DECOMPOSITION OF MODULES AND TENSOR PRODUCTS OVER PRINCIPAL SUBALGEBRAS OF TRUNCATED POLYNOMIAL RINGS

by

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ABSTRACT

DECOMPOSITION OF MODULES AND TENSOR PRODUCTS OVER PRINCIPAL SUBALGEBRAS OF TRUNCATED POLYNOMIAL RINGS

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The topic of my dissertation is to investigate the behavior of modules and tensor products over a truncated polynomial ring with prime characteristic. This investigation utilizes principal subalgebras of the truncated polynomial ring as the main tool for studying these objects. Then, we investigate if these modules and their tensor products have a similar behavior when viewed over more general truncated polynomial rings. In particular, we aim to investigate the behavior of these objects when we replace principal subalgebras over a field with prime characteristic by hypersurfaces over a field with no characteristic restriction.

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CHAPTER 1

Preliminaries

1.1 Historical Motivation and Set-Up

There are deep connections between algebra and geometry. In particular, there is a fundamental concept of support of various algebraic structures. In 1971, Quillen [13] started the idea of defining the support of various algebraic structures. In particular, he gives a description of the algebraic variety corresponding to the cohomolgy ring of a finite group called the support variety. In 1982, Carlson defines in [7] the notion of a rank variety, which is another type of support. Carlson [7] and Avrunin-Scott [3] then show that the rank variety is isomorphic to support variety, particularly for modules over group algebras of elementary abelian p-groups. Rank varieties give us a way of studying the geometry of modules. Carlson's and Quillen's work has resulted in the development of analogous theories in various contexts, mainly modules over commutative complete intersection rings, and over commutative Hopf algebras.

The purpose of this thesis is to continue Carlson's work in [7]. Moreover, we strive to understand the structure of modules over principal ideal rings. To do this, we apply the classification theorem for principal ideal domains to modules over truncated polynomial restricted to principal subalgebras with characteristic p. The classification theorem will allow us to decompose our models over certain principal subalgebras while holding onto the group algebra structure of our truncated polynomial ring. We can find the decompositions of our modules using their representation matrices. From here, we aim to understand the structure of the tensor product of these modules by looking at their rank varieties. This is where we introduce Carlson's identity

$$V(M \otimes_k N) = V(M) \cap V(N) \qquad (*)$$

and apply it to modules over our truncated polynomial ring. Carlson [7] uses this identity to show that every algebraic variety is the rank variety of some module over our truncated polynomial ring. We will use (*) to investigate a particular class of modules over our truncated polynomial ring with prime characteristic and compare them with similar class of modules over a truncated polynomial ring with not necessarily prime characteristic.

Let $A_p^n = k[X_1, \ldots, X_n]/(X_1^p, \ldots, X_n^p)$, that is, A_p^n is the truncated polynomial ring where k is a field and $char(k) = p$ for some prime p. We will use x_i to denote the coset of X_i. We investigate how modules decompose over the principal subalgebras $R_\lambda = k[u_\lambda]$ where $u_\lambda = \lambda_1 x_1 + \cdots + \lambda_n x_n$ for $\lambda = (\lambda_1, \ldots, \lambda_n) \in k^n$. In particular, we investigate how decompositions may, or may not, change due to the choice of the principal subalgebra. The modules whose decompositions don't change are referred to as modules of constant Jordan type [4], and these have been well-studied. The primary interest of this thesis is in the modules whose decompositions do change. In particular, we are interested in determining the decompositions for tensor products over the principal subalgebras. Understanding these decompositions reduces to the Clebsch-Gordan problem for a truncated polynomial ring in one variable.

1.2 Applying the Fundamental Theorem for Modules over PIDs to $k[X]/(X^p)$

Theorem 1.2.1. *[15] Let M be a finitely generated module over a principal ideal domain R. Then M is a direct sum of cyclic submodules. More precisely, there exist* *nonnegative integers h, m, irreducible elements $p_1, \ldots, p_m \in R$ and positive integers $t_1, \ldots, t_m$ such that*

$$M \cong R/Rp_1^{t_1} \oplus \cdots \oplus R/Rp_m^{t_m} \oplus R^h.$$

Theorem 1.2.2. *Let M be a finitely generated module over $k[x]/(x^p)$ and $D_i = k[x]/(x^i)$ for any $1 \leq i \leq p$. Then there exists nonnegative integers m_i, $i = 1, \ldots, p$ such that*

$$M \cong D_1^{m_1} \oplus D_2^{m_2} \oplus \cdots \oplus D_p^{m_p}.$$

Proof. There is a natural epimorphism $f : k[x] \to k[x]/(x^p)$ Thus, any finitely generated $k[x]/(x^p)$-module M can also be considered as a $k[x]$-module with

$$M \cong k[x]/(p_1^{t_1}) \oplus \cdots \oplus k[x]/(p_m^{t_m}) \oplus k[x]^h.$$

When viewing M as a $k[x]/(x^p)$-module, we have that $x^p M = 0$. This forces $h = 0$ and $x^p k[x]/(p_i^{t_i}) = 0$ for each i. This implies that $x^p \in (p_i^{t_i})$ for each i, so $(x^p) \subseteq (p_i^{t_i})$ for each i. Using radicals [8], we can see that $(x) \subseteq (p_i)$ for each i. Since each p_i is irreducible, it immediately follows that $(x) \supseteq (p_i)$, so we have that $(x) = (p_i)$ for each i. $\square$

1.3 Decomposition

This section will be dedicated to examples where $n > 1$ and each direct summand is of the form $k[u_\lambda]/(u_\lambda^i)$. It is worth noting that $\dim_k(A) = p^n$ and $\dim_k(R_\lambda) = p$ for each nonzero u_λ.

Example. Let $A = k[x_1, x_2]/(x_1^2, x_2^2)$ and $M = (x_1)$. We want to look at the decomposition of M as a

(i) $k[x_1]$ - module

(ii) $k[x_2]$ - module

(iii) $k[x_1 + x_2]$ - module

Note that a basis for M as a k-vector space is $\{x_1, x_1x_2\}$, so $\dim_k(M) = 2$.

For (i), M over $R = k[x_1]$ is generated by $\{x_1, x_1x_2\}$ so by our classification theorem, we have that

$$M \cong R/ann_R(x_1) \oplus R/ann_R(x_1x_2) = R/(x_1) \oplus R/(x_1) \cong k^2.$$

For (ii), M over $R = k[x_2]$ is generated by $\{x_1\}$ so we have that

$$M \cong R/ann_R(x_1) = R/\{0_R\} \cong R,$$

which means that M is free of rank 1 as a $k[x_2]$-module.

For (iii), M over $R = k[x_1 + x_2]$ is generated by $\{x_1\}$ so we have that

$$M \cong R/ann_R(x_1) = R/\{0_R\} \cong R,$$

so we also have that M is free of rank 1 as a $k[x_1 + x_2]$-module.

Example. Let $A = k[x_1, x_2]/(x_1^2, x_2^2)$ and $M = (x_1, x_2)$. We want to look at the composition of M as a

(i) $k[x_1]$ - module

(ii) $k[x_2]$ - module

(iii) $k[x_1 + x_2]$ - module

Note that a base for M as a k-vector space is $\{x_1, x_2, x_1x_2\}$ so $\dim_k(M) = 3$.

For (i), M over $R = k[x_1]$ is generated by $\{x_1, x_2\}$ so we have that

$$M \cong R/ann_R(x_1) \oplus R/ann_R(x_2) = R/(x_1) \oplus R/\{0\} \cong k \oplus R.$$

For (ii), M over $R = k[x_2]$ is generated by $\{x_1, x_2\}$ so we have that

$$M \cong R/ann_R(x_1) \oplus R/ann_R(x_2) = R/\{0\} \oplus R/(x_1) \cong R \oplus k.$$

For (iii), M over $R = k[x_1 + x_2]$ is generated by $\{x_1 + x_2, x_2\}$ so we have that

$$M \cong R/ann_R(x_1 + x_2) \oplus R/ann_R(x_2) = R/(x_1 + x_2) \oplus R/\{0\} \cong k \oplus R.$$

Notice that as we change the subalgebra on Example 2, the structure of the decomposition remains the same. We consider $M = (x_1, x_2)$ to be a module of constant Jordan type.

Example. Let $A = k[x_1, x_2, x_3]/(x_1^2, x_2^2, x_3^2)$ with $char(k) = 2$ and define $M = (x_1)$. We want to look at the composition of M as a

(i) $k[x_1]$ - module

(ii) $k[x_2]$ - module

(iii) $k[x_3]$ - module

(iv) $k[x_1 + x_2]$ - module

(v) $k[x_1 + x_3]$ - module

(vi) $k[x_2 + x_3]$ - module

(vii) $k[x_1 + x_2 + x_3]$ - module

Note that a base for M as a k-vector space is $\{x_1, x_1x_2, x_1x_3, x_1x_2x_3\}$ so $\dim_k(M) = 4$. For (i), M over $R = k[x_1]$ is generated by $\{x_1, x_1x_2, x_1x_3, x_1x_2x_3\}$ so by our classification theorem, we have that

$$M \cong R/ann_R(x_1) \oplus R/ann_R(x_1x_2) \oplus R/ann_R(x_1x_3) \oplus R/ann_R(x_1x_2x_3) \cong k^4.$$

For (ii), M over $R = k[x_2]$ is generated by $\{x_1, x_1x_3\}$ so we have that

$$M \cong R/ann_R(x_1) \oplus R/ann_R(x_1x_3) = R/\{0_R\} \oplus R/\{0_R\} \cong R^2,$$

which means that M is free of rank 2 as a $k[x_2]$-module.

For (iii), M over $R = k[x_3]$ is generated by $\{x_1, x_1x_2\}$ so we have that

$$M \cong R/ann_R(x_1) \oplus R/ann_R(x_1x_2) \cong R/\{0_R\} \oplus R/\{0_R\} \cong R^2$$

which means that M is free of rank 2 as a $k[x_3]$-module.

For (iv), M over $R = k[x_1 + x_2]$ is generated by $\{x_1, x_1x_3\}$ so we have that

$$M \cong R/ann_R(x_1) \oplus R/ann_R(x_1x_3) = R/\{0_R\} \oplus R/\{0_R\} \cong R^2,$$

so we also have that M is free of rank 2 as a $k[x_1 + x_2]$-module.

For (v), M over $R = k[x_1 + x_3]$ is generated by $\{x_1, x_1x_2\}$ so we have that

$$M \cong R/ann_R(x_1) \oplus R/ann_R(x_1x_2) = R/\{0_R\} \oplus R/\{0_R\} \cong R^2,$$

so we also have that M is free of rank 2 as a $k[x_1 + x_3]$-module.

For (vi), M over $R = k[x_2 + x_3]$ is generated by $\{x_1, x_1x_2\}$ so we have that

$$M \cong R/ann_R(x_1) \oplus R/ann_R(x_1x_2) = R/\{0_R\} \oplus R/\{0_R\} \cong R^2,$$

so we also have that M is free of rank 2 as a $k[x_2 + x_3]$-module.

For (vii), M over $R = k[x_1 + x_2 + x_3]$ is generated by $\{x_1, x_1x_2\}$ so we have that

$$M \cong R/ann_R(x_1) \oplus R/ann_R(x_1x_2) = R/\{0_R\} \oplus R/\{0_R\} \cong R^2,$$

so we also have that M is free of rank 2 as a $k[x_1 + x_2 + x_3]$-module.

Example. Let $A = k[x_1, x_2, x_3]/(x_1^2, x_2^2, x_3^2)$ with $char(k) = 2$ and define $M = (x_1, x_2)$. We want to look at the composition of M as a

(i) $k[x_1]$ - module

(ii) $k[x_2]$ - module

(iii) $k[x_3]$ - module

(iv) $k[x_1 + x_2]$ - module

(v) $k[x_1 + x_3]$ - module

(vi) $k[x_2 + x_3]$ - module

(vii) $k[x_1 + x_2 + x_3]$ - module

Note that a base for M as a k-vector space is $\{x_1, x_2, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3\}$ so $\dim_k(M) = 6$.

For (i), M over $R = k[x_1]$ is generated by $\{x_1, x_2, x_1x_3, x_2x_3\}$ so by our classification theorem, we have that

$$M \cong R/ann_R(x_1) \oplus R/ann_R(x_2) \oplus R/ann_R(x_1x_3) \oplus R/ann_R(x_2x_3) \cong k^2 \oplus R^2.$$

For (ii), M over $R = k[x_2]$ is generated by $\{x_1, x_2, x_1x_3, x_2x_3\}$ so we have that

$$M \cong R/ann_R(x_1) \oplus R/ann_R(x_2) \oplus R/ann_R(x_1x_3) \oplus R/ann_R(x_2x_3) \cong k^2 \oplus R^2.$$

For (iii), M over $R = k[x_3]$ is generated by $\{x_1, x_2, x_1x_2\}$ so we have that

$$M \cong R/ann_R(x_1) \oplus R/ann_R(x_2) \oplus R/ann_R(x_1x_2) \cong R^3$$

which means that M is free of rank 3 as a $k[x_3]$-module.

For (iv), M over $R = k[x_1 + x_2]$ is generated by $\{x_1, x_1 + x_2, x_1x_3, x_1x_3 + x_2x_3\}$ so we have that

$$M \cong R/ann_R(x_1) \oplus R/ann_R(x_1 + x_2) \oplus R/ann_R(x_1x_3) \oplus R/ann_R(x_1x_3 + x_2x_3) \cong k^2 \oplus R^2.$$

For (v), M over $R = k[x_1 + x_3]$ is generated by $\{x_1, x_2, x_2x_3\}$ so we have that

$$M \cong R/ann_R(x_1) \oplus R/ann_R(x_2) \oplus R/ann_R(x_2x_3) \cong R^3,$$

so we also have that M is free of rank 3 as a $k[x_1 + x_3]$-module.

For (vi), M over $R = k[x_2 + x_3]$ is generated by $\{x_1, x_2, x_1x_2\}$ so we have that

$$M \cong R/ann_R(x_1) \oplus R/ann_R(x_2) \oplus R/ann_R(x_1x_2) \cong R^3,$$

so we also have that M is free of rank 3 as a $k[x_2 + x_3]$-module.

For (vii), M over $R = k[x_1 + x_2 + x_3]$ is generated by $\{x_1, x_2, x_1x_2\}$ so we have that

$$M \cong R/ann_R(x_1) \oplus R/ann_R(x_2) \oplus R/ann_R(x_1x_2) \cong R^3,$$

so we also have that M is free of rank 3 as a $k[x_1 + x_2 + x_3]$-module.

1.4 Rank Varieties and Representation Matrices

Definition 1.4.1. The *rank variety* is defined as the set of $\lambda \in k^n$ such that an A-module M is not free when restricted to $k[u_\lambda]$. We will denote this as $V(M)$, which is the same notation that Carlson uses in [7].

Although it is not clear from the definition, Carlson proves that $V(M)$ is an algebraic variety. [Theorem 4.3 [7]]. Therefore, we can view the rank variety of a module as the zero set of certain polynomials. Consider the polynomial ring $k[\chi_1, \ldots, \chi_n]$ and the polynomials $f_1, \ldots, f_r \in k[\chi_1, \ldots, \chi_n]$. If we have that

$$V(M) = \{\lambda \in k^n : f_i(\lambda) = 0 \; \forall \; i\}$$

for some A-module M defined by $(f_1, \ldots, f_r)$, then we say the rank variety $V(M)$ is the affine variety $\mathbf{V}(f_1, \ldots, f_r)$. We will look at an example of a rank variety in this section.

Example. Choose $A_2^2 = k[x_1, x_2]/(x_1^2, x_2^2)$ and consider the ideal $I = (x_1)$ of A_2^2. Note that I has the following k-basis: $\{x_1, x_1x_2\}$. If $\lambda = (1, 0)$, then $I \cong k^2$ as an R-module. If $\lambda = (0, 1)$, then $I \cong R$ as an R-module. Notice for this example, we not only had different decompositions for I when restricted to $k[u_\lambda]$ for different λ, but with the right conditions on λ, I is a free $k[u_\lambda]$-module. In particular, we have that

$$V(I) = \{\lambda \in k^2 \; : \; \lambda_2 = 0\}.$$

We can also say that

$$V(I) = \mathbf{V}(\chi_2).$$

This is not always clear to see in most examples. In order to understand what the conditions on λ have to be in order for an A-module M to be free over R_λ, we look at the representation matrix of u_λ over M.

Definition 1.4.2. Let M be a finitely generated A-module with the following k-basis: $\{e_1, e_2, \ldots, e_m\}$. Suppose there exists $a_{ij} \in k$ such that

$$\begin{aligned} u_\lambda e_1 &= a_{11}e_1 + a_{21}e_2 + \cdots + a_{m1}e_m \\ u_\lambda e_2 &= a_{12}e_1 + a_{22}e_2 + \cdots + a_{m2}e_m \\ &\vdots \\ u_\lambda e_m &= a_{1m}e_1 + a_{2m}e_2 + \cdots + a_{mm}e_m. \end{aligned}$$

Then the following matrix is the *representation matrix* of u_λ on M with respect to the k-basis $\{e_1, e_2, \ldots, e_m\}$:

$$\begin{bmatrix} u_\lambda \end{bmatrix}_M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}$$

We will take the previous examples and look at the representation matrices of each of these modules over $k[u_\lambda]$.

Example. Let $A = k[x_1, x_2]/(x_1^2, x_2^2)$ and $M_1 = (x_1)$. So $u_\lambda = \lambda_1 x_1 + \lambda_2 x_2$. Note that the k-basis for M_1 in lexicographic order is $\{x_1, x_1x_2\}$. Note that

$$\begin{aligned} u_\lambda x_1 &= \lambda_2 x_1 x_2 \\ u_\lambda x_1 x_2 &= 0 \end{aligned}$$

so we have that

$$\begin{bmatrix} u_\lambda \end{bmatrix}_{M_1} = \begin{bmatrix} 0 & 0 \\ \lambda_2 & 0 \end{bmatrix}.$$

Let $M_2 = (x_1, x_2)$. Note that the k-basis for M_2 in lexicographic order is $\{x_1, x_2, x_1x_2\}$. So we have that

$$u_\lambda x_1 = \lambda_2 x_1 x_2$$

$$u_\lambda x_2 = \lambda_1 x_1 x_2$$

$$u_\lambda x_1 x_2 = 0$$

which gives us

$$\begin{bmatrix} u_\lambda \end{bmatrix}_{M_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_2 & \lambda_1 & 0 \end{bmatrix}.$$

Example. Let $A = k[x_1, x_2]/(x_1^2, x_2^2, x_3^2)$ and $M_1 = (x_1)$. So $u_\lambda = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$. Note that the k-basis for M_1 in lexicographic order is $\{x_1, x_1x_2, x_1x_3, x_1x_2x_3\}$. So we have that

$$u_\lambda x_1 = \lambda_2 x_1 x_2 + \lambda_3 x_1 x_3$$

$$u_\lambda x_1 x_2 = \lambda_3 x_1 x_2 x_3$$

$$u_\lambda x_1 x_3 = \lambda_2 x_1 x_2 x_3$$

$$u_\lambda x_1 x_2 x_3 = 0$$

which gives us

$$\begin{bmatrix} u_\lambda \end{bmatrix}_{M_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \lambda_2 & 0 & 0 & 0 \\ \lambda_3 & 0 & 0 & 0 \\ 0 & \lambda_3 & \lambda_2 & 0 \end{bmatrix}.$$

Let $M_2 = (x_1, x_2)$. Note that the k-basis for M_2 in dictionary order is $\{x_1, x_2, x_1x_2, x_1x_3, x_2x_3, x_1x_2, x_3\}$. So we have that

$$u_\lambda x_1 = \lambda_2 x_1 x_2 + \lambda_3 x_1 x_3$$

$$
\begin{aligned}
u_\lambda x_2 &= \lambda_1 x_1 x_2 + \lambda_3 x_2 x_3 \\
u_\lambda x_1 x_2 &= \lambda_3 x_1 x_2 x_3 \\
u_\lambda x_1 x_3 &= \lambda_2 x_1 x_2 x_3 \\
u_\lambda x_2 x_3 &= \lambda_1 x_1 x_2 x_3 \\
u_\lambda x_1 x_2 x_3 &= 0
\end{aligned}
$$

which gives us

$$
\begin{bmatrix} u_\lambda \end{bmatrix}_{M_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_2 & \lambda_1 & 0 & 0 & 0 & 0 \\ \lambda_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & \lambda_2 & \lambda_1 & 0 \end{bmatrix}.
$$

So far, we have just looked at some examples of representation matrices for A-modules over $k[u_\lambda]$. These matrices provide us with a nice tool in determining when M is free over $k[u_\lambda]$. Notice in all of these examples, every A-module's vector space dimension is a multiple of p. It is clear that if we have an A-module M whose vector space dimension is not a multiple of p, then M is never free over $k[u_\lambda]$. Thus, we will always assume that $\dim_k(M) = mp$ for some positive integer m. The following proposition tells us how we can use an A-module's representation matrix to determine its freeness over $k[u_\lambda]$.

Proposition 1.4.3. *Consider a finitely generated A-module M with $dim_k(M) = mp$ for some nonnegative integer m. Then M is free as a $k[u_\lambda]$-module if and only if*

$\text{rank}\left(\begin{bmatrix} u_\lambda \end{bmatrix}_M\right) = m(p-1).$

Proof. Assume $dim_k(M) = mp$ for some positive integer m. Suppose $M \cong k[u_\lambda]^m$ as a $k[u_\lambda]$-module. This implies that the Jordan normal form of $\begin{bmatrix} u_\lambda \end{bmatrix}_M$ consists of m Jordan blocks of size $p \times p$, which has rank $(p-1)m$. Thus, it immediately follows that $\text{rank}\left(\begin{bmatrix} u_\lambda \end{bmatrix}_M\right) = m(p-1)$. Now suppose that $\text{rank}\left(\begin{bmatrix} u_\lambda \end{bmatrix}_M\right) = m(p-1)$. We will prove this direction using contradiction by assuming that M is not free as a $k[u_\lambda]$-module. Then for the Jordan normal form of $\begin{bmatrix} u_\lambda \end{bmatrix}_M$, there is at least one Jordan block of size $r \times r$ where $r < p$. It is important to note that $\text{rank}(J_p) > \text{rank}(J_r) + \text{rank}(J_{p-r})$ for any $r < p$ since

$$\text{rank}(J_p) = p - 1 > p - 2 = r - 1 + p - r - 1 = \text{rank}(J_r) + \text{rank}(J_{p-r}).$$

In order for dimensions to add up, we need at least one more Jordan block of size $p - r \times p - r$. Since ranks are invariant between a square matrix and its Jordan normal form, this means that

$$\begin{aligned}
\text{rank}\left(\begin{bmatrix} u_\lambda \end{bmatrix}_M\right) &= \sum_{i=1}^{m-1} \text{rank}(J_p) + \text{rank}(J_r) + \text{rank}(J_{p-r}) \\
&= \sum_{i=1}^{m-1} (p-1) + r - 1 + p - r - 1 \\
&= (m-1)(p-1) + p - 2 \\
&= mp - m - 2 \\
&\leq m(p-1)
\end{aligned}$$

which contradicts our original assumption. Therefore, we have that M is free as a $k[u_\lambda]$-module.

□

1.5 Tensor Products and Kronecker Products

One goal we have in this paper is to determine the structure of tensor products of A-modules. In particular, we want to understand the structure of their representation matrices. To do this, we first list some lemmas and propositions that will be useful for understanding certain properties of these tensor products. We then will use those to help us understand the structure of the Kronecker product of two matrices. This will assist us in determining the representation matrices of A-modules. In Chapter 2, we will define the group algebra and coalgebra structure of A, so we can construct a tensor product for A-modules. We will then look at some explicit examples of tensor products of A-modules and their representation matrices. In Chapter 3, we will refocus on the decomposition of A-modules. In particular, we will examine how to determine the decomposition of $M \otimes_k N$ if we know the decompositions for A-modules M and N. In Chapter 4, we will introduce the idea of modules over more general truncated polynomials and how to construct a "faux" tensor product once we lose the coalgebra structure of A. In the rest of this section, we will list lemmas and propositions that will allow us to find the representation matrices of $M \otimes_k N$, given the representation matrices of M and N.

Lemma 1.5.1. *Let $T : U \to V$ and $T' : U' \to V'$ be linear maps of vector spaces over a field k. Suppose the $\operatorname{rank}(T) = r$ and $\operatorname{rank}(T') = s$. Then the $\operatorname{rank}(T \otimes_k T') = rs$.*

Proof. Note that the $r = \dim_k(\operatorname{Im} T)$ and $s = \dim_k(\operatorname{Im} T')$. Let $B_T = \{e_1, \ldots, e_r\}$ be the k-basis for $\operatorname{Im} T$ and $B_{T'} = \{f_1, \ldots, f_s\}$ be the k-basis for $\operatorname{Im} T'$. Consider the set $B = \{e_1 \otimes f_1, \ldots, e_1 \otimes f_s, e_2 \otimes f_1, \ldots, e_2 \otimes f_s, \ldots, e_r \otimes f_1, \ldots, e_r \otimes f_s\}$. Note that $|B| = rs = \operatorname{rank}(T) \cdot \operatorname{rank}(T')$. We want to prove that B is a basis for $Im(T \otimes_k T')$. Note that for any $u \in U$ and $u' \in U'$, we have that $(T \otimes T')(u \otimes u') = T(u) \otimes T'(u')$.

Note that $T(u) = \sum_{i=1}^{r} a_i e_i$ and $T'(u') = \sum_{j=1}^{s} b_j f_j$ where all $a_i, b_j \in k$. So we have that

$$(T \otimes_k T')(u \otimes u') = T(u) \otimes T'(u') = \left(\sum_{i=1}^{r} a_i e_i \right) \otimes \left(\sum_{j=1}^{s} b_j f_j \right) = \sum_{i=1}^{r} \sum_{j=1}^{s} a_i b_j (e_i \otimes f_j)$$

which shows that B spans $\mathrm{Im}(T \otimes_k T')$. Since B spans $\mathrm{Im}(T \otimes_k T')$ and $|B| = rs = \dim_k(T \otimes_k T')$, it follows that B must be a basis for $\mathrm{Im}(T \otimes_k T')$. Therefore, $\mathrm{rank}(T \otimes_k T') = rs$. $\square$

Definition 1.5.2. If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is a $p \times q$ matrix, then the Kronecker Product $A \bigotimes B$ is the $pm \times qn$ block matrix

$$A \bigotimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \dots & a_{11}b_{1q} & \dots & \dots & a_{1n}b_{11} & a_{1n}b_{12} & \dots & a_{1n}b_{1q} \\ a_{11}b_{21} & a_{11}b_{22} & \dots & a_{11}b_{2q} & \dots & \dots & a_{1n}b_{21} & a_{1n}b_{22} & \dots & a_{1n}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ a_{11}b_{p1} & a_{11}b_{p2} & \dots & a_{11}b_{pq} & \dots & \dots & a_{1n}b_{p1} & a_{1n}b_{p2} & \dots & a_{1n}b_{pq} \\ \vdots & \vdots & & \vdots & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \ddots & \vdots & \vdots & & \vdots \\ a_{m1}b_{11} & a_{m1}b_{12} & \dots & a_{m1}b_{1q} & \dots & \dots & a_{mn}b_{11} & a_{mn}b_{12} & \dots & a_{mn}b_{1q} \\ a_{m1}b_{21} & a_{m1}b_{22} & \dots & a_{m1}b_{2q} & \dots & \dots & a_{mn}b_{21} & a_{mn}b_{22} & \dots & a_{mn}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{p1} & a_{m1}b_{p2} & \dots & a_{m1}b_{pq} & \dots & \dots & a_{mn}b_{p1} & a_{mn}b_{p2} & \dots & a_{mn}b_{pq} \end{bmatrix}.$$

Consider a linear transformation $T : k^m \to k^n$ that is represented by A, an $n \times m$ matrix. Now consider $T' : k^{m'} \to k^{n'}$ that is represented by B, an $n' \times m'$ matrix. We have the linear transformation $T \otimes T'$ defined by

$$T \otimes T' : k^m \otimes k^{m'} \to k^n \otimes k^{n'}$$

$$v \otimes w \mapsto T(v) \otimes T(w)$$

for all $v \in k^m$ and $w \in k^{m'}$. Let $\{v_1, \ldots, v_m\}$ be the standard basis for k^m and $\{w_1, \ldots, w_{m'}\}$ be the standard basis for $k^{m'}$. Then the standard basis for $k^m \otimes k^{m'}$ is $\{v_i \otimes w_j \mid 1 \le i \le m, 1 \le j \le m'\}$. We call this the lexicographic order of this basis. Suppose that

$$A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & \ldots & b_{1n'} \\ \vdots & \ddots & \vdots \\ b_{m'1} & \ldots & b_{m'n'} \end{bmatrix}.$$

The question we ask ourselves is what is the matrix representation for $T \otimes T'$? For $k^m \otimes k^{m'}$, consider using the basis that is in lexicographic order. Note that $(T \otimes T')(v_p \otimes w_q) = T(v_p) \otimes T(w_q)$ for every base element $v_p \otimes w_q$. Let $\{v'_1, \ldots, v'_n\}$ be the standard basis for k^n and $\{w'_1, \ldots, w'_{n'}\}$ be the standard basis for $k^{n'}$. Then the standard basis for $k^n \otimes k^{n'}$ in lexicographic order is $\{v'_1 \otimes w'_1, \ldots, v'_1 \otimes w'_{n'}, v'_2 \otimes w'_1, \ldots, v'_2 \otimes w'_{n'}, \ldots, v'_{n'} \otimes w'_1, \ldots, v'_{n'} \otimes w'_{n'}\}$. Note that

$$\begin{aligned}(T \otimes T')(v_p \otimes w_q) &= T(v_p) \otimes T'(w_q) \\ &= \left(\sum_{i=1}^{n} a_{ip} v'_i\right) \otimes \left(\sum_{j=1}^{n'} b_{iq} w'_j\right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n'} a_{ip} b_{jq} (v'_i \otimes w'_j).\end{aligned}$$

Let C be the matrix representation for $T \otimes T'$. Consider $v_1 \otimes w_1$. Note that

$$(T \otimes T')(v_1 \otimes w_1) = \sum_{i=1}^{n} \sum_{j=1}^{n'} a_{i1} b_{j1} (v'_i \otimes w'_j).$$

Taking $(T \otimes T')(v_1 \otimes w_1)$ represents the first column of C. Since our basis for $k^n \otimes k^{n'}$ is in lexicographic order, the first column of C will be

$$C_{*,1} = \begin{bmatrix} a_{11}b_{11} \\ a_{11}b_{21} \\ \vdots \\ a_{11}b_{m'1} \\ \vdots \\ \vdots \\ a_{m1}b_{11} \\ a_{m1}b_{21} \\ \vdots \\ a_{m1}b_{m'1} \end{bmatrix}.$$

What if we wanted to look at the c^{th} column of C? Notice that we can rewrite c as $c = (p-1)n' + q$ for the respective $v_p \otimes w_q$ element, i.e., $(T \otimes T')(v_p \otimes w_q)$ represents the $(p-1)n' + q$ column of C. So the c^{th} column can be written as

$$C_{*,c} = \begin{bmatrix} a_{1p}b_{1q} \\ a_{1p}b_{2q} \\ \vdots \\ a_{1p}b_{m'q} \\ \vdots \\ \vdots \\ a_{mp}b_{1q} \\ a_{mp}b_{2q} \\ \vdots \\ a_{mp}b_{m'q} \end{bmatrix}.$$

Since we have all of the columns of C, we can see that

$$C = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \dots & a_{11}b_{1n'} & \dots & \dots & a_{1n}b_{11} & a_{1n}b_{12} & \dots & a_{1n}b_{1n'} \\ a_{11}b_{21} & a_{11}b_{22} & \dots & a_{11}b_{2n'} & \dots & \dots & a_{1n}b_{21} & a_{1n}b_{22} & \dots & a_{1n}b_{2n'} \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ a_{11}b_{p1} & a_{11}b_{m'2} & \dots & a_{11}b_{m'n'} & \dots & \dots & a_{1n}b_{m'1} & a_{1n}b_{m'2} & \dots & a_{1n}b_{m'n'} \\ \vdots & \vdots & & \vdots & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \ddots & \vdots & \vdots & & \vdots \\ a_{m1}b_{11} & a_{m1}b_{12} & \dots & a_{m1}b_{1n'} & \dots & \dots & a_{mn}b_{11} & a_{mn}b_{12} & \dots & a_{mn}b_{1n'} \\ a_{m1}b_{21} & a_{m1}b_{22} & \dots & a_{m1}b_{2n'} & \dots & \dots & a_{mn}b_{21} & a_{mn}b_{22} & \dots & a_{mn}b_{2n'} \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{m'1} & a_{m1}b_{m'2} & \dots & a_{m1}b_{m'n'} & \dots & \dots & a_{mn}b_{m'1} & a_{mn}b_{m'2} & \dots & a_{mn}b_{m'n'} \end{bmatrix}$$

which is equal to $A \bigotimes B$. This tells us that the matrix representation for $T \otimes T'$ is $A \bigotimes B$.

Definition 1.5.3. Let $T : U \to V$ be a linear map of vector spaces over a field k. If A is the matrix of T relative to some pair of ordered bases, then the rank of T is equal to the rank of A.

For an A-module M, Proposition 1.4.3 suggests that the rank of its representation matrix of u_λ determines its freeness over $k[u_\lambda]$. Thus, it would be beneficial to understand how to find the rank of the Kronecker product of two matrices. The next lemma provides that.

Lemma 1.5.4. *Let A be a $n \times m$ matrix with* $\operatorname{rank} r$ *and let B be a $p \times q$ matrix with rank s. Then* $\operatorname{rank}(A \bigotimes B) = rs = \operatorname{rank}(A)\operatorname{rank}(B)$.

Proof. Suppose there exists linear maps T, T' such that A is the matrix of T relative to some pair of ordered bases and B is the matrix of T' relative to some pair of ordered bases. Then by Definition 1.5.3, we have that $\text{rank}(A) = \text{rank}(T)$ and $\text{rank}(B) = \text{rank}(T')$. Therefore, we have that

$$\begin{aligned} \text{rank}(A \bigotimes B) &= \text{rank}(T \otimes T') && \text{(Definition 1.5.3)} \\ &= \text{rank}(T)\,\text{rank}(T') \\ &= \text{rank}(A)\,\text{rank}(B) && \text{(Definition 1.5.3).} \end{aligned}$$

□

1.6 Group Algebras and Hopf Algebras

This section will include definitions of group algebras and Hopf algebras. We will use these definitions to describe the group algebra and Hopf algebra structures of A in Chapter 2. For this section, we will let k be a field with identity 1_k. Definitions for group algebras are supplied by [11] and [12].

Definition 1.6.1. Suppose G is a group with identity element 1. The **group algebra** kG is the set of all finite linear combinations of elements of G with coefficients in k. The elements of kG are of the form

$$a_1 g_1 + \cdots + a_m g_m$$

where $a_i \in k$ and $g_i \in G$ for all $i = 1, \ldots, m$. This element can also be written as

$$\sum_{g \in G} a_g g,$$

where $a_g = 0$ for all but finitely many elements g. Addition and multiplication in kG are given by

$$\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g) g$$

$$\left(\sum_{g \in G} a_g g\right)\left(\sum_{h \in G} b_h h\right) = \sum_{g \in G} \sum_{h \in G} (a_g b_h)(gh)$$

for all $a_g, b_g, b_h \in k$ and $g, h \in G$.

Definition 1.6.2. A **k-algebra** is a k-vector space with two linear maps

$$m : A \otimes_k A \to A$$
$$u : k \to A$$

such that the following diagrams commute:

$$\begin{tikzcd}
A \otimes_k A \otimes_k A \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"] & A \otimes A \arrow[d, "m"] \\
A \otimes_k A \arrow[r, "m"] & A
\end{tikzcd}$$

and

$$\begin{tikzcd}
& A \otimes_k A \arrow[d, "m"] & \\
k \otimes_k A \arrow[ur, "u \otimes id"] \arrow[r, "s"] & A & A \otimes_k k \arrow[ul, "id \otimes u"'] \arrow[l, "s"]
\end{tikzcd}$$

where s denotes scalar multiplication.

Definition 1.6.3. A **k-coalgbera** is a k-vector space, C, with two k-linear maps, Δ (coproduct) and ϵ (counit), with

$$\Delta : C \to C \otimes_k C \quad and \quad \epsilon : C \to k,$$

such that the following diagrams commute:

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes_k C \arrow[d, "id \otimes \Delta"] \\
C \otimes_k C \arrow[r, "\Delta \otimes id"] & C \otimes_k C \otimes_k C
\end{tikzcd}$$

and

$$\begin{tikzcd}
k \otimes_k C & C \otimes_k C \arrow[l, "\epsilon \otimes id"'] \arrow[r, "id \otimes \epsilon"] & C \otimes_k k \\
& C \arrow[ul, "1 \otimes_k _"] \arrow[u, "\Delta"] \arrow[ur, "_ \otimes_k 1"'] &
\end{tikzcd}$$

where $1 \otimes \_$ is the map $x \mapsto 1 \otimes x$.

Definition 1.6.4. A **k-algebra homomorphism** is a ring homomorphism that is also a k-module homomorphism. [Definition 7.3 [11]]

Definition 1.6.5. Let C and D be k-coalgebras. A linear map $f : C \to D$ is a **k-coalgebra homomorphism** if the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_A"] & D \arrow[d, "\Delta_B"] \\ C \otimes_k C \arrow[r, "f \otimes_k f"] & D \otimes_k D \end{tikzcd}$$

and

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\epsilon_C"] & D \arrow[d, "\epsilon_D"] \\ k \arrow[r, leftrightarrow] & k \end{tikzcd}$$

Definition 1.6.6. A **bialgebra** A is a k-vector space, $A = (A, m, u, \Delta, \epsilon)$ where (A, m, u) is an algebra, (A, Δ, ϵ) is a coalgebra, and both of the following conditions hold:

1. Δ and ϵ are algebra homomorphisms
2. m and u are coalgebra homomorphims.

Definition 1.6.7. Let $A = (A, m, u, \Delta, \epsilon)$ be a bialgebra. Then a linear endomorphism S from A to A is an antipode for A if the following diagram commutes:

$$\begin{tikzcd} A \otimes_k A \arrow[r, "m"] & A & A \otimes_k A \arrow[l, "m"'] \\ A \otimes_k A \arrow[u, "id \otimes S"] & A \arrow[l, "\Delta"] \arrow[r, "\Delta"] \arrow[u, "u \circ \epsilon"] & A \otimes_k A \arrow[u, "S \otimes id"'] \end{tikzcd}$$

Definition 1.6.8. A **Hopf Algebra** is a bialgebra with an antipode.

CHAPTER 2

Translation to Truncated Polynomial Rings

This chapter will focus on describing the coalgebra and Hopf algebra structures of A, so we can understand the structure of tensor products of A-modules. In particular, we will use these structures to determine how to build the representation matrix for a tensor product of two A-modules.

2.1 Group Algebra and Coalgebra Structures of A

In section 2.4, we will show the properties that make A a Hopf algebra. For this section, we will focus on the coalgebra structure of A. To do this, we need to describe the group algebra structure of A. The next lemma will allow us to do so.

Lemma 2.1.1. *Consider the multiplicative group $G = \langle g_1 \rangle \times \cdots \times \langle g_n \rangle$ where $|g_i| = p$ for each $i = 1, \ldots, n$. Denote kG as the the group algebra where $char(k) = p$. Then $kG \cong A$ as k-algebras.*

Proof. Let $\psi_1 : kG \to A$ be defined by $\psi_1(g_i) = x_i + 1$ for each $i = 1, \ldots, n$ and extended by linearity. We will show that ψ_1 is a k-algebra isomorphism. Note that ψ_1 is well-defined since we sending each generator of kG to a generator of A. It is clear that ψ_1 is a k-algebra homomorphism by construction. Now we want to show that ψ_1 is onto and one-to-one. Note that $\{x_1^{\ell_1} \cdots x_n^{\ell_n}\}$, where the powers run from 0 to $p - 1$, is a k-basis for A. Note that

$$\psi[(g_1 - 1)^{\ell_1} \cdots (g_n - 1)^{\ell_n}] = x_1^{\ell_1} \cdots x_n^{\ell_n}$$

by construction of ψ_1, so every element of A has a preimage in kG. Thus, ψ_1 is onto. Note that $\dim_k(kG) = p^n = \dim_k(A)$, so it follows that $\ker(\psi_1) = \{0\}$. Thus, ψ_1 is one-to-one. Therefore, we have that $kG \cong A$ as k-algebras. $\square$

Suppose $\psi_2 : A \to kG$ is defined by $\psi_2(x_i) = g_i - 1$ for each $i = 1, \ldots, n$. It is important to note that $\psi_1 = \psi_2^{-1}$. We want to determine a suitable coproduct map, $\Delta : A \to A \otimes_k A$, to see how an element in A acts on $A \otimes_k A$. For this, we need the following diagram to commute:

$$\begin{tikzcd} A \arrow[r, "\psi_2"] \arrow[d, "\Delta"] & kG \arrow[d, "\Delta"] \\ A \otimes_k A \arrow[r, "\psi_2 \otimes_k \psi_2"] & kG \otimes_k kG \end{tikzcd}$$

so we need

$$(\psi_2 \otimes_k \psi_2 \circ \Delta)(x_1^{i_1} \cdots x_n^{i_n}) = (\Delta \circ \psi_2)(x_1^{i_1} \cdots x_n^{i_n}).$$

Note that

$$\begin{aligned}
(\Delta \circ \psi_2)(x_1^{i_1} \cdots x_n^{i_n}) &= \Delta\left((g_1 - 1)^{i_1} \cdots (g_n - 1)^{i_n}\right) \\
&= \Delta\left(\sum_{j=0}^{i_1} \binom{i_1}{j} g_1^j (-1)^{i_1 - j} \cdots \sum_{j=0}^{i_n} \binom{i_n}{j} g_n^j (-1)^{i_n - j}\right) \\
&= \sum_{j=0}^{i_1} \binom{i_1}{j} (g_1 \otimes g_1)^j (-1 \otimes 1)^{i_1 - j} \cdots \sum_{j=0}^{i_n} \binom{i_n}{j} (g_n \otimes g_n)^j (-1 \otimes 1)^{i_n - j} \\
&= (g_1 \otimes g_1 - 1 \otimes 1)^{i_1} \cdots (g_n \otimes g_n - 1 \otimes 1)^{i_n}.
\end{aligned}$$

By using $\psi_1 \otimes_k \psi_1$, we can see that

$$\begin{aligned}
\Delta(x_1^{i_1} \cdots x_n^{i_n}) &= \psi_1 \otimes_k \psi_1\left((g_1 \otimes g_1 - 1 \otimes 1)^{i_1} \cdots (g_n \otimes g_n - 1 \otimes 1)^{i_n}\right) \\
&= ((x_1 + 1) \otimes (x_1 + 1) - 1 \otimes 1)^{i_1} \cdots ((x_n + 1) \otimes (x_n + 1) - 1 \otimes 1)^{i_n} \\
&= (1 \otimes x_1 + x_1 \otimes 1 + x_1 \otimes x_1)^{i_1} \cdots (1 \otimes x_n + x_n \otimes 1 + x_n \otimes x_n)^{i_n}
\end{aligned}$$

In section 2.4, we will show that $\Delta : A \to A \otimes_k A$ defined by $\Delta(x_i) = 1 \otimes x_i + x_i \otimes 1 + x_i \otimes x_i$ is a suitable coproduct map when extended by linearity. In [7], Carlson utilizes a usual coalgebra structure for our truncated polynomial, where the coproduct map is $\Delta' : A \to A \otimes_k A$ defined by $\Delta'(x_i) = 1 \otimes x_i + x_i \otimes 1$. We will focus on the differences of these diagonal maps in Chapter 3.

2.2 Tensor Products of A-modules and their Representation Matrices

Suppose we have two A-modules M, N. Once we understand the structure of these two individual A-modules, we then investigate the structure of $M \otimes_k N$. In [7], Carlson proves that

$$V(M \otimes_k N) = V(M) \cap V(N)$$

for any modules M, N over an elementary abelian p-group. We translate this to a similar statement for modules over A. To show this, we must understand how to form a tensor product for $A \otimes_k A$, i.e., how each x_i acts on $A \otimes_k A$. For this, we must show that (A, Δ, ϵ) is a k-coalgebra. The definition for this is provided in section 2.4.

Now we will look at some examples of the representation matrices of certain tensor products. To do this, we must define

$$\begin{aligned}
\Delta(u_\lambda) &= \Delta\left(\sum_{i=1}^{c} \lambda_i x_i\right) \\
&= \sum_{i=1}^{n} \lambda_i \Delta(x_i) && (\Delta : k - \text{linear}) \\
&= \sum_{i=1}^{n} \lambda_i (1 \otimes x_i + x_i \otimes 1 + x_i \otimes x_i) \\
&= \sum_{i=1}^{n} 1 \otimes \lambda_i x_i + \sum_{i=1}^{n} \lambda_i x_i \otimes 1 + \sum_{i=1}^{n} \lambda_i x_i \otimes x_i
\end{aligned}$$

$$= 1 \otimes u_\lambda + u_\lambda \otimes 1 + \sum_{i=1}^{n} \lambda_i x_i \otimes x_i.$$

Example. Let $A = k[x_1, x_2]/(x_1^2, x_2^2)$, $M_1 = (x_1)$, $M_2 = (x_2)$ and $M_3 = (x_1, x_2)$. Recall that the k-basis for M_1, M_2 and M_3 are $\{x_1, x_1x_2\}$, $\{x_2, x_1x_2\}$ and $\{x_1, x_2, x_1x_2\}$, respectively, so we have that

$$\begin{bmatrix} u_\lambda \end{bmatrix}_{M_1} = \begin{bmatrix} 0 & 0 \\ \lambda_2 & 0 \end{bmatrix}, \begin{bmatrix} u_\lambda \end{bmatrix}_{M_2} = \begin{bmatrix} 0 & 0 \\ \lambda_1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} u_\lambda \end{bmatrix}_{M_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda_2 & \lambda_1 & 0 \end{bmatrix}.$$

Note that

$$\Delta(u_\lambda) = \lambda_1(1 \otimes x_1 + x_1 \otimes 1 + x_1 \otimes x_1) + \lambda_2(1 \otimes x_2 + x_2 \otimes 1 + x_2 \otimes x_2).$$

(1) Consider $M = M_1 \otimes_k M_2$. The k-basis for M, in dictionary order, is $\{x_1 \otimes x_2, x_1 \otimes x_1x_2, x_1x_2 \otimes x_2, x_1x_2 \otimes x_1x_2\}$. So we have that

$$\begin{aligned}
\Delta(u_\lambda)(x_1 \otimes x_2) &= \lambda_1(x_1 \otimes x_1x_2) + \lambda_2(x_1x_2 \otimes x_2) \\
\Delta(u_\lambda)(x_1 \otimes x_1x_2) &= \lambda_2(x_1x_2 \otimes x_1x_2) \\
\Delta(u_\lambda)(x_1x_2 \otimes x_2) &= \lambda_1(x_1x_2 \otimes x_1x_2) \\
\Delta(u_\lambda)(x_1x_2 \otimes x_1x_2) &= 0.
\end{aligned}$$

Thus, we have that

$$\begin{bmatrix} u_\lambda \end{bmatrix}_{M} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ \lambda_2 & 0 & 0 & 0 \\ 0 & \lambda_2 & \lambda_1 & 0 \end{bmatrix}.$$

(2) Consider $M = M_1 \otimes_k M_3$. The k-basis for M, in dictionary order, is $\{x_1 \otimes x_1, x_1 \otimes x_2, x_1 \otimes x_1x_2, x_1x_2 \otimes x_1, x_1x_2 \otimes x_2, x_1x_2 \otimes x_1x_2\}$. So we have that

$$\Delta(u_\lambda)(x_1 \otimes x_1) = \lambda_2(x_1 \otimes x_1x_2 + x_1x_2 \otimes x_1 + x_1x_2 \otimes x_1x_2)$$

$$
\begin{aligned}
\Delta(u_\lambda)(x_1 \otimes x_2) &= \lambda_1(x_1 \otimes x_1x_2) + \lambda_2(x_1x_2 \otimes x_2)\\
\Delta(u_\lambda)(x_1 \otimes x_1x_2) &= \lambda_2(x_1x_2 \otimes x_1x_2)\\
\Delta(u_\lambda)(x_1x_2 \otimes x_1) &= \lambda_2(x_1x_2 \otimes x_1x_2)\\
\Delta(u_\lambda)(x_1x_2 \otimes x_2) &= \lambda_1(x_1x_2 \otimes x_1x_2)\\
\Delta(u_\lambda)(x_1x_2 \otimes x_1x_2) &= 0.
\end{aligned}
$$

Thus, we have that

$$
\Big[u_\lambda\Big]_M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ \lambda_2 & \lambda_1 & 0 & 0 & 0 & 0\\ \lambda_2 & 0 & 0 & 0 & 0 & 0\\ 0 & \lambda_2 & 0 & 0 & 0 & 0\\ \lambda_2 & 0 & \lambda_2 & \lambda_2 & \lambda_1 & 0 \end{bmatrix}.
$$

(3) Consider $M = M_3 \otimes_k M_3$. The k-basis for M, in dictionary order, is $\{x_1 \otimes x_1, x_1 \otimes x_2, x_1 \otimes x_1x_2, x_2 \otimes x_1, x_2 \otimes x_2, x_2 \otimes x_1x_2, x_1x_2 \otimes x_1, x_1x_2 \otimes x_2, x_1x_2 \otimes x_1x_2\}$. So we have that

$$
\begin{aligned}
\Delta(u_\lambda)(x_1 \otimes x_1) &= \lambda_2(x_1 \otimes x_1x_2 + x_1x_2 \otimes x_1 + x_1x_2 \otimes x_1x_2)\\
\Delta(u_\lambda)(x_1 \otimes x_2) &= \lambda_1(x_1 \otimes x_1x_2) + \lambda_2(x_1x_2 \otimes x_2)\\
\Delta(u_\lambda)(x_1 \otimes x_1x_2) &= \lambda_2(x_1x_2 \otimes x_1x_2)\\
\Delta(u_\lambda)(x_2 \otimes x_1) &= \lambda_1(x_1x_2 \otimes x_1) + \lambda_2(x_2 \otimes x_1x_2)\\
\Delta(u_\lambda)(x_2 \otimes x_2) &= \lambda_1(x_2 \otimes x_1x_2 + x_1x_2 \otimes x_2 + x_1x_2 \otimes x_1x_2)
\end{aligned}
$$

$$\Delta(u_\lambda)(x_2 \otimes x_1x_2) = \lambda_1(x_1x_2 \otimes x_1x_2)$$

$$\Delta(u_\lambda)(x_1x_2 \otimes x_1) = \lambda_2(x_1x_2 \otimes x_1x_2)$$

$$\Delta(u_\lambda)(x_1x_2 \otimes x_2) = \lambda_1(x_1x_2 \otimes x_1x_2)$$

$$\Delta(u_\lambda)(x_1x_2 \otimes x_1x_2) = 0.$$

Thus, we have that

$$\left[u_\lambda\right]_M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_2 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & \lambda_1 & 0 & 0 & 0 & 0 \\ \lambda_2 & 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ \lambda_2 & 0 & \lambda_2 & 0 & \lambda_1 & \lambda_1 & \lambda_2 & \lambda_1 & 0 \end{bmatrix}.$$

2.3 Representation Matrices of Tensor Products of A-Modules

In the last two sections, we were able to determine the structure of tensor products of A-modules and look at some examples of their representation matrices. In this section, we will generalize what the representation matrix of u_λ for $M \otimes_k N$ is, given the representation matrices of u_λ for A-modules M and N. Since $u_\lambda = \sum_{i=1}^{n} \lambda_i x_i$, it suffices to understand the structure of the representation matrix of x_i over $M \otimes_k N$. For this, we need the following lemma:

Lemma 2.3.1. *Suppose M and N are A-modules with $dim_k(M) = d_1$ and $dim_k(N) = d_2$. Then for all $i = 1, \ldots, n$, we have*

$$\left[x_i\right]_{M\otimes_k N} = I_{d_1} \bigotimes \left[x_i\right]_N + \left[x_i\right]_M \bigotimes I_{d_2} + \left[x_i\right]_M \bigotimes \left[x_i\right]_N$$

with respect to the k-basis of $M \otimes_k N$ in lexicographic order, where $\bigotimes$ represents the Kronecker product.

Proof. It suffices to show the equality holds for just one variable since we can reindex later. Fix i. Suppose M has the k-basis $\{v_1, \ldots, v_{d_1}\}$ and N has the k-basis $\{w_1, \ldots, w_{d_2}\}$. Then a k-basis for $M \otimes_k N$ in lexicographic order is $\{v_1 \otimes w_1, \ldots, v_1 \otimes w_{d_2}, \ldots, v_{d_1} \otimes w_1, \ldots, v_{d_1} \otimes w_{d_2}\}$. Consider the following k-linear transformations:

$T_1 : M \otimes_k N \to M \otimes_k N$ defined by $T_1(v_j \otimes w_\ell) = (1 \otimes x_i) \cdot (v_j \otimes w_\ell) = v_j \otimes x_i w_\ell$

$T_2 : M \otimes_k N \to M \otimes_k N$ defined by $T_1(v_j \otimes w_\ell) = (x_i \otimes 1) \cdot (v_j \otimes w_\ell) = x_i v_j \otimes w_\ell$

$T_3 : M \otimes_k N \to M \otimes_k N$ defined by $T_1(v_j \otimes w_\ell) = (x_i \otimes x_i) \cdot (v_j \otimes w_\ell) = x_i v_j \otimes x_i w_\ell$.

We can see that the matrix representation for T_1 is $I_{d_1} \bigotimes \left[x_i\right]_N$, the matrix representation for T_2 is $\left[x_i\right]_M \bigotimes I_{d_2}$ and the matrix representation for T_3 is $\left[x_i\right]_M \bigotimes \left[x_i\right]_N$. Note that $T = T_1 + T_2 + T_3$ is also a k-linear transformation on $M \otimes_k N$ and is defined by

$$T(v_j \otimes w_\ell) = \Delta(x_i) \cdot (v_j \otimes w_\ell).$$

It is also clear that the matrix representation for T is

$$I_{d_1} \bigotimes \left[x_i\right]_N + \left[x_i\right]_M \bigotimes I_{d_2} + \left[x_i\right]_M \bigotimes \left[x_i\right]_N.$$

$\square$

From here, since $u_\lambda = \sum_{i=1}^{n} \lambda_i x_i$, it is clear that

$$\left[u_\lambda\right]_{M\otimes_k N} = \sum_{i=1}^{n} \lambda_i \left[x_i\right]_{M\otimes_k N}$$

for any A-modules M, N. One of the goals of this paper is to determine the decomposition of $M \otimes_k N$ when restricted to $k[u_\lambda]$. We can do this by finding the Jordan normal form of the representation matrix of u_λ for $M \otimes_k N$. In section 1.3, we formed a decomposition theorem for modules over $R = k[X]/(X^p)$. Since we have that $k[u_\lambda] \cong R$ as rings, we can use the decomposition theorem and treat each $D_i = R/(x^i)$ as $k[u_\lambda]$. Note that a k-basis for D_i is $\{1, x, \ldots, x^{p-1}\}$, so the representation matrix of x for D_i is of size $i \times i$ where there are $1's$ on the subdiagonal and $0's$ everywhere else. Notice that these matrices have the structure of Jordan blocks with 0 as the lone eigenvalue. Throughout the paper, we will refer to these blocks as Jordan blocks and denote them as J_i, where i represents the size of the Jordan block. The question we ask ourselves is how much information do we need to determine the Jordan normal form of a given representation matrix for some A-module M? Integral parts of being able to determine the Jordan normal form of a matrix is being able to determine its rank, nilpotency degree and the rank of it's powers. We will look at some examples to understand why this is the case.

Example. Suppose there are $k[x]/(x^p)$-modules M, N with the following representation matrices:

$$\begin{bmatrix} x \end{bmatrix}_M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \end{bmatrix}_N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is clear to see that both matrices have rank 2. However, their nilpotency degrees will be different. Note that we can write the matrices as

$$\begin{bmatrix} x \end{bmatrix}_M = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \text{and} \quad \begin{bmatrix} x \end{bmatrix}_N = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

The nilpotency degrees of the biggest Jordan blocks will end up being the nilpotency degree for the representation matrix. So from that, we have that $nd(M) = 2$ and $nd(N) = 3$. From looking at these two matrices and their Jordan blocks, we can see that

$$M \cong k[x]/(x^2) \oplus k[x]/(x^2) \ \text{ and } \ N \cong k[x]/(x^3) \oplus k.$$

Clearly, these decompositions are different but with both action matrices having the same rank. What if they also have the same nilpotency degree?

Example. Consider the following 8×8 representation matrices:

$$\begin{bmatrix} x \end{bmatrix}_M = \left[\begin{array}{cccc|ccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{and} \quad \begin{bmatrix} x \end{bmatrix}_N = \left[\begin{array}{cccc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Note that both matrices have rank 5 and nilpotency degree 4 since their biggest Jordan blocks both have $nd = 4$. However, over $k[x]/(x^p)$, we can clearly see from the Jordan blocks that

$$M \cong k[x]/(x^4) \oplus k[x]/(x^3) \oplus k \text{ and } N \cong k[x]/(x^4) \oplus k[x]/(x^2) \oplus k[x]/(x^2).$$

Here we see that the representation matrices for M and N have the same nilpotency degree and rank, but M and N have different decompositions over $k[x]/(x^p)$. So it is not enough to simply have the same ranks and nilpotency degrees to determine if two modules have the same decomposition over $k[x]/(x^p)$. Notice that as you look at the powers of the representation matrices, the ranks will differ. We have that

$$\left[x^2\right]_M = \left[\begin{array}{cccc|ccc|c} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ \hline 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ \hline 0&0&0&0&0&0&0&0 \end{array}\right] \text{ and } \left[x^2\right]_N = \left[\begin{array}{cccc|cc|cc} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ \hline 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ \hline 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{array}\right].$$

Note that $\left[x^2\right]_M$ have rank 3 and $\left[x^2\right]_N$ have rank 2.

The question we ask ourselves is do we have enough to determine if two modules have the same decomposition over $k[x]/(x^p)$ if they have the same nilpotency degrees and all of the powers of their representation matrices have the same rank? We will answer this question with the following lemma:

Lemma 2.3.2. *Let $A = k[x]/(x^p)$ for some p. Suppose M and N are A-modules such that* $\dim_k(M) = \dim_k(N)$ *and* $\operatorname{rank}([x^i]_M) = \operatorname{rank}([x^i]_N)$ *for each $i = 1, \ldots, p$ and . Then $M \cong N$ over $k[x]/(x^p)$.*

Proof. Since $\operatorname{rank}([x^i]_M) = \operatorname{rank}([x^i]_N)$ for each $i = 1, \ldots, p$, it follows that the representation matrices of x for M and N must have the same nilpotency degree. Let $d \leq p$ be the common nilpotency degree. Write $M \cong D_d^{m_d} \oplus D_{d-1}^{m_{d-1}} \oplus \cdots \oplus D_2^{m_2} \oplus D_1^{m_1}$ and $N \cong D_d^{n_d} \oplus D_{d-1}^{n_{d-1}} \oplus \cdots \oplus D_2^{n_2} \oplus D_1^{n_1}$ where each $D_i = k[x]/(x^i)$ and $i \leq d$. We want to show that $m_i = n_i$ for each $i = 1, \ldots, d$. Let J_i be the Jordan block that represents the representation matrix of x for D_i. Each J_i block has rank $i - 1$. Note that

$$\operatorname{rank}([x]_M) = \sum_{i=1}^{d} m_i(i-1) = \sum_{i=1}^{d} n_i(i-1) = \operatorname{rank}([x]_N).$$

From this, we can see that

$$\sum_{i=1}^{d} m_i(i-1) - \sum_{i=1}^{d} n_i(i-1) = 0$$

which implies that

$$\sum_{i=1}^{d} (m_i - n_i)(i-1) = 0.$$

Now consider $[x^2]_M$ and $[x^2]_N$. Since the rank of each J_1 block is already 0, it is not affected in the overall representation matrices when we square them. Note that $xD_p \cong D_{p-1}$, so the rank of a Jordan block will decrease by one when we multiply our representation matrix by $[x]_M$ as long as our Jordan block was not equal to J_1. This implies that

$$\sum_{i=2}^{d} (m_i - n_i)(i-2) = 0$$

since

$$\operatorname{rank}([x^2]_M) = \sum_{i=2}^{d} m_i(i-2) = \sum_{i=2}^{d} n_i(i-2) = \operatorname{rank}([x^2]_N).$$

In general, for any $[x^i]_M$ or $[x^i]_N$, any Jordan block J_j for any $j < i$ will have rank 0. Thus, we can see that

$$\operatorname{rank}([x^j]_M) = \sum_{i=j}^{d} m_i(i-j) = \sum_{i=j}^{d} n_i(i-j) = \operatorname{rank}([x^j]_N)$$

which implies that

$$\sum_{i=j}^{d}(m_i - n_i)(i - j) = 0$$

for all $j = 1, \ldots, d$. By letting $j = d - 1$, we can see that

$$0 = \sum_{j=d-1}^{d}(m_i - n_i)(i - d + 1) = (m_{d-1} - n_{d-1})(d - 1 - d + 1) + (m_d - n_d)(d - d + 1) = m_d - n_d$$

which implies that $m_d = n_d$. By letting $j = d - 2$, we can see that $m_{d-1} = n_{d-1}$ since $m_d = n_d$. We can continue this process to show that $m_i = n_i$ for all $i = 2, \ldots, d$. It remains to show that $m_1 = n_1$. Recall that $\dim_k(M) = \dim_k(N)$ and $\dim_k(D_i) = i$ for each $1 \leq i \leq d$, so we have that

$$\dim_k(M) = \sum_{i=1}^{d} m_i i = \sum_{i=1}^{d} n_i i = \dim_k(N).$$

Note that Since we know that $m_i = n_i$ for all $i = 2, \ldots, d$ and $\dim_k(D_1) = 1$, it is clear to see that $m_1 = n_1$. Therefore, $M \cong N$ over $k[\lambda x]$ for any nonzero $\lambda \in k$. $\square$

Now we want to talk more about these Jordan blocks that we labeled as J_i. Let $A = k[X]/(X^p)$ for a prime p and $char(k) = p$. Then the only possible Jordan blocks we have are $J_1, J_2, \ldots, J_p$ where J_i is the Jordan block representation for $D_i \cong k[u_\lambda]/(u_\lambda^i)$. Each J_i block is an $i \times i$ matrix with $1's$ on the sub-diagonal and $0's$ everywhere else. For example, we have

$$J_1 = [0], J_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, J_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and so on. Note that the rank of a J_i block is $i - 1$. Consider A-modules M and N. Note that M and N can be decomposed as

$$M \cong \bigoplus_{i=1}^{p} D_i^{m_i}$$

and

$$N \cong \bigoplus_{i=1}^{p} D_i^{n_i}$$

where each m_i, n_i is a nonnegative integer. Then we have that

$$M \otimes N \cong \left(\bigoplus_{i=1}^{p} D_i^{m_i} \right) \otimes \left(\bigoplus_{i=1}^{p} D_i^{n_i} \right).$$

To break this down, we want to be able to determine what the decomposition for $D_i \otimes D_j$ looks like for some arbitrary $i, j \leq p$. We refer to this as the Clebsch-Gordan problem [10] for $k[x]/(x^p)$ in Chapter 3.

2.4 Hopf Algebra Structure of A

It is well known that A is a Hopf algebra, so this section is dedicated to deriving and describing the properties that make A a Hopf Algebra. Recall that the set $\{x_1^{\ell_1} \cdots x_n^{\ell_n}\}$, where the powers run from 0 to $p-1$, is a k-basis for A. So we define the maps, extended by linearity, as follows:

$m : A \otimes A \to A$ defined by

$$x_1^{\ell_1} \cdots x_n^{\ell_n} \otimes x_1^{\ell'_1} \cdots x_n^{\ell'_n} \longmapsto x_1^{\ell_1+\ell'_1} \cdots x_n^{\ell_n+\ell'_n}$$

$u : k \to A$ defined by

$$1_k \longmapsto 1_A$$

$\Delta : A \to A \otimes_k A$ defined by

$$x_1^{\ell_1} \cdots x_n^{\ell_n} \longmapsto (1 \otimes x_1 + x_1 \otimes 1 + x_1 \otimes x_1)^{\ell_1} \cdots (1 \otimes x_n + x_n \otimes 1 + x_n \otimes x_n)^{\ell_n}$$

$\epsilon : A \to k$ defined by

$$1_A \longmapsto 1_k$$

$$x_1^{\ell_1} \cdots x_n^{\ell_n} \longmapsto 0 \text{ when } 1 \leq \ell_i \leq p-1 \text{ for each } \ell_i$$

$S : A \to A$ defined by

$$x_1^{\ell_1} \cdots x_n^{\ell_n} \longmapsto \prod_{i=j}^{n} \left(\sum_{s=1}^{p-1} (-1)^s x_j^s \right)^{\ell_j}$$

First, we want to show that (A, m, u) is a k-algebra. For m, we want to verify that

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m\otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id}\otimes m"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

commutes, i.e., showing that

$$(m \circ m{\otimes}\mathrm{Id})(x_1^{i_1} \cdots x_n^{i_n} {\otimes} x_1^{j_1} \cdots x_n^{j_n} {\otimes} x_1^{h_1} \cdots x_n^{h_n}) = (m \circ \mathrm{Id} \otimes m)(x_1^{i_1} \cdots x_n^{i_n} {\otimes} x_1^{j_1} \cdots x_n^{j_n} {\otimes} x_1^{h_1} \cdots x_n^{h_n}).$$

Note that

$$\begin{aligned}(m \ \circ \ m \otimes \mathrm{Id})(x_1^{i_1} \cdots x_n^{i_n} \otimes x_1^{j_1} \cdots x_n^{j_n} \otimes x_1^{h_1} \cdots x_n^{h_n}) &= m(x_1^{i_1+j_1} \cdots x_n^{i_n+j_n} \otimes x_1^{h_1} \cdots x_n^{h_n}) \\ &= x_1^{i_1+j_1+h_1} \cdots x_n^{i_n+j_n+h_n}\end{aligned}$$

and

$$\begin{aligned}(m \ \circ \ Id \otimes m)(x_1^{i_1} \cdots x_n^{i_n} \otimes x_1^{j_1} \cdots x_n^{j_n} \otimes x_1^{h_1} \cdots x_n^{h_n}) &= m(x_1^{i_1} \cdots x_n^{i_n} \otimes x_1^{j_1+h_1} \cdots x_n^{j_n+h_n}) \\ &= x_1^{i_1+j_1+h_1} \cdots x_n^{i_n+j_n+h_n}.\end{aligned}$$

For u, we want to verify that

$$\begin{tikzcd} & A \otimes_k A \arrow[d, "m"] & \\ k \otimes_k A \arrow[ur, "u\otimes id"] \arrow[r, "s"] & A & A \otimes_k k \arrow[ul, "id\otimes u"'] \arrow[l, "s"'] \end{tikzcd}$$

commutes, i.e., showing that

$$m \circ \mathrm{Id} \otimes u(x_1^{i_1} \cdots x_n^{i_n} \otimes 1_k) = m \circ u \otimes \mathrm{Id}(1_k \otimes x_1^{i_1} \cdots x_n^{i_n}).$$

Note that

$$m \circ \mathrm{Id} \otimes u(x_1^{i_1} \cdots x_n^{i_n} \otimes 1_k) = m(x_1^{i_1} \cdots x_n^{i_n} \otimes 1_A) = x_1^{i_1} \cdots x_n^{i_n} \cdot 1_A = x_1^{i_1} \cdots x_n^{i_n}$$

and

$$m \circ u \otimes \mathrm{Id}(1_k \otimes x_1^{i_1} \cdots x_n^{i_n}) = m(1_A \otimes x_1^{i_1} \cdots x_n^{i_n}) = 1_A \cdot x_1^{i_1} \cdots x_n^{i_n} = x_1^{i_1} \cdots x_n^{i_n}.$$

Now we want to show that (A, Δ, ϵ) is a k-coalgebra. For Δ, we want to verify

$$\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "\Delta \otimes \mathrm{Id}"'] \\
A \otimes A \arrow[u, "\mathrm{Id} \otimes \Delta"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

commutes, i.e., showing that

$$(\Delta \otimes \mathrm{Id} \circ \Delta)(x_1^{i_1} \cdots x_n^{i_n}) = (\mathrm{Id} \otimes \Delta \circ \Delta)(x_1^{i_1} \cdots x_n^{i_n}).$$

Note that

$$(\Delta \otimes \mathrm{Id}) \circ \Delta(x_j) = (\Delta \otimes \mathrm{Id})(1 \otimes x_j + x_j \otimes 1 + x_j \otimes x_j), \tag{1}$$

which is equal to

$$1 \otimes 1 \otimes x_j + x_j \otimes 1 \otimes 1 + 1 \otimes x_j \otimes 1 + x_j \otimes x_j \otimes 1 + x_j \otimes 1 \otimes x_j + 1 \otimes x_j \otimes x_j + x_j \otimes x_j \otimes x_j,$$

and

$$(\mathrm{Id} \otimes \Delta) \circ \Delta(x_j) = (\mathrm{Id} \otimes \Delta)(1 \otimes x_j + x_j \otimes 1 + x_j \otimes x_j), \tag{2}$$

which is equal to

$$1 \otimes x_j \otimes 1 + 1 \otimes 1 \otimes x_j + 1 \otimes x_j \otimes x_j + x_j \otimes 1 \otimes 1 + x_j \otimes 1 \otimes x_j + 1 \otimes x_j \otimes x_j + x_j \otimes x_j \otimes x_j.$$

Since (1) and (2) are equal, we can use this fact to show that

$$(\Delta \otimes \mathrm{Id} \circ \Delta)(x_1^{i_1} \cdots x_n^{i_n}) = (\Delta \otimes \mathrm{Id}) \left[\prod_{j=1}^{n} (1 \otimes x_j + x_j \otimes 1 + x_j \otimes x_j)^{i_j} \right]$$

$$
\begin{aligned}
&= \prod_{j=1}^{n} (\Delta \otimes \mathrm{Id})[(1 \otimes x_j + x_j \otimes 1 + x_j \otimes x_j)^{i_j}] \\
&= \prod_{j=1}^{n} [(\Delta \otimes \mathrm{Id})(1 \otimes x_j + x_j \otimes 1 + x_j \otimes x_j)]^{i_j} \qquad (1) \\
&= \prod_{j=1}^{n} [(\mathrm{Id} \otimes \Delta)(1 \otimes x_j + x_j \otimes 1 + x_j \otimes x_j)]^{i_j} \qquad (2) \\
&= \prod_{j=1}^{n} (\mathrm{Id} \otimes \Delta)[(1 \otimes x_j + x_j \otimes 1 + x_j \otimes x_j)^{i_j}] \\
&= (\mathrm{Id} \otimes \Delta) \left[\prod_{j=1}^{n} (1 \otimes x_j + x_j \otimes 1 + x_j \otimes x_j)^{i_j} \right] \\
&= (\mathrm{Id} \otimes \Delta \circ \Delta)(x_1^{i_1} \cdots x_n^{i_n}).
\end{aligned}
$$

For ϵ, we want to show that

$$
\begin{tikzcd}
k \otimes_k A & A \otimes_k A \arrow[l, "\epsilon\otimes id"'] \arrow[r, "id\otimes\epsilon"] & A \otimes_k k \\
& A \arrow[ul, "1\otimes_k -"] \arrow[u, "\Delta"] \arrow[ur, "-\otimes_k 1"'] &
\end{tikzcd}
$$

commutes, i.e., verifying that

$$(\mathrm{Id} \otimes \epsilon \circ \Delta)(x_1^{i_1} \cdots x_n^{i_n}) = x_1^{i_1} \cdots x_n^{i_n} \otimes 1$$

and

$$(\epsilon \otimes \mathrm{Id} \circ \Delta)(x_1^{i_1} \cdots x_n^{i_n}) = 1 \otimes x_1^{i_1} \cdots x_n^{i_n}.$$

Note that

$$
\begin{aligned}
(\mathrm{Id} \otimes \epsilon \circ \Delta)(x_1^{i_1} \cdots x_n^{i_n}) &= (\mathrm{Id} \otimes \epsilon) \prod_{j=1}^{n} (1 \otimes x_j + x_j \otimes 1 + x_j \otimes x_j)^{i_j} \\
&= \prod_{j=1}^{n} (\mathrm{Id} \otimes \epsilon)[(1 \otimes x_j + x_j \otimes 1 + x_j \otimes x_j)^{i_j}] \\
&= \prod_{j=1}^{n} [(\mathrm{Id} \otimes \epsilon)(1 \otimes x_j + x_j \otimes 1 + x_j \otimes x_j)]^{i_j}
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{j=1}^{n} (x_j \otimes 1)^{i_j} \\
&= x_1^{i_1} \cdots x_n^{i_n} \otimes 1
\end{aligned}
$$

and

$$
\begin{aligned}
(\epsilon \otimes \mathrm{Id} \ \circ \Delta)(x_1^{i_1} \cdots x_n^{i_n}) &= (\epsilon \otimes \mathrm{Id}) \prod_{j=1}^{n} (1 \otimes x_j + x_j \otimes 1 + x_j \otimes x_j)^{i_j} \\
&= \prod_{j=1}^{n} (\epsilon \otimes \mathrm{Id})[(1 \otimes x_j + x_j \otimes 1 + x_j \otimes x_j)^{i_j}] \\
&= \prod_{j=1}^{n} [(\epsilon \otimes \mathrm{Id})(1 \otimes x_j + x_j \otimes 1 + x_j \otimes x_j)]^{i_j} \\
&= \prod_{j=1}^{n} (1 \otimes x_j)^{i_j} \\
&= 1 \otimes x_1^{i_1} \cdots x_n^{i_n}.
\end{aligned}
$$

By the definition of the maps above, it is clear that Δ, ϵ are algebra homomorphisms and m, u are coalgebra homomorphisms, so A is a bialgebra. Now we want to show that S is an antipode for A. This means showing that

$$
\begin{tikzcd}
A \otimes_k A \arrow[r, "m"] & A & A \otimes_k A \arrow[l, "m"'] \\
A \otimes_k A \arrow[u, "id \otimes S"] & A \arrow[l, "\Delta"] \arrow[r, "\Delta"'] \arrow[u, "u \circ \epsilon"] & A \otimes_k A \arrow[u, "S \otimes id"]
\end{tikzcd}
$$

commutes. For S, we have that

$$
u \circ \epsilon(1_A) = u(1_k) = 1_A
$$

and

$$
u \circ \epsilon(x_1^{i_1} \cdots x_n^{i_n}) = u(0) = 0
$$

when $i_j \geq 1$ for all $j = 1, \ldots, n$, so we want to verify the following:

$$
m \circ S \otimes \mathrm{Id} \circ \Delta(1_A) = 1_A
$$

$$m \circ S \otimes \mathrm{Id} \circ \Delta(x_1^{i_1} \cdots x_n^{i_n}) = 0 \qquad (i_j \geq 1)$$

$$m \circ \mathrm{Id} \otimes S \circ \Delta(1_A) = 1_A$$

$$m \circ \mathrm{Id} \otimes S \circ \Delta(x_1^{i_1} \cdots x_n^{i_n}) = 0. \qquad (i_j \geq 1)$$

Note that

$$\begin{aligned} m \circ S \otimes \mathrm{Id} \circ \Delta(1_A) &= m \circ S \otimes \mathrm{Id}(1 \otimes 1) \\ &= m(1 \otimes 1) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} m \circ \mathrm{Id} \otimes S \circ \Delta(1_A) &= m \circ \mathrm{Id} \otimes S(1 \otimes 1) \\ &= m(1 \otimes 1) \\ &= 1. \end{aligned}$$

Now we verify the other two compositions. Assume $i_j \geq 1$ for all $j = 1, \ldots, n$. Then

$$\begin{aligned} m \circ S \otimes \mathrm{Id} \circ \Delta(x_1^{i_1} \cdots x_n^{i_n}) &= m \circ S \otimes \mathrm{Id} \left[\prod_{j=1}^{n} (1 \otimes x_j + x_j \otimes 1 + x_j \otimes x_j)^{i_j} \right] \\ &= \prod_{j=1}^{n} \left[m \circ S \otimes \mathrm{Id}(1 \otimes x_j + x_j \otimes 1 + x_j \otimes x_j) \right]^{i_j} \\ &= \prod_{j=1}^{n} [m(S(1) \otimes x_j + S(x_j) \otimes 1 + S(x_j) \otimes x_j)]^{i_j} \\ &= \prod_{j=1}^{n} \left[m \left(1 \otimes x_j + \left(\sum_{s=1}^{p-1} (-1)^s x_j^s \right) \otimes 1 + \left(\sum_{s=1}^{p-1} (-1)^s x_j^s \right) \otimes x_j \right) \right]^{i_j} \\ &= \prod_{j=1}^{n} \left[x_j + \sum_{s=1}^{p-1} (-1)^s x_j^s + \sum_{s=1}^{p-1} (-1)^s x_j^{s+1} \right]^{i_j} \\ &= \prod_{j=1}^{n} \left[x_j - x_j + \sum_{s=2}^{p-1} (-1)^s x_j^s - \sum_{s=1}^{p-1} (-1)^{s+1} x_j^{s+1} \right]^{i_j} \end{aligned}$$

$$= 0$$

and

$$\begin{aligned}
m \circ \mathrm{Id} \otimes S \circ \Delta(x_1^{i_1} \cdots x_n^{i_n}) &= m \circ \mathrm{Id} \otimes S \left[\prod_{j=1}^{n} (1 \otimes x_j + x_j \otimes 1 + x_j \otimes x_j)^{i_j} \right] \\
&= \prod_{j=1}^{n} \left[m \circ \mathrm{Id} \otimes S (1 \otimes x_j + x_j \otimes 1 + x_j \otimes x_j) \right]^{i_j} \\
&= \prod_{j=1}^{n} [m(x_j \otimes S(1) + 1 \otimes S(x_j) + x_j \otimes S(x_j))]^{i_j} \\
&= \prod_{j=1}^{n} \left[m \left(x_j \otimes 1 + 1 \otimes \left(\sum_{s=1}^{p-1} (-1)^s x_j^s \right) + x_j \otimes \left(\sum_{s=1}^{p-1} (-1)^s x_j^s \right) \right) \right]^{i_j} \\
&= \prod_{j=1}^{n} \left[x_j + \sum_{s=1}^{p-1} (-1)^s x_j^s + \sum_{s=1}^{p-1} (-1)^s x_j^{s+1} \right]^{i_j} \\
&= \prod_{j=1}^{n} \left[x_j - x_j + \sum_{s=2}^{p-1} (-1)^s x_j^s - \sum_{s=1}^{p-1} (-1)^{s+1} x_j^{s+1} \right]^{i_j} \\
&= 0.
\end{aligned}$$

CHAPTER 3

Clebsch-Gordan Problem for $k[X]/(X^p)$

3.1 What is the Clebsch-Gordan Problem for $k[X]/(X^p)$?

The Clebsch-Gordan Problem (CGP) provides formulae that have many applications in mathematics and physics. This problem has been solved for various classes of algebras, particularly in Lie algebra theory. We first came across this problem in [10], where the problem is studied for the algebra $k[X]$. Recall that $k[u_\lambda] \cong k[X]/(X^p)$, so we simplify our problem to solving the decomposition over $R = k[X]/(X^p)$. Consider two R-modules M and N. Knowing the decompositions of M and N over R, we aim to derive the decomposition of $M \otimes_k N$ over R. The fundamental theorem for modules over principal ideal domains gives a decomposition theorem for modules over R. Let $D_i = R/(x^i)$ for $i = 1, \ldots, p$. We see that D_i represents all of our indecomposables over R. Given two R-modules M and N, we have that

$$M \cong D_1^{m_1} \oplus D_2^{m_2} \oplus \cdots \oplus D_p^{m_p}$$
$$N \cong D_1^{n_1} \oplus D_2^{n_2} \oplus \cdots \oplus D_p^{n_p}$$

for some nonegative integers $m_1, \ldots, m_p, n_1, \ldots, n_p$. Note that $M \otimes_k N$ is an R-module, so it will have a decomposition as well, i.e.,

$$M \otimes_k N \cong D_1^{\ell_1} \oplus D_2^{\ell_2} \oplus \cdots \oplus D_p^{\ell_p}.$$

The goal is to see if ℓ_i can be determined if we know each m_i, n_i. Since tensor products commute with direct sums, it suffices to understand how $D_i \otimes_k D_j$ decomposes for some arbitrary $i, j \leq p$. This is mentioned in [10] for the polynomial ring $k[X]$.

3.2 Using Representation Matrices to Solve the Clebsch-Gordan Problem

To understand better how $D_i \otimes_k D_j$ decomposes over R, we use their representation matrices. Equivalently, we aim to find the Jordan Canonical form of the representation matrix of x for $D_i \otimes_k D_j$. Due to the Hopf algebra structure of R, we determined that the diagonal map $\Delta : R \to R \otimes_k R$ is defined by $\Delta(x) = 1 \otimes x + x \otimes 1 + x \otimes x$. When translating to representation matrices, we see that

$$\Big[x\Big]_{D_i \otimes_k D_j} = I_i \bigotimes J_j + J_i \bigotimes I_j + J_i \bigotimes J_j$$

where J_i represents the $i \times i$ Jordan block with sole eigenvalue 0 and $\bigotimes$ represents the Kronecker product. By finding the Jordan Canonical form of this matrix, we can determine the decompositon for $D_i \otimes_k D_j$ since each J_i will correspond to D_i. In [7], Carlson talks about a common diagonal map that is also used for A_p^n. The diagonal map he provides is $\Delta'(x) = 1 \otimes_k x + x \otimes_k 1$. When translating to representation matrices, we see that

$$\Big[x\Big]_{D_i \otimes_k' D_j} = I_i \bigotimes J_j + J_i \bigotimes I_j.$$

The goal is to compare these diagonal maps and see if they result in different decompositions. The $i = 1$ case for both diagonal maps is trivial, i.e., $D_i \otimes_k D_1 \cong D_i$ for any $i = 1, \ldots, p$. The $i = p$ case tells us that if we tensor a free-module with another module, we get a free module back, i.e., $D_p \otimes_k D_j \cong D_p^j$ for any $j = 1, \ldots, p$. We will expand on this in the next section.

3.3 Criteria for $k[X]/(X^p)$-modules to be free

We want to examine when the tensor product of two R-modules are free. Suppose M and N are R-modules. Since we are working in one variable, then from earlier in the section, there exists nonnegative integers $m_1, \ldots, m_p, n_1, \ldots, n_p$ such that

$$M \cong D_1^{m_1} \oplus \cdots \oplus D_p^{m_p}$$

$$N \cong D_1^{n_1} \oplus \cdots \oplus D_p^{n_p}$$

where $D_i \cong k[u_\lambda]/(u_\lambda^i)$ for $i = 1, \ldots, p$. From here, we aim to determine when $M \otimes_k N$ is free, in particular, do we have that

$$M \otimes_k N \cong D_p^{\ell_p}$$

for some positive integer ℓ_p? Since tensor products commute with direct sums, it is equivalent to determining the conditions on i, j such that

$$D_i \otimes_k D_j \cong D_p^{\ell_p}.$$

We can also ask the same questions using $\otimes_k'$. From here, we can gather the following proposition:

Proposition 3.3.1. *$D_p \otimes_k D_j \cong D_p^j$ and $D_p \otimes_k' D_j \cong D_p^j$ for any $j = 1, \ldots, p$*

This is a known fact for modules over Hopf algebras, but we will use representation matrices and dimension to prove it.

Proof. Let $M = D_p \otimes_k D_j$ for any $j = 1, \ldots, p$. By Proposition 1.5.1, we know that M is free if $rank([x]_M) = (p-1)j$. By Lemma 2.3.1, we have that the following $p \times p$ matrix with $j \times j$ square matrices as entries:

$$[x]_M = I_p \bigotimes J_j + J_p \bigotimes I_j + J_p \bigotimes J_j = \begin{bmatrix} J_j & 0 & \cdots & 0 \\ I_j + J_j & J_j & & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & I_j + J_j & J_j \end{bmatrix}.$$

We know that the maximal rank of this matrix is $(p-1)j$, so it is sufficient to show that the minimal rank of this matrix is $(p-1)j$. Consider the submatrix where you remove the first j rows and the last j columns, then we have

$$\begin{bmatrix} I_j + J_j & J_j & \cdots & 0 \\ 0 & I_j + J_j & \cdots & \vdots \\ \vdots & \ddots & \ddots & J_j \\ 0 & \cdots & 0 & I_j + J_j \end{bmatrix}.$$

I claim that this matrix has full rank. Consider the submatrix

$$\begin{bmatrix} I_j + J_j & J_j \end{bmatrix}.$$

Note that using a series of elementary row operations, we can get the matrix

$$\begin{bmatrix} I_j & \sum_{r=1}^{j-1} (-1)^{r-1} J_j^r \end{bmatrix}.$$

Let $J'_j = \sum_{r=1}^{j-1} (-1)^{r-1} J_j^r$. So we have that

$$\begin{bmatrix} I_j + J_j & J_j & & 0 \\ & \ddots & \ddots & \\ & & \ddots & J_j \\ 0 & & & I_j + J_j \end{bmatrix} \to \begin{bmatrix} I_j & J'_j & & 0 \\ & \ddots & \ddots & \\ & & \ddots & J'_j \\ 0 & & & I_j \end{bmatrix}$$

which is an upper triangular matrix with $1's$ on the diagonal. Thus, it has determinant 1, which implies it has full rank. Therefore, the rank of $[x]_M$ is $(p-1)j$.

Now we show the proof for $\otimes'_k$.

Let $M = D_p \otimes_k' D_j$ for any $j = 1, \ldots, p$. By Proposition 1.5.1, we know that M is free if $rank([x]_M) = (p-1)j$. So we have that the following $p \times p$ matrix with $j \times j$ square matrices as entries :

$$[x]_M = \begin{bmatrix} J_j & 0 & \cdots & 0 \\ I_j & J_j & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & I_j & J_j \end{bmatrix}.$$

We know that the maximal rank of this matrix is $(p-1)j$, so it is sufficient to show that the minimal rank of this matrix is $(p-1)j$. Consider the submatrix where you remove the first j rows and the last j columns, then we have

$$\begin{bmatrix} I_j & J_j & \cdots & 0 \\ 0 & I_j & \cdots & \vdots \\ \vdots & \ddots & \ddots & J_j \\ 0 & \cdots & 0 & I_j \end{bmatrix}$$

which is a square matrix of size $(p-1)j \times (p-1)j$. This matrix is of full rank since it is upper triangular and has all $1's$ on the diagonal. Thus, the minimal rank of $[x]_M$ is $(p-1)j$. Therefore, we can conclude that the rank of $[x]_M$ is $(p-1)j$. This tells us that there are j Jordan blocks in the decomposition of M, so it immediately follows that each Jordan block is a J_p block. Therefore, we have that

$$M = D_p \otimes_k' D_j \cong D_p^j.$$

$\square$

From here, we get an immediate corollary for tensoring an arbitrary A-module with a free A-module when restricted to $k[u_\lambda]$.

Corollary 3.3.2. *Suppose M and N are R-modules. If M or N is free, then $M \otimes_k N$ is free and $M \otimes'_k N$ is free.*

Proof. This proof will heavily rely on dimension and use of tensor products with direct sums, so we will prove it holds for $\otimes_k$ without a loss of generality. Assume M and N have the following decompositions:

$$M \cong \bigoplus_{i=1}^{p} D_i^{m_i}$$
$$N \cong \bigoplus_{j=1}^{p} D_j^{n_j}$$

where each m_i, n_j is a nonnegative integer. Without a loss of generality, suppose M is a free-module, that is, $m_i = 0$ for all $i = 1, \ldots, p-1$ and $m_p > 0$. Then we have

$$\begin{aligned} M \otimes_k N &\cong D_p^{m_p} \otimes_k \left(\bigoplus_{j=1}^{p} D_j^{n_j} \right) \\ &\cong \bigoplus_{j=1}^{p} (D_p^{m_p} \otimes_k D_j^{n_j}) \\ &\cong \bigoplus_{i=1}^{m_p} \bigoplus_{j=1}^{p} (D_p \otimes_k D_j)^{n_j} \\ &\cong \bigoplus_{i=1}^{m_p} \bigoplus_{j=1}^{p} D_p^{jn_j}, \end{aligned}$$

which is a free module. □

This corollary tells us that if we tensor a free module with another module, we get a free module. The question we now ask ourselves is if we tensor two non-free modules, can we still get a free module? In particular, can we have a scenario where

$i, j \neq p$ and $D_i \otimes D_j \cong D_p^{\ell_p}$ for some positive integer ℓ_p? Suppose that $i \neq p$ and $j \neq p$. If $M = D_i \otimes_k D_j$, note that $dim_k(M) = ij$. We want $ij = p\ell_p$. This means that $p \mid ij$. Since p is prime, we know that $p \mid i$ or $p \mid j$. This only happens if $i = p$ or $j = p$, so we need one of the modules to be free in order to get a free module in the one-variable case. Since this argument focuses on dimension, it works for $\otimes_k$ and $\otimes_k'$. Now we want to expand our argument to the following lemma:

Lemma 3.3.3. *Let M, N be R-modules. Then $M \otimes_k N$ and $M \otimes_k' N$ is free if and only if M is free or N is free.*

Proof. This proof will heavily rely on dimension, so we will prove it holds for $\otimes_k$. Then it will hold for $\otimes_k'$ as well. Using Proposition 3.3.1, the backwards direction is clear since tensor products distribute over direct sums. For the forward direction, suppose there exists nonnegative numbers m_i, n_j such that

$$M \cong \bigoplus_{i=1}^{p} D_i^{m_i}$$

and

$$N \cong \bigoplus_{j=1}^{p} D_j^{n_j}.$$

Assume $M \otimes_k N$ is free and M is not free. We want to show that N is free. Since M is not free, there exists an $i < p$ such that $m_i \neq 0$. So we have that $D_i \otimes_k N$ is a direct summand of $M \otimes_k N$. Since $M \otimes_k N$ is free, then $D_i \otimes_k N$ must be free, which is only possible if $n_j = 0$ for all $j = 1, \ldots, p-1$ and $n_p \neq 0$. Therefore, we have that N is free.

$\square$

This lemma immediately gives us the following theorem:

Theorem 3.3.4. *Let M, N be R-modules. Then $M \otimes_k N$ is free if and only if $M \otimes_k' N$ is free.*

Proof. Without a loss of generality, assume $M \otimes_k N$ is free. Then by Lemma 3.3.2, it follows that either M or N is free. By Proposition 3.3.1, it immediately follows that $M \otimes'_k N$ is free. $\square$

3.4 Clebsch-Gordan Problem with $\Delta(x) = 1 \otimes x + x \otimes 1 + x \otimes x$

We will look at some examples and results for the decomposition of $D_i \otimes_k D_j$, using the three-term diagonal map $\Delta(x) = 1 \otimes x + x \otimes 1 + x \otimes x$.

Example. Consider $D_1 \otimes D_2$. This case is trivial since $D_1 \cong k$ so we get that

$$D_1 \otimes D_2 \cong k \otimes D_2 \cong D_2.$$

For a proof that we will need later, it is worth it to note that $D_1 \otimes D_2$ has the same matrix structure as D_2, i.e.,

$$[x]_{D_1 \otimes D_2} = J_1 \bigotimes I_2 + I_1 \bigotimes J_2 + J_1 \bigotimes J_2 = J_2$$

since $J_1 = [0]$. This means that the representation matrix (or the operator $T : k^2 \to k^2$ represented by the representation matrix) will have minimal polynomial $\rho(\lambda) = \lambda^2$.

Example. Consider $D_2 \otimes D_2$. Note that

$$\begin{bmatrix} x \end{bmatrix}_{D_2 \otimes D_2} = I_2 \bigotimes J_2 + J_2 \bigotimes I_2 + J_2 \bigotimes J_2.$$

A k-basis for D_2 is $\{1, x\}$ so a k-basis for $D_2 \otimes D_2$ is $\{1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x\}$. We will label our basis elements $\{e_1, e_2, e_3, e_4\}$ in the standard lexicographic order. We have $\Delta(x) = 1 \otimes x + x \otimes 1 + x \otimes x$ so we have the following calculations:

$$\Delta(x)e_1 = \Delta(x)(1 \otimes 1) = 1 \otimes x + x \otimes 1 - x \otimes x = e_2 + e_3 + e_4$$

$$\Delta(x)e_2 = \Delta(x)(1 \otimes x) = x \otimes x = e_4$$

$$\Delta(x)e_3 = \Delta(x)(x \otimes 1) = x \otimes x = e_4$$

$$\Delta(x)e_4 = \Delta(x)(x \otimes x) = 0.$$

This gives us the following representation matrix:

$$[x]_{D_2 \otimes D_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Now we want to determine the Jordan Canonical Form of this matrix. Let $T(e_i) = \Delta(x)e_i$ so that

$$T(e_1) = e_2 + e_3 + e_4$$
$$T(e_2) = e_4$$
$$T(e_3) = e_4$$
$$T(e_4) = 0.$$

We can also see that

$$e_1 \xrightarrow{T} e_2 + e_3 + e_4 \xrightarrow{T} 2e_4 \xrightarrow{T} 0$$

so we can use this to form a new k-basis for $D_2 \otimes D_2$. Note that a new k-basis for $D_2 \otimes D_2$ is $\{e_1, T(e_1), T^2(e_1), e_2 - e_3\}$. From this, we can see that the Jordan canonical form of our representation matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

which tells us that $D_2 \otimes D_2 \cong D_3 \oplus D_1$.

Another way to look at this is to consider the representation matrix we found at first:

$$A = [x]_{D_2 \otimes_k D_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Consider the operator $T : k^4 \to k^4$ represented by the matrix $A = [x]_{D_2 \otimes D_2}$. Let v_i represent the ith standard basis element of k^4. Since the only eigenvalue of T is 0, we need to look at $Null(A)$ to find our eigenvectors. Note that $Null(A) = span\{v_2 - v_3, v_4\}$. This tells us that the Jordan canonical form of A will contain only two Jordan blocks. Also, since our only eigenvalue is 0, the characteristic polynomial of T is $f(\lambda) = \lambda^4$. However, since $T^3(v_1) = 0$, we have that the minimal polynomial of T is $\rho(\lambda) = \lambda^3$. This tells us that our biggest Jordan block of A is J_3. Thus, the only option for the other block is a J_1 since A is a 4x4 matrix and J_3 is a $3x3$. Thus, we can see that the Jordan canonical form of A is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

which tells us that $D_2 \otimes D_2 \cong D_3 \oplus D_1$.

Example. Consider $D_3 \otimes D_2$. Note that

$$\Big[x\Big]_{D_3 \otimes D_2} = I_3 \bigotimes J_2 + J_3 \bigotimes I_2 + J_3 \bigotimes J_2.$$

A k-basis for D_3 is $\{1, x, x^2\}$ and a k-basis for D_2 is $\{1, x\}$, so a k-basis for $D_3 \otimes D_2$ is $\{1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x, x^2 \otimes 1, x^2 \otimes x\}$. We will label our basis elements in the following order:

$$
\begin{aligned}
e_1 &= 1 \otimes 1 \\
e_2 &= 1 \otimes x \\
e_3 &= x \otimes 1 \\
e_4 &= x \otimes x \\
e_5 &= x^2 \otimes 1 \\
e_6 &= x^2 \otimes x.
\end{aligned}
$$

We have $\Delta(x) = 1 \otimes x + x \otimes 1 - x \otimes x$ so we have the following calculations:

$$
\begin{aligned}
\Delta(x)e_1 &= \Delta(x)(1 \otimes 1) = 1 \otimes x + x \otimes 1 - x \otimes x = e_2 + e_3 + e_4 \\
\Delta(x)e_2 &= \Delta(x)(1 \otimes x) = x \otimes x = e_4 \\
\Delta(x)e_3 &= \Delta(x)(x \otimes 1) = x \otimes x + x^2 \otimes 1 - x^2 \otimes x = e_4 + e_5 + e_6 \\
\Delta(x)e_4 &= \Delta(x)(x \otimes x) = x^2 \otimes x = e_6 \\
\Delta(x)e_5 &= \Delta(x)(x^2 \otimes 1) = x^2 \otimes x = e_6 \\
\Delta(x)e_6 &= \Delta(x)(x^2 \otimes x) = 0.
\end{aligned}
$$

This gives us the following representation matrix:

$$
[x]_{D_3 \otimes D_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.
$$

We want to determine the Jordan Canonical Form of this matrix because that will give us the decomposition for $D_3 \otimes D_2$. Let $T(e_i) = \Delta(x)e_i$ so that

$$
\begin{aligned}
T(e_1) &= e_2 + e_3 + e_4 \\
T(e_2) &= e_4 \\
T(e_3) &= e_4 + e_5 + e_6 \\
T(e_4) &= e_6 \\
T(e_5) &= e_6 \\
T(e_6) &= 0.
\end{aligned}
$$

We can also see that

$$e_1 \xrightarrow{T} e_2 + e_3 + e_4 \xrightarrow{T} e_4 + e_4 + e_5 + e_6 + e_6 \xrightarrow{T} e_6 + e_6 + e_6 \xrightarrow{T} 0$$

so we can use this to form a new k-basis for $D_3 \otimes D_2$. Consider the elements $e_1, T(e_1), T^2(e_1), T^3(e_1)$. We have that $\{e_1, T(e_1), T^2(e_1), T^3(e_1), e_2, T(e_2)\}$ is also a k-basis for $D_3 \otimes D_2$. From this, we can see that the Jordan Canonical Form for our representation matrix is

$$\left[\begin{array}{cccc|cc}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right].$$

This tells us that $D_3 \otimes D_2 \cong D_4 \oplus D_2$.

Another way we can verify this is by viewing our representation matrix as the matrix representation of some linear operator. Consider the linear operator

$T : k^6 \to k^6$ represented by the matrix A where $A = [x]_{D_3 \otimes D_2}$. Let v_i represent the ith standard basis element of k^6. Since the only eigenvalue of A is 0, we will need to find the $Null(A)$. Note that $Null(A) = span\{v_4 - v_5, v_6\}$. This tells us that the Jordan normal form of A will contain only two Jordan Blocks. Since the only eigenvalue is 0, we can see that the characteristic polynomial of A is $f(\lambda) = \lambda^6$. In our previous work of this example, we showed that $T^4(e_1) = 0$. This tells us that $A^4 = 0$ so this and the characteristic polynomial shows the minimal polynomial of A is $\rho(\lambda) = \lambda^4$. This tells us that our biggest Jordan block of A will be a J_4. Since A is of size 6×6 and J_4 is of size 4×4, our other Jordan block must be a J_2. This also makes sense because $\text{rank}(J_4) + \text{rank}(J_2) = 3 + 1 = 4 = \text{rank}(A)$.

Example. Now consider $D_4 \otimes D_2$. We can see that a k-basis for $D_4 \otimes D_2$ is $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ where

$$e_1 = 1 \otimes 1$$

$$e_2 = 1 \otimes x$$

$$e_3 = x \otimes 1$$

$$e_4 = x \otimes x$$

$$e_5 = x^2 \otimes 1$$

$$e_6 = x^2 \otimes x$$

$$e_7 = x^3 \otimes 1$$

$$e_8 = x^3 \otimes x.$$

Then we have the following calculations:

$$\Delta(x)(e_1) = e_2 + e_3 + e_4$$

$$\Delta(x)(e_2) = e_4$$

$$\Delta(x)(e_3) = e_4 + e_5 + e_6$$

$$\Delta(x)(e_4) = e_6$$

$$\Delta(x)(e_5) = e_6 + e_7 + e_8$$

$$\Delta(x)(e_6) = e_8$$

$$\Delta(x)(e_7) = e_8$$

$$\Delta(x)(e_8) = 0$$

which gives us the following representation matrix:

$$[x]_{D_4 \otimes D_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Let $T : k^8 \to k^8$ be represented by $B = [x]_{D_4 \otimes D_2}$ and v_i be the ith standard basis element of k^8. Our only eigenvalue of B is 0 so $Null(B) = span\{v_6 - v_7, v_8\}$. Our Jordan decomposition of B will contain two Jordan blocks. Note that the characteristic polynomial of T is $f(\lambda) = \lambda^8$ and $T^5(v_1) = 0$ so we have that the minimal polynomial of T is $\rho(\lambda) = \lambda^5$. This tells us that J_5 will be the biggest block of the Jordan decomposition of B. Since there are only two blocks in the decomposition and B

is an $8x8$ matrix, it follows that the other block must be a J_3. Thus, the Jordan canonical form of B is

$$\left[\begin{array}{ccccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

which tells us that $D_4 \otimes D_2 \cong D_5 \oplus D_3$.

Looking at these previous examples, we notice a common theme for $D_j \otimes_k D_2$ and express it as a theorem.

Theorem 3.4.1. *$D_j \otimes_k D_2 \cong D_{j+1} \oplus D_{j-1}$ for any $2 \leq j \leq p-1$.*

Proof. Note that a k-basis for $D_j \otimes D_2$ is $\{e_1, \ldots, e_{2j-1}, e_{2j}\}$ where

$$\begin{aligned} e_1 &= 1 \otimes 1 \\ e_2 &= 1 \otimes x \\ e_3 &= x \otimes 1 \\ &\vdots \\ e_{2j-1} &= x^j \otimes 1 \\ e_{2j} &= x^j \otimes x. \end{aligned}$$

Recall that the matrix representation for D_j is J_j which is the jxj matrix with $1's$ on the subdiagonal and $0's$ everywhere else. Note that

$$[x]_{D_j \otimes D_2} = I_j \bigotimes J_2 + J_j \bigotimes I_2 - J_j \bigotimes J_2$$

so it follows that the only eigenvalue of $M = [x]_{D_j \otimes D_2}$ is 0. Let $T : k^{2j} \to k^{2j}$ be represented by the matrix $M = [x]_{D_j \otimes D_2}$ and $\{v_1, v_2, \ldots, v_{2j}\}$ be the standard k-basis for k^{2j}. Then we have the following calculations:

$$T(v_i) = \begin{cases} v_{i+1} + v_{i+2} + v_{i+3}, & i \leq 2j-3, \text{i odd} \\ v_{i+2}, & i \neq 2j, \text{i even} \\ v_{2j}, & i = 2j-2 \text{ or } i = 2j-1 \\ 0, & i = 2j \end{cases}$$

We want to find the Jordan canonical form of M. First, we want to determine how many Jordan blocks there are. Since our only eigenvalue is 0, then we just need to find $Null(T)$. Let v_i represent the ith standard basis element of k^{2j}. I claim that $Null(T) = span\{v_{2j-2} - v_{2j-1}, v_{2j}\}$.

It is clear that $Null(T) \supseteq span\{v_{2j-2} - v_{2j-1}, v_{2j}\}$ since $T(v_{2j}) = 0$ and $T(v_{2j-2} - v_{2j-1}) = v_{2j} - v_{2j} = 0$. Let $v \in Null(T)$. We want to show that $v = a(v_{2j-2} - v_{2j-1}) + bv_{2j}$ for some $a, b \in k$. Since $v \in Null(T)$, we have that

$$\begin{aligned} 0 &= T(v) \\ &= T(a_1 v_1 + \cdots + a_{2j-3}v_{2j-3} + a_{2j-2}v_{2j-2} + a_{2j-1}v_{2j-1} + a_{2j}v_{2j}) \\ &= a_1 T(v_1) + \cdots + +a_{2j-3}T(v_{2j-3}) + a_{2j-2}T(v_{2j-2}) + a_{2j-1}T(v_{2j-1}) + a_{2j}T(v_{2j}) \\ &= a_1 T(v_1) + \cdots + +a_{2j-3}T(v_{2j-3}) + (a_{2j-2} + a_{2j-1})v_{2j}. \end{aligned}$$

Note that $a_1 T(v_1)$ will generate a sole $a_1 v_2$ term so we need $a_1 = 0$ for v_2 to vanish. Likewise, $a_2 T(v_2)$ will generate a sole $a_2 v_3$ term so we need $a_2 = 0$ for v_3 to vanish. We can continue in this fashion all the way to a_{2j-3}. Then we get

$$0 = T(v) = (a_{2j-2} + a_{2j-1})v_{2j} \Rightarrow a_{2j-2} = -a_{2j-1}.$$

Let $a = a_{2j-2}$ and $b = a_{2j}$. Thus, we have that

$$v = a_{2j-2}v_{2j-2} + a_{2j-1}v_{2j-1} + a_{2j}v_{2j} = a(v_{2j-2} - v_{2j-1}) + bv_{2j}.$$

Thus, we have that $Null(T) = span\{v_{2j-2} - v_{2j-1}, v_{2j}\}$.

From the $Null(T)$, we can see that our Jordan decomposition for M will contain two Jordan blocks. Thus, if we figure out one block, we will have the other for free. Note that the characteristic polynomial for T is $f(\lambda) = \lambda^{2j}$. To determine the minimal polynomial, we want to find the smallest positive integer i such that $T^i(v_1) = 0$ since M is lower triangular with $0's$ on the diagonal. We can see from the previous examples that $i = j + 1$ for $j = 2, 3, 4$. I claim that this holds for all $j \geq 1$. We can see from the examples above that this holds for $j = 1, 2, 3, 4$. Assume $j > 4$. We need to show that $T^{j+1}(v_1) = 0$ and $T^j(v_1) \neq 0$. Note that

$$\begin{aligned} T^j(v_1) &= T^{j-1}(T(v_1)) \\ &= T^{j-1}(v_2 + v_3 - v_4) \\ &= T^{j-1}(v_2) + T^{j-1}(v_3) - T^{j-1}(v_4). \end{aligned}$$

By our previous calculations of $T(v_i)$, we can see that

$$T^{j-1}(v_2) = v_{2+2(j-1)} = v_{2j}$$

and

$$T^{j-1}(v_4) = 0$$

since $4 + 2(j-1) = 2j + 2 > 2j$. Now we need to focus on $T^{j-1}(v_3)$. Recall that $T(v_i) = v_{i+1} + v_{i+2} - v_{i+3}$ if i is odd and $i \leq 2j - 3$. If $i > 2j - 3$ and i is odd, then $i = 2j - 1$ which implies that $T(v_i) = v_{2j}$. So we have that

$$T^j(v_{2j-1}) = T^{j-2}(v_{2j}) = 0.$$

Since $j > 4$, we know that there will be at least 8 basis elements so we have that

$$T(v_3) = v_4 + v_5 - v_6$$

which implies that

$$T^{j-1}(v_3) = T^{j-2}(v_4) + T^{j-2}(v_5) - T^{j-2}(v_6).$$

Since 4 and 6 are even, we can see that

$$T^{j-2}(v_4) = v_{2j} \;\; (4 + 2(j-2) = 2j)$$

and

$$T^{j-2}(v_6) = 0 \;\; (6 + 2(j-2) = 2j + 2 > 2j).$$

So we have

$$T^{j-1}(v_3) = v_{2j} + T^{j-2}(v_5)$$

which implies that

$$\begin{aligned} T^j(v_1) &= T^{j-1}(v_2) + T^{j-1}(v_3) - T^{j-1}(v_4) \\ &= 2v_{2j} + T^{j-2}(v_5). \end{aligned}$$

Now notice that

$$\begin{aligned} T^{j-2}(v_5) &= T^{j-3}(v_6) + T^{j-3}(v_7) - T^{j-3}(v_8) \\ &= v_{2j} + T^{j-3}(v_7) - T^{j-3}(v_8) && (6 + 2(j-3) = 2j) \\ &= v_{2j} + T^{j-3}(v_7) && (8 + 2(j-3) = 2j + 2 > 2j) \end{aligned}$$

which implies that

$$\begin{aligned} T^j(v_1) &= T^{j-1}(v_2) + T^{j-1}(v_3) - T^{j-1}(v_4) \\ &= 2v_{2j} + T^{j-2}(v_5) \end{aligned}$$

$$= 3v_{2j} + T^{j-3}(v_7).$$

We can continue this process until we get a $T(v_{2j-1})$. To get this term, we will need to operate on v_1 $j-1$ times. This means that we will have

$$\begin{aligned} T(v_1) &= (j-1)v_{2j} + T(v_{2j-1}) \\ &= jv_{2j} \\ &\neq 0. \end{aligned}$$

From this we can see that

$$\begin{aligned} T^{j+1}(v_1) &= T(jv_{2j}) \\ &= jT(v_{2j}) \\ &= 0. \end{aligned}$$

Thus, $j+1$ is the smallest integer i such that $T^i(v_1) = 0$. This tells us that $\rho(\lambda) = \lambda^{j+1}$ is the minimal polynomial for T. Thus, the biggest Jordan block in our decomposition will be a J_{j+1} block. Note that our matrix M is a $2j \times 2j$ matrix so we can figure out what the second block by subtracting the dimensions. Thus, our second Jordan block must be a J_{j-1} block. We can also verify this using ranks. Note that since $\dim_k Null(T) = 2$ and $\dim_k(k^{2j}) = 2j$, we have that $rank(M) = \dim_k Range(T) = 2j-2$. The rank of a J_{j+1} block is j and the rank of a J_{j-1} block is $j-2$ so the Jordan decomposition of M will also have *rank* $2j-2$. Therefore, we can conclude that

$$D_j \otimes D_2 \cong D_{j+1} \oplus D_{j-1}.$$

$\square$

In the next section, we will show a similar result and other examples using the two-term diagonal map.

3.5 Clebsch-Gordan Problem with $\Delta'(x) = 1 \otimes x + x \otimes 1$

Proposition 3.5.1. *$D_i \otimes'_k D_1 \cong D_i$ for any $1 \le i \le p$*

Proof. Let $M = D_i \otimes'_k D_1$. Then we have that

$$\begin{bmatrix} x \end{bmatrix}_M = I_i \bigotimes J_1 + J_i \bigotimes I_1 = J_i$$

which is the Jordan normal form for $\begin{bmatrix} x \end{bmatrix}_M$. Therefore, we have

$$D_i \otimes_k D_1 \cong D_i.$$

$\square$

Proposition 3.5.2. *$D_i \otimes'_k D_2 \cong D_{i+1} \oplus D_{i-1}$ for any $2 \le i \le p-1$*

Proof. Let $M = D_i \otimes'_k D_2$. Note that

$$\begin{bmatrix} x \end{bmatrix}_M = I_i \bigotimes J_2 + J_i \bigotimes I_2 = \begin{bmatrix} J_2 & & & 0 \\ I_2 & J_2 & & \\ & \ddots & \ddots & \\ 0 & & I_2 & J_2 \end{bmatrix}.$$

I claim that the Jordan normal form of this matrix contains a J_{i+1} block and a J_{i-1}. To show this, we first need to prove that the Jordan normal form of the matrix contains two Jordan blocks. Note that our only eigenvalue is 0. Thus, to find the number of Jordan blocks, we need to show $\dim_k(Null([x]_M - 0I)) = \dim_k(Null([x]_M)) = 2$. This is equivalent to showing that $rank([x]_M) = 2i - 2$. Consider the submatrix that results from cutting out the first two rows and the last two columns. Then we have

$$\begin{bmatrix} I_2 & J_2 & & 0 \\ & I_2 & \ddots & \\ & & \ddots & J_2 \\ 0 & & & I_2 \end{bmatrix}.$$

This submatrix is an upper triangular matrix with $1's$ on the diagonal, so it clearly has full rank. Thus, $rank([x]_M) \geq 2i-2$. Now consider $[x]_M$. Note that the first row is all zeroes and second row is exactly the same as the third row. Thus, the first two rows are linearly dependent on the $2i-2$ rows below them. Thus, $rank([x]_M) \leq 2i-2$ which implies that $rank([x]_M) = 2i-2$. This implies that $\dim_k(Null([x]_M)) = 2$. This tells us that there are two blocks in our Jordan normal form. Since there are only two blocks, it is enough to find the biggest Jordan block. We can do this by determining the minimal polynomial. Since the characteristic polynomial of $[x]_M$ is α^{2i}, it is equivalent to find the smallest positive integer j such that $[x]_M^j = 0$. Note that

$$\begin{aligned}
[x]_M^i &= \sum_{r=0}^{i} \binom{i}{r} (I_i \bigotimes J_2)^r (J_i \bigotimes I_2)^{i-r} \\
&= \sum_{r=0}^{i} \binom{i}{r} (I_i \bigotimes J_2^r)(J_i^{i-r} \bigotimes I_2) \\
&= \sum_{r=0}^{i} \binom{i}{r} (J_i^{i-r} \bigotimes J_2^r) \\
&= \binom{i}{1} J_i^{i-1} \bigotimes J_2 && (J_2^r = 0 \text{ for any } r \geq 2)
\end{aligned}$$

which is a nonzero $2i \times 2i$ matrix. However, we have that

$$\begin{aligned}
[x]_M^{i+1} &= [x]_M^i \cdot [x]_M \\
&= (J_i^{i-1} \bigotimes J_2) \cdot (I_i \bigotimes J_2 + J_i \bigotimes I_2) \\
&= J_i^{i-1} \bigotimes J_2^2 + J_i^i \bigotimes J_2 && (J_i^i = 0) \\
&= 0.
\end{aligned}$$

Thus, $j = i+1$ is the smallest positive integer such that $[x]_M^j = 0$. This implies that α^j is the minimal polynomial of $[x]_M$, which tells us that biggest Jordan block for $[x]_M$ is a J_{i+1} block. Hence, it is clear that the other block must be a J_{i-1} block since the dimensions and ranks are invariant with respect to Jordan decompositions of square matrices. Therefore, we have that $D_i \otimes_k D_2 \cong D_{i+1} \oplus D_{i-1}$. $\square$

Example. Consider $M = D_3 \otimes_k' D_3$. We will assume $p \geq 5$. Note that

$$\begin{bmatrix} x \end{bmatrix}_M = \begin{bmatrix} J_3 & 0 & 0 \\ I_3 & J_3 & 0 \\ 0 & I_3 & J_3 \end{bmatrix}$$

which has rank 6. This implies that the decomposition consist of 3 D_i blocks. Let's determine the biggest D_i block in the decomposition. Note that

$$\begin{bmatrix} x^4 \end{bmatrix}_M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ J_2^2 & 0 & 0 \end{bmatrix},$$

which has rank 1 and

$$\begin{bmatrix} x^5 \end{bmatrix}_M = 0,$$

so there is one D_5 block. Now, we need to determine the ranks of the representation matrices of x^2, x^3, and x^4. This will give us our decomposition for $M = D_3 \otimes_k D_3$. We have the following representation matrices:

$$\begin{bmatrix} x^2 \end{bmatrix}_M = \sum_{k=0}^{2} \binom{2}{k} \left(J_3^{2-k} \bigotimes J_3^k \right) = \begin{bmatrix} J_3^2 & 0 & 0 \\ 2J_3 & J_3^2 & 0 \\ I_3 & 2J_3 & J_3^2 \end{bmatrix} \qquad (rank = 3)$$

$$\begin{bmatrix} x^3 \end{bmatrix}_M = \sum_{k=0}^{3} \binom{3}{k} \left(J_3^{3-k} \bigotimes J_3^k \right) = \begin{bmatrix} 0 & 0 & 0 \\ 3J_3^2 & 0 & 0 \\ J_3 & 3J_3^2 & 0 \end{bmatrix} \qquad (rank = 2)$$

$$\left[x^4\right]_M = \sum_{k=0}^{4} \binom{4}{k} \left(J_3^{4-k} \bigotimes J_3^k\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 6J_3^2 & 0 & 0 \end{bmatrix} \qquad (rank = 1).$$

As we go through the representation matrices of x to x^2, we see that the rank stays the same, so there must be three D_i blocks of at least size two. Going through the representation matrices of x^2 and x^3, the rank decreases by 1, so there must be at least one block of at least size three. We know that block must be D_5 from our findings earlier in the example. Since we have to have three blocks of at least size two, but only one block of at least size three, this implies that the other two blocks are D_2. Therefore, we get that

$$D_3 \otimes_k' D_3 \cong D_5 \oplus D_2^2.$$

Finding this result will help us generalize a theorem for $D_i \otimes_k' D_3$. In particular,

Proposition 3.5.3. *$D_i \otimes_k' D_3 \cong D_{i+2} \oplus D_{i-1}^2$ for any $3 \leq i \leq p-2$.*

Proof. Let $M = D_i \otimes_k' D_3$ where $3 \leq i \leq p-2$. Note that the representation matrix for x over M is an $i \times i$ matrix whose entries are 3×3 matrices, i.e.,

$$\left[x\right]_M = \begin{bmatrix} J_3 & 0 & \cdots & 0 \\ I_3 & J_3 & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & I_3 & J_3 \end{bmatrix}.$$

First, we want to find the rank of this representation matrix, so we can determine the number of blocks in the decomposition. Note that the first three rows are linearly dependent on the rest of the rows of the matrix. The first row consists of all zeroes, the second row is the same as the fourth row and the third row is the same as the fifth row minus the seventh row. Thus, the first three rows are linearly dependent on the bottom $3i-3$ rows, which says that the rank can't be any bigger than $3i-3$. Note that the following $(3i-3)\times(3i-3)$ submatrix

$$\begin{bmatrix} I_3 & J_3 & \cdots & 0 \\ 0 & I_3 & \cdots & \vdots \\ \vdots & \ddots & \ddots & J_3 \\ 0 & \cdots & 0 & I_3 \end{bmatrix}$$

has full rank, so the rank of $\begin{bmatrix} x \end{bmatrix}_M$ has rank $3i-3$. Since $dim_k(M)=3i$, then the nullity of $\begin{bmatrix} x \end{bmatrix}_M$ is 3, which implies that the Jordan normal form of $\begin{bmatrix} x \end{bmatrix}_M$ has 3 Jordan blocks. Next, we will determine the biggest Jordan block. In particular, we will determine the smallest positive integer r such that $\begin{bmatrix} x^r \end{bmatrix}_M = 0$. We can see that $r\geq 3$ since $J_3^3=0$. Note that

$$\begin{aligned}
[x^r]_M &= [x]_M^r \\
&= \sum_{k=0}^{r}\binom{r}{k}\left(I_i\bigotimes J_3\right)^k\left(J_i\bigotimes I_3\right)^{r-k} \\
&= \sum_{k=0}^{r}\binom{r}{k}\left(I_i\bigotimes J_3^k\right)\left(J_i^{r-k}\bigotimes I_3\right) \\
&= \sum_{k=0}^{r}\binom{r}{k}\left(J_i^{r-k}\bigotimes J_3^k\right) \\
&= \sum_{k=0}^{2}\binom{r}{k}\left(J_i^{r-k}\bigotimes J_3^k\right). && (J_3^3=0)
\end{aligned}$$

In order for $\begin{bmatrix} x^r \end{bmatrix}_M = 0$ to hold, we need $r-k\geq i$ for $k=0,1,2$ since $J_i^i=0$. Thus, the smallest integer r that has this property is $r=i+2$. From here, we can see that $\Big[x^{i+2}\Big]_M = 0$. To ensure that J_{i+2} is the biggest Jordan block, we need to see if $\Big[x^{i+1}\Big]_M \neq 0$. Note that

$$\begin{aligned}
[x^{i+1}]_M &= [x]_M^{i+1} \\
&= \sum_{k=0}^{i+1} \binom{i+1}{k} \left(J_i^{i+1-k} \bigotimes J_3^k\right) \\
&= \sum_{k=0}^{2} \binom{i+1}{k} \left(J_i^{i+1-k} \bigotimes J_3^k\right) && (J_3^3 = 0) \\
&= \binom{i+1}{2} J_i^{i-1} \bigotimes J_3^2
\end{aligned}$$

which is nonzero and has rank 1. This verifies that J_{i+2} is the biggest Jordan block of $\Big[x\Big]_M$, and there is only one of them. We will determine the other blocks by working backwards. Note that

$$\begin{aligned}
[x^{i}]_M &= [x]_M^{i} \\
&= \sum_{k=0}^{i} \binom{i}{k} \left(J_i^{i-k} \bigotimes J_3^k\right) \\
&= \sum_{k=0}^{2} \binom{i}{k} \left(J_i^{i-k} \bigotimes J_3^k\right) && (J_3^3 = 0) \\
&= \binom{i}{1} J_i^{i-1} \bigotimes J_3 + \binom{i}{2} J_i^{i-2} \bigotimes J_3^2
\end{aligned}$$

which has rank 2. $\Big[x^i\Big]_M$ having rank 2 and $\Big[x^{i+1}\Big]_M$ having rank 1 implies that there is one Jordan block of at least size $i+1$, which we already know is of size $i+2$. Now, let's check $i-1$. Note that

$$[x^{i-1}]_M = [x]_M^{i-1}$$

$$= \sum_{k=0}^{i-1} \binom{i-1}{k} \left(J_i^{i-1-k} \bigotimes J_3^k\right)$$

$$= \sum_{k=0}^{2} \binom{i-1}{k} \left(J_i^{i-1-k} \bigotimes J_3^k \right) \qquad (J_3^3 = 0)$$

$$= \binom{i-1}{0} J_i^{i-1} \bigotimes I_3 + \binom{i-1}{1} J_i^{i-2} \bigotimes J_3 + \binom{i-1}{2} J_i^{i-3} \bigotimes J_3^2$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ J_3^2 & 0 & 0 & 0 & \cdots & 0 \\ J_3 & J_3^2 & 0 & 0 & \cdots & 0 \\ I_3 & J_3 & J_3^2 & 0 & \cdots & 0 \end{bmatrix}.$$

Note that this matrix has at least rank 3 due to the submatrix formed by the I_3 in the bottom left corner. We will show that the rank must be 3. Let R_s correspond to the s^{th} row of the matrix. Note that we have the following:

$$R_{3i-8} = 0R_{3i-2}$$

$$R_{3i-7} = 0R_{3i-2}$$

$$R_{3i-6} = R_{3i-2}$$

$$R_{3i-5} = 0R_{3i-2}$$

$$R_{3i-4} = R_{3i-2}$$

$$R_{3i-3} = R_{3i-1}.$$

This tells us that rows $3i-8$ to $3i-3$ are linearly dependent on the last three rows of the matrix. Since all of the rows above row $3i-8$ are all zeroes, we have that the rank of this matrix can be at most 3. Thus, $rank([x^{i-1}]_M) = 3$. This combined with the fact that $rank([x^i]_M) = 2$ shows us that there is at least one Jordan block of size i, which we know is of size $i+2$. Thus, we will determine $rank([x^{i-2}]_M)$. Note that

$$
\begin{aligned}
[x^{i-2}]_M &= [x]_M^{i-2} \\
&= \sum_{k=0}^{i-2} \binom{i-2}{k} \left(J_i^{i-2-k} \bigotimes J_3^k \right) \\
&= \sum_{k=0}^{2} \binom{i-2}{k} \left(J_i^{i-2-k} \bigotimes J_3^k \right) && (J_3^3 = 0) \\
&= \binom{i-2}{0} J_i^{i-2} \bigotimes I_3 + \binom{i-2}{1} J_i^{i-3} \bigotimes J_3 + \binom{i-2}{2} J_i^{i-4} \bigotimes J_3^2 \\
&= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
J_3^2 & 0 & 0 & 0 & 0 & \cdots & 0 \\
J_3 & J_3^2 & 0 & 0 & 0 & \cdots & 0 \\
I_3 & J_3 & J_3^2 & 0 & 0 & \cdots & 0 \\
0 & I_3 & J_3 & J_3^2 & 0 & \cdots & 0
\end{bmatrix}.
\end{aligned}
$$

Note that the following submatrix

$$
\begin{bmatrix}
I_3 & J_3 \\
0 & I_3
\end{bmatrix}
$$

has full rank and is of size 6, so this implies $rank([x^{i-2}]_M) \geq 6$. Consider the following submatrix:

$$\begin{bmatrix} J_3^2 & 0 & 0 \\ J_3 & J_3^2 & 0 \\ I_3 & J_3 & J_3^2 \end{bmatrix}.$$

We will do the following row operations in this order:

$$R_3 - R_7 \to R_3 \tag{3.1}$$

$$R_5 - R_7 \to R_5 \tag{3.2}$$

$$R_6 - R_8 \to R_6 \tag{3.3}$$

which gives us the following matrix:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_3 & J_3 & J_3^2 \end{bmatrix}.$$

We can do similar row operations on $[x^{i-2}]_M$ to get

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ I_3 & J_3 & J_3^2 & 0 & 0 & \cdots & 0 \\ 0 & I_3 & J_3 & J_3^2 & 0 & \cdots & 0 \end{bmatrix}.$$

Putting this matrix in row echelon form gives us

$$\begin{bmatrix} I_3 & J_3 & J_3^2 & 0 & 0 & \cdots & 0 \\ 0 & I_3 & J_3 & J_3^2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

which tells us that $rank([x^{i-2}]_M) = 6$. This combined with the fact that $rank([x^{i-1}]) = 3$ implies that there are 3 Jordan blocks of at least size $i-1$. Since there are 3 total Jordan blocks in the decomposition and one of them is of size $i+2$, this implies that the other two blocks must be of size $i-1$. Therefore, we have that

$$D_i \otimes_k' D_3 \cong D_{i+2} \oplus D_{i-1}^2.$$

$\square$

This next lemma will help us confirm the results for D_2 and D_3 by determining the largest Jordan block of the Jordan normal form of the representation matrix of x over $D_i \otimes_k' D_j$.

Lemma 3.5.4. *Let $M = D_i \otimes_k' D_j$ where $i \leq j$ and $i+j-1 < p$. Then $i+j-1$ is the smallest integer r such that $[x^r]_M = 0$.*

Proof. Let r be some positive integer. Note that

$$\begin{aligned} [x^r]_M &= [x]_M^r \\ &= \sum_{k=0}^{r} \binom{r}{k} \left(I_i \bigotimes J_j\right)^k \left(J_i \bigotimes I_j\right)^{r-k} \\ &= \sum_{k=0}^{r} \binom{r}{k} \left(I_i \bigotimes J_j^k\right) \left(J_i^{r-k} \bigotimes I_j\right) \\ &= \sum_{k=0}^{r} \binom{r}{k} \left(J_i^{r-k} \bigotimes J_j^k\right) \end{aligned}$$

$$= \sum_{k=0}^{j-1} \binom{r}{k} \left(J_i^{r-k} \bigotimes J_j^k \right). \qquad (J_j^j = 0)$$

Thus, in order for $[x^r]_M = 0$, we need $r - k \geq i$, or equivalently, $r \geq i + k$ for all $k = 1, \dots, j-1$. This is true only if $r \geq i + j - 1$. Therefore, the smallest integer r such that $[x^r]_M = 0$ is $r = i + j - 1$. □

This tells us that the biggest block of an arbitrary $M = D_i \otimes'_k D_j$ where $i \geq j \geq 2$ and $i + j - 1 \leq p$ is a D_{i+j-1} block. We can also show that there is only one D_{i+j-1} block by looking at the rank of $[x^{i+j-2}]_M$. Note that

$$[x^{i+j-2}]_M = \sum_{k=0}^{i+j-2} \binom{i+j-2}{k} \left(J_i^{i+j-2-k} \bigotimes J_j^k \right)$$

$$= \sum_{k=1}^{j-1} \binom{i+j-2}{k} \left(J_i^{i+j-2-k} \bigotimes J_j^k \right)$$

$$= \binom{i+j-2}{j-1} \left(J_i^{i-1} \bigotimes J_j^{j-1} \right), \qquad (i + j - 2 - k \geq i \;\; \forall \;\; k = 0, 1, \dots, j-2)$$

which has rank 1, so this confirms that there is only one D_{i+j-1} block in the decomposition of $D_i \otimes_k D_j$ when $i \geq j \geq 2$. The next theorem will show us that there are j blocks in the decomposition of $D_i \otimes'_k D_j$.

Theorem 3.5.5. *Let $M = D_i \otimes'_k D_j$ where $i \leq j$ and $i + j - 1 < p$. Then $rank([x]_M) = j(i-1)$.*

Proof. Let $r = rank([x]_M)$. First, we will show that $r \geq j(i-1)$. Note that

$$[x]_M = \begin{bmatrix} J_j & 0 & \cdots & 0 \\ I_j & J_j & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & I_j & J_j \end{bmatrix},$$

which is a $i \times i$ square matrix with $j \times j$ square matrix entries. Consider the $j(i-1) \times j(i-1)$ submatrix

$$\begin{bmatrix} I_j & J_j & \cdots & 0 \\ 0 & I_j & \ddots & \vdots \\ \vdots & \ddots & \ddots & J_j \\ 0 & \cdots & 0 & I_j \end{bmatrix},$$

which can be constructed by cutting out the first i rows and last i columns of $[x]_M$. This matrix clearly has full rank since it is upper triangular with all $1's$ on the diagonal. Thus, $r \geq j(i-1)$.

Now we must show that $r \leq j(i-1)$. It is equivalent to show that the first j rows of $\begin{bmatrix} x \end{bmatrix}_M$ are linearly dependant on the other rows of the matrix. Recall that

$$[x]_M = \begin{bmatrix} J_j & 0 & \cdots & 0 \\ I_j & J_j & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & I_j & J_j \end{bmatrix},$$

which is a $i \times i$ square matrix with $j \times j$ square matrix entries. Consider the submatrix

$$\begin{bmatrix} J_j & 0 \\ I_J & J_j \end{bmatrix}.$$

By using row elementary row operations on this submatrix, we can get the new matrix

$$\begin{bmatrix} 0 & -J_j^2 \\ I_j & J_j \end{bmatrix}.$$

In particular, this is what you get when you do the row operation $R_s - R_{s+j-1} \to R_s$ for all $s = 2, \ldots, j$. Doing those row operations on the first j rows of $[x]_M$ gives us

$$\begin{bmatrix} 0 & -J_j^2 & \cdots & 0 \\ I_j & J_j & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & I_j & J_j \end{bmatrix}.$$

Now we can create the following submatrix:

$$\begin{bmatrix} -J_j^2 & 0 \\ I_j & J_j \end{bmatrix}.$$

We can do the following row operations, in this order,

$$\begin{gathered} R_3 + R_{j+1} \to R_3 \\ R_4 + R_{j+2} \to R_4 \\ \vdots \\ R_{j-1} + R_{2j-3} \to R_{j-1} \\ R_j + R_{2j-2} \to R_j \end{gathered}$$

to get the new submatrix:

$$\begin{bmatrix} 0 & J_j^3 \\ I_j & J_j \end{bmatrix}.$$

Since the other matrices in corresponding rows of $[x]_M$ are zero matrices, you can use a similar set of row operations to get

$$\begin{bmatrix} 0 & 0 & J_j^3 & 0 & \cdots & 0 \\ I_j & J_j & 0 & 0 & \cdots & 0 \\ 0 & I_j & J_j & 0 & \cdots & 0 \\ 0 & 0 & I_j & J_j & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & I_j & J_j \end{bmatrix}.$$

In general, consider the submatrix

$$\begin{bmatrix} (-1)^{s+1}J_j^s & 0 \\ I_j & J_j. \end{bmatrix}$$

We can do a series of row operations similar to the ones above to get the submatrix

$$\begin{bmatrix} 0 & (-1)^{s+2}J_j^{s+1} \\ I_j & J_j \end{bmatrix}.$$

Since $i \geq j$, we can continue in this fashion until we turn $[x]_M$ into the following row-equivalent matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ I_j & J_j & 0 & 0 & \cdots & 0 \\ 0 & I_j & J_j & 0 & \cdots & 0 \\ 0 & 0 & I_j & J_j & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & I_j & J_j \end{bmatrix}.$$

By turning this matrix into row echelow form, we can get the following row-equivalent matrix:

$$\begin{bmatrix} I_j & J_j & 0 & 0 & \cdots & 0 \\ 0 & I_j & J_j & 0 & \cdots & 0 \\ 0 & 0 & I_j & J_j & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & I_j & J_j \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

which has $j(i-1)$ nonzero rows. Therefore, we have that $rank([x]_M) = j(i-1)$. $\square$

This tells us that there are j Jordan blocks in the Jordan normal form of $[x]_M$. This matches with our results when $j = 1, 2, 3$. Lemma 3.5.4 and Theorem 3.5.5 give us the number of Jordan blocks and the size of the biggest Jordan block of the Jordan normal form of

$$\begin{bmatrix} x \end{bmatrix}_{D_i \otimes'_k D_j}$$

for some i, j where $j \le i \le p - j + 1$. Finding the decompostion for $D_i \otimes'_k D_j$ for an arbitrary i, j is left as an open problem.

CHAPTER 4

More General Truncated Polynomial Rings

Section 3.3 tells us that it does not matter which diagonal map we choose to use to determine freeness of a tensor product of A-modules over $k[u_\lambda]$. Thus, for this chapter, we will associate the tensor product $\otimes_k$ with the diagonal map $\Delta(x_i) = 1 \otimes x_i + x_i \otimes 1$.

4.1 Introducing A' and hypersurfaces H_λ

Suppose $char(k) \neq p$. Then we get a different truncated polynomial ring, which is no longer a group algebra, nor a coalgebra. We will call it A'. Let

$$H_\lambda = k[X_1, \ldots, X_n]/(\tilde{u_\lambda}^{(p)})$$

where $\tilde{u}_\lambda = \sum_{i=1}^{n} \lambda_i X_i$ and $\tilde{u_\lambda}^{(p)} = \sum_{i=1}^{n} \lambda_i^p X_i^p$ are elements in $k[X_1, \ldots, X_n]$. When $char(k) = p$, there exists a map μ such that we have the following commutative diagram

$$\begin{tikzcd} & H_\lambda \arrow[d] \\ k[u_\lambda] \arrow[ur, "\mu"] \arrow[r, hook] & A. \end{tikzcd}$$

where the vertical map is the natural projection and μ sends u_λ to $\tilde{u_\lambda}$. We have the following known result:

Theorem 4.1.1. *Given an A-module M, M is free as a $k[u_\lambda]$-module if and only if M has finite projective dimension over H_λ ([1], [5]).*

This is no longer true once $char(k) \neq p$. In particular, we are not able to decompose A'-modules over $k[u_\lambda]$ since $k[u_\lambda]$. Thus, in order to determine the rank variety of certain A'-modules, we use a different, but related notion.

Definition 4.1.2. The *rank variety*, $W(M')$, of an A'-module M' is the set of $\lambda \in k^n$ affine space such that M' has infinite projective dimension over H_λ. In set notation, we have that

$$W(M') = \{\lambda \in k^n : pd_{H_\lambda}(M') = \infty\}$$

Although it is not clear from the definition, it is known that $W(M)$ is indeed an algebraic variety [2]. Thus, just like $V(M)$ for an A'-module, we will view $W(M')$ as the zero-set of certain polynomials in $k[\chi_1, \ldots, \chi_n]$.

Throughout this chapter, we will investigate the rank varieties of a particular class of A-modules and compare them to the rank varieties of 'similar' class of A'-modules. Note that A' does not have a coalgebra structure, so we also don't have a notion of a tensor product of A'-modules. For this, we will build a faux tensor product structure for certain A'-modules, show that it is isomorphic to an ordinary tensor product structure with a certain multiplication property, and show that Carlson's identity (*) applies to a particular class of A'-modules that are similar to the aforementioned particular A-modules. It is important to note that $char(k) = p$ for A-modules and $char(k) \neq p$ for A'-modules.

4.2 Class of A-Modules of the form $A/(x_{i_1}, \ldots, x_{i_r})$

We will define a class of A-modules that have a particular behavior over $k[u_\lambda]$, in terms of their rank variety. In section 4.3, we will then show that a 'similar' class of A'-modules have 'similar' rank varieties over H_λ. To do this, we first need to determine the rank variety of these modules. Let us look at an example of an

A-module of this structure.

Example. Consider A_2^2 and let $M = A/(X_1)$. A k-basis for M is $\{1, x_2\}$. Then we get the following representation matrix of u_λ over M with respect to the k-basis $\{1, x_2\}$.

$$\Big[u_\lambda\Big]_M = \begin{array}{c} \begin{array}{cc} 1 & x_2 \end{array} \\ \begin{bmatrix} 0 & 0 \\ \lambda_2 & 0 \end{bmatrix} \end{array}.$$

Proposition 1.4.3 tells us that M will be free as an $k[u_\lambda]$-module if and only if $rank\left(\Big[u_\lambda\Big]_M\right) = 1$. This is true if and only if $\lambda_2 \neq 0$. This means that

$$V(M) = \{\lambda \in k^2 : \lambda_2 = 0\} = \mathbf{V}(\chi_2).$$

Notice that for this example, we have that M is free as a $k[u_\lambda]$-module if and only if $u_\lambda \in (x_1)$. Our next theorem generalizes the rank variety for A-modules of this structure.

Theorem 4.2.1. *Suppose r is an integer between 0 and n. Let $M = A/(x_{i_1}, \ldots, x_{i_r})$ where $1 \leq i_1 < i_2 < \cdots < i_r \leq n$. Then M is not free over $k[u_\lambda]$ if and only if $u_\lambda \in (x_{i_1}, \ldots, x_{i_r})$.*

Proof. If $r = 0$, then $M = A$. It is a well-known result that A is free over $k[u_\lambda]$ for any nonzero u_λ. Suppose $r \neq 0$. Without loss of generality, we can reindex the variables to assume that $(x_{i_1}, \ldots, x_{i_r}) = (x_1, \ldots, x_r)$. We will prove the backwards direction first. Assume that $u_\lambda \in (x_1, \ldots, x_r)$. Then $\lambda_{r+1} = \cdots = \lambda_n = 0$. Note that a k-basis for M is $B = \{x_{r+1}^{\ell_{r+1}} \cdots x_n^{\ell_n}\}$ where $0 \leq \ell_j \leq p-1$ for each ℓ_j, so $\dim_k(M) = p^{n-r}$. We can see that for any basis element $b \in B$, we have that $u_\lambda \cdot b = 0$ in M, so the representation matrix for u_λ over M is the $p^{n-r} \times p^{n-r}$ zero matrix. Thus, the rank of that matrix is 0, which, by Proposition 1.5.1, implies that M is not free over $k[u_\lambda]$.

For the forward direction, assume that M is not free over $k[u_\lambda]$ for some nonzero u_λ. Suppose $u_\lambda \notin (x_1, \ldots, x_r)$. Then there exists a $j \in \{r+1, \ldots, n\}$ such that $\lambda_j \neq 0$. Without loss of generality, we can assume $j = r+1$. It is important to note that $dim_k(M) = p^{n-r}$, where a k-basis for M is $\{x_{r+1}^{\ell_{r+1}} \cdots x_n^{\ell_n}\}$, where the powers are nonnegative integers that run from 0 to $p-1$. Consider the monomial $x_{r+1}^{\ell_{r+1}} \cdots x_n^{\ell_n}$ where the powers are any integer between 0 and $p-1$. Note that $u_\lambda \cdot x_{r+1}^{\ell_{r+1}} \cdots x_n^{\ell_n}$ will contain a term of the form $\lambda_j x_{r+1}^{\ell_{r+1}+1} \cdots x_n^{\ell_n}$ if $\ell_{r+1} < p-1$. If $\ell_{r+1} = p-1$, that particular term will be 0. We will reorder the aformentioned k-basis of M in the following manner:

1. Order the elements $\{x_{r+2}^{\ell_{r+2}} \cdots x_n^{\ell_n}\}$ in lexicographic order.
2. Label them as $m_1, m_2, \ldots, m_{p^{n-r-1}}$. This means that $m_1 = 1$, $m_2 = x_{r+2}$ and so on.
3. Order the k-basis $\{x_{r+1}^{\ell_{r+1}} x_{r+2}^{\ell_{r+2}} \cdots x_n^{\ell_n}\}$ in the following way:

$$\{m_1, x_{r+1}m_1, x_{r+1}^2 m_1, \ldots, x_{r+1}^{p-1} m_1, m_2, x_{r+1}m_2, \ldots, x_{r+1}^{p-1} m_2, \ldots\}$$

By reordering the basis of M this way, we have that $\Big[u_\lambda\Big]_M$ is a lower triangular matrix with $0's$ on the diagonal and the subdiagonal of $\Big[u_\lambda\Big]_M$ contains $(p-1)p^{n-r-1}$ entries with λ_j and p^{n-r-1} entries with zeros. This is due to the fact that every basis element that shows up in the term $u_\lambda \cdot x_{r+1}^{\ell_{r+1}} \cdots x_n^{\ell_n}$ is ordered after the basis element $x_{r+1}^{\ell_{r+1}+1} \cdots x_n^{\ell_n}$. Thus, we can see that $\Big[u_\lambda\Big]_M$ contains a nonzero minor of size $(p-1)p^{n-r-1}$, which implies that

$$rank\left(\Big[u_\lambda\Big]_M\right) \geq (p-1)p^{n-r-1}.$$

However, $(p-1)p^{n-r-1}$ is the maximal rank for $\left[u_\lambda\right]_M$, so we have that

$$rank\left(\left[u_\lambda\right]_M\right) = (p-1)p^{n-r-1}.$$

Therefore, by Proposition 1.5.1, M is free over $k[u_\lambda]$, which is a contradiction.

□

Consider the A-module $M = A/(x_{i_1}, \ldots, x_{i_r})$ where $1 \leq i_1 < \cdots < i_r \leq n$. Theorem 4.2.1 tells us that

$$V(M) = \{\lambda \in k^n : u_\lambda \in (x_{i_1}, \ldots, x_{i_r})\}.$$

As stated back in section 1.4, $V(M)$ is also an algebraic variety, so we have that

$$V(M) = \mathbf{V}(x_{j_1}, \ldots, x_{j_{n-r}})$$

where $\{j_1, \ldots, j_{n-r}\} = \{1, \ldots, n\} - \{i_1, \ldots, i_r\}$ and each χ_j is a polynomial in $k[\chi_1, \ldots, \chi_n]$. Looking at the forward direction of the proof of Theorem 4.2.1, we get an immediate corollary regarding the decomposition of M over $k[u_\lambda]$.

Corollary 4.2.2. *Let $M = A/(x_{i_1}, \ldots, x_{i_r})$ where $1 \leq i_j \leq n$ for each $j = 1, \ldots, r$. Then M is not free as a $k[u_\lambda]$-module if and only if $M \cong k^{p^{n-r}}$ as a $k[u_\lambda]$-module.*

Proof. This is immediate from the backwards direction of the proof of Theorem 4.2.1. □

Now we will investigate the tensor products of A-modules of this form. Consider the A-modules $M = A/(x_1)$ and $N = A/(x_2, \ldots, x_n)$. Note that

$$V(M) = \mathbf{V}(\chi_2, \ldots, \chi_n)$$

and

$$V(N) = \mathbf{V}(\chi_1),$$

so by Theorem 4.2.1 and Carlon's identity [7], we have that

$$V(M \otimes_k N) = V(M) \cap V(N) = \mathbf{V}(\chi_1, \chi_2, \ldots, \chi_n) = \{0\}.$$

This tells us that $M \otimes_k N$ is free over $k[u_\lambda]$ for any nonzero u_λ. It is also a well known result that A is free over $k[u_\lambda]$ for any nonzero u_λ [9]. Now we will prove this using the following theorem by creating an isomorphism between A and $M \otimes_k N$.

Theorem 4.2.3. *Consider the following A-modules $M = A/(x_1)$ and $N = A/(x_2, \ldots, x_n)$. Then $A \cong M \otimes_k N$ as A-modules.*

Proof. Let $\phi : A \to M \otimes_k N$ be defined by $x_1^{\ell_1} x_2^{\ell_2} \cdots x_n^{\ell_n} \mapsto x_2^{\ell_2} \cdots x_n^{\ell_n} \otimes x_1^{\ell_1}$, then extend by linearity. Clearly, we have that $1 \mapsto 1 \otimes 1$. Consider the element $x_1^{\ell_1} x_2^{\ell_2} \cdots x_n^{\ell_n} \in A$. Note that

$$\begin{aligned}
(x_1^{\ell_1} x_2^{\ell_2} \cdots x_n^{\ell_n}) \cdot \phi(1) &= \Delta(x_1^{\ell_1} x_2^{\ell_2} \cdots x_n^{\ell_n})(1 \otimes 1) \\
&= (1 \otimes x_1 + x_1 \otimes 1 + x_1 \otimes x_1)^{\ell_1} \cdots (1 \otimes x_n + x_n \otimes 1 + x_n \otimes x_n)^{\ell_n} (1 \otimes 1) \\
&= (1 \otimes x_1)^{\ell_1} \prod_{i=2}^{n} (x_i \otimes 1)^{\ell_2} \\
&= (1 \otimes x_1^{\ell_1}) \prod_{i=2}^{n} (x_i^{\ell_2} \otimes 1) \\
&= x_2^{\ell_2} \cdots x_n^{\ell_n} \otimes x_1^{\ell_1} \\
&= \phi(x_1^{\ell_1} x_2^{\ell_2} \cdots x_n^{\ell_n})
\end{aligned}$$

since $x_1 = 0$ in M and $x_i = 0$ in N for all $i = 2, \ldots, n$. This shows that ϕ is a well-defined A-module homomorphism. Note that $\{x_2^{\ell_2} \cdots x_n^{\ell_n} \otimes x_1^{\ell_1}\}$, where every power runs from 0 to $p-1$, is a k-basis for $M \otimes_k N$. Since each of these basis elements has a preimage in A, then ϕ is surjective. Thus, by the First Isomorphism Theorem, we have that $A/ker(\phi) \cong M \otimes_k N$. Note that

$$\dim_k(A) = p^n = p^{n-1} \cdot p = \dim_k(M \otimes_k N),$$

so $ker(\phi)$ cannot contain any of the variables or their powers. It is clear that $1 \notin \ker(\phi)$, so it immediately follows that $ker(\phi) = \{0\}$. Thus, we have that ϕ is injective. Therefore, $A \cong M \otimes_k N$ as A-modules.

$\square$

Due to Theorem 4.2.3, we have that $V(A) = V(M \otimes_k N) = \{0\}$, which shows that A is free over $k[u_\lambda]$ for any nonzero u_λ. Notice for Theorem 4.2.3, we investigated the tensor product of two specific A-modules of the form $A/(x_{i_1}, \ldots, x_{i_r})$. What happens if we investigate the tensor product in a more general setting? This next theorem will show that taking the tensor product of two A-modules of this form will result in another A-module of this form as long as certain conditions are met.

Theorem 4.2.4. *Let* $\alpha = \{\alpha_1, \ldots, \alpha_s\} \subseteq \{1, 2, \ldots, n\}$ *and* $\beta = \{\beta_1, \ldots, \beta_t\} \subseteq \{1, 2, \ldots, n\}$. *Suppose* $I_\alpha = (x_{\alpha_1}, \ldots, x_{\alpha_s})$ *and* $I_\beta = (x_{\beta_1}, \ldots, x_{\beta_t})$. *If* $\alpha \cup \beta = \{1, 2, \ldots, n\}$ *and* $\alpha \cap \beta = \gamma \neq \emptyset$, *then*

$$A/I_\alpha \otimes_k A/I_\beta \cong A/I_\gamma$$

as A*-modules.*

Proof. Without loss of generality, we can reindex the variables to assume that $\alpha = \{1, \ldots, r\}$, $\beta = \{s, \ldots, n\}$ and $1 \leq s \leq r \leq n$. It is important to note that $\dim_k(A/I_\alpha) = p^{n-r}$ and $\dim_k(A/I_\beta) = p^{s-1}$. Note that $A_\alpha \otimes_k A_\beta$ is cyclic with generator $1 \otimes 1$, so we can define a map $\phi : A \to A_\alpha \otimes_k A_\beta$ defined by $1 \mapsto 1 \otimes 1$. For any $a \in A$, this means that $\phi(a) = \Delta(a) \cdot (1 \otimes 1)$. By definition, it is clear that ϕ is an A-module homomorphism. Now we will show that ϕ is surjective. Note that a k-basis for $A_\alpha \otimes_k A_\beta$ is $\{x_{r+1}^{\ell_{r+1}} \cdots x_n^{\ell_n} \otimes x_1^{\ell_1} \cdots x_{s-1}^{\ell_{s-1}}\}$ where all of the powers run from 0 to $p - 1$. Note that

$$\phi(x_1^{\ell_1} \cdot x_n^{\ell_n}) = \Delta(x_1^{\ell_1} \cdot x_n^{\ell_n}) \cdot (1 \otimes 1)$$

$$
\begin{aligned}
&= (1 \otimes x_1 + x_1 \otimes 1)^{\ell_1} \cdots (1 \otimes x_n + x_n \otimes 1)^{\ell_n} \cdot (1 \otimes 1) \\
&= (1 \otimes x_1)^{\ell_1} \cdots (1 \otimes x_{s-1})^{\ell_{s-1}} (x_{r+1} \otimes 1)^{\ell_{r+1}} \cdots (x_n \otimes 1)^{\ell_n} \\
&= (1 \otimes x_1^{\ell_1}) \cdots (1 \otimes x_{s-1}^{\ell_{s-1}})(x_{r+1}^{\ell_{r+1}} \otimes 1) \cdots (x_n^{\ell_n} \otimes 1) \\
&= x_{r+1}^{\ell_{r+1}} \cdots x_n^{\ell_n} \otimes x_1^{\ell_1} \cdots x_{s-1}^{\ell_{s-1}},
\end{aligned}
$$

so we have that ϕ is surjective. By the First Isomorphism Theorem, we have that $A/ker(\phi) \cong A_\alpha \otimes_k A_\beta$. Let $I_\gamma = (x_s, \ldots, x_r)$. We want to show that $ker(\phi) = I_\gamma$. Note that

$$
\begin{aligned}
\phi(x_s^{\ell_s}, \ldots, x_r^{\ell_r}) &= \Delta(x_s^{\ell_s}, \ldots, x_r^{\ell_r}) \cdot (1 \otimes 1) \\
&= (1 \otimes x_s + x_s \otimes 1)^{\ell_s} \cdots (1 \otimes x_r + x_r \otimes 1)^{\ell_r} \cdot (1 \otimes 1) \\
&= (1 \otimes x_s + x_s \otimes 1)^{\ell_s} \cdots (1 \otimes x_r + x_r \otimes 1)^{\ell_r} \\
&= 0
\end{aligned}
$$

since $1 \otimes x_j + x_j \otimes 1 = 0$ for any $j \in \{s, \ldots, r\}$. Thus, we have that $\phi(a) = 0$ for any $a \in I_\gamma$, so it follows that $ker(\phi) \supseteq I_\gamma$. Since $ker(\phi) \supseteq I_\gamma$, then

$$
\dim_k(A/ker(\phi)) \leq p^n/p^{r-s+1} = p^{n-r+s-1} = p^{n-r} \cdot p^{s-1} = \dim_k(A/I_\alpha \otimes_k A/I_\beta).
$$

Thus, $\ker(\phi)$ cannot be any larger, so it follows that $\ker(\phi) = I_\alpha \cap I_\beta$. Therefore, we have that $A/I_\alpha \otimes_k A/I_\beta \cong A/I_\gamma$ as A-modules.

□

4.3 Class of A'-Modules of the form $A'/(x_{i_1}, \ldots, x_{i_r})$

Now, we will look at the behavior of A'-modules of the form $A'/(x_{i_1}, \ldots, x_{i_r})$. In particular, we will show that these modules have similar behavior, in terms of their rank varieties, to A-modules of the form $A/(x_{i_1}, \ldots, x_{i_r})$. Let's look at the following example:

Example. Suppose $n = p = 2$. Consider the A'-module $M' = A'/(x_1)$ and the hypersurface $H_\lambda = k[x_1, x_2]/(\tilde{u_\lambda}^{(2)})$ where $\tilde{u_\lambda} = \lambda_1 x_1 + \lambda_2 x_2$. Suppose $\lambda = (1, 0)$. Then M' has infinite projective dimension over H_λ with the following resolution:

$$\cdots H_\lambda \xrightarrow{[x_1]} H_\lambda \xrightarrow{[x_1]} H_\lambda \to M' \to 0.$$

Suppose $\lambda = (0, 1)$. Then M' has finite projective dimension over H_λ with the following resolution:

$$0 \to H_\lambda \xrightarrow{[x_1]} H_\lambda \to M' \to 0.$$

In particular, M' has infinite projective dimension over H_λ if and only $\lambda_2 = 0$. This means that

$$W(M') = \{\lambda \in k^2 : \lambda_2 = 0\} = \mathbf{V}(\chi_2).$$

Note that M' has the same algebraic variety associated with its rank variety that M, from the example in section 4.2, has with its rank variety. We will generalize the similarity between these modules and their rank varieties with the following theorem:

Theorem 4.3.1. *Let $M' = A'/(x_{i_1}, \ldots, x_{i_r})$. Then M' has infinite projective dimension over H_λ if and only if $\tilde{u_\lambda} \in (X_{i_1}, \ldots, X_{i_r})$.*

Proof. For simplicity, we will assume $M' = A'/(x_1, \ldots, x_r)$. Suppose $\tilde{u_\lambda} \in (X_1, \ldots, X_r)$. Let $H'_\lambda = k[X_1, \ldots, X_r]/(\tilde{u_\lambda}^{(p)})$. Note that H'_λ is not regular, so by the Serre-Auslander-Buchsbaum Theorem [Theorem 8.62 [14]] and the fact that the global dimension of H'_λ is equal to the projective dimension of k over H'_λ [Theorem 8.55 [14]], we have that k has infinite projective dimension over H'_λ. Take a minimal infinite resolution

$$\cdots \to (H'_\lambda)^{b_3} \to (H'_\lambda)^{b_2} \to (H'_\lambda)^{b_1} \to H'_\lambda \to k \to 0$$

and tensor with $\_ \otimes_k k[X_{r+1}, \ldots, X_n]$. We get the following exact sequence:

$$\cdots \to (H'_\lambda)^{b_1} \otimes_k k[X_{r+1}, \ldots, X_n] \to H'_\lambda \otimes_k k[X_{r+1}, \ldots, X_n] \to k \otimes_k k[X_{r+1}, \ldots, X_n] \to 0.$$

This is a minimal infinite resolution of $k[X_{r+1}, \ldots, X_n]$ over $H'_\lambda \otimes_k k[X_{r+1}, \ldots, X_n] \cong H_\lambda$. Note that $X_{r+1}^p, \ldots, X_n^p$ is $k[X_{r+1}, \ldots, X_n]$-regular, so

$$pd_{H_\lambda} \left(k[X_{r+1}, \ldots, X_n]/(X_{r+1}^p, \ldots, X_n^p) \right) = \infty$$

by 1.3.6 in [6]. Since

$$M' \cong k[X_{r+1}, \ldots, X_n]/(X_{r+1}^p, \ldots, X_n^p),$$

it follows that M' has infinite projective dimension over H_λ.

For the other direction, we will prove the contrapositive. Suppose $\tilde{u_\lambda} \notin (X_1, \ldots, X_r)$. Then, $\lambda_j \neq 0$ for some $r+1 \leq j \leq n$. Thus,

$$\tilde{u_\lambda}^{(p)}, X_{r+1}^p, \ldots, X_{j-1}^p, X_{j+1}^p, \ldots, X_n^p, X_1, \ldots, X_r$$

forms a regular sequence in $k[X_1, \ldots, X_n]$, so

$$X_{r+1}^p, \ldots, X_{j-1}^p, X_{j+1}^p, \ldots, X_n^p, X_1, \ldots, X_r$$

forms a regular sequence in H_λ. Hence, by Corollary 1.6.14 in [6], the Kozul complex forms a resolution for M' over H_λ. Therefore, M' has finite projective dimension over H_λ. $\square$

Consider the A'-module $M' = A'/(x_{i_1}, \ldots, x_{i_r})$ where $1 \leq i_1 < \cdots < i_r \leq n$. Theorem 4.3.1 tells us that

$$W(M') = \{\lambda \in k^n : \tilde{u_\lambda} \in (X_{i_1}, \ldots, X_{i_r})\}.$$

As stated in section 4.1, $W(M')$ is also an algebraic variety, so we have that

$$W(M') = \mathbf{V}(\chi_j \mid j \neq i_1, \ldots, i_r)$$

where χ_j are polynomials in $k[\chi_1, \ldots, \chi_n]$. Comparing to our results in section 4.2, we have that

$$V(M) = W(M')$$

as algebraic varieties where $M = A/(x_{i_1}, \ldots, x_{i_r})$ is an A-module and $M' = A'/(x_{i_1}, \ldots, x_{i_r})$ is an A'-module. In the final section of this thesis, we will show that Carlson's identity holds for this particular class of A'-modules by constructing a special tensor product for them.

4.4 Faux Tensor Product of A'-Modules of the form $A'/(x_{i_1}, \ldots, x_{i_r})$

In section 2.2, we mentioned how given two A-modules M, N, we can construct a new module, $M \otimes_k N$ and investigate its structure. In particular, we would look at its representation matrix for u_λ. As mentioned earlier in this section, when $char(k) \neq p$, we get a new truncated polynomial ring A'. In particular, we are not allowed to tensor two modules over A' and restrict them to $k[u_\lambda]$. Instead, we consider two A'-modules M', N', that are similar to M and N, respectively. Then, we create a faux tensor product, which we will denote as

$$M' \boxtimes_k N'$$

, to investigate the structure of a newly constructed A'-module from individual A'-modules M', N'. We create this faux tensor product by focusing on the representation matrices of x_i over M' and x_i over N'. We treat the representation matrices as

A'-modules themselves, and we create the faux tensor product by taking the following Kronecker product:

$$\begin{bmatrix} x_i \end{bmatrix}_{M' \boxtimes_k N'} = I_{d_1} \bigotimes \begin{bmatrix} x_i \end{bmatrix}_{N'} + \begin{bmatrix} x_i \end{bmatrix}_{M'} \bigotimes I_{d_2}$$

where d_1 is the k-dimension of M' and d_2 is the k-dimension of N'. From here, we can take this new representation matrix of x_i for $M' \boxtimes N'$ and investigate if $M' \boxtimes_k N'$ has the same rank variety as $M \otimes_k N$. In particular, we aim to see if Carlson's formula,

$$W(M' \boxtimes_k N') = W(M') \cap W(N'),$$

holds in this case. In order for this faux tensor product to be acceptable as an A'-module, we need certain properties to hold. In particular, we need $x_i^p(M' \boxtimes_k N') = 0$. Suppose $p > 2$. If we are viewing $M' \boxtimes_k N'$ through its representation matrix, then we have that

$$\begin{bmatrix} x_i^2 \end{bmatrix}_{M' \boxtimes_k N'} = \left(I_{d_1} \bigotimes \begin{bmatrix} x_i \end{bmatrix}_{N'} + \begin{bmatrix} x_i \end{bmatrix}_{M'} \bigotimes I_{d_2} \right)^2$$

$$= I_{d_1} \bigotimes \begin{bmatrix} x_i^2 \end{bmatrix}_{N'} + 2 \begin{bmatrix} x_i \end{bmatrix}_{M'} \bigotimes \begin{bmatrix} x_i \end{bmatrix}_{N'} + \begin{bmatrix} x_i^2 \end{bmatrix}_{M'} \bigotimes I_{d_2},$$

but we need

$$\begin{bmatrix} x_i^2 \end{bmatrix}_{M' \boxtimes_k N'} = I_{d_1} \bigotimes \begin{bmatrix} x_i^2 \end{bmatrix}_{N'} + \begin{bmatrix} x_i^2 \end{bmatrix}_{M'} \bigotimes I_{d_2}.$$

This only holds if $\begin{bmatrix} x_i \end{bmatrix}_{M'} = 0$ or $\begin{bmatrix} x_i \end{bmatrix}_{N'} = 0$. This means that x_i would have to kill either M or N. Our next theorem generalizes this statement.

Theorem 4.4.1. *Suppose M' and N' are A'-modules such that x_i is either in the annihilator of M or annihilator of N for each $i = 1, \ldots, n$. Then $M' \boxtimes_k N'$ is an A'-module.*

Proof. In order to show that $M' \boxtimes_k N'$ is an A'-module, we need to verify that two conditions for $M' \boxtimes_k N'$ hold:

1. $\Big[x_i^p\Big]_{M'\boxtimes_k N'} = 0$ for each $i = 1, \dots, n$
2. $\Big[x_i\Big]_{M'\boxtimes_k N'} \cdot \Big[x_j\Big]_{M'\boxtimes_k N'} = \Big[x_j\Big]_{M'\boxtimes_k N'} \cdot \Big[x_i\Big]_{M'\boxtimes_k N'}$ for each $i, j = 1, \dots, n$.

Suppose $\dim_k(M') = d_1$ and $\dim_k(N') = d_2$. For the first condition, fix i. By definition, we have that

$$\begin{aligned}
\Big[x_i^p\Big]_{M'\boxtimes_k N'} &= \Big[x_i\Big]^p_{M'\otimes'_k N'} \\
&= \left(I_{d_1} \bigotimes \Big[x_i\Big]_{N'} + \Big[x_i\Big]_{M'} \bigotimes I_{d_2}\right)^p \\
&= \sum_{j=0}^{p} \binom{p}{j} \left(I_{d_1} \bigotimes \Big[x_i\Big]_{N'}\right)^j \left(\Big[x_i\Big]_{M'} \bigotimes I_{d_2}\right)^{p-j} \\
&= \sum_{j=0}^{p} \binom{p}{j} \left(I_{d_1} \bigotimes \Big[x_i\Big]^j_{N'}\right) \left(\Big[x_i\Big]^{p-j}_{M'} \bigotimes I_{d_2}\right) \\
&= \sum_{j=0}^{p} \binom{p}{j} \left(\Big[x_i\Big]^{p-j}_{M'} \bigotimes \Big[x_i\Big]^j_{N'}\right).
\end{aligned}$$

Notice that we have $\Big[x_i\Big]^{p-j}_{M'} = 0$ when $j = 0$ and $\Big[x_i\Big]^{j}_{N'} = 0$ when $j = p$ since M', N' are A'-modules. Now we want to consider what happens when $j = 1, \dots, p-1$. By our assumption, we have that each variable is contained in at least one of the annihilators of either M' or N', so we immediately have that

$$\Big[x_i\Big]^{p-j}_{M'} \bigotimes \Big[x_i\Big]^{j}_{N'} = 0$$

whenever $j = 1, \dots, p-1$. Thus, we have that

$$\Big[x_i^p\Big]_{M'\boxtimes_k N'} = 0$$

for each $i = 1, \ldots, n$. For the second condition, fix i and j. Note that

$$\Big[x_i\Big]_{M' \boxtimes_k N'} \cdot \Big[x_j\Big]_{M' \boxtimes_k N'} = \left(I_{d_1} \bigotimes \Big[x_i\Big]_{N'} + \Big[x_i\Big]_{M'} \bigotimes I_{d_2}\right) \cdot \left(I_{d_1} \bigotimes \Big[x_j\Big]_{N'} + \Big[x_j\Big]_{M'} \bigotimes I_{d_2}\right)$$

which equates to

$$I_{d_1} \bigotimes \left(\Big[x_i\Big]_{N'} \cdot \Big[x_j\Big]_{N'}\right) + \Big[x_j\Big]_{M'} \bigotimes \Big[x_i\Big]_{N'} + \Big[x_i\Big]_{M'} \bigotimes \Big[x_j\Big]_{N'} + \left(\Big[x_i\Big]_{M'} \cdot \Big[x_j\Big]_{M'}\right) \bigotimes I_{d_2}.$$

Due to the commutativity of A'-modules M' and N', this then equates to

$$I_{d_1} \bigotimes \left(\Big[x_j\Big]_{N'} \cdot \Big[x_i\Big]_{N'}\right) + \Big[x_j\Big]_{M'} \bigotimes \Big[x_i\Big]_{N'} + \Big[x_i\Big]_{M'} \bigotimes \Big[x_j\Big]_{N'} + \left(\Big[x_j\Big]_{M'} \cdot \Big[x_i\Big]_{M'}\right) \bigotimes I_{d_2}$$

which is equivalent to

$$\left(I_{d_1} \bigotimes \Big[x_j\Big]_{N'} + \Big[x_j\Big]_{M'} \bigotimes I_{d_2}\right) \cdot \left(I_{d_1} \bigotimes \Big[x_i\Big]_{N'} + \Big[x_i\Big]_{M'} \bigotimes I_{d_2}\right),$$

but this equates to

$$\Big[x_j\Big]_{M' \boxtimes_k N'} \cdot \Big[x_i\Big]_{M' \boxtimes_k N'}.$$

Both conditions have been satisfied, so we have that $M' \boxtimes_k N'$ is an A'-module. $\square$

From here, we get an immediate corollary that the tensor product of the special A'-modules we investigated in Theorem 4.3.1 are indeed A'-modules, given that each x_i is modded out in one of the modules.

Corollary 4.4.2. *Let* $\alpha = \{\alpha_1, \ldots, \alpha_s\} \subseteq \{1, 2, \ldots, n\}$ *and* $\beta = \{\beta_1, \ldots, \beta_t\} \subseteq \{1, 2, \ldots, n\}$*. Suppose* $I_\alpha = (x_{\alpha_1}, \ldots, x_{\alpha_s})$ *and* $I_\beta = (x_{\beta_1}, \ldots, x_{\beta_t})$*. Consider the* *following A'-modules $M' = A'/I_\alpha$ and $N' = A'/I_\beta$. If $\alpha \cup \beta = \{1, 2, \ldots, n\}$, then $M' \boxtimes_k N'$ is an A'-module.*

Proof. This is immediate from the use of Theorem 4.4.1 and the fact that $\alpha \cup \beta = \{1, \ldots, n\}$. $\square$

Now that we know that the tensor product of these types of A'-modules is indeed an A'-module, we want to be able to prove that Carlson's identity, $(*)$, holds for these particular A'-modules. To do this, we need to dive more into the structure of the faux tensor product of these A'-modules. For that, we will compare it with the ordinary tensor product over k of two A'-modules with similar properties and a certain multiplication. In particular, we will show that this ordinary tensor product is an A'-module.

Theorem 4.4.3. *Consider A'-modules M', N' with $\dim_k(M') = d_1$ and $\dim_k(N') = d_2$. For each $i = 1, \ldots, n$, suppose x_i is either in the annihilator of M' or N'. Note that for any monomial $m \in A$ of degree 1 or higher, we can write $m = rs$ where r, s are both monomials in A and we have that $rM' = 0$ and $sN' = 0$. Consider the ordinary tensor product $M' \otimes_k N'$ with the following multiplication property:*

$$m(a \otimes b) = sa \otimes rb$$

and

$$m_0(a \otimes b) = m_0 a \otimes b = a \otimes m_0 b$$

for any $a \otimes b \in M' \otimes_k N'$ and any constant polynomial $m_0 \in A$. For any arbitrary $a \in A'$, extend the multiplication by linearity. Then $M' \otimes_k N'$ is an A'-module.

Proof. Without loss of generality, we can assume $x_i M' = 0$ for all $i = 1, \ldots, r$ and $x_i N' = 0$ for all $i = s, \ldots, n$ where $1 \leq s \leq r \leq n$. Then by definition, we have that

$$(x_1^{\ell_1} \cdots x_n^{\ell_n})(a \otimes b) = x_{r+1}^{\ell_{r+1}} \cdots x_n^{\ell_n} a \otimes x_1^{\ell_1} \cdots x_{s-1}^{\ell_{s-1}} b$$

for any $a \otimes b \in M' \otimes_k N'$. Since we are using the ordinary tensor product, we know that $M' \otimes_k N'$ is an additive abelian group. It is also clear that $1_{A'}(a \otimes b) = a \otimes b$ for any $a \otimes b \in M' \otimes_k N'$. Suppose $a \otimes b, c \otimes d \in M' \otimes_k N'$ and $\ell_i, \ell_i' \in \{1, \ldots, p-1\}$ for each $i = 1, \ldots, n$. Since the multiplication property is extended by linearity for polynomials in A', it is enough to show the following conditions hold for multiplying by monomials in A:

1. $(x_1^{\ell_1} \cdots x_n^{\ell_n})[(a \otimes b) + (c \otimes d)] = (x_1^{\ell_1} \cdots x_n^{\ell_n})(a \otimes b) + (x_1^{\ell_1} \cdots x_n^{\ell_n})(c \otimes d)$
2. $(x_1^{\ell_1} \cdots x_n^{\ell_n} + x_1^{\ell_1'} \cdots x_n^{\ell_n'})(a \otimes b) = (x_1^{\ell_1} \cdots x_n^{\ell_n})(a \otimes b) + (x_1^{\ell_1'} \cdots x_n^{\ell_n'})(a \otimes b)$
3. $(x_1^{\ell_1} \cdots x_n^{\ell_n})((x_1^{\ell_1'} \cdots x_n^{\ell_n'})(a \otimes b)) = ((x_1^{\ell_1} \cdots x_n^{\ell_n}) \cdot (x_1^{\ell_1'} \cdots x_n^{\ell_n'}))(a \otimes b)$.

For condition 1, it is clear

$$\begin{aligned}(x_1^{\ell_1} \cdots x_n^{\ell_n})[(a \otimes b) + (c \otimes d)] &= x_{r+1}^{\ell_{r+1}} \cdots x_n a \otimes x_1^{\ell_1} \cdots x_{s-1}^{\ell_{s-1}} b + x_{r+1}^{\ell_{r+1}} \cdots x_n c \otimes x_1^{\ell_1} \cdots x_{s-1}^{\ell_{s-1}} d \\ &= (x_1^{\ell_1} \cdots x_n^{\ell_n})(a \otimes b) + (x_1^{\ell_1} \cdots x_n^{\ell_n})(c \otimes d)\end{aligned}$$

since the multiplication was extended linearly, so condition 1 holds. Note that condition 2 holds by construction of the multiplication. For condition 3, we have that

$$\begin{aligned}(x_1^{\ell_1} \cdots x_n^{\ell_n})((x_1^{\ell_1'} \cdots x_n^{\ell_n'})(a \otimes b)) &= (x_1^{\ell_1} \cdots x_n^{\ell_n})(x_{r+1}^{\ell_{r+1}'} \cdots x_n^{\ell_n'} a \otimes x_1^{\ell_1'} \cdots x_{s-1}^{\ell_{s-1}'} b) \\ &= x_{r+1}^{\ell_{r+1}} \cdots x_n^{\ell_n} x_{r+1}^{\ell_{r+1}'} \cdots x_n^{\ell_n'} a \otimes x_1^{\ell_1} \cdots x_{s-1}^{\ell_{s-1}} x_1^{\ell_1'} \cdots x_{s-1}^{\ell_{s-1}'} b \\ &= x_{r+1}^{\ell_{r+1}+\ell_{r+1}'} \cdots x_n^{\ell_n+\ell_n'} a \otimes x_1^{\ell_1+\ell_1'} \cdots x_{s-1}^{\ell_{s-1}+\ell_{s-1}'} b \\ &= (x_1^{\ell_1+\ell_1'} \cdots x_n^{\ell_n+\ell_n'})(a \otimes b) \\ &= ((x_1^{\ell_1} \cdots x_n^{\ell_n}) \cdot (x_1^{\ell_1'} \cdots x_n^{\ell_n'}))(a \otimes b).\end{aligned}$$

This proves condition 3. Therefore, we have that $M' \otimes_k N'$ is an A'-module.

$\square$

Recall that the class of A'-modules that we are investigating in this chapter meet the criteria that x_i is in the annihilator of one of the A'-modules when utilizing the faux tensor product, so we can view it as the ordinary tensor product $M' \otimes_k N'$ with the certain aforementioned multiplication. We can use this to show that our faux tensor product of this particular class of A'-modules is isomorphic to one of these A'-modules.

Theorem 4.4.4. *Let $\alpha = \{\alpha_1, \ldots, \alpha_s\}, \beta = \{\beta_1, \ldots, \beta_t\}$, and $\gamma = \{\gamma_1, \ldots, \gamma_r\}$ all be subsets of $\{1, 2, \ldots, n\}$. Suppose $I_\alpha = (x_{\alpha_1}, \ldots, x_{\alpha_s})$, $I_\beta = (x_{\beta_1}, \ldots, x_{\beta_t})$ and $I_\gamma = (x_{\gamma_1}, \ldots, x_{\gamma_r})$. Consider the following A'-modules $M' = A'/I_\alpha$ and $N' = A'/I_\beta$. If $\alpha \cup \beta = \{1, 2, \ldots, n\}$ and $\alpha \cap \beta = \gamma$, then we have that $M' \boxtimes_k N' \cong A'/I_\gamma$ as A'-modules.*

Proof. Without loss of generality, suppose $\alpha = \{1, \ldots, s\}$, $\beta = \{t, \ldots, n\}$ and $t \leq s$. Then $I_\gamma = (x_t, \ldots, x_s)$. It is worth noting that

$$\dim_k(M' \otimes_k' N') = p^{n-s} \cdot p^{t-1} = p^{n-s+t-1} = p^{n-(s-t+1)} = \dim_k(A'/I_\gamma).$$

By Corollary 4.4.3, we know $M' \boxtimes_k N'$ is an A'-module. Note that the representation matrix of x_i over $M' \boxtimes_k N'$ is

$$\Big[x_i\Big]_{M' \boxtimes_k N'} = I_{p^{n-s}} \bigotimes \Big[x_i\Big]_{N'} + \Big[x_i\Big]_{M'} \bigotimes I_{p^{t-1}}$$

for each $i = 1, \ldots, n$ by the construction of $M' \boxtimes_k N'$. Consider the ordinary tensor product of M' and N' over k, denoted as $M' \otimes_k N'$, with the constructed multiplication property from Theorem 4.4.3. Note that $M' \otimes_k N'$ has the following k-basis: $\{x_{s+1}^{\ell_{s+1}} \cdots x_n^{\ell_n} \otimes x_1^{\ell_1} \cdots x_{t-1}^{\ell_{t-1}}\}$. Note that

$$x_i(x_{s+1}^{\ell_{s+1}} \cdots x_n^{\ell_n} \otimes x_1^{\ell_1} \cdots x_{t-1}^{\ell_{t-1}}) = \begin{cases} x_{s+1}^{\ell_{s+1}} \cdots x_n^{\ell_n} \otimes x_1^{\ell_1} \cdots x_i^{\ell_i+1} \cdots x_{t-1}^{\ell_{t-1}}, & \text{if } 1 \leq i < t \\ x_{s+1}^{\ell_{s+1}} \cdots x_i^{\ell_i+1} \cdots x_n^{\ell_n} \otimes x_1^{\ell_1} \cdots x_{t-1}^{\ell_{t-1}}, & \text{if } s < i \leq n. \end{cases}$$

Assume the aformentioned basis is in lexicographic order. Then the representation matrix of x_i on $M' \otimes_k N'$ is

$$\Big[x_i\Big]_{M' \otimes_k N'} = I_{p^{n-s}} \bigotimes \Big[x_i\Big]_{N'} + \Big[x_i\Big]_{M'} \bigotimes I_{p^{t-1}}$$

where $\Big[x_i\Big]_{M'} = 0$ if $1 \leq i \leq s$ and $\Big[x_i\Big]_{N'} = 0$ if $t \leq i \leq n$. This is, by definition, what the representation matrix for x_i on $M' \boxtimes_k N'$ is, so it immediately follows that $M' \boxtimes_k N' \cong M' \otimes_k N'$ as A'-modules. Now we will show that $M' \otimes_k N' \cong A'/I_\gamma$. Note that a k-basis for A' is $\{x_1^{\ell_1} \cdots x_n^{\ell_n}\}$ where all of the powers run from 0 to $p-1$. Consider the map $\phi' : A' \to M' \otimes_k N'$ defined by

$$x_1^{\ell_1} \cdots x_n^{\ell_n} \longmapsto x_{s+1}^{\ell_{s+1}} \cdots x_n^{\ell_n} \otimes x_1^{\ell_1} \cdots x_{t-1}^{\ell_{t-1}}$$

and extended linearly. Clearly, we have that $1 \mapsto 1 \otimes 1$, which are the respective multiplicative identities of each A'-module. Since each k-basis element of $M' \otimes_k N'$ has a preimage in A', it immediately follows that ϕ' is surjective. The multiplication defined on $M' \otimes_k N'$ combined with ϕ' being extended linearly are enough to prove that ϕ' is an A'-module homomorphism. Thus, by the First Isomorphism Theorem, we have that $A'/ker(\phi') \cong M' \otimes_k N'$. We want to show that $ker(\phi') = I_\gamma$. It is clear that $I_\gamma \subseteq ker(\phi')$ since

$$\phi'(x_i) = 0$$

for any $s \leq i \leq t$. Since $A'/ker(\phi') \cong M' \otimes_k N'$ and $\dim_k(M' \otimes_k N') = p^{n-s+t-1}$, it follows that $\dim_k(A'/ker(\phi') = p^{n-s+t-1}$. Note that $\dim_k(A') = p^n$, so that means that $\dim_k(ker(\phi')) = p^{s-t+1}$ which is exactly what the vector space dimension of I_γ is. Thus, we have that $ker(\phi') = I_\gamma$. Therefore, $A'/I_\gamma \cong M' \otimes_k N'$ as A'-modules, which implies that

$$M' \boxtimes_k N' \cong M' \otimes_k N'$$

as A'-modules. $\square$

This result leads us to our final and major theorem for this chapter.

Theorem 4.4.5. *Let $\alpha = \{\alpha_1, \ldots, \alpha_s\} \subseteq \{1, 2, \ldots, n\}$ and $\beta = \{\beta_1, \ldots, \beta_t\} \subseteq \{1, 2, \ldots, n\}$. Suppose $I_\alpha = (x_{\alpha_1}, \ldots, x_{\alpha_s})$ and $I_\beta = (x_{\beta_1}, \ldots, x_{\beta_t})$. Consider the* *following A'-modules $M' = A'/I_\alpha$ and $N' = A'/I_\beta$. If $\alpha \cup \beta = \{1, 2, \ldots, n\}$ and $\alpha \cap \beta \neq \emptyset$, then*

$$W(M' \boxtimes_k N') = W(M') \cap W(N').$$

Proof. Without loss of generality, we can reindex the variables to assume that $\alpha = \{1, \ldots, s\}$, $\beta = \{t, \ldots, n\}$ and $1 \leq t \leq s \leq n$. By Theorem 4.3.1, we have that

$$W(M') = \mathbf{V}(\chi_{s+1}, \ldots, \chi_n)$$

and

$$W(N') = \mathbf{V}(\chi_1, \ldots, \chi_{t-1}),$$

so it immediately follows that

$$W(M') \cap W(N') = \mathbf{V}(\chi_1, \ldots, \chi_{t-1}, \chi_{s+1}, \ldots, \chi_n).$$

By Theorem 4.4.3, we know that $M' \boxtimes_k N' \cong A'/I_\gamma$ as A'-modules where $I_\gamma = (x_t, \ldots, x_s)$. This implies that

$$W(M' \boxtimes_k N') = W(A'/I_\gamma) = \mathbf{V}(\chi_1, \ldots, \chi_{t-1}, \chi_{s+1}, \ldots, \chi_n).$$

Therefore, we have that

$$W(M' \boxtimes_k N') = W(M') \cap W(N').$$

□

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BIOGRAPHICAL STATEMENT

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