}^2$. Thus, by the Fundamental Existence-Uniqueness Theorem [37], there is $a > 0$ such that the intial value problem (4.2.2) has a unique solution $\vec{x}(t)$ on $[-a, a]$.
In the following theorem, we will show that the solutions of Model (4.2.1) are non-negative whenever $(S_0, B_0) \in \mathbb{R}^2_+$, and that solutions do not tend to infinity with increasing time.

Theorem 4.2.1. *If $B_0 > 0$ and $S_0 > 0$, then the solution $(S(t), B(t))$ of Model (4.2.1) are non-negative for all $t > 0$. Moreover, the dynamical system (4.2.1) is dissipative.*

Proof. Let $f_1(S, B) = D(S_{in} - S) - q\mu(S)B$, and $f_2(S, B) = D(B_{in} - B) + \mu(S)B - mB$, are the right-hand sides of Model (4.2.1). Since D, S_{in}, and $B_{in}0$ are all positive, then $f_1(0, B) = DS_{in} > 0$, and $f_2(S, 0) = DB_{in} > 0$. Therefore, $S(t) > 0$ and $B(t) > 0$ whenever $S_0 > 0$ and $B_0 > 0$. Next, we aim to show Model (4.2.1) is dissipative. Let $\Sigma = S + qB$, then

$$\begin{aligned}
\Sigma' &= D\Big(S_{in} - S\Big) - q\mu(S)B \\
&\quad + qD\Big(B_{in} - B\Big) + q\mu(S)B - qmB \\
&= DS_{in} + qDB_{in} - D(S + qB) - qmB \\
&< DS_{in} + qDB_{in} - D\Sigma.
\end{aligned}$$

If $U(t)$ is the solution of $U' = DS_{in} + qDB_{in} - DU$ satisfying $U(0) = \Sigma(0)$, then it follows by comparison that

$$\Sigma(t) < U(t). \tag{4.2.3}$$

One can easily solve

$$U' = DS_{in} + qDB_{in} - DU, \;\; U(0) = \Sigma(0),$$

and obtain the exact solution as

$$U(t) = S_{in} + qB_{in} + \Big(\Sigma(0) - S_{in} - qB_{in}\Big)e^{-Dt}.$$

Therefore, we can easily see that

$$\lim_{t\to\infty} U(t) = S_{in} + qB_{in}.$$

Above limit together with Inequality (4.2.3) implies

$$\limsup_{t\to\infty} S(t) + qB(t) \leq S_{in} + qB_{in}.$$

That implies the dynamycal system (4.2.1) is dissipative. □

4.2.2 Equilibria and stability analysis

The existence of one equilibrium for Model (4.2.1) is stated in the following proposition.

Proposition 4.2.1 (Existence of Equilibrium)**.** *System* (4.2.1) *has one equilibrium* $E^* = (S^*, B^*)$ *where:*

$$B^* = \frac{D}{d}B_{in} + \frac{1}{q}\frac{D}{d}(S_{in} - S^*), \;\; \textit{and}$$

$$S^* = \begin{cases} \frac{S_{in}+\frac{dK}{\mu^{max}-d}+\frac{q\mu^{max}B_{in}}{\mu^{max}-d}-\sqrt{(S_{in}+\frac{dK}{\mu^{max}-d}+\frac{q\mu^{max}B_{in}}{\mu^{max}-d})^2-4\frac{dS_{in}K}{\mu^{max}-d}}}{2}, & \textit{if } d < \mu^{max} \\ \frac{dS_{in}K}{dK+q\mu^{max}B_{in}}, & \textit{if } d = \mu^{max} \\ \frac{S_{in}+\frac{dK}{\mu^{max}-d}+\frac{q\mu^{max}B_{in}}{\mu^{max}-d}+\sqrt{(S_{in}+\frac{dK}{\mu^{max}-d}+\frac{q\mu^{max}B_{in}}{\mu^{max}-d})^2-4\frac{dS_{in}K}{\mu^{max}-d}}}{2}, & \textit{if } d > \mu^{max} \end{cases},$$

with $d = D + m$.

Proof. To find the equilibrium, we set the right-hand sides of Model (4.2.1) to zero, i.e.

$$D(S_{in} - S) - q\mu(S)B = 0, \tag{4.2.4}$$

$$D(B_{in} - B) + \mu(S)B - mB = 0. \tag{4.2.5}$$

Multiplying the second equation by q, and then adding it with the first equation yields

$$D(S_{in} - S) + qD(B_{in} - B) - qmB = 0.$$

Solving the above equation for B gives

$$B = \frac{D}{d}B_{in} + \frac{D}{qd}\Big(S_{in} - S\Big).$$

Next, to find S^*, we now substitute the value of B in Equation (4.2.4); we have

$$D(S_{in} - S) - \frac{qD}{d}B_{in}\frac{\mu^{max}S}{K+S} - \frac{D}{d}\frac{\mu^{max}S}{K+S}\Big(S_{in} - S\Big) = 0.$$

Now multiplying the above equation by $\frac{d}{D}(K + S)$ yields

$$d(S_{in} - S)(K + S) - qB_{in}\mu^{max}S - \mu^{max}S(S_{in} - S) = 0,$$

After rearranging the terms of the above equation, one can easily get the following quadratic equation

$$(\mu^{max} - d)S^2 - \Big((\mu^{max} - d)S_{in} + dK + qB_{in}\mu^{max}\Big)S + dS_{in}K = 0.$$

By Solving the above quadratic equation, one can easily find the positive root of S^* as follows

1. If $d = \mu^{max}$, then $S^* = \frac{dS_{in}K}{dK+q\mu^{max}B_{in}}$.
2. If $d > \mu^{max}$, then $S^* = \frac{S_{in}+\frac{dK}{\mu^{max}-d}+\frac{q\mu^{max}B_{in}}{\mu^{max}-d}+\sqrt{(S_{in}+\frac{dK}{\mu^{max}-d}+\frac{q\mu^{max}B_{in}}{\mu^{max}-d})^2-4\frac{dS_{in}K}{\mu^{max}-d}}}{2}$.

3. If $d < \mu^{max}$, then $S^* = \frac{S_{in}+\frac{dK}{\mu^{max}-d}+\frac{q\mu^{max}B_{in}}{\mu^{max}-d}-\sqrt{(S_{in}+\frac{dK}{\mu^{max}-d}+\frac{q\mu^{max}B_{in}}{\mu^{max}-d})^2-4\frac{dS_{in}K}{\mu^{max}-d}}}{2}$.

From the above one can easily see that $S^* > 0$. Moreover, we claim $S_{in} > S^*$. Notice, if $d = \mu^{max}$, then $S^* = \frac{dS_{in}K}{dK+q\mu^{max}B_{in}} < S_{in}$, (since $dK < dK + q\mu^{max}B_{in}$). If $d > \mu^{max}$, we note that

$$S_{in} - S^* = \frac{S_{in} + \frac{dK}{d-\mu^{max}} + \frac{q\mu^{max}B_{in}}{d-\mu^{max}} - \sqrt{(S_{in} + \frac{dK}{\mu^{max}-d} + \frac{q\mu^{max}B_{in}}{\mu^{max}-d})^2 - 4\frac{dS_{in}K}{\mu^{max}-d}}}{2},$$

multiplying by the square conjugate yields

$$S_{in} - S^* = \frac{4B_{in}\mu^{max}qS_{in}}{2\big(d-\mu^{max}\big)\Big(S_{in} + \frac{dK}{d-\mu^{max}} + \frac{q\mu^{max}B_{in}}{d-\mu^{max}} + \sqrt{(S_{in} + \frac{dK}{\mu^{max}-d} + \frac{q\mu^{max}B_{in}}{\mu^{max}-d})^2 + 4\frac{dS_{in}K}{d-\mu^{max}}}\Big)} > 0.$$

Similarly, one can show $S_{in} > S^*$ when $d < \mu^{max}$. Hence , $B^* > 0$, and therefore E^* always exists. □

The following theorem concerning the local stability of the equilibrium E^*.

Theorem 4.2.2. *The equilibrium E^* of Model* (4.2.1) *is locally asymptotically stable.*

Proof. The Jacobian evaluated at E^* is

$$J(E^*) = \begin{bmatrix} -D - qB^*\frac{\mu^{max}K}{(K+S^*)^2} & -q\mu(S^*) \\ \frac{B^*\mu^{max}K}{(K+S^*)^2} & \mu(S^*) - (D+m) \end{bmatrix}.$$

Next, we find the trace and the determinant (det) of $J(E^*)$, we have

$$trace(J(E^*)) = -2D - m - qB^*\frac{\mu^{max}K}{(K+S^*)^2} + \mu(S^*),$$

$$det(J(E^*)) = D\Big(D + m - \mu(S^*)\Big) + (D+m)\frac{qB^*\mu^{max}K}{(K+S^*)^2}.$$

We note $D(S_{in} - S^*) = q\mu(S^*)B^*$. Thus, $\mu(S^*) = \frac{D}{q}\frac{S_{in}-S^*}{B^*} = D\frac{S_{in}-S^*}{\frac{qD}{d}B_{in}+\frac{D}{d}(S_{in}-S^*)} < D\frac{S_{in}-S^*}{\frac{D}{d}(S_{in}-S^*)} = d = D + m$. Therefore, $det(J(E^*)) > 0$, and $-trace(J(E^*)) > 0$. As a result, the equilibrium E^* is locally asymptotically stable. □

For the global stability of the unique equilibrium E^*, we have the following result.

Theorem 4.2.3. *The equilibrium E^* attracts any solution initiated in*

$$\triangle = \{(S, B) : S + qB \leq S_{in} + qB_{in}\}.$$

Proof. Since E_1 is locally asymptotically stable, and it is the only equilibrium of Model (4.2.1), and System (4.2.1) is bounded (Proposition (4.2.1)), then by Poincaré-Bendixson theorem it suffices to show that System (4.2.1) does not have periodic solutions. To prove that we use Dulac criterion, let

$$f_1(S, B) = D(S_{in} - S) - q\mu(S)B,$$
$$f_2(S, B) = D(B_{in} - B) + \mu(S)B - mB.$$

We define

$$\phi(S, B) = \frac{1}{B}.$$

Since $\frac{\partial(\phi f_1)}{\partial S} = -\frac{D}{B} - q\frac{\mu_{max}K}{(K+S)^2}$, and $\frac{\partial(\phi f_2)}{\partial B} = -\frac{DB_{in}}{B^2}$, then

$$\frac{\partial(\phi f_1)}{\partial S} + \frac{\partial f_2(S, B)}{\partial B} = -\Big(\frac{D}{B} + q\frac{\mu_{max}K}{(K+S)^2} + \frac{DB_{in}}{B^2}\Big) < 0.$$

Hence, there are no nontrivial periodic solutions, and the proof is completed. □

4.3 Competition of two microbial with one limited resource and different constant death rates and one of the bacteria is in the input flow

Next, we consider a competition chemostat model of bacterial competition in the presence of a constant homogeneous toxin affecting the death rates of the organisms. The competition in the chemostat model is based on several essential assumptions:

1. The micro-organisms introduced in the vessel are of two species.
2. The substrate (of concentration S) is the single limiting resource for growth.
3. The vessel is perfectly mixed.

$$\begin{aligned}
\frac{dS}{dt} &= \underbrace{DS_{in}}_{\text{input}} - \underbrace{DS}_{\text{dilution}} - \underbrace{q_1\mu_1(S)B_1}_{\text{consumption by } B_1} - \underbrace{q_2\mu_2(S)B_2}_{\text{consumption by } B_2}, \\
\frac{dB_1}{dt} &= \underbrace{DB_{in}}_{input} - \underbrace{DB_1}_{\text{dilution}} + \underbrace{\mu_1(S)B_1}_{\text{growth of } B_1} - \underbrace{m_1B_1}_{\text{death of } B_1}, \\
\frac{dB_2}{dt} &= \underbrace{\mu_2(S)B_2}_{\text{growth of } B_2} - \underbrace{DB_2}_{\text{dilution}} - \underbrace{m_2B_2}_{\text{death of } B_2},
\end{aligned} \tag{4.3.1}$$

where D is a constant dilution rate, S_{in} is the limiting nutrient concentration, m_i is the death rate of the bacteria B_i, and B_{in} is the input concentration for B_1. Here, B_1 is the population density of the plasmid-bearing bacteria, B_2 is the concentration of the plasmid-free bacteria, and S is the nutrient concentration. The function $\mu_i(S)$ is bounded and increasing $\mu_i(0) = 0$, $0 \leq \mu_i(S) \leq \mu_i^{max}$. For growth rate, the well-known Monod function is used $\mu_i(S) = \frac{\mu_i^{max} S}{K_i + S}$ where μ_i^{max} is the maximal growth rate as $S \to \infty$, and K_i is the half-saturation constant. Both μ_i^{max} and K_i can be measured experimentally. The parameter q_i is the nutrient uptake efficiency by B_i.

4.3.1 Basic properties

4.3.1.1 Model well-posedness

Model (4.3.1) represents a biological system. Therefore, all solutions should remain non-negative and bounded. The Cauchy problem associated with the system (4.3.1) is

$$\vec{x}' = \vec{f}(\vec{x}); \quad \vec{x}(t_0) = \vec{x}_0, \tag{4.3.2}$$

where $\vec{x} = (S, B_1, B_2)$, $\vec{x}_0 = (S_0, B_1^0, B_2^0)$ and $\vec{f}(\vec{x}) \in \mathbb{R}^3$. Here, $\vec{f}(\vec{x})$ is the right-hand side in Equation (4.3.1). Clearly, the function $\vec{f}(\vec{x}(t))$ is continuously differentiable in $\mathbb{R}^3$. Thus, by the Fundamental Existence-Uniqueness Theorem [37], there is $a > 0$

such that the initial value problem (4.3.2) has a unique solution $\vec{x}(t)$ on $[-a, a]$. In the following theorem, we will show that the solutions of Model (4.3.1) are non-negative whenever $(S_0, B_1^0, B_2^0) \in \mathbb{R}^3_+$, and that solutions are bounded.

Theorem 4.3.1. *If $S_0 > 0$, $B_1^0 > 0$ and $B_2^0 > 0$, then the solution $(S(t), B_1(t), B_2(t))$ of Model (4.3.1) are non-negative for all $t > 0$. Moreover, the dynamical system (4.3.1) is dissipative.*

Proof. Let

$$\begin{aligned} f_1(S, B_1, B_2) &= D\Big(S_{in} - S\Big) - q_1\mu_1(S)B_1 - q_2\mu_2(S)B_2, \\ f_2(S, B_1, B_2) &= D\Big(B_{in} - B_1\Big) + \mu_1(S)B_1 - m_1B_1, \\ f_3(S, B_1, B_2) &= \Big(\mu_2(S) - D - m_2\Big)B_2, \end{aligned}$$

are the right-hand sides of Model (4.3.1). Since D, S_{in} and B_{in} are all positive, then $f_1(0, B_1, B_2) = DS_{in} > 0$, $f_2(S, 0, B_2) = DB_{in} > 0$, and $f_3(S, B_1, 0) = 0$. Therefore, $S(t) \geq 0$, $B_1(t) \geq 0$ and $B_2(t) \geq 0$ whenever $S_0 \geq 0$, $B_1^0 \geq 0$ and $B_2^0 \geq 0$. To complete the proof we will show that

$$\limsup_{t\to\infty} S(t) + q_1B_1(t) + q_2B_2(t) \leq M, \quad \text{for some} \quad M > 0.$$

Let $\Sigma = S + q_1B_1 + q_2B_2$, then

$$\begin{aligned} \Sigma' &= DS_{in} + q_1DB_{in} - D\Big(S + q_1B_1 + q_2B_2\Big) - q_1m_1B_1 - q_2m_2B_2 \\ &< DS_{in} + q_1DB_{in} - D\Sigma. \end{aligned}$$

If $U(t)$ is the solution of

$$U'(t) = DS_{in} + q_1DB_{in} - DU(t), \quad U(0) = \Sigma(0), \tag{4.3.3}$$

then by comparison we have

$$\Sigma(t) < U(t). \tag{4.3.4}$$

One can easily verify that

$$U(t) = S_{in} + q_1 B_{in} + \Big(\Sigma(0) - S_{in} - q_1 B_{in}\Big) e^{-Dt}.$$

is the exact solution to Equation (4.3.3). Hence,

$$\lim_{t\to\infty} U(t) = S_{in} + q_1 B_{in}.$$

The above limit and the inequality (4.3.4) yields

$$\limsup_{t\to\infty} \Sigma(t) \leq S_{in} + q_1 B_{in}.$$

Therefore, Model (4.3.1) is dissipative. □

4.3.2 Equilibria and stability analysis

We first find the equilibria, for simplicity, let $d_i = D + m_i$, for all $i = 1, 2$, and setting the right-hand sides of Model (4.3.1) to zeros gives:

$$\begin{aligned}
&D(S_{in} - S) - q_1\mu_1(S)B_1 - q_2\mu_2(S)B_2 = 0,\\
&D(B_{in} - B_1) + \mu_1(S)B_1 - m_1 B_1 = 0,\\
&B_2(\mu_2(S) - D - m_2) = 0.
\end{aligned}$$

Therefore, System (4.3.1) has two equilibria. One is the boundary equilibrium $E_0 = (S_0^*, B_{10}^*, B_{20}^*)$, and the other is the interior equilibrium $E_1 = (S_1^*, B_{11}^*, B_{21}^*)$, where $B_{10}^* = \frac{D}{d_1} B_{in} + \frac{1}{q_1}\frac{D}{d_1}(S_{in} - S^*)$, $B_{20}^* = 0$, and

$$S_0^* = \begin{cases} \frac{S_{in} + \frac{dK}{\mu^{max} - d} + \frac{q\mu^{max} B_{in}}{\mu^{max} - d} - \sqrt{(S_{in} + \frac{dK}{\mu^{max} - d} + \frac{q\mu^{max} B_{in}}{\mu^{max} - d})^2 - 4\frac{dS_{in}K}{\mu^{max} - d}}}{2}, & \text{if } d < \mu^{max} \\ \frac{dS_{in}K}{dK + q\mu^{max} B_{in}}, & \text{if } d = \mu^{max} \\ \frac{S_{in} + \frac{dK}{\mu^{max} - d} + \frac{q\mu^{max} B_{in}}{\mu^{max} - d} + \sqrt{(S_{in} + \frac{dK}{\mu^{max} - d} + \frac{q\mu^{max} B_{in}}{\mu^{max} - d})^2 - 4\frac{dS_{in}K}{\mu^{max} - d}}}{2}, & \text{if } d > \mu^{max} \end{cases}.$$

Moreover, the components of the interior equilibrium are $S_1^* = \mu_2^{-1}(d_2)$, $B_{11}^* = \frac{DB_{in}}{d_1 - \mu_1(S_1^*)}$, and $B_{21}^* = \frac{D(S_{in} - S_1^*) - q_1\mu_1(S_1^*)B_{11}^*}{q_2 d_2}$.

Here, $\mu_2^{-1}(d_2) = \frac{d_2 K_2}{\mu_2^{max} - d_2}$. Using similar arguments used in the proof of Proposition (4.2.1), we clearly see that E_0 always exists. It remains to find the existence conditions of the positive equilibrium E_1, which is stated in the following proposition.

Proposition 4.3.1. *The equilibrium E_1 exists if and only if $d_2 < \mu_2(S_{in})$, and $d_1 > \mu_1(S_1^*)(1 + \frac{q_1 B_{in}}{S_{in} - S_1^*})$.*

Proof. First, suppose the equilibrium E_1 exists, then $S_1^* > 0$, $B_{11}^* > 0$ and $B_{21}^* > 0$. We clearly see that $B_{21}^* > 0$ implies $S_{in} > S_1^* = \mu_2^{-1}(d_2)$, i.e. $d_2 < \mu_2(S_{in})$. Also, since $B_{21}^* > 0$ implies

$$D(S_{in} - S_1^*) > q_1 \mu_1(S_1^*) B_{11}^* = q_1 \mu_1(S_1^*) \frac{D B_{in}}{d_1 - \mu_1(S_1^*)}. \tag{4.3.5}$$

But since $B_{11}^* > 0$, i.e. $d_1 > \mu_1(S_1^*)$, then one can easily rewrite Inequality (4.3.5) as

$$(S_{in} - S_1^*)(d_1 - \mu_1(S_1^*)) > q_1 \mu_1(S_1^*) B_{in},$$

which yields

$$d_1 > \mu_1(S_1^*)\Big(1 + \frac{q_1 B_{in}}{S_{in} - S_1^*}\Big).$$

To prove the other direction, we now assume $\mu_1(S_1^*)\left(1 + \frac{q_1 B_{in}}{S_{in} - S_1^*}\right) < d_1$, and $d_2 < \mu_2(S_{in})$, then since $d_2 < \mu_2(S_{in}) < \mu_2^{max}$ implies $S_1^* > 0$. Also, since

$$\mu_1(S_1^*)\left(1 + \frac{q_1 B_{in}}{S_{in} - S_1^*}\right) < d_1,$$

then $d_1 > \mu_1(S_1^*)$ which implies $B_{11}^* > 0$. Moreover, since $d_2 < \mu_2(S_{in})$, then $S_1^* < S_{in}$. Therefore, the inequality $\mu_1(S_1^*)\left(1 + \frac{q_1 B_{in}}{(S_{in} - S_1^*)}\right) < d_1$ implies

$$\mu_1(S_1^*)\Big((S_{in} - S_1^*) + q_1 B_{in}\Big) < d_1(S_{in} - S_1^*),$$

which is equivalent to

$$(S_{in} - S_1^*)(d_1 - \mu_1(S_1^*)) > q_1 B_{in} \mu_1(S_1^*),$$

dividing both sides by $(d_1 - \mu_1(S_1^*))$ yields

$$(S_{in} - S_1^*) > \mu_1(S_1^*) q_1 \frac{B_{11}^*}{D}.$$

Therefore, $B_{21}^* > 0$, and this completes the proof. □

The local stability of the boundary equilibrium E_0 is stated in the following theorem.

Theorem 4.3.1. *The equilibrium E_0 is locally asymptotically stable if and only if $d_2 > \mu_2(S_0^*)$.*

Proof. The Jacobian evaluated at E_0 is

$$J(E_0) = \begin{bmatrix} -D - A_1 & -q_1\mu_1(S_0^*) & -q_2\mu_2(S_0^*) \\ \frac{A_1}{q_1} & \mu_1(S_0^*) - d_1 & 0 \\ 0 & 0 & \mu_2(S_0^*) - d_2 \end{bmatrix},$$

where $A_1 = q_1 B_{10}^* \frac{\mu_1^{max} K_1}{(K_1+S)^2}$. The characteristic equation of $J(E_0)$ is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where $a_1 = -tr(J(E_0))$, $a_3 = -det(J(E_0))$ ($tr(J(E_0))$ and $det(J(E_0))$ are the trace and the determinant of $J(E_0)$, respectively), and

$$\begin{aligned} a_2 = &\Big(D + A_1\Big)\Big(d_1 - \mu_1(S_0^*)\Big) + \Big(D + A_1\Big)\Big(d_2 - \mu_2(S_0^*)\Big) \\ &+ \Big(\mu_1(S_0^*) - d_1\Big)\Big(\mu_2(S_0^*) - d_2\Big) + A_1\mu_1(S_0^*). \end{aligned}$$

An easy computation yields:

$$\begin{aligned} -tr(J(E_0) &= D + d_1 + d_2 + A_1 - \mu_1(S_0^*) - \mu_2(S_0^*), \text{ and} \\ -det(J(E_0)) &= \Big(d_2 - \mu_2(S_0^*)\Big)\Big(d_1 D + d_1 A_1 - D\mu_1(S_0^*)\Big). \end{aligned}$$

Also, we note that since $D(S_{in} - S_0^*) - q_1\mu_1(S_0^*)B_{10}^* = 0$, then

$$\mu_1(S_0^*) = \frac{D(S_{in} - S_0^*)}{q_1 B_{10}^*} = \frac{D(S_{in} - S_0^*)}{q_1\frac{D}{d_1}B_{in} + \frac{D}{d_1}(S_{in} - S_0^*)} < d_1. \tag{4.3.6}$$

We wish to prove all eigenvalues of $J(E_0)$ have a negative real part. Using the Routh-Hurwitz conditions, all eigenvalues have a negative real part is equivalent to show $-tr(J(E_0)) > 0$, $-det(J(E_0)) > 0$, and $a_1a_2 > a_3$. Since Inequality (4.3.6) holds, we have $d_1 > \mu_1(S_0^*)$, and since by the assumption $d_2 > \mu_2(S_0^*)$, we clearly see $-tr(J(E_0)) > 0$, and $-det(J(E_0)) > 0$. It now remains to prove $a_1a_2 > a_3$.
Since

$$\begin{aligned}
a_2 &= \Big(D + A_1\Big)\Big(d_1 - \mu_1(S_0^*)\Big) + \Big(D + A_1\Big)\Big(d_2 - \mu_2(S_0^*)\Big) \\
&\quad + \Big(\mu_1(S_0^*) - d_1\Big)\Big(\mu_2(S_0^*) - d_2\Big) + A_1\mu_1(S_0^*) \\
&= Dd_1 - D\mu_1(S_0^*) + A_1d_1 - A_1\mu_1(S_0^*) + \Big(D + A_1\Big)\Big(d_2 - \mu_2(S_0^*)\Big) \\
&\quad + \Big(\mu_1(S_0^*) - d_1\Big)\Big(\mu_2(S_0^*) - d_2\Big) + A_1\mu_1(S_0^*) \geq Dd_1 - D\mu_1(S_0^*) + A_1d_1,
\end{aligned}$$

then $a_1a_2 > a_1\Big(Dd_1 - D\mu_1(S_0^*) + A_1d_1\Big)$. Since by the inequality (4.3.6), we have $d_1 > \mu_1(S_0^*)$. Therefore, $D + d_1 + A_1 - \mu_1(S_0^*) > 0$, and $Dd_1 - D\mu_1(S_0^*) + A_1d_1 > 0$. From here one can easily verify

$$\begin{aligned}
a_1a_2 &> -det(J(E_0)) + \Big(D + d_1 + A_1 - \mu_1(S_0^*)\Big)\Big(Dd_1 - D\mu_1(S_0^*) + A_1d_1\Big) \\
&\geq -det(J(E_0)) + 0 = a_3.
\end{aligned}$$

□

The local stability of the positive equilibrium E_1 is stated in the following theorem.

Theorem 4.3.2. *If the interior equilibrium E_1 exists, then it is locally asymptotically stable.*

Proof. The Jacobian evaluated at E_1 is

$$J(E_1) = \begin{bmatrix} -D - A_1 - A_2 & -q_1\mu_1(S_1^*) & -q_2 d_2 \\ \frac{A_1}{q_1} & \mu_1(S_1^*) - d_1 & 0 \\ \frac{A_2}{q_2} & 0 & 0 \end{bmatrix},$$

where $A_1 = q_1 B_{11}^* \frac{\mu_1^{max} K_1}{(K_1 + S_1^*)^2}$, $A_2 = q_2 B_{21}^* \frac{\mu_2^{max} K_2}{(K_2 + S_1^*)^2}$. An easy computation yields

$$-det(J(E_1)) = A_2 d_2 \Big(d_1 - \mu_1(S_1^*)\Big), \text{ and}$$

$$-tr(J(E_1)) = A_1 + A_2 + D + d_1 - \mu_1(S_1^*).$$

and $-tr(J(E_1)) = A_1 + A_2 + D + d_1 - \mu_1(S_1^*)$. The characteristic equation is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where $a_1 = -tr(J(E_1))$, $a_2 = (D + A_1 + A_2)(d_1 - \mu_1(S_1^*)) + A_1\mu_1(S_1^*) + A_2 d_2$, and $a_3 = -det(J(E_1))$. Also, we note that since

$$B_{11}^* = \frac{DB_{in}}{d_1 - \mu_1(S_1^*)},$$

then we clearly see that $d_1 > \mu_1(S_1^*)$, (since $B_{11}^* > 0$). Therefore, $-tr(J(E_1)) > 0$ and $-det(J(E_1)) > 0$. Now it remains to prove $a_1 a_2 > a_3$. Since

$$a_2 = A_2 d_2 + (D + A_1 + A_2)(d_1 - \mu_1(S_1^*)) + A_1\mu_1(S_1^*),$$

then by multiplying the above equation by a_1 yields:

$$\begin{aligned} a_1 a_2 &= A_2 d_2 \Big(d_1 - \mu_1(S_1^*)\Big) \\ &\quad + \Big(d_1 - \mu_1(S_1^*)\Big)\Big((D + A_1 + A_2)(d_1 - \mu_1(S_1^*)) + A_1\mu_1(S_1^*)\Big) + \Big(A_1 + A_2 + D\Big)a_2 \\ &\quad > -det(J(E_1)) + 0 + 0 = a_3. \end{aligned}$$

Therefore, $a_1 a_2 > a_3$. Using Routh-Hurwitz conditions we can conclude that E_1 is locally asymptotically stable.

□

Table 4.1 summarizes the equilibria and their corresponding stability conditions. Next, we aim to find relations between the existence and stability conditions stated

Equilibrium Points	Existence Conditions	Local Stability
$E_0 = (S_0^*, B_{10}^*, B_{20}^*)$	always	$d_2 > \mu_2(S_0^*)$
$E_1 = (S_1^*, B_{11}^*, B_{21}^*)$	$d_2 < \mu_2(S_{in})$, and $d_1 > \mu_1(S_1^*)(1 + \frac{q_1 B_{in}}{S_{in} - S_1^*})$	stable

Table 4.1: Existence and stability conditions for the equilibrium points.

in Table (4.1). The following two propositions are used to find the relations between the boundary equilibrium E_0 and the interior equilibrium E_1.

Proposition 4.3.2. *If* $d_2 < \mu_2(S_{in})$ *and* $d_1 > \mu_1(S_1^*)(1 + \frac{q_1 B_{in}}{S_{in} - S_1^*})$, *then* $d_2 < \mu_2(S_0^*)$.

Proof. We wish to show $d_2 < \mu_2(S_0^*)$ which is equivalent to show $\mu_2^{-1}(d_2) < S_0$. From the second equation in (4.3.1), we have

$$B_{10}^* = \frac{DB_{in}}{d_1 - \mu_1(S_0^*)},$$

and from the first equation in (4.3.1) together with the above value for B_{10}^* we have

$$S_0^* = S_{in} - q_1 \frac{\mu_1(S_0^*) B_{in}}{d_1 - \mu_1(S_0^*)}.$$

That is S_0^* is a fixed point of the following function

$$g(S) = S_{in} - q_1 \frac{\mu_1(S) B_{in}}{d_1 - \mu_1(S)}.$$

The function g is differentiable on $(0, \mu_1^{-1}(d_1))$ and

$$g'(S) = -q_1 \frac{B_{in} \mu_1'(S)}{(d_1 - \mu_1(S))^2} < 0, \text{ since } \mu_1'(S) > 0.$$

Moreover, we clearly see

$$\lim_{S \to \mu_1^{-1}(d_1)} g(S) = -\infty.$$

Therefore, $S_0^* \in (0, \mu_1^{-1}(d_1))$. It remains to show that $S_0^* > \mu_2^{-1}(d_2)$. Notice by the assumption we have

$$d_1 > \mu_1(\mu_2^{-1}(d_2))\Big(1 + \frac{q_1 B_{in}}{S_{in} - \mu_2^{-1}(d_2)}\Big),$$

multiplying both sides by $S_{in} - \mu_2^{-1}(d_2)$ yields

$$d_1\Big(S_{in} - \mu_2^{-1}(d_2)\Big) > \mu_1(\mu_2^{-1}(d_2))\Big(S_{in} - \mu_2^{-1}(d_2)\Big) + q_1 B_{in}\mu_1(\mu_2^{-1}(d_2)),$$

which is equivalent to

$$S_{in}\Big(d_1 - \mu_1(\mu_2^{-1}(d_2))\Big) - q_1 B_{in}\mu_1(\mu_2^{-1}(d_2)) > \mu_2^{-1}(d_2)\Big(d_1 - \mu_1(\mu_2^{-1}(d_2))\Big).$$

Therefore, we have

$$\mu_2^{-1}(d_2) < g(\mu_2^{-1}(d_2)), \tag{4.3.7}$$

and since g is decreasing, we see that $S_0^* \in (\mu_2^{-1}(d_2), \mu_1^{-1}(d_1))$, to see this suppose for the sake of a contradiction not, then, $S_0^* \leq \mu_2^{-1}(d_2)$, but g is decreasing thus $g(S_0^*) \geq g(\mu_2^{-1}(d_2))$. By Inequality (4.3.7),

$$g(S_0^*) > \mu_2^{-1}(d_2),$$

but S_0^* is a fixed point of g, thus $S_0^* > \mu_2^{-1}(d_2)$. Hence, $S_0^* = \mu_2^{-1}(d_2)$. By the inequality (4.3.7) we have

$$S_0^* < g(S_0^*) = S_0^*.$$

That is a contradiction, therefore, $S_0^* > \mu_2^{-1}(d_2)$.

□

Proposition 4.3.3. *If $d_2 < \mu_2(S_{in})$ and $d_1 < \mu_1(S_1^*)(1 + \frac{q_1 B_{in}}{S_{in} - S_1^*})$, then $d_2 > \mu_2(S_0^*)$.*

Proof. The proof uses the same arguments as the proof for the previous proposition and it is omitted here. □

The following table summarizes the behavior of System (4.3.1).

Parameter subset	Local stability
$d_2 > \mu_2(S_{in})$	Only E_0 exists and locally stable.
$d_2 < \mu_2(S_{in})$	i. If $d_1 > \mu_1(S_1^*)(1 + \frac{q_1 B_{in}}{S_{in} - S_1^*})$, then both E_0 and E_1 exist. E_1 is locally stable while E_0 is unstable. ii. If $d_1 < \mu_1(S_1^*)(1 + \frac{q_1 B_{in}}{S_{in} - S_1^*})$, then only E_0 exists and it is stable.

Table 4.2: Parameter relationship to local stability.

4.4 Competition of two microbial with one limited resource and one of the bacteria is in the input flow in the presence of a constant homogeneous plasmid

Competition of the plasmid-bearing bacteria B_1 and the non-plasmid beaning bacteria B_2 in the presence of a constant homogeneous plasmid. Moreover, the model is based on several essential assumptions:

1. The micro- organisms introduced in the vessel are of two species.
2. The substrate (of concentration S) is the single limiting resource for growth.
3. The vessel is perfectly mixed.

$$\begin{aligned}
\frac{dS}{dt} &= \underbrace{DS_{in}}_{\text{input}} - \underbrace{DS}_{\text{dilution}} - \underbrace{q_1\mu_1(S)B_1}_{\text{consumption by } B_1} - \underbrace{q_2\mu_2(S)B_2}_{\text{consumption by } B_2}, \\
\frac{dB_1}{dt} &= \underbrace{DB_{in}}_{\text{input}} - \underbrace{DB_1}_{\text{dilution}} + \underbrace{\mu_1(S)B_1}_{\text{growth}} + \underbrace{\eta B_2}_{B_2 \text{ becomes } B_1}, \\
\frac{dB_2}{dt} &= \underbrace{\mu_2(S)B_2}_{\text{growth}} - \underbrace{DB_2}_{\text{dilution}} - \underbrace{\eta B_2}_{B_2 \text{ becomes } B_1}.
\end{aligned} \tag{4.4.1}$$

We assume $q_1 \leq q_2$. The parameter q_i are assumed constants and represent the efficiency at which the organisms B_i harvests nutrient for population growth. Here, D is a constant dilution rate, S_{in} is the concentration of the limiting nutrient, and B_{in} is the input concentration for B_1. Here, B_1 is the population density of the

plasmid-bearing bacteria, B_2 is the concentration of the plasmid-free bacteria, and S is the nutrient concentration. The function $\mu_i(S)$ is bounded and increasing $\mu_i(0) = 0$, $0 \leq \mu_i(S) \leq \mu_i^{max}$. For growth rate, the well-known Monod function is used $\mu_i(S) = \frac{\mu_i^{max} S}{K_i + S}$ where μ_i^{max} is the maximal growth rate as $S \to \infty$, and K_i is the half-saturation constant. Both μ_i^{max} and K_i can be measured experimentally. We assume plasmids are constant and homogeneous across the environment. The plasmid-free bacteria, B_2 takes in the plasmid and becomes the plasmid-bearing bacteria, B_1 at a rate η.

4.4.1 Basic properties

4.4.1.1 Model well-posedness

All solutions should remain non-negative and bounded. The Cauchy problem associated to system (4.4.1) is

$$\vec{x}' = \vec{f}(\vec{x}); \quad \vec{x}(t_0) = \vec{x}_0, \tag{4.4.2}$$

where $\vec{x} = (S, B_1, B_2)$, and $\vec{x}_0 = (S_0, B_1^0, B_2^0)$ and $\vec{f}(\vec{x}) \in \mathbb{R}^3$. Here, $\vec{f}(\vec{x})$ is the right-hand side in Equation (4.4.1). Clearly, the function $\vec{f}(\vec{x}(t))$ is continuously differentiable in $\mathbb{R}^3$. Thus, by the Fundamental Existence-Uniqueness, there is $a > 0$ such that the initial value problem (4.4.2) has a unique solution $\vec{x}(t)$ on $[-a, a]$.
In the following theorem we will show that the solutions of Model (4.4.1) are non-negative whenever $(S_0, B_1^0, B_2^0) \in \mathbb{R}^3_+$ and that solutions do not tend to infinity with increasing time.

Theorem 4.4.1. *If $S^0 > 0$, $B_1^0 > 0$ and $B_2^0 > 0$, then the solution $(S(t), B_1(t), B_2(t))$ of Model (4.4.1) are non-negative for all $t > 0$. Moreover, the dynamical system (4.4.1) is dissipative.*

Proof. Let

$$f_1(S, B_1, B_2) = D\Big(S_{in} - S\Big) - q_1\mu_1(S)B_1 - q_2\mu_2(S)B_2,$$
$$f_2(S, B_1, B_2) = D\Big(B_{in} - B_1\Big) + \mu_1(S)B_1 + \eta B_2,$$
$$f_3(S, B_1, B_2) = \Big(\mu_2(S) - D - \eta\Big)B_2.$$

are the right-hand sides of Model (4.4.1). Since D, S_{in} and B_{in} are all positive, then $f_1(0, B_1, B_2) = DS_{in} > 0$, $f_2(S, 0, B_2) = DB_{in} + \eta B_2 > 0$ whenever $B_2 > 0$, and $f_3(S, B_1, 0) = 0$. Therefore, $S(t) \geq 0$, $B_1(t) \geq 0$ and $B_2(t) \geq 0$ whenever $S_0 \geq 0$, $B_1^0 \geq 0$ and $B_2^0 \geq 0$. To complete the proof we will show that

$$\limsup_{t\to\infty} S(t) + q_1B_1(t) + q_2B_2(t) \leq M, \text{ for some } M > 0.$$

Let $\Sigma = S + q_1B_1 + q_2B_2$, then

$$\begin{aligned}\Sigma' &= DS_{in} + q_1DB_{in} - D\Big(S + q_1B_1 + q_2B_2\Big) + \eta q_1B_2 - \eta q_2B_2 \\ &= DS_{in} + q_1DB_{in} - D\Big(S + q_1B_1 + q_2B_2\Big) + \eta\Big(q_1 - q_2\Big)B_2 \quad (\text{since } q_1 \leq q_2) \\ &\leq DS_{in} + q_1DB_{in} - D\Sigma.\end{aligned}$$

If $U(t)$ is the solution of

$$U'(t) = DS_{in} + q_1DB_{in} - DU(t), \quad U(0) = \Sigma(0), \tag{4.4.3}$$

then by comparison we have

$$\Sigma(t) \leq U(t). \tag{4.4.4}$$

One can easily verify that

$$U(t) = S_{in} + q_1B_{in} + \Big(\Sigma(0) - S_{in} - q_1B_{in}\Big)e^{-Dt},$$

is the exact solution to Equation (4.4.3). Hence,

$$\lim_{t\to\infty} U(t) = S_{in} + q_1B_{in},$$

The above limit and the inequality (4.4.4) yield

$$\limsup_{t\to\infty} \Sigma(t) \leq S_{in} + q_1 B_{in}.$$

Therefore, Model (4.4.1) is dissipative. □

4.4.2 Equilibria and stability analysis

Model (4.4.1) has two equilibria; a boundary equilibrium $E_0 = (S_0^*, B_{10}^*, B_{20}^*)$ and an interior equilibrium $E_1 = (S_1^*, B_{11}^*, B_{21}^*)$, where:

$B_{10}^* = \frac{DB_{in}}{D-\mu_1(S_0^*)}$, $B_{20}^* = 0$, and

$$S_0^* = \begin{cases} \frac{S_{in}+\frac{DK_1}{\mu_1^{max}-D}+\frac{q_1\mu_1^{max}B_{in}}{\mu_1^{max}-D}-\sqrt{\left(S_{in}+\frac{DK_1}{\mu_1^{max}-D}+\frac{q_1\mu_1^{max}B_{in}}{\mu_1^{max}-D}\right)^2-4\frac{DS_{in}K_1}{\mu_1^{max}-D}}}{2}, & \text{if } D<\mu^{max} \\ \frac{DS_{in}K_1}{DK_1+q_1\mu_1^{max}B_{in}}, & \text{if } D=\mu^{max} \\ \frac{S_{in}+\frac{DK_1}{\mu_1^{max}-D}+\frac{q_1\mu_1^{max}B_{in}}{\mu_1^{max}-D}+\sqrt{\left(S_{in}+\frac{DK_1}{\mu_1^{max}-D}+\frac{q_1\mu_1^{max}B_{in}}{\mu_1^{max}-D}\right)^2-4\frac{DS_{in}K_1}{\mu_1^{max}-D}}}{2}, & \text{if } D>\mu^{max} \end{cases}.$$

The components of the interior equilibrium are; $S_1^* = \mu_2^{-1}(D+\eta)$, $B_{11}^* = \frac{DB_{in}}{D-\mu_1(S_1^*)} + \frac{\eta B_{21}^*}{D-\mu_1(S_1^*)}$, and $B_{21}^* = \frac{D(S_{in}-S_1^*)(D-\mu_1(S_1^*))-q_1\mu_1(S_1^*)DB_{in}}{q_1\mu_1(S_1^*)\eta+q_2\mu_2(S_1^*)(D-\mu_1(S_1^*))}$, and the function $\mu_2^{-1}(D+\eta) = \frac{(D+\eta)K_2}{\mu_2^{max}-(D+\eta)}$. Using similar arguments used in Proposition (4.2.1), one can easily show that the boundary equilibrium E_0 always exists. The following theorem summarizes the existence condition for the interior equilibrium E_1.

Theorem 4.4.2. *If $\mu_2^{-1}(D+\eta) < \mu_1^{-1}(D) < S_{in}$, then the interior equilibrium exists if and only if*

$$D > \mu_1(S_1^*)\Big(1 + \frac{q_1 B_{in}}{S_{in} - S_1^*}\Big). \tag{4.4.5}$$

Proof. Notice that $\mu_2^{-1}(D+\eta) < \mu_1^{-1}(D) < S_{in}$, implies $S_{in} > S_1^*$ and $D > \mu_1(S_1^*)$. Then,

$$D > \mu_1(S_1^*)\Big(1 + \frac{q_1 B_{in}}{S_{in} - S_1^*}\Big),$$

which is equivalent to

$$D - \mu_1(S_1^*) > \mu_1(S_1^*)\frac{q_1 B_{in}}{S_{in} - S_1^*},$$

rewriting the above inequality yields

$$S_{in} - S_1^* > \mu_1(S_1^*)\frac{q_1 B_{in}}{D - \mu_1(S_1^*)},$$

Multiplying the above inequality by $D\Big(D - \mu_1(S_1^*)\Big)$ implies

$$D\Big(S_{in} - S_1^*\Big)\Big(D - \mu_1(S_1^*)\Big) > D\mu_1(S_1^*)q_1 B_{in}.$$

The above inequality together with $S_{in} > S_1^*$ and $D > \mu_1(S_1^*)$ imply $B_{21}^* > 0$, and hence $B_{11}^* > 0$.

□

The following theorem shows that the boundary equilibrium E_0 always exists and the relations between the boundary and the interior equilibria.

Theorem 4.4.3. *Assume $\mu_2^{-1}(D+\eta) < \mu_1^{-1}(D) < S_{in}$ is satisfied, then System (4.4.1) has a unique equilibrium in $\partial\mathbb{R}^3_+$:*

$$E_0 = (S_0^*, B_{10}^*, 0) \in \partial\mathbb{R}^3_+,$$

and $S_0^* \in (0, \mu_1^{-1}(D))$ is the unique fixed point of $g : [0, \mu_1^{-1}(D)) \to R_+$ given by:

$$g(S) = S_{in} - q_1\frac{\mu_1(S)B_{in}}{D - \mu_1(S)}.$$

Moreover, $S_0^* \in (\mu_2^{-1}(D+\eta), \mu_1^{-1}(D))$ when (4.4.5) is satisfied and $S_0^* \in (0, \mu_2^{-1}(D+\eta))$ when it is strictly not satisfied.

Proof. Notice that the function g is continuous on $[0, \mu_1^{-1}(D))$ and that

$$\lim_{S \to \mu_1^{-1}(D)} g(S) = -\infty.$$

Also, we note that $g(0) = S_{in}$ and an easy calculation yields

$$g'(S) = -q_1 \frac{B_{in}\mu_1'(S)}{(D-\mu_1(S))^2} < 0, \text{ since } \mu_1'(S) > 0.$$

Hence, g is strictly decreasing on $[0, \mu_1^{-1}(D))$, and therefore, the function g will intersect the $y = S$ only once at (S_0^*, S_0^*). Thus, $S_0^* \in (0, \mu_1^{-1}(D))$. Next, if the inequality (4.4.5) is satisfied. Then, we have

$$D > \mu_1(S_1^*)\Big(1 + \frac{q_1 B_{in}}{S_{in} - S_1^*}\Big),$$

which is equivalent to

$$S_{in} - S_1^* > \frac{\mu_1(S_1^*)q_1 B_{in}}{D - \mu_1(S_1^*)},$$

That implies

$$S_{in} - \frac{\mu_1(S_1^*)q_1 B_{in}}{D - \mu_1(S_1^*)} > S_1^*,$$

Hence,

$$g(S_1^*) > S_1^*, \tag{4.4.6}$$

recall that $S_1^* = \mu_2^{-1}(D + \eta)$, and since g is decreasing, we clearly see that $S_0^* \in (\mu_2^{-1}(D), \mu_1^{-1}(D))$, to see this suppose for the sake of a contradiction not, then, $S_0^* \leq \mu_2^{-1}(D)$, but g is decreasing thus $g(S_0^*) \geq g(\mu_2^{-1}(D))$. By Inequality (4.4.6),

$$g(S_0^*) > \mu_2^{-1}(D),$$

but S_0^* is a fixed point of g, thus $S_0^* > \mu_2^{-1}(D)$. Hence, $S_0^* = \mu_2^{-1}(D)$. By the inequality (4.4.6) we have

$$S_0^* < g(S_0^*) = S_0^*.$$

That is a contradiction, therefore, $S_0^* > \mu_2^{-1}(D)$. Similarly, $S_0^* \in (0, \mu_2^{-1}(D + \eta))$ when the inequality (4.4.5) is strictly not satisfied. □

The following theorem states the condition when the boundary equilibrium E_0 is locally asymptotically stable.

Theorem 4.4.4. *The equilibrium E_0 is locally asymptotically stable if and only if $\mu_2^{-1}(D+\eta) > S_0^*$.*

Proof. Introducing the notation $A_1 = q_1 B_{10}^* \frac{\mu_1^{max} K_1}{(K_1+S_0^*)^2}$, $A_2 = q_2 B_{20}^* \frac{\mu_2^{max} K_2}{(K_2+S_0^*)^2}$, then the Jacobian evaluated at E_0 is

$$J(E_0) = \begin{bmatrix} -D - A_1 & -q_1\mu_1(S_0^*) & -q_2\mu_2(S_0^*) \\ \frac{A_1}{q_1} & \mu_1(S_0^*) - D & \eta \\ 0 & 0 & \mu_2(S_0^*) - D - \eta \end{bmatrix}.$$

Also, one can have

$$\begin{aligned} -tr(J(E_0) &= 3D + \eta + A_1 - \mu_1(S_0^*) - \mu_2(S_0^*), \\ -det(J(E_0)) &= \Big(D + \eta - \mu_2(S_0^*)\Big)\Big(D^2 + DA_1 - D\mu_1(S_0^*)\Big). \end{aligned}$$

Also, we note since $D(S_{in} - S_0^*) - q_1\mu_1(S_0^*)B_{10}^* = 0$, then

$$\mu_1(S_0^*) = \frac{D(S_{in} - S_0^*)}{q_1 B_{10}^*} = \frac{D(S_{in} - S_0^*)}{q_1 B_{in} + (S_{in} - S_0^*)} < D. \tag{4.4.7}$$

The characteristic equation of $J(E_0)$ is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where $a_1 = -tr(J(E_0))$, $a_3 = -det(J(E_0))$, and

$$\begin{aligned} a_2 = &\Big(D + A_1\Big)\Big(D - \mu_1(S_0^*)\Big) + \Big(D + A_1\Big)\Big(D + \eta - \mu_2(S_0^*)\Big) + \\ &\Big(\mu_1(S_0^*) - D\Big)\Big(\mu_2(S_0^*) - D - \eta\Big) + A_1\mu_1(S_0^*). \end{aligned}$$

Using the Routh-Hurwitz conditions, we need to show $a_1 > 0$, $a_3 > 0$, and $a_1 a_2 > a_3$. Using the assumption, $D + \eta > \mu_2(S_0^*)$, and since by inequality (4.4.7), we have

$D > \mu_1(S_0^*)$, then we clearly see $-tr(J(E_0)) > 0$ and $-det(J(E_0)) > 0$. It now remains to show $a_1 a_2 > a_3$.

Since

$$\begin{aligned} a_2 &= \Big(D + A_1\Big)\Big(D - \mu_1(S_0^*)\Big) + \Big(D + A_1\Big)\Big(D + \eta - \mu_2(S_0^*)\Big) + \\ &\Big(\mu_1(S_0^*) - D\Big)\Big(\mu_2(S_0^*) - D - \eta\Big) + A_1\mu_1(S_0^*) \\ &= D^2 - D\mu_1(S_0^*) + A_1 D - A_1\mu_1(S_0^*) + \Big(D + A_1\Big)\Big(D + \eta - \mu_2(S_0^*)\Big) + \\ &\Big(\mu_1(S_0^*) - D\Big)\Big(\mu_2(S_0^*) - D - \eta\Big) + A_1\mu_1(S_0^*) > D^2 - D\mu_1(S_0^*) + A_1 D + 0 + 0. \end{aligned}$$

Now since $a_1 = -tr(J(E_)) > 0$, then $a_1 a_2 > a_1\Big(D^2 - D\mu_1(S_0^*) + A_1 D\Big)$. Since by the inequality (4.4.7), $D > \mu_1(S_0^*)$, then $D + D + A_1 - \mu_1(S_0^*) > 0$ and $D^2 - D\mu_1(S_0^*) + A_1 D > 0$. Therefore, we have

$$\begin{aligned} a_1 a_2 &\geq -det(J(E_0)) + \Big(D + D + A_1 - \mu_1(S_0^*)\Big)\Big(D^2 - D\mu_1(S_0^*) + A_1 D\Big) \\ &\geq -det(J(E_0)) + 0 = a_3. \end{aligned}$$

□

The local stability of the interior equilibrium E_1 is stated in the following theorem.

Theorem 4.4.5. *If the interior equilibrium exists, then it is locally asymptotically stable.*

Proof. The Jacobian matrix evaluated at E_1 is

$$J(E_1) = \begin{bmatrix} -D - q_1 B_{11}^* \mu_1'(S_1^*) - q_2 B_{21}^* \mu_2'(S_1^*) & -q_1\mu_1(S_1^*) & -q_2(D+\eta) \\ B_{11}^* \mu_1'(S_1^*) & \mu_1(S_1^*) - D & \eta \\ B_{21}^* \mu_2'(S_1^*) & 0 & 0 \end{bmatrix}.$$

The characteristic equation is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \text{where}$$

$$
\begin{aligned}
a_1 &= -tr(J(E_1)) = D + q_1 B_{11}^* \mu_1'(S_1^*) + q_2 B_{21}^* \mu_2'(S_1^*) + D - \mu_1(S_1^*),\\
a_2 &= q_2 B_{21}^* \mu_2'(S_1^*)\Big(D+\eta\Big) + \Big(D + q_1 B_{11}^* \mu_1'(S_1^*) + q_2 B_{21}^* \mu_2'(S_1^*)\Big)\Big(D - \mu_1(S_1^*)\Big)\\
&\quad + q_1 B_{11}^* \mu_1'(S_1^*)\mu_1(S_1^*),\\
a_3 &= -det(J(E_1)) = q_2 B_{21}^* \mu_2'(S_1^*) D(D - \mu_1(S_1^*)) + q_2 \eta B_{21}^* \mu_2'(S_1^*)(D - \mu_1(S_1^*))\\
&\quad + q_1 \eta B_{21}^* \mu_2'(S_1^*)\mu_1(S_1^*).
\end{aligned}
$$

Since $\mu_2^{-1}(D+\eta) < \mu_1^{-1}(D) < S_{in}$, then $D > \mu_1(S_1^*)$. Therefore, $-tr(J(E_1)) > 0$, $-det(J(E_1)) > 0$ and $a_2 > 0$. Now it remains to prove $a_1 a_2 > a_3$.

$$
\begin{aligned}
a_1 a_2 - a_3 &= \Big[D + q_1 B_{11}^* \mu_1'(S_1^*) + q_2 B_{21}^* \mu_2'(S_1^*) + D - \mu_1(S_1^*)\Big]\Big[q_2 B_{21}^* \mu_2'(S_1^*)\Big(D+\eta\Big)\\
&\quad + \Big(D + q_1 B_{11}^* \mu_1'(S_1^*) + q_2 B_{21}^* \mu_2'(S_1^*)\Big)\Big(D - \mu_1(S_1^*)\Big) + q_1 B_{11}^* \mu_1'(S_1^*)\mu_1(S_1^*)\Big]\\
&\quad - q_2 B_{21}^* \mu_2'(S_1^*)\Big(D+\eta\Big)\Big(D - \mu_1(S_1^*)\Big) - q_1 \eta B_{21}^* \mu_2'(S_1^*)\mu_1(S_1^*)\\
&= \Big[D + q_1 B_{11}^* \mu_1'(S_1^*) + q_2 B_{21}^* \mu_2'(S_1^*)\Big]\Big[q_1 B_{11}^* \mu_1'(S_1^*)\mu_1(S_1^*)\\
&\quad + \Big(D + q_1 B_{11}^* \mu_1'(S_1^*) + q_2 B_{21}^* \mu_2'(S_1^*)\Big)\Big(D - \mu_1(S_1^*)\Big)\Big]\\
&\quad + \Big[D + q_1 B_{11}^* \mu_1'(S_1^*) + q_2 B_{21}^* \mu_2'(S_1^*)\Big] q_2 B_{21}^* \mu_2'(S_1^*)\Big(D+\eta\Big)\\
&\quad + q_2 B_{21}^* \mu_2'(S_1^*)\Big(D+\eta\Big)\Big(D - \mu_1(S_1^*)\Big) + q_1 B_{11}^* \mu_1'(S_1^*)\mu_1(S_1^*)\Big(D - \mu_1(S_1^*)\Big)\\
&\quad + \Big(D + q_1 B_{11}^* \mu_1'(S_1^*) + q_2 B_{21}^* \mu_2'(S_1^*)\Big)\Big(D - \mu_1(S_1^*)\Big)^2\\
&\quad - q_2 B_{21}^* \mu_2'(S_1^*)\Big(D+\eta\Big)\Big(D - \mu_1(S_1^*)\Big) - q_1 \eta B_{21}^* \mu_2'(S_1^*)\mu_1(S_1^*)\\
&> \Big[D + q_1 B_{11}^* \mu_1'(S_1^*) + q_2 B_{21}^* \mu_2'(S_1^*)\Big]\Big[q_1 B_{11}^* \mu_1'(S_1^*)\mu_1(S_1^*)\\
&\quad + \Big(D + q_1 B_{11}^* \mu_1'(S_1^*) + q_2 B_{21}^* \mu_2'(S_1^*)\Big)\Big(D - \mu_1(S_1^*)\Big)\Big] + D q_2 B_{21}^* \mu_2'(S_1^*)\Big(D+\eta\Big)\\
&\quad + q_2 B_{21}^* \mu_2'(S_1^*)\Big(D+\eta\Big)\Big(D - \mu_1(S_1^*)\Big) + q_1 B_{11}^* \mu_1'(S_1^*)\mu_1(S_1^*)\Big(D - \mu_1(S_1^*)\Big)\\
&\quad + \Big(D + q_1 B_{11}^* \mu_1'(S_1^*) + q_2 B_{21}^* \mu_2'(S_1^*)\Big)\Big(D - \mu_1(S_1^*)\Big)^2\\
&\quad - q_2 B_{21}^* \mu_2'(S_1^*)\Big(D+\eta\Big)\Big(D - \mu_1(S_1^*)\Big) - q_1 \eta B_{21}^* \mu_2'(S_1^*)\mu_1(S_1^*).
\end{aligned}
$$

Since $q_1 \leq q_2$ and $D > \mu_1(S_1^*)$, then $a_1a_2 - a_3 \geq 0$. Therefore, E_1 is locally asymptotically stable. □

The following table summarizes the behavior of System (4.4.1).

Parameter subset	Local stability
$D + \eta > \mu_2(S_{in})$	Only E_0 exists and locally stable.
$D + \eta < \mu_2(S_{in})$	i. If $D > \mu_1(S_1^*)(1 + \frac{q_1 B_{in}}{S_{in} - S_1^*})$, then both E_0 and E_1 exist. E_1 is locally stable while E_0 is unstable. ii. If $D < \mu_1(S_1^*)(1 + \frac{q_1 B_{in}}{S_{in} - S_1^*})$, then only E_0 exists and it is stable.

Table 4.3: Parameter relationship to local stability.

CHAPTER 5

Numerical Simulations

5.1 Numerical Simulations for one-dimensional differential equations

The performance of the new SOPESN1 (3.2.14) and SOPESN2 (3.3.12) methods is numerically compared with the second-order modified nonstandard explicit Euler (SONSEE) method [24], the standard second-order two-stage explicit Runge-Kutta (ERK2) method [38], and the PESN1 (2.0.5) and PESN2 (2.0.6) methods for three different equations. We first consider the following productive-destructive equation:

$$\frac{dx}{dt} = k_1 bx - k_{-1}x^2, \tag{5.1.1}$$

which models a simple autocatalytic reaction (autocatalysis), where a chemical with concentration x is involved in its own production using a reactant [33]. Here, k_1 is the forward reaction rate constant, k_{-1} is the reverse reaction rate constant, and b is the constant concentration of the reactant. In the setting of Equation (2.0.2), the production function is $P(x) = k_1 bx$ and the destruction function $D(x) = k_{-1}x^2$. Equation (5.1.1) is also considered the chemical equivalent of the logistic growth equation [33]:

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{K}\right), \tag{5.1.2}$$

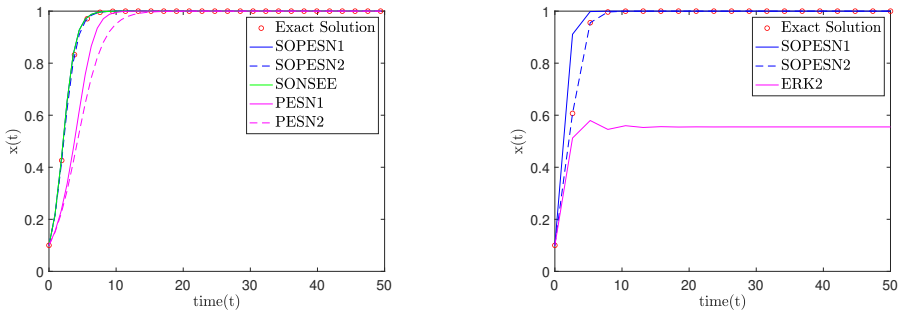
where the intrinsic growth rate $a = k_1 b$ is the equivalent of the unit production rate, while the carrying capacity of the environment $K = \dfrac{k_1 b}{k_{-1}}$ is the equivalent of the reaction's steady-state.

Equation (5.1.2) has two hyperbolic equilibria: $x^* = K$, which is stable, and $x^* = 0$, which is unstable, provided that $a > 0$. In all examples, we set $|a| = 1$ and

$K = 1$. In Figure 5.1(a), for $h = 0.9$ and initial condition $x_0 = 0.1$, we see that while all the methods behave well and similar to the exact solution, the new SOPESN methods (3.2.14), (3.3.12), and the SONSEE approach the exact solution at a faster rate than the PESN methods. For the two PESN methods, we use the nonstandard denominator function

$$\varphi(h) = \frac{1 - e^{-qh}}{q},$$

with $q = 1$. Figure 5.1(b), for $h = 2.6$, we see that the ERK2 method initially oscillates and introduces an artificial fixed point, and indeed with a larger enough step-size e.g. $h > 2.9$, the ERK2 method will not converge to any point in $\mathbb{R}$. We see that there is no such behavior from SOPESN methods (3.2.14) and (3.3.12).



(a) Comparison of SOPESN, SONSEE and PESN

(b) Comparison of SOPESN and ERK2

Figure 5.1: Numerical solutions of Equation (5.1.2) for $a = 1$ and $K = 1$, and using $h = 0.9$ in (a) and $h = 2.6$ in (b).

Next, we consider the following modified logistic growth equation ([43], p. 124):

$$\frac{dx}{dt} = ax\Big(\frac{x}{\theta} - 1\Big)\Big(1 - \frac{x}{K}\Big), \tag{5.1.3}$$

with $0 < \theta < K$. In the second example, we set $a = 1$, $\theta = 0.5$ and $K = 1$. For this set of parameter values, the right-hand side function of Equation (5.1.3) is $f(x) = x(2x - 1)(1 - x)$, and the equation has $x^* = \frac{1}{2}$ as an unstable equilibrium

while $x^* = 0, 1$ are stable equilibria. In the productive-destructive setting of Equation (2.0.2), the function $f(x)$ can be rewritten as $f(x) = P(x) - D(x)$, where $P(x) = 3x^2$ and $D(x) = 2x^3 + x$. Next, to better visualize the second-order accuracy of the new SOPESN methods (3.2.14) and (3.3.12), we denote the numerical solution for a given mesh size h as x^h. Let us define the l^∞ error as

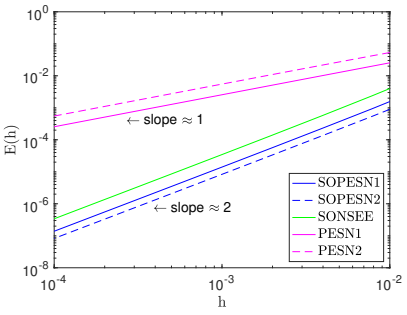
$$E(h) = \|x^h - x\|_\infty,$$

where

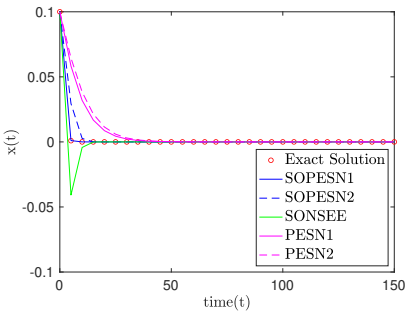
$$\|y\|_\infty = \max_{k=0,\cdots,N_t} |y_k|$$

represents the discrete l_∞ norm of the vector y, and x represents the exact solution of Equation (2.0.1). As can be seen in Figure 5.2(a), the error lines for the PESN methods (2.0.5) and (2.0.6) have slopes 1, whereas both SOPESN methods (3.2.14) and (3.3.12), including the SONSEE method, have error lines of slope 2. For small values of the time step h, all of the above numerical methods approximate the solution of Equation (5.1.3) very well. However, for large values of h, the SONSEE method produces a numerical solution that is initially before correcting itself back to the stable equilibrium $x^* = 0$, while both the second-order (SOPESN) and first-order (PESN) positive and elementary stable methods always preserve the positivity of the solution, as can be seen in Figure 5.2(b) for $h = 5$ and initial condition $x_0 = 0.1$.

Lastly, for the third example, we set $a = -1$, $\theta = 0.5$ and $K = 1$ in Equation (5.1.3). In this case, the right-hand side function becomes $f(x) = x(2x - 1)(x - 1)$, and Equation (5.1.3) has three hyperbolic equilibria, where $x^* = 0, 1$ are unstable and $x^* = \frac{1}{2}$ is stable. Similarly to the previous example, the right-hand side function $f(x)$ of Equation (5.1.3) can be rewritten in the productive-destructive setting of Equation (2.0.2) as $f(x) = P(x) - D(x)$, where $P(x) = 2x^3 + x$ and $D(x) = 3x^2$. In Figure 5.3(a), for $h = 0.2$ and initial condition $x_0 = 0.9$, we see that the numerical solutions

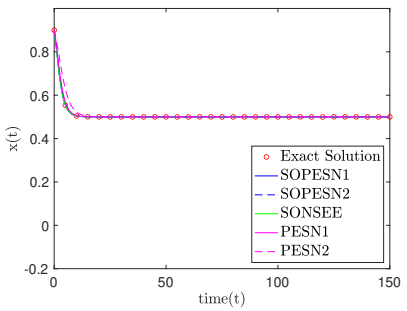


(a) Error plot: SOPESN vs PESN

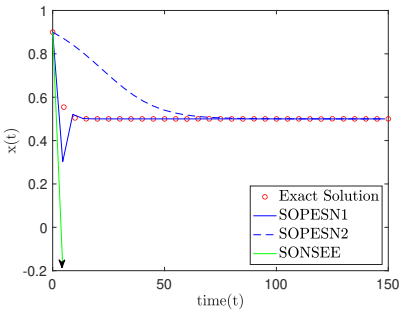


(b) Comparison of SOPESN, SONSEE and PESN

Figure 5.2: Numerical solutions of Equation (5.1.3) for $a = 1$, $\theta = 0.5$ and $K = 1$, and using $h = 5$ in (b).



(a) Comparison of SOPESN, SONSEE and PESN



(b) Comparison of SOPESN and SONSEE

Figure 5.3: Numerical solutions of Equation (5.1.3) for $a = -1$, $\theta = 0.5$ and $K = 1$, and using $h = 0.2$ in (a) and $h = 4.5$ in (b).

produced by all five methods accurately resemble the exact solution and converge to the stable equilibrium $x^* = \frac{1}{2}$. However, for the large time-step $h = 4.5$, the solution produced by the SONSEE method, which is not designed to be positivity preserving, goes negative and therefore eventually blows up while the numerical solutions of the SOPESN methods always remain positive and approach the stable equilibrium, as seen in Figure 5.3(b).

5.2 SOPESN methods applied to the chemostat models

5.2.1 SOPESN methods applied to the simple chemostat model with microbial input and a constant death rate

In this subsection, the SOPESN methods are applied to approximate the solution of the simple chemostat model with microbial input and a constant death rate (4.2.1) discussed in Chapter (4). Also, SOPESN methods are compared with several standard and nonstandard finite difference methods in different settings. As was illustrated in Chapter (4), Model (4.2.1) has a unique interior equilibrium $E^* = (S^*, B^*)$ that is asymptotically stable. The SOPESN1 method for Model (4.2.1) is constructed as follows:

$$\begin{aligned}
\frac{S^{k+1} - S^k}{\varphi_1(h, S^k, B^k)} &= w_1^k\Big(D\Big(S_{in} - S^k\Big) - q\mu(S^k)B^k\Big) = f_1(S^k, B^k), \\
\frac{B^{k+1} - B^k}{\varphi_2(h, S^k, B^k)} &= w_2^k\Big(D\Big(B_{in} - B^k\Big) + \mu(S^k)B^k - mB^k\Big) = f_2(S^k, B^k),
\end{aligned} \tag{5.2.1}$$

where

$$w_1^k = \begin{cases} 1, & \text{if } f_1(S^k, B^k) \geq 0 \\ \dfrac{S^{k+1}}{S^k}, & \text{if } f_1(S^k, B^k) < 0 \end{cases}, \quad \text{and} \quad w_2^k = \begin{cases} 1, & \text{if } f_2(S^k, B^k) \geq 0 \\ \dfrac{B^{k+1}}{B^k}, & \text{if } f_2(S^k, B^k) < 0 \end{cases}.$$

The denominator functions are chosen as follows

$$\varphi_i(h, S^k, B^k) = \begin{cases} \dfrac{1 - e^{-q_i(S^k, B^k)h}}{q_i(S^k, B^k)}, & \text{if } q_i(S^k, B^k) \neq 0 \\ \\ h, & \text{if } q_i(S^k, B^k) = 0 \end{cases},$$

where $q_i(S^k, B^k)$ as in Theorem (3.2.1) for all $i = 1, 2$. In the productive-destructive setting of System (2.0.2), the functions $f_1(S, B)$ and $f_2(S, B)$ can be rewritten as $f_i(S, B) = P_i(S, B) - D_i(S, B)$, for all $i = 1, 2$, where $P_1(S, B) = DS_{in}$, $D_1(S, B) =$

$DS+q\mu(S)B$, $P_2(S,B) = DB_{in}+\mu(S)B$, and $D_2(S,B) = DB+mB$. The SOPESN2 method is constructed as follows

$$\begin{aligned} \frac{S^{k+1}-S^k}{\varphi_1(h,S^k,B^k)} &= P_1(S^k,B^k) - D_1(S^k,B^k)\frac{S^{k+1}}{S^k}, \\ \frac{B^{k+1}-B^k}{\varphi_2(h,S^k,B^k)} &= P_2(S^k,B^k) - D_2(S^k,B^k)\frac{B^{k+1}}{B^k}. \end{aligned} \tag{5.2.2}$$

The denominator functions are chosen as follows

$$\varphi_i(h,S^k,B^k) = \begin{cases} \dfrac{1-e^{-q_i(S^k,B^k)h}}{q_i(S^k,B^k)}, & \text{if } q_i(S^k,B^k) \neq 0 \\ \\ h, & \text{if } q_i(S^k,B^k) = 0 \end{cases}.$$

where $q_i(S^k,B^k)$ as in Theorem (3.3.1) for all $i = 1,2$, and $\mu(S) = \frac{\mu^{max}S}{K+S}$.

For Model (4.2.1), we consider the following set of parameter values: $D = 0.4$, $S_{in} = 1.5$, $B_{in} = 0.5$, $q = 10^{-8}$, $m = 0.2$, $\mu^{max} = 0.3$ and $K = 0.1$. As was shown in Chapter (4) that Model (4.2.1) has unique positive equilibrium $E^* = (1.5, 0.627451)$ that is asymptotically stable. Moreover, the eigenvalues of the Jacobian matrix evaluated at E^* are: $\lambda_1 = -0.4$, and $\lambda_2 = -0.31875$. The PESN1 method (2.0.1) is constructed as follows:

$$\begin{aligned} \frac{S^{k+1}-S^k}{\varphi(h)} &= w_1^k\Big(D\Big(S_{in}-S^k\Big) - q\mu(S^k)B^k\Big) = w_1^k f_1(S^k,B^k), \\ \frac{B^{k+1}-B^k}{\varphi(h)} &= w_2^k\Big(D\Big(B_{in}-B^k\Big) + \mu(S^k)B^k - mB^k\Big) = w_2^k f_2(S^k,B^k), \end{aligned} \tag{5.2.3}$$

where

$$w_1^k = \begin{cases} 1, & \text{if } f_1(S^k,B^k) \geq 0 \\ \dfrac{S^{k+1}}{S^k}, & \text{if } f_1(S^k,B^k) < 0 \end{cases}, \quad \text{and} \quad w_2^k = \begin{cases} 1, & \text{if } f_2(S^k,B^k) \geq 0 \\ \dfrac{B^{k+1}}{B^k}, & \text{if } f_2(S^k,B^k) < 0 \end{cases}.$$

The choice of the denominator function φ is chosen as described in [49]:

$$\varphi(h) = \frac{1 - e^{-0.3h}}{0.3}, \text{ here, } 0.3 > \max\left\{\frac{|\lambda_1|}{2}, \frac{|\lambda_2|}{2}\right\} = 0.2.$$

In order to apply the PESN2 method to Model (4.2.1), the model is rewritten in the form of System (2) presented in [48], such as:

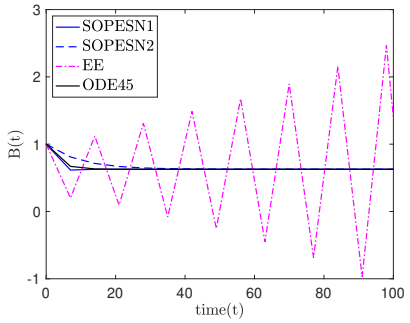
$$\begin{aligned} \frac{dS}{dt} &= DS_{in} - \Big(D + q\tfrac{\mu^{max}B}{K+S}\Big)S, \\ \\ \frac{dB}{dt} &= DB_{in} + \tfrac{\mu^{max}S}{K+S}B - \Big(D+m\Big)B. \end{aligned}$$

Then, the PESN2 method (2.0.1) is constructed as follows:

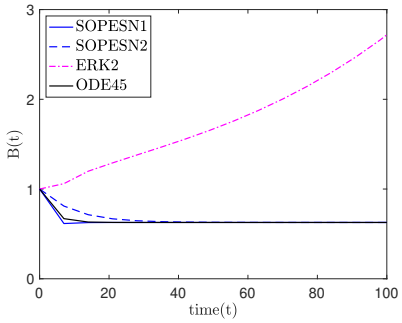
$$\begin{aligned} \frac{S^{k+1} - S^k}{\varphi(h)} &= DS_{in} - \Big(D + q\tfrac{\mu^{max}B^k}{K+S^k}\Big)S^{k+1}, \\ \\ \frac{B^{k+1} - B^n}{\varphi(h)} &= DB_{in} + \tfrac{\mu^{max}S^k}{K+S^k}B^k - \Big(D+m\Big)B^{k+1}. \end{aligned} \tag{5.2.4}$$

The denominator function is $\varphi(h) = h$. The choice of the denominator function $\varphi(h)$ was determined using the methodology in Appendix A [48]. Figure 5.4(a) compares the SOPESN methods (3.2.3) and (3.3.3) with the standard explicit Euler (EE) method [38] for $h = 7$. It is known that the EE method neither preserves the positivity of the solutions nor the local stability of the equilibria independently of the step size h. As we can see, the EE method keeps oscillating and does not converge to any point, while the SOPESN methods behave well and converge to the exact solution. Figure 5.4(b), for $h = 7$, compares the SOPESN methods with the standard second-order two-stage explicit Runge-Kutta (ERK2) method [38], and it can be seen that the ERK2 method does not converge to the exact solution and will eventually blow up to infinity. In contrast, the SOPESN (3.2.3) and (3.3.3) methods behave well and converge to the exact solution. Figure 5.4(c) and Figure 5.4(d) compare the SOPESN1 method with the PESN1 method (2.0.5) and the SOPESN2

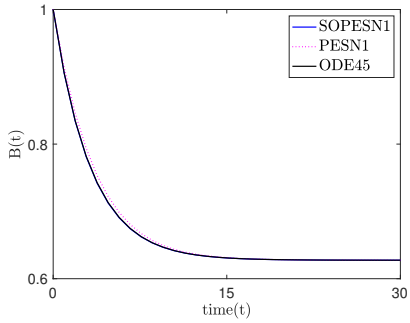
method with the PESN2 method (2.0.6), respectively, for $h = 0.95$. one can see that all numerical solutions converge to the exact solution. However, the SOPESN methods are much more accurate than their corresponding first-order PESN methods. Moreover, the PESN1 method converges to the exact solution faster than the PESN2 method. Therefore, we compare the SOPESN methods with the PESN1 method from now on. Figure 5.5(a) and Figure 5.5(b) show the global asymptotic stability of the positive equilibrium E^* as established in Theorem (4.2.3). Figure 5.5(c) and Figure 5.5(d) support the global asymptotic stability of the positive equilibrium E^*.



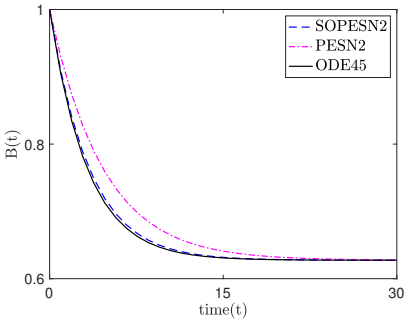
(a) Comparison of SOPESN1, SOPESN2, ODE45, and EE methods



(b) Comparison of SOPESN1, SOPESN2, ODE45, and ERK2 methods

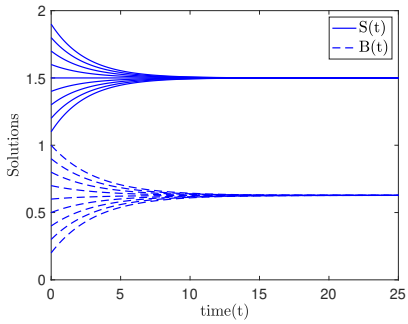


(c) Comparison of SOPESN1, ODE45, and PESN1 methods

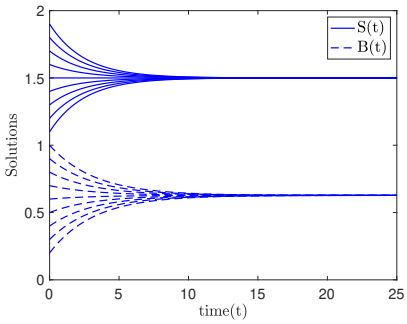


(d) Comparison of SOPESN2, ODE45, and PESN2 methods

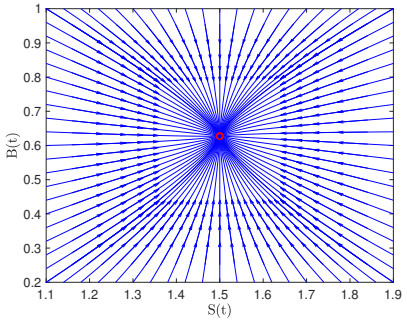
Figure 5.4: Numerical solutions of Model (4.2.1) using $h = 7$ in (a) and (b), and $h = 0.95$ in (c) and (d).



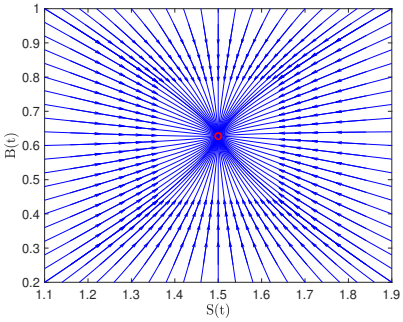
(a) Numerical solutions of Model (4.2.1) by SOPESN1 method



(b) Numerical solutions of Model (4.2.1) by SOPESN2 method



(c) Numerical solutions of Model (4.2.1) by SOPESN1 method



(d) Numerical solutions of Model (4.2.1) by SOPESN2 method

Figure 5.5: Stability of E^* (see Theorem (4.2.3)).

5.2.2 SOPESN methods applied to the competition model with microbial input and constant death rates

In this subsection, the SOPESN methods are applied to approximate the solutions of the competition model (4.3.1). Moreover, comparisons between the SOPESN methods and other standard and nonstandard finite difference methods are provided. We recall that System (4.3.1) has a boundary equilibrium, $E_0 = (S_0^*, B_{10}^*, 0)$, and a unique positive equilibrium $E_1 = (S_1^*, B_{11}^*, B_{21}^*)$. The positive equilibrium is locally asymptotically stable whenever it exists. While the boundary equilibrium is

locally asymptotically stable if and only if $d_2 > \mu_2(S_0^*)$. The SOPESN1 method for Model (4.3.1) is constructed as follows:

$$\begin{aligned}\frac{S^{k+1}-S^k}{\varphi_1(h,S^k,B_1^k,B_2^k)} &= w_1^k\Big(D\Big(S_{in}-S^k\Big)-q_1\mu_1(S^k)B_1^k-q_2\mu_2(S^k)B_2^k\Big)\\ &= w_1^k f_1(S^k,B_1^k,B_2^k),\end{aligned}$$

$$\begin{aligned}\frac{B_1^{k+1}-B_1^k}{\varphi_2(h,S^k,B_1^k,B_2^k)} &= w_2^k\Big(D\Big(B_{in}-B_1^k\Big)+\mu_1(S^k)B_1^k-m_1B_1^k\Big)\\ &= w_2^k f_2(S^k,B_1^k.B_2^k),\end{aligned}$$

$$\begin{aligned}\frac{B_2^{k+1}-B_2^k}{\varphi_3(h,S^k,B_1^k,B_2^k)} &= w_3^k\Big(B_2^k\Big(\mu_2(S^k)-D-m_2\Big)\Big)\\ &= w_3^k f_3(S^k,B_1^k.B_2^k),\end{aligned}$$

where

$$w_1^k=\begin{cases}1, & \text{if } f_1(S^k,B_1^k,B_2^k)\geq 0\\ \dfrac{S^{k+1}}{S^k}, & \text{if } f_1(S^k,B_1^k,B_2^k)<0\end{cases},\quad w_2^k=\begin{cases}1, & \text{if } f_2(S^k,B_1^k,B_2^k)\geq 0\\ \dfrac{B_1^{k+1}}{B_1^k}, & \text{if } f_2(S^k,B_1^k,B_2^k)<0\end{cases},$$

$$\text{and } w_3^k=\begin{cases}1, & \text{if } f_3(S^k,B_1^k,B_2^k)\geq 0\\ \dfrac{B_2^{k+1}}{B_2^k}, & \text{if } f_3(S^k,B_1^k,B_2^k)<0\end{cases}.$$

The denominator functions are chosen as

$$\varphi_i(h,S^k,B_1^k,B_2^k)=\begin{cases}\dfrac{1-e^{-q_i(S^k,B_1^k,B_2^k)h}}{q_i(S^k,B_1^k,B_2^k)}, & \text{if } q_i(S^k,B_1^k,B_2^k)\neq 0\\ h, & \text{if } q_i(S^k,B_1^k,B_2^k)=0\end{cases}.$$

where $q_i(S^k,B_1^k,B_2^k)$ as in Theorem (3.2.1) for all $i=1,2,3$. In order to apply the SOPESN2 method (3.3.3) to the system (4.3.1), the right-hand sides of System (4.3.1)

are rewritten in the productive-destructive setting of System (2.0.2). The functions $f_1(S, B_1, B_2)$, $f_2(S, B_1, B_2)$ and $f_3(S, B_1, B_2)$ can be rewritten as

$$f_i(S, B_1, B_2) = P_i(S, B_1, B_2) - D_i(S, B_1, B_2), \quad \text{for all } i = 1, 2, 3,$$

where

$$P_1(S, B_1, B_2) = DS_{in},\ D_1(S, B_1, B_2) = DS + q_1\mu_1(S)B_1 + q_2\mu_2(S)B_2,$$

$$P_2(S, B_1, B_2) = DB_{in} + \mu_1(S)B_1\ , D_2(S, B_1, B_2) = DB_1 + m_1B_1,$$

$$P_3(S, B_1, B_2) = \mu_2(S)B_2, \text{ and } D_3(S, B_1, B_2) = DB_2 + m_2B_2.$$

The SOPESN2 method is constructed as follows

$$\frac{S^{k+1} - S^k}{\varphi_1(h, S^k, B_1^k, B_2^k)} = P_1(S^k, B_1^k, B_2^k) - D_1(S^k, B_1^k, B_2^k)\frac{S^{k+1}}{S^k},$$

$$\frac{B_1^{k+1} - B_1^k}{\varphi_2(h, S^k, B_1^k, B_2^k)} = P_2(S^k, B_1^k, B_2^k) - D_2(S^k, B_1^k, B_2^k)\frac{B_1^{k+1}}{B_1^k}, \tag{5.2.5}$$

$$\frac{B_2^{k+1} - B_2^k}{\varphi_3(h, S^k, B_1^k, B_2^k)} = P_3(S^k, B_1^k, B_2^k) - D_3(S^k, B_1^k, B_2^k)\frac{B_2^{k+1}}{B_2^k}.$$

The denominator functions are chosen as

$$\varphi_i(h, S^k, B_1^k, B_2^k) = \begin{cases} \dfrac{1 - e^{-q_i(S^k, B_1^k, B_2^k)h}}{q_i(S^k, B_1^k, B_2^k)}, & \text{if } q_i(S^k, B_1^k, B_2^k) \neq 0 \\ \\ h, & \text{if } q_i(S^k, B_1^k, B_2^k) = 0 \end{cases}.$$

where $q_i(S^k, B_1^k, B_2^k)$ as in Theorem (3.3.1) for all $i = 1, 2, 3$.

For Model (4.3.1), we first consider the following set of parameter values: $D = 0.2$, $S_{in} = 1.5$, $B_{in} = 0.5$, $q_1 = q_2 = 10^{-2}$, $m_1 = m_2 = 0.1$, $\mu_1^{max} = 0.1$, $\mu_2^{max} = 0.3$, and $K_1 = K_2 = 0.1$, with initial conditions

$$(S(0), B_1(0), B_2(0)) = (0.5, 1, 0.5).$$

For this set of parameter values, and since $d_2 = 0.3 > 0.2812 = \mu_2(S_{in})$, then by Proposition (4.3.1) the interior equilibrium does not exist. Thus, Model (4.3.1) has only the boundary equilibrium $E_0 = (1.4977, 0.4848, 0)$, and it is locally asymptotically stable, since $d_2 = 0.3 > 0.2812 = \mu_2(S_0^*)$ (Theorem (4.3.1)). Moreover, the eigenvalues of the Jacobian matrix evaluated at E_0 are: $\lambda_1 = -0.204376 + 0.0108459\ i$, $\lambda_2 = -0.204376 - 0.0108459\ i$, and $\lambda_3 = -0.0221533$. The PESN1 method (2.0.5) for Model (4.3.1) is:

$$\frac{S^{k+1} - S^k}{\varphi(h)} = w_1^k f_1(S^k, B_1^k, B_2^k),$$

$$\frac{B_1^{k+1} - B_1^k}{\varphi(h)} = w_2^k f_2(S^k, B_1^k.B_2^k), \tag{5.2.6}$$

$$\frac{B_2^{k+1} - B_2^k}{\varphi(h)} = w_3^k f_3(S^k, B_1^k.B_2^k),$$

where

$$w_1^k = \begin{cases} 1, & \text{if } f_1(S^k, B_1^k, B_2^k) \geq 0 \\ \dfrac{S^{k+1}}{S^k}, & \text{if } f_1(S^k, B_1^k, B_2^k) < 0 \end{cases}, \quad w_2^k = \begin{cases} 1, & \text{if } f_2(S^k, B_1^k, B_2^k) \geq 0 \\ \dfrac{B_1^{k+1}}{B_1^k}, & \text{if } f_2(S^k, B_1^k, B_2^k) < 0 \end{cases},$$

$$\text{and } w_3^k = \begin{cases} 1, & \text{if } f_3(S^k, B_1^k, B_2^k) \geq 0 \\ \dfrac{B_2^{k+1}}{B_2^k}, & \text{if } f_3(S^k, B_1^k, B_2^k) < 0 \end{cases}.$$

Following the approach in [49], the denominator function is

$$\varphi(h) = \frac{1 - e^{-h0.2}}{0.2}, \text{ here, } 0.2 > \max\left\{\frac{|\lambda_j|^2}{2\,\mathrm{Re}(|\lambda_j|)} : j = 1, 2, 3\right\} = 0.102476.$$

Figure (5.6a) compares the SOPESN methods with the EE method for $h = 10$. EE method does not preserve the local stability of the equilibria for any step size h. We can see that the EE method keeps oscillating and does not converge to the exact

solution, while the SOPESN methods converge to the exact solution. Figure 5.6(b) compares the SOPESN methods with the ERK2 method for $h = 10$. Even though the ERK2 method is second-order accurate, it does not preserve the local stability of the equilibria. We can easily see that the ERK2 method diverges from the exact solution, while the SOPESN methods converge. Figure 5.6(c) and Figure 5.6(d) compare the SOPESN methods with the PESN1 method for $h = 0.95$. While all methods behave well and converge to the boundary equilibrium $E_0 = (1.4977, 0.4848, 0)$. We can see that the SOPESN methods are more accurate than the PESN1 method.

Next, we consider the following set of parameters values: $D = 0.1$, $S_{in} = 1.5$, $B_{in} = 0.1$, $q_1 = q_2 = 1$, $m_1 = 0.4$, $m_2 = 0.1$, $\mu_1^{max} = 0.4$, $\mu_2^{max} = 0.5$, and $K_1 = K_2 = 0.1$, with initial conditions:

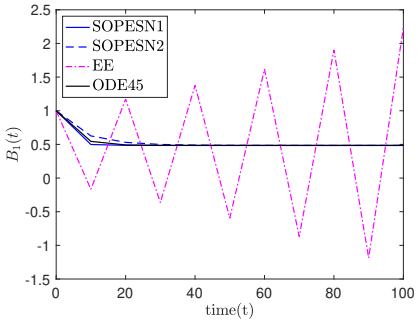
$$(S(0), B_1(0), B_2(0)) = (0.01, 0.2, 1.4).$$

For this set of parameter values, Model (4.3.1) has two equilibria, a boundary equilibrium $E_0 = (1.2165, , 0.0767, 0)$, and an interior equilibrium $E_1 = (0.0667, 0.0294, 0.6931)$. Since $d_2 = 0.2 < 0.4620 = \mu_2(S_0^*)$, then the boundary equilibrium E_0 is unstable (Theorem (4.3.1)). Moreover, since the interior equilibrium exists, then it is locally asymptotically stable (Theorem(4.3.2)). Furthermore, the eigenvalues of the Jacobian matrix evaluated at E_0 are: $\lambda_{10} = 0.262021$, $\lambda_{20} = -0.116077 + 0.0212038\ i$, and $\lambda_{30} = -0.116077 - 0.0212038\ i$. While the eigenvalues of the Jacobian matrix evaluated at E_1 are: $\lambda_{1,1} = -1.3894$, $\lambda_{2,1} = -0.160048$ and $\lambda_{3,1} = -0.20006$. Here, the denominator function for the PESN1 method (5.2.6) is:

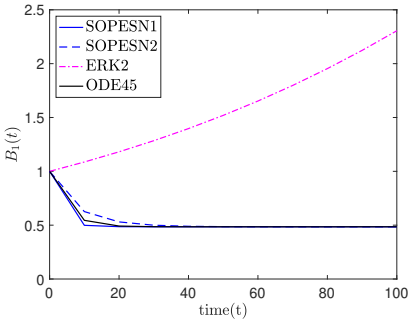
$$\varphi(h) = \frac{1 - e^{-0.6h}}{0.6},$$

where $0.6 > \max\left\{\frac{|\lambda_{jk}|^2}{2\,\mathrm{Re}(|\lambda_{jk}|)} : j = 1, 2, 3 \text{ and } k = 0, 1\right\} = 0.583768$. Figure 5.7(a) and Figure 5.7(b) compare the SOPESN methods with the PESN1 method for $h = 0.95$.

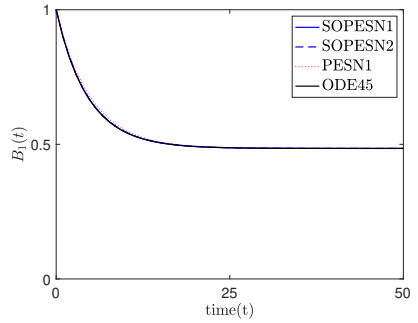
As discussed previously, the interior equilibrium $E_1 = (0.0667, 0.0294, 0.6931)$ is locally asymptotically stable. We can see all methods converge to the interior equilibrium E_1. However, the SOPESN methods converge to the exact solution faster than the PESN method.



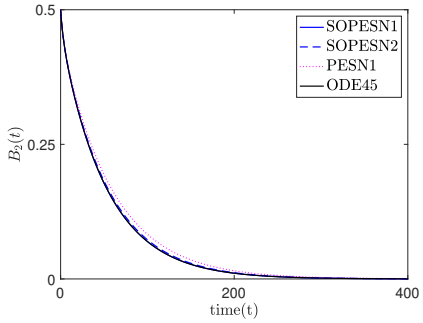
(a) Comparison of SOPESN1, SOPESN2, ODE45, and EE methods



(b) Comparison of SOPESN1, SOPESN2, ODE45, and ERK2 methods

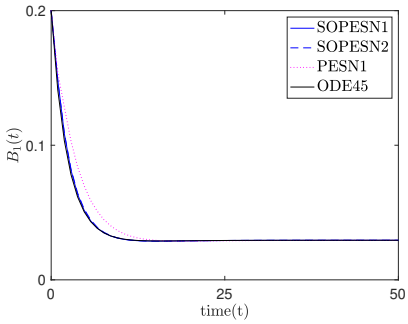


(c) Comparison of SOPESN1, SOPESN2, ODE45, and PESN1 methods

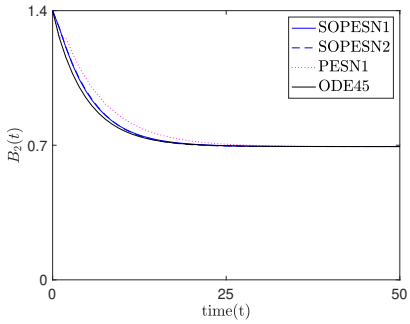


(d) Comparison of SOPESN1, SOPESN2, ODE45, and PESN2 methods

Figure 5.6: Numerical solutions of Model (4.3.1) using $h = 10$ in (a) and (b), and $h = 0.95$ in (c) and (d).



(a) Comparison of SOPESN1, SOPESN2, PESN1, and ODE45 methods



(b) Comparison of SOPESN1, SOPESN2, PESN1, and ODE45 methods

Figure 5.7: Numerical solutions of Model (4.3.1) using $h = 0.95$.

5.2.3 SOPESN methods applied to the competition model with microbial input and in the presence of a constant homogeneous plasmid

In this subsection, the SOPESN methods are applied to approximate the solution to the competition model in the presence of plasmid (4.4.1). Moreover, comparisons between the SOPESN methods, and the PESN1 method are also provided. We recall that System (4.4.1) has a boundary equilibrium, $E_0 = (S_0^*, B_{10}^*, 0)$, and a unique positive equilibrium $E_1 = (S_1^*, B_{11}^*, B_{21}^*)$. The positive equilibrium is locally asymptotically stable whenever it exists. While the boundary equilibrium is locally

asymptotically stable if and only if $D + \eta > \mu_2(S_0^*)$. The SOPESN1 method for Model (4.3.1) is constructed as follows:

$$\frac{S^{k+1} - S^k}{\varphi_1(h, S^k, B_1^k, B_2^k)} = w_1^k\Big(D\Big(S_{in} - S^k\Big) - q_1\mu_1(S^k)B_1^k - q_2\mu_2(S^k)B_2^k\Big) = f_1(S^k, B_1^k, B_2^k),$$

$$\frac{B_1^{k+1} - B_1^k}{\varphi_2(h, S^k, B_1^k, B_2^k)} = w_2^k\Big(D\Big(B_{in} - B_1^k\Big) + \mu_1(S^k)B_1^k + \eta B_2^k\Big) = f_2(S^k, B_1^k.B_2^k),$$

$$\frac{B_2^{k+1} - B_2^k}{\varphi_3(h, S^k, B_1^k, B_2^k)} = w_3^k\Big(B_2^k\Big(\mu_2(S^k) - D - \eta\Big)\Big) = f_3(S^k, B_1^k.B_2^k), \tag{5.2.7}$$

where

$$w_1^k = \begin{cases} 1, & \text{if } f_1(S^k, B_1^k, B_2^k) \geq 0 \\ \dfrac{S^{k+1}}{S^k}, & \text{if } f_1(S^k, B_1^k, B_2^k) < 0 \end{cases}, \quad w_2^k = \begin{cases} 1, & \text{if } f_2(S^k, B_1^k, B_2^k) \geq 0 \\ \dfrac{B_1^{k+1}}{B_1^k}, & \text{if } f_2(S^k, B_1^k, B_2^k) < 0 \end{cases},$$

$$\text{and } \ w_3^k = \begin{cases} 1, & \text{if } f_3(S^k, B_1^k, B_2^k) \geq 0 \\ \dfrac{B_2^{k+1}}{B_2^k}, & \text{if } f_3(S^k, B_1^k, B_2^k) < 0 \end{cases}.$$

The denominator functions are chosen as

$$\varphi_i(h, S^k, B_1^k, B_2^k) = \begin{cases} \dfrac{1 - e^{-q_i(S^k, B_1^k, B_2^k)h}}{q_i(S^k, B_1^k, B_2^k)}, & \text{if } q_i(S^k, B_1^k, B_2^k) \neq 0 \\ h, & \text{if } q_i(S^k, B_1^k, B_2^k) = 0 \end{cases}.$$

where $q_i(S^k, B_1^k, B_2^k)$ as in Theorem (3.2.1) for all $i = 1, 2, 3$. In order to apply the SOPESN2 method (3.3.3) to the system (4.3.1), the right-hand sides of System (4.3.1) are rewritten in the productive-destructive setting of System (2.0.2). The functions $f_1(S, B_1, B_2)$, $f_2(S, B_1, B_2)$ and $f_3(S, B_1, B_2)$ can be rewritten as

$$f_i(S, B_1, B_2) = P_i(S, B_1, B_2) - D_i(S, B_1, B_2), \quad \text{for all } \ i = 1, 2, 3,$$

where

$$P_1(S, B_1, B_2) = DS_{in},\ D_1(S, B_1, B_2) = DS + q_1\mu_1(S)B_1 + q_2\mu_2(S)B_2,$$

$$P_2(S, B_1, B_2) = DB_{in} + \mu_1(S)B_1 + \eta B_2,\ D_2(S, B_1, B_2) = DB_1,$$

$$P_3(S, B_1, B_2) = \mu_2(S)B_2, \text{ and } D_3(S, B_1, B_2) = DB_2 + \eta B_2.$$

The SOPESN2 method is constructed as follows

$$\frac{S^{k+1} - S^k}{\varphi_1(h, S^k, B_1^k, B_2^k)} = P_1(S^k, B_1^k, B_2^k) - D_1(S^k, B_1^k, B_2^k)\frac{S^{k+1}}{S^k},$$

$$\frac{B_1^{k+1} - B_1^k}{\varphi_2(h, S^k, B_1^k, B_2^k)} = P_2(S^k, B_1^k, B_2^k) - D_2(S^k, B_1^k, B_2^k)\frac{B_1^{k+1}}{B_1^k}, \tag{5.2.8}$$

$$\frac{B_2^{k+1} - B_2^k}{\varphi_3(h, S^k, B_1^k, B_2^k)} = P_3(S^k, B_1^k, B_2^k) - D_3(S^k, B_1^k, B_2^k)\frac{B_2^{k+1}}{B_2^k}.$$

The denominator functions are chosen as

$$\varphi_i(h, S^k, B_1^k, B_2^k) = \begin{cases} \dfrac{1 - e^{-q_i(S^k, B_1^k, B_2^k)h}}{q_i(S^k, B_1^k, B_2^k)}, & \text{if } q_i(S^k, B_1^k, B_2^k) \neq 0 \\ h, & \text{if } q_i(S^k, B_1^k, B_2^k) = 0 \end{cases}.$$

where $q_i(S^k, B_1^k, B_2^k)$ as in Theorem (3.3.1) for all $i = 1, 2, 3$.

For Model (4.4.1), we first consider the following set of parameter values: $D = 0.1$, $S_{in} = 2$, $B_{in} = 0.1$, $q_1 = 0.1$, $q_2 = 0.2$, $\mu_1^{max} = 0.1$, $\mu_2^{max} = 0.2$, $\eta = 0.1$, and $K_1 = K_2 = 0.1$, with the following initial conditions:

$$(S(0), B_1(0), B_2(0)) = (0.5, 0.5, 0.1).$$

With this set of parameter values, Model (4.4.1) has only the boundary equilibrium $E_0 = (1.8182, 1.9182, 0)$. Since $D + \eta = 0.2 > 0.1905 = \mu_2(S_{in})$, then the interior equilibrium E_1 does not exist (Theorem (4.4.2)). Moreover, since

$D + \eta = 0.2 > 0.1896 = \mu_2(S_0^*)$, the boundary equilibrium E_0 is locally asymptotically stable (Theorem (4.4.4)). Furthermore, the eigenvalues of the Jacobian matrix evaluated at the equilibrium E_0 are $\lambda_1 = -0.1$, $\lambda_2 = -0.0104264$, and $\lambda_3 = -0.00573454$. The PESN1 method (2.0.5) for Model (4.4.1) is constructed as follows:

$$\frac{S^{k+1} - S^k}{\varphi(h)} = w_1^k f_1(S^k, B_1^k, B_2^k),$$

$$\frac{B_1^{k+1} - B_1^k}{\varphi(h)} = w_2^k f_2(S^k, B_1^k.B_2^k), \tag{5.2.9}$$

$$\frac{B_2^{k+1} - B_2^k}{\varphi(h)} = w_3^k f_3(S^k, B_1^k.B_2^k),$$

where

$$w_1^k = \begin{cases} 1, & \text{if } f_1(S^k, B_1^k, B_2^k) \geq 0 \\ \dfrac{S^{k+1}}{S^k}, & \text{if } f_1(S^k, B_1^k, B_2^k) < 0 \end{cases}, \quad w_2^k = \begin{cases} 1, & \text{if } f_2(S^k, B_1^k, B_2^k) \geq 0 \\ \dfrac{B_1^{k+1}}{B_1^k}, & \text{if } f_2(S^k, B_1^k, B_2^k) < 0 \end{cases},$$

$$\text{and } w_3^k = \begin{cases} 1, & \text{if } f_3(S^k, B_1^k, B_2^k) \geq 0 \\ \dfrac{B_2^{k+1}}{B_2^k}, & \text{if } f_3(S^k, B_1^k, B_2^k) < 0 \end{cases}.$$

Following the approach in [49], the denominator function is

$$\varphi(h) = \frac{1 - e^{-0.1h}}{0.1}, \text{ here } 0.1 > \max\left\{\frac{|\lambda_j|}{2} : j = 1, 2, 3\right\} = 0.05.$$

Figure 5.8(a) and Figure 5.8(b) compare the SOPESN methods with the PESN1 method for $h = 0.95$. As discussed above, the interior equilibrium does not exist, while the boundary equilibrium exists, and it is locally asymptotically stable. One can easily see that all numerical methods converge to the boundary equilibrium E_0. However, the SOPESN methods converge to the exact solution at a faster rate than the PESN1 method.

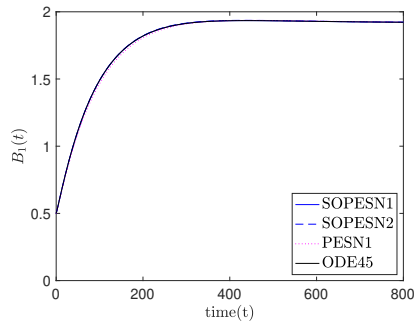
Next, we consider the following set of parameter values: $D = 0.1$, $S_{in} = 2$, $B_{in} = 0.1$, $q_1 = q_2 = 1$, $\mu_1^{max} = 0.2$, $\mu_2^{max} = 0.6$, $\eta = 0.1$, and $K_1 = K_2 = 0.1$, with the following initial conditions:

$$(S(0), B_1(0), B_2(0)) = (0.02, 0.8, 0.1).$$

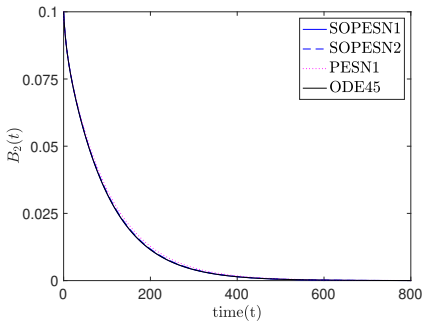
In this case, Model (4.4.1) has two equilibria. One is the boundary equilibrium $E_0 = (0.0905, 2.0095, 0)$ and the other is the interior equilibrium $E_1 = (0.05, 0.4375, 1.6125)$. Since $D + \eta = 0.2 < 0.2851 = \mu_2(S_0^*)$, the boundary equilibrium is unstable (Theorem (4.4.4)). Moreover, since the interior equilibrium E_1 exists, it is locally asymptotically stable (Theorem (4.4.5)). Furthermore, the eigenvalues of the Jacobian matrix evaluated at E_0 are: $\lambda_{10} = -1.11245$, $\lambda_{20} = -0.1$, and $\lambda_{30} = 0.0850394$. While the eigenvalues of the Jacobian matrix evaluated at E_1 are: $\lambda_{11} = -2.57287$, $\lambda_{21} = -0.1$, and $\lambda_{31} = -0.0604599$. Here, the denominator function $\varphi(h)$ of the PESN1 method (5.2.9) is

$$\varphi(h) = \frac{1 - e^{-1.5h}}{1.5},$$

where $1.5 > \max\left\{\frac{|\lambda_{jk}|^2}{2\,\mathrm{Re}(|\lambda_{jk}|)} : j = 1, 2, 3 \text{ and } k = 0, 1\right\} = 1.28644$. Figure 5.9(a) and Figure 5.9(b) compare the SOPESN methods with the PESN1 method for $h = 0.95$. In this case, we know that the interior equilibrium is locally asymptotically stable while the boundary equilibrium is not stable. Both figures show that all numerical converge to the interior equilibrium E_1. However, the SOPESN methods are more accurate than the PESN1 method.

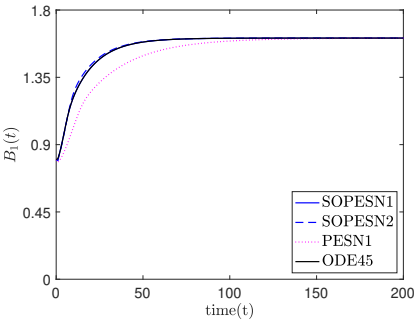


(a) Comparison of SOPESN1, SOPESN2, PESN1, and ODE45 methods

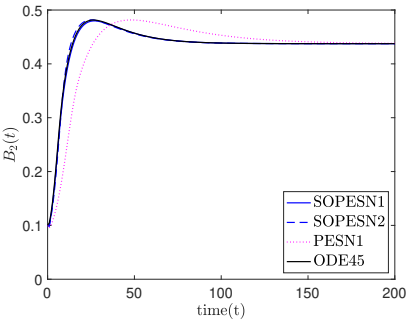


(b) Comparison of SOPESN1, SOPESN2, PESN1, and ODE45 methods

Figure 5.8: Numerical solutions of Model (4.4.1) using $h = 0.95$.



(a) Comparison of SOPESN1, SOPESN2, PESN1, and ODE45 methods



(b) Comparison of SOPESN1, SOPESN2, PESN1, and ODE45 methods

Figure 5.9: Numerical solutions of Model (4.4.1) using $h = 0.95$.

CHAPTER 6

Conclusions

In the first part of this dissertation, two modified nonstandard numerical methods, namely SOPESN1 and SOPESN2 methods, were constructed and analyzed. SOPESN1 method was formulated to approximate solutions of general n-dimensional autonomous differential equations (2.0.1), while SOPESN2 method was formulated to approximate solutions of general n-dimensional productive-destructive equations (2.0.2). This work presented a novel approach that modifies the nonstandard denominator functions in the PESN numerical methods to increase the order of accuracy of existing underlying NSFD methods, resulting in a second-order accuracy of the corresponding methods. The numerical methods are dynamically consistent with respect to the equilibria's local stability and the positivity of the solution. Moreover, they are second-order accurate and can be written in an explicit form that makes them easy to implement, so they can be used to solve numerous problems arising in economics, engineering, and science.

Next, we analyzed three chemostat models with microbial input. The first model aims to represent the growth of a bacteria with the bacteria in the input inflow and in the presence of a constant homogeneous constant. Since competition is crucial in nature. A detailed analysis of a competition model with a microbial input and a constant death rate due to a toxin was presented. Also, since the plasmid plays an essential role in antibiotic resistance, we analyzed a competition model with microbial input in the presence of a constant homogeneous plasmid. As was expected, all models have an interior equilibrium that is stable whenever it exists. This ensures

the coexistence of all species in the chemostat. When there is no competition in the chemostat with microbial input and a constant death rate, the analysis of Model(4.2.1) reveals that the donor bacteria will always be present. However, in the case of the competition, either in the presence of a toxin Model (4.3.1) or a plasmid Model (4.4.1). We note that the coexistence of both bacteria depends strongly on the relationship between the growth rate of the donor bacteria and its dilution rate.

Finally, before applying the SOPESN methods to approximate the solutions of the chemostat models, the SOPESN methods were compared with several standard and nonstandard finite difference methods. The numerical simulations illustrate the importance of the elementary stability, positivity-preserving properties of the SOPESN methods and the SOPESN methods are more accurate than the first-order numerical methods. Then, SOPESN methods were used to approximate the solutions of the chemostat models and compare with other standard and nonstandard finite difference methods. As shown, the SOPESN methods perform better than the other methods. Also, by using the SOPESN methods, the theoretical results presented in Chapter (4) were validated.

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BIOGRAPHICAL STATEMENT

Fawaz Karhan Alalhareth was born in Najran, Saudi Arabia in 1990. He earned his B.S. and M.S. degrees in Mathematics at Najran University and at Ball State University, Muncie, Indiana, in 2013 and 2017, respectively.