A STUDY IN THE FREENESS OF FINITELY GENERATED A_p^n-MODULES UPON RESTRICTION TO PRINCIPAL SUBALGEBRAS

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Abstract

A STUDY IN THE FREENESS OF FINITELY GENERATED A_p^n-MODULES UPON RESTRICTION TO PRINCIPAL SUBALGEBRAS

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We are interested in quantitative information on the freeness of modules over a truncated polynomial ring when restricting to subalgebras generated by a linear form. After investigating the structure of the truncated polynomial ring, subalgebras generated by a linear form, and corresponding vector spaces, we construct a generic representation and discuss its connection to a certain affine space. We quantify the abundance of freeness of modules using a certain variety called the rank variety. For any possible dimension we construct a module whose rank variety has that dimension. Finally, we define another variety, called the module variety, and show that the dimension of this variety is invariant under a change of subalgebra.

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CHAPTER 1

A_p^n-modules, Decompositions, and Matrix Representations

1.1 Introduction

An open area of study is the understanding of module categories for a particular ring. A 1961 paper by Heller and Reiner [14] pointed out that in most cases the module category is wild. This means that it is hopeless to try to classify all indecomposable modules up to isomorphism. One focus of research has been to classify modules in terms of invariants, which yield a weaker classification than isomorphism. A breakthrough in the construction of such invariants to study modules was made by Quillen [18] [19] [20] in a series of three papers. The method proposed by Quillen was to associate to modules certain geometric objects, called the support or cohomological variety. Alperin [1] proposed the study of modules via complexity, which is a generalization of the dimension of Quillen's varieties. Kroll [17] gave an effective method for computing the complexity of modules over a group algebra of an elementary abelian p-group.

Carlson proposed that another invariant could be used to further the study of modules over group algebras of an elementary abelian p-group. To this end Carlson introduced the rank variety in [7]. The rank variety involves restriction of the modules being studied to subalgebras of the group algebra of the module. The rank variety proved to be a useful invariant for a number of reasons. For one, the rank variety of a module characterizes projectivity as a result of Dade's Lemma [9]. Also, the tensor product property, that the rank variety of a tensor product is the intersection of the rank varieties of the two modules, holds [4]. Due to the effectiveness of Carlson's rank variety and support varieties for group algebras, the theory has been applied in a broader context, such as to p-restricted Lie algebras in [12]. More recently, rank varieties have been used to study the property of constant Jordan type for modules [8].

We are motivated to study the abundance of freeness, which we will later define in Definition 3.1.1. The study of constant rank by Carlson and others [8] [5] provided motivation for the topics of this thesis. We develop machinery (see Definition 2.2.4) that simultaneously recovers Carlson's rank variety (Definition 3.2.4) and also defines a new variety called the module variety (Definition 3.2.5). In the process, we concretely construct modules whose rank variety is any possible dimension (Corollary 4.2.16), define a canonical representation matrix (Definition 4.2.12), and study the invariance of the module variety (Theorem 5.1.4). The modules studied in this thesis are modules over a group algebra of an elementary abelian p-group. These group algebras are truncated polynomial rings, which we will be calling A_p^n.

The main object of this thesis is the truncated polynomial ring A_p^n where each variable is nilpotent with nilpotency index of fixed prime p (see Definition 1.2.1). The characteristic of the coefficient field is also p. Next, we discuss principal subalgebras of the truncated polynomial ring, which are generated by a linear form in Definition 1.2.5. We exploit the connections between finitely generated modules over the truncated polynomial ring and their underlying vector spaces over the coefficient field (see Fact 1.2.3).

In Section 1.3, we study the module decomposition of a finitely generated A_p^n-module when restricted to the principal subalgebras. We recall the decomposition theorem for modules over principal ideal domains, and show that the module decomposition over a principal ideal domain can be modified to work over our principal subalgebras in the case of A_p^n-modules (Corollary 1.3.4). This module decomposition allows us to comment on the various options for the module structure in general.

Since A_p^n modules have an underlying vector space, multiplication by a linear form defines a linear transformation and therefore can be represented by a matrix, which we call the representation matrix. The goal is to use the representation matrix to understand the module decomposition. In particular, the Jordan canonical form of the representation matrix tells us exactly how the module decomposes as stated in Fact 1.4.7. Armed with a matrix representation we are ready to ask questions about how decompositions change when varying the subalgebra. With a matrix representation, we are also able to connect a choice of matrix to a point in affine space. This is similarly done for a choice of subalgebra.

To further analyze which modules hold certain properties, for example, a specific module decomposition, we define a generic module. This is in effect a finite set of generic matrices representing all possible A_p^n-modules of a fixed dimension. Each generic matrix corresponds to a generator of A_p^n. These generic matrices are required to reflect the commutativity and nilpotency conditions held by the generators of A_p^n. To ensure that the generic matrices hold these properties, we define the ideal Q and corresponding algebraic variety $V(Q)$ in Definition 2.1.7. A point in $V(Q)$ then corresponds to an A_p^n-module and vice versa.

Following the construction of the generic matrices, we employ them to study freeness. In this thesis, freeness upon restriction to principal subalgebras is encoded in terms of a certain ideal described in Corollary 3.1.8, and its corresponding variety. The points in the rank variety correspond to subalgebras where the module is not free.

Next, it is shown that there are modules that are both free upon restriction to infinitely many subalgebras and not free upon restriction to infinitely many subalgebras. To this end, we employ the Zariski topology to quantify their abundance. This leads to a general statement quantifying freeness based on the module variety and the rank variety in Theorem 3.2.6. Since the points of the rank variety correspond to points of the subalgebra where the module is not free, the property of freeness is more abundant, i.e. the condition of freeness is an open condition.

In Section 4.1 we show by example (see Fact 4.1.1) that the rank variety can be nonzero and not the whole space. We explore how the rank variety encodes non-freeness over different subalgebras for fixed modules. We give many examples (see Example 4.1.3, Example 4.1.4 and Example 4.1.5) exploring the behavior of the rank variety.

After fixing a special ordered basis of the underlying vector space of A_p^n in Definition 4.2.1, we describe the canonical representation matrices in Definition 4.2.12. We use this description throughout Section 4.2 to show there are modules whose rank variety achieve any possible dimension. The existence of such modules has been proposed in [6], but in this thesis we give concrete examples for any rank variety.

In Chapter 5 we study the module variety by looking at the case of a fixed subalgebra and generic module. This direction does not prove to be as interesting, but nevertheless we obtain a theorem on the invariance of the dimension of the module variety for any principal subalgebra as Theorem 5.1.4.

1.2 Defining A_p^n and $\Bbbk[u_\lambda]$

The starting point for this thesis is A_p^n, a truncated commutative polynomial ring. We are interested in finitely generated A_p^n-modules that are restricted to principal subalgebras, namely subalgebras that are generated by a single homogeneous linear form of A_p^n.

Definition 1.2.1. Let A_p^n be the commutative ring

$$A_p^n = \Bbbk[Z_1, Z_2, ..., Z_n]/(Z_1^p, Z_2^p, ..., Z_n^p)$$

where $\Bbbk$ is a field, $char(\Bbbk) = p$, and p is a prime integer. Define z_i to be the coset of Z_i in A_p^n. The field is assumed to be algebraically closed when necessary.

The ring A_p^n is a finite dimensional vector space over the coefficient field $\Bbbk$. One such basis of this vector space consists of the monomials

$$\{z_1^{k_1} z_2^{k_2} ... z_n^{k_n} | 0 \leq k_i \leq p-1, 1 \leq i \leq n\}.$$

One counts easily the monomials in the basis to find the number of elements in a $\Bbbk$-basis of A_p^n. We state this in the form of the following fact.

Fact 1.2.2. The dimension of A_p^n as a $\Bbbk$-vector space is p^n.

Proof. Notice that the set

$$\{z_1^{k_1} z_2^{k_2} ... z_n^{k_n} | 0 \leq k_i \leq p-1, 1 \leq i \leq n\}$$

is both $\Bbbk$-linearly independent and spans A_p^n as a $\Bbbk$-vector space. Thus, this set is a basis for V. The basis has p^n elements because there are p choices of k_i for all n choices of i. $\square$

Additionally, A_p^n is a finite dimensional $\Bbbk$-algebra. The modules that are the focus of this thesis are all finitely generated A_p^n-modules making the following a critical component of this study. The majority of A_p^n-modules constructed for use in examples throughout are defined by an ideal generated by the elements that are listed in the example. Notationally, an A_p^n-module M defined by an ideal generated by z_1 and z_2 is denoted $M = (z_1, z_2)$.

Fact 1.2.3. A_p^n is a finite dimensional $\Bbbk$-algebra, and as such, finitely generated A_p^n-modules are finite dimensional $\Bbbk$-vector spaces. Equivalently, every finitely generated A_p^n-module has a finite $\Bbbk$-basis.

To justify this fact we will come up with a $\Bbbk$-basis of a finitely generated module. Given a finitely generated A_p^n-module M generated by $a_1, a_2, ..., a_m$ we find that the elements $a_i z_1^{k_1} z_2^{k_2} ... z_n^{k_n}$, where $0 \leq k_j \leq p-1$ and $1 \leq i \leq m$, span M as a $\Bbbk$-vector space. This is a finite spanning set from which a basis can be chosen. In cases where the elements of the module have a degree, the basis can be ordered in terms of this degree. However, there are cases where elements of the module do not have a degree. At this point, we know a basis can always be found and we formalize a canonical basis order for A_p^n in Chapter 4. The following example looks into the $\Bbbk$-basis of the underlying vector space of an A_p^n-module.

Example 1.2.4. Suppose that we use A_2^3 as the underlying ring and look at M as an A_2^3-module.

1. If $M = A_2^3$, then the underlying $\Bbbk$-vector space has a 9 element basis
$$\{1, z_1, z_2, z_1 z_2, z_1^2, z_2^2, z_1^2 z_2, z_1 z_2^2, z_1^2 z_2^2\}.$$
2. If $M = (z_1)$, the ideal generated by z_1, then the underlying 6-dimensional $\Bbbk$-vector space has basis
$$\{z_1, z_1 z_2, z_1^2, z_1^2 z_2, z_1 z_2^2, z_1^2 z_2^2\}.$$
3. If $M = (z_2)$, then the underlying $\Bbbk$-vector space is again 6-dimensional and has basis
$$\{z_2, z_1 z_2, z_2^2, z_1^2 z_2, z_1 z_2^2, z_1^2 z_2^2\}.$$
4. If $M = (z_1, z_2)$, then the underlying $\Bbbk$-vector space is 8-dimensional and has basis
$$\{z_1, z_2, z_1 z_2, z_1^2, z_2^2, z_1^2 z_2, z_1 z_2^2, z_1^2 z_2^2\}.$$
5. If $M = (z_1 z_2)$, then the underlying $\Bbbk$-vector space is 4-dimensional and has basis
$$\{z_1 z_2, z_1^2 z_2, z_1 z_2^2, z_1^2 z_2^2\}.$$

6. Finally, if $M = (z_1 + z_2)$, then the underlying $\Bbbk$-vector space is 6-dimensional and has basis

$$\{z_1 + z_2, z_1^2 + z_1z_2, z_1z_2 + z_2^2, z_1^2z_2 + z_1z_2^2, z_1^2z_2, z_1^2z_2^2\}.$$

In both (2) and (3), we find 6-dimensional $\Bbbk$-vector spaces. They are therefore isomorphic as $\Bbbk$-vector spaces. However, they are not isomorphic as A_p^n-modules since (z_1) and (z_2) have different annihilators in A_p^n.

The structure of finitely generated A_p^n-modules can be extremely complicated. We introduce subalgebras to better understand their structure. The subalgebras are generated by a linear form of A_p^n. It is worth pointing out that the results of this thesis depend on the subalgebra being a single homogeneous linear form and it is not obvious what would happen otherwise.

Definition 1.2.5. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ where $\lambda_i \in \Bbbk$. Additionally, define

$$u_\lambda = \sum_{i=1}^n \lambda_i z_i \text{ for } \lambda_i \in \Bbbk$$

and let $\Bbbk[u_\lambda]$ denote the principal subalgebra of A_p^n generated by u_λ.

By definition, u_λ is a homogeneous linear form, and henceforth we assume that u_λ is nonzero. In other words, a choice of λ that is entirely zero is not permissible. There is a bijective correspondence between nonzero linear forms in A_p^n and nonzero $\lambda \in \mathbb{A}^n$. A critical behavior of A_p^n is that any homogeneous linear form to the p^{th} power is 0.

Fact 1.2.6. For $x_i \in A_p^n$ and $k > 0$,

$$(x_1 + x_2 + ... + x_k)^p = x_1^p + x_2^p + ... + x_k^p.$$

Proof. We proceed by induction on k. The result is obvious for $k = 1$, so we prove the result with $k = 2$ as the base case. By binomial expansion, we find that

$$(x_1 + x_2)^p = \binom{p}{0}x_1^p + \binom{p}{1}x_1^{p-1}x_2 + ... + \binom{p}{p-1}x_1x_2^{p-1} + \binom{p}{p}x_2^p.$$

All coefficients other than $\binom{p}{0}$ and $\binom{p}{p}$ are a multiple of p. In other words, $\binom{p}{p'} = mp$ for some positive integer m when $0 < p' < p$. Since the characteristic of $\Bbbk$ is p, each of these terms is zero. This means that

$$(x_1 + x_2)^p = \tbinom{p}{0}x_1^p + \tbinom{p}{p}x_2^p = x_1^p + x_2^p$$

and thus the $k = 2$ case holds. Assume the fact is true for $k - 1 > 0$. Now $(x_1 + x_2 + ... + x_k)^p = (x_1 + x_2 + ... + x_{k-1})^p + x_k^p$ by the $k = 2$ case. By induction, $(x_1 + x_2 + ... + x_{k-1})^p = x_1^p + x_2^p + ... + x_{k-1}^p$ and we find

$$(x_1 + x_2 + ... + x_k)^p = x_1^p + x_2^p + ... + x_k^p.$$

□

Applying Fact 1.2.6 to u_λ we have the following.

Fact 1.2.7. For any λ, $u_\lambda^p = 0$.

Proof. We have $u_\lambda = \lambda_1 z_1 + ... + \lambda_n z_n$ for $\lambda_i \in \Bbbk$. Then $u_\lambda^p = \lambda_1^p z_1^p + \lambda_2^p z_2^p + ... + \lambda_n^p z_n^p$ by the previous fact. Since $z_i^p = 0$ for all i, we conclude $u_\lambda^p = 0$. □

The next example highlights the structure of the subalgebra $\Bbbk[u_\lambda]$ of A_p^n.

Example 1.2.8. Consider the ring A_3^2 and $a, b \in \Bbbk[u_\lambda]$.

Let $a = a_0 + u_\lambda a_1 + u_\lambda^2 a_2$ and $b = b_0 + u_\lambda b_1 + u_\lambda^2 b_2$ for a_i and b_i in $\Bbbk$.

The addition and multiplication of $\Bbbk[u_\lambda]$ are inherited from A_p^n. For example, when multiplying a and b we obtain the following.

$$\begin{aligned} a \cdot b &= a_0 b_0 + u_\lambda a_0 b_1 + u_\lambda^2 a_0 b_2 + u_\lambda a_1 b_0 + u_\lambda^2 a_1 b_1 + u_\lambda^2 a_2 b_0 \\ &= \mathrm{a}_0 b_0 + u_\lambda(a_0 b_1 + a_1 b_0) + u_\lambda^2(a_0 b_2 + a_1 b_1 + a_2 b_0) \end{aligned}$$

The previous example can be extended to a general case with arbitrary n and p while functioning in a similar manner. The key observation is the simplicity of the structure of $\Bbbk[u_\lambda]$. Simply defining the coefficients of each power of u_λ^k for $0 \leq k \leq p - 1$ uniquely defines an element of $\Bbbk[u_\lambda]$. As a subalgebra of a finite dimensional $\Bbbk$-vector space, $\Bbbk[u_\lambda]$ is likewise a finite dimensional $\Bbbk$-vector space. The structure of $\Bbbk[u_\lambda]$ is explicitly described in the following fact.

Fact 1.2.9. The principal subalgebra $\Bbbk[u_\lambda]$ and $\Bbbk[x]/(x^p)$ are isomorphic as $\Bbbk$-algebras. Indeed, the natural map $\Bbbk[x] \to \Bbbk[u_\lambda]$ where $x \mapsto u_\lambda$ is surjective with kernel (x^p). Note that $\Bbbk[x]/(x^p)$ is a principal ideal ring since $\Bbbk[x]$ is a principal ideal ring. Therefore $\Bbbk[u_\lambda]$ is also a principal ideal ring.

A natural $\Bbbk$-basis of $\Bbbk[x]/(x^p)$ is

$$\{1, x, x^2, x^3, ..., x^{p-1}\}$$

and similarly a natural $\Bbbk$-basis for $\Bbbk[u_\lambda]$ is

$$\{1, u_\lambda, u_\lambda^2, u_\lambda^3, ..., u_\lambda^{p-1}\}.$$

Next, we give an important fact using u_λ as a linear transformation.

Fact 1.2.10. Let M be an A_p^n-module. Multiplication by a fixed ring element $a \in A_p^n$ can be regarded as a linear transformation $M \xrightarrow{a} M$ defined by $x \mapsto ax$ on the underlying vector space of an A_p^n-module M. In particular, multiplication by u_λ defines a linear transformation on the underlying $\Bbbk$-vector space of an A_p^n-module M.

We now explore examples of various u_λ acting as a linear transformation on A_p^n-modules.

Example 1.2.11. We investigate u_λ as a linear transformation on M where M is an A_p^n-module.

1. Let $M = A_3^2$, where M is an A_3^2-module. The underlying $\Bbbk$-vector space is 9 dimensional with basis

$$\{1, z_1, z_2, z_1z_2, z_1^2, z_2^2, z_1^2z_2, z_1z_2^2, z_1^2z_2^2\}.$$

If $u_\lambda = z_1$, then the image $u_\lambda M$ of the linear transformation is a 6 dimensional $\Bbbk$-vector space with basis

$$\{z_1, z_1^2, z_1z_2, z_1^2z_2, z_1z_2^2, z_1^2z_2^2\}$$

If $u_\lambda = z_2$, then the image $u_\lambda M$ of the linear transformation is a 6 dimensional $\Bbbk$-vector space with basis

$$\{z_2, z_1z_2, z_2^2, z_1z_2^2, z_1^2z_2, z_1^2z_2^2\}$$

If $u_\lambda = z_1 + z_2$, then the image $u_\lambda M$ of the linear transformation is a 6 dimensional $\Bbbk$-vector space with basis

$$\{z_1 + z_2, z_1^2 + z_1z_2, z_1z_2 + z_1z_2^2, z_1^2z_2 + z_1z_2^2, z_1^2z_2, z_1^2z_2^2\}.$$

2. Let $M = A_2^3$, where M is an A_2^3-module. Then the underlying $\Bbbk$-vector space is 8 dimensional with basis

$$\{1, z_1, z_2, z_3, z_1z_2, z_1z_3, z_2z_3, z_1z_2z_3\}.$$

If $u_\lambda = z_1$, then the image $u_\lambda M$ of the linear transformation is a 4 dimensional $\Bbbk$-vector space with basis

$$\{z_1, z_1z_2, z_1z_3, z_1z_2z_3\}$$

If $u_\lambda = z_1 + z_2$, then the image $u_\lambda M$ of the linear transformation is a 4 dimensional $\Bbbk$-vector space with basis

$$\{z_1 + z_2, z_1z_2, z_1z_3 + z_2z_3, z_1z_2z_3\}$$

If $u_\lambda = z_1 + z_2 + z_3$, then the image $u_\lambda M$ of the linear transformation is a 4 dimensional $\Bbbk$-vector space with basis

$$\{z_1 + z_2 + z_3, z_1z_2 + z_2z_3, z_1z_3 + z_2z_3, z_1z_2z_3\}$$

How exactly the dimension of $A_p^n \downarrow \Bbbk[u_\lambda]$ changes as u_λ varies will be a topic of further discussion. The previous facts will be important in the algebraic decomposition of A_p^n-modules after restriction. A main objective of this thesis is to understand finitely generated A_p^n-modules after restriction to the principal subalgebras $\Bbbk[u_\lambda]$. Such a restriction is possible as outlined in the following fact.

Fact 1.2.12. The natural embedding $\Bbbk[u_\lambda] \to A_p^n$ realizes A_p^n as a $\Bbbk[u_\lambda]$-algebra. Every A_p^n-module is a $\Bbbk[u_\lambda]$-module via restriction of scalars.

Such an embedding exists for any λ and thus an A_p^n-module can be regarded as a $\Bbbk[u_\lambda]$ module for any λ. Obviously, $\Bbbk[u_\lambda]$ is not an integral domain, due to the nonzero nilpotent elements. Even though $\Bbbk[u_\lambda]$ is not a principal ideal domain, there still exists a module decomposition theorem for its finitely generated modules. We will discuss this next.

1.3 Module Decomposition

In this section, we discuss the decomposition of finitely generated A_p^n-modules when restricted to $\Bbbk[u_\lambda]$. We begin by looking at the well-known decompositions of modules over a principal ideal domain, and derive the decompositions of modules over $k[x]/(x^p)$.

Theorem 1.3.1 (Decomposition of Modules over PIDs [15])**.** *Let M be a finitely generated module over a principal ideal domain R. Then there exist nonnegative integers h and m, positive integers t_i, and irreducible elements p_i such that*

$$M \cong R/Rp_1^{t_1} \oplus \cdots \oplus R/Rp_m^{t_m} \oplus R^h.$$

We extend the theorem of module decomposition over a principal ideal domain to a decomposition theorem over $\Bbbk[x]/(x^p)$. This is possible since $\Bbbk[x]$ is a principal ideal domain and there is a natural epimorphism from $\Bbbk[x] \to \Bbbk[x]/(x^p)$, and thus any finitely generated $\Bbbk[x]/(x^p)$-module can be viewed as a $\Bbbk[x]$-module. We now find the $k[x]/(x^p)$-module is isomorphic to

$$\Bbbk[x]/(p_1^{t_1}) \oplus \cdots \oplus \Bbbk[x]/(p_m^{t_m}) \oplus \Bbbk[x]^h.$$

In general, identifying the irreducible elements p_i in a principal ideal ring is a nontrivial task. In the case of $\Bbbk[x]/(x^p)$, the only irreducible element is x. The decomposition of $\Bbbk[x]/(x^p)$ will be relevant to our study of finitely generated A_p^n-modules after restriction to $\Bbbk[u_\lambda]$ since a map sending $x \to u_\lambda$ induces the isomorphism from Fact 1.2.9

$$\Bbbk[x]/(x^p) \cong \Bbbk[u_\lambda].$$

Fact 1.3.2. When M is a $\Bbbk[x]/(x^p)$-module viewed as a $\Bbbk[x]$-module, we have $x^p M = 0$.

We can use a series of observations to obtain another decomposition. We find that $x^p M = 0$ when M is a $\Bbbk[x]/(x^p)$-module. Since $x^p \Bbbk[x] \neq 0$ we must have $h = 0$. We know $x^p \in (p_i^{t_i})$. Thus $(x^p) \subseteq (p_i^{t_i})$ and by taking radicals we get $(x) \subseteq (p_i)$. Since (x) is maximal we have equality. Thus $x = p_i$ up to a unit for all i. This leads us to our main theorem for this section stated once in general terms and then again within the context of $\Bbbk[u_\lambda]$ in the subsequent corollary.

Theorem 1.3.3 (Decomposition of Modules over $\Bbbk[x]/(x^p)$)**.** *Let M be a finitely generated module over $\Bbbk[x]/(x^p)$. Then there exist nonnegative integers m_i such that*

$$M \cong (\Bbbk)^{m_1} \oplus (\Bbbk[x]/(x^2))^{m_2} \oplus ... \oplus (\Bbbk[x]/(x^p))^{m_p}.$$

Reformulating this theorem in terms of u_λ yields the following corollary.

Corollary 1.3.4. *Let M be a finitely generated A_p^n-module restricted to $\Bbbk[u_\lambda]$ and m_i be integers ≥ 0. Then*

$$M \cong (\Bbbk)^{m_1} \oplus (\Bbbk[u_\lambda]/(u_\lambda^2))^{m_2} \oplus ... \oplus (\Bbbk[u_\lambda]/(u_\lambda^{p-1}))^{m_{p-1}} \oplus (k[u_\lambda])^{m_p}$$

The form of decomposition in Corollary 1.3.4 is used when finding the module decomposition of finitely generated A_p^n-modules after restriction to $\Bbbk[u_\lambda]$. The following example shows this in practice.

Example 1.3.5. Consider $M = (z_1)$ as an A_3^2-module. We restrict to $k[u_\lambda]$ with $\lambda = (1,0)$ and $(0,1)$, and then find the module decomposition. The underlying $\Bbbk$-vector space of M has basis $B = \{z_1, z_1z_2, z_1^2, z_1^2z_2, z_1z_2^2, z_1^2z_2^2\}$. Applying corollary 1.3.4, we need to find m_1, m_2, and m_3 for each choice of λ. Recall that there are three m_i variables because $p = 3$. To determine the m_i for each λ, we multiply each element of the basis needed to generate M as a $\Bbbk[u_\lambda]$-module by u_λ and observe how many terms are annihilated.

1. If $\lambda = (1, 0)$, then $u_\lambda = z_1$. We find that M is generated as a $\Bbbk[u_\lambda]$-module by three elements. Namely, z_1, z_1z_2 and $z_1z_2^2$. We find that none of the three elements are annihilated by u_λ and all three are annihilated by u_λ^2. Thus $m_1 = 0$, $m_2 = 3$, and $m_3 = 0$.
2. If $\lambda = (0, 1)$, then $u_\lambda = z_2$. We find M as a $\Bbbk[u_\lambda]$-module is generated by z_1 and z_1^2. Neither element is annihilated by u_λ or u_λ^2 so we find that $m_1 = 0, m_2 = 0$, and $m_3 = 2$.

In the previous example we see the need for a more systematic way to determine the m_i in the module decomposition. Now that we have the module decompositions of finitely generated A_p^n-modules restricted to $\Bbbk[u_\lambda]$ we investigate representation matrices of the same modules.

1.4 Representation Matrix Decompositions

Now that the algebraic decomposition of an A_p^n-module has been defined, we can identify the corresponding matrix form of such a decomposition. We start with the representation matrix of each z_i as a linear transformation and expand to the matrix representation of u_λ.

Definition 1.4.1. Consider z_i as a multiplication map on the underlying vector space M of a finitely generated A_p^n-module. Let $[z_i]_M$ denote the matrix representing z_i as a linear transformation on M with respect to some fixed basis B of M. We call $[z_i]_M$ the *representation matrix* of z_i with respect to B.

It is important to note that the representation matrix $[z_i]_M$ depends upon the choice of basis for M. In other words, the representation matrix of $[z_i]_M$ is determined uniquely up to a change of basis. More about the change of basis matrix and the following fact can be found in [16].

Fact 1.4.2. The representation matrix $[z_i]_M$ is well-defined up to conjugation by an invertible matrix. Specifically, if $[z_i]'_M$ is another representation matrix with respect to a different basis B', and P is the change of basis matrix, then $P[z_i]_M = [z_i]'_M P$.

The following example calculates $[z_i]_M$ in a specific case.

Example 1.4.3. Let $M = (z_1^2 z_2)$ be an A_3^3-module. Suppose we use the ordered basis $\{z_1^2 z_2, z_1^2 z_2^2\}$ and find $[z_2]_M$. We need to multiply the basis elements by z_2. $z_1^2 z_2 \cdot z_2 = z_1^2 z_2^2$ and $z_1^2 z_2^2 \cdot z_2 = 0$. Thus

$$[z_2]_M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

If we instead want to determine $[z_1]_M$ for the same M and basis, we get

$$[z_1]_M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We can use the representation matrix for z_i to construct a representation matrix for u_λ. Our goal is to study finitely generated A_p^n-modules after restriction to $\Bbbk[u_\lambda]$ and having a representation matrix for u_λ will be an important tool. Recall that

$$u_\lambda = \lambda_1 z_1 + \lambda_2 z_2 + ... + \lambda_n z_n \text{ where } \lambda_i \in \Bbbk.$$

We can construct the representation matrix for u_λ by scaling the representation matrix for each z_i by λ_i and finding the sum.

Definition 1.4.4. Fix M to be a finitely generated A_p^n-module. Let $[u_\lambda]_M = \lambda_1[z_1]_M + \lambda_2[z_2]_M + ... + \lambda_n[z_n]_M$. We call this the *representation matrix for u_λ*. By construction, u_λ can be seen as a linear transformation on the underlying vector space of M.

We find that $[u_\lambda]_M$ inherits some properties of A_p^n in the following fact.

Fact 1.4.5. For a fixed M, since $z_i^p = 0$ we find $[z_i]_M^p = 0$. Furthermore, we know z_i commutes with z_j. We find that $[z_i]_M$ commutes with $[z_j]_M$.

We offer an example of finding $[u_\lambda]_M$.

Example 1.4.6. Let M be the A_3^2-module (z_1z_2). We fix $B = \{z_1z_2, z_1^2z_2, z_1z_2^2, z_1^2z_2^2\}$, a basis for M. We find $[u_\lambda]_M$ by multiplying u_λ by each element of B. The results $u_\lambda(z_1z_2) = \lambda_1z_1^2z_2 + \lambda_2z_1z_2^2, u_\lambda(z_1^2z_2) = \lambda_2z_1^2z_2^2, u_\lambda(z_1z_2^2) = \lambda_1z_1^2z_2^2$ and $u_\lambda(z_1^2z_2^2) = 0$ vary when u_λ varies. Using $u_\lambda B$, we now find $[z_1]_M, [z_2]_M$, and $[u_\lambda]_M$. The representation matrices are

$$[z_1]_M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, [z_2]_M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \text{ and } [u_\lambda]_M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ \lambda_2 & 0 & 0 & 0 \\ 0 & \lambda_2 & \lambda_1 & 0 \end{bmatrix}.$$

As mentioned in the definition of $[z_i]_M$ and $[u_\lambda]_M$, fixing a basis is necessary in order to have a unique representation matrix. For now we point out that the set of all representation matrices that result from different choices of basis is a conjugacy class. Later in the thesis, we further unpack this idea, but presently we want to be able to pick a representative of the conjugacy class formed by every possible basis that yields a standard representation matrix. For this reason, we introduce the Jordan canonical form of representation matrices. It turns out the Jordan canonical form of $[u_\lambda]_M$ reveals the module decomposition of $M \downarrow \Bbbk[u_\lambda]$. Conversely, if we know the module decomposition of $M \downarrow \Bbbk[u_\lambda]$, then we know the Jordan canonical form of $[u_\lambda]_M$. Note that the Jordan canonical form used in this thesis is always lower triangular. This is out of convenience as natural choices of basis lead to the lower triangular form. For example, a natural basis for $\Bbbk[x]/(x^p)$ is

$$\{1, x, x^2, ..., x^{p-1}\}$$

leading to the lower triangular form. This decision is purely cosmetic as any $[u_\lambda]_M$ in lower triangular form can be reformulated under a change of basis to become upper triangular.

Fact 1.4.7. Since $[u_\lambda]_M$ is nilpotent, we know the eigenvalues of $[u_\lambda]_M$ are identically 0 (see [3, 8.19]) and as such there is a basis for M such that $[u_\lambda]_M$ is in Jordan canonical form. The module decomposition of M corresponds to the representation matrix $[u_\lambda]_M$ in Jordan canonical form. More specifically, each Jordan block will correspond to a summand of the module decomposition.

Recall that the module decomposition of a finitely generated A_p^n-module restricted to $\Bbbk[u_\lambda]$ is determined by m_i for $1 \leq i \leq p$. The dimension and multiplicity of the $\Bbbk[u_\lambda]/(u_\lambda^i)^{m_i}$ terms in the module decomposition determine the Jordan blocks in Jordan canonical form. The dimension and multiplicity of the i^{th} term in the decomposition is simply m_i. In other words, each m_i is the number of Jordan blocks of size $i \times i$ in the Jordan canonical form.

If the module decomposition of $[u_\lambda]_M$ is known we have no trouble finding the Jordan canonical form. However, this thesis focuses on cases where the exact module decomposition is not yet calculated. Through the representation matrix we can infer the module decomposition. Theoretically, the Jordan canonical form of $[u_\lambda]_M$ can always be obtained under a change of basis. In practice, this can be very computationally expensive. Luckily, the following proposition allows for the inference of the Jordan canonical form of $[u_\lambda]_M$ in a different way.

Proposition 1.4.8. *Let $[u_\lambda]_M$ be the representation matrix of an A_p^n-module M restricted to $k[u_\lambda]$. Then the Jordan canonical form of $[u_\lambda]_M$ has precisely*

$$-2(rank([u_\lambda]_M^j))+rank([u_\lambda]_M^{j-1})+rank([u_\lambda]_M^{j+1})$$

blocks of size $j \times j$ for $j > 0$.

Proof. Let $[u_\lambda]_M$ be the representation matrix of a finitely generated A_p^n-module restricted to $\Bbbk[u_\lambda]$. We know that the only eigenvalue of $[u_\lambda]_M$ is 0 because $[u_\lambda]_M$ is nilpotent. Since the eigenvalues are 0, [16, Lemma 1.3.18] implies that the number of Jordan blocks of size $j \times j$ or larger is

$$\dim \ker([u_\lambda]_M^j) - \dim \ker([u_\lambda]_M^{j-1}).$$

The number of blocks of size $j \times j$ is then the number of blocks of size $j \times j$ or larger minus the number of blocks of size $(j+1) \times (j+1)$ or larger. Thus the number of blocks of size $j \times j$ in the Jordan canonical form of $[u_\lambda]_M$ is

$$\begin{gathered}\dim \ker([u_\lambda]_M^j) - \dim \ker([u_\lambda]_M^{j-1}) - \dim \ker([u_\lambda]_M^{j+1}) + \dim \ker([u_\lambda]_M^j) = \\ 2\dim \ker([u_\lambda]_M^j) - \dim \ker([u_\lambda]_M^{j-1}) - \dim \ker([u_\lambda]_M^{j+1})\ .\end{gathered}$$

This is equivalent to

$$-2(\mathrm{rank}([u_\lambda]_M{}^{j}_{M})) + \mathrm{rank}([u_\lambda]_M{}^{j-1}_{M}) + \mathrm{rank}([u_\lambda]_M{}^{j+1}_{M}),$$

as desired. □

We provide a fact to explain how we calculate the rank of a matrix when the rank is not immediately clear, as is the case for a matrix in Jordan canonical form.

Fact 1.4.9. The rank of a matrix is the size of the largest nonzero minor. This fact comes from the proof of Theorem 1 in [11].

We now continue with an example showing various representation matrices and their Jordan canonical form. In part of this example, we calculate the rank of a matrix by finding the largest nonzero ideal generated by minors. After the example, we will formalize the process of calculating rank. For now, the focus is on how the Jordan canonical form of $[u_\lambda]_M$ is affected by changes in λ.

Example 1.4.10. Throughout this example the underlying field $\Bbbk$ is $\mathbb{Z}/p\mathbb{Z}$ and λ is chosen such that $\lambda_i = 1$ for all i. Let $M = (z_1)$ be an A_3^2-module and fix the basis of M as $\{z_1, z_1^2, z_1z_2, z_1^2z_2, z_1z_2^2, z_1^2z_2^2\}$. We first calculate $[u_\lambda]_M$ and then the rank of the powers of $[u_\lambda]$ needed to determine the Jordan canonical form. Since $p = 3$, the Jordan canonical form depends on the rank of $[u_\lambda]_M^0, [u_\lambda]_M, [u_\lambda]_M^2$ and $[u_\lambda]_M^3$. We find that in this case

$$[u_\lambda]_M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

and $\mathrm{rank}[u_\lambda]_M = 4$. The rank of $[u_\lambda]_M^0$ is the size of $[u_\lambda]_M$ or 6 in this case. We find

$$[u_\lambda]_M^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \end{bmatrix}$$

and calculate that $\mathrm{rank}([u_\lambda]_M^2) = 2$. In this case $[u_\lambda]_M^3 = 0$. The Jordan canonical form then has $-2(4) + 6 + 2 = 0$ blocks of size 1×1, $-2(2) + 4 + 0 = 0$ blocks of size 2×2, $-2(0) + 2 + 0 = 2$ blocks of size 3×3 and no blocks of any larger size. In summary, the Jordan canonical form of $[u_\lambda]_M$ is

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

which corresponds to the module decomposition $M \cong \Bbbk[u_\lambda]^2$.

Next, we let $M = A_2^3$ as an A_2^3-module, and fix the basis as $\{1, z_1, z_2, z_3, z_1z_2, z_1z_3, z_2z_3, z_1z_2z_3\}$. We again seek to find $[u_\lambda]_M$, the rank of the powers of $[u_\lambda]_M$, and the Jordan canonical form. We calculate

$$[u_\lambda] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

with $\operatorname{rank}[u_\lambda]_M = 4$ and since $p = 2$, $[u_\lambda]_M^2 = 0$. The decomposition has $-2(4)+8+0 = 0$ blocks of size 1×1 and $-2(0)+4+0 = 4$ blocks of size 2×2. The Jordan canonical form is then

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

which corresponds to a module decomposition of $\Bbbk[u_\lambda]^4$.

Finally, we let $M = A_2^4$ as an A_2^4-module and fix the basis as $\{1, z_1, z_2, z_3, z_4, z_1z_2, z_1z_3, z_1z_4, z_2z_3, z_2z_4, z_3z_4, z_1z_2z_3, z_1z_3z_4, z_1z_2z_4, z_2z_3z_4, z_1z_2z_3z_4\}$. We calculate that

$$[u_\lambda]_M = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&1&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&1&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0 \end{bmatrix}$$

with $[u_\lambda]_M^2 = 0$. The rank of $[u_\lambda]_M$ is 8 and thus there are $-2(10)+16+1 = 0$ blocks of size 1×1 and $-2(0)+8+0 = 8$ blocks of size 2×2. The module decomposition is

$$M \cong \Bbbk[u_\lambda]^8.$$

Each part of the previous example relied heavily on the calculation of the rank of the powers of $[u_\lambda]$. We use the following notation in situations where the rank of a matrix is calculated.

Definition 1.4.11. Let $I_g(X)$ be the ideal generated by the $g \times g$ minors of a $d \times d$ matrix X where $1 \leq g \leq d$.

We now employ $I_g(X)$ to determine the number of Jordan blocks. The following example gives the ideal that determines the rank of the powers of $[u_\lambda]_M$.

Example 1.4.12. Let $M = (z_1 z_2)$ be an A_3^2-module with fixed basis $\{z_1z_2, z_1^2z_2, z_1z_2^2, z_1^2z_2^2\}$. To determine the Jordan canonical form of $[u_\lambda]_M$ we need to calculate $[u_\lambda]_M, [u_\lambda]_M^2$ and $[u_\lambda]_M^3$.

$$[u_\lambda]_M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ \lambda_2 & 0 & 0 & 0 \\ 0 & \lambda_2 & \lambda_1 & 0 \end{bmatrix}, [u_\lambda]_M^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\lambda_1\lambda_2 & 0 & 0 & 0 \end{bmatrix},$$

and $[u_\lambda]_M^3 = 0$. Here $I_3([u_\lambda]_M) = 0$, $I_2([u_\lambda]_M) = (\lambda_1\lambda_2, \lambda_1^2, \lambda_2^2)$, $I_1([u_\lambda]_M) = (\lambda_1, \lambda_2)$, $I_3([u_\lambda]_M^2) = 0$, $I_2([u_\lambda]_M^2) = 0$, and $I_1([u_\lambda]_M^2) = (2\lambda_1\lambda_2)$. Now the choice of u_λ determines the Jordan canonical form. Suppose the underlying field is $\mathbb{Z}/3\mathbb{Z}$. If $u_\lambda = z_2$, then $[u_\lambda]_M$ has rank 2 and $[u_\lambda]_M^2$ has rank 0. The Jordan canonical form here is

$$\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

If $u_\lambda = z_1 + z_2$, then $[u_\lambda]_M$ has rank 2 and $[u_\lambda]_M^2$ has rank 1. The Jordan canonical form is then

$$\left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

There is no u_λ here that will yield a matrix of a single 4×4 block.

By changing the choice of u_λ, we can change the Jordan canonical form of $[u_\lambda]_M$ in some cases but not in others. Similarly, changing n and p can change the Jordan canonical form of $[u_\lambda]_M$ in some cases but not in others. In the next chapter we create generic matrices in order to study the changes that a choice of A_p^n-module or λ can make on $[u_\lambda]_M$.

CHAPTER 2

Constructing a Generic Representation Matrix

2.1 Construction and Notation

For each $1 \leq i \leq n$, we will construct a $d \times d$ generic matrix that will represent $[z_i]_M$, where M is an unspecified d-dimensional module. We begin by discussing generic matrices in general. The construction of generic matrices is based on a similar construction found in [5] . To begin, we offer an example of the construction of an insufficient matrix and highlight why it is insufficient.

Example 2.1.1. Consider the polynomial ring $\Bbbk[x_i | 1 \leq i \leq d^2]$ where the x_i's are indeterminates and suppose X is a square matrix with entries x_i. If we fix $d = 2$, we have

$$X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}.$$

This matrix is of the right size to represent all linear transformations on a two dimensional $\Bbbk$-vector space. For an A_p^n-module, we know the entries of $[z_i]_M$ are in $\Bbbk$. Replacing the x_i by an arbitrary choice of elements of $\Bbbk$ would not be enough to guarantee that $X^p = 0$, which is a defining property of A_p^n-modules. Additionally, if X represents $[z_i]_M$ for some i, then we need notation that denotes all of the other $[z_j]_M$ where $i \neq j$. Our objective is to construct a generic $[z_i]_M$ in order to later build a generic $[u_\lambda]_M$. Note that using this X as a representation matrix is not a good way to represent $[u_\lambda]_M$ since this choice of X does not encode the impact of changing the values of the λ_i's in u_λ. Creating a single matrix of indeterminates of the right size is not sufficient to represent all $[z_i]_M$, where M is of a fixed dimension.

This leads us to our definition of X_i.

Definition 2.1.2. For $1 \leq i \leq n$, let X_i be a $d \times d$ matrix of indeterminates from the polynomial ring $\Bbbk[x_{i,r,s} | 1 \leq i \leq n, 1 \leq r, s \leq d]$. Thus the indeterminate $x_{i,r,s}$ is the entry in row r and column s of the matrix X_i. We display X_i below.

$$X_i = \begin{bmatrix} x_{i,1,1} & x_{i,1,2} & x_{i,1,3} & \cdots & x_{i,1,d-2} & x_{i,1,d-1} & x_{i,1,d} \\ x_{i,2,1} & x_{i,2,2} & x_{i,2,3} & \cdots & x_{i,2,d-2} & x_{i,2,d-1} & x_{i,2,d} \\ x_{i,3,1} & x_{i,3,2} & x_{i,3,3} & \cdots & x_{i,3,d-2} & x_{i,3,d-1} & x_{i,3,d} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{i,d-2,1} & x_{i,d-2,2} & x_{i,d-2,3} & \cdots & x_{i,d-2,d-2} & x_{i,d-2,d-1} & x_{i,d-2,d} \\ x_{i,d-1,1} & x_{i,d-1,2} & x_{i,d-1,3} & \cdots & x_{i,d-1,d-2} & x_{i,d-1,d-1} & x_{i,d-1,d} \\ x_{i,d,1} & x_{i,d,2} & x_{i,d,3} & \cdots & x_{i,d,d-2} & x_{i,d,d-1} & x_{i,d,d} \end{bmatrix}$$

The X_i have a fixed size of $d \times d$ since they will represent a module of dimension of d. We give an example of constructing X_i where $d = 3$.

Example 2.1.3. First, we let $d = 3$ and display X_1.

$$X_1 = \begin{bmatrix} x_{1,1,1} & x_{1,1,2} & x_{1,1,3} \\ x_{1,2,1} & x_{1,2,2} & x_{1,2,3} \\ x_{1,3,1} & x_{1,3,2} & x_{1,3,3} \end{bmatrix}$$

We want X_1 to represent all possible $[z_1]_M$ where the underlying vector space of M is 3-dimensional. Notice that X_1 has nine entries and therefore corresponds to a point in $\mathbb{A}^9$.

We want to establish the correspondence between a generic matrix and a point in affine space. To achieve this we introduce the following notation.

Definition 2.1.4. Let α be a point in nd^2-dimensional affine space, $\mathbb{A}^{nd^2}$. More specifically, let α be the ordered nd^2-tuple $(\alpha_{i,r,s})$ where $1 \leq i \leq n$ and $1 \leq r, s \leq d$. We take the lexicographic order on α with priority on i, r and then s. Keep in mind we will only introduce a specific α in the context of a fixed basis.

This definition purposely mirrors the definition of X_i. Hence if we want to specify n matrices of size $d \times d$, we can replace the indeterminates $x_{i,r,s}$ from the X_i with $\alpha_{i,r,s}$. The following example illustrates this substitution of α for the indeterminates.

Example 2.1.5. Let M be an A_p^1-module of dimension 2 with representation matrix

$$[z_1]_M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

In other words, $[z_1]_M$ represents the linear transformation $M \xrightarrow{z_1} M$ on the underlying vector space of M for a fixed basis. This matrix corresponds to the point $\alpha = (0, 0, 1, 0)$ in $\mathbb{A}^4$ as follows. Since $d = 2$,

$$X_1 = \begin{bmatrix} x_{1,1,1} & x_{1,1,2} \\ x_{1,2,1} & x_{1,2,2} \end{bmatrix}.$$

Choose α such that $\alpha_{1,1,1} = 0, \alpha_{1,1,2} = 0, \alpha_{1,2,1} = 1$ and $\alpha_{1,2,2} = 0$. Now we can replace the indeterminates of X_1 with the corresponding values of α. The resulting matrix after replacement is $[z_1]_M$. If we instead choose α such that $\alpha_{1,1,1} = 1, \alpha_{1,1,2} = 0, \alpha_{1,2,1} = 0$ and $\alpha_{1,2,2} = 1$, then after replacing the indeterminates of X_1 we have the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This matrix is not the representation matrix $[z_1]_M$ for a finitely generated A_p^n-module M, since if it were, then $[z_1]_M^p = 0$. Therefore we need to restrict α to guarantee the result is a representation matrix of some A_p^n-module.

Recall that we defined $I_g(X)$ in Definition 1.4.11. Let Q' be the homogeneous ideal $I_1(X_i)$ of $\Bbbk[x_{i,r,s} | 1 \leq i \leq n, 1 \leq r, s \leq d]$, in other words Q' is the ideal generated by the entries of X_i. Let $V(Q')$ be the affine subvariety of $\mathbb{A}^{nd^2}$ corresponding to Q'. We introduce Q' now informally as any ideal and will later use the same idea to define a specific homogeneous ideal Q that allows us to ensure a chosen α meets required conditions such that α corresponds to an A_p^n-module. The following example illustrates Q'.

Example 2.1.6. We fix $d = 3$ and consider $Q' = I_1(X)$.

$$Q' = (x_{1,1,1}, x_{1,1,2}, x_{1,1,3}, x_{1,2,1}, x_{1,2,2}, x_{1,2,3}, x_{1,3,1}, x_{1,3,2}, x_{1,3,3})$$

We observe that this is indeed a homogeneous ideal. In this case the affine variety of Q' has 9 defining equations. They are all of the form

$$x_{1,r,s} = 0.$$

Therefore the corresponding variety is an intersection of nine hyperplanes.

We need to construct an ideal Q using the defining equations of a finitely generated A_p^n-module. To this end, the generic representation matrix needs to exhibit commutativity and the property that each $X_i^p = 0$. Up until this point, the parameter p has not played a role in this construction of the generic matrix. We will use p in the ideal Q in the definition that follows.

Definition 2.1.7. Let Q be the homogeneous ideal of $\Bbbk[x_{i,r,s} | 1 \leq i \leq n, 1 \leq r, s \leq d]$ generated by the entries of the matrices $X_iX_j - X_jX_i$ for $i < j$ and X_i^p, where $1 \leq i \leq n$ for both. The variety in $\mathbb{A}^{nd^2}$ of Q is denoted $V(Q)$.

We construct Q with $X_iX_j - X_jX_i$ to ensure commutativity and with X_i^p to ensure the X_i are nilpotent. Recall that the z_i in A_p^n commute and are nilpotent. The following example illustrates the conditions $X_iX_j - X_jX_i$ and X_i^p defining Q.

Example 2.1.8. Fix $d = 2, n = 2$, and $p = 2$. Then

$$X_1 = \begin{bmatrix} x_{1,1,1} & x_{1,1,2} \\ x_{1,2,1} & x_{1,2,2} \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} x_{2,1,1} & x_{2,1,2} \\ x_{2,2,1} & x_{2,2,2} \end{bmatrix}.$$

The X_i matrices represent the $[z_i]_M$. Recall from Fact 1.4.5 that the $[z_i]_M$ are necessarily nilpotent and commute with each other. With only X_1 and X_2, having the entries of $X_1X_2 - X_2X_1$ in Q is sufficient to guarantee that $[z_1]_M$ and $[z_2]_M$ commute. The four entries in the matrix $X_1X_2 - X_2X_1$ are:

$$x_{1,1,1}x_{2,1,1} - x_{1,1,2}x_{2,2,1}$$

$$\mathrm{x}_{1,1,2}x_{2,1,2} - x_{1,1,2}x_{2,2,2}$$

$$\mathrm{x}_{1,2,1}x_{2,1,1} - x_{1,2,2}x_{2,2,1}$$

$$\mathrm{x}_{1,2,1}x_{2,1,2} - x_{1,2,2}x_{2,2,2}.$$

With these entries included as generators of the ideal Q, the linear transformations represented are guaranteed to have the commutativity desired. The other condition imposed by Q is that $X_1^2 = 0$ and $X_2^2 = 0$. In this case, X_1^2 and X_2^2 determine a further eight generators of Q, namely,

$$x_{1,1,1}^2 - x_{1,1,2}x_{1,2,1}$$

$$x_{1,1,1}x_{1,2,1} - x_{1,1,2}x_{1,2,2}$$

$$x_{1,2,1}x_{1,1,1} - x_{1,2,2}x_{1,2,1}$$

$$x_{1,2,1}x_{1,1,2} - x_{1,2,2}^2$$

$$x_{2,1,1}^2 - x_{2,1,2}x_{2,2,1}$$

$$x_{2,1,1}x_{2,2,1} - x_{2,1,2}x_{2,2,2}$$

$$x_{2,2,1}x_{2,1,1} - x_{2,2,2}x_{2,2,1}$$

$$x_{2,2,1}x_{2,1,2} - x_{2,2,2}^2.$$

Using all twelve of these elements of $\Bbbk[x_{i,r,s} | 1 \leq i \leq n, 1 \leq r, s \leq d]$, we obtain

$$Q = I_1(X_1X_2 - X_2X_1) + I_1(X_1^2) + I_1(X_2^2).$$

We can now guarantee that a chosen α corresponds to a valid A_p^n-module if $\alpha \in V(Q)$.

Next, we give an example looking at a case of $\alpha \in V(Q)$.

Example 2.1.9. Fix $d = 3, p = 3$, and $n = 2$. Choose α such that $\alpha_{1,3,2} = 1, \alpha_{2,2,1} = 1$, and $\alpha_{2,3,2} = 1$ with all other $\alpha_{i,r,s} = 0$. After replacing the indeterminates of X_1 and X_2 with the corresponding components of α,

$$X_1 \text{ becomes } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } X_2 \text{ becomes } \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

We want to check if $\alpha \in V(Q)$. We find that $X_1^3 = 0$ and $X_2^3 = 0$. However, $X_1X_2 - X_2X_1 \neq 0$. If we instead choose α such that either $\alpha_{1,3,2} = 1, \alpha_{2,2,1} = 0$, and $\alpha_{2,3,2} = 1$, or if $\alpha_{1,3,2} = 1, \alpha_{2,2,1} = 1$, and $\alpha_{2,3,2} = 0$, we find that $X_1X_2 - X_2X_1 = 0$ and $\alpha \in V(Q)$. This means that for α to be in $V(Q)$ either $\alpha_{1,3,2} = 0$ or $\alpha_{2,2,1} = 0$.

In the next section we will improve the generic matrix to represent $[u_\lambda]_M$.

2.2 Generic Matrices after Restriction

In this section, we discuss how the generic representation matrix for $[z_i]_M$ can be used to construct a generic representation matrix for $[u_\lambda]_M$. We can both add the X_i together and scale the X_i by an element of $\Bbbk$. We intentionally defined α so that when every $\alpha_{i,r,s}$ is specified for all i, r, and s, the number of components in α is enough to substitute all indeterminates in $X_1, X_2, ..., X_n$. We offer an example augmenting a sum of X_i with a specific λ.

Example 2.2.1. Let $d = 2, n = 2$, fix λ, and suppose the A_p^n-module M has representation matrix

$$[u_\lambda]_M = \begin{bmatrix} 0 & 0 \\ \lambda_1 + \lambda_2 & 0 \end{bmatrix}.$$

The matrix $[u_\lambda]_M$ can be obtained from $X_1 + X_2$ by substituting in

$$\alpha = (0, 0, \lambda_1, 0, 0, 0, \lambda_2, 0)$$

for the indeterminates $x_{i,r,s}$. However, this is not a desirable choice of α because we want to be able to choose an α corresponding to an A_p^n-module and then account separately for alternative choices of λ. To achieve this, suppose we consider $\lambda_1 X_1 +$

$\lambda_2 X_2$ instead of $X_1 + X_2$. Now we can substitute $\alpha_{i,r,s}$ for the indeterminates $x_{i,r,s}$ where

$$\alpha = (0, 0, 1, 0, 0, 0, 1, 0)$$

and have the result equal $[u_\lambda]_M$.

We can now define the generic representation matrix U_λ for u_λ.

Definition 2.2.2. For a chosen λ, we define the $d \times d$ matrix

$$U_\lambda = \lambda_1 X_1 + \lambda_2 X_2 + ... + \lambda_n X_n.$$

We display U_λ below.

$$U_\lambda = \begin{bmatrix} \lambda_1 x_{1,1,1} + \lambda_2 x_{2,1,1} + ... + \lambda_n x_{n,1,1} & \cdots & \lambda_1 x_{1,1,d} + ... + \lambda_n x_{n,1,d} \\ \lambda_1 x_{1,2,1} + \lambda_2 x_{2,2,1} + ... + \lambda_n x_{n,2,1} & \cdots & \lambda_1 x_{1,2,d} + ... + \lambda_n x_{n,2,d} \\ \cdots & \cdots & \cdots \\ \lambda_1 x_{1,d,1} + \lambda_2 x_{2,d,1} + ... + \lambda_n x_{n,d,1} & \cdots & \lambda_1 x_{1,d,d} + ... + \lambda_n x_{n,d,d} \end{bmatrix}$$

We can do something similar for a fixed module and generic u_λ.

Definition 2.2.3. Let $\alpha \in V(Q)$ and let Λ_i be indeterminates where $1 \leq i \leq n$. We define the $d \times d$ matrix

$$U_\Lambda(\alpha) = \Lambda_1 X_1(\alpha) + \Lambda_2 X_2(\alpha) + ... + \Lambda_n X_n(\alpha)$$

where $X_i(\alpha)$ is X_i after substitution by the corresponding entries of α. We display $U_\Lambda(\alpha)$ below.

$$U_\Lambda(\alpha) = \begin{bmatrix} \Lambda_1 \alpha_{1,1,1} + \Lambda_2 \alpha_{2,1,1} + ... + \Lambda_n \alpha_{n,1,1} & \cdots & \Lambda_1 \alpha_{1,1,d} + ... + \Lambda_n \alpha_{n,1,d} \\ \Lambda_1 \alpha_{1,2,1} + \Lambda_2 \alpha_{2,2,1} + ... + \Lambda_n \alpha_{n,2,1} & \cdots & \Lambda_1 \alpha_{1,2,d} + ... + \Lambda_n \alpha_{n,2,d} \\ \cdots & \cdots & \cdots \\ \Lambda_1 \alpha_{1,d,1} + \Lambda_2 \alpha_{2,d,1} + ... + \Lambda_n \alpha_{n,d,1} & \cdots & \Lambda_1 \alpha_{1,d,d} + ... + \Lambda_n \alpha_{n,d,d} \end{bmatrix}$$

We have a final definition for the case of a generic module and a generic subalgebra.

Definition 2.2.4. Fix d, n, and p. Let Λ_i for $1 \leq i \leq n$ be indeterminates. We define the $d \times d$ matrix

$$U_\Lambda = \Lambda_1 X_1 + \Lambda_2 X_2 + ... + \Lambda_n X_n.$$

Below we display U_Λ.

$$U_\Lambda = \begin{bmatrix} \Lambda_1 x_{1,1,1} + \Lambda_2 x_{2,1,1} + ... + \Lambda_n x_{n,1,1} & \cdots & \Lambda_1 x_{1,1,d} + ... + \Lambda_n x_{n,1,d} \\ \Lambda_1 x_{1,2,1} + \Lambda_2 x_{2,2,1} + ... + \Lambda_n x_{n,2,1} & \cdots & \Lambda_1 x_{1,2,d} + ... + \Lambda_n x_{n,2,d} \\ \cdots & \cdots & \cdots \\ \Lambda_1 x_{1,d,1} + \Lambda_2 x_{2,d,1} + ... + \Lambda_n x_{n,d,1} & \cdots & \Lambda_1 x_{1,d,d} + ... + \Lambda_n x_{n,d,d} \end{bmatrix}$$

The relationship between the indeterminate Λ and $\lambda \in \mathbb{A}^n$ is the same as that of $x_{i,r,s}$ to $\alpha_{i,r,s}$. We construct U_Λ using Λ and $x_{i,r,s}$, and U_Λ is the set of all possible $[u_\lambda]_M$. Using λ and α, we can replace indeterminates with specific values and either select a single $[u_\lambda]_M$ from the set of all possible $[u_\lambda]_M$ or replace only a few of the indeterminates in U_λ and get a subset of all possible $[u_\lambda]_M$.

Definition 2.2.5. If the entries of a generic matrix X_i, for $1 \leq i \leq n$, are replaced by a specific choice of $\alpha \in V(Q)$, then each X_i represents a linear transformation $\mathbb{V} \to \mathbb{V}$ with respect to some fixed basis of a d-dimensional vector space $\mathbb{V}$. Since $\alpha \in V(Q)$, this gives the vector space $\mathbb{V}$ the structure of an A_p^n-module, which we call M_α. In other words, M_α is the A_p^n-module corresponding to this choice of α. Now $[z_i]_{M_\alpha} = X_i(\alpha)$ and consequently, $[u_\lambda]_{M_\alpha} = U_\lambda(\alpha)$.

The following examples unpack the notation of the prior definitions. The first shows how a choice of α refers to a matrix and finds ideals of that matrix. Next, we highlight that the process of substituting an α into an X_i and then taking an ideal of minors commutes with first finding the ideal of minors of X_i and then substituting in α.

Example 2.2.6. Fix $d = 2$ and $n = 2$. Let $\alpha \in V(Q)$ where

$$\alpha = (\alpha_{1,1,1}, \alpha_{1,1,2}, \alpha_{1,2,1}, \alpha_{1,2,2}, \alpha_{2,1,1}, \alpha_{2,1,2}, \alpha_{2,2,1}, \alpha_{2,2,2}).$$

Here $U_\Lambda = \Lambda_1 X_1 + \Lambda_2 X_2$ and we have

$$U_\Lambda(\alpha) = \begin{bmatrix} \Lambda_1\alpha_{1,1,1} + \Lambda_2\alpha_{2,1,1} & \Lambda_1\alpha_{1,1,2} + \Lambda_2\alpha_{2,1,2} \\ \Lambda_1\alpha_{1,2,1} + \Lambda_2\alpha_{2,2,1} & \Lambda_1\alpha_{1,2,2} + \Lambda_2\alpha_{2,2,2} \end{bmatrix}.$$

The ideals $I_1(U_\Lambda(\alpha))$ and $I_2(U_\Lambda(\alpha))$ can be calculated here. The reason for why we are interested in such ideals will be discussed later, throughout Chapter 3. $I_1(U_\Lambda(\alpha)) =$

$$(\Lambda_1\alpha_{1,1,1} + \Lambda_2\alpha_{2,1,1}, \Lambda_1\alpha_{1,1,2} + \Lambda_2\alpha_{2,1,2}, \Lambda_1\alpha_{1,2,1} + \Lambda_2\alpha_{2,2,1}, \Lambda_1\alpha_{1,2,2} + \Lambda_2\alpha_{2,2,2}).$$

$I_2(U_\Lambda(\alpha)) =$

$$(\Lambda_1^2\alpha_{1,1,1}\alpha_{1,2,2} + \Lambda_1\Lambda_2\alpha_{1,2,2}\alpha_{2,1,1} + \Lambda_1\Lambda_2\alpha_{1,1,1}\alpha_{2,2,2} + \Lambda_2^2\alpha_{2,1,1}\alpha_{2,2,2}$$
$$-\Lambda_1^2\alpha_{1,1,2}\alpha_{1,2,1} - \Lambda_1\Lambda_2\alpha_{1,2,1}\alpha_{2,1,2} - \Lambda_1\Lambda_2\alpha_{1,1,2}\alpha_{2,2,1} - \Lambda_2^2\alpha_{2,1,2}\alpha_{2,2,1}).$$

Now if we specify $\alpha = (0,0,1,0,0,0,1,0)$, then

$$I_1(U_\Lambda(\alpha)) = (\Lambda_1 + \Lambda_2) \text{ and } I_2(U_\Lambda(\alpha)) = (0).$$

No part of this example depended upon the choice of p.

The second example examines the process of choosing an α that lies in $V(Q)$. Recall that choosing $\alpha \in V(Q)$ guarantees that $U_\Lambda(\alpha)$ leads to an A_p^n-module.

Example 2.2.7. Fix $n = 2, p = 2$, and $d = 2$. Suppose we want to find $U_\Lambda(\alpha)$ where $\alpha = (0,0,1,0,0,0,1,0) \in \mathbb{A}^8$. Thus

$$X_1X_2 - X_2X_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0, X_1^2 = 0, \text{ and } X_2^2 = 0.$$

This confirms that this choice of α is indeed in $V(Q)$. Hence, we obtain

$$U_\Lambda(\alpha) = \begin{bmatrix} 0 & 0 \\ \lambda_1 + \lambda_2 & 0 \end{bmatrix}.$$

Before discussing further concepts in algebraic geometry it is worth highlighting the dimension of, and relationship between, the objects that are being studied. The entries of U_Λ involve the variables $x_{i,r,s}$ and Λ_i. Thus U_Λ corresponds to the affine space $\mathbb{A}^{nd^2} \times \mathbb{A}^n$. For a fixed α, $U_\Lambda(\alpha)$ corresponds to the affine space $\mathbb{A}^n$. Instead of fixing an α, if we instead fix λ, then U_λ corresponds to the affine space $\mathbb{A}^{nd^2}$. This correspondence is illustrated by the diagrams below.

$$\begin{tikzcd} & \mathbb{A}^{nd^2} \times \mathbb{A}^n \arrow[dl] \arrow[dr] & \\ \mathbb{A}^{nd^2} & & \mathbb{A}^n \end{tikzcd} \qquad \begin{tikzcd} & U_\Lambda \arrow[dl] \arrow[dr] & \\ U_\lambda & & U_\Lambda(\alpha) \end{tikzcd}$$

Recall that U_Λ is equal to a specific $[u_\lambda]_M$ after substituting both the nd^2 indeterminates $x_{i,r,s}$ and the n indeterminates Λ_i. A point in $\mathbb{A}^{nd^2} \times \mathbb{A}^n$ can be used to substitute the $nd^2 + n$ indeterminates. Additionally, we can consider the underlying polynomial ring from which the generic matrices draw indeterminates. More specifically, U_Λ corresponds to

$$\Bbbk[x_{i,r,s}, \Lambda_i | 1 \leq i \leq n, 1 \leq r, s \leq d],$$

a polynomial ring in $nd^2 + n$ variables. Similarly, U_λ corresponds to the polynomial ring in nd^2 variables,

$$\Bbbk[x_{i,r,s} | 1 \leq i \leq n, 1 \leq r, s \leq d].$$

We also find $U_\Lambda(\alpha)$ corresponds to the polynomial ring in n variables,

$$\Bbbk[\Lambda_i | 1 \leq i \leq n].$$

The chart below illustrates the polynomial ring corresponding to the generic matrices.

$$\begin{tikzcd} & \Bbbk[x_{i,r,s}, \Lambda_i] \arrow[dl] \arrow[dr] & \\ \Bbbk[x_{i,r,s}] & & \Bbbk[\Lambda_i] \end{tikzcd}$$

The U_Λ, U_λ, $U_\Lambda(\alpha)$, and $U_\lambda(\alpha)$ notation will be used extensively for the rest of the thesis as tools to analyze freeness. In Chapter 3, we define freeness.

CHAPTER 3

Freeness of A_p^n-modules restricted to $\Bbbk[u_\lambda]$

3.1 Analyzing the Freeness of Modules After Restriction

From this point in the thesis we focus on the freeness of A_p^n-modules after restriction to $\Bbbk[u_\lambda]$. For this reason we are interested in $[u_\lambda]_M$ and the generic representation matrices rather than an individual $[z_i]_M$. To be clear, freeness will be studied only for A_p^n-modules after restriction to $\Bbbk[u_\lambda]$. We show in Proposition 3.1.7 that if a module is free as an A_p^n-module, then it is free at every restriction, and therefore is not interesting. We know from Chapter 1 that after restriction to $\Bbbk[u_\lambda]$, an A_p^n-module decomposes as a direct sum of cyclic submodules. This is the key idea behind the following definition. When we described module decompositions in Section 1.3, the m_i for $1 \leq i \leq p$ entirely determined the decomposition. The following definition states that freeness is equivalent to $m_i = 0$ for all $i \neq p$ in the decomposition.

Definition 3.1.1. For an A_p^n-module M, $M \downarrow \Bbbk[u_\lambda]$ is *free* if the decomposition is a direct sum of copies of $\Bbbk[u_\lambda]$. The corresponding representation matrix will have a Jordan canonical form of only blocks of size $p \times p$. Throughout the thesis we refer to a free decomposition after restriction to $\Bbbk[u_\lambda]$ simply as *freeness.*

In the following example we take an A_p^n-module and investigate the freeness of the module after restriction $\Bbbk[u_\lambda]$ as λ varies.

Example 3.1.2. Consider the 3-dimensional A_3^3-module M_α defined by $\alpha \in V(Q) \subseteq \mathbb{A}^{27}$, where

$$\alpha = (0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

This choice of α yields

$$U_\Lambda(\alpha) = \begin{bmatrix} 0 & 0 & 0 \\ \Lambda_1 + \Lambda_2 & 0 & 0 \\ 0 & \Lambda_1 + \Lambda_2 & 0 \end{bmatrix}.$$

Now we can choose various λ and observe freeness. Suppose $\lambda = (1, 0, 0)$. Then

$$U_\lambda(\alpha) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

This matrix is already in Jordan canonical form so we know

$$M_\alpha \downarrow \Bbbk[u_\lambda] \cong \Bbbk[u_\lambda]$$

showing M_α is free after restriction to $\Bbbk[u_\lambda]$. We get the same decomposition and Jordan canonical form if $\lambda = (0, 1, 0)$. However if we instead use $\lambda = (0, 0, 1)$ then

$$U_\lambda(\alpha) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and we find that

$$M_\alpha \downarrow \Bbbk[u_\lambda] \cong \Bbbk \oplus \Bbbk \oplus \Bbbk.$$

Hence, this choice of λ demonstrates that the restricted module need not be free.

The following example unpacks the relationship between freeness and the module decomposition for a specific module.

Example 3.1.3. We will approach the concept of freeness after restriction from two different perspectives. First, suppose we have

$$[u_\lambda]_M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We purposely do not specify p, n, or u_λ for the moment. We will observe freeness for various values of p. Since p is necessarily prime, we consider the cases of $p = 2, 3$, or 5. If $p = 5$, then from Propostion 1.4.8 and the fact that $[u_\lambda]_M^2 = 0$ we find

$$M \cong [\Bbbk[u_\lambda]/(u_\lambda^2)]^3 \ .$$

Thus when $p = 5$, $M \downarrow k[u_\lambda]$ is not free. If $p = 3$, we again have $[u_\lambda]_M^2 = 0$ and find

$$M \downarrow k[u_\lambda] \cong [\Bbbk[u_\lambda]/(u_\lambda^2)]^3.$$

Hence $M \downarrow k[u_\lambda]$ is not free. If $p = 2$, then

$$M \downarrow k[u_\lambda] \cong \Bbbk[u_\lambda]^3$$

so $M \downarrow \Bbbk[u_\lambda]$ is free.

For the second perspective on freeness in this example, let M be an A_p^n-module where

$$M \downarrow k[u_\lambda] \cong \Bbbk[u_\lambda]^2.$$

We want to consider possibilities for $[u_\lambda]_M$ and n, p, and d. We know that the matrix must consist of two $p \times p$ blocks, and this decomposition necessitates having $d = 2p$.

If $p = 2$ then $d = 4$ and so on. When $p = 3$ then $d = 6$ and the Jordan canonical form of $[u_\lambda]_M$ is

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Generalizing some of the findings from the previous example, we have the following result.

Proposition 3.1.4. *In order to have a free decomposition after restriction to $\Bbbk[u_\lambda]$, the underlying $\Bbbk$-vector space of an A_p^n-module must have dimension a multiple of p.*

For this reason, going forward, we consider only modules that have dimension a multiple of p.

Proof. Let M be an A_p^n-module restricted to $\Bbbk[u_\lambda]$ of dimension d. We know from Corollary 1.3.4 that

$$M \cong (\Bbbk)^{m_1} \oplus (\Bbbk[u_\lambda]/(u_\lambda^2))^{m_2} \oplus ... \oplus (\Bbbk[u_\lambda]/(u_\lambda^{p-1}))^{m_{p-1}} \oplus (k[u_\lambda])^{m_p}$$

In order for $M \downarrow \Bbbk[u_\lambda]$ to be free, we need

$$M \downarrow k[u_\lambda] \cong (k[u_\lambda])^{m_p}.$$

In other words, $m_i = 0$ for all $i \neq p$ and $m_p \neq 0$. Therefore $d = \dim(M) = m_p p$. In other words, for $M \downarrow \Bbbk[u_\lambda]$ to be free, then $d = \nu p$ for some positive integer $\nu = m_p$. □

In terms of $[u_\lambda]_M$, $M \downarrow \Bbbk[u_\lambda]$ is free if the Jordan canonical form consists entirely of $p \times p$ blocks. Note that having the dimension be a multiple of p does not mean that an A_p^n-module restricted to $\Bbbk[u_\lambda]$ has a free decomposition. This is simply a requirement for freeness to potentially occur. Recall that for the rest of the thesis we assume that d is a multiple of p.

Definition 3.1.5. Let ν be the unique integer where $d = p\nu$.

When $M \downarrow k[u_\lambda]$ is free then the Jordan canonical form of $[u_\lambda]_M$ consists entirely of ν blocks of size $p \times p$. The newly defined ν is immediately useful in the following proposition.

Proposition 3.1.6. *Fix $d = \nu p$. Then $[u_\lambda]_M$ is free if and only if rank$([u_\lambda]_M^{p-1}) = \nu$.*

Proof. From Proposition 1.4.8 we know that the number of $j \times j$ blocks in the Jordan canonical form of $[u_\lambda]_M$ is

$$-2\ \mathrm{rank}([u_\lambda]_M^j) + \mathrm{rank}([u_\lambda]_M^{j-1}) + \mathrm{rank}([u_\lambda]_M^{j+1}).$$

We also know that $[u_\lambda]^p = 0$ and thus $[u_\lambda]^{p+1} = 0$. Applying Proposition 1.4.8 where $j = p-1$, we find the number of $p \times p$ blocks in the Jordan canonical form of $[u_\lambda]_M$ is

$$\mathrm{rank}([u_\lambda]_M^{p-1}).$$

This means that $\mathrm{rank}([u_\lambda]_M^{p-1})$ entirely determines the number of $p \times p$ blocks in the Jordan canonical form. Knowing that freeness requires ν blocks of size $p \times p$ and that $rank([u_\lambda]_M^{p-1})$ is the number of $p \times p$ blocks, we conclude $M \downarrow \Bbbk[u_\lambda]$ is free if and only if $\mathrm{rank}([u_\lambda]_M^{p-1}) = \nu$. □

This gives us a powerful tool for calculating whether or not $[u_\lambda]_M$ exhibits freeness. At this point we can make a statement on the freeness of A_p^n as a module over itself.

Proposition 3.1.7. *Let $M = A_p^n$ be the free A_p^n-module of rank one. Then for any λ, $M \downarrow \Bbbk[u_\lambda]$ is free.*

Proof. Let B be an ordered basis for M and fix i and j where $i \neq j$ and $1 \leq i, j \leq n$. Let $\phi : M \to M$ be the algebra homomorphism defined by $\phi(z_i) = z_j$, $\phi(z_j) = z_i$, and $\phi(z_k) = z_k$ for all $k \neq i, j$. We call $B' = \phi(B)$ a reordering of B. Note that if $B = (b_1, b_2, ..., b_{p^n})$, then $\phi(B)$ is the ordered p^n-tuple $(\phi(b_1), \phi(b_2), ..., \phi(b_{p^n}))$. Under this reordered basis, we determine that $[z_i]_M$ (using basis B) is the same matrix as $[z_j]'_M$ (using basis B'). Referencing Fact 1.4.2, $[z_j]_M$ (using basis B) is conjugate to $[z_j]'_M$ (using basis B'). Thus $[z_i]_M$ is conjugate to $[z_j]_M$. Next, we show $[z_i]_M$ is conjugate to $[u_\lambda]_M$ for any λ where $\lambda_i \neq 0$. Note that u_λ is required to have a nonzero λ_k for some k, so a choice of such an i is possible. Let $\psi : M \to M$ be the algebra homomorphism defined by $\psi(z_i) = u_\lambda$ and $\psi(z_k) = z_k$ for all $k \neq i$. Let $\psi(B) = B'$ be the ordered p^n-tuple $(\psi(b_1), \psi(b_2), ..., \psi(b_{p^n}))$. Then $[z_i]_M$ (using basis B) is the same matrix as $[u_\lambda]'_M$ (using basis B'). Fact 1.4.2 shows that $[u_\lambda]'_M$ is conjugate to $[u_\lambda]_M$. Thus $[z_i]_M$ is conjugate to $[u_\lambda]_M$ when $\lambda_i \neq 0$. Due to conjugacy, we know that the Jordan canonical form of $[z_i]_M$ is the same as the Jordan canonical form of both $[z_j]_M$ and $[u_\lambda]_M$. Recall that the dimension of A_p^n is p^n. In the basis of monomials for M, one can easily check that for any i there are precisely p^{n-1} terms in the basis for M containing z_i. Consequently, we find that the rank of $[z_i]_M$ is always $\nu = p^n/p = p^{n-1}$. Thus $M = A_p^n$ is free as a $\Bbbk[z_i]$-module and therefore also as a $\Bbbk[u_\lambda]$-module for any λ. □

Corollary 3.1.8. *For any α and λ, $I_\nu(U_\lambda(\alpha)^{p-1})$ is nonzero if and only if $M_\alpha \downarrow \Bbbk[u_\lambda]$ is free.*

Since $U_\lambda(\alpha)^{p-1}$ is a matrix with entries in $\Bbbk$ we find that $I_\nu(U_\lambda(\alpha)^{p-1})$ is an ideal of a field. We recognize that an ideal of a field must either be 0 or the entire field. In the previous proposition and corollary we utilized the ideal $I_\nu(U_\lambda(\alpha)^{p-1})$ and pointed out how this ideal determines freeness. For this reason, we refer to this as the ideal determining freeness. Later, we also consider ideals $I_\nu(U_\Lambda(\alpha)^{p-1})$ and $I_\nu(U_\lambda^{p-1})$ of the polynomial rings $\Bbbk[\Lambda_i | 1 \leq i \leq n]$ and $\Bbbk[x_{i,r,s} | 1 \leq i \leq n, 1 \leq r, s \leq d]$, respectively, and discuss how they determine freeness.

3.2 The Main Theorem on Freeness and the Zariski Topology

In the category of A_p^n-modules where the underlying field is infinite, for example when the field is algebraically closed, the number of non-isomorphic modules that are free, as well as the number of non-isomorphic modules that are not free is vast and certainly infinite in both cases. Therefore we need another means of discerning when there are more modules satisfying freeness than not. To this end we employ the Zariski topology.

Fact 3.2.1. The category of A_p^n-modules has infinite representation type when $\Bbbk$ is an algebraically closed field and $n \geq 2$.

We offer this fact with an explanation rather than a formal proof as this fact is a combination of previous results in representation theory. Suppose we take a minimal free resolution of $\Bbbk$. We know that the ranks of the free modules in the resolution increase and that the ranks strictly increase after a certain point from part 2 in Theorem 7.3 of [2]. Each of the syzygy modules in the resolution is then a module with more and more generators. In fact, there is no bound to the growth of the number of generators. It is well-known that A_p^n is self-injective. The syzygies of an indecomposable module over self-injective algebras are indecomposable. For these reasons, the fact is true.

Consequently we will proceed by measuring the abundance of modules using the Zariski Topology.

Definition 3.2.2 (pg. 676, [10]). In affine k-space, $\mathbb{A}^k$, we define the *Zariski closed* sets to be those of the form

$$V(S) = \{x \in \mathbb{A}^k | f(x) = 0, \forall f \in S\}$$

where S is a set of polynomials in k variables over $\Bbbk$. The complement of a Zariski closed set is a *Zariski open* set. Additionally, $V(S) = V((S))$ where (S) is the ideal generated by the elements of S.

When the Zariski open set is nonempty (or nonzero in the homogeneous case) we regard it as a large set. When the Zariski closed set is not the whole space we regard it as a small set. We will use the Zariski Topology to declare a subset of $\mathbb{A}^{nd^2}$ or $\mathbb{A}^n$ as Zariski open or closed. Recall that points in $\mathbb{A}^{nd^2}$ and $\mathbb{A}^n$ are denoted by α or λ, respectively. The following example shows how to determine if a set is Zariski open.

Example 3.2.3. We want to determine if a set $A \subset \mathbb{A}^4$ is Zariski open or closed where

$$A = \{\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{A}^4 | \alpha_1 = \alpha_4\}.$$

We find that A is a Zariski closed set in $\mathbb{A}^4$ by Definition 3.2.2 because $A = V(x_1 - x_4)$.

Before getting to the main theorem of this chapter we introduce a helpful notation from Carlson [6].

Definition 3.2.4. Let M be an A_p^n-module. We define $W(M)$ to be

$$W(M) = \{\lambda \in \mathbb{A}^n | M \downarrow \Bbbk[u_\lambda] \text{ is not free }\}.$$

We call $W(M)$ the *rank variety* of M.

In what follows we are interested in finding $W(M_\alpha)$ for $\alpha \in V(Q)$. We define a similar notion in the case of a fixed $\lambda \in \mathbb{A}^n$ instead of a fixed $\alpha \in V(Q)$.

Definition 3.2.5. For a fixed $\lambda \in \mathbb{A}^n$, the *module variety* $Y(\lambda)$ of λ is

$$Y(\lambda) = \{\alpha \in V(Q) | M_\alpha \downarrow \Bbbk[u_\lambda] \text{ is not free}\}.$$

We now introduce a theorem directing the rest of the thesis.

Theorem 3.2.6. *The following subsets of $\mathbb{A}^n$ and $V(Q) \subseteq \mathbb{A}^{nd^2}$ are Zariski closed sets:*

1) For a fixed $\alpha \in V(Q), W(M_\alpha)$.

2) For a fixed $\lambda \neq 0$ in $\mathbb{A}^n, Y(\lambda)$.

Proof. For the first case, let $\alpha \in V(Q)$. We know from Corollary 3.1.8 that M_α is not free after substituting λ_i for the Λ_i in $U_\Lambda(M_\alpha)$ if and only if $I_\nu([U_\lambda(\alpha)]^{p-1}) = 0$. The expressions that define $I_\nu([U_\Lambda(\alpha)]^{p-1})$ are polynomials in n variables over $\Bbbk$ with indeterminates that are precisely the Λ_i. Therefore, the set of all choices of λ that do not result in freeness after substitution is a Zariski closed subset of $\mathbb{A}^n$. In other words, $W(M_\alpha)$ is Zariski closed.

In the second case we fix λ. Again, we know from Corollary 3.1.8 that U_λ represents an A_p^n-module that is not free after substituting $\alpha_{i,r,s}$ for the $x_{i,r,s}$ if and only if $I_\nu([U_\lambda(\alpha)]^{p-1}) = 0$. The expressions that define $I_\nu([U_\lambda]^{p-1})$ are polynomials in nd^2 variables over $\Bbbk$ with indeterminates precisely the $x_{i,r,s}$. Therefore, the set of α that do not result in freeness after substitution is a Zariski closed subset of $V(Q)$. In sum, $Y(\lambda)$ is Zariski closed. $\square$

Theorem 3.2.6 quantifies the abundance of freeness as is our goal in this chapter. We offer an example applying Theorem 3.2.6 that shows the relevant ideals in great detail.

Example 3.2.7. Fix $d = 3, n = 2$, and $p = 3$. We investigate the ideal defining freeness $I_\nu(U_\Lambda^{p-1}) \subset \Bbbk[x_{i,r,s}, \Lambda_i]$, the rank variety $W(M_\alpha)$, and the module variety $Y(\lambda)$. To generate the ideal defining freeness, we first find

$$U_\Lambda = \begin{bmatrix} x_{1,1,1}\Lambda_1 + x_{2,1,1}\Lambda_2 & x_{1,1,2}\Lambda_1 + x_{2,1,2}\Lambda_2 & x_{1,1,3}\Lambda_1 + x_{2,1,3}\Lambda_2 \\ x_{1,2,1}\Lambda_1 + x_{2,2,1}\Lambda_2 & x_{1,2,2}\Lambda_1 + x_{2,2,2}\Lambda_2 & x_{1,2,3}\Lambda_1 + x_{2,2,3}\Lambda_2 \\ x_{1,3,1}\Lambda_1 + x_{2,3,1}\Lambda_2 & x_{1,3,2}\Lambda_1 + x_{2,3,2}\Lambda_2 & x_{1,3,3}\Lambda_1 + x_{2,3,3}\Lambda_2 \end{bmatrix}.$$

Here $p-1=2$ and $\nu=1$ so the ideal defining freeness is $I_1(U_\Lambda^2) \neq 0$.

More specifically,

$$I_1(U_\Lambda^2) = ((x_{1,1,1}\Lambda_1 + x_{2,1,1}\Lambda_2)^2 + (x_{1,1,2}\Lambda_1 + x_{2,1,2}\Lambda_2)(x_{1,2,1}\Lambda_1 + x_{2,2,1}\Lambda_2) + (x_{1,1,3}\Lambda_1 + x_{2,1,3}\Lambda_2)(x_{1,3,1}\Lambda_1 + x_{2,3,1}\Lambda_2),$$

$$(x_{1,1,1}\Lambda_1 + x_{2,1,1}\Lambda_2)(x_{1,1,2}\Lambda_1 + x_{2,1,2}\Lambda_2) + (x_{1,1,2}\Lambda_1 + x_{2,1,2}\Lambda_2)(x_{1,2,2}\Lambda_1 + x_{2,2,2}\Lambda_2) + (x_{1,1,3}\Lambda_1 + x_{2,1,3}\Lambda_2)(x_{1,3,2}\Lambda_1 + x_{2,3,2}\Lambda_2),$$

$$(x_{1,1,1}\Lambda_1 + x_{2,1,1}\Lambda_2)(x_{1,1,2}\Lambda_1 + x_{2,1,2}\Lambda_2) + (x_{1,1,2}\Lambda_1 + x_{2,1,2}\Lambda_2)(x_{1,2,2}\Lambda_1 + x_{2,2,2}\Lambda_2) + (x_{1,1,3}\Lambda_1 + x_{2,1,3}\Lambda_2)(x_{1,3,3}\Lambda_1 + x_{2,3,3}\Lambda_2),$$

$$(x_{1,2,1}\Lambda_1 + x_{2,2,1}\Lambda_2)(x_{1,1,1}\Lambda_1 + x_{2,1,1}\Lambda_2) + (x_{1,2,2}\Lambda_1 + x_{2,2,2}\Lambda_2)(x_{1,2,1}\Lambda_1 + x_{2,2,1}\Lambda_2) + (x_{1,2,3}\Lambda_1 + x_{2,2,3}\Lambda_2)(x_{1,3,1}\Lambda_1 + x_{2,3,1}\Lambda_2),$$

$$(x_{1,2,1}\Lambda_1 + x_{2,2,1}\Lambda_2)(x_{1,1,2}\Lambda_1 + x_{2,1,2}\Lambda_2) + (x_{1,2,2}\Lambda_1 + x_{2,2,2}\Lambda_2)(x_{1,2,2}\Lambda_1 + x_{2,2,2}\Lambda_2) + (x_{1,2,3}\Lambda_1 + x_{2,2,3}\Lambda_2)(x_{1,3,2}\Lambda_1 + x_{2,3,2}\Lambda_2),$$

$$(x_{1,2,1}\Lambda_1 + x_{2,2,1}\Lambda_2)(x_{1,1,2}\Lambda_1 + x_{2,1,2}\Lambda_2) + (x_{1,2,2}\Lambda_1 + x_{2,2,2}\Lambda_2)(x_{1,2,2}\Lambda_1 + x_{2,2,2}\Lambda_2) + (x_{1,2,3}\Lambda_1 + x_{2,2,3}\Lambda_2)(x_{1,3,3}\Lambda_1 + x_{2,3,3}\Lambda_2),$$

$$(x_{1,3,1}\Lambda_1 + x_{2,3,1}\Lambda_2)(x_{1,1,1}\Lambda_1 + x_{2,1,1}\Lambda_2) + (x_{1,3,2}\Lambda_1 + x_{2,3,2}\Lambda_2)(x_{1,2,1}\Lambda_1 + x_{2,2,1}\Lambda_2) + (x_{1,3,3}\Lambda_1 + x_{2,3,3}\Lambda_2)(x_{1,3,1}\Lambda_1 + x_{2,3,1}\Lambda_2),$$

$$(x_{1,3,1}\Lambda_1 + x_{2,3,1}\Lambda_2)(x_{1,1,2}\Lambda_1 + x_{2,1,2}\Lambda_2) + (x_{1,3,2}\Lambda_1 + x_{2,3,2}\Lambda_2)(x_{1,2,2}\Lambda_1 + x_{2,2,2}\Lambda_2) + (x_{1,3,3}\Lambda_1 + x_{2,3,3}\Lambda_2)(x_{1,3,2}\Lambda_1 + x_{2,3,2}\Lambda_2),$$

$$(x_{1,3,1}\Lambda_1 + x_{2,3,1}\Lambda_2)(x_{1,1,2}\Lambda_1 + x_{2,1,2}\Lambda_2) + (x_{1,3,2}\Lambda_1 + x_{2,3,2}\Lambda_2)(x_{1,2,2}\Lambda_1 + x_{2,2,2}\Lambda_2) + (x_{1,3,3}\Lambda_1 + x_{2,3,3}\Lambda_2)^2)$$

We know from Theorem 3.2.6 that $W(M_\alpha)$ is a Zariski closed set and we proceed by finding a choice of α that lies in the closed set. Choose a specific element of $V(Q)$, say

$$\alpha = (0,1,2,0,0,2,0,0,0,0,0,1,0,0,0,0,0,0),$$

that satisfies the 27 equations defining $V(Q)$. We find that

$$I_1(U_\Lambda(M_\alpha)^2) = (\Lambda_1\Lambda_2, \Lambda_1, 2\Lambda_1\Lambda_2).$$

We find from $I_1(U_\Lambda(M_\alpha)^2)$ that freeness occurs only if λ_1 is zero but λ_2 can be arbitrary.

Now suppose instead of fixing an α, we instead fix $u_\lambda = z_1 + 2z_2$. With this fixed λ and generic α, $I_1(U_\lambda^2) =$

$$((x_{1,1,1}+2x_{2,1,1})^2+(x_{1,1,2}+2x_{2,1,2})(x_{1,2,1}+2x_{2,2,1})+(x_{1,1,3}+2x_{2,1,3})(x_{1,3,1}+2x_{2,3,1}),$$
$$(x_{1,1,1}+2x_{2,1,1})(x_{1,1,2}+2x_{2,1,2})+(x_{1,1,2}+2x_{2,1,2})(x_{1,2,2}+2x_{2,2,2})+(x_{1,1,3}+2x_{2,1,3})(x_{1,3,2}+2x_{2,3,2}),$$
$$(x_{1,1,1}+2x_{2,1,1})(x_{1,1,2}+2x_{2,1,2})+(x_{1,1,2}+2x_{2,1,2})(x_{1,2,2}+2x_{2,2,2})+(x_{1,1,3}+2x_{2,1,3})(x_{1,3,3}+2x_{2,3,3}),$$
$$(x_{1,2,1}+2x_{2,2,1})(x_{1,1,1}+2x_{2,1,1})+(x_{1,2,2}+2x_{2,2,2})(x_{1,2,1}+2x_{2,2,1})+(x_{1,2,3}+2x_{2,2,3})(x_{1,3,1}+2x_{2,3,1}),$$
$$(x_{1,2,1}+2x_{2,2,1})(x_{1,1,2}+2x_{2,1,2})+(x_{1,2,2}+2x_{2,2,2})(x_{1,2,2}+2x_{2,2,2})+(x_{1,2,3}+2x_{2,2,3})(x_{1,3,2}+2x_{2,3,2}),$$
$$(x_{1,2,1}+2x_{2,2,1})(x_{1,1,2}+2x_{2,1,2})+(x_{1,2,2}+2x_{2,2,2})(x_{1,2,2}+2x_{2,2,2})+(x_{1,2,3}+2x_{2,2,3})(x_{1,3,3}+2x_{2,3,3}),$$
$$(x_{1,3,1}+2x_{2,3,1})(x_{1,1,1}+2x_{2,1,1})+(x_{1,3,2}+2x_{2,3,2})(x_{1,2,1}+2x_{2,2,1})+(x_{1,3,3}+2x_{2,3,3})(x_{1,3,1}+2x_{2,3,1}),$$
$$(x_{1,3,1}+2x_{2,3,1})(x_{1,1,2}+2x_{2,1,2})+(x_{1,3,2}+2x_{2,3,2})(x_{1,2,2}+2x_{2,2,2})+(x_{1,3,3}+2x_{2,3,3})(x_{1,3,2}+2x_{2,3,2}),$$
$$(x_{1,3,1}+2x_{2,3,1})(x_{1,1,2}+2x_{2,1,2})+(x_{1,3,2}+2x_{2,3,2})(x_{1,2,2}+2x_{2,2,2})+(x_{1,3,3}+2x_{2,3,3})^2)).$$

Recall that if $I_1(U_\lambda(\alpha)^2) = 0$ then the corresponding module is not free. Theorem 3.2.6 also proves that the α that yield freeness when substituted into $U_\lambda(\alpha)$ form a closed set.

We end the chapter with a corollary to Theorem 3.2.6.

Corollary 3.2.8. *An A_p^n-module M is free as an A_p^n-module if and only if the restriction of M to $\Bbbk[u_\lambda]$ is a free $\Bbbk[u_\lambda]$-module for every λ. In other words, $W(M_\alpha) =$*

0 when $M_\alpha = A_p^n$ *as an* A_p^n*-module. Furthermore, when* M_α *is isomorphic to* $\bigoplus_i \Bbbk$*, then* $W(M_\alpha) = \mathbb{A}^n$.

By Proposition 3.1.7, A_p^n (viewed as an A_p^n-module) is free over $\Bbbk[u_\lambda]$ for any λ, therefore any free A_p^n-module is also free over $\Bbbk[u_\lambda]$ for any λ. If $M_\alpha \downarrow \Bbbk[u_\lambda]$ is isomorphic to a direct sum of a finite number of copies of $\Bbbk$ as an A_p^n-module, then we know α is identically 0 and thus $M_\alpha \downarrow \Bbbk[u_\lambda]$ is not free for any λ. In the next chapter we will fix an α and study the freeness of $U_\Lambda(\alpha)$.

CHAPTER 4

Fixed Module Freeness

4.1 Non-trivial Rank Varieties and Existence of Concrete Examples of Theorem 3.2.6

Theorem 3.2.6 shows that the $W(M_\alpha)$ and $Y(\lambda)$ corresponding to non-freeness are Zariski closed sets of their respective affine spaces. The question remains whether these sets are nonzero and not the entire affine space. If the sets from this theorem are indeed only zero, then there is not much to discuss or analyze. To show the sets are nonzero, we need to be able to produce specific examples to show they contain more than just zero. Recall from Section 3.1 that the dimension of the modules considered is a multiple of p.

Fact 4.1.1. For a fixed α with M_α having dimension that is a multiple of p, it is possible that $W(M_\alpha)$ is zero. One could choose M_α to be A_p^n as shown in Proposition 3.1.7. Corollary 3.2.8 states that $W(M_\alpha) = \mathbb{A}^n$ when α is zero. For a fixed λ, $Y(\lambda)$ is nonzero since a choice of α such that $M_\alpha \cong \bigoplus \Bbbk \oplus A_p^n$ results in non-freeness. Additionally, $Y(\lambda)$ cannot be $\mathbb{A}^{nd^2}$ due to Proposition 3.1.7.

One easy example of an α that never corresponds to a free module after restriction is an α where $U_\lambda(\alpha)$ is already in Jordan canonical form and the Jordan blocks are not all maximally sized. This idea appears in the next example.

Example 4.1.2. Suppose $d = 4, p = 2$ and $n = 1$. Then if

$$\alpha = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0),$$

$$U_\Lambda(\alpha) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Lambda_1 & 0 \end{bmatrix}.$$

Regardless of the choice of λ, $U_\lambda(\alpha)$ does not have a Jordan canonical form of two blocks of size 2×2.

Even though $W(M_\alpha)$ is zero for some α, using α where $W(M_\alpha) \neq 0$, we can still explore some interesting questions. What if we want the dimension of $W(M_\alpha)$ to be a certain value? We know the dimension of $W(M_\alpha)$ is between 0 and n. We look for an α where the λ's resulting in non-freeness after restriction form a line in $\mathbb{A}^2$, for example.

Example 4.1.3. Consider the case of $n = 2, d = 2$, and $p = 2$. Here, we choose $\lambda \in \mathbb{A}^2$. Can we find an α where the λ resulting in non-freeness is the Λ_1-axis? Recall that the set of modules that are free after restriction is a Zariski open set in the set of all A_p^n-modules where $\lambda \notin W(M_\alpha)$. The generic representation matrix under consideration is

$$U_\Lambda = \begin{bmatrix} x_{1,1,1}\Lambda_1 + x_{2,1,1}\Lambda_2 & x_{1,1,2}\Lambda_1 + x_{2,1,2}\Lambda_2 \\ x_{1,2,1}\Lambda_1 + x_{2,2,1}\Lambda_2 & x_{1,2,2}\Lambda_1 + x_{2,2,2}\Lambda_2 \end{bmatrix}. \tag{4.1}$$

We need to consider a specific choice of α for the desired λ's to correspond to non-freeness. Suppose $\alpha = (0, 0, 1, 0, 0, 0, 1, 0)$ yielding

$$U_\Lambda(\alpha) = \begin{bmatrix} 0 & 0 \\ \Lambda_1 + \Lambda_2 & 0 \end{bmatrix}.$$

We find the ideal defining freeness here, $I_1(U_\Lambda(\alpha))$, to be $(\Lambda_1 + \Lambda_2)$. Our $U_\Lambda(\alpha)$ corresponds to a non-free module after restriction if $\lambda_1 + \lambda_2 = 0$. In $\mathbb{A}^2$, this corresponds to the line $\Lambda_2 = -\Lambda_1$.

As another attempt, suppose $\alpha = (0, 0, 1, 0, 0, 0, 0, 0)$. In this case we have

$$U_\Lambda(\alpha) = \begin{bmatrix} 0 & 0 \\ \Lambda_1 & 0 \end{bmatrix}.$$

Now the ideal defining freeness is generated by only Λ_1 meaning the module is not free on the Λ_2-axis or when $\lambda_1 = 0$. We could easily switch α to $(0, 0, 0, 0, 0, 0, 1, 0)$ to instead have the Λ_1-axis be where non-freeness occurs. For two dimensions, we found choices of α where a choice of λ on the line $\Lambda_2 = -\Lambda_1$, the Λ_2-axis, or the Λ_1-axis result in a non-free $M_\alpha \downarrow \Bbbk[u_\lambda]$.

The next example is similar to the last with $d = 3$.

Example 4.1.4. Let $n = 2$ and $p = d = 3$. Thus we have

$$U_\Lambda = \begin{bmatrix} x_{1,1,1}\Lambda_1 + x_{2,1,1}\Lambda_2 & x_{1,1,2}\Lambda_1 + x_{2,1,2}\Lambda_2 & x_{1,1,3}\Lambda_1 + x_{2,1,3}\Lambda_2 \\ x_{1,2,1}\Lambda_1 + x_{2,2,1}\Lambda_2 & x_{1,2,2}\Lambda_1 + x_{2,2,2}\Lambda_2 & x_{1,2,3}\Lambda_1 + x_{2,2,3}\Lambda_2 \\ x_{1,3,1}\Lambda_1 + x_{2,3,1}\Lambda_2 & x_{1,3,2}\Lambda_1 + x_{2,3,2}\Lambda_2 & x_{1,3,3}\Lambda_1 + x_{2,3,3}\Lambda_2 \end{bmatrix}.$$

Can we find α such that $U_\Lambda(\alpha)$ will correspond to a non-free module after restriction for a choice of (λ_1, λ_2) on the Λ_1-axis in $\mathbb{A}^2$? Inspired by Example 4.1.3, let $\alpha = (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$, and thus

$$U_\Lambda(\alpha) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Lambda_1 & 0 & 0 \end{bmatrix}.$$

This time since $p = 3$, the ideal determining freeness is $I_1(U_\Lambda(\alpha)^2)$. However, for this choice of α we find $U_\Lambda(\alpha)^2 = 0$. Thus, we need to choose a different α. Instead, if we use $\alpha = (0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ we have

$$U_\Lambda(\alpha) = \begin{bmatrix} 0 & 0 & 0 \\ \Lambda_1 & 0 & 0 \\ 0 & \Lambda_1 & 0 \end{bmatrix}$$

so the ideal defining freeness is $I_1(U_\Lambda(\alpha)^2) = (\Lambda_1)$. This ideal will only be 0 after substitution if $\lambda_1 = 0$. This means $U_\Lambda(\alpha)$ corresponds to a non-free module after restriction if we choose (λ_1, λ_2) on the Λ_1-axis in $\mathbb{A}^2$, that is, $\lambda_1 \neq 0$ and $\lambda_2 = 0$.

In the previous example found the α that result in non-freeness for any λ on a single axis. What if instead, we are looking for freeness on a two dimensional plane in $\mathbb{A}^3$? We offer an example where $n = 3$ and we find an α where freeness occurs for λ on a plane of $\mathbb{A}^3$.

Example 4.1.5. Suppose $n = 3, d = 3$, and $p = 3$. We want a point α in $\mathbb{A}^{18}$ such that $U_\Lambda(\alpha)$ corresponds to non-freeness after substitution by λ only when λ is on the $\Lambda_1\Lambda_2$-plane of $\mathbb{A}^3$. Since freeness is directly connected to Jordan canonical form it makes sense to choose α such that $U_\lambda(\alpha)$ is a matrix in Jordan canonical form. Suppose we choose α such that

$$U_\Lambda(\alpha) = \begin{bmatrix} 0 & 0 & 0 \\ \Lambda_1\alpha_{1,2,1} + \Lambda_2\alpha_{2,2,1} + \Lambda_3\alpha_{3,2,1} & 0 & 0 \\ 0 & \Lambda_1\alpha_{1,3,2} + \Lambda_2\alpha_{2,3,2} + \Lambda_3\alpha_{3,3,2} & 0 \end{bmatrix}.$$

The ideal we want to consider in order to determine freeness is

$$I_1(U_\Lambda(\alpha)^2) = (\Lambda_1^2\alpha_{1,2,1}\alpha_{1,3,2} + \Lambda_1\Lambda_2\alpha_{2,2,1}\alpha_{1,3,2} + \Lambda_1\Lambda_3\alpha_{3,2,1}\alpha_{1,3,2} + \Lambda_1\Lambda_2\alpha_{1,2,1}\alpha_{2,3,2} + \Lambda_2^2\alpha_{2,2,1}\alpha_{2,3,2} + \Lambda_2\Lambda_3\alpha_{3,2,1}\alpha_{2,3,2} + \Lambda_1\Lambda_3\alpha_{1,2,1}\alpha_{3,3,2} + \Lambda_2\Lambda_3\alpha_{2,2,1}\alpha_{3,3,2} + \Lambda_3^2\alpha_{3,2,1}\alpha_{3,3,2}).$$

Knowing the ideal determining freeness, we can give more specific values for $\alpha_{i,r,s}$ and easily check freeness. If $\alpha = (0,0,0,1,0,0,0,1,0,0,0,0,1,0,0,0,1,0,0,0,0,0,0,0,0,0,0)$ then

$$I_1(U_\Lambda^2) = (\Lambda_1^2 + \Lambda_1\Lambda_2 + \Lambda_2^2).$$

Here $I_1(U_\Lambda^2)$ corresponds to freeness in a nontrivial way but not on the $\Lambda_1\Lambda_2$-plane as desired. If instead

$$\alpha = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,1,0)$$

then $I_1(U_\Lambda^2)$ is simply (Λ_3^2). With this α, $(\lambda_1, \lambda_2, \lambda_3)$ corresponds to a non-free module after restriction if and only if $\lambda_3 = 0$. In other words, the module is not free on the $\Lambda_1\Lambda_2$-plane.

In $\mathbb{A}^2$ and $\mathbb{A}^3$ we were able to find modules where the choice of λ made $U_\lambda(\alpha)$ non-free if λ was on a line or on a plane, respectively. For such cases, being able to find a module that is free on the Λ_1-axis in $\mathbb{A}^2$ is really not a different problem than finding a module that is non-free on the Λ_2-axis in $\mathbb{A}^2$. This leads us to the main guiding question for this chapter. Can we find a specific α such that the corresponding module is not free after restriction on a j-dimensional linear subspace for any $0 \leq j \leq n$? The following section begins to address the answer to this question.

4.2 Dimensions of Rank Varieties

In this section we construct and use a certain ordered basis of A_p^n with respect to which the representation matrices are easier to understand. Throughout this section this is the only ordered basis of A_p^n we use. The notation for this ordered basis involves the following: consider the ordered k-tuple $B = (b_1, b_2, ..., b_k)$, $b_i \in A_p^n$. For $x \in A_p^n$ we write Bx for the k-tuple $(b_1 x, b_2 x, ..., b_k x)$. For two tuples $B = (b_1, b_2, ..., b_k)$ and $B' = (b'_1, b'_2, ..., b'_{k'})$, we write $B \sqcup B'$ to mean $(b_1, b_2, ..., b_k, b'_1, b'_2, ..., b'_{k'})$. The aforementioned ordered basis is defined recursively as follows.

Definition 4.2.1. Define B_k to be an ordered basis of A_p^k where

$$B_1 = (1, z_1, z_1^2, ..., z_1^{p-1})$$

$$B_2 = B_1 \sqcup z_2 B_1 \sqcup z_2^2 B_1 \sqcup ... \sqcup z_2^{p-1} B_1$$

$$...$$

$$B_k = B_{k-1} \sqcup z_k B_{k-1} \sqcup z_k^2 B_{k-1} \sqcup ... \sqcup z_k^{p-1} B_{k-1}.$$

We emphasize that B_k is an ordered tuple.

Note that B_k is a basis of monomials for A_p^k. We know the module decomposition of A_p^n as an A_p^n-module after restriction to any $\Bbbk[u_\lambda]$ from Proposition 3.1.7. We have not yet shown the form of the representation matrix of $A_p^n \downarrow \Bbbk[u_\lambda]$ using the ordered basis. The next example finds B_k for the case where $n = 3$ and $p = 3$.

Example 4.2.2. Suppose we want to write the basis B_3 for the underlying $\Bbbk$-vector space of A_3^3. Displayed below are B_1, B_2, and B_3.

$$B_1 = (1, z_1, z_1^2)$$

$$B_2 = (1, z_1, z_1^2, z_2, z_1z_2, z_1^2z_2, z_2^2, z_1z_2^2, z_1^2z_2^2)$$

$$B_3 = (1, z_1, z_1^2, z_2, z_1z_2, z_1^2z_2, z_2^2, z_1z_2^2, z_1^2z_2^2,$$

$$z_3, z_1z_3, z_1^2z_3, z_2z_3, z_1z_2z_3, z_1^2z_2z_3, z_2^2z_3, z_1z_2^2z_3, z_1^2z_2^2z_3,$$

$$z_3^2, z_1z_3^2, z_1^2z_3^2, z_2z_3^2, z_1z_2z_3^2, z_1^2z_2z_3^2, z_2^2z_3^2, z_1z_2^2z_3^2, z_1^2z_2^2z_3^2)$$

In this case B_1 has 3 elements, B_2 has 9 and B_3 has 27.

The number of elements in each B_k is p^k as expected, since the $\Bbbk$-vector space dimension of A_p^k is p^k.

Fact 4.2.3. $A_p^n/(z_1, ..., z_i)$ has $\Bbbk$-vector space dimension p^{n-i} for $1 \leq i \leq n$. The dimension of $A_p^n/(z_1, ..., z_n)$ is $p^0 = 1$.

This fact is true because B_{n-i} can be used as a basis for $A_p^n/(z_1, ..., z_i)$. Note that due to the construction of B_k, B_1 has p elements, B_2 has p^2 elements and so on. This is because B_1 is constructed to have p elements, namely the constant term and the powers of z_1 up to $p-1$. Each subsequent B_k will multiply the terms of B_{k-1} by the k^{th} variable and the powers of the k^{th} variable up to $p-1$. It is important that we have not only the dimension of $A_p^n/(z_k, z_{k+1}, ..., z_n)$, but also a defined basis of $A_p^n/(z_k, z_{k+1}, ..., z_n)$ for any n, p, and k. We offer a definition to formalize A_p^n-modules of the form $A_p^n/(z_k, z_{k+1}, ..., z_n)$.

Definition 4.2.4. For an integer $1 \leq i \leq n$, let γ_i be the ideal $(z_i, z_{i+1}, ..., z_n)$ in A_p^n. We take γ_0 to be the zero ideal.

We are now ready to restate the main objective for this section. For any γ_i, can we find a specific α such that the representation matrix of M_α after restriction to

$\Bbbk[u_\lambda]$ is not free if and only if $\lambda_j = 0$ for $j > i$? The previous section explores some specific examples of this, but here we solve the problem in the general case. We do this using the module structure of A_p^n/γ_i. Any such module can be seen as a point α in $\mathbb{A}^{nd^2}$. We give a name to a choice of α that corresponds to modules of this form.

Definition 4.2.5. Let α_i be the α in $V(Q)$ corresponding to A_p^n/γ_i, and recall that $V(Q) \subset \mathbb{A}^{(i-1)p^{2(i-1)}}$. Note that if $i \neq j$, then α_i and α_j belong to different dimensional affine spaces. We only introduce specific α_i in the context of the fixed ordered basis B_k.

One objective of this section is to provide the representation matrix of A_p^n/γ_i for any permissible choice of n, p, or i. After finding a general form for these representation matrices, we comment on their freeness after restriction. The following fact highlights why A_p^n/γ_i is useful.

Fact 4.2.6. For $1 \leq i \leq n$ we have that $A_p^n/\gamma_i \cong A_p^{i-1}$ as rings.

To see this, consider the natural surjection from A_p^n onto A_p^{i-1} where $z_j \mapsto z_j$ for $1 \leq j \leq i-1$ and $z_j \mapsto 0$ otherwise. Clearly the kernel of this map is γ_i. We can use B_{i-1} as a basis for A_p^n/γ_i. We offer an example in the case of A_2^2.

Example 4.2.7. Let $n = 2$ and $p = 2$. Then $M = A_2^2/\gamma_2$ has ordered basis $(1, z_1)$ and representation matrices

$$[z_1]_M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } [z_2]_M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Here $M \downarrow \Bbbk[u_\lambda]$ will be free if and only if $\lambda_1 \neq 0$. We find that $W(M) = V(\Lambda_1)$ in $\mathbb{A}^2$, which has dimension one.

As another example, we find representation matrices over rings with multiple possible choices of γ_i.

Example 4.2.8. Let $n = 3$ and $p = 2$. We explore the $[z_1]_M, [z_2]_M$, and $[z_3]_M$ representation matrices and rank varieties for γ_3 and γ_2.

1. The case of $M = A_2^3/\gamma_3$ has ordered basis $B_2 = (1, z_1, z_2, z_1z_2)$. We find
$$[z_1]_M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ [z_2]_M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \text{ and } [z_3]_M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
We find that $W(M) = V(\Lambda_1, \Lambda_2)$ in $\mathbb{A}^3$, which has dimension one.

2. The case of $M = A_2^3/\gamma_2$ has ordered basis $B_1 = (1, z_1)$.
$$[z_1]_M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ [z_2]_M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } [z_3]_M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$
We find that $W(M) = V(\Lambda_1)$ in $\mathbb{A}^3$, which has dimension two.

Next, we let $n = 2, p = 3$, and again find the $[z_k]_M$ matrices for $k = 1$ and $k = 2$. We use the module $M = A_3^2/\gamma_2$. Here we have ordered basis $B_1 = (1, z_1, z_1^2)$,
$$[z_1]_M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and } [z_2]_M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
We find that $W(M) = V(\Lambda_1)$ in $\mathbb{A}^2$, which has dimension one.

The matrices are changing in a predictable way as n and p increase. We see the matrices are taking a block matrix form with an increase of n increasing the number of blocks and an increase of p increasing the size of the blocks. Notice how the matrices of the same size take a similar form. The fact given below is a key ingredient in finding the representation matrix of A_p^n/γ_i for any permissible n, p, or i.

Fact 4.2.9. We find $U_\Lambda(\alpha_1) = [0]$, the 1×1 zero matrix, since $A_p^n/\gamma_1 \cong \Bbbk$.

The representation matrix for $M = A_p^n/\gamma_2$ is

$$[z_1]_M = \begin{bmatrix} 0 & 0 & ... & 0 & 0 \\ 1 & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 1 & 0 \end{bmatrix}.$$

The representation matrix for z_i where $i \neq 1$ is the $p \times p$ zero matrix. Note that $[u_\lambda]_M$ is a $p \times p$ matrix when $M = A_p^n/\gamma_2$.

At this point this fact is only shown through examples. Theorem 4.2.13 will prove this fact in general. We offer yet another example that will involve grouping the representation matrices according to dimension.

Example 4.2.10. For this example, all matrices shown are $U_\Lambda(\alpha_i)$ matrices. In practice, one can first find the $[z_i]_M$ if it is not yet clear what the $U_\Lambda(\alpha_i)$ matrix is. We observe the changing of $U_\Lambda(\alpha_i)$ for $2 \leq n \leq 4$ and $p = 2$ for different choices of γ_i where $2 \leq i \leq n$. As before, $M = A_p^n/\gamma_i$. The $U_\Lambda(\alpha_i)$ matrices shown are grouped by dimension this time to highlight an emerging pattern.

1. There are three matrices where $d = 2$. Namely,

$$\text{when } i = 2 \text{ and } n = 2 \begin{bmatrix} 0 & 0 \\ \Lambda_1 & 0 \end{bmatrix},$$

$$\text{when } i = 2 \text{ and } n = 3 \begin{bmatrix} 0 & 0 \\ \Lambda_1 & 0 \end{bmatrix},$$

$$\text{and finally when } i = 2 \text{ and } n = 4 \begin{bmatrix} 0 & 0 \\ \Lambda_1 & 0 \end{bmatrix}.$$

2. There are two matrices where $d = 4$. Namely,

$$\text{when } i=3 \text{ and } n=3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ \Lambda_1 & 0 & 0 & 0 \\ \Lambda_2 & 0 & 0 & 0 \\ 0 & \Lambda_2 & \Lambda_1 & 0 \end{bmatrix},$$

$$\text{when } i=3 \text{ and } n=4 \begin{bmatrix} 0 & 0 & 0 & 0 \\ \Lambda_1 & 0 & 0 & 0 \\ \Lambda_2 & 0 & 0 & 0 \\ 0 & \Lambda_2 & \Lambda_1 & 0 \end{bmatrix}.$$

3. There is one matrix where $d = 8$. This comes from $i = 4$ and $n = 4$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Lambda_2 & \Lambda_1 & 0 & 0 & 0 & 0 & 0 \\ \Lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Lambda_3 & 0 & 0 & \Lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \Lambda_3 & 0 & \Lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Lambda_3 & 0 & \Lambda_2 & \Lambda_1 & 0 \end{bmatrix}.$$

Below, we examine the rank variety of the $2 \times 2, 4 \times 4$, and 8×8 matrices.

1. When $i = 2$ and $n = 2$, $W(M) = V(\Lambda_1)$ which has dimension 1.
2. When $i = 2$ and $n = 3$, $W(M) = V(\Lambda_1)$ which has dimension 2.
3. When $i = 2$ and $n = 4$, $W(M) = V(\Lambda_1)$ which has dimension 3.
4. When $i = 3$ and $n = 3$, $W(M) = V(\Lambda_1, \Lambda_2)$ which has dimension 1.
5. When $i = 3$ and $n = 4$, $W(M) = V(\Lambda_1, \Lambda_2)$ which has dimension 2.
6. When $i = 4$ and $n = 4$, $W(M) = V(\Lambda_1, \Lambda_2, \Lambda_3)$ which has dimension 1.

Notice that in the cases where $i = n$, the variety is generated by a different number of Λ_i, but the dimension of the variety remained the same. Next, we conduct a similar exploration of $U_\Lambda(\alpha_i)$ using the same M where n is fixed at 2 but $p = 2, 3$, or 5 and the $i = 2$ in γ_i. The dimension of these matrices is a multiple of p just like when p was fixed.

1. When $p = 2$ we have $\begin{bmatrix} 0 & 0 \\ \Lambda_1 & 0 \end{bmatrix}$.
2. When $p = 3$ we have $\begin{bmatrix} 0 & 0 & 0 \\ \Lambda_1 & 0 & 0 \\ 0 & \Lambda_1 & 0 \end{bmatrix}$.
3. When $p = 5$ we have $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \Lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \Lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \Lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \Lambda_1 & 0 \end{bmatrix}$.

For $n = 2$ as p increases the size of the matrix increases, but the matrix keeps the same form. We find for any of the three choices of p, $W(M) = V(\Lambda_1)$. The dimension of $V(\Lambda_1)$ is 1 when $p = 2, 3$, or 5.

We offer a proposition on the structure of A_p^n/γ_i-modules.

Proposition 4.2.11. *For fixed n and p,*

$$U_\Lambda(\alpha_2) = \begin{bmatrix} 0 & 0 & 0 & ... & 0 & 0 & 0 \\ \Lambda_1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & \Lambda_1 & 0 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & \Lambda_1 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & \Lambda_1 & 0 \end{bmatrix}.$$

This is a matrix with Λ_1 on the lower sub-diagonal and zeroes elsewhere. The matrix is of size $p \times p$.

Proof. Fix n and p. Let $M = A_p^n/\gamma_2$. We find the matrix size of $U_\Lambda(\alpha_2)$ to be $p \times p$ since $U_\Lambda(\alpha_2) \cong A_p^1$ according to Fact 4.2.6. We know from Definition 2.2.3 that

$$U_\Lambda(\alpha_2) = \Lambda_1 X_1(\alpha_2) + \Lambda_2 X_2(\alpha_2) + ... + \Lambda_n X_n(\alpha_2).$$

For $k > 2$ we know that $X_k = 0$ since $z_k \in \gamma_2$. Thus $U_\Lambda(\alpha_2) = \Lambda_1 X_1(\alpha_2)$. Using ordered basis B_1 we can calculate $U_\Lambda(\alpha_2)$. We conclude that $U_\Lambda(\alpha_2)$ has the desired form as stated in the proposition because $\Lambda_1 B_1 = \{\Lambda_1 z_1, \Lambda_1 z_1^2, ..., \Lambda_1 z_1^{p-1}\}$. $\square$

Next, we define a notation for important representation matrices.

Definition 4.2.12. Let $\mathcal{B}_n$ be the generic representation matrix $U_\Lambda(\alpha_0)$ using the ordered basis from 4.2.1. In other words, $\mathcal{B}_n$ is the *canonical representation matrix* of A_p^n restricted to the generic u_Λ.

The n in $\mathcal{B}_n$ is the same as the n of the corresponding A_p^n-module. We now give a theorem showing the form of $\mathcal{B}_n$.

Theorem 4.2.13. *For a fixed p and using ordered basis B_k, we recursively construct all $\mathcal{B}_k$ as follows for any $1 \leq k \leq n$. We have shown $\mathcal{B}_1$ in 4.2.11.*

$$\mathcal{B}_k = \begin{bmatrix} \mathcal{B}_{k-1} & 0 & 0 & ... & 0 & 0 & 0 \\ \Lambda_k I & \mathcal{B}_{k-1} & 0 & ... & 0 & 0 & 0 \\ 0 & \Lambda_k I & \mathcal{B}_{k-1} & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \mathcal{B}_{k-1} & 0 & 0 \\ 0 & 0 & 0 & ... & \Lambda_k I & \mathcal{B}_{k-1} & 0 \\ 0 & 0 & 0 & ... & 0 & \Lambda_k I & \mathcal{B}_{k-1} \end{bmatrix}$$

Here $\Lambda_i I$ is the identity matrix with each entry multiplied by Λ_i. For any k, $\mathcal{B}_k$ is size $p^k \times p^k$.

Proof. We know $\mathcal{B}_1$ from Proposition 4.2.11 since $A_p^1 \cong A_p^n/\gamma_2$ as $\Bbbk$-algebras. Furthermore, the dimension of A_p^1 is p and the size of $\mathcal{B}_1$ is clearly $p \times p$. We proceed by induction, assuming that $\mathcal{B}_{k-1}$ has the representation matrix described in the theorem. By definition, $\mathcal{B}_k$ is the generic representation matrix $U_\Lambda(\alpha_0)$ where $\alpha_0 \in \mathbb{A}^{kd^2}$ is the point corresponding to A_p^k. We find that $A_p^{k-1} \cong A_p^k/\gamma_k$. From Definition 2.2.4, we find $U_\Lambda(\alpha_0) = \Lambda_1 X_1(\alpha_0) + \Lambda_2 X_2(\alpha_0) + ... + \Lambda_{k-1} X_{k-1}(\alpha_0) + \Lambda_k X_k(\alpha_0)$. We use B_k as the ordered basis and show $U_\Lambda(\alpha_0)$ below. To better understand $U_\Lambda(\alpha_0)$, recall $B_k = B_{k-1} \sqcup z_k B_{k-1} \sqcup z_k^2 B_{k-1} \sqcup ... \sqcup z_k^{p-1} B_{k-1}$. The p groups listed that compose B_k each yield a $\mathcal{B}_{k-1}$ block in $U_\Lambda(\alpha_0)$. More specifically, using B_{k-1} as the ordered basis for A_p^{k-1} we find $\Lambda_1 X_1(\alpha_0) + \Lambda_2 X_2(\alpha_0) + ... + \Lambda_{k-1} X_{k-1}(\alpha_0)$ is the matrix

$$\begin{bmatrix} \mathcal{B}_{k-1} & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & \mathcal{B}_{k-1} & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & \mathcal{B}_{k-1} & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \mathcal{B}_{k-1} & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & \mathcal{B}_{k-1} & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & \mathcal{B}_{k-1} \end{bmatrix}.$$

To obtain $U_\Lambda(\alpha_0)$, we first need $\Lambda_k X_k(\alpha_0)$. We find $\Lambda_k X_k(\alpha_0)$ by multiplying B_k by z_k, which yields

$$z_k B_k = z_k B_{k-1} \sqcup z_k^2 B_{k-1} \sqcup ... \sqcup z_k^{p-1} B_{k-1}$$

and

$$\Lambda_k X_k(\alpha_0) = \begin{bmatrix} 0 & 0 & 0 & ... & 0 & 0 & 0 \\ \Lambda_k I & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & \Lambda_k I & 0 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & \Lambda_k I & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & \Lambda_k I & 0 \end{bmatrix}.$$

Thus $U_\Lambda(\alpha_0)$ is the desired matrix

$$\mathcal{B}_k = \begin{bmatrix} \mathcal{B}_{k-1} & 0 & 0 & ... & 0 & 0 & 0 \\ \Lambda_k I & \mathcal{B}_{k-1} & 0 & ... & 0 & 0 & 0 \\ 0 & \Lambda_k I & \mathcal{B}_{k-1} & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \mathcal{B}_{k-1} & 0 & 0 \\ 0 & 0 & 0 & ... & \Lambda_k I & \mathcal{B}_{k-1} & 0 \\ 0 & 0 & 0 & ... & 0 & \Lambda_k I & \mathcal{B}_{k-1} \end{bmatrix}.$$

Since the size of each $\mathcal{B}_{k-1}$ block in $U_\Lambda(\alpha_0)$ is assumed to be of size $p^{k-1} \times p^{k-1}$, we find that $U_\Lambda(\alpha_0)$ is of size $p^k \times p^k$. $\square$

We make a similar statement about A_p^n/γ_i because of Fact 4.2.6.

Corollary 4.2.14. *Fix p and n and use the ordered basis B_k. For any i where $1 \leq i < n-1$ and $M_\alpha = A_p^n/\gamma_i$,*

$$U_\Lambda(\alpha_i) = \begin{bmatrix} U_\Lambda(\alpha_{i+1}) & 0 & 0 & ... & 0 & 0 & 0 \\ \Lambda_{i+1}I & U_\Lambda(\alpha_{i+1}) & 0 & ... & 0 & 0 & 0 \\ 0 & \Lambda_{i+1}I & U_\Lambda(\alpha_{i+1}) & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & U_\Lambda(\alpha_{i+1}) & 0 & 0 \\ 0 & 0 & 0 & ... & \Lambda_{i+1}I & U_\Lambda(\alpha_{i+1}) & 0 \\ 0 & 0 & 0 & ... & 0 & \Lambda_{i+1}I & U_\Lambda(\alpha_{i+1}) \end{bmatrix}$$

where $\Lambda_{i+1}I$ is the identity matrix with every entry multiplied by Λ_{i+1}.

As a result of this corollary, we are now able to draw our final conclusions for the chapter and answer our guiding question.

Theorem 4.2.15. *For any n and p we find $W(A_p^n/\gamma_i) = V(\Lambda_1, \Lambda_2, ..., \Lambda_{i-1})$ which has dimension $n - i + 1$.*

Proof. We know the $U_\Lambda(\alpha_i)$ representation matrix of A_p^n/γ_i due to Corollary 4.2.14. Using the known representation matrix we can determine the rank variety. The rank variety for any $1 \leq i \leq n - 1$ is best understood recursively. In the trivial case of $i = 1$, we find $W(A_p^n/\gamma_1) = V(0)$, which is indeed of dimension n. When $i = 2$, we use $U_\Lambda(\alpha_2)$, which is shown in Proposition 4.2.11, to conclude $W(A_p^n/\gamma_2) = V(\Lambda_1)$, which has dimension $n - 1$. We continue by using induction on i, assuming that $W(A_p^n/\gamma_{k-1}) = V(\Lambda_1, \Lambda_2, ..., \Lambda_{k-2})$ and that $V(\Lambda_1, \Lambda_2, ..., \Lambda_{k-2})$ is of dimension $n-k+2$. Applying Corollary 4.2.14, in order for $U_\Lambda(\alpha_k)$ not to be of maximal rank we must have that $\Lambda_{k-1} = 0$. Thus if $\Lambda_{k-1} = 0$, then $U_\Lambda(\alpha_k)$ has maximal rank if and only if $\Lambda_j \neq 0$ for some $1 \leq j \leq k - 2$. We conclude that $W(A_p^n/\gamma_k) = V(\Lambda_1, \Lambda_2, ..., \Lambda_{k-1})$. The dimension of $V(\Lambda_1, \Lambda_2, ..., \Lambda_{k-1})$ is $n-k+1$. Thus $W(A_p^n/\gamma_i) = V(\Lambda_1, \Lambda_2, ..., \Lambda_{i-1})$ which has dimension $n - i + 1$. □

We offer a final corollary to answer the guiding question of this chapter.

Corollary 4.2.16. *For any positive integer, there is a choice of n, p and i such that the rank variety of A_p^n/γ_i is that integer. Additionally we can calculate the representation matrix of this module.*

With these conclusions drawn our investigation in the case of a fixed module and generic λ is concluded.

CHAPTER 5

Fixed Subalgebra Freeness

5.1 Dimension of Module Varieties

In the previous chapter we explored the freeness from Theorem 3.2.6 with a fixed module. In this chapter we instead focus on freeness with a fixed subalgebra. In other words, we are looking at module variety rather than rank variety. Ultimately, this chapter will address how various choices of subalgebra interact with the ideal defining the freeness of a generic module.

We are going to fix a subalgebra and consider which α lead to freeness. The key to determining freeness is the ideal $I_\nu(U_\lambda^{p-1})$. We offer the following definition so that we can investigate the dimension of the underlying ring.

Definition 5.1.1. Let S_λ be the ring

$$\Bbbk[x_{i,r,s} \mid 1 \leq i \leq n, 1 \leq r, s \leq d]/(Q + I_\nu(U_\lambda^{p-1})).$$

This definition uses the language from Chapter 2 defining a generic A_p^n-module. In total there are nd^2 of the $x_{i,r,s}$. We are interested in how the Krull dimension of S_λ, $\dim(S_\lambda)$, relates to the Krull dimension of S'_λ, $\dim(S_{\lambda'})$, for $\lambda, \lambda' \in \mathbb{A}^n$. In other words, does the Krull dimension of the module variety change when we change λ? Before approaching S_λ as a whole, we investigate the dimension of $V(Q)$. When Q was introduced in Definition 2.1.7, we gave an example for $d = 2$, but the following example goes into further depth.

Example 5.1.2. Suppose we observe the Krull dimension of $V(Q)$ in various cases. After finding Q, we can calculate the Krull dimension of $\Bbbk[x_{i,r,s} \mid 1 \leq i \leq n, 1 \leq r, s \leq d]/Q$.

If $p = 2, n = 2$, and $d = 2$, we know from Example 2.1.8 that Q can be defined by 12 equations. For this n, p, and d, the Krull dimension of the ring $\Bbbk[x_{i,r,s} \mid 1 \leq i \leq n, 1 \leq r, s \leq d]$ is 8. Furthermore, the Krull dimension of $\Bbbk[x_{i,r,s} \mid 1 \leq i \leq n, 1 \leq r, s \leq d]/Q$ is 3. This Krull dimension can be computed in Macaulay2 [13]. On the other hand, if $p = 2, n = 3$, and $d = 2$, there are now 24 defining equations for Q. The height of Q here is 4. Continuing to increase n, if $p = 2, n = 4$, and $d = 2$, there are now 40 defining equations and $\dim(V(Q)) = 5$. As p increases we quickly find that this calculation is incredibly computationally expensive, but we are at least able to see the effect of increasing n.

In general, we know how many equations define Q, but cannot always compute the Krull dimension of the underlying ring. To better understand S_λ, we need to understand the ideal $I_\nu(U_\lambda^{p-1})$. The following example looks at this for some simple cases.

Example 5.1.3. Suppose $p = 2, n = 2$, and $d = 2$, and we want to observe the Krull dimension of the underlying ring of $I_1(U_\lambda)$. We know $U_\lambda = \lambda_1 X_1 + \lambda_2 X_2$ is the matrix

$$\begin{bmatrix} \lambda_1 x_{1,1,1} + \lambda_2 x_{2,1,1} & \lambda_1 x_{1,1,2} + \lambda_2 x_{2,1,2} \\ \lambda_1 x_{1,2,1} + \lambda_2 x_{2,2,1} & \lambda_1 x_{1,2,2} + \lambda_2 x_{2,2,2} \end{bmatrix}.$$

Thus our ideal defining freeness is

$$I_1(U_\Lambda) = (\lambda_1 x_{1,1,1} + \lambda_2 x_{2,1,1}, \lambda_1 x_{1,1,2} + \lambda_2 x_{2,1,2}, \lambda_1 x_{1,2,1} + \lambda_2 x_{2,2,1}, \lambda_1 x_{1,2,2} + \lambda_2 x_{2,2,2}).$$

For a given choice of λ, we are looking for the conditions that make $I_1(U_\Lambda) = 0$

1. If $u_\lambda = x_1$, then $I_1(U_\Lambda) = (x_{1,1,1}, x_{1,1,2}, x_{1,2,1}, x_{1,2,2})$ requiring $X_1 = 0$ for $I_1(U_\Lambda)$ to be 0.
2. If $u_\lambda = x_2$, then $I_1(U_\Lambda) = (x_{2,1,1}, x_{2,1,2}, x_{2,2,1}, x_{2,2,2})$ requiring $X_2 = 0$ for $I_1(U_\Lambda)$ to be 0.
3. If $u_\lambda = x_1 + x_2$, then $I_1(U_\Lambda) = (x_{1,1,1} + x_{2,1,1}, x_{1,1,2} + x_{2,1,2}, x_{1,2,1} + x_{2,2,1}, x_{1,2,2} + x_{2,2,2})$ requiring $x_{1,j,k} - x_{2,j,k} = 0$ where $j, k \in \{1, 2\}$ for $I_1(U_\Lambda)$ to be 0.

In the test cases from the preceding example, the Krull dimension of S_λ did not change for any of the choices of λ.

Theorem 5.1.4. *For any λ, λ' in $\mathbb{A}^n$, $S_\lambda \cong S_{\lambda'}$. Additionally, $Y(\lambda) \cong Y(\lambda')$.*

Proof. Choose a nonzero λ in $\mathbb{A}^n$. Then λ_i is nonzero for some $1 \leq i \leq n$. We want to show that S_{e_i} is isomorphic to S_λ as k-algebras where e_i is the n-tuple that is entirely zero except for a 1 in the i^{th} component. Define $\phi : P \to P$ by $\phi(x_{i,r,s}) = \lambda_1 x_{1,r,s} + ... + \lambda_n x_{n,r,s}$ and $\phi(x_{j,r,s}) = x_{j,r,s}$ for $j \neq i$ and all $1 \leq r, s \leq d$. By construction, ϕ is a homomorphism that preserves powers and minors of a matrix. This definition can be described in the shorthand notation utilizing matrices, $\phi(X_i) = \lambda_1 X_1 + ... + \lambda_n X_n$ and $\phi(X_j) = X_j$ for $i \neq j$. This clearly extends to an automorphism of P. In order to prove the theorem, we need to show that $\phi(Q) = Q$ and $\phi(I_\nu(U_{e_i}^{p-1})) = I_\nu(U_\lambda^{p-1})$.

First, we show $\phi(Q) = Q$. For any $j_1, j_2 \neq i$, we find

$$\phi(X_{j_1}X_{j_2} - X_{j_2}X_{j_1}) = X_{j_1}X_{j_2} - X_{j_2}X_{j_1} \in \phi(Q) \text{ and } \phi(X_{j_1})^p = X_{j_1}^p \in \phi(Q).$$

The only remaining terms of Q that we need to check involve X_i. We observe for any $j \neq i$ that

$$\phi(X_iX_j - X_jX_i) = (\lambda_1X_1 + ... + \lambda_iX_i + ... + \lambda_nX_n)X_j - X_j(\lambda_1X_1 + ... + \lambda_iX_i + ... + \lambda_nX_n) =$$
$$\lambda_1X_1X_j + ... + \lambda_iX_iX_j + ... + \lambda_nX_nX_j - \lambda_1X_jX_1 - ... - \lambda_iX_jX_i - ... - \lambda_nX_jX_n =$$
$$\lambda_1(X_1X_j - X_jX_1) + ... + \lambda_i(X_iX_j - X_jX_i) + ... + \lambda_n(X_nX_j - X_jX_n) \in \phi(Q).$$

Since $\lambda_{j_1}(X_{j_1}X_{j_2} - X_{j_2}X_{j_1}) \in \phi(Q)$ for $j_1, j_2 \neq i$ we conclude that $\lambda_i(X_iX_j - X_jX_i)$ must be in $\phi(Q)$. Since λ_i is nonzero, we have $X_iX_j - X_jX_i \in \phi(Q)$. Now we consider $\phi(X_i^p)$. Here, we first observe that

$$\phi(X_i^p) = \phi(X_i)^p = (\lambda_1X_1 + ... + \lambda_iX_i + ... + \lambda_nX_n)^p =$$
$$\lambda_1^pX_1^p + ... + \lambda_i^pX_i^p + ... + \lambda_n^pX_n^p \in \phi(Q).$$

This depends on Fact 1.2.6, where we showed that in the context of characteristic p the power of a sum is equivalent to the sum of the powers. We already found that $X_j^p \in \phi(Q)$ for $i \neq j$. Therefore, we conclude that $\lambda_i^p X_i^p \in \phi(Q)$ and since λ_i is nonzero, $X_i^p \in \phi(Q)$.

Now we have shown that $Q \subset \phi(Q)$. It is clear that $\phi(Q) \subset Q$ and therefore $\phi(Q) = Q$.

Next, we show that $\phi(I_\nu(U_{e_i})) = I_\nu(U_\lambda)$. In fact,

$$\phi(U_{e_i}) = \phi(X_i) = \lambda_1 X_1 + ... + \lambda_n X_n = U_\lambda.$$

Since this is true, we know that

$$\phi(I_\nu(U_{e_i}^{p-1})) = I_\nu(U_\lambda^{p-1}).$$

Since both $\phi(Q) = Q$ and $\phi(I_\nu(U_{e_i}^{p-1})) = I_\nu(U_\lambda^{p-1})$ then

$$\phi(Q + I_\nu(U_{e_i}^{p-1})) = Q + I_\nu(U_\lambda^{p-1}).$$

This is sufficient to conclude that $S_{e_1} \cong S_\lambda$. In order to prove the more general statement, we now need to show that $S_{e_i} \cong S_{e_j}$ for $i \neq j$. To this, end we define $\psi : P \to P$ by $\psi(X_i) = X_j$, $\psi(X_j) = X_i$ and $\psi(X_{k_1}) = X_{k_1}$ for $i, j \neq k_1$. Similar to what we did with ϕ, we are going to show that $\psi(Q) = Q$ and $\psi(I_\nu(U_{e_i}^{p-1}) = I_\nu(U_{e_j}^{p-1})$. To show $\psi(Q) = Q$, we let k_1 and k_2 be positive integers other than i or j. We know $\psi(X_{k_1}X_{k_2} - X_{k_2}X_{k_1}) = X_{k_1}X_{k_2} - X_{k_2}X_{k_1} \in \psi(Q)$ and that $\psi(X_{k_1}^p) = X_{k_1}^p \in \psi(Q)$. Additionally,

$$\psi(X_{k_1}X_i - X_iX_{k_1}) = X_jX_{k_1} - X_{k_1}X_j \in \psi(Q).$$

For the same reason, $X_iX_{k_1} - X_{k_1}X_i \in \psi(Q)$. This leaves the case of

$$\psi(X_jX_i - X_iX_j) = X_jX_i - X_iX_j \in \psi(Q)$$

which shows that X_i and X_j have their commutativity condition from Q encoded in $\psi(Q)$. We also find

$$\psi(X_i^p) = X_j^p \in \psi(Q) \text{ and } \psi(X_j^p) = X_i^p \in \psi(Q).$$

In summary, $Q \subset \psi(Q)$. The other direction is easily verifiable and thus $\psi(Q) = Q$. Now we show $\psi(I_\nu(U_{e_i}^{p-1}) = I_\nu(U_{e_j}^{p-1})$. This is a direct result of the equation

$$\psi(U_{e_i}) = \psi(X_i) = \lambda_j X_j = U_{e_j}.$$

This equation implies that $\psi(U_{e_i}^{p-1}) = I_\nu(U_{e_j}^{p-1})$. Similarly, we show that $\psi(U_{e_j}^{p-1}) = I_\nu(U_{e_i}^{p-1})$. Combining this with $\psi(Q) = Q$ we conclude that $S_{e_i} \cong S_{e_j}$.
Finally, let λ and λ' be two nonzero elements of $\mathbb{A}^n$. Then λ_i is nonzero for some i and λ'_j is nonzero for some j. $S_{e_i} \cong S_\lambda$ and $S_{e_j} \cong S_{\lambda'}$. But we know $S_{e_i} \cong S_{e_j}$ so $S_\lambda \cong S_{\lambda'}$. Since $S_\lambda \cong S_{\lambda'}$ we can conclude $Y(\lambda) \cong Y(\lambda')$. □

The following example applies the idea of this proof to the $n = 2$ and $p = 2$ setting and shows the ideal of freeness.

Example 5.1.5. Suppose that ϕ is a homomorphism from $\Bbbk[X_1, X_2] \to \Bbbk[X_1, X_2]$ where $\phi(X_1) = \lambda_1 X_1 + \lambda_2 X_2$ and $\phi(X_2) = X_2$ with λ_1, and $\lambda_2 \in \Bbbk$. Here, Q is generated by $X_1X_2 - X_2X_1, X_1^2$, and X_2^2. The twelve elements of Q were shown for this n, p, and d in Example 2.1.8. We want to compare those twelve generators to the elements that generate $\phi(Q)$, where

$$\phi(Q) = ((\lambda_1 X_1 + \lambda_2 X_2)X_2 + X_2(\lambda_1 X_1 + \lambda_2 X_2), (\lambda_1 X_1 + \lambda_2 X_2)^2, X_2^2).$$

The generators of Q and $\phi(Q)$ compared are as follows.

1. For the commutativity requirement we have $X_1X_2 - X_2X_1$ for Q and $(\lambda_1 X_1 + \lambda_2 X_2)X_2 - X_2(\lambda_1 X_1 + \lambda_2 X_2)$ for $\phi(Q)$
2. The generators of X_1^2 in Q become $(\lambda_1 X_1 + \lambda_2 X_2)^2$ in $\phi(Q)$.
3. The generators of the X_2^2 component will remain identical in both P and $\phi(Q)$.

Although these matrices are different we know the ideals generated by their minors have the same height.

We generalize the example in the following corollary.

Corollary 5.1.6. *The Krull dimension of S_λ is invariant under the choice of λ. In other words, the dimension of the module variety of λ is invariant under a change of λ. In other words, this means dim$(Y(\lambda))$ =dim$(Y(\lambda'))$ for all nonzero $\lambda, \lambda' \in \Bbbk^n$.*

The isomorphism from $S_\lambda \to S_{\lambda'}$ is a stronger result and so this corollary follows. This completes our study of freeness for a fixed subalgebra and our study of freeness in general.

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