FORMING COALITIONS AND SHARING PAYOFFS IN n-PERSON NORMAL FORM GAMES

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To my husband Ebenezer without whose support this journey would not have been a successful one. And to my daughters Afia and Nanayaa, I hope this encourages you to know that the you can do whatever you put your minds to.

ABSTRACT

FORMING COALITIONS AND SHARING PAYOFFS IN n-PERSON NORMAL FORM GAMES

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For a given n-person normal form game, we form all possible sets of mutually exclusive and collectively exhaustive coalitions of the n players. For each set of coalitions, we define a coalitional semi-cooperative game as one in which these coalitions are taken as the players of this new game, each coalition tries to maximize the sum of its individual players' payoffs, and the players within a coalition cooperate to do so. For any coalitional semi-cooperative game, the goal of the original n players is to improve their individual payoffs obtained in a Greedy Scalar Equilibrium (GSE) of the original game, where a GSE is an analog of the Nash equilibrium but always exists in pure strategies. We define a "best" such coalitional game as one that gives the n players their "best" possible payoffs among all possible coalitional semi-cooperative games. We present an algorithm for selecting such a "best" set of coalitions and present examples.

Also, for a given n-person normal form game, we consider the situation where, for each strategy profile in the game, every player gives a pre-determined fraction of his payoff selfishly to himself and altruistically to the remaining $n-1$ players. We show that the Nash equilibrium and Berge equilibrium are extreme cases of this situation.

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TABLE OF CONTENTS

CHAPTER 1

Introduction

This dissertation strives to show how to form coalitions in normal form games as well as sharing payoffs in such games. The dissertation is presented in the format of an article-based dissertation comprising two papers.

The first paper is entitled "Forming Coalitions in Normal Form Games". It is a study about all possible sets of mutually exclusive and collective exhaustive coalitions of the n players in an n-person normal form game. For each set of coalitions, a coalitional semi-cooperative game is defined as one in which these coalitions are taken as the players of this new game and each coalition tries to maximize the sum of its individual players' payoffs, and the players within a coalition cooperate to do so. For any coalitional semi-cooperative game, the goal of the original n players is to improve their individual payoffs obtained in a Greedy Scalar Equilibrium (GSE) of the original game, where a GSE is an analog of the Nash equilibrium but always exists in pure strategies. A "best" such coalitional game is defined as one that gives the n players their "best" possible payoffs among all possible coalitional semi-cooperative games. An algorithm for selecting such a "best" set of coalitions is presented and examples given.

The second paper is entitled "Modeling Degrees of Player Selfishness or Altruism in Mormal Form Games". In this paper, we consider the situation where for an n-person game in normal form and for each strategy profile in the game, every player gives a pre-determined fraction of his payoff selfishly to himself and altruistically to the remaining $n-1$ players. We show that the Nash equilibrium and Berge equilibrium are extreme cases of this situation and an example given.

The dissertation is organized as follows. Chapter 2 includes the study on forming coalitions in normal form games. In Chapter 3, sharing payoffs in normal form games is presented and general conclusions stated in Chapter 4.

Forming Coalitions in Normal Form Games

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Abstract

For a given n-person normal form game, we form all possible sets of mutually exclusive and collectively exhaustive coalitions of the n players. For each set of coalitions, we define a coalitional semi-cooperative game as one in which these coalitions are taken as the players of this new game, each coalition tries to maximize the sum of its individual players' payoffs, and the players within a coalition cooperate to do so. For any coalitional semi-cooperative game, the goal of the original n players is to improve their individual payoffs obtained in a Greedy Scalar Equilibrium (GSE) of the original game, where a GSE is an analog of the Nash equilibrium but always exists in pure strategies. We define a "best" such coalitional game as one that gives the n players their "best" possible payoffs among all possible coalitional semi-cooperative games. We present an algorithm for selecting such a "best" set of coalitions and present examples.

Keywords

Game Theory; Normal Form Game; Coalitional Semi-Cooperative Game.

CHAPTER 2

Forming Coalitions in Normal Form Games

2.1 Introduction

Game theory is the study of mathematical decision-making made by a finite number of players either as individuals or in coalitions. A player's choices are made according to the player's personal approach to treating oneself and others, as well as the expectation of the actions of the other players. In cooperative games, there is competition between groups of players, whereas in non-cooperative games there is competition between individual players. Cooperative games may also involve an arbitrator who selects the players' strategies in a manner agreed upon by the players.

Historically, Borel informally introduced two-person zero-sum game theory beginning in 1921 as summarized in [1] and discussed by Fréchet in [2]. Borel did not establish the minimax theorem, which von Neuman later did in [3]. Indeed, [3] considered cooperative coalitional games as well as non-cooperative zero-sum games. Subsequently in [4], Nash as-

sumed each player was selfish and established that an equilibrium for non-cooperative games always exists in mixed strategies. In [5], he further formulated and solved a game-theoretic bargaining problem.

Specifically, von Neuman and Morgenstern [3] considered cooperative games with coalitions in which the players within a coalition collaborated for their mutual benefit in competing with other coalitions. They defined the notion of a solution to such a game as the core, a certain set of undominated rewards for the game's players reminiscent of Pareto maxima. This definition was refined by Gillies in [6]. Later, Shapley provided an alternative solution concept for cooperative games, and the so-called Shapley value gives more equitable solutions than does the core [7],[8]. On the other hand, Bacharach [9] formalized an axiomatic theory of solutions for normal form games.

Although von Neumann and Morgenstern defined the core of a cooperative game in [3], they provided no method for the players to agree on a particular member of the core. One method for doing so is for the players to agree to an arbitrator's choice. In [10] Raiffa defined an arbitration scheme

for a given generalized two-person game to yield a unique mixed strategy for the arbitrator to enforce. Unlike Raiffa's mixed-strategy model for two-person games, Rosenthal [11] presented a single Pareto-optimal pure strategy reflecting the relative strengths of the players for the arbitrator to select. Later, Kalai and Rosenthal [12] devised schemes for an arbitrator to assign a fair outcome in the game where two players possess complete information.

More recently, Bacharach [13] expressed the interactions between players who choose to act as individuals sometimes and as teams at other times with the possibility of being in more than one team. Unlike the model of [13] where players vacillate between acting as individuals and as teams, Bacharach et al. [14] considered a team reasoning approach where players made the "best" decisions for their respective teams and not for themselves. While the previous models considered individuals or teams making decisions to benefit themselves or their respective teams, Colman et al. [15] showed how the Berge equilibrium models altruistic cooperation and related it to the Nash equilibrium. For a semi-cooperative approach in which there is some level of selfishness and some level of altruism, Kalai and Kalai [16] decompose a two-person game with transferable utility into cooperative and competitive components by using a formula. In addition, Corley [17] defines the notion of scalar equilibria for all these categories of games using multiple-objective optimization techniques and maximizing particular scalar objective functions. He also discusses the difficulties of mixed strategies in games and justifies the use of only pure strategies.

In this paper, for a given n-person normal form game, we form all possible sets of mutually exclusive and collectively exhaustive coalitions of the n players. For each set of coalitions, we define a coalitional semi-cooperative game as one in which these coalitions are taken as the players of this new game, each coalition tries to optimize the sum of its individual players' payoffs, and the players within a coalition cooperate to do so. For any coalitional semi-cooperative game, the goal of the original n players is to improve their individual payoffs obtained in a Greedy Scalar Equilibrium (GSE) of the original game, where a GSE is an analog of the Nash equilibrium but always exists in pure strategies. We define a "best" such coalitional game as one that gives the n players payoffs that are closest together. It is assumed that the n players apriori desire equitable payoffs with the choice of such a "best" coalitional game being enforced by an arbitrator. We present an algorithm for selecting such a "best" set of coalitions and present two examples. A classic example of a situation where the work done in this paper is applied is forming two teams from one soccer team at training sessions. During each session, there is cooperation within each team but competition between the two teams even though the coach would want to minimize the overall variation among the teams.

The paper is organized as follows. We present preliminary notation, definitions, and results in Section 2. In Section 3 we formally define the notion of a coalitional semi-cooperative game and present an algorithm to divide the payoff of each coalition among its members. We then define a metric to select a "best" coalitional game and hence "best" set of coalitions. Two examples are given in Section 4, and conclusions stated in Section 5.

2.2 Preliminaries

Let $G_n = \left\langle I, (S_i)_{i\in I}, (u_i)_{i\in I} \right\rangle$ **be an** n**-player game where** $I = \{1, \ldots, n\}$ **is the set of players,** $S_i = \{s_i^1, \ldots, s_i^{m_i}\}$ **is the finite set of** $m_i \geq 2$ **pure strategies for player** i **and** $u_i(s)$ **is the utility of player** i **for an action profile** $s = (s_1, \ldots, s_n) \in \times_{j\in I} S_j = S$. **The** $m_1 \times \ldots \times m_n$ **matrix of** $\Big(u_i(s), \ldots, u_n(s)\Big)$, $s \in S$ **is called the payoff matrix for** G **and a game given in terms of a payoff matrix is called a normal form game.**

Given that player i **chooses each strategy** s_i^j **with probability** $\sigma_i(s_i^j)$, **a mixed strategy for player** i **denoted** $\sigma_i = (\sigma_i^1, \ldots, \sigma_i^{m_i})$, **is a probability distribution over the player's pure strategies set, where** $\sum_{j=1}^{m_i} \sigma_i(s_i^j) = 1$ **and** $\sigma_i(s_i^j) \geq 0,\ j = 1, \ldots, m_i$. **A strategy** σ^* **is a Nash equilibrium(NE) if no player with a unilateral change of strategy can increase his expected payoff. An NE always exists in mixed strategies but not in pure strategies.**

Let $\Gamma_n = \left\langle I, (S_i)_{i\in I}, (u_i)_{i\in I}, \Omega, T \right\rangle$ **be the** n**-person game in normal form corresponding to** G_n, **where** I, S, S_i, s_i, u_i **are as in** G_n **and where only the action profiles in** Ω **are acceptable to the** n **players. Now assume that each player is greedy, desiring**

a payoff as high as jointly possible. Consider the scalar utility function $T_G : u(\Omega) \to R^1$ **in [17] defined as**

$$T_G[u(s)] = \prod_{i \in I} \frac{1}{M_i - u_i(s) + 1}, \; s \in \Omega, \tag{2.1}$$

where $M_i = \max_{i \in I} u_i(s)$. **A pure strategy profile** s^* **is called a Greedy Scalar Equilibrium (GSE) for** Γ_n **if and only if** s^* **maximizes** $T_G[u(s)]$ **over** Ω**. The GSE is a scalar analog of the NE since each player is greedy and wants a payoff as high as jointly possible. Although an NE does not always exist in pure strategies, a GSE always does.**

Moreover, from (2.1), it follows that maximizing $T_G[u(s)]$ **over** Ω **requires that each be as close as jointly possible. The reason is that the maximization of (2.1) is a discrete version of the continuous problem of maximizing** $\prod_{i \in I} \frac{1}{x_i+1}$ **over** $x_i \geq 0, \; i \in I$ **in which case the** x_i**'s are equal.**

Note that a GSE from one payoff matrix cannot be compared to a GSE from a different payoff matrix (i.e., a different game). In addition, Corley [17] proves that the GSE is Pareto maximal over all the strategies of Γ_n**. In other words, the** n**-tuple of payoffs** $(u_1^*, \ldots, u_n^*)$ **associated with a GSE are not dominated by any other** $(u_1, \ldots, u_n)$ **in the payoff matrix.**

Next, we define a coalitional semi-cooperative game and also present an algorithm for creating coalitions from a normal form game.

2.3 Forming a Coalitional Semi-Cooperative Game from a Normal Form Game

In this section, we define a coalitional semi-cooperative game and give examples to illustrate the definition. We also present a metric and an algorithm for obtaining a best coalition from an n-person normal form game. Note that the average payoff of a player i is $\frac{\sum_{k=1}^{m_i} u_k(s)}{m_i}$, where m_i is the number of strategies of player i. The best coalition is selected according to a criterion described below which can be enforced by an arbitrator as necessary.

We define a coalitional semi-cooperative game as follows. Let G be an n-player game in normal form. A game G' consisting of C mutually exclusive and collectively exhaustive fixed coalitions formed from the n-players of G is a coalitional semi-cooperative game if there is competition among the C fixed coalitions of G' and cooperation within each coalition. G' is a distinct normal form game where the players are the coali-

tions and strategies for a given coalition is a combination of individual strategies for players in the given coalition.

Suppose a coalitional semi-cooperative game of Q coalitions is formed from an n-player game. In this coalitional game, suppose coalition k has C_k players such that $\sum_{k=1}^{Q} C_k = n$. The coalitions in the coalitional semi-cooperative game are ordered in ascending order of the first player in each coalition and the players in each coalition ordered in ascending order. The players in each coalition are separated by commas and the various coalitions separated by vertical bars. Denote the coalitional semi-cooperative game by $G_{1_1,\ldots,1_{C_1}|2_1,\ldots,2_{C_2}|\ldots|Q_1,\ldots,Q_{C_Q}}$, where there are Q coalitions depicted by the subscripts $1,\ldots,Q$ and coalition k has C_k players depicted by the double subscripts $1,\ldots,C_k$. Each strategy profile of $G_{1_1,\ldots,1_{C_1}|2_1,\ldots,2_{C_2}|\ldots|Q_1,\ldots,Q_{C_Q}}$ is represented as $((s_{1_1},\ldots,s_{1_{C_1}}),\ldots,(s_{Q_1},\ldots,s_{Q_{C_Q}}))$ with corresponding payoff $(u_{1_1}+\ldots+u_{1_{C_1}},\ldots,u_{Q_1}+\ldots+u_{Q_{C_Q}})$ of this game's payoff matrix.

As an example, consider the case where 5 coalitions are formed from a 12-player normal form game G_{12}. Assume players 1, 3, and 4 form one coalition; players 2 and 11 form a second coalition; and players 5, 8, and 9 form a third; players 6, 10, and 12 form a fourth; and finally player 7 forms a coalition of a single player. This coalitional semi-cooperative game is represented by $G_{1,3,4|2,11|5,8,9|6,10,12|7}$, where the coalitions are separated by vertical bars and the players in each coalition arranged in ascending order and separated by commas. Each strategy profile is depicted as $((s_1, s_3, s_4), (s_2, s_{11}), (s_5, s_8, s_9),$ $(s_6, s_{10}, s_{12}), (s_7))$ with corresponding payoff $(u_1 + u_3 + u_4, u_2 + u_6, u_5 + u_8 + u_9, u_7)$.

We next describe a metric for selecting a "best" separation of the players of a normal form game into one or more coalitions. For each coalitional semi-cooperative game $G_{1_1,\ldots,1_{C_1}|2_1,\ldots,2_{C_2}|\ldots|Q_1,\ldots,Q_{C_Q}}$ and for each payoff matrix cell of this game corresponding to a GSE, divide the payoff for each coalition among its members according to their corresponding levels of contribution. The GSE is used because (i) each coalition unto itself is greedy and (ii) the players within a coalition cooperate. We then "fairly" divide this payoff among players of the coalition in an attempt to make the coalition's payoff as large as possible. The new payoffs of the players together is called the modified payoff. For every GSE of each coalitional game of G, we compute the geometric mean of its modified payoff. A coalitional game with the largest geometric mean yields a best set of coalitions (i.e., the ones for that game). Obviously there could be multiple such best sets, and any such set might be selected by an arbitrator arbitrarily

We now present Algorithm 1 to determine the coalitional games giving the largest geometric mean among all coalitional games for a given n-person game in normal form.

Algorithm 1

Step 1. Compute the average payoff A_i for player i, $i = 1, \ldots, n$, over the original game G.

Step 2. Consider a coalitional semi-cooperative game $G_{1_1,\ldots,1_{C_1}|\ \ldots\ |Q_1,\ldots,Q_{C_Q}}$ of G. For each strategy profile of $G_{1_1,\ldots,1_{C_1}|\ \ldots\ |Q_1,\ldots,Q_{C_Q}}$, sum the payoffs of players in each coalition and form the payoff matrix for this coalitional game.

Step 3. Identify all GSEs of the game in Step 2.

Step 4. For each coalition in Step 2, sum the averages in Step 1 of all the players in that coalition to obtain the sum V_j for the coalitions $j = 1, \ldots, Q$.

Step 5. If player i, $i = 1, \ldots, n$, is a member of coalition j, compute $R_{ij} = \frac{A_i}{V_j}$.

Step 6. For each GSE of Step 3 and each coalition's payoff for this GSE, divide the payoff of each coalition among its members by multiplying the ratio of their contribution R_{ij} by the payoff for that coalition from Step 2. This resulting payoff is called the modified payoff.

Step 7. Compute the geometric mean of the modified payoff obtained from Step 6.

Step 8. Repeat Steps 2-7 for the remaining coalitional semi-cooperative games of G.

Step 9. Select a best set of coalitions as one associated with a coalitional game having the largest geometric mean computed in Step 7 among all coalitional games.

2.4 Examples

In this section, we present two examples to illustrate Algorithm 1. In Example 1, the GSE of the original game G is different from the GSE of the coalitional game for the best set of coalitions in Step 9. Unlike Example 1, in Example 2, the GSE of the coalitional game for a best set of coalitions in Step 9 coincides with the GSE of the original game G.

2.4.1 Example 1

Let G be a 3-player game in normal form where the strategies for player i are s_i and t_i, $i = 1, 2, 3$ as shown in the payoff matrix of Table 2.1.

Table 2.1. Payoff Matrix for G

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	(6,5,5)	(1,-2,4)	(6,1,3)	(4,1,4)
t_1	(4,2,3)	(3,1,4)	(3,6,7)	(1,0,5)

To determine a best set (in the sense of Section 3) of coalitions to form using Algorithm 1, consider the coalitional semi-cooperative games of G. There are five possibilities. The first possibility consists of two coalitions - coalition I consisting of player 1 alone and coalition II consisting of players 2 and 3. We model the associated coalitional semi-cooperative game consisting of coalition I versus coalition II as a two-player normal form game $G_{1|2,3}$, where coalition I is construed as the first player and coalition II as the second player. Similarly, the other possibilities are $G_{1,2|3}$, $G_{1,3|2}$, $G_{1,2,3}$, and $G_{1|2|3} = G$.

In Step 1 of Algorithm 1, $A_1 = 3.5$, $A_2 = 1.75$ **and** $A_3 = 4.375$. **Now consider the coalitional semi-cooperative** $G_{1|2,3}$ **with payoff matrix in Table 2.2 as stated in Step 2.**

Table 2.2. Payoff Matrix for $G_{1|2,3}$

	(s_2, s_3)	(t_2, s_3)	(s_2, t_3)	(t_2, t_3)
s_1	(6,10)	(1,2)	(6,4)	(4,5)
t_1	(4,5)	(3,5)	(3,13)	(1,5)

Coalition II in $G_{1|2,3}$ **has four strategies, which are simply all combinations of the strategies of players 2 and 3. For example, the cell** $(s_1, (s_2, s_3))$ **in Table 2.2 has the payoff vector** $(6, 10)$ **where the payoff** 10 **for coalition II comes from cell** (s_1, s_2, s_3) **in Table 2.1 by adding the payoffs 5 for player 2 and 5 for player 3. Note that in Table 2.2, none of the original strategies of** G **has been omitted. There are still 8 possibilities.**

Table 2.3 represents the GSE matrix of $G_{1|2,3}$ **as in Step 3 with GSEs** $(s_1, (s_2, s_3))$ **and** $(t_1, (s_2, t_3))$ **with corresponding payoffs in** $G_{1|2,3}$ **as** $(6, 10)$ **and** $(3, 13)$ **respectively.**

In Step 4, $V_1 = 3.5$ **for player 1 in coalition I and the sum of the averages of players 2 and 3 in coalition II is** $V_2 = 1.75 + 4.375 = 6.125$. **In Step 5,** $R_{11} = \frac{3.5}{3.5} = 1$, $R_{22} = \frac{1.75}{6.125} = 0.29$ **and**

Table 2.3. GSE Matrix for $G_{1|2,3}$

	(s_2, s_3)	(t_2, s_3)	(s_2, t_3)	(t_2, t_3)
s_1	**0.2500**	0.0139	0.1000	0.0370
t_1	0.0370	0.0278	**0.2500**	0.0185

$R_{32} = \frac{4.375}{6.125} = 0.71$. **In Step 6, the new payoffs corresponding to the first GSE of** $G_{1|2,3}$ **for player 1 is** 6 **because player 1's ratio of contribution to coalition I is** $R_{11} = 1$. **The payoff for coalition II corresponding to this GSE of** $G_{1|2,3}$ **is divided between players 2 and 3 using their ratios of contribution** $R_{22} = 0.29$ **and** $R_{32} = 0.71$ **respectively. Player 2's new payoff is** $0.29 \times 10 = 2.9$ **and the new payoff of player 3 is** $0.71 \times 10 = 7.1$. **The modified payoff corresponding to this GSE of** $G_{1|2,3}$ **is** $(6, 2.9, 7.1)$ **and the geometric mean of this payoff as in Step 7 is** $\sqrt[3]{(6 \times 2.9 \times 7.1)} = 4.9805$. **The new payoff corresponding to the second GSE of** $G_{1|2,3}$ **for player 1 is** 3 **since** $R_{11} = 1$. **The payoff for coalition II corresponding to the GSE of** $G_{1|2,3}$ **is again divided between players 2 and 3 using** $R_{22} = 0.29$ **and** $R_{32} = 0.71$ **respectively. Player 2's new payoff is** $0.29 \times 13 = 3.8$ **and the new payoff of player 3 is** $0.71 \times 13 = 9.2$. **The modified payoff corresponding to this GSE of** $G_{1|2,3}$ **is** $(3, 3.8, 9.2)$ **and the geometric mean of this payoff as in Step 7 is** $\sqrt[3]{(3 \times 3.8 \times 9.2)} = 4.7159$.

In Step 8, we consider the remaining coalitional games and repeat Steps 2 to 7 for each of these games. Next is the game $G_{1,2|3}$ made of two coalitions where coalition I consists of players 1 and 2 and coalition II consists of player 3 alone. In Step 2, the payoff matrix for $G_{1,2|3}$ is Table 2.4.

Table 2.4. Payoff Matrix for $G_{1,2|3}$

	s_3	t_3
(s_1, s_2)	(11,5)	(7,3)
(t_1, s_2)	(6,3)	(9,7)
(s_1, t_2)	(-1,4)	(5,4)
(t_1, t_2)	(4,4)	(1,5)

Table 2.5 represents the GSE matrix of $G_{1,2|3}$ as in Step 3 with GSEs $((s_1, s_2), s_3)$ and $((t_1, s_2), t_3)$ with corresponding payoffs in $G_{1,2|3}$ as $(11, 5)$ and $(9, 7)$ respectively.

Table 2.5. GSE Matrix for $G_{1,2|3}$

	s_3	t_3
(s_1, s_2)	**0.3333**	0.0400
(t_1, s_2)	0.0333	**0.3333**
(s_1, t_2)	0.0192	0.0357
(t_1, t_2)	0.0313	0.0303

The sum of the averages of players 1 and 2 in Step 4 is $V_1 = 3.5 + 1.75 = 5.25$ **and player 3's in coalition II is** $V_2 = 4.375$.

In Step 5, $R_{11} = \frac{3.5}{5.25} = 0.67$, $R_{21} = \frac{1.75}{5.25} = 0.33$ **and** $R_{32} = \frac{4.375}{4.375} = 1$**. In Step 6, player 1's new payoff corresponding to the first GSE** $((s_1, s_2), s_3)$ **is** $0.67 \times 11 = 7.4$, **and player 2's is** $0.33 \times 11 = 3.6$. **In coalition II,** $R_{32} = 1$ **since player 3 is the only player in it. Thus, player 3's new payoff corresponding to this GSE of** $G_{1,2|3}$ **is** 5. **The modified payoff of all three players that corresponds to this GSE of** $G_{1,2|3}$ **is** $(7.4, 3.6, 5)$ **in Step 6 and the geometric mean of this modified payoff in Step 7 is** $\sqrt[3]{(7.4 \times 3.6 \times 5)} = 5.1070$. **Again in Step 6, player 1's new payoff corresponding to the second GSE** $((t_1, s_2), t_3)$ **is** $0.67 \times 9 = 6$, **and player 2's is** $0.33 \times 9 = 3$. **In coalition II,** $R_{32} = 1$ **since player 3 is the only player in it. Thus, player 3's new payoff corresponding to this GSE of** $G_{1,2|3}$ **is** 7. **The modified payoff of all three players that corresponds to this GSE of** $G_{1,2|3}$ **is** $(6, 3, 7)$ **and the geometric mean of the modified payoff in Step 7 is** $\sqrt[3]{(6 \times 3 \times 7)} = 5.0132$.

Now consider the game $G_{1,3|2}$ **with two coalitions where coalition I comprises players 1 and 3 and coalition II comprises player 2 alone. The payoff matrix for** $G_{1,3|2}$ **in Step 2 is Table 2.6.**

Table 2.6. Payoff Matrix for $G_{1,3|2}$

	s_2	t_2
(s_1, s_3)	(11,5)	(5,-2)
(t_1, s_3)	(7,2)	(7,1)
(s_1, t_3)	(9,1)	(8, 1)
(t_1, t_3)	(10,6)	(6,0)

In Step 3, Table 2.7 is the GSE matrix for $G_{1,3|2}$ **with GSEs** $((s_1, s_3), s_2)$ **and** $((t_1, t_3), s_2)$ **with corresponding payoffs** $(11, 5)$ **and** $(10, 6)$ **respectively.**

Table 2.7. GSE Matrix for $G_{1,3|2}$

	s_2	t_2
(s_1, s_3)	**0.5000**	0.0159
(t_1, s_3)	0.0400	0.0333
(s_1, t_3)	0.0556	0.0417
(t_1, t_3)	**0.5000**	0.0238

The sum of the averages of players 1 and 3 in Step 4 is $V_1 = 3.5 + 4.375 = 7.875$ **and the sum of the average of player 2 in coalition II is** $V_2 = 1.75$. **In Step 5,** $R_{11} = \frac{3.5}{7.875} = 0.44$, $R_{22} = \frac{1.75}{1.75} = 1$, **and** $R_{31} = \frac{4.375}{7.875} = 0.56$. **In Step 6, player 1's new payoff corresponding to the first GSE** $((s_1, s_3), s_2)$ **is** $0.44 \times 11 = 4.8$, **player 2's new payoff is** 5 **since** $R_{22} = 1$, **and player 3's is** $0.56 \times 11 = 6.2$. **The modified payoff of all three players corresponding**

to this GSE of $G_{1,3|2}$ **is** $(4.8, 5, 6.2)$, **and the geometric mean of the modified payoff in Step 7 is** $\sqrt[3]{(4.8 \times 5 \times 6.2)} = 5.2991$. **Again in Step 6, player 1's new payoff corresponding to the second GSE** $((t_1, t_3), s_2)$ **is** $0.44 \times 10 = 4.4$, **player 2's new payoff is** 6 **since** $R_{22} = 1$, **and player 3's is** $0.56 \times 10 = 5.6$. **The modified payoff of all three players corresponding to this GSE of** $G_{1,3|2}$ **is** $(4.4, 6, 5.6)$, **and the geometric mean of the modified payoff in Step 7 is** $\sqrt[3]{(4.4 \times 6 \times 5.6)} = 5.2877$.

We next consider the grand coalition in which all the players of G **form the single coalition** $G_{1,2,3}$. **Table 2.8 gives the corresponding payoff matrix from Step 2.**

Table 2.8. Payoff Matrix for $G_{1,2,3}$

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	16	3	10	9
t_1	9	8	16	6

The GSEs of $G_{1,2,3}$ **in Step 3 are** $((s_1, s_2, s_3))$ **and** $((t_1, s_2, t_3))$ **with the corresponding payoffs in Table 2.9 as** 16.

In Step 4, $V_1 = 3.5 + 1.75 + 4.375 = 9.625$. **Moreover,** $R_{11} = \frac{3.5}{9.625} = 0.36$, $R_{21} = \frac{1.75}{9.625} = 0.18$, **and** $R_{31} = \frac{4.375}{9.625} = 0.46$ **in Step 5. In Step 6, Player 1's new payoff corresponding to the first GSE**

Table 2.9. GSE Matrix for $G_{1,2,3}$

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	**1.0000**	0.1875	0.6250	0.5625
t_1	0.5625	0.5000	**1.000**	0.3750

$((s_1, s_2, s_3))$ **is** $0.36 \times 16 = 5.8$**, player 2's new payoff is** $0.18 \times 16 = 2.9$, **and player 3's is** $0.46 \times 16 = 7.3$. **The modified payoff of all three players corresponding to the first GSE of** $G_{1,2,3}$ **is** $(5.8, 2.9, 7.3)$, **and the geometric mean of the modified payoff in Step 7 is** $\sqrt[3]{(5.8 \times 2.9 \times 7.3)} = 4.9703$. **Similarly, the modified payoff and geometric mean corresponding to the second GSE** $((t_1, s_2, t_3))$ **are** $(5.8, 2.9, 7.3)$ **and** 4.9703 **respectively.**

Finally, consider the game G **with three coalitions in which each player constitutes a coalition unto himself. This game** G **as a coalitional semi-cooperative game is represented by** $G_{1|2|3}$. **The payoff matrix of** $G_{1|2|3}$ **in Step 2 is Table 2.1. In Step 3, the GSE matrix of** $G_{1|2|3}$ **is depicted in Table 2.10 with GSE** $((t_1), (s_2), (t_3))$ **and corresponding payoff in Table 2.11 as** $(3, 6, 7)$. **Thus, each player's new payoff corresponding to the GSE of** $G_{1|2|3}$ **is the corresponding payoff in** G.

Table 2.10. GSE Matrix for G or $G_{1|2|3}$

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	0.1667	0.0046	0.0333	0.0139
t_1	0.0133	0.0104	**0.2500**	0.0079

In Step 4, $V_1 = 3.5$, $V_2 = 1.75$ **and** $V_3 = 4.375$. **Since each player is a coalition onto himself,** $R_{11} = R_{22} = R_{33} = 1$ **from Step 5. The modified payoff of all three players that corresponds to the GSE of** $G_{1|2|3}$ **is** $(3, 6, 7)$, **and the geometric mean of this modified payoff in Step 7 is** $\sqrt[3]{(3 \times 6 \times 7)} = 5.0133$.

The geometric means of all coalitional semi-cooperative games from the game G **are** 4.9805 **and** 4.7159 **for** $G_{1|2,3}$, 5.1070 **and** 5.0132 **for** $G_{1,2|3}$, **and** 5.2991 **and** 5.2877 **for** $G_{1,3|2}$. **In addition, the geometric mean of** $G_{1|2|3}$ **is** 5.0132 **and that of** $G_{1,2,3}$ **is** 4.9703.

According to Step 9, the best set of coalitions is given by the two coalitions from $G_{1,3|2,}$, **where coalition I consists of players 1 and 3 and coalition II consists of players 2 alone. In particular, the geometric mean associated with** $G_{1,3|2}$ **is the largest among the geometric means of all 5 coalitional semi-cooperative games of** G.

Example 1 depicts the three players select the modified payoff $(4.8, 5, 6.2)$ from the coalitional game $G_{1,3|2}$ instead of the payoff $(3, 6, 7)$ from $G = G_{1|2|3}$. Although players 2 and 3 seem to do worse when comparing these two payoffs, fairness is enforced in the following ways. The R_{ij}'s in Step 4 of Algorithm 1 ensure a player's new payoff after joining a coalition is proportional to their payoff over the game G. In addition, the geometric mean minimizes the variability in the modified payoff from the best coalition. and as mentioned earlier, gives the n players payoffs that are closest together.

In Example 1, the GSE $((s_1, s_3), s_2)$ of the coalitional game which corresponds to the best coalitions to form and that of G $((t_1), (s_2), (t_3))$ are different. We next present an example where the GSE from G and the GSE of the coalitional game with the best set of coalitions coincide.

2.4.2 Example 2

Let G be a 3-player game in normal form where the strategies for player i are s_i and t_i, $i = 1, 2, 3$ as shown in the payoff matrix of Table 2.11.

Table 2.11. Payoff Matrix for G

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	(1,1,0)	(1,2,3)	(2,4,5)	(0,1,2)
t_1	(4,3,2)	(5,7,2)	(7,3,2)	(4,1,3)

To determine a best set (in the sense of Section 3) of coalitions to form using Algorithm 1, consider the coalitional semi-cooperative games of G. **There are five possibilities namely** $G_{1|2,3}$, $G_{1,3|2}$**,** $G_{1,2,3}$, **and** $G_{1|2|3} = G$.

In Step 1 of Algorithm 1, $A_1 = 3$, $A_2 = 2.75$ **and** $A_3 = 2.375$. **Now consider the coalitional semi-cooperative** $G_{1|2,3}$ **with payoff matrix Table 2.12 as stated in Step 2.**

Table 2.12. Payoff Matrix for $G_{1|2,3}$

	(s_2, s_3)	(t_2, s_3)	(s_2, t_3)	(t_2, t_3)
s_1	(1,1)	(1,5)	(2,9)	(0,3)
t_1	(4,5)	(5,9)	(7,5)	(4,4)

Table 2.13 represents the GSE matrix of $G_{1|2,3}$ **as in Step 3 with GSE** $(t_1, (t_2, t_3))$ **and corresponding payoff in** $G_{1|2,3}$ **as** $(5, 9)$.

In Step 4, $V_1 = 3$ **for player 1 in coalition I and the sum of the averages of players 2 and 3 in coalition II is** $V_2 = 2.75 +$

Table 2.13. GSE Matrix for $G_{1|2,3}$

	(s_2, s_3)	(t_2, s_3)	(s_2, t_3)	(t_2, t_3)
s_1	0.0159	0.0286	0.1667	0.0178
t_1	0.0500	**0.3333**	0.2000	0.0417

$2.375 = 5.125$. **In Step 5,** $R_{11} = \frac{3}{3} = 1$, $R_{22} = \frac{2.75}{5.125} = 0.54$ **and** $R_{32} = \frac{2.375}{5.125} = 0.46$. **In Step 6, the new payoff corresponding to the GSE of** $G_{1|2,3}$ **for player 1 is** 5 **because** $R_{11} = 1$. **The payoff for coalition II corresponding to the GSE of** $G_{1|2,3}$ **is divided between players 2 and 3 using** $R_{22} = 0.54$ **and** $R_{32} = 0.46$ **respectively. Player 2's new payoff is** $0.54 \times 9 = 4.86$ **and the new payoff of player 3 is** $0.46 \times 9 = 4.14$. **The modified payoff corresponding to the GSE of** $G_{1|2,3}$ **is** $(5, 4.86, 4.14)$ **and the geometric mean of this payoff as in Step 7 is** $\sqrt[3]{(5 \times 4.86 \times 4.14)} = 4.6509$.

In Step 8, we consider the remaining coalitional games and repeat Steps 2 to 7 for each of these games. Next is the game $G_{1,2|3}$ **made of two coalitions where coalition I consists of players 1 and 2 and coalition II consists of player 3 alone. In Step 2, the payoff matrix for** $G_{1,2|3}$ **is Table 2.14.**

Table 2.15 represents the GSE matrix of $G_{1,2|3}$ **as in Step 3 with GSE** $((t_1, t_2), s_3)$ **and corresponding payoff in** $G_{1,2|3}$ **as** $(12, 2)$.

Table 2.14. Payoff Matrix for $G_{1,2|3}$

	s_3	t_3
(s_1, s_2)	(2,0)	(6,5)
(t_1, s_2)	(7,2)	(10,2)
(s_1, t_2)	(3,3)	(1,2)
(t_1, t_2)	(12,2)	(5,3)

Table 2.15. GSE Matrix for $G_{1,2|3}$

	s_3	t_3
(s_1, s_2)	0.0152	0.1428
(t_1, s_2)	0.0147	0.0833
(s_1, t_2)	0.0333	0.0208
(t_1, t_2)	**0.2500**	0.0417

The sum of the averages of players 1 and 2 in Step 4 is $V_1 = 3 + 2.75 = 5.75$ **and player 2's in coalition II is** $V_2 = 2.375$. **In Step 5,** $R_{11} = \frac{3}{5.75} = 0.52$, $R_{21} = \frac{2.75}{5.75} = 0.48$ **and** $R_{32} = \frac{2.375}{2.375} = 1$. **Player 1's new payoff in Step 6 is** $0.52 \times 12 = 6.24$, **and player 2's is** $0.48 \times 12 = 5.76$. **In coalition II,** $R_{32} = 1$ **since player 3 is the only player in it. Thus, player 3's new payoff corresponding to the GSE of** $G_{1,2|3}$ **is** 2. **The modified payoff of all three players that corresponds to the GSE of** $G_{1,2|3}$ **is** $(6.24, 5.76, 2)$ **and the geometric mean of the modified payoff in Step 7 is** $\sqrt[3]{(6.24 \times 5.76 \times 2)} = 4.1579$.

Now consider the game $G_{1,3|2}$ **with two coalitions, where coalition I comprises players 1 and 3 and coalition II comprises**

player 2 alone. The payoff matrix for $G_{1,3|2}$ from Step 2 is Table 2.16.

Table 2.16. Payoff Matrix for $G_{1,3|2}$

	s_2	t_2
(s_1, s_3)	(1,1)	(4,2)
(t_1, s_3)	(6,3)	(7,7)
(s_1, t_3)	(7,4)	(2, 1)
(t_1, t_3)	(9,3)	(7,1)

Table 2.17 from Step 3 shows the GSE matrix for $G_{1,3|2}$ **with GSE** $((t_1, s_3), t_2)$ **and corresponding payoff** $(7, 7)$.

Table 2.17. GSE Matrix for $G_{1,3|2}$

	s_2	t_2
(s_1, s_3)	0.0158	0.0278
(t_1, s_3)	0.0500	**0.3333**
(s_1, t_3)	0.0833	0.0179
(t_1, t_3)	0.2000	0.0476

The sum of the averages of players 1 and 3 in Step 4 is $V_1 = 3 + 2.375 = 5.375$ **and the sum of the average of player 2 in coalition II is** $V_2 = 2.75$. **In Step 5,** $R_{11} = \frac{3}{5.375} = 0.56$, $R_{22} = \frac{2.75}{2.75} = 1$, **and** $R_{31} = \frac{2.375}{5.375} = 0.44$. **In Step 6, player 1's new payoff is** $0.56 \times 7 = 3.92$, **player 2's new payoff is** 7 **since** $R_{22} = 1$, **and player 3's is** $0.44 \times 7 = 3.08$. **The modified payoff of all**

three players corresponding to the GSE of $G_{1,3|2}$ **is** $(3.92, 7, 3.08)$, **and the geometric mean of the modified payoff in Step 7 is** $\sqrt[3]{(3.92 \times 7 \times 3.08)} = 4.3885$.

We next consider the grand coalition in which all the players of G **form the single coalition** $G_{1,2,3}$. **Table 2.18 gives the corresponding payoff matrix from Step 2.**

Table 2.18. Payoff Matrix for $G_{1,2,3}$

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	2	6	11	3
t_1	9	14	12	8

The GSE of $G_{1,2,3}$ **in Step 3 is** $((t_1, t_2, s_3))$ **in Table 2.19 with corresponding payoff** 14.

Table 2.19. GSE Matrix for $G_{1,2,3}$

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	0.0769	0.1111	0.2500	0.0833
t_1	0.1667	**1.000**	0.3333	0.1429

In Step 4, $V_1 = 3 + 2.75 + 2.375 = 8.125$. **Moreover,** $R_{11} = \frac{3}{8.125} = 0.37$, $R_{21} = \frac{2.75}{8.125} = 0.34$, **and** $R_{31} = \frac{2.375}{8.125} = 0.29$ **in Step 5. In Step 6, Player 1's new payoff corresponding to the GSE of** $G_{1,2,3}$

is $0.37 \times 14 = 5.18$**, player 2's new payoff is** $0.34 \times 14 = 4.76$**, and player 3's is** $0.29 \times 14 = 4.06$**. The modified payoff of all three players corresponding to the GSE of** $G_{1,2,3}$ **is** $(5.18, 4.76, 4.06)$**, and the geometric mean of the modified payoff in Step 7 is** $\sqrt[3]{(5.18 \times 4.76 \times 4.06)} = 4.6432$.

Finally, consider the game G **with three coalitions in which each player constitutes a coalition unto himself. This game** G **as a coalitional semi-cooperative game is represented by** $G_{1|2|3}$**. The payoff matrix of** $G_{1|2|3}$ **in Step 2 is Table 2.11. In Step 3, the GSE matrix of** $G_{1|2|3}$ **is depicted in Table 2.20 with GSE** $((t_1), (t_2), (s_3))$ **and corresponding payoff in Table 2.11 as** $(5, 7, 2)$**. Thus, each player's new payoff corresponding to the GSE of** $G_{1|2|3}$ **is the corresponding payoff in** G.

Table 2.20. GSE Matrix for G or $G_{1|2|3}$

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	0.0034	0.0079	0.0417	0.0045
t_1	0.0125	**0.0833**	0.0500	0.0119

In Step 4, $V_1 = 3,\ V_2 = 2.75$ **and** $V_3 = 2.375$**. Since each player is a coalition onto himself,** $R_{11} = R_{22} = R_{33} = 1$ **from Step 5. The modified payoff of all three players that corresponds to**

the GSE of $G_{1|2|3}$ in Step 6 is $(5, 7, 2)$, and the geometric mean of this modified payoff in Step 7 is $\sqrt[3]{(5 \times 7 \times 2)} = 4.1212$.

The geometric means of all coalitional semi-cooperative games from the game G are $4.6509, 4.1579, 4.3885, 4.6432,$ and 4.1212 for $G_{1|2,3}, G_{1,2|3}, G_{1,3|2}, G_{1|2|3},$ and $G_{1|2|3}$ respectively. According to Step 9, the best set of coalitions is given by the two coalitions from $G_{1|2,3}$, where coalition I consists of players 1 alone and coalition II consists of players 2 and 3. In particular, the geometric mean associated with $G_{1|2,3}$ is the largest among the geometric means of all 5 coalitional semi-cooperative games of G.

2.5 Conclusion

The notion of a coalitional semi-cooperative game of a normal form game, as presented here, models a given normal form game as a cooperative game rather than a non-cooperative game by forming coalitions via Algorithm 1. In this approach, we use the GSE, which is a reasonable facsimile for the NE. The NE may not exist in pure strategies, but the GSE always exists and can be easily computed. Future work

could focus on incorporating the other scalar equilibria given in [17] in forming coalitions from a normal form game.

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Modeling Degrees of Player Selfishness or Altruism in Normal Form Games

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Abstract

For a given n-person normal form game, we consider the situation where, for each strategy profile in the game, every player gives a pre-determined fraction of his payoff selfishly to himself and altruistically to the remaining $n-1$ players. We show that the Nash equilibrium and Berge equilibrium are extreme cases of this situation.

Game Theory; Normal Form Game; Markov Chains.

CHAPTER 3

Modeling Degrees of Player Selfishness or Altruism in Normal Form Games

3.1 Introduction

The current use of game theory primarily involves the Nash equilibrium (NE), which for an n-person game models selfishness because in an NE, every player's strategy maximizes his payoff for the strategies of the other $n-1$ players. In an NE, no player can unilaterally improve his payoff by a strategy change without a strategy change by the other players. Secondarily, game theory considers the Berge equilibrium (BE), which models altruism. In an n-person game, a pure BE is a strategy profile in which every $n-1$ players have strategies that maximize the payoff of the remaining player. Therefore in a BE, no unilateral change in the strategy of any player can improve the payoff of any other player. In this paper, we consider the situation where for a given n-person game in normal form and each strategy profile in the game, every player gives a pre-determined fraction of his payoff corresponding to that

strategy profile both to himself and to the remaining $n - 1$ players. We show that the NE and BE are special cases of this model.

As background, we note that [Nash [1950]] established the NE for non-cooperative games in which each player is assumed to be selfish and cannot obtain a better payoff by unilaterally changing his strategy. Using fixed point theorems, [Nash [1951]] proved that an NE always exists in mixed strategies. NEs for two-person nonzero-sum games have been characterized by [Mills [1960]] as solutions of a system of linear inequalities in which some variables are restricted to be integers while [Mangasarian and Stone [1964]] modeled the problem of finding an NE for a two-person nonzero-sum game as a quadratic programming problem. Extending these results, [Batbileg and Enkhbat [2010]] formulated the problem of finding an NE for a three-person nonzero-sum game as a nonlinear programming problem and later generalized in [Batbileg and Enkhbat [2011]] for n-person games. Overviews of the NE can be found in [Holt and Roth [2004]] and [Hoang [2012]]. A polynomial algorithm for finding all pure NEs was given by [Corley [2020]].

Unlike the NE where each player is assumed to be selfish, an altruistic solution concept was introduced by [Berge [1957]] and formally defined in [Zhukovskiy [1985]]. This solution concept is called a Berge equilibrium (BE), where a unilateral change of strategy by any one player cannot increase the payoff of another player. [Colman *et al.* [2011]] provided a theorem on the existence of the BE, how it models altruistic cooperation and its relationship with the NE for two-player games by permuting utility functions. [Pottier and Nessah [2014]] defined more general transformations of games that led to a correspondence between the NE and BE. Disregarding the relationship between the BE and NE in [Colman *et al.* [2011]], [Musy *et al.* [2012]] restated the previous theorem, proved the existence of the BE without using the NE and deduced a method for computing the BE. [Corley and Kwain [2015]] developed a polynomial algorithm for obtaining all BEs. [Corley [2015]] then extended the existence of the BE from pure strategies to mixed strategies to provide a dual to the NE referred to as the mixed BE. He also showed that for each $n > 2$ there exists a game for which no mixed BE exists. Incorporating both the NE and BE, [Abalo and Kostreva [2004]]

proved the existence of both the NE and BE and later in [Abalo and Kostreva [2005]], proved the existence of the Berge-Nash equilibria for a finite or an infinite number person games under certain assumptions.

Other people have considered other general equilibria different from what we did in this paper. A generalized equilibrium (GE) for an n-person normal form game is presented in [Nahhas and Corley [2017]]. The GE involved a collection of mixed strategies in which no player in some subset B of the players can achieve a better expected payoff if players in an associated set G change strategies unilaterally. This work here appears to be the only one that incorporates both the NE and BE. We offer a simpler approach with an intuitive but practical interpretation.

The paper is organized as follows. We present preliminary notation in Section 2. In Section 3 we give the relationship of the one-step transition probability matrix of a Markov chain to the matrix of pre-determined fractions that any player gives of his payoff to any other player. An example for a 3-player game is given in Section 4, and conclusions are stated in Section 5.

3.2 Preliminaries

Let $G_n = \left\langle I, (S_i)_{i\in I}, (u_i)_{i\in I} \right\rangle$ **be an** n**-player game where** $I = \{1, \ldots, n\}$ **is the set of players,** $S_i = \{s_i^1, \ldots, s^{m_i}\}$ **is the finite set of** $m_i \geq 2$ **pure strategies for player** i **and** $u_i(s)$ **is the utility of player** i **for an action profile** $s = (s_1, \ldots, s_n) \in \times_{j\in I} S_j = S$. **The** $m_1 \times \ldots \times m_n$ **matrix of** $\Big(u_i(s), \ldots, u_n(s)\Big)$, $s \in S$ **is called the payoff matrix for** G **and a game given in terms of a payoff matrix is called a normal form game.**

Given that player i **chooses each strategy** s_i^j **with probability** $\sigma_i(s_i^j)$**, a mixed strategy for player** i **denoted** $\sigma_i = (\sigma_i^1, \ldots, \sigma_i^{m_i})$**, is a probability distribution over the player's pure strategies set, where** $\sum_{j=1}^{m_i} \sigma_i(s_i^j) = 1$ **and** $\sigma_i(s_i^j) \geq 0,\ j = 1, \ldots, m_i$. **A strategy** σ^* **is a Nash equilibrium(NE) if no player with a unilateral change of strategy can increase his expected payoff. An NE always exists in mixed strategy but not in pure strategy. A strategy** $\hat{\sigma}$ **is a Berge Equilibrium (BE) if a unilateral change of strategy by any one player cannot increase the payoff of another player.**

In [Corley and Kwain [2015]], the regret and disappointment matrices of a two-person game with payoff matrix (A, B) **are denoted as** $R(A, B)$ **and** $D(A, B)$ **respectively. A pure strat-**

egy pair of a two-person game is an NE if and only if (0, 0) is the corresponding entry in $R(A, B)$ whereas a pure strategy pair is a DE if and only if $(0, 0)$ is the corresponding entry in $D(A, B)$.

We next provide some background on Markov chains as that will be helpful in our later development. See [Gallager [2013]] for details. A stochastic process is a collection of random variables $\{X(t) : t \in T\}$, where T is an index set. A Markov chain is a stochastic process with a discrete index set and a discrete state space in which the probability of predicting a future event depends only on the state attained in the previous event. A one-step transition probability from state i to state j is defined as

$$p_{ij} = P[X_{n+1} = j | X_n = i], \forall i, j, \ n = 0, 1, 2, \ldots, \tag{3.1}$$

while the one-step transition matrix is defined by

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1M} \\ \vdots & \vdots & \ddots & \vdots \\ p_{M1} & p_{M2} & \cdots & p_{MM} \end{bmatrix}, \tag{3.2}$$

where M is the number of states and $\sum_{j=1}^{M} p_{ij} = 1, \ \forall i$. In particular, a homogeneous Markov chain is one in which the transi-

tion probabilities are independent of of n in (3.1). A one-step transition matrix is applied m times to obtain the m-step transition matrix defined by

$$\mathbf{P}^{(m)} = \begin{bmatrix} p_{11}^{(m)} & p_{12}^{(m)} & \cdots & p_{1M}^{(m)} \\ \vdots & \vdots & \ddots & \vdots \\ p_{M1}^{(m)} & p_{M2}^{(m)} & \cdots & p_{MM}^{(m)} \end{bmatrix}, \tag{3.3}$$

where M is the number of states and $\sum_{j=1}^{M} p_{ij}^{(m)} = 1,\ \forall i$. The element $p_{ij}^{(m)}$ in (3.3) is the probability of being in the state j from the state i after m transitions.

The absolute probabilities of a Markov chain are given as follows. Let $a_j^{(m)} = P[X_m = j]$, then $a_j^{(m)} = \sum_i a_i^{(0)} p_{ij}^{(m)} = \sum_i a_i^{(k)} p_{ij}^{(m-k)},\ k = 1, \ldots, m$ or $\mathbf{a}^{(m)} = \mathbf{a}^{(0)}\mathbf{P}^{(m)} = \mathbf{a}^{(k)}\mathbf{P}^{(m-k)},\ 0 \le k \le m$ where $\mathbf{a}^{(m)}$ is a row vector. The long-run probabilities of a Markov chain are given by $lim_{m\to\infty} a_j^{(m)} = \pi_j$, where π_j is the probability of being in state j as $m \to \infty$. $\vec{\pi} = (\pi_1, \ldots, \pi_n)$ is the vector of the components j. Since $\mathbf{a}^{(m)} = \mathbf{a}^{(m-1)}\mathbf{P},\ m = 1, 2, \ldots,$ taking the limits gives

$$\vec{\pi} = \vec{\pi}\mathbf{P} \tag{3.4}$$

where $\sum_j \pi_j = 1$ and $\pi_j \ge 0$. Solving (3.4) together with $\sum_j \pi_j = 1$ and $\pi_j \ge 0$ gives probability of being in state j as $m \to \infty$.

Next, we define the α and α-transformed matrices that model the degree to which each player gives all players, including himself, portions of his payoff. The result will be a new set of payoffs.

3.3 Degrees of Selfishness and Altruism

Let G_n be an n-person game in normal form. For a given strategy profile $s = (s_1, \ldots, s_n)$ with corresponding payoff $(u_1, \ldots, u_n)$, let α_{ij} represent the fraction of player i's payoff given to player j with $\sum\limits_{j=1}^{n} \alpha_{ij} = 1, \ \forall i$. The α_{ij}'s for all players of the game are represented in a matrix called the α-matrix A represented by

$$\mathbf{A} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}. \tag{3.5}$$

These α_{ij}s in the α-matrix A are somewhat analogous to the one-step transition probabilities in a Markov process as well as the level of optimism used in the Hurwicz criterion for decision making (see [Taha [2017]]). For a strategy profile $s = (s_1, \ldots, s_n)$ with payoff $(u_1(s), \ldots, u_n(s))$, apply A to the payoff vector to obtain the new set of payoffs. The resulting

payoff matrix after applying A to each of the payoffs in G_n is called the α-transformed payoff matrix. Note that the α-matrix can either be applied once or infinitely many times. Fix a strategy profile $s = (s_1, \ldots, s_n)$ of G_n with corresponding payoff vector $\mathbf{U_0}(s) = (u_1(s), \ldots, u_n(s))$, the new set of payoffs after a left matrix multiplication of the payoff vector $\mathbf{U_0}(s)$ by the α-matrix A resulting in a new set of payoff $\mathbf{U_1}(s)$. In other words, $\mathbf{U_1}(s) = \mathbf{U_0}(s)\mathbf{A}$. Next apply α-matrix to $\mathbf{U_1}(s)$ to obtain $\mathbf{U_2}(s)$, i.e., $\mathbf{U_2}(s) = \mathbf{U_1}(s)\mathbf{A} = \mathbf{U_0}(s)\mathbf{A}^2$. Applying A to $\mathbf{U_0}(s)$ m-times gives

$$\mathbf{U_m}(s) = \mathbf{U_0}(s)\mathbf{A}^m = \mathbf{U_{m-1}}(s)\mathbf{A}, \tag{3.6}$$

where $\mathbf{U_m}(s) = (u_1^{(m)}(s), \ldots, u_n^{(m)}(s))$.

Even though the α-matrix is not a probability matrix like the one-step transition matrix for a Markov chain, it has numerical properties like a probability matrix. Let m represent the number of times that A is applied to a given payoff vector in G_n. If the limit as the $m \to \infty$ exists, then we can obtain

$$\vec{u}(s) = \vec{u}(s)\mathbf{A}, \tag{3.7}$$

where $\vec{u}(s) = \lim_{m\to\infty} \mathbf{U}_m(s)$ much as in the derivation of (3.4). In addition, for every strategy profile $s = (s_1, \ldots, s_n)$,

$$\sum_{i=1}^{n} u_i(s) = K(s), \tag{3.8}$$

where $K(s)$ is a constant. Solving (3.7) and (3.8) gives the payoff each player receives in the long run for each strategy profile $s = (s_1, \ldots, s_n)$. Note that one of the equations in (3.7) is always redundant since the sum of each row in the α-matrix equals 1. Alternatively, we can obtain the long run payoffs by solving for the πs in 3.4 and using the α-matrix A. For each strategy profile $s = (s_1, \ldots, s_n)$ multiply $\sum_{i=1}^{n} u_i(s) = K(s)$ by each component of π to obtain the payoff that corresponding player receives in the long run.

Thus, for a strategy profile $(s_1, \ldots, s_n)$, the corresponding payoffs in the α-transformed payoff matrix of G_n is represented by

$$\Big(\sum_{i=1}^{n} \alpha_{i1} u_i(s), \sum_{i=1}^{n} \alpha_{i2} u_i(s), \ldots, \sum_{i=1}^{n} \alpha_{in} u_i(s)\Big). \tag{3.9}$$

For a 2-person game, we make the following observations for (3.9). If $\alpha_{ii} = 1$ and $\alpha_{ij} = 0$, for $i \neq j$, then the $(0,0)$ entry in $R(A,B)$ of the α-transformed matrix is an NE. Similarly, if $\alpha_{ij} = 1$, for $i \neq j$, and $\alpha_{ii} = 0$, then the $(0,0)$ entry in $R(A,B)$ of the α-transformed matrix is a BE. Thus the NE and BE are

extreme cases for the method presented in this section. We next present an example to show how the method works.

3.4 Example

Let G_3 **be a 3-player game in normal form where the strategies for player** i **are** s_i **and** t_i**,** $i = 1, 2, 3$ **as shown in the payoff matrix of Table 3.1.**

Table 3.1. Payoff Matrix for G_3

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	(3,1,2)	(3,4,0)	(6,0, 3)	(3,5,1)
t_1	(2,4,4)	(1,4,5)	(7,1,3)	(4,2,3)

For G_3, **the** α**-matrix which is in (3.10) shows the fraction of each player's payoff they are willing to give to themselves and to the other players. For example, the first row of** A **shows that for each payoff vector in** G_3, **player 1 is willing to give** 0.4, 0.3, **and** 0.3 **of his payoff in that cell to himself, player 2, and player 3 respectively.**

$$\mathbf{A} = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.1 & 0.5 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix} \tag{3.10}$$

If A is applied to each payoff vector in G_3 one time, then the α-transformed payoff matrix of G_3 is

Table 3.2. α-Transformed Payoff Matrix for G_3 when $m = 1$

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	(1.9, 1.8, 2.3)	(1.6, 2.9, 2.5)	(3.3, 2.4, 3.3)	(2.0, 3.6, 3.4)
t_1	(2.4, 3.4, 4.2)	(2.3, 3.3, 4.4)	(3.8, 3.2, 4.0)	(2.7, 2,8, 3.5)

From (3.10), $\alpha_{11} = 0.4$, $\alpha_{21} = 0.1$, $\alpha_{31} = 0.3$. $\alpha_{12} = 0.3$, $\alpha_{22} = 0.5$, $\alpha_{32} = 0.2$, $\alpha_{13} = 0.3$, $\alpha_{23} = 0.4$, **and** $\alpha_{33} = 0.5$. **In Table 3.2, the payoff corresponding to the strategy profile** (s_1, s_2, s_3) **is obtained as follows**

$$\begin{aligned} 3\alpha_{11} + 1\alpha_{21} + 2\alpha_{31} &= 3(0.4) + 1(0.1) + 2(0.3) = 1.9 \\ 3\alpha_{12} + 1\alpha_{22} + 2\alpha_{32} &= 3(0.3) + 1(0.5) + 2(0.2) = 1.8 \\ 3\alpha_{13} + 1\alpha_{23} + 2\alpha_{33} &= 3(0.3) + 1(0.4) + 2(0.5) = 2.3 \end{aligned}$$

We next consider the case when A is applied to the payoff vectors in G_3 infinitely many times, i.e., the α-transformed

matrix as $m \to \infty$. Table 3.3 gives the α-transformed payoff matrix of G_3 as $m \to \infty$.

Table 3.3. α-Transformed Payoff Matrix for G_3 as $m \to \infty$

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	(1.6, 1.9, 2.5)	(1.8, 2.3, 2.9)	(2.4, 2.9, 3.7)	(2.4, 2.9, 3.7)
t_1	(2.6, 3.2, 4.2)	(2.6, 3.2, 4.2)	(2.9, 3.5, 4.6)	(2.4, 2.9, 3.7)

The payoffs in the payoff matrix in Table 3.3 are obtained by solving $\vec{u}(s) = \vec{u}(s)\mathbf{A}$ and $\sum_{i=1}^{n} u_i(s) = K(s)$ for each strategy profile $s = (s_1, \ldots, s_n)$. So, for the strategy profile (s_1, s_2, s_3),

$$
\begin{aligned}
(u_1(s)\ u_2(s)\ u_3(s)) &= (u_1(s)\ u_2(s)\ u_3(s)) \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.1 & 0.5 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix} \\
u_1(s) + u_2(s) + u_3(s) &= 6
\end{aligned}
$$

Keeping the last equation and dropping one of the others gives the solution

$$u_1(s) = 1.6,\ u_2(s) = 1.9; u_3(s) = 2.5.$$

Repeat the same process for the rest of the strategy profiles to obtain the payoff matrix in Table 3.3. Alternatively, we use the process of Markov chain as in (3.4) with its conditions to

obtain the payoff of each player in the long-run. $\vec{\pi} = \vec{\pi}\mathbf{A}$ **where** $\sum_j \pi_j = 1$ **and** $\pi_j \geq 0$.

$$\begin{aligned}
(\pi_1\ \pi_2\ \pi_3) &= (\pi_1\ \pi_2\ \pi_3)\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.1 & 0.5 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix} \\
\pi_1 + \pi_2 + \pi_3 &= 1
\end{aligned}$$

Simplifying the above system gives

$$\begin{aligned}
\pi_1 &= 0.4\pi_1 + 0.1\pi_2 + 0.3\pi_3 \\
\pi_2 &= 0.3\pi_1 + 0.5\pi_2 + 0.2\pi_3 \\
\pi_3 &= 0.3\pi_1 + 0.4\pi_2 + 0.5\pi_3 \\
\pi_1 + \pi_2 + \pi_3 &= 1
\end{aligned}$$

Solving the above system of equations results in the following solution

$$\pi_1 = \frac{17}{65} = 0.2615,\ \pi_2 = \frac{21}{65} = 0.3231,\ \pi_3 = \frac{27}{65} = 0.4154.$$

The payoff in Table 3.3 corresponding to the strategy profile (s_1, s_2, s_3) **is obtained again by first considering the payoff corresponding to the strategy profile** (s_1, s_2, s_3) **in Table 3.1 which is** $(3, 1, 2)$. **Thus, the total payoff corresponding to** (s_1, s_2, s_3) **is**

$K(s) = 3 + 1 + 2 = 6$. **Multiplying each $\pi_j,\ j = 1, 2, 3$ to this total payoff results in**

$$6\pi_1 = 6\left(\frac{17}{65}\right) = 1.6$$
$$6\pi_2 = 6\left(\frac{21}{65}\right) = 1.9$$
$$6\pi_3 = 6\left(\frac{27}{65}\right) = 2.5.$$

3.5 Conclusion

For a given n-person normal form game, we discussed the situation where for each strategy profile in the game, every player gives a pre-determined fraction of his payoff corresponding to that strategy profile to himself and the remaining $n-1$ **players. Even though there is not a one-to-one correspondence between our method and Markov chains, we developed a model using transition probability matrices and an analog of it and also showed that the NE and BE are extreme cases of this model.**

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CHAPTER 4

General Conclusion

This dissertation comprises two papers. It is a comprehensive study on forming new coalitions in an n-person game in normal form and also sharing payoff among the players in such games.

In the first paper, we focused on all the possible sets of mutually exclusive and collective exhaustive coalitions of the n players in an n-person normal form game. A "best" set of coalitions is selected among all possible coalitional semi-cooperative games. In other words, we answer the queston: what is the "best" set of coalition to form in an n-person game in normal form.

We next presented sharing payoffs among the players in an n-person normal form game. Here, we defined how every player gives a pre-determined fraction of his payoff selfishly to himself and altruistically to the remaining $n-1$ players. The NE and BE are shown to be the extreme cases of our model. Examples are presented to explain the model further.

BIOGRAPHICAL STATEMENT

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