CHIRALITY IN QUANTUM FIELD THEORY AND ITS ROLE IN THE STANDARD MODEL AND BEYOND

by

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To my parents

For their endless love, patience, and support.

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ABSTRACT

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Chirality as a symmetry of particle physics occupies a unique role in the standard model. It arises from general space-time principles yet remains central to the formulation of local gauge theories. This work explores the complexities arising from a physically comprehensive treatment of this topic. We present the origins of the property from first principles by deriving the chiral Dirac equation (CDE). We demonstrate how the resulting chiral degrees of freedom for spin-1/2 objects may be employed in constructing composite chiral objects through the Bargmann-Wigner formalism, leading to novel couplings. We then consider means by which the degrees of chiral freedom in the standard model may be spontaneously broken through left-chiral Majorana neutrino fields. The resulting modifications to standard model processes are explored before a final exploration of chirality in curved space-time.

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CHAPTER 1

INTRODUCTION

1.1 Brief Historical Remarks and Motivation

Many historians of physics date the inception of modern Quantum Field Theory (QFT) to 1927. That year, Dirac's paper on "The quantum theory of the emission and absorption of radiation" was published [Dirac (1927)]. That paper would signal the beginning of quantum electrodynamics (QED). Just one year later, being dissatisfied with previous attempts to construct a relativistic wave equation – already Klein (1926) and Gordon (1926) had separately published their eponymous equation – Dirac published "The quantum theory of the electron" [Dirac (1928)]. In this work, he presented the now famous equation which bears his name, forever associating him with that nascent subbranch of relativistic quantum mechanics. This branch soon sprouted fertile fruit in the form of quantum field theory. Years later, titans of twentieth-century physics – Wigner, Feynman, Schwinger, and others – would build upon the theory to become the most precisely experimentally verified physical theory yet devised by humans. Many complex challenges, frustrations, triumphs, and discoveries have decorated the landscape of this nearly hundred-year-old theory. Yet, through all these years, Dirac's original equation remains deep-set in the foundations of the theory.

Among the fundamental equations of QFT, the Dirac equation [Dirac (1928)] plays a unique role. It describes the free fermionic fields of which the universe appears principally constructed. But this utility came at the cost of a good deal of confusion. Because Dirac had been unable to develop a first-order equation using simple numbers, he had been compelled to introduce four component objects called spinors (more accurately *bi*-spinors). The validity of the Dirac equation was quickly established soon after its introduction, with confidence in its predictive power buoyed by the verified prediction of antiparticles by Anderson (1932). However, the question persisted: What was the physical reason for the necessity of spinors?

The work of Wigner (1939) was among the first to highlight the role of group theory in QFT. By classifying the irreducible representations of the Poincaré group and identifying these as single particle states of elementary fields [Kim and Noz (2012)], much of the cloud of confusion surrounding Dirac's theory began to disperse. In 1954 the CPT theorem, one of the central theorems of QFT, was proven [Lüders (1954)], which states that the physical law is invariant under the simultaneous reversals of space (parity), time and charge. This seemed even more evidence in favor of Dirac's equation since the symmetry of parity was baked into Dirac's spinors. Soon after, years of mounting hints culminated in the theoretical proposal by Lee and Yang (1956) and subsequent verification of Wu et al. (1957) parity violation in fundamental weak interactions, challenging decades of presupposed notions. After proving their combined action's invariance, the universe's discrete symmetries were now in doubt.

This dilemma brought to the fore a curious property of Dirac's spinors known as *chirality*. In the following decade, torrents of discoveries washed over the field of particle physics, both experimental and theoretical. The work of Glashow (1959), Salam (1964), and Weinberg (1967) finally clarified the issue of parity violation, setting up chirality as a central property within the standard model of particle physics (SM).

Today it is understood that the concept of chirality plays an important part in QFT (e.g., Ryder (1996); Peskin and Schroeder (2018)). Still, an open question remains: To what extent can chirality and the specification of a field's chiral basis be considered a physical property, as opposed to a convenient mathematical description? Therefore, this dissertation's main motivation is understanding the chiral degrees of freedom allotted to physical states. More specifically, we will uncover where the degrees of freedom from chirality exist within the Dirac equation and how these might be applied to higher spin fields. We will then consider chirality in a broader context and investigate the physical implications of chiral fixing within the Standard Model. Lastly, we will seek to extend our knowledge from flat to curved space-time.

1.2 Aims of This Dissertation

At its core, this dissertation is an attempt to answer the following questions:

1. To what extent do the degrees of freedom introduced by isomorphic ambiguities of the irreducible representations of the Poincaré group (e.g., chirality) affect the fundamental laws of physics?
2. If endowed with a postulated physical significance, what are the observable implications of extending or restricting these degrees?
3. In what ways can a comprehensive and consistent treatment of chirality advance our knowledge of the universe?

In seeking to answer these questions, we will derive the most general set of Poincaré invariant equations governing four-component bi-spinors and find that the specification of the chiral basis is essential in unambiguously defining a representation. We then extend these concepts to new equations for "chiral" vector fields and find that the field separates into two identifiable components, one vector, and one scalar, for misaligned chiral bases underlying the representation. Next, we will present a means of promoting the chiral bases to a novel level of definite physical reality, resulting in new interactions and implications for why and how neutrinos are massive. And finally, we will conclude by exploring the possibility of applying the concepts uncovered to quantum field theories in curved spaces, a necessary precursor to a quantum theory of gravity.

1.3 Organization

This document is organized as follows:

Chapter 2 begins with an overview of the foundational principles and mathematical tools required for the succeeding chapters.

Chapter 3 presents our published derivations of the most general Poincaré invariant equations governing four-component bi-spinor fields. We explore several avenues of deriving the chiral Dirac equation before commenting on its symmetries and significance.

In Chapter 4, we present our published derivation of the chiral Bargmann-Wigner equations for spin-1 massive fields and use the properties of the Clifford basis.

Chapter 5 investigates chirality and its relation to the standard model. In doing so, we obtain results presented here for the first time, which will follow in peer-reviewed form shortly. We consider how symmetry violation in the form of Majorana neutrinos might determine the local chiral basis of fermionic fields and lead to new interactions. These interactions are then studied, and observables discussed.

In Chapter 6, we outline the difficulties of quantum field theories in curved space-time and observe the unique position the concept of chirality holds in relation to resolving these challenges.

Chapter 7 is devoted to conclusions.

Additionally, an appendix summarizes conventions and notation for the reader.

CHAPTER 2

FOUNDATIONS

2.1 Symmetries of the Standard Model

The Standard Model (SM) of modern particle physics grew out of a concerted effort in the middle of the last century to understand the structure and dynamic evolution of matter and energy at the most fundamental level. Early in its development, the work of Wigner, Bargmann, and others made evident that the appropriate mathematical language for the task was that of group theory.

At first glance, the symmetries of the standard model appear to separate neatly into two distinct classes: the induced external symmetries of space and time, which follow locally from the observed isotropy and homogeneity of the universe, and the so-called internal symmetries which arise from the gauge invariance of the constituent quanta of matter. To this first class belongs the Poincaré group and the discrete symmetries of C, P, and T, while to the second belong the gauge theories responsible for the known forces of nature (gravity being a notable exception). Somewhere in between these classes fall the chiral symmetries.

2.2 The Poincaré Group

The Poincaré Group is most accurately defined as the inhomogeneous group of Lorentz transformations. Perhaps a more useful working definition is as the group of isometric transformations of Minkowski space-time contiguous with the identity. The general structure of the Poincaré group is $\mathcal{P} = SO(3,1) \otimes_s T(3+1)$, from which we are led immediately to the non-invariant Lorentz subgroup $SO(3,1)$ of rotations and boosts, though the second makes apparent the role of the invariant subgroup $T(3+1)$ of space-time translations; this structure includes reversal of parity and time. Furthermore, implicit in $T(3+1)$ is the Lie algebraic structure of the group.

To begin, we consider the effects of the Poincaré group on a four-vector field $v_\mu(x)$ in Minkowski space-time. If the space-time is homogeneous, we may derive the four-translation operators by observing that physics is invariant under the infinitesimal transformation

$$v_\mu(x) \to v_\mu(x+\delta x) = \left(1 + \delta x^\nu \frac{\partial}{\partial x^\nu}\right) v_\mu(x),$$

which for finite transformations is only approximately true. An exact expression for a finite translation Δx^μ is obtained via Lie exponentiation through the limit

$$\begin{aligned} v_\mu(x) \to v_\mu(x+\Delta x) &= \lim_{n\to\infty} \left(1 + \frac{\Delta x^\nu}{n}\frac{\partial}{\partial x^\nu}\right)^n v_\mu(x) \\ &= \exp\left(-iP_\mu \Delta x^\mu\right) v_\mu(x) \\ &= \hat{T}(\Delta x) v_\mu(x) \end{aligned}$$

where $\hat{T}(\Delta x)$ is an element of $T(3+1)$, and we have defined the generators of the translations as

$$P_\mu \equiv i\frac{\partial}{\partial x^\mu}.$$

Analogously, we may derive the rotation and boost operators by expanding about the identity for infinitesimal transformations. Finite transformations are then found using the exponential map. We find the appropriate operators to be

$$\begin{aligned} R(\theta) &= \exp(-i\vec{J}\cdot\vec{\theta}) = 1 - (i\vec{J}\cdot\hat{\theta})\sin\theta + (i\vec{J}\cdot\hat{\theta})^2(1-\cos\theta) \\ B(\phi) &= \exp(-i\vec{K}\cdot\vec{\phi}) = 1 - (i\vec{K}\cdot\hat{\phi})\sinh\phi - (i\vec{K}\cdot\hat{\phi})^2(1-\cosh\phi) \end{aligned}$$

for the matrix generators of rotations (J^i) whose elements are given by

$$[J^i]^\mu_\nu = \frac{i}{2}\epsilon^{ijk}(\eta^{j\mu}\delta^k_\nu - \eta^{k\mu}\delta^j_\nu)$$

and the boost matrices (K^i) given by

$$[K^i]^\mu_\nu = i(\eta^{i\mu}\delta^0_\nu - \eta^{0\mu}\delta^i_\nu).$$

The definition of these generators thereby allows us to summarize the Poincaré algebra via the commutation relations

$$\begin{aligned} [J^i, P^0] &= 0 \\ [J^i, P^j] &= i\epsilon^{ijk}P^k \\ [J^i, J^j] &= i\epsilon^{ijk}J^k \\ [J^i, K^j] &= i\epsilon^{ijk}K^k \end{aligned} \qquad \begin{aligned} [K^i, P^0] &= -iP^i \\ [K^i, P^j] &= i\eta^{ij}P^0 \\ [K^i, K^j] &= -i\epsilon^{ijk}J^k \end{aligned}$$

As will be shown, spinors provide an isomorphism between the Lorentz group $SO(1,3)$ and the complexified special linear group $SL(2,\mathbb{C})$ which allows for the natural identification of spin-1/2 fields.

2.3 Spinors and the Weyl Equation

The study of spinors historically dates back to Cartan (1913) when it was first appreciated that, given a complex vector $\vec{z} = z^n\hat{e}_n \in \mathbb{E}_{2N+1}$ for $n \in \{0, 1, 1', ..., N', N\}$ and the associated quadratic form $F = (z^0)^2 + z^1z^{1'} + ... + z^Nz^{N'}$, then by setting $F = 0$ one obtains 2^N independent, linear equations of an auxiliary set of variables ζ_i. These equations define both the complex $2N$-component spinor ζ and the $2N \times 2N$ matrix Z whose elements are linear arrangements of the initial z coordinates. The matrix Z (which acts on the spinor ζ) then serves as a representation of the vector $\vec{z}$.

To understand how the statements above may be put to use, we will now consider the space $\mathbb{R}_3$. A matrix representation of $\vec{x}$ may be constructed by considering the equation $\vec{x} \cdot \vec{x} = 0$. To construct our linear equations for this quadratic form, we define two complex numbers ξ_1 and ξ_2, which satisfy

$$x_1 = \xi_1^2 - \xi_2^2$$

$$x_2 = i(\xi_1^2 + \xi_2^2)$$

$$x_3 = -2\xi_1\xi_2.$$

This allows us to write the linear system

$$\begin{bmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = (\vec{\sigma} \cdot \vec{x})\, \xi = 0 \tag{2.1}$$

where the object ξ is referred to as a spinor and the 2-by-2 matrix (which we will refer to as the matrix associated with the vector $\vec{x}$,) is–not coincidentally–given as the product of $\vec{x}$ with the Pauli matrices $\vec{\sigma}$. A given vector-associated matrix exhibits several properties which make it of mathematical utility; a few of these are enumerated here:

1. The matrix associated with a real-valued vector is Hermitian.
2. The determinant of an associated matrix equals minus the square of its respective vector.
3. The scalar product of two vectors equals half the sum of the symmetrized product of their associated matrices. E.g.

$$\vec{x} \cdot \vec{y} = \frac{1}{2} \left((\vec{\sigma} \cdot \vec{x})(\vec{\sigma} \cdot \vec{y}) + (\vec{\sigma} \cdot \vec{y})(\vec{\sigma} \cdot \vec{x}) \right) \tag{2.2}$$

4. The scalar product of two vectors equals minus half the trace of the product of their associated matrices.

Note that the last property may be derived as a corollary of the second using the Jacobi formula for derivatives of determinants.

Now, because spinors are defined with respect to a fundamental quadratic form (eg. [Eq. 2.1]), it follows that any transformation of the matrix associated with a vector must also induce a transformation of the corresponding spinor. As an illustration consider again $\mathbb{R}_3$. Let $\mathcal{T}$ be the transformation matrix corresponding to the three-vector transformation $\vec{a} \to \hat{T}\vec{a}$. Now suppose this transformation is a symmetry (i.e. leaves invariant) of the eigenvalue equation

$$(\vec{\sigma} \cdot \vec{a})\chi = \lambda\chi.$$

Then for the transformed vector a' there must correspond a transformed spinor χ' satisfying

$$(\vec{\sigma} \cdot \vec{a'})\chi' = \lambda\chi'.$$

To solve for χ', we operate from the left on the untransformed equation with $\mathcal{T}$

$$[\mathcal{T}(\vec{\sigma} \cdot \vec{a})\mathcal{T}^\dagger][\mathcal{T}\chi] = (\vec{\sigma} \cdot \vec{a'})[\mathcal{T}\chi] = \lambda[\mathcal{T}\chi].$$

Thus $\chi' = \mathcal{T}\chi$ and we say the vector transformation $\vec{a} \to \hat{T}\vec{a}$ has induced the transformation $\chi \to \mathcal{T}\chi$. By classifying and exploiting these induced transformations, it is possible to construct fundamental wave equations for representations of spin-half particles.

Using the properties of the vector-associated matrices $\vec{\sigma} \cdot \vec{x}$, we may derive the relationship between rotations in three dimensional space. Given a three-vector $\vec{x}$ associated with the matrix X, the rotated vector $\exp\left(-i\vec{J} \cdot \vec{\theta}\right)\vec{x}$ is associated with the matrix $\mathcal{R}(\theta)X\mathcal{R}^\dagger(\theta)$ where

$$\mathcal{R}(\theta) = \exp\left(-\frac{i}{2}\vec{\sigma} \cdot \vec{\theta}\right) = \cos\frac{\theta}{2} - i\vec{\sigma} \cdot \hat{\theta}\sin\frac{\theta}{2}.$$

The proof of the equivalence follows from the properties above, but the equality of the algebraic representations may be deduced from the commutation properties of the Pauli matrices, $[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$. From these, it is easily verified that the terms

$+\sigma^i/2$ generate an algebra isomorphic to that of J^i. The group of spinor rotations is SU(2) and provides a double coverage of the rotation group.

The above observations for rotations suggest representations of boosts can be constructed by noting the isomorphism which also exists between the generators K^i and the matrices $\pm i\sigma^i/2$. Pulling this thread, we are led to define the spinor boost operator

$$\mathcal{B}_\pm(\phi) = \exp\left(\mp\frac{1}{2}\vec{\sigma}\cdot\vec{\phi}\right) = \cosh\frac{\phi}{2} \mp i\vec{\sigma}\cdot\hat{\phi}\sinh\frac{\phi}{2}$$

corresponding to the boost $\exp\left(-i\vec{K}\cdot\vec{\phi}\right)$. There are a few apparent problems with this definition. First, $\mathcal{B}_\pm$ is clearly not unitary, which appears troubling. Second, we have a worrying sign ambiguity in our definition. In fact, both seeming problems hint at deeper symmetries, as will be shown. First, let us consider the following eigenvalue equation for χ

$$(\vec{\sigma}\cdot\vec{p})\chi = \lambda\chi.$$

Since our stated aim is to use spinors to construct a fundamental quantum mechanical wave equation–and for any equation to be considered fundamental, it must transform covariantly under boosts and rotations–this is what we now must verify. The rotational covariance of the above eigenvalue equation may be explicitly demonstrated. It follows directly from the rotation operator's unitarity and three-dimensional spinors' rotation properties.

$$(\vec{\sigma}\cdot\vec{p})\,\chi = \lambda\chi \qquad \xrightarrow{\text{Rotation}} \qquad \left[\mathcal{R}\,(\vec{\sigma}\cdot\vec{p})\,\mathcal{R}^\dagger\right][\mathcal{R}\chi] = \lambda\,[\mathcal{R}\chi]$$

Which reduces to $\left(\vec{\sigma}\cdot\vec{p'}\right)\chi' = \lambda\chi'$ for $\vec{p'} = \exp\left(-i\vec{J}\cdot\vec{\theta}\right)\vec{p}$ and $\chi' = \mathcal{R}(\theta)\chi$.

Under our boost operator, we find the non-unitary nature of the transformation complicates the resulting expression. We have

$$(\vec{\sigma}\cdot\vec{p})\,\chi = \lambda\chi \qquad \xrightarrow{\text{Boost}} \qquad \left[\mathcal{B}_\pm\,(\vec{\sigma}\cdot\vec{p})\,\mathcal{B}_\pm^\dagger\right]\left[\mathcal{B}_\pm^{\dagger-1}\chi\right] = \lambda\left[\mathcal{B}_\pm\mathcal{B}_\pm^\dagger\right]\left[\mathcal{B}_\pm^{\dagger-1}\chi\right].$$

Substituting the explicit exponential form for a given ϕ and taking $\chi' = \mathcal{B}_{\pm}^{\dagger -1}\chi$, the boosted eigenvalue equation may be written as

$$\vec{\sigma} \cdot \left[\vec{p} - \hat{\phi}(\hat{\phi} \cdot \vec{p})\,(1 - \cosh\phi) \mp \lambda\hat{\phi}\sinh\theta\right]\chi' = \left[\lambda\cosh\phi \mp \hat{\phi}\cdot\vec{p}\sinh\phi\right]\chi'.$$

Lorentz covariance thereby requires any spinor eigenvector equation to have eigenvalues which are invariant under rotations but that transform non-trivially under boosts. Moreover, by comparing the form of the required transformation with the transformation properties of the energy and momentum parameters under boosts given by

$$E \to E' = E\cosh\phi - \hat{\phi}\cdot\vec{p}\sinh\phi$$
$$\vec{p} \to \vec{p'} = \vec{p} - \hat{\phi}(\hat{\phi}\cdot\vec{p})\,(1-\cosh\phi) - E\hat{\phi}\sinh\theta,$$

we find two equally valid Lorentz covariant solutions, $\lambda = \pm E$. Here the sign of the energy is contingent on the transformation properties of the spinor considered. Employing the conventional subscripts L (for left-handed) and R (for right-handed) to differentiate how the given spinor transforms under boosts, we define

$$\chi_L \to \chi_L' = \mathcal{B}_{-}^{\dagger -1}\chi_L = \exp\left(-\frac{\vec{\sigma}\cdot\vec{\phi}}{2}\right)\chi_L$$
$$\chi_R \to \chi_R' = \mathcal{B}_{+}^{\dagger -1}\chi_R = \exp\left(+\frac{\vec{\sigma}\cdot\vec{\phi}}{2}\right)\chi_R.$$

We then find two distinct Lorentz-covariant equations, which may be written as

$$(E + \vec{\sigma}\cdot\vec{p})\chi_L = 0 \qquad (E - \vec{\sigma}\cdot\vec{p})\chi_R = 0.$$

In promoting these to proper quantum-mechanical wave functions, all that remains is to make the canonical substitutions for $\vec{p}$ and E with the generators of translations $-i\nabla$ and $i\partial_t$, respectively. To simplify the notation we write the two-by-two identity as σ^0 and write the adjugate Pauli matrices $\bar{\sigma}^\mu = \eta^{\mu\mu}\sigma^\mu$. We thereby obtain the *Weyl Equations*:

$$i\bar{\sigma}^\mu\partial_\mu\chi_L = 0 \qquad i\sigma^\mu\partial_\mu\chi_R = 0.$$

Note that under a parity inversion ($x \to -x$), we find $i\bar{\sigma}^\mu\partial_\mu \to i\sigma^\mu\partial_\mu$ and $i\sigma^\mu\partial_\mu \to i\bar{\sigma}^\mu\partial_\mu$. Thus, the necessity of having two "types" of particles which transform oppositely under boosts and satisfy two separate equations is no accident, but instead a consequence of discrete symmetry conservation. This is the origin of chiral fields. Having made these identifications we may identify the plane-wave solutions for the right-chiral Weyl equations by taking the three-momentum to be

$$\vec{p} = p\ (\cos\varphi \sin\theta\ \hat{\mathrm{x}} + \sin\varphi\sin\theta\ \hat{\mathrm{y}} + \cos\theta\ \hat{\mathrm{z}}).$$

We then obtain two solutions $\chi_R \in \{\chi_\uparrow^+, \chi_\downarrow^-\}$ for

$$\chi_\uparrow^+ = \begin{bmatrix} \cos\frac{\theta}{2} \\ \mathrm{e}^{i\varphi}\sin\frac{\theta}{2} \end{bmatrix} \exp\left(-ip_\mu x^\mu\right), \qquad \chi_\downarrow^- = \begin{bmatrix} \mathrm{e}^{-i\varphi}\sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{bmatrix} \exp\left(+ip_\mu x^\mu\right)$$

where the notation is such that $i\partial_0\chi_\updownarrow^\pm = \pm E\chi_\updownarrow^\pm$ for $E = |E| > 0$. We may therefore interpret $\chi_\uparrow^+$ as a positive energy spinor and $\chi_\downarrow^-$ as having negative energy in the sense that $\partial_t\chi_\downarrow^- = -E\chi_\downarrow^-$. The physical interpretation of this negative energy state is supplied by the Feynman-Stueckelberg interpretation [Thomson (2013)] and leads us to consider such states as positive energy antiparticles. Observe that the left-chiral spinor solutions are then given by $\chi_L \in \{\chi_\uparrow^-, \chi_\downarrow^+\}$ which are obtained from the above via a change in the sign of the energy.

2.4 Dirac Spinors and the Dirac Equation

As demonstrated above, Weyl spinors are eigenstates of the energy and momentum operators, which transform as three-dimensional spinors under rotations and isomorphically to four-vectors under boosts. They also constitute *masssless* spin-half representations of particles in the sense that, if χ_L and χ_R are solutions to their respective Weyl equations, they are also solutions to the masslesss Klein–Gordon (KG) equation[1]

$$-i\bar{\sigma}^{\nu}\partial_{\nu}\left(i\sigma^{\mu}\partial_{\mu}\chi_R\right) = \partial^{\mu}\partial_{\mu}\chi_R = 0$$

$$-i\sigma^{\nu}\partial_{\nu}\left(i\bar{\sigma}^{\mu}\partial_{\mu}\chi_L\right) = \partial^{\mu}\partial_{\mu}\chi_L = 0.$$

In order to introduce a mass to the fields χ_L and χ_R, we may couple these left- and right-handed fields symmetrically. We may then write the coupled Weyl equations in matrix form as [Peskin and Schroeder (2018)]

$$\begin{bmatrix} -m & i\sigma^{\mu}\partial_{\mu} \\ i\bar{\sigma}^{\mu}\partial_{\mu} & -m \end{bmatrix} \begin{bmatrix} \chi_L \\ \chi_R \end{bmatrix} = 0.$$

Ever seeking to simplify the notation, we define the Dirac gamma matrices in the chiral basis as

$$\gamma^{\mu} = \begin{bmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma} & 0 \end{bmatrix}$$

These matrices satisfy the Clifford algebraic anticommutation relations $\gamma^{\mu}, \gamma^{\nu} = 2\eta^{\mu\nu}$. We also define the Dirac bi-spinor as

$$\psi = \begin{bmatrix} \chi_L \\ \chi_R \end{bmatrix}.$$

These definitions allow us to identify the two, coupled Weyl equations with a single equation for a massive Dirac fermion satisfying the Dirac Equation [Peskin and Schroeder (2018)]

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi = 0.$$

In addition to the four γ-matrices above, it is useful to define the fifth matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ which anti-commutes with all other γ-matrices. This matrix allows

[1] The KG equation itself follows from direct application of the canonical quantum four-momentum operators $(i\partial_{\mu})$ to the Einstein energy-momentum relation $E^2 = p^2 + m^2$ [Maggiore (2005)].

us to “extract” the left- and right-chiral components of a Dirac spinor through the projection operations defined by

$$\frac{1}{2}\left(1+\gamma^5\right)\psi = \begin{bmatrix} 0 \\ \chi_R \end{bmatrix} \equiv \psi_R \qquad , \qquad \frac{1}{2}\left(1-\gamma^5\right)\psi = \begin{bmatrix} \chi_L \\ 0 \end{bmatrix} \equiv \psi_L .$$

This choice of γ-matrices are not algebraically unique, but are connected via a unitary spin transformation to other bases. That is, physical equivalence is maintained under the combined transformations

$$\gamma^\mu \to U\gamma^\mu U^\dagger \qquad \psi \to U\psi$$

for any unitary matrix U, thereby defining the transformation of γ-basis.

Lastly, we note that the four-component Dirac spinors are not four-dimensional spinors since they do not transform correctly under the larger class of four-dimensional rotations. Instead, they are reducible objects transforming like composite three-dimensional spinors under rotations and block-diagonally under boosts. For this reason, they are often called bi-spinors. Strictly speaking, these Dirac bi-spinors constitute induced irreducible representations of Minkowski space-time extended by parity [Ryder (1996)]. However, as has been demonstrated, Dirac bi-spinors are reducible objects in the context of more general parity considerations.

2.5 Charge, Parity, and Time

In addition to the continuous symmetries associated with the Poincare group, there are also those which arise from discrete reversals. Parity, which has been mentioned above, is one such symmetry. To make the remaining symmetries explicit, we may define the operations of charge conjugation ($\mathcal{C}$), spatial inversion ($\mathcal{P}$), and time reversal ($\mathcal{T}$) acting on a operator $\hat{O}(x,t)$ and bi-spinor $\psi(x,t)$ as given by

$$\mathcal{C}[\hat{O}(x,t)\psi(x,t)]\mathcal{C}^{-1} = \hat{O}^*(x,t)\psi^*(x,t)$$
$$\mathcal{P}[\hat{O}(x,t)\psi(x,t)]\mathcal{P}^{-1} = \hat{O}(-x,t)\psi(-x,t)$$
$$\mathcal{T}[\hat{O}(x,t)\psi(x,t)]\mathcal{T}^{-1} = \hat{O}^*(x,t)\psi(x,-t).$$

Note that, in keeping with Wigner's theorem [Wigner (1939)], $\mathcal{C}$ and $\mathcal{P}$ are unitary operators while $\mathcal{T}$ is *anti*-unitary. With these operations defined, it is straightforward to calculate their effects on the Dirac operator acting on a bi-spinor ψ.

$$\mathcal{C}[(i\gamma^\mu\partial_\mu - m)\,\psi(x,t)]\mathcal{C}^{-1} = (-i(\gamma^\mu)^*\partial_\mu - m)\,\psi^*(x,t)$$
$$\mathcal{P}[(i\gamma^\mu\partial_\mu - m)\,\psi(x,t)]\mathcal{P}^{-1} = \left(i\gamma^0\partial_0 - i\gamma^k\partial_k - m\right)\psi(-x,t)$$
$$\mathcal{T}[(i\gamma^\mu\partial_\mu - m)\,\psi(x,t)]\mathcal{T}^{-1} = \left(i(\gamma^0)^*\partial_0 - i(\gamma^k)^*\partial_k - m\right)\psi(x,-t)$$

It is then a matter of determining the appropriate unitary spinor transformation that carries these equations back to the basis of the original gamma matrices. In the chiral basis we may write these unitary matrices as

$$C = e^{i\xi}\gamma^2 \qquad P = e^{i\eta}\gamma^5 \qquad T = e^{i\zeta}\gamma^1\gamma^3$$

where the phases are arbitrary. We then find

$$C\left\{\mathcal{C}[(i\gamma^\mu\partial_\mu - m)\,\psi(x,t)]\mathcal{C}^{-1}\right\} = (i\gamma^\mu\partial_\mu - m)\,\psi^{\mathcal{C}}(x,t)$$
$$P\left\{\mathcal{P}[(i\gamma^\mu\partial_\mu - m)\,\psi(x,t)]\mathcal{P}^{-1}\right\} = (i\gamma^\mu\partial_\mu - m)\,\psi^{\mathcal{P}}(x,t)$$
$$T\left\{\mathcal{T}[(i\gamma^\mu\partial_\mu - m)\,\psi(x,t)]\mathcal{T}^{-1}\right\} = (i\gamma^\mu\partial_\mu - m)\,\psi^{\mathcal{T}}(x,t).$$

Where we have defined the charge-, parity-, and time-conjugated fields to be

$$\psi^{\mathcal{C}}(x,t) = C\psi^*(x,t) \qquad \psi^{\mathcal{P}}(x,t) = P\psi(-x,t) \qquad \psi^{\mathcal{T}}(x,t) = T\psi(x,-t).$$

Because the free Dirac equation is invariant under each of these transformations individually, it is also invariant under any combination of them. When interactions are included this is no longer true, however it is demonstrable that the simultaneous action of $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ must leave any local, Lorentz-invariant quantum system invariant [Streater and Wightman (2000)].

2.6 Electroweak Interactions and Spontaneous Symmetry Breaking

We now move from the external space-time symmetries to the internal gauge symmetries of electroweak theory. The Electroweak model predicts the effective unification of the weak and electromagnetic fields at high energies through the gauging of the $SU(2)_L \times U(1)_y$ symmetry group of weak isospin and weak hypercharge [Peskin and Schroeder (2018), Ryder (1996)]. At low energies – such as exist naturally in the universe today – the larger symmetry group is broken by the non-zero vacuum expectation value of the Higgs field. In the standard model, the Higgs gives mass to the mediator bosons of the weak interaction as well as the fermions. We will consider the electroweak model for two fermion fields, (ψ_1 and ψ_2) which are reducible under parity to two left-chiral ($\psi_{1L},\ \psi_{2L}$) and two right-chiral ($\psi_{1R},\ \psi_{2R}$) fields. It is helpful to write the electroweak Lagrangian in four distinct terms as $\mathcal{L}_{EW} = \mathcal{L}_f + \mathcal{L}_h + \mathcal{L}_m + \mathcal{L}_g$. Each term may then be written fully as [Kaku (1993)]

$$
\begin{aligned}
\mathcal{L}_f &= i\bar{\Psi}_L\gamma^\mu D_\mu \Psi_L + i\bar{\psi}_{1R}\gamma^\mu D_\mu \psi_{1R} + i\bar{\psi}_{2R}\gamma^\mu D_\mu \psi_{2R} \\
\mathcal{L}_h &= (D_\mu\Phi)^\dagger D_\mu\Phi + \frac{\mu^2}{2}\Phi^\dagger\Phi - \frac{\lambda_0}{4}(\Phi^\dagger\Phi)^2 \\
\mathcal{L}_m &= -i\lambda_1(\bar{\Psi}_L\sigma^2\Phi^*\psi_{1R} - \bar{\psi}_{1R}\Phi^T\sigma^2\Psi_L) - \lambda_2(\bar{\Psi}_L\Phi\psi_{2R} + \bar{\psi}_{2R}\Phi^\dagger\Psi_L) \\
\mathcal{L}_g &= -\frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.
\end{aligned}
$$

Here the terms $\mu,\ \lambda_0,\ \lambda_1$, and λ_2 are constants of the theory. The constituent symmetry elements of this theory are the left-chiral fermion doublet (Ψ_L), the complex scalar doublet (Φ), and the right-chiral singlet states ($\psi_{1R},\ \psi_{2R}$) which we write in our simple two-fermion formulation as

$$\Psi_L = \begin{bmatrix} \psi_{1L} \\ \psi_{2L} \end{bmatrix} = \frac{1}{2}(1-\gamma^5)\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \qquad \Phi = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix}, \qquad \psi_{iR} = \frac{1}{2}(1+\gamma^5)\psi_i.$$

The doublet Φ is often referred to as the "Higgs doublet".

In addition to these fields, there are four gauge fields $W^{1,2,3}$ and B which appear as connection terms in the gauge-covariant derivatives given explicitly for the fermions by

$$D_\mu \Psi_L = \left[\partial_\mu - \frac{ig' y_L}{2} B_\mu - \frac{ig}{2}\vec{\sigma}\cdot\vec{W}_\mu\right]\Psi_L \qquad D_\mu \psi_{iR} = \left[\partial_\mu - \frac{ig' y_{iR}}{2} B_\mu\right]$$

for the coupling constants g' and g, and the weak hypercharges y_L and y_{iR}. The relationship between electric charge and the weak hypercharge parameters are given in the pseudo Gell-Mann Nishijima relations [Weinberg (1995a)]

$$q_1 = \frac{1}{2} + \frac{y_L}{2} = \frac{y_{1R}}{2}, \qquad q_2 = -\frac{1}{2} + \frac{y_L}{2} = \frac{y_{2R}}{2}. \tag{2.3}$$

The corresponding field strength tensors for the gauge fields, expressing their kinetic contributions to the energy, are

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{abc} W^b_\mu W^c_\nu$$
$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu.$$

The complete Lagrangian $\mathcal{L}_{EW}$ is invariant under the set of $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ symmetry transformations taken as

$$\Psi_L \to \exp\left[\frac{i}{2}(y_L\beta + \vec{\sigma}\cdot\vec{\theta})\right]\Psi_L \qquad \psi_{iR} \to \exp\left[\frac{i}{2}y_{iR}\beta\right]\psi_{iR}.$$

It is also taken for granted that X_μ and W^a_μ transform in the appropriate Yang-Mills covariant manner. Also while we have provided definitions and transformations in terms of Ψ_L and ψ_R, because Φ is defined as a doublet within the symmetry group it transforms identically to Ψ_L throughout.

Now, spontaneous symmetry breaking (SSB) occurs when the Higgs doublet Φ takes on a non-zero vacuum expectation value (VEV) at the minimum of the effective potential, given by the non-kinetic terms in $\mathcal{L}_h$. The degenerate minima conditions are satisfied for $\Phi_0^\dagger \Phi_0 = \mu^2/\lambda_0$. By exploiting the local isospin symmetry transformations we may expand Φ about Φ_0 at each space-time point as

$$\Phi_{min}(x) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ v + h(x) \end{bmatrix} \tag{2.4}$$

for the real-valued Higgs field h and the VEV $v \equiv \sqrt{2\mu/\lambda_0}$. Next, we introduce four new fields constructed from the linear combinations of the original gauge fields.

$$A_\mu = \cos\theta_W B_\mu + \sin\theta_W W_\mu^3$$
$$Z_\mu = \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu$$
$$W_\mu^\pm = \frac{1}{\sqrt{2}} \left(W_\mu^1 + i W_\mu^2 \right).$$

Here the weak mixing angle θ_W may be taken as defined by the equality

$$\cos\theta_W = \frac{g}{\sqrt{g'^2 + g^2}}$$

Then by substituting $\Phi = \Phi_{min}$ into $\mathcal{L}_h$ we obtain from the kinetic terms

$$\left(D_\mu \Phi\right)^\dagger D_\mu \Phi = \frac{1}{2} \partial_\mu h \partial^\mu h + \frac{g^2 (v+h)^2}{8\cos^2\theta_W} Z_\mu Z^\mu + \frac{g^2(v+h)^2}{8} \left(W_\mu^+ W^{-\mu} + W_\mu^- W^{+\mu} \right).$$

and so find the fields W and Z have acquired the masses

$$M_W = \frac{vg}{2} \qquad M_Z = \frac{vg}{2\cos\theta_W} = \frac{M_W}{\cos\theta_W},$$

while the photon (A) has remained massless. The Higgs field also obtains a mass through self couplings in the potential and has a value of $M_h = \mu\sqrt{2}$.

The fermion couplings to the gauge fields are now found by expanding the covariant derivatives of Eq. (2.3) in terms of the newly defined bosons. We find the interaction terms separate into electromagnetic, neutral current, and charged current interactions

$$
\begin{aligned}
\mathcal{L}_{EM} &= -q_1 g \sin\theta_w A_\mu \bar{\psi}_1 \gamma^\mu \psi_1 - q_2 g \sin\theta_w A_\mu \bar{\psi}_2 \gamma^\mu \psi_2 \\
\mathcal{L}_{NC} &= -\frac{g}{2\cos\theta_W} Z_\mu \bar{\psi}_1 \gamma^\mu \left(c_{V1} - c_{A1}\gamma^5\right)\psi_1 - \frac{g}{2\cos\theta_W} Z_\mu \bar{\psi}_2 \gamma^\mu \left(c_{V2} - c_{A2}\gamma^5\right)\psi_2 \\
\mathcal{L}_{CC} &= -\frac{g}{2\sqrt{2}} W_\mu^+ \bar{\psi}_2 \gamma^\mu \left(1-\gamma^5\right)\psi_1 - \frac{g}{2\sqrt{2}} W_\mu^- \bar{\psi}_1 \gamma^\mu \left(1-\gamma^5\right)\psi_2 .
\end{aligned}
$$

The fundamental electric charge e is now found to be given by $e = g\sin\theta_w$ and the axial coupling constants c_{Vi} and c_{Ai} may be written

$$
\begin{aligned}
c_{V1} &= +\frac{1}{2} - 2q_1 \sin^2\theta_W \qquad & c_{A1} &= +\frac{1}{2} \\
c_{V2} &= -\frac{1}{2} - 2q_2 \sin^2\theta_W \qquad & c_{A2} &= -\frac{1}{2}.
\end{aligned}
$$

Finally, the fermion mass terms now appear from the Yukawa couplings in $\mathcal{L}_m$ after SSB as

$$
\mathcal{L}_m = -m_1 \bar{\psi}_1 \psi_1 - m_2 \bar{\psi}_2 \psi_2
$$

for the mass values $m_i = \lambda_i v/\sqrt{2}$.

2.7 Majorana Mass Terms

As has been shown, Dirac mass terms add mass through a coupling between left- and right-chiral representations and can be derived naturally through Yukawa couplings to the Higgs. However, this is not the only means of introducing massive fermions. It is possible to incorporate Majorana mass terms through a self-coupling between a field and its charge conjugate. This is most naturally expressed using Weyl spinors, where the resulting equation of motion may be written as

$$
i\bar{\sigma}^\mu \partial_\mu \chi + m\chi_c = 0
$$

where χ_c is the charge-conjugated spinor field given by

$$\chi_c \equiv \mathrm{e}^{i\xi}\sigma^2\chi^*$$

for a choice of the phase ξ. This coupling to the charge-conjugate will explicitly break any U(1) symmetry associated with the phase of χ. In Chapter 5, this will play an important role in the understanding of chiral-fixing.

2.8 Quantum Field Theory in Curved space-time

Quantum field theory in flat Minkowski space-time is limited by its inability to account properly for the gravitational interaction. A first step to understanding the quantum theory of gravitation is a consistent description of the quantum field theory in curved spaces. In previous work, QFT in curved space-time was mainly formulated for scalar fields (e.g., Elizalde (1987), Nicolas (1995) Strohmaier (2000)); however, there are also exceptions when vector and spinor fields are considered (e.g., Wald (1994), Nyambuya (2008), Alhaidari and Jellal (2015)). The background curvature of space-time is specified and remains fixed, which means that the introduced matter and energy in the form of fields and particles do not influence space-time. This is a significant constraint on the theory, allowing only the so-called 'test-fields' or 'test-particles' [Wald (1994)].

Curved space-time of GR is described by a 4D, smooth, pseudo-Riemannian manifold $\mathcal{M}$ endowed with the metric $ds^2 = g_{\mu\nu}(x)\ dx^\mu dx^\nu$, where $g_{\mu\nu}$ is the metric tensor, μ and ν are 0, 1, 2 and 3, and the usual summation conventions are employed [Wald (1994)]. Thus GR replaces the flat space-time of SR by $\mathcal{M}$ and requires that the Poincaré group, which is the group of transformations that leave dynamical equations invariant in SR, is replaced by a general group of coordinate transformations. This general group is known as the diffeomorphism group and it is denoted as Diff($\mathcal{M}$) (e.g., Weinberg (1972), Heller (1992)); this is a spatio-temporal group that is non-restrictive and carries limited information [Auyang et al. (1995)]. Let us denote coordinates by x and introduce a diffeomorphism η, which is considered to be close to the identity. Coordinatization of Diff($\mathcal{M}$) assigns to each η a set of functions $\tilde{\eta}$ that allows us to relate x to other coordinates x' by the following relationship $x' = x + \tilde{\eta}$.

Let the coordinates systems of all observers be related by the above continuous linear transformation between an observer with coordinates x and another with coordinates x' be called the GR observers. Using this definition of GR observers, the principle of general covariance–used by Einstein (1915) as a basis for developing his GR–can be established. A modern view is that general covariance is a tool but not a principle. Specifically, Weinberg (1972) claims that 'the principle of general covariance is not an invariance principle.' To make a clear distinction between general covariance and invariance, in this dissertation, we call a physical theory covariant if its basic equations are written in the coordinate-free form. However, we consider such a theory invariant if its equations are left unchanged by symmetry operations.

2.9 Dark Matter

Numerous measurements of rotation curves of different galaxies were performed (e.g., Persic et al. 1996; Boriello & Salucci 2001; Livio 2003; Bertone et al. 2005) with the main conclusion that the galactic rotation curves can only be explained if the existence of DM distributed in a spherical galactic halo around these galaxies is postulated [Freeman and McNamara (2006), Navarro et al. (2011)]. Additional strong evidence for the existence of DM was given by NASA's WMAP [Benini et al. (2013), Larson et al. (2011)] and by some gravitational lensing measurements [Ellis (2010)] as well as by NASA's Bullet Cluster [Barrena et al. (2002)]. The WMAP data established the total amount of DM in the Universe. The amount of ordinary matter (OM), DM and Dark Energy (DE) found from the WMAP data was refined by the ESA's Planck mission, and the currently accepted values are: $\Omega_{OM} = 0.049 \pm 0.04$, $\Omega_{DM} = 0.268$ and $\Omega_{OM} = 0.683$ [Aghanim et al. (2020)].

Models of DM can be divided into three groups: hot, warm and cold. The main candidates for hot DM are electron (ν_e), muon (ν_μ) and tau (ν_τ) neutrinos. However, based on the current limits on their masses, there are too few of them to solve the DM problem, and they are too fast to be bound and so cannot explain the structure formation in the observed Universe [White et al. (1983), Freeman and McNamara (2006)]. Regarding the warm DM, the main candidates are sterile (right-handed) neutrinos, which have not been discovered. Then there is the long list of candidates for the cold DM [Sugita et al. (2008)]. An incomplete list of these candidates may include weakly interacting massive particles (WIMPs) as originally suggested by Peebles (1982), supersymmetric (SUSY) particles like gravitino [Pagels and Primack (1982)] or neutralino [Barbier et al. (2005)], axions [Treiman and Wilczek (1978), Rosenberg and Van Bibber (2000)], the Klein-Kaluza particles emerging from theories with extra dimensions [Cheng et al. (2002))], and extremely light bosonic particles ELBPs [Sin (1994)]. Detailed description of different DM candidates can be found in many reviews, such as Overduin and Wesson (2004), Freeman and McNamara (2006) and Sugita et al. (2008).

As of today, neither the origin of CDM nor its nature is known. Hence, this has become one of modern science's most urgent and challenging problems. This dissertation presents another possible solution to the DM problem.

CHAPTER 3

THE CHIRAL DIRAC EQUATION

3.1 Background

Since its introduction in 1928 numerous modifications of the Dirac operator [Dirac (1928)] have been proposed resulting in a number of so-called generalized Dirac equations. Such generalizations have been invoked for the purposes of unifying leptons and quarks [Sogami (1981), Kruglov (2006), Marsch and Narita (2015)], accounting for the three families of elementary particles [Pfister (1994), Kruglov (2007), 2012], including ad hoc a pseudoscalar mass [Leiter and Szamosi (1972)], and extending the Dirac equation to distances comparable to the Planck length [Nozari (2007)].

With the benefit of hindsight and the accumulation of experimental evidence, it may be said that one of Dirac's great insights was his observation that elementary particles exhibit a quantization of both angular momentum (spin-up/spin-down) and sign of the energy (matter/antimatter). Modern particle physics has reinforced this picture for free particles of a given flavor independent of internal (gauge) symmetries. Here we demonstrate how the assumption of two quantized characteristics is equivalent to the chiral form of the Dirac equation (CDE), Dirac's original equation being a special case.

In an attempt to demonstrate these results holistically, the present chapter is organized as follows: Section 2 presents three different methods of deriving the CDE operator beginning with the standard group theory and Lagrange formalism derivations before presenting a novel derivation from the assumption of the existence of independent physical states and their corresponding orthogonal idempotents (projection operators) parameterized by physically meaningful quantities; Section 3 then sets out a detailed investigation into the space-time (continuous) and CPT (discrete) symmetries underlying the derived CDE, their role and constraints in the derivation procedures, and the physical implications resulting from this equation; Section 4 is devoted to conclusions.

3.2 Methods of Derivation

Group theoretical notions may be utilized to define a fundamental theory as satisfying the following principles: the principle of invariance, the principle of locality, and the principle of least action [Bargmann and Wigner (1948), Fushchich and Nikitin (1994), Musielak and Fry (2009), Fry et al. (2011)]. We apply these principles under the assumption of the Poincaré group as the fundamental symmetry group of local space-time and demonstrate the derivation of generalized first order Poincaré-invariant dynamical equation governing the evolution of irreducible representations of spin-1/2 particle states containing the Dirac equation. The resulting generalized Dirac equation extends the original Dirac equation to include chiral symmetries for massive elementary particles and it shows that the Dirac equation is not a unique factorization of the Klein-Gordon equation [Klein (1926), Gordon (1926)]; the presented results significantly differ from the previous attempts to generalize the Dirac equations [Kruglov (2006), Nozari (2007), Kruglov (2012), Huegele et al. (2013)].

Since the irreps of the invariant subgroup are the irreps of the entire group [Bargmann (1954)], the condition that a wave function transforms as one of the irreps of the subgroup of the Poincaré group is found in the following eigenvalue equation

$$i\partial_\mu \phi \;=\; k_\mu \phi \tag{3.1}$$

originally obtained for ϕ being a scalar wavefunction [Fry et al. (2011)]. As proved by Wigner (1939), the proper irreps of spin-1/2 elementary particles that are considered here are the four-component bi-spinors ψ for which the eigenvalue equation 3.1 becomes

$$iA^{\mu}\partial_{\mu}\psi \;=\; A^{\mu}k_{\mu}\psi \; , \tag{3.2}$$

where A^{μ} is an arbitrary constant matrix of 4×4. Defining $X^{\mu}=iA^{\mu}$ and $Y=A^{\mu}k_{\mu}$, we obtain

$$(X^{\mu}\partial_{\mu}+Y)\psi=0 \tag{3.3}$$

where X^{μ} and Y are to be determined. This equation is consistent with the fact that the generators of translations in quantum mechanics are equivalent to the Hermitian momentum operator ($i\partial_{\mu}$) and that, by definition, these operators give rise to dynamical equations of motion. For these reasons, we take Eq. (3.3) as our starting point. The four-component bi-spinors are given by

$$\psi=\begin{bmatrix}\chi_L\\ \chi_R\end{bmatrix}, \tag{3.4}$$

where χ_L and χ_R are two-component spinors.

The necessity of coupling these spinors is understood mathematically as accommodating the sign ambiguity introduced in the construction of the isomorphism between boosts in SO(3,1) and those in SU(2). As a result, we find χ_L and χ_R to transform identically under rotations but oppositely under boosts. We may thus write our Lorentz transformation for the bi-spinors

$$\Lambda=\begin{bmatrix}\Lambda_L & 0\\ 0 & \Lambda_R\end{bmatrix}=\begin{bmatrix}\exp\left(-\frac{i\vec{\sigma}\cdot(\vec{\theta}+i\vec{\phi})}{2}\right) & 0\\ 0 & \exp\left(-\frac{i\vec{\sigma}\cdot(\vec{\theta}-i\vec{\phi})}{2}\right)\end{bmatrix} \tag{3.5}$$

where $\vec{\theta}$ and $\vec{\phi}$ parameterize our rotations and boosts, respectively, and are related to the transformations of four vectors via the four-by-generators $\vec{J}$ and $\vec{K}$ such that

$$\hat{\Lambda} = \exp\bigg(- i\vec{J}\cdot\vec{\theta} - i\vec{K}\cdot\vec{\phi})\bigg).$$

Applying the Lorentz transformation of (3.5) to (3.3) and the inverse transformation from the left yields:

$$\big((\Lambda^{-1}\hat{\Lambda}^{\nu}_{\mu}X^{\mu}\Lambda)\partial_{\nu} + (\Lambda^{-1}Y\Lambda)\big)\psi = 0. \tag{3.6}$$

This leads to the necessary conditions for invariance.

$$\hat{\Lambda}^{\nu}_{\mu}X^{\mu} = \Lambda X^{\nu}\Lambda^{-1} \qquad Y = \Lambda Y \Lambda^{-1}$$

. Solving these, we find the most general form of our matrix coefficients written in block form.

$$X^{\mu} = \begin{bmatrix} 0 & x_R(\sigma^0\delta^{\mu}_0 + \sigma^k\delta^{\mu}_k) \\ x_L(\sigma^0\delta^{\mu}_0 - \sigma^k\delta^{\mu}_k) & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} y_L\sigma^0 & 0 \\ 0 & y_R\sigma^0 \end{bmatrix}.$$

where x_R, x_L, y_R, and y_L are free parameters.

We may simplify this by taking the Dirac γ matrices in the chiral basis.

$$\gamma^0 = \begin{bmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{bmatrix} \qquad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}$$

and identifying the chiral projection operators as

$$P_L = \begin{bmatrix} \sigma^0 & 0 \\ 0 & 0 \end{bmatrix} \qquad P_R = \begin{bmatrix} 0 & 0 \\ 0 & \sigma^0 \end{bmatrix}.$$

Then, the equation becomes

$$\big((x_L P_R + x_R P_L)\gamma^{\mu}\partial_{\mu} + (y_L P_L + y_R P_R)\big)\psi = 0 \tag{3.7}$$

Now, under the assumption that x_L and x_R are nonzero, we are free to multiply from left with $i(x_L P_R + x_R P_L)^{-1}$, and obtain

$$\left(i\gamma^\mu \partial_\mu + i\left(\frac{y_L}{x_R} P_L + \frac{y_R}{x_L} P_R\right)\right)\psi = 0 \tag{3.8}$$

which leads naturally to the Hamiltonian

$$\mathcal{H}\psi = -i\gamma^0\left(\gamma^k \partial_k + \left(\frac{y_L}{x_R} P_L + \frac{y_R}{x_L} P_R\right)\right)\psi \tag{3.9}$$

Calculating the Hamiltonian squared operator, we find

$$\mathcal{H}^2\psi = \left(\partial^k \partial_k - \frac{y_L y_R}{x_L x_R}\right)\psi \tag{3.10}$$

Note the propagation mass term

$$m \equiv \pm i\sqrt{\frac{y_L y_R}{x_L x_R}} \tag{3.11}$$

arises naturally from our degrees of freedom in (3.9). Additionally, we observe the restriction of the square of (3.11) to positive real numbers is equivalent to the physical restriction of the Einstein energy-momentum relationship.

Let us point out that an immediate consequence of (3.8) is the simultaneous permissibility of both fundamental scalar and pseudoscalar mass terms. We may see this explicitly by expanding P_L and P_R in terms of $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ and collecting terms

$$\left(i\gamma^\mu \partial_\mu - M - \widetilde{M}\gamma^5\right)\psi = 0 \tag{3.12}$$

where we have defined the scalar mass

$$M \equiv -\frac{i}{2}\left(\frac{y_R}{x_L} + \frac{y_L}{x_R}\right) \tag{3.13}$$

and the coefficient of the pseudoscalar mass

$$\widetilde{M} \equiv -\frac{i}{2}\left(\frac{y_R}{x_L} - \frac{y_L}{x_R}\right) \tag{3.14}$$

The propagation mass then takes the form

$$m = \sqrt{M^2 - \widetilde{M}^2} \tag{3.15}$$

3.3 Chiral symmetries

To understand the degrees of freedom in the generalized equations, let us consider a global chiral rotation of the field variable

$$\psi \to \psi' = e^{\frac{i}{2}\gamma^5\alpha}\psi. \tag{3.16}$$

At this point, it will be both more natural and more convenient to discuss the chiral transformation of (3.16) in the context of the Lagrangian formalism. The Lagrangian (3.12) may now be written as

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - \bar{\psi}(M + \widetilde{M}\gamma^5)\psi. \tag{3.17}$$

Under the transformation of (3.16) The Lagrangian (3.17) becomes

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - \bar{\psi}(M + \widetilde{M}\gamma^5)e^{i\gamma^5\alpha}\psi \tag{3.18}$$

so we find the effect of (3.16) is equivalent to the rotation of our mass parameters

$$\begin{bmatrix} M' \\ i\widetilde{M}' \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} M \\ i\widetilde{M} \end{bmatrix} \tag{3.19}$$

Significantly, this is precisely the transformations required to leave invariant the propagation mass in Eq. (3.15). Thus we find that a chiral rotation of a massive field is equivalent to a choice of factorization of the Klein-Gordon equation (i.e., the Einstein energy relationship). We may thus consider the chiral angle as determining the fraction of mass distributed between our field's left- and right-chiral components. To our knowledge, these observations were first made by Leiter and Szamosi (1972), but without physical motivation for introducing pseudoscalar terms. The present work differs substantially from all those previous in that we have demonstrated how both scalar and pseudoscalar terms arise as the necessary consequence of considering fundamental physical symmetries in flat space-time. Therefore, we must conclude that the existence (or non-existence) of such terms is wholly determined by those physical mechanisms governing the generation of fermion masses.

3.4 Derivation from Lagrangian formalism

The Lagrangian formalism is a powerful and independent way to derive a dynamical equation. The Lagrangian for the Dirac equation ($\alpha = 0$ in Eq. 3.3) is very well-known and presented in textbooks (e.g., [Ryder (1996), Frampton (2008)]) without derivation. In fact, the Lagrangian was not a part of Dirac's original paper where the equation first appeared [Dirac (1928)]. An interesting attempt to obtain the Dirac Lagrangian is presented and discussed in [Doughty (2018)]. Let us briefly review the main points of this attempt and then use them to obtain the Lagrangian for Eq. (3.3).

In case $\alpha = 0$, Eq. (3.3) reduces to the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \ , \tag{3.20}$$

which describes a free, massive, non-chiral and spin 1/2 relativistic elementary particle [Dirac (1928), Doughty (2018), Frampton (2008),Ryder (1996)]. To obtain the Lagrangian density for this equation, we follow [Doughty (2018)] and require that the Lagrangian is a Hermitian, single-valued proper scalar or pseudo-scalar in ψ and $\partial_\mu \psi$. Since ψ has the double-valued properties under rotations, the terms in the Lagrangians must have even numbers of ψ. The simplest proper scalar is $\bar{\psi}\psi$, where $\bar{\psi}$ is the Dirac adjoint. Now, the construction of a scalar kinetic term has to be done with caution as $\partial_\mu \psi$ requires saturation of the index μ, which another derivative cannot do since the result would be a second-order equation. Therefore, the Dirac matrices γ^μ are used to saturate the index μ, and write the kinetic term as $i\bar{\psi}\gamma^\mu \partial_\mu \psi$. Since the physical units of the kinetic term are different from $\bar{\psi}\psi$, the latter must be multiplied by an inverse length dimension, which in natural units is mass. Then, the Dirac Lagrangian can be written in the following form

$$\mathcal{L}_D = \frac{1}{2}\bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{1}{2}\bar{\psi}(i\gamma^\mu \overleftarrow{\partial}_\mu + m)\psi = 0 \ . \tag{3.21}$$

This fully symmetric form of the Lagrangian shows that when evaluated along a stationary path, the Dirac Lagrangian vanishes [Doughty (2018)]. Both the Dirac equation and its Lagrangian are Poincaré invariant.

Using the above procedure, the Lagrangian for the CDE (see Eq. 3.3) can also be obtained and written as

$$\mathcal{L}_{CD} = \frac{1}{2}\bar{\psi}(i\gamma^\mu \partial_\mu - me^{-i\alpha\gamma^5})\psi - \frac{1}{2}\bar{\psi}(i\gamma^\mu \overleftarrow{\partial}_\mu + me^{-i\alpha\gamma^5})\psi = 0 \ . \tag{3.22}$$

Similarly to $\mathcal{L}_D$, the Lagrangian is also fully symmetric, hermitian and single-valued proper scalar or pseudo-scalar, and it vanishes when evaluated along a stationary path. Moreover, the CDE and its Lagrangian are Galilean-invariant. By substituting $\mathcal{L}_{CD}$ into the Euler-Lagrange equation for variations with respect to $\bar{\psi}$, the CDE given by Eq. (3.3) is obtained. This method of deriving the CDE is independent of the group theory derivation and demonstrates that the equation satisfies a least-action principle requisite of any fundamental theory. Our derivation based on projection operators is now presented.

3.5 Derivation from Orthogonal Idempotents

We may define a set of projection operators operating on an N-dimensional complex vector space with any set of N-by-N orthogonal idempotent matrices satisfying

$$\hat{P}_i\hat{P}_j = \hat{P}_j\hat{P}_i = \delta_{ij}\hat{P}_i \qquad \sum_{i=1}^{N}\hat{P}_i = 1 \tag{3.23}$$

with the total number of projection operators of a given vector space is maximally equal to the dimensions of the space considered. Therefore, for $N = 2$, we may expand the most general operators acting on a spinor in terms of the Pauli matrices and the two-by-two identity matrix. Let

$$\hat{P}_1 = a_0 I_2 + \vec{\sigma} \cdot \vec{a} \qquad \hat{P}_2 = b_0 I_2 + \vec{\sigma} \cdot \vec{b} \ . \tag{3.24}$$

Then enforcing the orthogonal idempotent conditions, we obtain the following constraints.

$$a_0 = b_0 = \frac{1}{2} \qquad \vec{a} \cdot \vec{a} = \vec{b} \cdot \vec{b} = \frac{1}{4} \qquad \vec{a} = -\vec{b} \ . \tag{3.25}$$

Solving this, we find two projection operators for our symmetry group whose degrees of freedom may be parameterized in terms of the unit vector $\hat{a}$. We write these projection operators succinctly as

$$\hat{P}_{\pm}(\hat{a}) = \frac{1}{2}(I_2 \pm \vec{\sigma} \cdot \hat{a}) \ . \tag{3.26}$$

For a fixed $\hat{a}$, these operators allow us to define two types of objects in our two-dimensional vector space. For any such element $\chi \in \mathbb{C}^2$ we may define $\chi_{\pm}(\vec{a}) \equiv \hat{P}_{\pm}(\hat{a})\chi(\vec{a})$. It necessarily follows that

$$(\vec{\sigma} \cdot \hat{a})\chi_{\pm}(\vec{a}) = \pm\chi_{\pm}(\vec{a}) \ . \tag{3.27}$$

It is then a simple matter to extend these projection operators to projections in 2^N-dimensional vector spaces. In general, we may write

$$\hat{P}_{s_1,s_2,...s_N}(\hat{a}_1, \hat{a}_2, ...\hat{a}_N) = \bigotimes_{i=1}^{N} \hat{P}_{s_i}(\hat{a}_i) \ , \tag{3.28}$$

for $s_i \in \{\pm\}$. It is easy to see that these projection operators satisfy our orthogonal idempotent constraints. We then similarly define our set of eigenvectors.

$$\chi_{s_1,s_2,...s_N}(\vec{a}_1, \vec{a}_2, ...\vec{a}_N) = \bigotimes_{i=1}^{N} \chi_{s_i}(\vec{a}_i) \ . \tag{3.29}$$

By restricting our considerations to the vector space of spinors, we can give these abstract considerations physical significance. Recall that the rotation operator corresponding to a rotation of θ about the axis defined by $\hat{a}$ for the vector space of two-component spinors takes the form

$$\begin{aligned} \hat{R}(\theta, \hat{a}) &= \cos\frac{\theta}{2} + i(\vec{\sigma}\cdot\hat{a})\sin\frac{\theta}{2} \\ &= e^{\frac{i\theta}{2}}\hat{P}_{+}(\hat{a}) + e^{-\frac{i\theta}{2}}\hat{P}_{-}(\hat{a}) \ . \end{aligned} \tag{3.30}$$

It follows that eigenstates of our projection operators $\hat{P}_{\pm}(\hat{a})$ are physically invariant under rotations about $\hat{a}$, differing only by a phase. We therefore identify $\hat{P}_{\pm}(\hat{a})$ as projecting out the portion of the state vector with spin parallel (+) or anti-parallel (−) to the $\hat{a}$-axis. By choosing $\vec{a} = \vec{p}$, where $\vec{p}$ is the three-momentum of the particle, we find $\hat{P}_{\pm}(\hat{p})$ to be the helicity projection operators. This most neatly encapsulates the experimental observance of binary spin states in mathematical terms. We now wish to use our projection operator methodology to classify states of positive and negative energies, e.g., matter/anti-matter. Including an additional two-valued quantum property necessitates (at minimum) a four-dimensional vector space. We, therefore, construct the projection operators of the form

$$\hat{P}_{s_1,s_2}(\hat{q},\hat{p}) = \hat{P}_{s_1}(\hat{q}) \otimes \hat{P}_{s_2}(\hat{p}) \ , . \tag{3.31}$$

We have introduced the vector $\hat{q}$ about which we will have more to say shortly. For the time being, $\hat{q}$ is simply a set of three complex numbers and satisfies $\hat{q}\cdot\hat{q} = 1$. Next, we define the operand

$$\chi_{s_1,s_2}(\hat{q},\hat{p}) = \chi_{s_1}(\hat{q}) \otimes \chi_{s_2}(\hat{p}) \ . \tag{3.32}$$

The corresponding generalization of Eq. (3.27) yields

$$(\vec{\sigma}\cdot\hat{q} \otimes I_2)\chi_{s_1,s_2}(\hat{q},\hat{p}) = s_2\chi_{s_1,s_2}(\hat{q},\hat{p}) \tag{3.33}$$

$$(I_2 \otimes \vec{\sigma}\cdot\hat{p})\chi_{s_1,s_2}(\hat{q},\hat{p}) = s_1\chi_{s_1,s_2}(\hat{q},\hat{p}) \ . \tag{3.34}$$

Exploiting the fact that $s_1 = \pm s_2$, we may construct the equations

$$(\vec{\sigma} \cdot \hat{q} \otimes I_2 + I_2 \otimes \vec{\sigma} \cdot \hat{p})\chi_{\pm,\mp}(\hat{q}, \hat{p}) = (\vec{\sigma} \cdot \hat{q} \oplus \vec{\sigma} \cdot \hat{p})\chi_{\pm,\mp}(\hat{q}, \hat{p}) = 0 \tag{3.35}$$

$$(\vec{\sigma} \cdot \hat{q} \otimes I_2 - I_2 \otimes \vec{\sigma} \cdot \hat{p})\chi_{\pm,\pm}(\hat{q}, \hat{p}) = (\vec{\sigma} \cdot \hat{q} \ominus \vec{\sigma} \cdot \hat{p})\chi_{\pm,\pm}(\hat{q}, \hat{p}) = 0 \ . \tag{3.36}$$

We now identify the set of gamma matrices in the chiral representation.

$$\gamma^0 = \sigma^1 \otimes I_2$$

$$\gamma^k = i\sigma^2 \otimes \sigma^k$$

satisfying the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_4 \ .$$

Taking the usual definition $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ as the matrix that anticommutes with all γ^μ, let us write

$$i\gamma^5\gamma^0 = \sigma^2 \otimes I_2$$

$$-\gamma^5 = \sigma^3 \otimes I_2$$

which give

$$(\vec{\sigma} \cdot \hat{q} \otimes I_2) = \frac{1}{|\vec{q}|}\gamma^0\gamma^5(\gamma^5 q_1 - iI_4 q_2 + \gamma^0 q_3) \tag{3.37}$$

$$(I_2 \otimes \vec{\sigma} \cdot \hat{p}) = -\frac{1}{|\vec{p}|}\gamma^0\gamma^5\vec{\gamma} \cdot \vec{p} \ . \tag{3.38}$$

Rewriting the operators of Eqs. (3.31) and (3.32) in terms of the gamma matrices, we find

$$(\vec{\sigma} \cdot \hat{q} \oplus \vec{\sigma} \cdot \hat{p}) = \frac{1}{|\vec{p}||\vec{q}|}\gamma^0\gamma^5(\gamma^0|\vec{p}|q_3 - |\vec{q}|\vec{\gamma} \cdot \vec{p} + \gamma^5|\vec{p}|q_1 - iI_4|\vec{p}|q_2)$$
$$(\vec{\sigma} \cdot \hat{q} \ominus \vec{\sigma} \cdot \hat{p}) = \frac{1}{|\vec{p}||\vec{q}|}\gamma^0\gamma^5(\gamma^0|\vec{p}|q_3 + |\vec{q}|\vec{\gamma} \cdot \vec{p} + \gamma^5|\vec{p}|q_1 - iI_4|\vec{p}|q_2) \ .$$

It is our goal to identify the vector $\vec{q}$ with our physical quantities. Scaling these operators from the left with $\pm|\vec{p}||\vec{q}|\gamma^5\gamma^0$, we now restrict $\vec{q}$ and $\vec{p}$ to be of equal-magnitude (this is equivalent to the on-mass-shell asssumption). We then obtain equivalent equations to Eqs. (3.31) and (3.32) of the form

$$(-\gamma^0 q_3 + \vec{\gamma}\cdot\vec{p} + \gamma^5 q_1 - iI_4 q_2)\chi_{\pm,\mp} = 0 \tag{3.39}$$

$$(+\gamma^0 q_3 + \vec{\gamma}\cdot\vec{p} - \gamma^5 q_1 + iI_4 q_2)\chi_{\pm,\pm} = 0 \ . \tag{3.40}$$

It is now possible to make the identifications with our physical quantities explicit. For the vector $\vec{q}$, we identify:

$$q_1 = im\sin\alpha \qquad q_2 = -im\cos\alpha \qquad q_3 = p_0 = E \tag{3.41}$$

The identification of $q_3 = E$ ($E > 0$) is chosen to align with our definition of the Dirac gamma matrices though equivalent linear combinations of the vector may be found through unitary transformations. It may appear curious upon first inspection that the vector $\vec{q}$ is seemingly compelled to take on imaginary values in two components and real values in the third. It is, however, a simple matter to absolve ourselves of this inhomogeneity by performing a Wick-rotation of the energy axis in the complex plane and thereby considering the four vectors of Minkowski space-time in purely Euclidean terms. In this way, the connection between our projection operators as a basis of $SL(2,\mathbb{C})$ and the Lorentz group $SO(1,3)$ may be made explicit. These simplifications notwithstanding, we will continue to consider real-valued energies in Minkowski space-time. Substituting our terms of Eqs. (3.41), into Eqs. (3.39) and (3.40) we obtain

$$(-\gamma^0 E + \gamma^k p_k + me^{i\alpha\gamma^5})\chi_{\pm,\mp} = 0 \tag{3.42}$$

$$(+\gamma^0 E + \gamma^k p_k - me^{i\alpha\gamma^5})\chi_{\pm,\pm} = 0 \ . \tag{3.43}$$

It is clear that Eqs. (3.42) and (3.43) are the momentum-space analogues of the CDE for positive and negative energies and therefore are equivalent to plane wave solutions of Eq. (3.3). The eigenstates therefore satisfy

$$E^2\chi_{s_1,s_2} = (p^2 + m^2)\chi_{s_1,s_2} \ . \tag{3.44}$$

The method of projection operators for deriving fundamental equations has the potential to extend beyond the $\mathbb{C}^4$ vector space of bi-spinors. While outside the scope at present, the possibility of extending the concepts and methodologies presented here to investigate the algebraic structure inherent in three particle flavors and their mass spectrum remains a tantalizing possibility.

3.6 Space-Time Symmetries

From the Special Theory of Relativity, we know that for an equation to be considered fundamental, it must remain invariant in all inertial frames of reference. Such frames may be defined as those frames in which the symmetries of Minkowski space-time are agreed to hold. These symmetries are represented by the Poincaré group $\mathcal{P} = SO(3,1) \otimes_s T(3+1)$ consisting of rotations, boosts, and translations in space and time (see Section 2.1) which carry one inertial frame to another. We may therefore refer to the class of inertial observers to whom fundamental equations must remain invariant as Poincaré observers. It follows as a necessary condition for any equation to be considered fundamental that it preserve its form for all Poincaré observers and hence remain invariant with respect to all transformations given by $\mathcal{P}$. Only such equations can make physical predictions about which all Poincaré observers will agree. It is easy to verify that the derived CDE above is one such Poincaré invariant equation, and so satisfies a necessary condition to be called a *fundamental equation of physics*.

The other characteristics required of a fundamental theory may be summarized as locality, gauge invariance, and the satisfaction of a least-action principle (equivalent to the existence of Lagrangian density for the equation). Since the CDE is a first-order partial differential equation, it is local. Here only free particles are considered, and gauge invariance is not included. Above, we demonstrated that the Lagrangian density for the CDE exists and may be written in the fully symmetric form given by Eq. (3.22). In general, Lagrangians possess less symmetry than the dynamical equations resulting from them due to the assumptions on which the Noether theorem is based [Hojman (1984), Hojman (1992)]. The best-known example is the law of inertia, whose dynamical equation is Galilean invariant but whose Lagrangian is not [Landau and Lifshitz (2000), Lévy-Leblond (1969)]. However, a method to restore Galilean invariance of the Lagrangian was developed and applied to the law of inertia [Musielak and Watson (2020)].

We followed [Doughty (2018)] to construct the Dirac Lagrangian, which is already Poincaré invariant. Similarly, the Lagrangian for the CDE equation is also Poincaré invariant. Therefore the form of the CDE equation and its Lagrangian, as well as theoretical predictions resulting from them, are the same for all observers who accept the Principle of Relativity underlying the Special Theory of Relativity and its Poincaré group $\mathcal{P}$. Therefore, the presented results combined with the above discussion imply that any theory of physics based on the CDE and its Lagrangian may be rightly called a fundamental equation of physics.

3.7 CPT symmetries

Beyond the symmetries of the Poincaré group, we may also inquire into which discrete symmetries of nature hold for the CDE. From the discrete operations of charge conjugation ($\mathcal{C}$), spatial inversion ($\mathcal{P}$), and time-reversal ($\mathcal{T}$) given in Chapter 2, we may determine the parity-conjugated forms of the CDE. Defining $\hat{D}(\alpha) = i\gamma^\mu\partial_\mu - me^{i\alpha\gamma}$, taking the symmetry operation $\mathcal{U} \in \{\mathcal{C}, \mathcal{P}, \mathcal{T}\}$, and writing the corresponding unitary transformation acting on the spinor indices $\hat{U} \in \{\hat{C}, \hat{P}, \hat{T}\}$, we obtain

$$\hat{U}[\mathcal{U}\hat{D}(\alpha)\mathcal{U}^{-1}]\hat{U}^\dagger\psi_U(x,t) = 0 \tag{3.45}$$

where $\psi_U(x,t) \equiv \hat{U}[\mathcal{U}\psi(x,t)\mathcal{U}^{-1}]$. Then, the resulting transformations may be summarized as

$$\hat{C}[\mathcal{C}\hat{D}(\alpha)\mathcal{C}^{-1}]\hat{C}^\dagger = \hat{D}(\alpha^*) \tag{3.46}$$

$$\hat{P}[\mathcal{P}\hat{D}(\alpha)\mathcal{P}^{-1}]\hat{P}^\dagger = \hat{D}(-\alpha) \tag{3.47}$$

$$\hat{T}[\mathcal{T}\hat{D}(\alpha)\mathcal{T}^{-1}]\hat{T}^\dagger = \hat{D}(-\alpha^*)\ , \tag{3.48}$$

and the following conditions can be identified:

1. If ψ exhibits C-invariance ($\psi = \psi_C$), then $\alpha \in \mathbb{R}$.
2. If ψ exhibits CP-invariance ($\psi = \psi_{CP}$), then $\alpha \in \mathbb{I}$.
3. If ψ exhibits CPT-invariance ($\psi = \psi_{CPT}$), then $\alpha \in \mathbb{C}$.

These conditions, combined with the CPT theorem, reinforce our conclusion that the Dirac equation with chiral freedom is the most general first-order differential equation derivable from the irreps of the Poincaré group. While it is quite conceivable that the constraints imposed by observed classes of discrete symmetries in interactions (and their fundamental violations) may contribute to constraints on the CDE, it is essential to emphasize that the regimes of validity presented here are derived for free particles in the absence of interactions and therefore a result of extrinsic and not intrinsic symmetries. In this way, the above discrete constraints are foundational for any extended theoretical considerations.

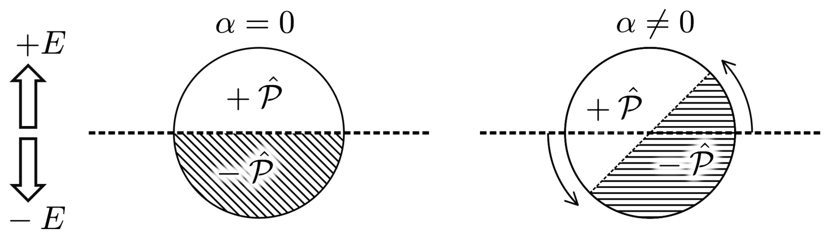

Figure 3.1. A diagrammatic representation of the effects of a unitary chiral rotations on the set of solutions to the Dirac equation ($\alpha = 0$) and the general form ($\alpha \neq 0$) presented in the text. The bijectivity of the unitary transformation results in positive energy solutions of mixed parity in the general case. The chiral Dirac equation is thereby seen to be a special case in which eigenstates of intrinsic parity and energy align. (Note the antiparticle diagram follows via a reversal of the parity signs..

3.8 On the Distinctions of the Dirac Equation and its Chiral Form

A regrettable amount of confusion has historically plagued the literature surrounding attempts to include ad hoc pseudoscalar mass terms, particularly as regards the relationship between the resulting solutions and those of the Dirac equation [Leiter and Szamosi (1972), Da Silveira (1976), Trzetrzelewski (2011)]. For the purpose of clarifying the matter, we will now set out to sketch a proof of the physical inequivalence between the representations related by a chiral rotation.

Given the Dirac equation

$$(i\gamma^{\mu}\partial_{\mu} - m)\,\psi = 0$$

We define a particle state in the chiral basis $\alpha = 0$ as the positive-energy ($+E$) solutions:

$$\psi^{+}\left(x, t; 0\right) = u_s\left(p; 0\right) e^{-ip\cdot x}$$

Similar antiparticle spinors we take to be given by the $+E$ solutions:

$$\psi^{-}\left(x, t; 0\right) = v_s\left(p; 0\right) e^{+ip\cdot x}$$

Now, there will always exist a unitary transformation which allows one to transform the Dirac equation into the CDE, and vice versa. It is given by

$$\psi \to U\psi = e^{\frac{i\alpha}{2}\gamma^5}\psi$$

and constitutes an alternate choice of chiral basis[1]. Applying the unitary chiral rotation operator above to our $+E$ Dirac solutions, we find

$$\psi^+(x,t;0) \to \psi^+(x,t;\alpha) = e^{\frac{i\alpha}{2}\gamma^5}\psi^+(x,t) = u_s(p;\alpha)\,e^{-ip\cdot x}$$

and

$$\psi^-(x,t;0) \to \psi^-(x,t;\alpha) = e^{\frac{i\alpha}{2}\gamma^5}\psi^-(x,t) = v_s(p;\alpha)\,e^{+ip\cdot x}$$

for

$$\begin{aligned} u_s(p;\alpha) &\equiv \left[\cos\frac{\alpha}{2}u_s(p;0) + i\sin\frac{\alpha}{2}v_{3-s}(p;0)\right] \\ v_s(p;\alpha) &\equiv \left[\cos\frac{\alpha}{2}v_s(p;0) + i\sin\frac{\alpha}{2}u_{3-s}(p;0)\right] \end{aligned}$$

These chiral-rotated states now satisfy the CDE of the form:

$$\left(i\gamma^\mu\partial_\mu - me^{-i\alpha\gamma^5}\right)\psi^\pm(x,t;\alpha) = 0$$

In investigating the parity properties of these representations, recall that the intrinsic parity of a fermionic field is defined by the action of the parity operator ($\hat{P} = \gamma^0$) on the at-rest solutions. Because a chiral rotation is not a unitary spin transformation, $\hat{P}$ remains invariant. Therefore, in the rest frame the CDE equations reduce to:

$$\hat{P}u_s(0;\alpha) = +e^{-i\alpha\gamma^5}u_s(0;\alpha)$$

$$\hat{P}v_s(0;\alpha) = -e^{-i\alpha\gamma^5}v_s(0;\alpha)$$

[1]Unlike all previously considered transformations, U is not a unitary *spin* transformation of the type $\psi \to V\psi$ and $\gamma^\mu \to V\gamma^\mu V^\dagger$. Rather, U acts only on the composite bi-spinor

and so $u_s(p;\alpha)$ and $v_s(p;\alpha)$ are not generally eigenstates of parity. This is significant as it implies a unitary chiral rotation will carry $+E$, parity-eigenstate solutions of the DE into $+E$ solutions of the CDE with mixed parity.

We are, of course, free to construct states with well-defined intrinsic parity in the $\alpha \neq 0$ basis. These are not difficult to obtain and are given simply in the rest frame by:

$$\psi^A(x,t;\alpha) = e^{-i\alpha\gamma^5/2} u_s(0;\alpha)\, e^{-imt} = u_s(0;0)\, e^{-imt}$$
$$\psi^B(x,t;\alpha) = e^{-i\alpha\gamma^5/2} v_s(0;\alpha)\, e^{+imt} = v_s(0;0)\, e^{+imt}$$

However, as we are currently working in the basis where $\alpha \neq 0$, these particle/anti-particle states are related to the solutions in the basis where $\alpha = 0$ via the inverse transformation $U^\dagger = \exp(-i\alpha\gamma^5/2)$. Hence the parity eigenstates $\psi^A(x,t;\alpha)$ and $\psi^B(x,t;\alpha)$ correspond in the $\alpha = 0$ basis to the at-rest solutions given by:

$$\begin{aligned}
\psi^A(x,t;\alpha) \to \psi^A(x,t;0) &= e^{-i\alpha\gamma^5/2}\psi^A(x,t;\alpha) = e^{-i\alpha\gamma^5} u_s(0;0)\, e^{-imt} \\
&= \left(\cos\frac{\alpha}{2} u_s(0;0) - i\sin\frac{\alpha}{2} v_{3-s}(0;0)\right) e^{-imt} \\
\psi^B(x,t;\alpha) \to \psi^B(x,t;0) &= e^{-i\alpha\gamma^5/2}\psi^B(x,t;\alpha) = e^{-i\alpha\gamma^5} v_s(0;0)\, e^{+imt} \\
&= \left(\cos\frac{\alpha}{2} v_s(0;0) - i\sin\frac{\alpha}{2} u_{3-s}(0;0)\right) e^{+imt}
\end{aligned}$$

Note the appearance of $v_{3-s}(p;0)\, e^{-imt}$ and $u_{3-s}(p;0)\, e^{+imt}$. These are not identifiable as positive energy states in the $\alpha = 0$ basis and so must taken as negative energy solutions. Thus, we have proven the following:

No unitary transformation exists which connects the set of positive-energy, parity-eigenstate solutions of the Chiral Dirac Equation to the set of positive-energy solutions of the Dirac Equation.[2]

[2]The only exception to this is the trivial case of the identity matrix when $\alpha = 0$.

The implications of this statement are physically meaningful when taken in conjunction with the requirement that any well-defined QFT should possess only states with bounded energies. It is then the combination of these facts that amount to the physical inequivalence of the representations. This is the central point which many authors have failed to properly appreciate, and is the reason we argue the CDE constitutes a non-trivial generalization of the Dirac Equation whose solutions form physically inequivalent irreducible representations of the Poincaré group.

3.9 Summary

The goal of this Chapter has been to highlight the importance of chirality in the context of identifying physically foundational theories. We began, motivated by underappreciated degrees of freedom present in the standard representations of the Poincaré group, with the derivation of the chiral form of the Dirac equation. The presence of chiral mass terms raised intriguing possibilities. The second half of the chapter was then devoted to employing heuristic methodologies and new techniques in deriving the chiral Dirac equation, with the Dirac equation proving to be a special case. Finally, a discussion regarding the discrete symmetries of nature was undertaken to assess the validity of our results.

CHAPTER 4

THE CHIRAL BARGMANN-WIGNER EQUATIONS

4.1 Background

In the previous chapter, we derived a generalized form of the Dirac equation exhibiting an internal degree of chiral freedom using the irreducible representations (irreps) of the Poincaré group $\mathcal{P} = SO(3,1) \otimes_s T(3+1)$, with $SO(3,1)$ being the group of rotations and boosts and $T(3+1)$ an invariant subgroup of space-time translations [Kim and Noz (2012)]. The derived chiral form of the Dirac equation (CDE) [Watson and Musielak (2020)] may be written as

$$\hat{D}\psi \equiv (i\gamma^\mu \partial_\mu - me^{-2i\alpha\gamma^5})\psi = 0 \ , \tag{4.1}$$

where α is the chiral angle specifying the basis and ψ represents a four-component spinor that transforms as one of the irreps of $T(3+1) \in \mathcal{P}$, and each of its components satisfies the Klein-Gordon equation [Klein (1926),Watson and Musielak (2020)]. We demonstrated that the chiral rotation of a massive field is equivalent to an alternative choice of chiral basis and that the Dirac equation is obtained by factorization of the Klein-Gordon equation if, and only if, a specific selection of chiral basis is made [Watson and Musielak (2020),Watson and Musielak (2021b)].

In the present chapter, we extend the chiral-asymmetric form of the Dirac equation beyond spin-1/2 particles utilizing the Bargmann-Wigner (BW) formalism [Bargmann and Wigner (1948)] in which all possible relativistic equations (with the exception of spin zero) are derived through the unitary representations of the Poincaré group as classified by Wigner [Wigner (1939), Kim and Noz (2012)]. The BW equations are coupled first-order partial differential equation with the original Dirac operator ($\hat{D} = i\gamma^\mu \partial_\mu - m$) acting on symmetric multispinor wavefunctions which are taken to describe a field of rest mass m and spin $s \leq 1/2$. For special cases of $s = 1/2$, $s = 1$ and $s = 3/2$, the BW equations reduce to the Dirac [Dirac (1928)], Proca [Proca (1936)] and Rarita-Schwinger [Rarita and Schwinger (1941)] equations, respectively; however, for $s = 2$, see [Dvoeglazov (2000)].

Using this formalism, it is possible to begin with the CDE and extend the concept of chirality to higher spin particles, thereby generalizing the BW equations to include chiral degrees of freedom. Much like the original technique of Bargmann and Wigner, our method of deriving the chiral (CBW) equations is based on the irreps of the Poincaré group $\mathcal{P}$. The derived CBW equations are valid for spin-1 massive fields (though higher-order generalizations are, in principle, possible). By specifying the chiral basis, we will demonstrate how our spin-1 equations reduce to a Proca-like equation [Proca (1936)] is coupled to an auxiliary equation for a spin-0 massive field. The resulting coupling is due to a misalignment of the chiral bases of the constituent representations and, therefore, chirally induced. This is a new phenomenon, whose physical implications are discussed.

4.2 Bargmann-Wigner equations with chiral freedom

We begin by observing that the at-rest solutions of Eq. (4.1) may be written as

$$\psi^{(\pm)} = \omega^{(\pm)}(\alpha) e^{\mp imt} \ , \tag{4.2}$$

where $(+) \in \{(1),(2)\}$ indicate positive energy solutions and $(-) \in \{(3),(4)\}$ indicate negative energies. Then, these spinors take the form

$$\omega^{(1)}(\alpha) = \begin{bmatrix} \cos\alpha \\ 0 \\ i\sin\alpha \\ 0 \end{bmatrix},\ \omega^{(2)}(\alpha) = \begin{bmatrix} 0 \\ \cos\alpha \\ 0 \\ i\sin\alpha \end{bmatrix},\ \omega^{(3)}(\alpha) = \begin{bmatrix} i\sin\alpha \\ 0 \\ \cos\alpha \\ 0 \end{bmatrix},\ \omega^{(4)}(\alpha) = \begin{bmatrix} 0 \\ i\sin\alpha \\ 0 \\ \cos\alpha \end{bmatrix} \tag{4.3}$$

It is easily to verify that spinors $\omega^{(1)}$ and $\omega^{(3)}$ are $+\frac{1}{2}$ eigenstates of the spin projection operator $\hat{S}^3$ and $\omega^{(2)}$ and $\omega^{(4)}$ are similarly $-\frac{1}{2}$ eigenstates. Moreover, all of these states have a definite propagation mass.

We may now define the general set of positive energy multispinors of spin-1 from the tensor product of the spinors of spin-1/2 as follows

$$\omega^{(1,1)}(\alpha,\beta) = \omega^{(1)}(\alpha) \otimes \omega^{(1)}(\beta)\ , \tag{4.4}$$

$$\omega^{(1,2)}(\alpha,\beta) = \omega^{(2,1)}(\alpha,\beta) = \omega^{(1)}(\alpha) \otimes \omega^{(2)}(\beta) + \omega^{(2)}(\alpha) \otimes \omega^{(1)}(\beta)\ , \tag{4.5}$$

and

$$\omega^{(2,2)}(\alpha,\beta) = \omega^{(2)}(\alpha) \otimes \omega^{(2)}(\beta)\ , \tag{4.6}$$

where β is the chiral angle associated with the chiral basis of the second bi-spinor of our representation. It must be noted that different chiral angles can be paired together in a single spinor as they are all valid eigenstates of momentum and spin, as can be seen by observing that these multispinors are eigenstates of the following spin operator (with indices included for clarity)

$$\left(\hat{S}^3\right)^{\mu\nu}_{\mu'\nu'} = \left(\hat{S}^3\right)^{\mu}_{\mu'}\delta^{\nu}_{\nu'} + \left(\hat{S}^3\right)^{\nu}_{\nu'}\delta^{\mu}_{\mu'}\ , \tag{4.7}$$

and satisfy their respective at-rest chiral Dirac equations

$$(\gamma^0 - e^{-2i\alpha})^{\mu}_{\mu'}\omega^{(+,+)}_{\mu\nu}(\alpha,\beta) = 0\ , \tag{4.8}$$

and

$$(\gamma^0 - e^{-2i\beta})^{\nu}_{\nu'}\omega^{(+,+)}_{\mu\nu}(\alpha,\beta) = 0 \ , \tag{4.9}$$

for $(+,+) \in \{(1,1),(1,2),(2,2)\}$. We may then move from the rest frame to an arbitrary momentum state by boosting each spinor

$$\omega^{(+,+)}_{\mu\nu}(\alpha,\beta;p^\mu) = \Lambda(\alpha,p^\mu)^{\mu}_{\mu'}\Lambda(\beta,p^\mu)^{\nu}_{\nu'}\omega^{(+,+)}_{\mu\nu}(\alpha,\beta) \ . \tag{4.10}$$

where, the chiral form of the Lorentz transformation is related via a similarity transformation to the case when $\alpha = 0$ given by $\Lambda(\alpha,p^\mu) = e^{i\alpha\gamma^5}\Lambda(0,p^\mu)e^{-i\alpha\gamma^5}$. Further, since the chiral rotation $e^{i\alpha\gamma^5}$ commutes with all generators of the Lorentz group it commutes with $\Lambda(0,p^\mu)$ and thus $\Lambda(\alpha,p^\mu) = \Lambda(0,p^\mu)$. We next define the symmetric and anti-symmetric positive energy multispinors and find

$$\Omega^{(+,+)}_{\mu\nu}(\alpha,\beta;p^\mu) = \frac{1}{2}\big(\omega^{(+,+)}_{\mu\nu}(\alpha,\beta;p^\mu) + \omega^{(+,+)}_{\mu\nu}(\beta,\alpha;p^\mu)\big) \ , \tag{4.11}$$

and

$$\widetilde{\Omega}^{(+,+)}_{\mu c}(\alpha,\beta;p^\mu) = \frac{1}{2}\big(\omega^{(+,+)}_{\mu\nu}(\alpha,\beta;p^\mu) - \omega^{(+,+)}_{\mu\nu}(\beta,\alpha;p^\mu)\big) \ , \tag{4.12}$$

such that the full symmetric and anti-symmetric positive energy solutions are given by

$$\psi^{(+)}_{\mu\nu}(\alpha,\beta,x^\mu) = \sum_{(+,+)} \int C_1^{(+,+)}(p^\mu)\Omega^{(+,+)}_{\mu\nu}(\alpha,\beta;p^\mu)e^{-ip_\mu x^\mu} d^3p \ , \tag{4.13}$$

and

$$\widetilde{\psi}^{(+)}_{\mu\nu}(\alpha,\beta,x^\mu) = \sum_{(+,+)} \int C_2^{(+,+)}(p^\mu)\widetilde{\Omega}^{(+,+)}_{\mu\nu}(\alpha,\beta;p^\mu)e^{-ip_\mu x^\mu} d^3p \ , \tag{4.14}$$

where the sum is over $(+,+) \in \{(1,1),(1,2),(2,2)\}$.

Similar arguments for negative energy solutions lead to the construction of the negative energy multispinors. Notationally, this is achieved via a simple index substitution $(+,+) \to (-,-) \in \{(3,3),(3,4),(4,4)\}$ and flipping the sign of the exponential. With this complete set of states, we may identify the most general symmetric and anti-symmetric multispinors

$$\Psi_{\mu\nu}(\alpha,\beta;x^\mu) = a_1\psi^{(+)}_{\mu\nu}(\alpha,\beta,x^\mu) + b_1\psi^{(-)}_{\mu\nu}(\alpha,\beta,x^\mu), \tag{4.15}$$

and

$$\widetilde{\Psi}_{\mu\nu}(\alpha,\beta;x^\mu) = a_2\widetilde{\psi}^{(+)}_{\mu\nu}(\alpha,\beta,x^\mu) + b_2\widetilde{\psi}^{(-)}_{\mu\nu}(\alpha,\beta,x^\mu) \tag{4.16}$$

that satisfy the following four coupled equations

$$\begin{aligned}&[i\gamma^\mu\partial_\mu - m\cos(\alpha-\beta)e^{-i(\alpha+\beta)\gamma^5}]^{\mu/\nu}_{\mu'/\nu'}\Psi_{\mu\nu}(\alpha,\beta;x^\mu)\\ &\quad = [-im\sin(\alpha-\beta)\gamma^5 e^{-i(\alpha+\beta)\gamma^5}]^{\mu/\nu}_{\mu'/\nu'}\widetilde{\Psi}_{\mu\nu}(\alpha,\beta;x^\mu)\ ,\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}&[i\gamma^\mu\partial_\mu - m\cos(\alpha-\beta)e^{-i(\alpha+\beta)\gamma^5}]^{\mu/\nu}_{\mu'/\nu'}\widetilde{\Psi}_{\mu\nu}(\alpha,\beta;x^\mu)\\ &\quad = [-im\sin(\alpha-\beta)\gamma^5 e^{-i(\alpha+\beta)\gamma^5}]^{\mu/\nu}_{\mu'/\nu'}\Psi_{\mu\nu}(\alpha,\beta;x^\mu)\ ,\end{aligned} \tag{4.18}$$

where the summed indices have been combined for brevity.

Isolating either multispinor yields

$$[\partial^\mu\partial_\mu + m^2]^\mu_{\mu'}\Psi_{\mu\nu}(\alpha,\beta;x^\mu) = 0\ , \tag{4.19}$$

and

$$[\partial^\mu\partial_\mu + m^2]^\mu_{\mu'}\widetilde{\Psi}_{\mu\nu}(\alpha,\beta;x^\mu) = 0\ , \tag{4.20}$$

which demonstrates that each element of our multispinors satisfy the Klein-Gordon equation. These are new chiral BW equations (CBW equations) for spin-1 massive fields; they reduce to the BW equations when $\alpha = \beta$ and thus $\widetilde{\Psi}$ vanishes. This shows that chiral degrees of freedom require an additional equation for $\widetilde{\Psi}$. The effects of this additional equation on the Proca equation [Proca (1936)] are now considered and discussed.

4.3 New insights into the Proca Equation

The CBW equations allows us to identify representations of spin-1 fields consistent with those degrees of freedom observed in the CDE. Using Renton (1990), we relate these representations to those known in the particle physics. Let $\hat{C} = i\gamma^2\gamma^0$ and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, then the Clifford basis is the following set of ten symmetric matrices $\{\gamma^\mu\hat{C}, \hat{\sigma}^{\mu\nu}\hat{C}\}$, and the six anti-symmetric matrices $\{\gamma^\mu\gamma^5\hat{C}, i\gamma^5\hat{C}, \hat{C}\}$ that allow expanding the multi-spinors in this basis, and obtain

$$\Psi = mA_\sigma\gamma^\sigma\hat{C} + \frac{1}{2}F_{\sigma\tau}\hat{\sigma}^{\sigma\tau}\hat{C} \ , \tag{4.21}$$

and

$$\widetilde{\Psi} = \rho e^{-i\theta\gamma^5}\hat{C} + mB_\sigma\gamma^\sigma\gamma^5\hat{C} \ , \tag{4.22}$$

where ρ is a scalar field, θ is a scalar parameter, A^μ and B^μ are vector fields, and $F^{\mu\nu}$ is an antisymmetric tensor.

Writing the CBW equations in matrix form, we find the four linearly independent combinations

$$\begin{aligned} i\partial_\mu\big(\gamma^\mu\Psi + \Psi(\gamma^\mu)^T\big) - m\cos(\alpha-\beta)\big\{\Psi, e^{-i(\alpha+\beta)\gamma^5}\big\} \\ -im\sin(\alpha-\beta)\big[\widetilde{\Psi}, \gamma^5 e^{-i(\alpha+\beta)\gamma^5}\big] = 0 \ , \end{aligned} \tag{4.23}$$

$$\begin{aligned} i\partial_\mu\big(\gamma^\mu\widetilde{\Psi} + \widetilde{\Psi}(\gamma^\mu)^T\big) - m\cos(\alpha-\beta)\big\{\widetilde{\Psi}, e^{-i(\alpha+\beta)\gamma^5}\big\} \\ -im\sin(\alpha-\beta)\big[\Psi, \gamma^5 e^{-i(\alpha+\beta)\gamma^5}\big] = 0 \ , \end{aligned} \tag{4.24}$$

$$\begin{aligned} i\partial_\mu\big(\gamma^\mu\Psi - \Psi(\gamma^\mu)^T\big) + m\cos(\alpha-\beta)\big[\Psi, e^{-i(\alpha+\beta)\gamma^5}\big] \\ +im\sin(\alpha-\beta)\big\{\widetilde{\Psi}, \gamma^5 e^{-i(\alpha+\beta)\gamma^5}\big\} = 0 \ , \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} i\partial_\mu\big(\gamma^\mu\widetilde{\Psi} - \widetilde{\Psi}(\gamma^\mu)^T\big) + m\cos(\alpha-\beta)\big[\widetilde{\Psi}, e^{-i(\alpha+\beta)\gamma^5}\big] \\ +im\sin(\alpha-\beta)\big\{\Psi, \gamma^5 e^{-i(\alpha+\beta)\gamma^5}\big\} = 0 \ . \end{aligned} \tag{4.26}$$

Then, substituting the matrix expansions of the multispinors and exploiting the linear independence of the basis matrices, we obtain our constraints. Among these are the requirement that the fields ρ and B_μ vanish and $\theta = 0$ unless we enforce $\theta = \pi/2 - \alpha - \beta$. With this condition, we define the rotated fields

$$\begin{bmatrix} A'_\mu \\ iB'_\mu \end{bmatrix} = \begin{bmatrix} \cos(\alpha-\beta) & \sin(\alpha-\beta) \\ -\sin(\alpha-\beta) & \cos(\alpha-\beta) \end{bmatrix} \begin{bmatrix} A_\mu \\ iB_\mu \end{bmatrix} \tag{4.27}$$

and summarize the set of constraint equations as

$$\begin{aligned} F_{\sigma\tau}\cos(\alpha+\beta) + \frac{1}{2}\partial^\sigma F^{\mu\nu}\epsilon_{\mu\nu\sigma\tau}\sin(\alpha+\beta) - (\partial_\sigma A'_\tau - \partial_\tau A'_\sigma) &= 0 \ , \\ \partial_\sigma B'_\tau - \partial_\tau B'_\sigma &= 0 \ , \\ \partial^\sigma F_{\sigma\tau} + m^2 A'_\tau \cos(\alpha+\beta) &= 0 \ , \\ \frac{1}{2}\partial^\sigma F^{\mu\nu}\epsilon_{\mu\nu\sigma\tau} + m^2 A'_\tau \sin(\alpha+\beta) &= 0 \ , \\ \partial^\mu A'_\mu + i\rho\sin(2\alpha-2\beta) &= 0 \ , \\ \partial^\mu B'_\mu + \rho\cos(2\alpha-2\beta) &= 0 \ , \\ \partial_\sigma\rho - m^2 B'_\sigma &= 0 \ . \end{aligned} \tag{4.28}$$

Taking the divergence of the first of these constraints and eliminating explicit dependence on $F_{\mu\nu}$ and B'_μ, we find the basic equations governing the fields reduce to three coupled equations of A'_μ and ρ:

$$\begin{aligned} \partial^\nu\left(\partial_\nu A'_\mu - \partial_\mu A'_\nu\right) + m^2 A'_\mu &= 0 \ , \\ \left[\partial^\mu\partial_\mu + m^2\cos(2\alpha-2\beta)\right]\rho &= 0 \ , \\ \partial^\mu A'_\mu + i\rho\sin(2\alpha-2\beta) &= 0 \ . \end{aligned} \tag{4.29}$$

It is evident that the vanishing of ρ is equivalent to the reduction of these equations to the Proca equation [Proca (1936)].

A more symmetric form of these equations is obtained by the definition of the constant $\kappa \equiv \sqrt{m^2\cos 2(\alpha-\beta) - \mu^2}$ (where μ appears as a free parameter), and the scalar field φ defined in terms of the divergence of the vector field $\varphi \equiv \kappa^{-1}\partial^\mu A'_\mu$. The constraint equations then reduce to

$$\begin{aligned} \left[\partial^\nu \partial_\nu + m^2\right] A'_\mu &= +\kappa \partial_\mu \varphi \ , \\ \left[\partial^\nu \partial_\nu + \mu^2\right] \varphi &= -\kappa \partial^\mu A'_\mu \ , \end{aligned} \tag{4.30}$$

where the first equation is the Proca-like equation and the second is the auxiliary equation for φ. Note that m and μ are generally not equal and correspond to the masses of the vector and scalar fields, respectively. In the Lorentz gauge $\partial^\mu A'_\mu = 0$, the scalar wavefunction $\varphi = 0$ and the Proca-like equation becomes the Proca equation [Proca (1936)]. The coupling between the spin-1 and spin-0 massive fields is a new phenomenon whose physical implications are now discussed.

4.4 Physical implications

The chiral Bargmann-Wigner (CBW) equations for spin-1 massive fields and the Proca-like equation with its required auxiliary equation for spin-0 massive fields lead to several physical implications.

The degrees of freedom introduced by the choice of chiral basis allowed from Poincaré invariance admit an asymmetry to the defined representations of the considered spin-1 massive fields described by the multispinors, thereby allowing for generalization of the BW equations, which admit internal chiral degrees of freedom. As a result, two CBW equations are obtained, one for Ψ and the other for $\widetilde{\Psi}$, and they are reduced to the original BW equations when $\widetilde{\Psi} = 0$, which is equivalent to the case when chirality is neglected. Thus, the main physical implication of the CBW equations is the presence of the additional field described by the multispinor $\widetilde{\Psi}$.

To explore this additional field in detail, we specified the chiral basis so that the CBW equations reduce to the Proca-like equation coupled to a spin-0 massive field.

We demonstrated that the asymmetry of the defined representations manifests itself as the coupled scalar and vector fields, with the total spin consistent with our choice of representations. By committing to a specific chiral basis, one fixes the coupling between these fields and restricts the system to the same total number of degrees of freedom as when the chiral bases coincide. The coupling is described by the coefficient κ that depends on the chiral angles α and β as well as on masses m and μ of the scalar and vector fields, respectively.

The coupling between the scalar and vector fields caused by the presence of internal chiral freedoms is a new phenomenon. While the fundamental spin-1 massive fields describing bosons $W^{\pm}$ and Z^0 are well-known in the Standard Model (SM) [Paschos (2007)], the only experimentally verified fundamental scalar field is the Higgs field [Paschos (2007), Aad et al. (2013)]. Experimental searches of this field have produced strong evidence in favor of an elementary, massive, neutral spin-0 particle of positive parity and [Aad et al. (2013), Chatrchyan et al. (2012)]. For this to be identifiable with the proposed field ρ necessitates an additional mechanism for the produce the self-coupling potential necessitated for the spontaneous breaking of the Electroweak model. The source and implications of the Higgs potential remain an active area of investigation [Sher (1989), Elias-Miro et al. (2012)]. It therefore remains plausible that such a mechanism is obtainable to reconcile ρ with the observed Higgs and so is worth entertaining the implications of ρ or φ as the Higgs boson. The most immediate necessary consequence of these is the coupled nature of ρ and φ to a vector boson which completes the spin representation of the field. Such a coupling has been shown to necessarily arise proportional to the divergence of the vector field, with κ being the proportionality coefficient representing chirality, this implies that the Higgs field would then be related to the divergence of the vector wavefunction and that chirality of a spin-1 massive elementary particle plays a dominant role in its representation. Notice that for values of $\alpha - \beta \approx \pi/4$ the mass of this vector field may be quite large while the mass of ρ is taken to the observed Higgs value.

Another, more plausible consequence of the existence of ρ is as a weakly-coupled massive scalar field constituting the currently unexplained dark matter (DM) [Freeman and McNamara (2006)]. The physical properties requisite of DM fields are well-explained by a spin-0 massive elementary particle [Freeman and McNamara (2006), Sugita et al. (2008)], whose existence has not yet been verified experimentally [Barbier et al. (2005), Ackermann et al. (2011), Ibarra et al. (2013)]. The presence of such a DM field and its possible coupling to spin-1 massive fields of ordinary matter (OM) through chirality would allow both OM and DM to be coupled. Moreover, as recently shown, in the nonrelativistic limit, DM may have both scalar [Musielak (2021)] and vector [Adshead and Lozanov (2021)] components that could be coupled by chirality. Further studies of these interesting phenomena are necessary, and they will be described elsewhere.

Finally, let us point out that the form of the dependence of the coupling κ on the chiral angles α and β illustrates a critical result of our derivation that we postulate in all physical systems, namely that in the absence of a mechanism to fix the chiral basis of the constituent fields, it is only *differences* in chiral bases which are experimentally observable.

4.5 Summary

We have demonstrated how the Chiral-Bargmann Wigner equations may be derived from massive Dirac bi-spinor representations whose chiral bases are not aligned. By carefully considering the necessary physical field requirements (symmetry, Lorentz invariance, etc.), we have further shown how it is possible to derive a system of equations for the composite representation of spin-1 fields whose form generally differs from that of the well-known Proca equation. These equations notably contain an interesting coupling to a massive scalar field with potential implications for dark matter searches.

CHAPTER 5

CHIRAL SYMMETRY BREAKING IN AND BEYOND THE STANDARD MODEL

5.1 Introduction

Thus far, all considerations have been made regarding chiral representations of *free* fields. While of fundamental interest, such fields are largely contrived as the limiting cases of more complicated, interacting systems of particles. For this reason, it is helpful to carry forward our investigations into the physical effects of chirality within the context of an interacting field theory.

As shall be shortly demonstrated, the mechanism which specifies the relative chiral phases (e.g., determines the differences of chiral basis) in the standard model is explicated in the theory of electroweak interactions. In the following sections, a mechanism for the spontaneous symmetry breaking of the $U(1)_Y$ weak hypercharge symmetry by left-chiral Majorana neutrinos is introduced and discussed. The implications are derived for effects arising from this specific means of $U(1)_Y$ spontaneous symmetry breaking with some derived results necessarily mechanism-agnostic. Finally, calculations are presented in which it is shown that nontrivial, observable differences from Standard Model (SM) neutrino scattering cross sections arise from the proposed mechanism. The viability of the model is then discussed.

5.2 Chirality in Electroweak

In order to understand how the chiral bases of fields are specified in electroweak theory we begin with spontaneous symmetry breaking. In standard electroweak theory the $SU(2) \times U(1)$ symmetry is spontaneously broken when the Higgs doublet Φ takes on a non-zero expectation value at the minimum of an effective potential[1]. In general we may expand Φ about Φ_0 at each space-time point as

$$\Phi_{min}(x) = \frac{1}{\sqrt{2}} \mathrm{e}^{i\alpha(x)/v} \begin{bmatrix} 0 \\ v + h(x) \end{bmatrix}, \tag{5.1}$$

where v is the Higgs VEV, h is the Higgs field and α is a real-valued Nambu-Goldstone boson (NGB), transforming linearly under weak hypercharge rotations

$$\alpha \xrightarrow{\mathrm{U}(1)_Y} \alpha + \frac{v\beta}{2}.$$

With α included in the definition of the Higgs field, the mass terms of the fermions after spontaneous symmetry breaking (SSB) become

$$\mathcal{L}_m = -m_1 \bar{\psi}_1 \mathrm{e}^{-i\alpha\gamma^5/v} \psi_1 - m_2 \bar{\psi}_2 \mathrm{e}^{+i\alpha\gamma^5/v} \psi_2$$

We have therefore identified the parameter which determines the chiral basis of particles in the standard model, the NGB α. Yet, in most circumstances α is omitted from the resulting symmetry-broken Lagrangian by means of a redefinition of the Z boson. To see how this works we fully expand the Higgs portion of the Lagrangian after SSB and include α. We find

$$\begin{aligned} \mathcal{L}_h = &\frac{1}{2}\partial_\mu h \, \partial^\mu h + \frac{1}{2} M_H^2 h^2 + \frac{M_H^2}{8v^2} h^3 (h + 4v) \\ &+ \frac{(v+h)^2}{v^2} \left[\frac{1}{2} M_Z^2 \left(Z_\mu + \frac{1}{M_Z}\partial_\mu \alpha \right) \left(Z^\mu + \frac{1}{M_Z}\partial^\mu \alpha \right) + M_W^2 W_\mu^+ W^{-\mu} \right] \end{aligned}$$

Thus by taking $Z_\mu \to Z'_\mu = Z_\mu - (1/M_Z)\partial_\mu \alpha$, the kinetic contribution of α disappears, and we are free to choose the $\mathrm{U}(1)_Y$ gauge in which $\alpha = 0$. Of course, this choice is only possible if the weak hypercharge symmetry is conserved. The question of the physical significance of the chiral basis is therefore intimately connected to the weak hypercharge symmetry of the standard model.

[1]See Chapter 2 for a review of electroweak theory

5.3 Weak Hypercharge: Noether Currents

Both before and after spontaneous symmetry breaking the electroweak Lagrangian exhibits a global $\mathrm{U}(1)_Y$ weak hypercharge symmetry under the transformation given by

$$\Psi_L \to \exp\left[\frac{i}{2}y_L\beta\right]\Psi_L \qquad \psi_{iR} \to \exp\left[\frac{i}{2}y_{iR}\beta\right]\psi_{iR} \qquad \Phi \to \exp\left[\frac{i}{2}y_\phi\beta\right]\Phi.$$

The conserved $\mathrm{U}(1)_Y$ Noether current is then

$$j_Y^\mu = \frac{y_L}{2}\bar{\Psi}_L\gamma^\mu\Psi_L + \frac{y_{1R}}{2}\psi_{1R}\gamma^\mu\psi_{1R} + \frac{y_{2R}}{2}\psi_{2R}\gamma^\mu\psi_{2R} + \frac{iy_\phi}{2}\left(\Phi^\dagger(\partial_\mu\Phi) - (\partial_\mu\Phi)^\dagger\Phi\right)$$

Now, because the complete form of symmetry currents survive the process of spontaneous symmetry breaking, we may expand Φ about the minimum of the effective potential as in [Eq. 5.1] and so find the conserved current after spontaneous symmetry breaking to be

$$j_Y^\mu = \frac{y_L}{2}\bar{\Psi}_L\gamma^\mu\Psi_L + \frac{y_{1R}}{2}\psi_{1R}\gamma^\mu\psi_{1R} + \frac{y_{2R}}{2}\psi_{2R}\gamma^\mu\psi_{2R} - \frac{y_\phi v}{2}\partial^\mu\alpha.$$

We may decompose this into the electromagnetic and purely left-chiral parts by using the relationship

$$q_1 = \frac{1}{2} + \frac{y_L}{2} = \frac{y_{1R}}{2} \qquad q_2 = -\frac{1}{2} + \frac{y_L}{2} = \frac{y_{2R}}{2} \tag{5.2}$$

then, since $y_\phi = 1$ we have

$$\begin{aligned} j_{EM}^\mu &= q_1(\bar{\psi}_{1L}\gamma^\mu\psi_{1L} + \bar{\psi}_{1R}\gamma^\mu\psi_{1R}) + q_2(\bar{\psi}_{2L}\gamma^\mu\psi_{2L} + \bar{\psi}_{2R}\gamma^\mu\psi_{2R}) \\ j_{\Delta L}^\mu &= -\frac{1}{2}\bar{\psi}_{1L}\gamma^\mu\psi_{1L} + \frac{1}{2}\bar{\psi}_{2L}\gamma^\mu\psi_{2L}. \end{aligned}$$

And so we may write the Noether current as

$$j_Y^\mu = j_{EM}^\mu + j_{\Delta L}^\mu - \frac{v}{2}\partial^\mu\alpha.$$

The direct proof of the conservation of this current is carried out explicitly by calculating the divergences directly from the Lagrangian. We may write the relevant portion of $\mathcal{L}_{EW}$ as

$$\Delta\mathcal{L}_{EW} = \frac{1}{2}\partial_\mu\alpha\partial^\mu\alpha + \bar{\psi}_1\left[i\gamma^\mu\partial_\mu - m_1 \mathrm{e}^{-\frac{i\alpha}{v}\gamma^5}\right]\psi_1 + \bar{\psi}_2\left[i\gamma^\mu\partial_\mu - m_2 \mathrm{e}^{+\frac{i\alpha}{v}\gamma^5}\right]\psi_2.$$

We find $\partial_\mu j^\mu_{EM} = 0$ identically. For the remaining terms, upon variation of $\mathcal{L}_{EW}$ we obtain

$$\begin{aligned}
\partial_\mu(\bar{\psi}_{1L}\gamma^\mu\psi_{1L}) &= \frac{i\lambda_1 v}{\sqrt{2}}\left(\mathrm{e}^{\frac{i\alpha}{v}}\bar{\psi}_{1R}\psi_{1L} - \mathrm{e}^{-\frac{i\alpha}{v}}\bar{\psi}_{1L}\psi_{1R}\right)\\
\partial_\mu(\bar{\psi}_{2L}\gamma^\mu\psi_{2L}) &= \frac{i\lambda_2 v}{\sqrt{2}}\left(\mathrm{e}^{-\frac{i\alpha}{v}}\bar{\psi}_{2R}\psi_{2L} - \mathrm{e}^{\frac{i\alpha}{v}}\bar{\psi}_{2L}\psi_{2R}\right)\\
\partial^\mu\partial_\mu\alpha &= -\frac{i\lambda_1}{\sqrt{2}}\left(\mathrm{e}^{\frac{i\alpha}{v}}\bar{\psi}_{1R}\psi_{1L} - \mathrm{e}^{-\frac{i\alpha}{v}}\bar{\psi}_{1L}\psi_{1R}\right)\\
&\quad + \frac{i\lambda_2}{\sqrt{2}}\left(\mathrm{e}^{-\frac{i\alpha}{v}}\bar{\psi}_{2R}\psi_{2L} - \mathrm{e}^{\frac{i\alpha}{v}}\bar{\psi}_{2L}\psi_{2R}\right).
\end{aligned}$$

And therefore

$$\partial^\mu\partial_\mu\alpha = \frac{2}{v}\partial_\mu j^\mu_{\Delta L},$$

which is precisely the condition for the divergence of j^μ_Y to vanish. Note the crucial step in this conservation was the cancellation of $\partial_\mu\partial^\mu$ by the sum of the divergence of the left-chiral fermion currents, $\partial_\mu(\bar{\psi}_{1L}\gamma^\mu\psi_{1L})$. In turn, crucial to this cancellation is the coupling of the fermions to the Higgs. Consequently, any (tree-level) mass term for weakly interacting fermions must have their origins in couplings to the Higgs VEV or else violate $\mathrm{U}(1)_Y$. In the next section we consider a type of mass term which explicitly violates this symmetry: left-chiral Majorana fermions.

5.4 Left-Chiral Majorana Neutrinos Effects on $\mathrm{U}(1)_Y$

A now well-established body of evidence points towards neutrinos having non-zero masses [Aker et al. (2022)]. From a SM perspective, the smallness of these masses injects a naturalness problem into the usual fermion mass generation mechanism[2]. One means of reconciling this lightness of neutrinos is to posit the Majorana nature of neutrino masses. In this model a single chiral species is coupled to its charge conjugate. Such a Majorana mass term may be written as

$$\Delta\mathcal{L}_M = \frac{im}{2}(\chi_L^T\sigma^2\chi_L - \chi_L^\dagger\sigma^2\chi_L^*)$$

for the two-component left-chiral spinor χ_L. It should be observed that $\Delta\mathcal{L}_M$ does not exhibit U(1) symmetry, i.e. is *not* invariant under $\chi \to \chi' = \mathrm{e}^{i\theta}\chi$ for $\theta \in \mathbb{R}$. As a result, tree level terms which take the form of $\Delta\mathcal{L}_M$ are forbidden within the standard model as they explicitly violate weak hypercharge [Ramond et al. (1999)].

We now consider the possibility that the electroweak model is somehow spontaneously broken by the presence of non-zero neutrino masses. At this point we wish to consider a model with only left-chiral neutrinos and which have tree-level masses generated exclusively from Majorana self-couplings. In the Majorana description it is more convenient to express our fields in the form of two-component chiral spinors. We write this new Lagrangian after spontaneous symmetry breaking as

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\partial_\mu\alpha\partial^\mu\alpha + i\chi_{1L}^\dagger\bar{\sigma}^\mu\partial_\mu\chi_{1L} + i\chi_{2L}^\dagger\bar{\sigma}^\mu\partial_\mu\chi_{2L} + i\chi_{2R}^\dagger\sigma^\mu\partial_\mu\chi_{2R}\\ &- m_2\left(\mathrm{e}^{\frac{i\alpha}{v}}\chi_{2L}^\dagger\chi_{2R} + \mathrm{e}^{-\frac{i\alpha}{v}}\chi_{2R}^\dagger\chi_{2L}\right) + \frac{im_1}{2}\left(\chi_{1L}^T\sigma^2\chi_{1L} - \chi_{1L}^\dagger\sigma^2\chi_{1L}^*\right)\end{aligned}$$

Because this tree-level Majorana mass term does not couple to the Higgs, it does not couple to α directly. As a result, defining our transformation exactly as before,

[2]To illustrate the problem one need only compare the mass ratios from a single generation of quarks and leptons: Current measurements give $(m_u/m_e) = 4.2^{+1.0}_{-0.5}$ and $(m_d/m_e) = 9.14^{+0.9}_{-0.33}$ while $(m_\nu/m_e) \approx 10^{-6}$ [Particle Data Group (2020)].

we find a nonzero divergence of j_Y^μ which follows from the non-vanishing divergence of $j_{\Delta L}^\mu$. For our one-generation model, this may be written as

$$\partial_\mu j_Y^\mu = \frac{1}{2}\partial_\mu \left(\chi_{1L}^\dagger \bar{\sigma}^\mu \chi_{1L}\right) = -\frac{m_1}{2}\left(\chi_{1L}^T \sigma^2 \chi_{1L} + \chi_{1L}^\dagger \sigma^2 \chi_{1L}^*\right).$$

We find that the weak hypercharge symmetry violation is proportional to the sum over these terms for left-chiral Majorana fermions.

5.5 Chiral Implications of Spontaneously Broken $\mathrm{U}(1)_Y$

The growing number of anomalous results from neutrino experiments suggest new physics and make a strong case for fully exploring all viable theoretical models capable of predicting modifications to observable processes currently accessible by experiments. Particularly, a theory of left-chiral Majorana neutrino masses arising from some hitherto unknown spontaneous symmetry breaking mechanism is a possibility whose implications should be thoroughly and intensively considered. Understanding the resulting phenomenology will validate or nullify its viability as a physical theory. As shall be demonstrated, the presence of left-chiral Majorana masses introduces subtle but non-zero effects.

We begin by defining the (formally real) pseudoscalar and scalar field operators for the left-chiral Weyl neutrino field χ

$$S = +\frac{1}{2}\left(\chi^\dagger \chi_c + \chi_c^\dagger \chi\right) = \mathrm{Re}\left[\chi^\dagger \chi_c\right]$$
$$P = -\frac{i}{2}\left(\chi^\dagger \chi_c - \chi_c^\dagger \chi\right) = \mathrm{Im}\left[\chi^\dagger \chi_c\right]$$

We omit the subscript "L" for brevity, but it should be borne in mind that this is a purely left-chiral neutrino. Now, because $\chi^\dagger \chi_c$ is a complex scalar, we may write it as

$$\chi^\dagger \chi_c = S + iP = N\mathrm{e}^{i\varphi}$$

for

$$N = \sqrt{P^2 + S^2}, \qquad \varphi = \tan^{-1}\left(\frac{P}{S}\right).$$

As has been stated above, any Lagrangian containing the quadratic self-term $\chi^\dagger\chi_c$ will explicitly violate the $\mathrm{U}(1)_Y$ weak hypercharge symmetry of electroweak theory which requires the action remain unchanged for $\chi \to \exp(-i\beta/2)\chi$. Under this transformation we find P and S transform like

$$\begin{aligned} S &\xrightarrow{\mathrm{U}(1)_Y} S\cos\beta + P\sin\beta \\ P &\xrightarrow{\mathrm{U}(1)_Y} P\cos\beta - S\sin\beta, \end{aligned}$$

which is as expected for the components of a complex number. As a result of these transformation properties it is easy to see that N is a conserved quantity under $\mathrm{U}(1)_Y$. However, the same is not true of the phase which transforms as $\varphi \to \varphi - \beta$. Comparing this to the transformation rule of the NGB α we find that at any point α and φ are linearly related. Therefore we have

$$\alpha = \alpha_0 - \frac{v}{2}\varphi = \alpha_0 - \frac{v}{2}\tan^{-1}\left(\frac{P}{S}\right) \tag{5.3}$$

for a constant α_0. This expresssion is important as it relates the NGB α directly to the neutrino field through S and P. Now, consider the $\mathrm{U}(1)_Y$-invariant Lagrangian

$$\mathcal{L} = i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + \mu\sqrt{P^2 + S^2} - \frac{\lambda}{2}\left(P^2 + S^2\right) \tag{5.4}$$

for constant terms μ and λ (not to be confused with the analgous terms in the Higgs potential). The effective potential of this Lagrangian is minimized when the neutrino field satisfies $\sqrt{P^2 + S^2} = \mu/\lambda$. Within the regime for which perturbations may be constrained to the minima, we may therefore break the symmetry and define the effective field χ' such that the real and imaginary parts of $(\chi'^{\dagger}\chi'_c)$ can be written as $S' = S$ and $P' = P + \mu/\lambda$. The Lagrangian written in terms of χ' becomes

$$\mathcal{L} = i\chi'^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\chi' + \mu\ P' - \frac{\lambda}{2}\left(P'^2 + S'^2\right) + \mu\sqrt{S'^2 + \left(P' - \frac{\mu}{\lambda}\right)^2} - \frac{\mu^2}{2\lambda}$$

with $\mu P'$ now appearing as a Majorana mass term. Thus, by the redefinition of our field χ about the effective self-coupling minima the Majorana neutrino has acquired mass at the expense of the spontaneous breaking of the weak hypercharge symmetry. In fact, the breaking of this symmetry has implications for more than just the neutrino field. The value of the chiral phase α (that is, the hitherto unconstrained NGB field) is now determined by the local values of the field operators P' and S'. Furthermore, from the full electroweak theory above we find the chiral couplings between the field α and the charged leptons may be given in the form

$$-\mathcal{L}_{int} = m_f\bar{\psi}\mathrm{e}^{\mp i\alpha\gamma^5/v}\psi$$

where m_f is the mass of the charged fermion. Since we have an expression for α in terms of the neutrino field χ (not χ' whose transformation properties may generally differ), we may substitute and upon reducing find (up to a phase)

$$-\mathcal{L}_{int} = \frac{m_f}{\left(P^2 + S^2\right)^{1/4}}\bar{\psi}\left(\mathrm{Re}\left[e^{\pm i\alpha_0/v}\sqrt{S + iP}\right] \pm i\mathrm{Im}\left[e^{\pm i\alpha_0/v}\sqrt{S + iP}\right]\gamma^5\right)\psi. \quad (5.5)$$

Notice that this expression is a necessary consequence of the breaking of the $\mathrm{U}(1)_Y$ symmetry and the local fixing of the field α by the neutrino field χ. As such, Eq.(5.5) remains valid regardless of the posited mass mechanism provided it is symmetry-violating. For our specific case[3], we have expanded about $\sqrt{P^2 + S^2} = \mu/\lambda$. There-

[3]To achieve generality in the resulting calculations one has simply to replace μ/λ with the desired vacuum expectation.

fore, substituting this into $\mathcal{L}_{int}$ and expanding to first order in the constant λ (or, equivalently, to first order in S) yields

$$-\mathcal{L}_{int} = \pm i m_f \bar{\psi}\gamma^5\psi + \frac{m_f\lambda}{2\mu}\left(\bar{\psi}S\psi\right).$$

where we have absorbed constant factors into our definition of α_0 and absorbed it into the chiral basis of the field ψ. We know from previous considerations that the first term is a standard chiral mass and will contribute a simple pole to the fermion propagator and may be omitted from our definition of the interaction terms. The second term is more interesting as it introduces a quartic scalar-scalar point-like coupling between the charged fermions and the neutrino. We may put this in a somewhat more standard form by taking

$$P = -\frac{i}{2}\bar{\nu}\gamma^5\nu \qquad \text{and} \qquad S = \frac{1}{2}\bar{\nu}\nu$$

for the Majorana bi-spinor field

$$\nu = \begin{bmatrix} \chi \\ \chi_c \end{bmatrix}.$$

We may then write the lowest-order, non-trival term as

$$\mathcal{L}_{int} = -\frac{m_f\lambda}{4\mu}\left[\bar{\nu}\nu\right]\left[\bar{\psi}\psi\right]. \tag{5.6}$$

We see immediately that the coupling constant for this interaction depends inversely on the neutrino mass parameter μ, linearly on the fermion mass m_f and linearly in the parameter λ which we may take to govern the strength of the coupling. It is clear that any searches for this quartic interaction will either strengthen or invalidate the details of the above theory. Moreover, searches for new neutrino couplings of the type given in Eq. (5.5) will serve as a probe of standard model symmetry violations. In the next section we will discuss the observable effects on neutrino scattering.

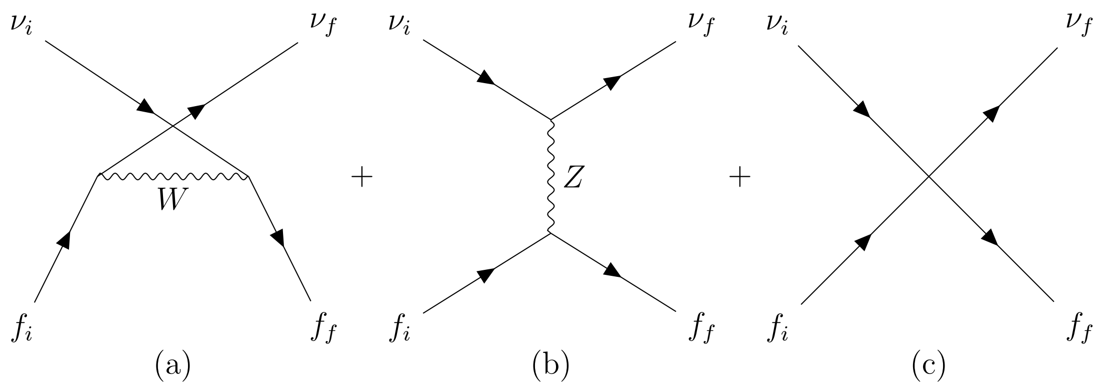

Figure 5.1. Contributions to $\nu f \to \nu f$ scattering amplitude for the model presented in the text. In addition to the two SM charged current (a) and neutral current (b) processes, a new four-fermion vertex (c) is postulated arising from spontaneously broken $\mathrm{U}(1)_Y$.

5.6 Modifications To Neutrino Scattering

As a consequence of taking the appearance of the neutrino mass to spontaneously break the $\mathrm{U}(1)_Y$ gauge symmetry of the electroweak theory, we have found the appearance of additional point-like couplings between charged lepton fields and left-chiral Majorana neutrinos. To further investigate the physical significance of this term, it is helpful to consider the specific case of neutrino elastic and quasi-elastic scattering. The tree-level Feynman diagrams for this process in our theory are shown in Fig. 5.1. To lowest order, the new contribution to the matrix element is

$$\mathcal{M}_\lambda = -\frac{i m_f \lambda}{4\mu} \left[\bar{\nu}\nu\right] \left[\bar{\psi}\psi\right] .$$

The contributions from the neutral current $\mathcal{M}_{NC}$ and charged current $\mathcal{M}_{CC}$ are well-known [Halzen and Martin (1984), Paschos (2007)]. These may be combined in the low-energy effective limit as a single term given by

$$\mathcal{M}_{SM} = -\frac{i G_F}{\sqrt{2}} \left[\bar{\nu}\gamma^\mu \left(1-\gamma^5\right)\nu\right] \left[\bar{\psi}\gamma_\mu \left(g_V - g_A\gamma^5\right)\psi\right] ,$$

where G_F is the Fermi constant related directly to the Higgs vev by $v^{-2} = G_F\sqrt{2}$ and g_A a g_V are the effective axial and vector couplings prescribed in the electroweak theory and their values depend upon the specific flavor states involved in the reaction. Squaring and performing the corresponding traces for the standard model amplitude yields the differential cross section in terms of the energy of the outgoing neutrino E'_f [Paschos (2007), Fukugita and Yanagida (2013)] which we write as

$$\frac{d\sigma_{SM}}{dE'_f} = \frac{G_F^2 m_f}{2\pi}\Bigg[(g_V + g_A)^2 + (g_V - g_A)^2\left(\frac{E'_\nu}{E_\nu}\right)^2 \tag{5.7}$$

$$+\left(g_V^2 - g_A^2\right)\frac{m_f}{E_\nu}\left(\frac{E'_\nu - E_\nu}{E_\nu}\right)\Bigg] \tag{5.8}$$

for E_ν the energy of the incident neutrino, and E'_ν the energy of the scattered neutrino. Next, to calculate our modification to this cross section, we add $\mathcal{M}_\lambda$ and $\mathcal{M}_{SM}$ coherently. In squaring we find that the cross terms $\mathcal{M}_{SM}\mathcal{M}^*_\lambda$ and $\mathcal{M}^*_{SM}\mathcal{M}_\lambda$ vanish identically when averaged over spins from the trace of an odd number of γ matrices. The resulting spin-averaged differential cross section then separates into a sum over the cross sections for the individual amplitudes. The quartic contribution from diagram (c) in Fig. (5.1) enters as

$$\frac{d\sigma_\lambda}{dE'_f} = \frac{G_F^2 \kappa_f^2 m_f}{2\pi}\left[\left(\frac{E'_\nu - E_\nu}{E_\nu}\right)^2 + \frac{5m_f}{4E_\nu}\left(\frac{E'_\nu - E_\nu}{E_\nu}\right)\right], \tag{5.9}$$

where we have defined the unitless parameter

$$\kappa_f \equiv \frac{\lambda m_f}{2\sqrt{2}G_F\mu}.$$

We see that, taking the average $\langle E'_\nu\rangle \approx E_\nu/2$ this becomes

$$\frac{d\sigma_\lambda}{dE'_f} = \frac{G_F^2 \kappa_f^2 m_f}{2\pi}\left[\frac{1}{4} + \frac{5m_f}{8E_\nu}\right]$$

which gives an overall finite increase which becomes more pronounced at smaller E_ν values due to the inverse dependence on E_ν, though this is suppressed by the mass m_f.

5.7 Mixing and Flavor Violation

By implicitly taking the interaction and mass bases to coincide, we have effectively avoided a discussion of mixing. A full treatment of this aspect requires modification of Eq.(5.4). One simple extension involves a simple tripling of the above Lagrangian, but the proliferation of unknown variables makes this solution unappealing. Another possible modification follows from $\mathrm{U}(1)_Y$ invariance allowing for the conservation of the properties of S and P is given by taking

$$\begin{aligned} P &= -\frac{im_i^\nu}{2\mu}\delta_{ij}\bar{\nu}_i^M\gamma^5\nu_j^M = -\frac{i}{2\mu}\left[m_k^\nu U_{nk}^* U_{km}\right]\bar{\nu}_n^I\gamma^5\nu_m^I \\ S &= \frac{m_i^\nu}{2\mu}\delta_{ij}\bar{\nu}_i^M\nu_j^M = \frac{1}{2\mu}\left[m_k^\nu U_{nk}^* U_{km}\right]\bar{\nu}_n^I\nu_m^I \end{aligned}$$

where the sum spans the three generations of neutrinos, m_i^ν are the neutrino masses and U_{nk} are the elements of the PMNS matrix [Pontecorvo (1958), Maki et al. (1962)] which accounts for flavor mixing of neutrinos. Note that these now amount to field-strength normalization conditions which set the allowable scale of perturbations about the minima.

In addition to mixing among the neutrinos, the mass bases of the charged fermions are defined by the diagonalization of the Yukawa couplings. In the interaction basis, the Yukawa matrix need not be diagonal for any species of fermions [see Akhmedov (2007)]. The modified form of Eq. (5.6) which accounts for mixing is then

$$\begin{aligned} \mathcal{L}_{int} &= -\frac{\lambda m_{\nu i} m_k}{4\mu^2}\delta_{ij}\delta_{kl}\left[\bar{\nu}_i^M\nu_j^M\right]\left[\bar{\psi}_k^M\psi_l^M\right] + h.c. \\ &= -\frac{v\lambda}{4\mu^2\sqrt{2}} m_{\nu n} U_{in}^* U_{nj} Y_{kl}\left[\bar{\nu}_i^I\nu_j^I\right]\left[\bar{\psi}_k^I\psi_l^I\right] + h.c., \end{aligned} \tag{5.10}$$

where the Y_{ij} are the Yukawa couplings in the interaction basis. It is clear that as a result of this interaction being diagonal in the mass basis it *cannot* be diagonal in the interaction basis (this is a general property of Eq.(5.5) when mixing is accounted for). Therefore, the form of Eq. (5.10) will permit flavor-violating reactions such

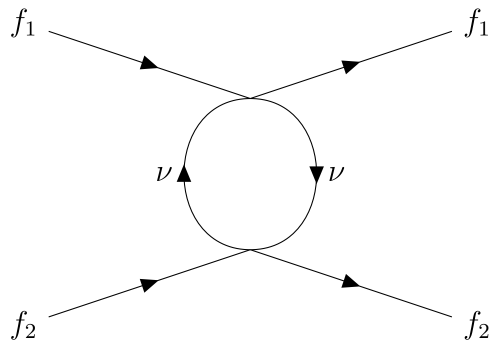

Figure 5.2. An example of one type of graph present at higher orders in perturbation theory. This term will contribute to the interactions of all charged fermions but may be expected to be overwhelmed by electromagnetic effects. However, nonzero contributions may arise when considering long-range interactions of neutral composite objects.

as $\nu_\mu + e^- \to \nu_\mu + \mu^-$. This is of particular significance to charged-current cross section experiments at relatively small ΔE_ν. If correct, Eq. (5.10) will contribute a background to charged current events in beam-line experiments with a differential cross section proportional to the kinematic terms in Eq. (5.9) with modifications to the couplings from coherent sums over the neutrino flavor content.

5.8 Discussion

Given recent anomalies in the neutrino sector [Athanassopoulos et al. (1997), Aguilar-Arevalo et al. (2018), Kostensalo et al. (2019), Giunti et al. (2022)], a strong case may be made for the full exploration of any testable, predictive, and physically well-motivated theory which consistently explains the origin of neutrino masses. We take this to be such a theory. However, this is not to say the above model is without challenges. Strict experimental limits exist which constrain the allowable values of our model parameters. Neutrino scattering has been mentioned as the most direct means of search for this process, but it is not the only experimental avenue sensitive to

$\mathcal{L}_{int}$. The proposed interaction necessarily contributes tree-level scattering diagrams which include radiated photons in the final state and thus would appear as a false-background in searches for the reaction $e^+ + e^- \to \nu + \bar{\nu} + \gamma$ which serves to constrain the number of interacting neutrinos in the standard model [Gaemers et al. (1979)]. The rate of background for the proposed interaction is first order in $\alpha_{EM}\lambda m_e/\mu$ (where for the purpose of establishing an order of magnitude estimate we have assumed no mixing). The leading order process at energies near the Z resonance enter in as $\alpha_{EM}G_F$. So, from observed rates [Acciarri et al. (1998), Janot and Jadach (2020)], we obtain the very approximate constraint $\lambda \ll G_F(1\text{eV})/m_e$, where 1eV estimates the neutrino mass scale. Additional physical searches which impose constraints on this process include orthoposotronium to invisible searches [Vigo et al. (2018)] with order of magnitude constraints similar to those from neutrino production limits above. Of course, with all of these the experimental considerations, an investigation into the renormalizability and radiative diagrams of the theory will yield more precise constraints and predictions.

In addition to experimental results which constrain the parameter space, several purely theoretical concerns exist. First, we have not addressed the issues of the larger symmetry group of the electroweak theory. Second, while many of the results of this chapter are independent of the exact SSB mechanism which generates the mass of the neutrinos, depending only on the VEV of the operators $P^2 + S^2$, careful considerations of the scale of the potential must be made to constrain the regime in which the expansion about the effective minima remains valid. Additionally a justification of the (admittedly derivative) potential found in Eq. (5.4) is wanting. Third, the form of the interactions in Eq. (5.5) and Eq. (5.6) admit higher order terms that should exist in perturbation theory (c.f. Fig. 5.2) which are observable in principle and may serve to facilitate an increase in long-range forces between neutral bodies (an analogous exchange is found in Feinberg and Sucher (1968) and recently investigated in Segarra and Bernabéu (2020)). Fourth and finally, our interaction has not been rigorously renormalized. Since severely unconstrained UV divergences famously plague the generic four-fermion vertex, this could be a source of challenges for the mechanism. However, as an effective model, an escape may exist via an appeal to the return to an unbroken $\mathrm{U}(1)_Y$ group at high energies.

Before concluding this section it is useful to place the proposed mechanism in context within the wider literature. Many theories in which SSB mechanisms adjacent to the Higgs give rise to masses for Majorana neutrinos have been explored in the literature. In many of these [e.g., Chikashige et al. (1981), Ma et al. (2017), Pisano and Sharma (1998) Latosinski et al. (2010),Gelmini and Roncadelli (1981)] the spontaneous breaking of lepton number or the Peccei–Quinn symmetry [Peccei and Quinn (1977)] is taken to generate a scalar field which couples to the neutrino fields to provide the Majorana mass terms. Our model differs from these in several respects. First, we take the mechanism which breaks the $\mathrm{U}(1)_Y$ symmetry to arise from the Majorana field itself. As a result, the degrees of freedom from the chiral NGB in the Standard Model become absorbed into the definition of the neutrino field itself. We are thereby absolved of the need for additional scalar fields. Second, through the mechanism of chiral fixing the Majorana neutrino fields induce characteristic couplings to the charged fermions which serve as a distinct signature of the interaction.

Finally, (gratuitously) we would be remiss to omit an interesting observation made in passing during the above analysis. Using the Fierz rearrangement theorems (found in, for example Okun (2013)) it is possible to write Eq. (5.6) as

$$\mathcal{L}_{int} = -\frac{m_f \lambda}{4\mu}(\bar{\nu}\sigma_{\mu\nu}\nu)(\bar{\psi}\sigma^{\mu\nu}) + V(\psi, \nu)$$

where the terms contained in $V(\psi, \nu)$ are of equal order and include vector, pseudovector, scalar, and pseudoscalar couplings. In our opinion, the form of this term is suggestive: a tensorial coupling proportional to the fermion mass. It is conceivable that such an interaction has implications for quantum theories of gravitation, though, at the moment, this claim remains speculative.

5.9 Addendum: Chirality in Quantum Chromodynamics

The above constitutes an entirely consistent development of the implications associated with a method by which the chiral basis of the standard model is fixed by SSB arising from left-chiral Majorana neutrinos. There are, however, additional means by which chiral symmetries are broken in the standard model. This section details the chiral symmetry breaking in Quantum Chromodynamics (QCD). Our reasons for presenting this are two-fold: First, to emphasize the differences between this symmetry breaking and that introduced in this chapter. Second, to understand how the QCD model fails to be satisfactory for our purposes.

In QCD–under certain conditions–pions obtain a non-zero expectation value from quark-antiquark pairs arising from the vacuum. This is of importance in consideration of the broken (approximate) chiral symmetry $SU(2)_L \times SU(2)_R$[4]. One finds the spontaneously broken symmetry induces a divergence of the three axial components of the Noether currents associated with the linear combinations of the symmetry currents from generators of $SU(2)_A$. The axial currents may be written as [Peskin and Schroeder (2018)]

$$j^{\mu 5a}(x) = \bar{Q}\gamma^{\mu}\gamma^{5}\tau^{a}Q$$

[4]In some formulations of the chiral symmetry in QCD, the strange quark is considered "light". The symmetry group then becomes $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$.

for the quark doublets Q and axial generators $\gamma^5\tau^a$. By identifying the pions as the NGBs of the theory, Lorentz invariance dictates that the amplitude of the axial current operating between the pion field and the vacuum can only be proportional to the four-momentum of the pion [Weinberg (1995a)]. Therefore, the divergence of the current is found to be [Peskin and Schroeder (2018)] Kaku (1993)].

$$\left\langle 0\middle|\partial_\mu j^{\mu 5a}\left(0\right)\middle|\pi^b\left(p\right)\right\rangle = -m_\pi^2 f_\pi \delta^{ab}$$

with the indices a and b running over the three pions and the three axial currents. The scale of the pion's mass is therefore directly proportional to the scale of validity of the symmetry. In the limit where the pions are massless, the symmetry is conserved. Notice that, as pseudoscalar mesons, the odd-parity of the pions makes this identification possible only for the *axial* currents [Weinberg (1995a).

We take the differences between the symmetries considered here and elsewhere in this chapter as self-evident. However, taken as an analogy, there are many parallels. For example, our previous result found the divergence of the weak hypercharge to be proportional to the sum over the neutrino masses and may indicate that $U(1)_Y$ is an approximate symmetry of the standard model, valid in the limit that the neutrino masses vanish. If true, $U(1)_Y$ may be conserved to an extremely high degree but still violated. The largest stumbling block to this interpretation is the existence of the Z boson. Taken holistically, the successes of the electroweak model clash against any theory in which the Z boson is not fundamental. This is the primary reason such an approximate $U(1)_Y$ symmetry was not considered in this chapter.

5.10 Summary and Future Work

In this chapter, we have brought together the postulated physical significance of the chiral basis with the electroweak theory of the standard model. Then by identifying a left-chiral Majorana neutrino field with the NGB of the electroweak symmetry, which specifies the absolute chiral basis, we were able to introduce a plausible mechanism by which the remaining $\mathrm{U}(1)_Y$ symmetry is subsequently broken, and the neutrino acquires mass. The resulting physical implications, including new couplings between neutrinos and charged fermions, were investigated, calculated, and discussed. Possible experimental verification was proposed through the measurement of neutrino scattering cross sections. Lastly, differences are observed between the proposed theory and the chiral symmetry breaking of QCD.

Future investigations into this topic must address those points raised in the discussion above. Additionally, a comprehensive phenomenological development of the underlying interactions remains crucial to assessing the validity of the proposed physical theory.

CHAPTER 6

CHIRALITY IN CURVED SPACE-TIME

6.1 Background

As described by Parker and Toms (2009), Wald (1994), Birrell and Davies (1984), and Fulling et al. (1989) in their monographs on QFT in curved space-time, as well as in the review papers by Benini et al. (2013), 2010, typical extensions of QFT from flat to to curved space-time are done using the methods already developed in flat space-time, with two major differences. First, no reference to the plane-wave basis is possible, and calculations must employ 'operator-valued distributions' [Wald (1994)]. Additionally, no preferred choice of Hilbert space exists, which indicates that no uniquely defined vacuum exists in the theory (e.g., Fulling et al. (1989)). An important requirement is that solutions to the QFT field equations have the same structure as in flat space-time, which is ensured by considering the so-called global hyperbolicity (e.g., Fulling et al. (1989), Wald (1994)). In general, the curved space-times are fixed in that no dynamics are of concern in constructing the QFT; however, see Wald (1994) for a semi-classical treatment of the backreaction problem.

All previous formulations of QFT in curved space-time fail at the level of identifying the proper physical vacuum. In this chapter, we posit that one contributing factor to these failings is the lack of emphasis placed on specifying the local chiral vacuum structure. We present the rationale for this conclusion and outline the relationship between chirality and the Bogolubov transformations.

6.2 The Vacuum Problem

Quantum field theory predicts that the vacuum is not empty but filled with virtual particles constantly being created and annihilated. Experimental confirmation of this picture is provided by the Casimir effect [Casimir (1948)]. The "standard" vacuum of QFT is the infinite Minkowski vacuum, defined as the state in which the expectation value of the operator describing the energy, momentum, and stresses of fields is zero. This definition allows the Minkowski vacuum to be uniquely defined and any theory of fields based on this definition to be self-consistent [Boi (2011), Milonni (2013)].

In the General Theory of Relativity, just as the metric is only locally Minkowski, the same principle holds for the vacuum. Only in the limit that the space is considered flat may the vacuum structure be taken as reducing to that of Minkowski QFT. As a result, the local vacuum properties vary from point to point with the curvature of space-time. Thus when considering the coupling of quantum fields to curvature, vacuum fluctuations pass a level of non-linearity into the definition of the vacuum. The geometry of the space determines the ground state, but the vacuum fluctuations modify this grounds state by contributing to the definition of the geometry and so on. A mathematical description of the quantum vacuum that effectively embodies this idea was given initially by Schwinger (1951).

Let $T^{\mu\nu}$ be a stress tensor of any combination of fields, including the gravitational field. Then, the vacuum-to-vacuum amplitude is given by

$$\langle \mathrm{out}, \mathrm{vac}|T^{\mu\nu}|\mathrm{in}, \mathrm{vac}\rangle = -i\frac{\delta}{\delta g_{\mu\nu}}\langle \mathrm{out}, \mathrm{vac}|\mathrm{in}, \mathrm{vac}\rangle$$

where the external field $g_{\mu\nu}$ serves as an arbitrary zero point for quantum fluctuations of the gravitational field. For QFTs in Minkowksi space-time, the expectation value of $T^{\mu\nu}$ is zero identically (otherwise this would not fit the definition of the vacuum given above), and the observed vacuum fluctuations arise from $\langle (T^{\mu\nu})^2 \rangle \neq 0$. The situation is very different in curved space-time, where curvature induces $\langle T^{\mu\nu} \rangle \neq 0$ and the expectation value $\langle (T^{\mu\nu})^2 \rangle$ is the source of gravitational vacuum polarization. Moreover, the topology of space-time plays an important role in gravitational vacuum polarization. This brief description shows that the concept of the quantum vacuum in QFT in curved space-time is not well-defined because there exist ambiguous 'vacuum states' appearing in the background. The ambiguity of vacuum states results in different numbers of particles, which prevents unique identification of the vacuum in such theories. This is known as the vacuum problem in curved space-time and so far its solution has remain elusive (Parker and Toms (2009), Wald (1994), Birrell and Davies (1984), and Fulling et al. (1989)). An investigation into the application of generalized chirality introduced in this dissertation in solving the vacuum problem in curved space-time is now explored.

6.3 Bogolubov Transformation

Quantum states with different numbers of particles appear naturally in QFT in curved space-time, and the problem becomes especially important in quantum theories dealing with the creation of particles in curved space-time, such as Hawking's theory of the production of particles near the event horizon of a Schwarzschild black hole [Hawking (1975)] and Parker's theory of creation of particles in early stages of the Universe, shortly after the Big Bang [Parker (1968), (1969), (1971)]. Let us now briefly describe Hawking's theory, with specific emphasis on its vacuum problem.

Hawking (1975) solved the Klein-Gordon equation in the Schwarzschild metric for massless particles, and used its solutions to determine a flux of particles emerging from the the black hole and reaching an external observer located far away from the black hole's event horizon. Following [Parker and Toms (2009)], the solutions in the entire space-time can be written as

$$\phi = \int (a_\omega f_\omega + a_\omega^\dagger f_\omega^*)\ d\omega, \tag{6.1}$$

where f_ω and f_ω^* are a complete set of solutions of the Klein-Gordon equation. In this Heisenberg picture, the operators a_ω and $a_\omega^\dagger$ are time-independent. These solutions are not unique as another set of solutions exists (p_ω and p_ω^*) and correspond to the field expansion in terms of positive frequencies, which is a well-defined set for the distant observer. Since this set of solutions does not apply directly to the event horizon, there is another set, q_ω and q_ω^*, which can be made valid at the event horizon. The ambiguity of solutions is caused by the vacuum problem in curved space-time [Fulling et al. (1989)]. Using these expansions, the solutions in the entire space-time can be written as [Parker and Toms (2009)]

$$\phi = \int (b_\omega p_\omega + b_\omega^\dagger p_\omega^* + c_\omega q_\omega + c_\omega^\dagger q_\omega^*)\ d\omega, \tag{6.2}$$

where b_ω are annihilation operators for particles reaching at late times the distant observer.

Thus, there are two sets of solutions given by Eqs (6.1) and (6.2), and corresponding to two different Fock spaces. It is the Bogolubov transformation that relates these solutions

$$p_\omega = \int (\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f_{\omega'}^*)\ d\omega', \tag{6.3}$$

where $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ are the Bogolubov coefficients given by $\alpha_{\omega\omega'} = (f_{\omega'}, p_\omega)$ and $\beta_{\omega\omega'} = -(f_{\omega'}^*, p_\omega)$. Now, a wave packet is formed from p_ω in a frequency range around ω and it reaches the distant observer; for details, see [Hawking (1975)] or [Parker and Toms (2009)].

6.4 Chirality and Vacuum Structure

We may understand the effects of a chiral rotation on the vacuum by considering a field before and after the rotation in the following way.

Let ψ be a quantized Dirac field solving the Dirac equation in flat space-time given by

$$\psi = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \sum_{s=1,2} a^s_{\mathbf{p}} u^s(p)\, \mathrm{e}^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} v^s(p)\, \mathrm{e}^{+ip\cdot x}$$

for the creation and annihilation operators $a_{s,p}$ and $b^\dagger_{p,s}$ satisfying the anticommutation relations $\{a^r_{\mathbf{p}}, a^{s\dagger}_{\mathbf{q}}\} = \{b^r_{\mathbf{p}}, b^{s\dagger}_{\mathbf{q}}\} = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs}$, with all others vanishing. In this construction all energies are positive, and so we may ostensibly interpret $a^{s\dagger}_{\mathbf{q}}$ fermions and $b^{s\dagger}_{\mathbf{q}}$ as creating anti-fermions. In both cases the particles are created with positive energies and with momentum $\mathbf{p}$ [Peskin and Schroeder (2018)].

Now, it is simple to show that if ψ solves the Dirac equation then $\psi' = \mathrm{e}^{i(\alpha/2)\gamma^5}\psi$ solves the corresponding chiral equation

$$\left(i\gamma^\mu\partial_\mu - m\mathrm{e}^{-i\alpha\gamma^5}\right)\psi' = 0.$$

We find that when an appropriate phase is chosen, our spinors $u^s(p)$ and $v^s(p)$ mix under chiral rotations as

$$\mathrm{e}^{i(\alpha/2)\gamma^5} u^s(p) = \cos\frac{\alpha}{2} u^s(p) + i\sin\frac{\alpha}{2} v^{3-s}(p)$$
$$\mathrm{e}^{i(\alpha/2)\gamma^5} v^s(p) = \cos\frac{\alpha}{2} v^s(p) + i\sin\frac{\alpha}{2} u^{3-s}(p)$$

The resulting solution may therefore be written in the original spinor basis as

$$\psi' = \mathrm{e}^{i(\alpha/2)\gamma^5}\psi = \cos\frac{\alpha}{2}\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \sum_{s=1,2}\left(a^s_{\mathbf{p}} u^s(p)\,\mathrm{e}^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} v^s(p)\,\mathrm{e}^{+ip\cdot x}\right)$$
$$+ i\sin\frac{\alpha}{2}\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \sum_{s=1,2}\left(a^{3-s}_{\mathbf{p}} v^s(p)\,\mathrm{e}^{-ip\cdot x} + b^{(3-s)\dagger}_{\mathbf{p}} u^s(p)\,\mathrm{e}^{+ip\cdot x}\right).$$

The first term is clearly just $\cos(\alpha/2)\psi$, the second term is in want of an interpretation. The spinors in it appear attached to the "wrong" exponential terms. We define the second term as $\widetilde{\psi}$ and write

$$\psi' = \cos\frac{\alpha}{2}\psi + i\sin\frac{\alpha}{2}\widetilde{\psi}.$$

Using the known properties of ψ and ψ' we find upon substitution of this form into the CDE that $\widetilde{\psi}$ satisfies the equation

$$\left(i\gamma^\mu\partial_\mu - m\mathrm{e}^{-i\alpha\gamma^5}\right)\widetilde{\psi} = im\cot\frac{\alpha}{2}\left(1-\mathrm{e}^{-i\alpha\gamma^5}\right)\psi$$

This suggests that ψ and $\widetilde{\psi}$ are not wholly independent. Also, in the limit $\alpha \to 0$, the right-hand side of this equation becomes formally infinite. Yet, operating from the left with $i\gamma^\mu\partial_\mu + m\mathrm{e}^{i\alpha\gamma^5}$ we find the right hand side goes identically to zero and the components of $\widetilde{\psi}$ satisfy the KG relation, $(\partial_\mu\partial^\mu - m^2)\widetilde{\psi} = 0$, independently of α. This suggests the field expansion $\widetilde{\psi}$ has some physical validity independently of ψ and α. The solution is to interpret these quanta as CPT-conjugated fields–negative energy states moving oppositely the direction implied by their creation operators.

The above highlights the unusual role chirality plays as a symmetry within the standard model. Unlike the unitary transformations that underlie the fundamental interactions' gauge fields, the choice of chiral basis provides inequivalent physical descriptions of the physical energies and hence the vacuum. To illustrate this point, consider two observers who choose different chiral bases in formulating a quantum field theory. These are not related by any unitary transformation and are not directly equatable. Therefore, what one takes to be valid positive energy states, the other necessarily views as mixtures of negative energy states and vice-versa[1]. Thus, a chiral

[1]One possibility which escapes this conclusion is if the system in question is invariant under a global chiral rotation. The validity of such a transformation in curved space is unlikely, though it remains an open question in curved space-time [Deser et al. (1980), Parker and Toms (2009)].

rotation is a vacuum transformation and has direct analogies with the Bogolubov transformations presented above.

6.5 Summary and Future Work

For the reasons described above, a full investigation into the chiral vacuum structure of curved space-time based on an axiomatic quantum field theory [Hollands and Wald (2010)] will likely yield additional insights into the vacuum problem. The mixing of negative modes through chiral means is a new observation indicative of a more significant role for chirality in the formulation QFT in curved space-time than has previously been appreciated [in, for example, Parker and Toms (2009)]. A full enunciation of the concept of chirality via the vierbein (or tetrad) formalism [Penrose and Rindler (1984a), Penrose and Rindler (1984b)] is a necessary next step in these investigations, with the goal being a complete covariant definition of chirality.

CHAPTER 7

CONCLUDING REMARKS

In the course of this document an attempt has been made to explore as comprehensively as possible the issues arising from aspects of chriality as they exist at the most fundamental levels of physics. In doing so several conclusions were ultimately reached regarding the subject. First, chiral freedoms are a necessary an inherent characteristic of any theory of fermions in which the nature of parity remains unconstrained [Watson and Musielak (2020) and Watson and Musielak (2021b)]. Second, these degrees of freedom have representational importance and must be considered when developing mathematical descriptions of free fields due to the underlying chiral representations having the ability to modify the physical properties of observed representations vis. a vis. the manifestation of auxiliary fields [Watson and Musielak (2021a)]. And third, the concept of chirality and chiral bases may be given physical significance through the spontaneous breaking of weak hypercharge symmetries within the standard model. The apparent violation, taken as proportional to the masses of the neutrino fields, gives rise to new couplings as a necessary consequence of the identification of the chiral bases dependence on neutrino field configurations. Finally, observations were made regarding the interpretation of second-quantized chiral-rotated fields. The negative energy modes which are manifest after rotation were shown to have significance to the vacuum problem in curved space-time and brief outline of determining the significance in curved spaces was presented.

APPENDIX A

Appendix

A.1 Notations and Conventions

Here we summarize our notational and mathematical conventions used throughout this dissertation:

Upper indexed vectors (v^μ) denote contravariant vectors which transform like the coordinate differentials dx^μ. Lower indexed vectors (v_μ) denote covariant vectors which transform like the gradient ∂_μ.

The Einstein summation convention is employed throughout: repeated Latin indices spanning $1, 2, 3$ are always summed over when repeated. Greek indices spanning $0, 1, 2, 3$ are summed over only when repeated in lower and upper positions.

The Minkowski metric is denoted by $\eta^{\mu\nu}$ and the metric signature is taken to be "mostly negative", e.g. $\eta^{00} = +1$ and $\eta^{11} = \eta^{22} = \eta^{33} = -1$ with all non-diagonal elements zero. Metrics differing from flat spacetimes are given by the more general $g^{\mu\nu}$.

The Pauli matrices are written contravariantly as σ^i and occasionally it will be useful to include among these the 2-by-2 identity which we denote as σ^0. We may then summarize the matrices as

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

A.2 Acronyms

An attempt is made to avoid confusion, but for convenience an incomplete list of some of the more frequently employed acronyms found in the text is given here:

BW = Bargmann-Wigner

CBW = Chiral Bargmann-Wigner

CDE = Chiral Dirac Equation

CPT = Charge, Parity, and Time

DE	=	Dirac Equation
DM	=	Dark Matter
EW	=	Electroweak
NGB	=	Nambu-Goldstone Boson
OM	=	Ordinary Matter
QCD	=	Quantum Chromodynamics
QED	=	Quantum Electrodynamics
QFT	=	Quantum Field Theory
SSB	=	Spontaneous Symmetry Breaking
SM	=	Standard Model (of Particle Physics)
VEV	=	Vacuum Expectation Value

APPENDIX B

Appendix

B.1 Raw Computations: Bargmann-Wigner Equations

Starting from the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0$$

We have proven (we suppose it is well known) that this equation remains Poincaré invariant under the chiral transformation which changes the field's discrete properties:

$$\gamma^\mu \to \gamma^\mu \qquad \psi \to e^{i\alpha\gamma^5}\psi$$

In which case we obtain:

$$\left(i\gamma^\mu \partial_\mu - m e^{-2i\alpha\gamma^5}\right)\psi' = 0$$

This is to say, the free Dirac equation exhibits a Poincaré-invariant chiral degree of freedom. Note the distinct difference between the above transformation and any unitary transformation of the form:

$$\gamma^\mu \to U\gamma^\mu U^\dagger \qquad \psi \to \psi' = U\psi$$

Such that yields:

$$\left(i\left[U\gamma^\mu U^\dagger\right]\partial_\mu - m\right)\psi' = 0$$

A chiral transformation is emphatically not a unitary spin transformation. The chiral degree of freedom for the free Dirac field is a necessary consequence of the two constituent fields contained in the Dirac bi-spinors. We may "tear apart" a free-Dirac spinor and obtain the constituent spinors (in the chiral basis):

$$\psi = \begin{bmatrix} \chi_L \\ \chi_R \end{bmatrix}$$

We now wish to solve the general equation

$$\left(i\gamma^\mu \partial_\mu - m e^{-2i\alpha\gamma^5}\right)\psi = 0$$

Moving to the standard basis, in the rest frame this equation takes the form

$$0 = \begin{bmatrix} E - m\cos 2\alpha & im\sin 2\alpha \\ im\sin 2\alpha & -E - m\cos 2\alpha \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}$$

Where $E = \pm m$. So:

$$0 = (\pm m - m\cos 2\alpha)\,\chi_1 + (im\sin 2\alpha)\,\chi_2$$

$$0 = (im\sin 2\alpha)\,\chi_1 + (\mp m - m\cos 2\alpha)\,\chi_2$$

So, for positive energies:

$$\psi^{(+)} = \begin{bmatrix} \cos\alpha\chi \\ i\sin\alpha\chi \end{bmatrix}$$

And, for negative energies:

$$\psi^{(+)} = \begin{bmatrix} i\sin\alpha\chi \\ \cos\alpha\chi \end{bmatrix}$$

This yields the simplest orthogonal solutions given by $\chi_\uparrow = \{1,0\}$ and $\chi_\downarrow = \{0,1\}$:

$$\omega^{(1)}(\alpha) = \begin{bmatrix} \cos\alpha \\ 0 \\ i\sin\alpha \\ 0 \end{bmatrix},\ \omega^{(2)}(\alpha) = \begin{bmatrix} 0 \\ \cos\alpha \\ 0 \\ i\sin\alpha \end{bmatrix},\ \omega^{(3)}(\alpha) = \begin{bmatrix} i\sin\alpha \\ 0 \\ \cos\alpha \\ 0 \end{bmatrix},\ \omega^{(4)}(\alpha) = \begin{bmatrix} 0 \\ i\sin\alpha \\ 0 \\ \cos\alpha \end{bmatrix}$$

Which satisfy:

$$\left(i\gamma^\mu\partial_\mu - me^{-2i\alpha\gamma^5}\right)\omega^{(+)}(\alpha)\,e^{imt} = 0$$

$$\left(i\gamma^\mu\partial_\mu - me^{-2i\alpha\gamma^5}\right)\omega^{(-)}(\alpha)\,e^{-imt} = 0$$

for "$+$" $\in \{1,2\}$ and "$-$" $\in \{3,4\}$ These equations are equivalent to:

$$\left(\gamma^0 - e^{-2i\alpha\gamma^5}\right)\omega^{(+)}(\alpha) = 0$$

$$\left(\gamma^0 + e^{-2i\alpha\gamma^5}\right)\omega^{(-)}(\alpha) = 0$$

The chiral angle connects these solutions (up to a normalization factor)

$$\omega^{(3)}(\alpha) = -i\omega^{(1)}\left(\alpha + \frac{\pi}{2}\right) = \gamma^5\omega^{(1)}(\alpha)$$
$$\omega^{(4)}(\alpha) = -i\omega^{(2)}\left(\alpha + \frac{\pi}{2}\right) = \gamma^5\omega^{(2)}(\alpha)$$

Note that since $e^{i\alpha\gamma^5}$ commutes with the spin operators we find that $\omega^{(\pm)}$ define particle states with definite spin and mass. This means we may apply the scheme of Bargmann and Wigner in constructing higher order equations. Note, however, that there is no need to sample strictly from spinors of equivalent chiral bases. Taking two sets of spinors $\omega^{(i)}(\alpha)$ and $\omega^{(i)}(\beta)$, we may define the neither symmetric or anti-symmetric at-rest multispinors:

$$\omega^{(1,1)}(\alpha,\beta) = \omega^{(1)}(\alpha) \otimes \omega^{(1)}(\beta)$$
$$\omega^{(1,2)}(\alpha,\beta) = \omega^{(2,1)}(\alpha,\beta) = \omega^{(1)}(\alpha) \otimes \omega^{(2)}(\beta) + \omega^{(2)}(\alpha) \otimes \omega^{(1)}(\beta)$$
$$\omega^{(2,2)}(\alpha,\beta) = \omega^{(2)}(\alpha) \otimes \omega^{(2)}(\beta)$$

These multispinors satisfy the following equations

$$\left(\gamma^\mu p_\mu - me^{-2i\alpha\gamma^5}\right)^{\mu}_{\mu'} \omega^{(+,+)}_{\mu\nu}(\alpha,\beta;p) = 0$$
$$\left(\gamma^\mu p_\mu - me^{-2i\beta\gamma^5}\right)^{\nu}_{\nu'} \omega^{(+,+)}_{\mu\nu}(\alpha,\beta;p) = 0$$

We then take the symmetric and anti-symmetric parts of these multispinors:

$$\Omega^{(+,+)}_{\mu\nu}(\alpha,\beta;p) = \frac{1}{2}\left(\omega^{(+,+)}_{\mu\nu}(\alpha,\beta;p) + \omega^{(+,+)}_{\nu\mu}(\alpha,\beta;p)\right)$$
$$\widetilde{\Omega}^{(+,+)}_{\mu\nu}(\alpha,\beta;p) = \frac{1}{2}\left(\omega^{(+,+)}_{\mu\nu}(\alpha,\beta;p) - \omega^{(+,+)}_{\nu\mu}(\alpha,\beta;p)\right)$$

These spinors must satisfy the coupled equations (suppressing the functional dependence for brevity):

$$\left[\gamma^\mu p_\mu - \frac{m}{2}\left(e^{-2i\alpha\gamma^5} + e^{-2i\beta\gamma^5}\right)\right]^{\mu}_{\mu'} \Omega^{(+,+)}_{\mu\nu} = \left[\frac{m}{2}\left(e^{-2i\alpha\gamma^5} - e^{-2i\beta\gamma^5}\right)\right]^{\mu}_{\mu'} \widetilde{\Omega}^{(+,+)}_{\mu\nu}$$
$$\left[\gamma^\mu p_\mu - \frac{m}{2}\left(e^{-2i\alpha\gamma^5} + e^{-2i\beta\gamma^5}\right)\right]^{\mu}_{\mu'} \widetilde{\Omega}^{(+,+)}_{\mu\nu} = \left[\frac{m}{2}\left(e^{-2i\alpha\gamma^5} - e^{-2i\beta\gamma^5}\right)\right]^{\mu}_{\mu'} \Omega^{(+,+)}_{\mu\nu}$$

Or, noting:

$$\frac{1}{2}\left(e^{-2i\alpha\gamma^5}+e^{-2i\beta\gamma^5}\right)=\cos(\alpha-\beta)e^{-i(\alpha+\beta)\gamma^5}$$
$$\frac{1}{2}\left(e^{-2i\alpha\gamma^5}-e^{-2i\beta\gamma^5}\right)=-i\sin(\alpha-\beta)\gamma^5e^{-i(\alpha+\beta)\gamma^5}$$

We find these coupled equations may be written:

$$\left[\gamma^\mu p_\mu - m\cos(\alpha-\beta)e^{-i(\alpha+\beta)\gamma^5}\right]^\mu_{\mu'}\Omega^{(+,+)}_{\mu\nu}=\left[-im\sin(\alpha-\beta)\gamma^5e^{-i(\alpha+\beta)\gamma^5}\right]^\mu_{\mu'}\widetilde{\Omega}^{(+,+)}_{\mu\nu}$$
$$\left[\gamma^\mu p_\mu - m\cos(\alpha-\beta)e^{-i(\alpha+\beta)\gamma^5}\right]^\mu_{\mu'}\widetilde{\Omega}^{(+,+)}_{\mu\nu}=\left[-im\sin(\alpha-\beta)\gamma^5e^{-i(\alpha+\beta)\gamma^5}\right]^\mu_{\mu'}\Omega^{(+,+)}_{\mu\nu}$$

And thus, we find the positive energy solutions corresponding to coupled symmetric and antisymmetric wavefunctions existing

$$\psi^{(+,+)}_{\mu\nu}(x^\mu)=\sum_{(+,+)}\int C_1^{(+,+)}(p^\mu)\ \Omega^{(+,+)}_{\mu\nu}(\alpha,\beta;p^\mu)\,e^{-ip_\mu x^\mu}\ d^3p$$
$$\widetilde{\psi}^{(+,+)}_{\mu\nu}(x^\mu)=\sum_{(+,+)}\int C_2^{(+,+)}(p^\mu)\ \widetilde{\Omega}^{(+,+)}_{\mu\nu}(\alpha,\beta;p^\mu)\,e^{-ip_\mu x^\mu}\ d^3p$$

Where $(+,+)$ ranges over (1,1), (1,2), (2,2). Then these fields must satisfy the coupled equations :

$$\left[i\gamma^\mu\partial_\mu - m\cos(\alpha-\beta)e^{-i(\alpha+\beta)\gamma^5}\right]^\mu_{\mu'}\psi^{(+,+)}_{\mu\nu}=\left[-im\sin(\alpha-\beta)\gamma^5e^{-i(\alpha+\beta)\gamma^5}\right]^\mu_{\mu'}\widetilde{\psi}^{(+,+)}_{\mu\nu}$$
$$\left[i\gamma^\mu\partial_\mu - m\cos(\alpha-\beta)e^{-i(\alpha+\beta)\gamma^5}\right]^\mu_{\mu'}\widetilde{\psi}^{(+,+)}_{\mu\nu}=\left[-im\sin(\alpha-\beta)\gamma^5e^{-i(\alpha+\beta)\gamma^5}\right]^\mu_{\mu'}\psi^{(+,+)}_{\mu\nu}$$

Similar arguments hold for negative energy solutions. So, we may construct the most general symmetric and anti-symmetric solutions:

$$\Psi_{\mu\nu}(\alpha,\beta;x^\mu)=a_1\psi^{(+,+)}_{\mu\nu}(\alpha,\beta,x^\mu)+b_1\psi^{(-,-)}_{\mu\nu}(\alpha,\beta,x^\mu), \tag{B.1}$$

and

$$\widetilde{\Psi}_{\mu\nu}(\alpha,\beta;x^\mu)=a_2\widetilde{\psi}^{(+,+)}_{\mu\nu}(\alpha,\beta,x^\mu)+b_2\widetilde{\psi}^{(-.-)}_{\mu\nu}(\alpha,\beta,x^\mu) \tag{B.2}$$

Which must satisfy the coupled equations:

$$\left[i\gamma^{\sigma}\partial_{\sigma} - m\cos(\alpha-\beta)e^{-i(\alpha+\beta)\gamma^5}\right]^{\mu}_{\mu'}\Psi_{\mu\nu}(\alpha,\beta;x^{\mu}) = \left[-im\sin(\alpha-\beta)\gamma^5 e^{-i(\alpha+\beta)\gamma^5}\right]^{\mu}_{\mu'}\widetilde{\Psi}_{\mu\nu}(\alpha,\beta;x^{\mu})$$
$$\left[i\gamma^{\sigma}\partial_{\sigma} - m\cos(\alpha-\beta)e^{-i(\alpha+\beta)\gamma^5}\right]^{\mu}_{\mu'}\widetilde{\Psi}_{\mu\nu}(\alpha,\beta;x^{\mu}) = \left[-im\sin(\alpha-\beta)\gamma^5 e^{-i(\alpha+\beta)\gamma^5}\right]^{\mu}_{\mu'}\Psi_{\mu\nu}(\alpha,\beta;x^{\mu})$$

These equations can be easily separated to yield:

$$\left[\partial_{\mu}\partial^{\mu} + m^2\right]^{\mu}_{\mu'}\Psi_{\mu\nu}(\alpha,\beta;x^{\mu}) = 0$$

$$\left[\partial_{\mu}\partial^{\mu} + m^2\right]^{\mu}_{\mu'}\widetilde{\Psi}_{\mu\nu}(\alpha,\beta;x^{\mu}) = 0$$

Thus, each component of the multispinors satisfies the Klein-Gordon equation. Considering the matrix form of the above equations:

$$\left(i\gamma^{\mu}\partial_{\mu}\Psi - m\cos(\alpha-\beta)e^{-i(\alpha+\beta)\gamma^5}\Psi\right) = -im\sin(\alpha-\beta)\gamma^5 e^{-i(\alpha+\beta)\gamma^5}\widetilde{\Psi} \tag{B.3}$$

$$\left(i\partial_{\mu}\Psi\left(\gamma^{\mu}\right)^T - m\cos(\alpha-\beta)\Psi e^{-i(\alpha+\beta)\gamma^5}\right) = +im\sin(\alpha-\beta)\widetilde{\Psi}\gamma^5 e^{-i(\alpha+\beta)\gamma^5} \tag{B.4}$$

$$\left(i\gamma^{\sigma}\partial_{\sigma}\widetilde{\Psi} - m\cos(\alpha-\beta)e^{-i(\alpha+\beta)\gamma^5}\widetilde{\Psi}\right) = -im\sin(\alpha-\beta)\gamma^5 e^{-i(\alpha+\beta)\gamma^5}\Psi \tag{B.5}$$

$$\left(i\partial_{\mu}\widetilde{\Psi}\left(\gamma^{\mu}\right)^T - m\cos(\alpha-\beta)\widetilde{\Psi}e^{-i(\alpha+\beta)\gamma^5}\right) = +im\sin(\alpha-\beta)\Psi\gamma^5 e^{-i(\alpha+\beta)\gamma^5} \tag{B.6}$$

And taking:

$$\Psi = mA_{\sigma}\gamma^{\sigma}\hat{C} + \frac{1}{2}F_{\sigma\tau}\hat{\sigma}^{\sigma\tau}\hat{C}$$
$$\widetilde{\Psi} = \rho e^{-i\theta\gamma^5}\hat{C} + mB_{\sigma}\gamma^{\sigma}\gamma^5\hat{C}$$

Where $\hat{C} = i\gamma^2\gamma^0$ and $\hat{\sigma}^{\mu\nu} \equiv \frac{i}{2}[\gamma^{\mu},\ \gamma^{\nu}]$. Then, noting $\hat{C}\left(\gamma^{\mu}\right)^T = -\gamma^{\mu}\hat{C}$ along with the relations

$$\{\gamma^{\mu},\gamma^{\nu}\} = 2\eta^{\mu\nu}$$

$$[\gamma^{\mu},\gamma^{\nu}] = -2i\sigma^{\mu\nu}$$

$$[\gamma^{\mu},\hat{\sigma}^{\sigma\tau}] = 2i\left(\eta^{\mu\sigma}\gamma^{\tau} - \eta^{\mu\tau}\gamma^{\sigma}\right)$$

$$\{\gamma^{\mu},\hat{\sigma}^{\sigma\tau}\} = -2\varepsilon^{\mu\sigma\tau\rho}\gamma_{\rho}\gamma^5$$

$$F_{\sigma\tau}\gamma^5\hat{\sigma}^{\sigma\tau} = \frac{i}{2}F^{\mu\nu}\epsilon_{\mu\nu\sigma\tau}\hat{\sigma}^{\sigma\tau}$$

$$F_{\sigma\tau}e^{-i(\alpha+\beta)\gamma^5}\hat{\sigma}^{\sigma\tau} = F_{\sigma\tau}\hat{\sigma}^{\sigma\tau}\cos(\alpha+\beta) + \frac{1}{2}F^{\mu\nu}\hat{\sigma}^{\sigma\tau}\epsilon_{\mu\nu\sigma\tau}\sin(\alpha+\beta)$$

$$F_{\sigma\tau}\gamma^5 e^{-i(\alpha+\beta)\gamma^5}\hat{\sigma}^{\sigma\tau} = -iF_{\sigma\tau}\hat{\sigma}^{\sigma\tau}\sin(\alpha+\beta) + \frac{i}{2}F^{\mu\nu}\hat{\sigma}^{\sigma\tau}\epsilon_{\mu\nu\sigma\tau}\cos(\alpha+\beta)$$

$$\left[\Psi, e^{-i(\alpha+\beta)\gamma^5}\right] = -2im\sin(\alpha+\beta)A_\sigma\gamma^\sigma\gamma^5\hat{C}$$

$$\left\{\Psi\ , e^{-i(\alpha+\beta)\gamma^5}\right\} = +2m\cos(\alpha+\beta)A_\sigma\gamma^\sigma\hat{C} + F_{\sigma\tau}e^{-i(\alpha+\beta)\gamma^5}\hat{\sigma}^{\sigma\tau}\hat{C}$$

$$\left[\Psi, \gamma^5 e^{-i(\alpha+\beta)\gamma^5}\right] = +2m\cos(\alpha+\beta)A_\sigma\gamma^\sigma\gamma^5\hat{C}$$

$$\left\{\Psi, \gamma^5 e^{-i(\alpha+\beta)\gamma^5}\right\} = -2im\sin(\alpha+\beta)A_\sigma\gamma^\sigma\hat{C} + F_{\sigma\tau}\gamma^5 e^{-i(\alpha+\beta)\gamma^5}\hat{\sigma}^{\sigma\tau}\hat{C}$$

$$\left[\widetilde{\Psi}, e^{-i(\alpha+\beta)\gamma^5}\right] = -2im\sin(\alpha+\beta)B_\sigma\gamma^\sigma\hat{C}$$

$$\left\{\widetilde{\Psi}, e^{-i(\alpha+\beta)\gamma^5}\right\} = +2m\cos(\alpha+\beta)B_\sigma\gamma^\sigma\gamma^5\hat{C} + 2\rho e^{-i(\alpha+\beta+\theta)\gamma^5}\hat{C}$$

$$\left[\widetilde{\Psi}, \gamma^5 e^{-i(\alpha+\beta)\gamma^5}\right] = +2mB_\sigma\cos(\alpha+\beta)\gamma^\sigma\ \hat{C}$$

$$\left\{\widetilde{\Psi}, \gamma^5 e^{-i(\alpha+\beta)\gamma^5}\right\} = -2im\sin(\alpha+\beta)B_\sigma\gamma^\sigma\gamma^5\hat{C} - 2\rho e^{-i(\alpha+\beta+\theta)\gamma^5}\gamma^5\hat{C}$$

$$i\gamma^\mu\Psi + i\Psi\left(\gamma^\mu\right)^T = imA_\sigma\left[\gamma^\mu,\gamma^\sigma\right]\hat{C} + \frac{i}{2}F_{\sigma\tau}\left[\gamma^\mu,\hat{\sigma}^{\sigma\tau}\right]\hat{C}$$

$$i\gamma^\mu\Psi - i\Psi\left(\gamma^\mu\right)^T = imA_\sigma\left\{\gamma^\mu\gamma^\sigma\right\}\hat{C} + \frac{i}{2}F_{\sigma\tau}\left\{\gamma^\mu,\hat{\sigma}^{\sigma\tau}\right\}\hat{C}$$

$$i\gamma^\mu\widetilde{\Psi} + i\widetilde{\Psi}\left(\gamma^\mu\right)^T = imB_\sigma\left\{\gamma^\mu,\gamma^\sigma\right\}\gamma^5\hat{C} + 2\rho\sin\theta\gamma^\mu\gamma^5\hat{C}$$

$$i\gamma^\mu\widetilde{\Psi} - i\widetilde{\Psi}\left(\gamma^\mu\right)^T = imB_\sigma\left[\gamma^\mu,\gamma^\sigma\right]\gamma^5\hat{C} + 2i\rho\cos\theta\gamma^\mu\hat{C}$$

And

$$\partial_\mu A_\sigma \left[\gamma^\mu, \gamma^\sigma\right] = -i \left(\partial_\mu A_\sigma - \partial_\sigma A_\mu\right) \sigma^{\mu\sigma}$$
$$\partial_\mu A_\sigma \left\{\gamma^\mu, \gamma^\sigma\right\} = 2\partial^\mu A_\mu$$
$$\partial_\mu B_\sigma \left[\gamma^\mu, \gamma^\sigma\right] \gamma^5 = -i \left(\partial_\mu B_\sigma - \partial_\sigma B_\mu\right) \gamma^5 \sigma^{\mu\sigma}$$
$$\partial_\mu B_\sigma \left\{\gamma^\mu, \gamma^\sigma\right\} \gamma^5 = 2\partial^\mu B_\mu \gamma^5$$
$$\partial_\mu F_{\sigma\tau} \left[\gamma^\mu, \hat{\sigma}^{\sigma\tau}\right] = 4i\partial^\sigma F_{\sigma\tau} \gamma^\tau$$
$$\partial_\mu F_{\sigma\tau} \left\{\gamma^\mu, \hat{\sigma}^{\sigma\tau}\right\} = -2\partial^\mu F^{\sigma\tau} \gamma^\rho \gamma^5 \varepsilon_{\mu\sigma\tau\rho}$$

We find the following reduction the linearly independent combinations of the multi-spinor matrix equations.

Eq. (B.3) + Eq. (B.4):

$$i\partial_\mu \left(\gamma^\mu \Psi + \Psi \left(\gamma^\mu\right)^T\right) = m\cos\left(\alpha - \beta\right)\{\Psi, e^{-i(\alpha+\beta)\gamma^5}\} + im\sin\left(\alpha - \beta\right)[\widetilde{\Psi}, \gamma^5 e^{-i(\alpha+\beta)\gamma^5}]$$

Reduces to:

$$F_{\sigma\tau}\cos\left(\alpha + \beta\right) + \frac{1}{2}F^{\mu\nu}\epsilon_{\mu\nu\sigma\tau}\sin\left(\alpha + \beta\right) = \frac{1}{\cos\left(\alpha - \beta\right)}\left(\partial_\sigma A_\tau - \partial_\tau A_\sigma\right)$$
$$\partial^\tau F_{\tau\sigma} = -m^2\cos\left(\alpha + \beta\right)\left(A_\sigma\cos\left(\alpha - \beta\right) + iB_\sigma\sin\left(\alpha - \beta\right)\right)$$

Eq. (B.3) − Eq. (B.4):

$$i\partial_\mu \left(\Psi \left(\gamma^\mu\right)^T - \gamma^\mu \Psi\right) = m\cos\left(\alpha - \beta\right)[\Psi, e^{-i(\alpha+\beta)\gamma^5}] + im\sin\left(\alpha - \beta\right)\{\widetilde{\Psi}, \gamma^5 e^{-i(\alpha+\beta)\gamma^5}\}$$

Reduces to:

$$\partial^\mu A_\mu = -i\rho\sin\left(\alpha - \beta\right)\sin\left(\alpha + \beta + \theta\right)$$
$$\rho\sin\left(\alpha - \beta\right)\cos\left(\alpha + \beta + \theta\right) = 0$$
$$-\frac{1}{2}\partial^\mu F^{\sigma\tau}\varepsilon_{\mu\sigma\tau\rho} = m^2\sin\left(\alpha + \beta\right)\left(A_\rho\cos\left(\alpha - \beta\right) + iB_\rho\sin\left(\alpha - \beta\right)\right)$$

Eq. (B.5) + Eq. (B.6):

$$i\partial_\mu \left(\widetilde{\Psi}\left(\gamma^\mu\right)^T - \gamma^\mu \widetilde{\Psi}\right) = m\cos\left(\alpha-\beta\right)[\widetilde{\Psi}, e^{-i(\alpha+\beta)\gamma^5}] + im\sin\left(\alpha-\beta\right)\{\Psi, \gamma^5 e^{-i(\alpha+\beta)\gamma^5}\}$$

Reduces to:

$$\begin{aligned}\partial_\sigma\left(\rho\cos\theta\right) &= im^2\sin\left(\alpha+\beta\right)\left(A_\sigma\sin\left(\alpha-\beta\right) - iB_\sigma\cos\left(\alpha-\beta\right)\right)\\ F_{\sigma\tau}\sin\left(\alpha+\beta\right) - \frac{1}{2}F^{\mu\nu}\epsilon_{\mu\nu\sigma\tau}\cos\left(\alpha+\beta\right) &= -\frac{i}{2\sin\left(\alpha-\beta\right)}\left(\partial^\mu B^\nu - \partial^\mu B^\nu\right)\epsilon_{\mu\nu\sigma\tau}\end{aligned}$$

Eq. (B.5) − Eq. (B.6):

$$i\partial_\sigma\left(\gamma^\sigma\widetilde{\Psi} + \widetilde{\Psi}\left(\gamma^\sigma\right)^T\right) = m\cos\left(\alpha-\beta\right)\{\widetilde{\Psi}, e^{-i(\alpha+\beta)\gamma^5}\} + im\sin\left(\alpha-\beta\right)[\Psi, \gamma^5 e^{-i(\alpha+\beta)\gamma^5}]$$

Reduces to:

$$\begin{aligned}\partial^\mu B_\mu &= -\rho\cos\left(\alpha-\beta\right)\sin\left(\alpha+\beta+\theta\right)\\ \rho\ \cos\left(\alpha-\beta\right)\cos\left(\alpha+\beta+\theta\right) &= 0\\ \partial_\sigma\left(\rho\sin\theta\right) = im^2\cos\left(\alpha+\beta\right)&\left(A_\sigma\sin\left(\alpha-\beta\right) - iB_\sigma\cos\left(\alpha-\beta\right)\right)\end{aligned}$$

Then, by making use of the identity $\epsilon^{\rho\lambda\sigma\tau}\epsilon_{\mu\nu\sigma\tau} = -2\delta^\rho_\mu\delta^\lambda_\nu$ and defining two new fields A'_μ and B'_μ as a rotation of our original fields:

$$\begin{bmatrix} A'_\mu \\ iB'_\mu \end{bmatrix} = \begin{bmatrix} \cos\left(\alpha-\beta\right) & \sin\left(\alpha-\beta\right) \\ -\sin\left(\alpha-\beta\right) & \cos\left(\alpha-\beta\right) \end{bmatrix}\begin{bmatrix} A_\mu \\ iB_\mu \end{bmatrix}$$

we observe that $\rho\sin\left(\alpha-\beta\right)\cos\left(\alpha+\beta+\theta\right) = 0$ and $\rho\cos\left(\alpha-\beta\right)\cos\left(\alpha+\beta+\theta\right) = 0$ require either $\rho = 0$ or $\alpha+\beta+\theta = \frac{\pi}{2}$, we find taking $\rho = 0$ yields:

$$\begin{aligned}\partial^\sigma\left(\partial_\sigma A'_\tau - \partial_\tau A'_\sigma\right) + m^2 A'_\tau &= 0\\ B'_\tau &= 0\end{aligned}$$

While taking $\theta = \frac{\pi}{2} - (\alpha+\beta)$ reduces the set of equations to:

$$F_{\sigma\tau}\cos(\alpha+\beta)+\frac{1}{2}\partial^{\sigma}F^{\mu\nu}\epsilon_{\mu\nu\sigma\tau}\sin(\alpha+\beta)-(\partial_{\sigma}A'_{\tau}-\partial_{\tau}A'_{\sigma})=0$$

$$\partial_{\sigma}B'_{\tau}-\partial_{\tau}B'_{\sigma}=0$$

$$\partial^{\sigma}F_{\sigma\tau}+m^{2}A'_{\tau}\cos(\alpha+\beta)=0$$

$$\frac{1}{2}\partial^{\sigma}F^{\mu\nu}\epsilon_{\mu\nu\sigma\tau}+m^{2}A'_{\tau}\sin(\alpha+\beta)=0\ ,$$

$$\partial^{\mu}A'_{\mu}+i\rho\sin(2\alpha-2\beta)=0$$

$$\partial^{\mu}B'_{\mu}+\rho\cos(2\alpha-2\beta)=0$$

$$\partial_{\sigma}\rho-m^{2}B'_{\sigma}=0$$

Then, taking the divergence of the first term and eliminating explicit dependence on $F_{\mu\nu}$ using the divergence relations for it and the dual tensor given by:

$$\partial^{\sigma}F_{\sigma\tau}=-m^{2}A'_{\tau}\cos(\alpha+\beta)$$
$$\frac{1}{2}\partial^{\sigma}F^{\mu\nu}\epsilon_{\mu\nu\sigma\tau}=-m^{2}A'_{\tau}\sin(\alpha+\beta)$$

we obtain the set of constraints:

$$\partial^{\mu}\left(\partial_{\mu}A'_{\nu}-\partial_{\nu}A'_{\mu}\right)+m^{2}A'_{\nu}=0$$

$$\partial_{\mu}B'_{\nu}-\partial_{\nu}B'_{\mu}=0$$

$$\partial_{\sigma}\rho-m^{2}B'_{\sigma}=0$$

$$\partial^{\mu}A'_{\mu}+i\rho\sin(2\alpha-2\beta)=0$$

$$\partial^{\mu}B'_{\mu}+\rho\cos(2\alpha-2\beta)=0$$

Which, upon elimination of B'_{μ} reduce to

$$\partial^{\nu}\left(\partial_{\nu}A'_{\mu}-\partial_{\mu}A'_{\nu}\right)+m^{2}A'_{\mu}=0$$

$$\left(\partial^{\mu}\partial_{\mu}+m^{2}\cos(2\alpha-2\beta)\right)\rho=0$$

$$\partial^{\mu}A'_{\mu}+i\rho\sin(2\alpha-2\beta)=0$$

B.2 Raw Computations: Cross Sections

Given the matrix element as the coherent sum

$$\mathcal{M}_{SM} + \mathcal{M}_{\lambda} = -\frac{iG_F}{2\sqrt{2}} \left[\bar{u}_\nu\left(k'\right)\gamma_\mu\left(1-\gamma^5\right)u_\nu\left(k\right)\right]\left[\bar{u}_f\left(p'\right)\gamma^\mu\left(c_V - c_A\gamma^5\right)u_f\left(p\right)\right] \\ -\frac{im\lambda}{4\mu}\left[\bar{u}_\nu\left(k'\right)u_\nu\left(k\right)\right]\left[\bar{u}_f\left(p'\right)u_f\left(p\right)\right]$$

We find that for the cross terms in the squared amplitude, we may write:

$$\Delta\left|\mathcal{M}\right|^2 = \left(\mathcal{M}^*_{SM}\mathcal{M}_\lambda + \mathcal{M}_{SM}\mathcal{M}^*_\lambda\right)$$

This is:

$$\Delta\left|\mathcal{M}\right|^2 = 2Re\Big(-\frac{G_F m\lambda}{8\mu\sqrt{2}}\left[\bar{u}_\nu\left(k'\right)\gamma_\mu\left(1-\gamma^5\right)u_\nu\left(k\right)\right]\left[u_\nu\left(k\right)\bar{u}_\nu\left(k'\right)\right] \\ \times\left[\bar{u}_f\left(p'\right)\gamma^\mu\left(c_V - c_A\gamma^5\right)u_f\left(p\right)\right]\left[u_f\left(p\right)\bar{u}_f\left(p'\right)\right]\Big)$$

And so:

$$\Delta\left|\mathcal{M}\right|^2 = -\frac{G_F m\lambda}{4\mu\sqrt{2}}Re\Big[Tr\left\{\left(\sum u_\nu\bar{u}_\nu\right)_{k'}\gamma_\sigma\left(1-\gamma^5\right)\left(\sum u_\nu\bar{u}_\nu\right)_k\right\} \\ \times Tr\left\{\left(\sum u_f\bar{u}_f\right)_{p'}\gamma^\sigma\left(c_V-c_A\gamma^5\right)\left(\sum u_f\bar{u}_f\right)_p\right\}\Big]$$

Using our completeness relations (taking the neutrino to be effectively massless) we find:

$$\Delta\left|\mathcal{M}\right|^2 = -\frac{G_F m\lambda}{4\mu\sqrt{2}}Re\Big[\eta_{\sigma\tau}k'_\mu k_\nu Tr\left\{\left(\gamma^\mu\gamma^\tau\gamma^\nu - \gamma^\mu\gamma^\tau\gamma^5\gamma^\nu\right)\right\} \\ \times Tr\left\{\left(\gamma^\mu p'_\mu + m\right)\gamma^\sigma\left(c_V - c_A\gamma^5\right)\left(\gamma^\nu p_\nu + m\right)\right\}\Big]$$

Performing the traces we find these terms disappear due to the odd number of gamma matrices present. So, moving on to the term $\mathcal{M}_\lambda$ by itself:

$$\left|\mathcal{M}\right|^2 = \frac{m^2\lambda^2}{16\mu^2}\left[\bar{u}_\nu\left(k'\right)u_\nu\left(k\right)\right]\left[\bar{u}_\nu\left(k\right)u_\nu\left(k'\right)\right]\left[\bar{u}_f\left(p'\right)u_f\left(p\right)\right]\left[\bar{u}_f\left(p\right)u_f\left(p'\right)\right] \tag{B.7}$$

$$= Tr\left\{\left(\sum u_\nu\bar{u}_\nu\right)_{k\prime}\left(\sum u_\nu\bar{u}_\nu\right)_k\right\} \times Tr\left\{\left(\sum u_f\bar{u}_f\right)_{p\prime}\left(\sum u_f\bar{u}_f\right)_p\right\}$$

$$k'_\mu k_\nu Tr\left\{\gamma^\mu\gamma^\nu\right\}\times Tr\left\{\left(\gamma^\mu\gamma^\nu p'_\mu p_\nu + m\gamma^\mu\left(p'_\mu + p_\mu\right) + m^2\right)\right\}$$

$$= p'_\sigma p_\tau k'_\mu k_\nu Tr\left\{\gamma^\mu\gamma^\nu\right\}\times Tr\left\{\gamma^\sigma\gamma^\tau\right\} + 4m^2 k'_\mu k_\nu Tr\left\{\gamma^\mu\gamma^\nu\right\}$$

Using the identity:

$$k'_\mu k_\nu Tr\left\{\gamma^\mu\gamma^\nu\right\} = 4k'\cdot k$$

We obtain

$$\left|\mathcal{M}_\lambda\right|^2 = \frac{m^2\lambda^2}{4\mu^2}\left(k'\cdot k\right)\left[4\left(p'\cdot p\right)+m_f^2\right] = \frac{m_f^2\lambda^2}{4\mu^2}\left(p_\nu^{in}\cdot p_\nu^{out}\right)\left[4\left(p_f^{in}\cdot p_f^{out}\right)+m_f^2\right]$$

The differential cross section for two-body scattering may be written as (Thomson (2013), Particle Data Group (2020)):

$$d\sigma = \frac{d^3p_\nu^{out}d^3p_f^{out}}{\sqrt{\left(p_\nu^{in}\cdot p_f^{in}\right)^2 - m_\nu^2 m_f^2}}\frac{\left|\mathcal{M}\right|^2}{64\pi^2 E_\nu^{out}E_f^{out}}\delta\left(E_\nu^{in}+E_f^{in}-E_\nu^{out}-E_f^{out}\right)$$

$$\times\delta^3\left(\vec{p}_\nu^{in}+\vec{p}_f^{in}-\vec{p}_\nu^{out}-\vec{p}_f^{out}\right)$$

For neutrino-fermion scattering in the lab frame, our kinematics are taken such that:

$$E_f^{in} = m_f \qquad \vec{p}_f^{in} = \vec{0}$$

And we take m_ν to be negligible. Then we find

$$\sqrt{\left(p_\nu^{in}\cdot p_f^{in}\right)^2 - m_\nu^2 m_f^2} = \left(p_\nu^{in}\cdot p_f^{in}\right) = E_\nu^{in} m_f$$

And so:

$$\sigma = \int \frac{\left|\bar{\mathcal{M}}\right|^2}{64\pi^2 E_\nu^{in} m_f E_\nu^{out} E_f^{out}}\delta\left(E_\nu^{in}+E_f^{in}-E_\nu^{out}-E_f^{out}\right)\delta^3\left(\vec{p}_\nu^{in}+\vec{p}_f^{in}-\vec{p}_\nu^{out}-\vec{p}_f^{out}\right)\ d^3p_\nu^{out}d^3p_f^{out}$$

Performing the integral over $d^3p_\nu^{out}$ with the help of the delta function we obtain the constraint $\vec{p}_\nu^{out} = \vec{p}_\nu^{in} - \vec{p}_f^{out}$. Therefore, neglecting the mass of the neutrino we find:

$$E_\nu^{out} = \left|\vec{p}_\nu^{out}\right| = \sqrt{\left|\vec{p}_\nu^{in}\right|^2 + \left|\vec{p}_f^{out}\right|^2 - 2\vec{p}_\nu^{in}\cdot\vec{p}_f^{out}}$$

Now, taking $\vec{p_\nu^{in}}$ to be along the z-axis and defining θ_f as the scattering angle of the charged fermion, we may expand $d^3 p_f^{out}$ as:

$$d^3 p_f^{out} = E_f^{out\,2} dE_f^{out}\; d\left(\cos\theta_f\right) d\phi_f$$

And take:

$$\left(\vec{p_\nu^{in}} \cdot \vec{p_f^{out}}\right) = E_\nu^{in}\sqrt{\left(E_f^{out}\right)^2 - m_f^2}\cos\theta_f$$

Then we have:

$$E_\nu^{out} = \left|\vec{p_\nu^{in}} - \vec{p_f^{out}}\right| = \sqrt{\left(E_\nu^{in}\right)^2 + \left(E_f^{out}\right)^2 - 2E_\nu^{in}\sqrt{\left(E_f^{out}\right)^2 - m_f^2}\cos\theta_f}$$

So we may write the cross section as:

$$\sigma = \int \frac{|\mathcal{M}|^2 E_f^{out}}{32\pi m_f \left|\vec{p_\nu^{in}} - \vec{p_f^{out}}\right| E_\nu^{in}} dE_f^{out} d\left(\cos\theta_f\right)$$
$$\times\, \delta\left(E_\nu^{in} - E_f^{out} + m_f - \sqrt{\left(E_\nu^{in}\right)^2 + \left(E_f^{out}\right)^2 - 2E_\nu^{in}\sqrt{\left(E_f^{out}\right)^2 - m_f^2}\cos\theta_f}\right)$$

Where we have performed the ϕ_ν integral from 0 to 2π. We then note the property of the delta function:

$$\delta\left(g\left(x\right)\right) = \sum \frac{\delta\left(x - x_i\right)}{\left|g'\left(x_i\right)\right|}$$

Where the sum is over the solutions to $g\left(x_i\right) = 0$ denoted x_i. We find the solutions to the term in the delta function to be given by the relation:

$$\cos\theta_f = \frac{1}{E_\nu^{in}}\frac{\left(m_f + E_\nu^{in}\right)}{\left(m_f + E_f^{in}\right)}\sqrt{\left(E_f^{out}\right)^2 - m_f^2}$$

Which is equivalent to keeping the particles on its mass shell. Then:

$$\delta\left(E_\nu^{in} + m_f - E_\nu^{out} - E_f^{out}\right) = \frac{E_\nu^{in} - E_f^{out} + m_f}{E_\nu^{in}\sqrt{\left(E_f^{out}\right)^2 - m_f^2}}$$
$$\delta\left(\cos\theta_f - \frac{1}{E_\nu^{in}}\frac{\left(m_f + E_\nu^{in}\right)}{\left(m_f + E_f^{out}\right)}\sqrt{\left(E_f^{out}\right)^2 - m_f^2}\right)$$

And so:

$$\frac{d\sigma}{dE_f^{out}} = \frac{|\bar{\mathcal{M}}|^2 E_f^{out}}{32\pi m_f \left(E_\nu^{in}\right)^2} \frac{1}{\sqrt{\left(E_f^{out}\right)^2 - m_f^2}}$$

Or, by taking $E_f^{out} \gg m_f$:

$$\frac{d\sigma}{dE_f^{out}} = \frac{|\bar{\mathcal{M}}|^2}{32\pi m_f \left(E_\nu^{in}\right)^2}$$

Now, taking the matrix element

$$|\mathcal{M}_\lambda|^2 = \frac{m^2\lambda^2}{4\mu^2} \left(k' \cdot k\right) \left[4 \left(p' \cdot p\right) + m_f^2\right] = \frac{m^2\lambda^2}{4\mu^2} \left(p_\nu^{in} \cdot p_\nu^{out}\right) \left[4 \left(p_f^{in} \cdot p_f^{out}\right) + m_f^2\right]$$

Substituting the the lab frame products:

$$p_\nu^{in} \cdot p_\nu^{out} = m_f \left(E_\nu^{in} - E_\nu^{out}\right)$$

$$p_\nu^{in} \cdot p_f^{in} = m_f E_\nu^{in}$$

$$p_\nu^{in} \cdot p_f^{out} = p_\nu^{out} \cdot p_f^{in} = m_f E_\nu^{out}$$

$$p_f^{in} \cdot p_f^{out} = m_f \left(E_\nu^{in} - E_\nu^{out} + m_f\right)$$

We find the differential cross section to be:

$$\frac{d\sigma_\lambda}{dE_f^{out}} = \frac{m_f^3\lambda^2}{32\pi\ \mu^2} \left[\left(\frac{E_\nu^{in} - E_\nu^{out}}{E_\nu^{in}}\right)^2 + \frac{5m_f}{4E_\nu^{in}} \left(\frac{E_\nu^{in} - E_\nu^{out}}{E_\nu^{in}}\right)\right]$$

Bibliography

Aad, G., Abajyan, T., Abbott, B., Abdallah, J., Khalek, S. A., Aben, R., Abi, B., Abolins, M., AbouZeid, O., Abramowicz, H., et al. (2013). Evidence for the spin-0 nature of the higgs boson using atlas data. *Physics Letters B*, 726(1-3):120–144.

Acciarri, M., Adriani, O., Aguilar-Benitez, M., Ahlen, S., Alcaraz, J., Alemanni, G., Allaby, J., Aloisio, A., Alviggi, M., Ambrosi, G., et al. (1998). Determination of the number of light neutrino species from single photon production at lep. *Physics Letters B*, 431(1-2):199–208.

Ackermann, M., Ajello, M., Albert, A. t., Atwood, W., Baldini, L., Ballet, J., Barbiellini, G., Bastieri, D., Bechtol, K., Bellazzini, R., et al. (2011). Constraining dark matter models from a combined analysis of milky way satellites with the fermi large area telescope. *Physical Review Letters*, 107(24):241302.

Adshead, P. and Lozanov, K. D. (2021). Self-gravitating vector dark matter. *Physical Review D*, 103(10):103501.

Aghanim, N., Akrami, Y., Ashdown, M., Aumont, J., Baccigalupi, C., Ballardini, M., Banday, A., Barreiro, R., Bartolo, N., Basak, S., et al. (2020). Planck 2018 results-vi. cosmological parameters. *Astronomy & Astrophysics*, 641:A6.

Aguilar-Arevalo, A., Brown, B., Bugel, L., Cheng, G., Conrad, J., Cooper, R., Dharmapalan, R., Diaz, A., Djurcic, Z., Finley, D., et al. (2018). Significant excess of electronlike events in the miniboone short-baseline neutrino experiment. *Physical review letters*, 121(22):221801.

Aker, M., Beglarian, A., Behrens, J., Berlev, A., Besserer, U., Bieringer, B., Block, F., Bobien, S., Boettcher, M., Bornschein, B., et al. (2022). Direct neutrino-mass measurement with sub-electronvolt sensitivity. *Nature Physics*, 18(2).

Akhmedov, E. K. (2007). Do charged leptons oscillate? *Journal of High Energy Physics*, 2007(09):116.

Alhaidari, A. and Jellal, A. (2015). Compatibility of symmetric quantization with general covariance in the dirac equation and spin connections. *Physics Letters A*, 379(45-46):2946–2950.

Anderson, C. D. (1932). The apparent existence of easily deflectable positives. *Science*, 76(1967):238–239.

Athanassopoulos, C., Metcalf, W., Yellin, S., Xiao, Y., Imlay, R., Church, E., Waltz, J., Tayloe, R., Federspiel, F., VanDalen, G. J., et al. (1997). Evidence for nu (mu)—> nu (e) neutrino oscillations from lsnd. *Phys. Rev. Lett.*, 81(UCRHEP-E-197):1774–1777.

Auyang, S. Y. et al. (1995). *How is Quantum Field Theory Possible?* Oxford University Press.

Barbier, R., Bérat, C., Besançon, M., Chemtob, M., Deandrea, A., Dudas, E., Fayet, P., Lavignac, S., Moreau, G., Perez, E., et al. (2005). R-parity-violating supersymmetry. *Physics reports*, 420(1-6):1–195.

Bargmann, V. (1954). On unitary ray representations of continuous groups. *Annals of Mathematics*, pages 1–46.

Bargmann, V. and Wigner, E. P. (1948). Group theoretical discussion of relativistic wave equations. *Proc. Nat. Acad. Sci.*, 34:211.

Barrena, R., Biviano, A., Ramella, M., Falco, E., and Seitz, S. (2002). The dynamical status of the cluster of galaxies 1e0657-56. *Astronomy & Astrophysics*, 386(3):816–828.

Barut, A. O. (1978). The mass of the muon. *Physics Letters B*, 73(3):310–312.

Barut, A. O., Cordero, P., and Ghirardi, G. C. (1969). Theory of the heavy electron. *Physical Review*, 182(5):1844.

Benini, M., Dappiaggi, C., and Hack, T.-P. (2013). Quantum field theory on curved backgrounds – a primer. *Int. Jour. Mod. Phys.*, 28(17).

Bennett, C., Hill, R., Hinshaw, G., Nolta, M., Odegard, N., Page, L., Spergel, D., Weiland, J., Wright, E., Halpern, M., et al. (2003). First-year wilkinson microwave anisotropy probe (wmap)* observations: foreground emission. *The Astrophysical Journal Supplement Series*, 148(1):97.

Birrell, N. D. and Davies, P. C. W. (1984). *Quantum Fields in Curved Space*. Cambridge university press.

Boi, L. (2011). *The quantum vacuum: A scientific and philosophical concept, from electrodynamics to string theory and the geometry of the microscopic world*. JHU Press.

Cartan, É. (1913). *The Theory of Spinors*. Dover.

Casimir, H. B. (1948). On the attraction between two perfectly conducting plates. In *Proc. Kon. Ned. Akad. Wet.*, volume 51, page 793.

Chatrchyan, S., Khachatryan, V., Sirunyan, A. M., Tumasyan, A., Adam, W., Aguilo, E., Bergauer, T., Dragicevic, M., Erö, J., Fabjan, C., et al. (2012). Observation of a new boson at a mass of 125 gev with the cms experiment at the lhc. *Physics Letters B*, 716(1):30–61.

Cheng, H.-C., Feng, J. L., and Matchev, K. T. (2002). Kaluza-klein dark matter. *Physical review letters*, 89(21):211301.

Chikashige, Y., Mohapatra, R. N., and Peccei, R. D. (1981). Are there real goldstone bosons associated with broken lepton number? *Physics Letters B*, 98(4):265–268.

Da Silveira, A. (1976). Mass term in the dirac electron theory. *Lett. Nuovo Cim*, 15(7):228–230.

Deser, S., Duff, M. J., and Isham, C. (1980). Gravitationally induced cp effects. *Physics Letters B*, 93(4):419–423.

Dirac, P. A. M. (1927). The quantum theory of the emission and absorption of radiation. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 114(767):243–265.

Dirac, P. A. M. (1928). The quantum theory of the electron. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 117(778):610–624.

Doughty, N. (2018). *Lagrangian interaction: an introduction to relativistic symmetry in electrodynamics and gravitation*. CRC Press.

Dvoeglazov, V. V. (2000). The bargmann-wigner formalism for spin 2 fields. *Advances in Applied Clifford Algebras*, 10(1):7–14.

Einstein, A. (1915). The field equations of gravitation. *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*, 1915:844–847.

Elias-Miro, J., Espinosa, J. R., Giudice, G. F., Isidori, G., Riotto, A., and Strumia, A. (2012). Higgs mass implications on the stability of the electroweak vacuum. *Physics Letters B*, 709(3):222–228.

Elizalde, E. (1987). Series solutions for the klein-gordon equation in schwarzschild space-time. *Physical Review D*, 36(4):1269.

Ellis, R. S. (2010). Gravitational lensing: a unique probe of dark matter and dark energy. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 368(1914):967–987.

Feinberg, G. and Sucher, J. (1968). Long-range forces from neutrino-pair exchange. *Physical Review*, 166(5):1638.

Frampton, P. H. (2008). *Gauge field theories*. John Wiley & Sons.

Freeman, K. and McNamara, G. (2006). *In Search of Dark Matter*. Springer.

Frieman, J., Turner, M., and Huterer, D. (2008). Dark energy and the accelerating universe. *arXiv preprint arXiv:0803.0982*.

Fry, J., Musielak, Z., and Chang, T.-w. (2011). Relativistic wave and particle mechanics formulated without classical mass. *Annals of Physics*, 326(8):1972–1982.

Fukugita, M. and Yanagida, T. (2013). *Physics of Neutrinos: and Application to Astrophysics*. Springer Science & Business Media.

Fulling, S. A. et al. (1989). *Aspects of Quantum Field Theory in Curved Spacetime*. Cambridge university press.

Fushchich, W. and Nikitin, A. (1994). *Symmetries of Equations of Quantum Mechanics*. Allerton Press.

Gaemers, K., Gastmans, R., and Renard, F. (1979). Neutrino counting in e+ e- collisions. *Physical Review D*, 19(5):1605.

Gell-Mann, M. and Lévy, M. (1960). The axial vector current in beta decay. *Il Nuovo Cimento (1955-1965)*, 16(4):705–726.

Gelmini, G. B. and Roncadelli, M. (1981). Left-handed neutrino mass scale and spontaneously broken lepton number. *Physics Letters B*, 99(5):411–415.

Giunti, C., Li, Y., Ternes, C., and Xin, Z. (2022). Reactor antineutrino anomaly in light of recent flux model refinements. *Physics Letters B*, 829:137054.

Glashow, S. L. (1959). The renormalizability of vector meson interactions. *Nuclear Physics*, 10:107–117.

Gordon, W. (1926). Der comptoneffekt nach der schrödingerschen theorie. *Zeitschrift für Physik*, 40(1):117–133.

Greiner, W. et al. (2000). *Relativistic Quantum Mechanics*, volume 2. Springer.

Gupta, N. S. (1967). The wave equation for zero rest-mass particles. *Nuclear Physics B*, 4(1):147–152.

Halzen, F. and Martin, A. D. (1984). *Quarks and Leptons*. John Wiley and Sons.

Hawking, S. W. (1975). Particle creation by black holes. *Communications in Mathematical Physics*, 43(3):199–220.

Heller, M. (1992). *Theoretical Foundations of Cosmology: Introduction to the Global Structure of Space-Time*. World Scientific.

Hojman, S. (1984). Symmetries of lagrangians and of their equations of motion. *Journal of Physics A: Mathematical and General*, 17(12):2399.

Hojman, S. A. (1992). A new conservation law constructed without using either lagrangians or hamiltonians. *Journal of Physics A: Mathematical and General*, 25(7):L291.

Hollands, S. and Wald, R. M. (2010). Axiomatic quantum field theory in curved spacetime. *Communications in Mathematical Physics*, 293(1):85–125.

Huegele, R., Fry, J., and Musielak, Z. (2013). Generalized levy-leblond and schrodinger equations for spinor wavefunctions. *Advanced Studies in Theoretical Physics*, 7:825–837.

Ibarra, A., Tran, D., and Weniger, C. (2013). Indirect searches for decaying dark matter. *International Journal of Modern Physics A*, 28(27):1330040.

Janot, P. and Jadach, S. (2020). Improved bhabha cross section at lep and the number of light neutrino species. *Physics Letters B*, 803:135319.

Kaku, M. (1993). *Quantum Field Theory: a Modern Introduction*. Oxford University Press.

Kayser, B., Gibrat-Debu, F., and Perrier, F. (1989). *The Physics of Massive Neutrinos*, volume 25. World Scientific.

Kim, Y. S. and Noz, M. (2012). *Theory and Applications of the Poincaré Group*, volume 17. Springer Science & Business Media.

Klein, O. (1926). Quantentheorie und fünfdimensionale relativitätstheorie. *Zeitschrift für Physik*, 37(12):895–906.

Kostensalo, J., Suhonen, J., Giunti, C., and Srivastava, P. C. (2019). The gallium anomaly revisited. *Physics Letters B*, 795:542–547.

Kragh, H. (1981). The genesis of dirac's relativistic theory of electrons. *Archive for history of exact sciences*, pages 31–67.

Kruglov, S. (2006). On the hamiltonian form of generalized dirac equation for fermions with two mass states. *EJTP*, 3(10):11–16.

Kruglov, S. (2007). Higher derivative fermionic field equation in the first-order formalism. *Canadian Journal of Physics*, 85(8):887–897.

Kruglov, S. (2012). Modified dirac equation with lorentz invariance violation and its solutions for particles in an external magnetic field. *Physics Letters B*, 718(1):228–231.

Landau, L. and Lifshitz, E. (2000). *Mechanics: Volume 1 (Course of Theoretical Physics Series).* Oxford: Pergamon Press.

Larson, D., Dunkley, J., Hinshaw, G., Komatsu, E., Nolta, M., Bennett, C., Gold, B., Halpern, M., Hill, R., Jarosik, N., et al. (2011). Seven-year wilkinson microwave anisotropy probe (wmap*) observations: power spectra and wmap-derived parameters. *The Astrophysical Journal Supplement Series*, 192(2):16.

Latosinski, A., Meissner, K. A., and Nicolai, H. (2010). Axions without peccei-quinn symmetry. *arXiv preprint arXiv:1010.5417*.

Lee, T.-D. and Yang, C.-N. (1956). Question of parity conservation in weak interactions. *Physical Review*, 104(1):254.

Leiter, D. and Szamosi, G. (1972). Pseudoscalar mass and its relationship to conventional scalar mass in the relativistic dirac theory of the electron. *Lettere al Nuovo Cimento (1971-1985)*, 5(12):814–816.

Lévy-Leblond, J.-M. (1969). Group-theoretical foundations of classical mechanics: the lagrangian gauge problem. *Communications in Mathematical Physics*, 12(1):64–79.

Lüders, G. (1954). On the equivalence of invariance under time reversal and under particle-antiparticle conjugation for relativistic field theories. *Dan. Mat. Fys. Medd.*, 28:1–17.

Ma, E. (1998). Pathways to naturally small neutrino masses. *Physical Review Letters*, 81(6):1171.

Ma, E., Ohata, T., and Tsumura, K. (2017). Majoron as the qcd axion in a radiative seesaw model. *Physical Review D*, 96(7):075039.

Maggiore, M. (2005). *A Modern Introduction to Quantum Field Theory*, volume 12. Oxford university press.

Magueijo, J. and Benincasa, D. M. (2011). Chiral vacuum fluctuations in quantum gravity. *Physical review letters*, 106(12):121302.

Maki, Z., Nakagawa, M., and Sakata, S. (1962). Remarks on the unifiedmodel of elementary particles. *Progress of Theoretical Physics*, 28(5):870–880.

Marsch, E. and Narita, Y. (2015). Fermion unification model based on the intrinsic su(8) symmetry of a generalized dirac equation. *Frontiers in Physics*, 3:82.

Milonni, P. W. (2013). *The quantum vacuum: an introduction to quantum electrodynamics*. Academic press.

Mohapatra, R. N. and Senjanović, G. (1980). Neutrino mass and spontaneous parity nonconservation. *Physical Review Letters*, 44(14):912.

Musielak, Z. (2021). New equation of nonrelativistic physics and theory of dark matter. *International Journal of Modern Physics A*, 36(03):2150042.

Musielak, Z. and Fry, J. (2009). Physical theories in galilean space-time and the origin of schrödinger-like equations. *Annals of Physics*, 324(2):296–308.

Musielak, Z. and Watson, T. B. (2020). Gauge functions and galilean invariance of lagrangians. *Physics Letters A*, 384(26):126642.

Navarro, J. F., Abadi, M. G., Venn, K. A., Freeman, K. C., and Anguiano, B. (2011). Through thick and thin: kinematic and chemical components in the solar neighbourhood. *Monthly Notices of the Royal Astronomical Society*, 412(2):1203–1209.

Nicolas, J.-P. (1995). Non-linear klein-gordon equation on schwarzschild-like metrics. *Journal de mathématiques pures et appliquées*, 74(1):35–58.

Nozari, K. (2007). Generalized dirac equation and its symmetries. *Chaos, Solitons & Fractals*, 32(2):302–311.

Nyambuya, G. (2007). Dirac equation in curved spacetime on the anomalous gyromagnetc ratio. *Electronic Journal of Theoretical Physics*, 4(14):105–116.

Nyambuya, G. (2008). New curved spacetime dirac equations. *Foundations of Physics*, 38(7):665–677.

Okun, L. B. (2013). *Leptons and Quarks*. Elsevier.

Overduin, J. M. and Wesson, P. S. (2004). Dark matter and background light. *Physics Reports*, 402(5-6):267–406.

Pagels, H. and Primack, J. R. (1982). Supersymmetry, cosmology, and new physics at teraelectronvolt energies. *Physical Review Letters*, 48(4):223.

Parker, L. (1968). Particle creation in expanding universes. *Physical Review Letters*, 21(8):562.

Parker, L. (1969). Quantized fields and particle creation in expanding universes. i. *Physical Review*, 183(5):1057.

Parker, L. (1971). Quantized fields and particle creation in expanding universes. ii. *Physical Review D*, 3(2):346.

Parker, L. and Toms, D. (2009). *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity*. Cambridge university press.

Particle Data Group (2020). Review of Particle Physics. *Progress of Theoretical and Experimental Physics*, 2020(8). 083C01.

Paschos, E. A. (2007). *Electroweak Theory*. Cambridge University Press.

Peccei, R. D. and Quinn, H. R. (1977). Cp conservation in the presence of pseudoparticles. *Physical Review Letters*, 38(25):1440.

Peebles, P. J. E. (1982). Primeval adiabatic perturbations-effect of massive neutrinos. *The Astrophysical Journal*, 258:415–424.

Penrose, R. and Rindler, W. (1984a). *Spinors and space-time: Volume 1, Two-spinor calculus and relativistic fields*, volume 1. Cambridge University Press.

Penrose, R. and Rindler, W. (1984b). *Spinors and space-time: Volume 2, Spinor and twistor methods in space-time geometry*, volume 2. Cambridge University Press.

Peskin, M. E. and Schroeder, D. (2018). *An Introduction to Quantum Field Theory*. CRC press.

Pfister, W. (1994). Exact solutions of generalized three-particle bargmann-wigner equations in the strong-coupling limit. *Il Nuovo Cimento A (1965-1970)*, 107(9):1523–1541.

Pisano, F. and Sharma, S. S. (1998). Majoron emitting neutrinoless double beta decay in the electroweak chiral gauge extensions. *Physical Review D*, 57(9):5670.

Pontecorvo, B. (1958). Inverse *beta* processe and nonconservation of lepton charge. *Zhur. Eksptl'. i Teoret. Fiz.*, 34.

Proca, A. (1936). Sur la théorie ondulatoire des électrons positifs et négatifs. *Journal de Physique et le Radium*, 7(8):347–353.

Ramond, P. et al. (1999). *Journeys Beyond the Standard Model*, volume 101. Perseus Books Cambridge.

Rarita, W. and Schwinger, J. (1941). On a theory of particles with half-integral spin. *Physical Review*, 60(1):61.

Renton, P. (1990). *Electroweak interactions: an introduction to the physics of quarks and leptons*. Cambridge University Press.

Rosenberg, L. J. and Van Bibber, K. A. (2000). Searches for invisible axions. *Physics Reports*, 325(1):1–39.

Ryder, L. H. (1996). *Quantum Field Theory*. Cambridge university press.

Salam, A. and Ward, J. (1964). Electromagnetic and weak interactions. *Physics Letters*, 13(2):168–171.

Schrödinger, E. (1926). An undulatory theory of the mechanics of atoms and molecules. *Physical Review*, 28(6):1049–1070.

Schwinger, J. (1951). On gauge invariance and vacuum polarization. *Physical Review*, 82(5):664.

Segarra, A. and Bernabéu, J. (2020). Absolute neutrino mass and the dirac/majorana distinction from the weak interaction of aggregate matter. *Physical Review D*, 101(9):093004.

Sher, M. (1989). Electroweak higgs potential and vacuum stability. *Physics Reports*, 179(5-6):273–418.

Sin, S.-J. (1994). Late-time phase transition and the galactic halo as a bose liquid. *Physical Review D*, 50(6):3650.

Sogami, I. (1981). Generalized dirac equation with four orthogonal families of spin 1/2 solutions. *Progress of Theoretical Physics*, 66(1):303–317.

Streater, R. F. and Wightman, A. S. (2000). *PCT, Spin and Statistics, and all that*, volume 52. Princeton University Press.

Strohmaier, A. (2000). On the local structure of the klein–gordon field on curved spacetimes. *Letters in Mathematical Physics*, 54(4):249–261.

Sugita, K., Okamoto, Y., and Sekine, M. (2008). Dark matter candidate particles, cp violation and higgs bosons. *International Journal of Theoretical Physics*, 47(11):2875–2881.

Thomson, M. (2013). *Modern Particle Physics*. Cambridge University Press.

Tokuoka, Z. (1967). A quantization of a spinor field in the generalized hilbert space. *Progress of Theoretical Physics*, 37(3):581–602.

Treiman, S. and Wilczek, F. (1978). Axion emission in decay of excited nuclear states. *Physics Letters B*, 74(4-5):381–383.

Trzetrzelewski, M. (2011). On the mass term of the dirac equation. *arXiv preprint arXiv:1101.3899*.

Vigo, C., Gerchow, L., Liszkay, L., Rubbia, A., and Crivelli, P. (2018). First search for invisible decays of orthopositronium confined in a vacuum cavity. *Physical Review D*, 97(9):092008.

Wald, R. M. (1994). *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*. University of Chicago press.

Watson, T. B. and Musielak, Z. (2021a). Chiral bargmann–wigner equations for spin-1 massive fields. *Modern Physics Letters A*, 36(38):2150276.

Watson, T. B. and Musielak, Z. E. (2020). Chiral symmetry in dirac equation and its effects on neutrino masses and dark matter. *International Journal of Modern Physics A*, 35(30):2050189.

Watson, T. B. and Musielak, Z. E. (2021b). Chiral dirac equation and its spacetime and cpt symmetries. *Symmetry*, 13(9):1608.

Weinberg, S. (1967). A model of leptons. *Physical review letters*, 19(21):1264.

Weinberg, S. (1969). Feynman rules for any spin. iii. *Physical Review*, 181(5):1893.

Weinberg, S. (1972). *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. Wiley.

Weinberg, S. (1995a). *The quantum theory of fields*, volume 2. Cambridge university press.

Weinberg, S. (1995b). *The quantum theory of fields*, volume 1. Cambridge university press.

White, S. D., Frenk, C. S., and Davis, M. (1983). Clustering in a neutrino-dominated universe. *The Astrophysical Journal*, 274:L1–L5.

Wigner, E. (1939). On unitary representations of the inhomogeneous lorentz group. *Annals of Mathematics*, pages 149–204.

Wu, C.-S., Ambler, E., Hayward, R. W., Hoppes, D. D., and Hudson, R. P. (1957). Experimental test of parity conservation in beta decay. *Physical review*, 105(4):1413.

BIOGRAPHICAL STATEMENT

Timothy Blake Watson was born in Dallas County in 1991. He received his Bachelor of Science degree from The University of Texas at Arlington in 2015. He has worked heavily in neutrino physics with former collaboration affiliations including the Deep Underground Neutrino Experiment and the IceCube Neutrino Observatory and has actively pursued theoretical research as part of the University of Texas at Arlington's Physics Theory Group.