SPELUNKING THROUGH A FOREST OF ROOTS: SOLUTIONS OF POLYNOMIALS IN DIFFERENT NUMBER SYSTEMS

by

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ABSTRACT

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Most advanced college students are familiar with the fact that an equation $p(x) = 0$, where $p(x)$ is a polynomial of degree n with real coefficients, will have at most n solutions. When the coefficients are complex, the Fundamental Theorem of Algebra (FTA) says that there are exactly n solutions, counting multiplicity. For example, $x^3 - x = 0$ has exactly three solutions, 0, 1, and -1. This thesis investigates how many solutions polynomial equations have in other number systems, particularly in hyperbolic and parabolic numbers. Our methods involve looking for how many solutions simple equations, such as $x^2 - k = 0$, possess for different choices of k in these distinct number systems. We then consider general polynomials. Overall, our results demonstrate that there can be more or less solutions than the degree of the polynomial, contradicting the outcomes expected from the FTA.

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CHAPTER 1

INTRODUCTION

1.1 My Rationale

Some who read this thesis may not be familiar with the mathematics and concepts foundational to the topics covered. Without sufficient knowledge or familiarity with the basics, how can one adequately understand concepts built upon them? For this reason alone, it is incumbent on me to provide information that can help any college-level reader comprehend the results shared in this paper. In other words, my goal is to make this paper accessible.

To accomplish this goal, the rest of this introduction section is dedicated to explaining background information. Some readers (especially non-STEM readers) may find the repetitive mathematical text quite easy to gloss over, as mathematics is technical in nature. Keeping this in mind, I intend to give more detailed explanations for mathematical conclusions, as well as an analogical visual, so that even those who study outside of STEM may be able to grasp the text. One may refer to this section if they get lost further in the paper.

1.2 Necessary Background

At some point in their academic career, most college students become familiarized with "polynomials" in an algebra-based class. They may see something like $x^2 + x - 1$, being told to find the "roots," or solutions, of this polynomial when it is set equal to 0. Their teacher or mentor may have shown how this polynomial looks graphically:

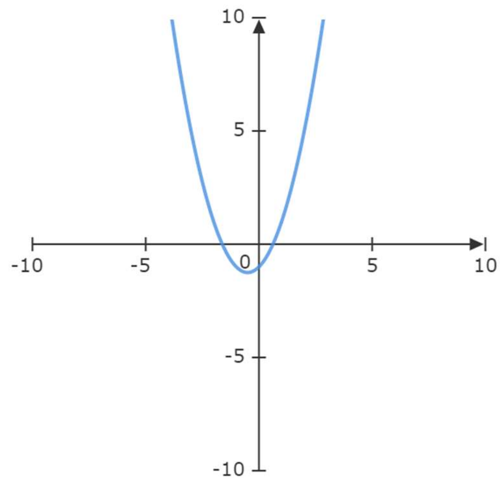


Figure 1.1: Graphical Representation of $p(x) = x^2 + x - 1$

Since we are finding the solutions x for which $p(x) = 0$, we are looking at the points where the graph intersects the x-axis. The quadratic formula may have even popped up when finding solutions to quadratic equations (equations where a polynomial has degree two and is set equal to zero). Many perspectives and methods to find the roots of such polynomials were taught. Finding roots for polynomials is precisely the topic of this paper, but we are now working with number systems that are not typically seen or known about.

For those who may not have a full picture of what a polynomial is or what it can look like, the general form of a polynomial is:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0,$$

where $n = 0, 1, 2, \dots$ (the counting numbers, including 0) and each a_j for $0 \leq j \leq n$ is a number. The numbers a_j are called the coefficients of the polynomial. The coefficients of a polynomial are numbers from a particular number set.

To understand the number sets we investigate, there are two sets whose definitions are useful to know.

Definition: The set of all whole numbers that can be positive, negative, or zero, is called the *set of integers*, denoted by $\mathbb{Z}$.

Definition: The set of all fractions of whole numbers where the denominator is not zero, expressed as

$$\{a/b \mid a, b \in \mathbb{Z}; b \neq 0\},$$

is called the *set of rational numbers*, denoted by $\mathbb{Q}$.

Note that the set of rational numbers contains all the integers, since we can let $b = 1$ and a be any integer. We are now ready to discuss the real numbers.

1.2.1 Real Numbers

There are quite a lot of rational numbers. Infinitely many, in fact. However, rational numbers do not compose every possible length. You may have taken a trigonometry class and were required to find the hypotenuse length of a right triangle such as:

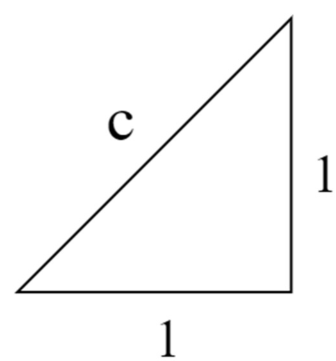


Figure 1.2: Right Triangle with Leg Lengths of 1

The solution can be found via the Pythagorean Theorem, which states that the sum of a right triangle's squared leg lengths is equal to the square of the hypotenuse. In other words, $a^2 + b^2 = c^2$. If we attempt to solve for c for the above right triangle, we get $\sqrt{2}$. This number cannot be expressed as a fraction like all rational numbers must. Therefore, it is not rational. Results like this one supposedly shocked those in the school of Pythagoras, as they initially believed all numbers could be expressed as a ratio of integers, or a fraction. Mathematicians have since come to accept this result and the existence of other non-rational numbers. To make every length along the number line correspond to an actual number, we need to extend the rational numbers to the *set of real numbers*, denoted by $\mathbb{R}$. That is, we may think of the real numbers as all locations along the number line, negative, positive, or zero. Numbers that are real but not rational are called irrational and belong in the *set of irrational numbers*. Therefore, the set of real numbers is the union of all rational and irrational numbers.

Without rigorous proof given here, I will share that the amount of numbers in the real set is *uncountably infinite*, as opposed to the *countably infinite* sets that are the integers and rational numbers. Essentially, while the set of integers (. . . -3, -2, -1, 0, 1, 2, 3, . . .) can be perfectly paired with the counting numbers (1, 2, 3, . . .), and the set of all rational numbers can be perfectly paired with the counting numbers, the real numbers cannot. There is so much infinitude in the set of real numbers that some of them will never be counted, no matter how you attempt to assign a unique counting number to each real number. Indeed, some infinities are greater than others (Abbott, 2015). This result is well-known amongst mathematicians, though more about the set of real numbers can be read in *Understanding Analysis* by Stephen Abbott.

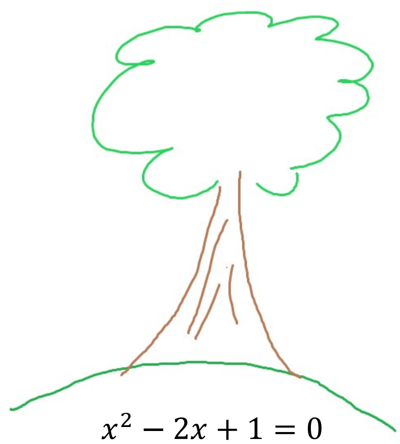
As described thus far, the set of real numbers is typically represented by the number line, or real line. Think of the x-axis one would see in a typical algebra or calculus class.

One may recall that not all polynomials are capable of being solved “normally.” Take for instance the following polynomial equation:

$$x^2 + 1 = 0$$

The solution to this equation requires a number whose square is negative. This is impossible when considering real numbers, as there does not exist any real number whose square is negative.

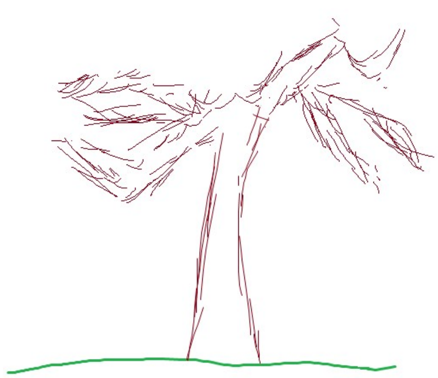
Exploring polynomials with real coefficients to find roots is comparable to hiking in a forest of real polynomial trees. Trees require roots to sustain themselves. Their *functionality*, so to speak, depends on the existence of roots. Similarly, polynomials can only have an output of zero if they possess roots. If one were hiking in such a forest, they may see normal trees like this:



$x^2 - 2x + 1 = 0$

Figure 1.3: Normal $x^2 - 2x + 1 = 0$ Tree

Or perhaps dead trees like this:



$x^2 + x + 1 = 0$

Figure 1.4: Dead $x^2 + x + 1 = 0$ Tree

The dead trees are dead precisely because they have no real roots. Consequently, they are incapable of sustaining themselves. Taking out one's trusty spade and attempting to uproot any roots is equivalent to seeking out roots of a real polynomial equation $p(x) = 0$.

How does one solve this problem of lacking roots? Could we envision a forest where dead trees, with no roots in the real numbers, now possess roots? In the real numbers, it is not possible to find roots of all polynomial equations $p(x) = 0$. However, there is yet hope.

1.2.2 Complex Numbers

Let us introduce a new number, call it i, and give it the following property:

$$i^2 = -1$$

It may not seem like it, but this *thoroughly* enhances our ability to find solutions for polynomial equations. Recall the previous polynomial equation $x^2 + 1 = 0$. Using our new number i, we are now capable of finding solutions to this. In particular, the solutions to the polynomial equation $x^2 + 1 = 0$ are i and -i. We can check this by seeing that:

$$i^2 + 1 = -1 + 1 = 0$$

$$(-i)^2 + 1 = -1 + 1 = 0$$

The equation $x^2 + 1 = 0$, which previously had no solutions in the real numbers, now has two solutions when including the number i. In mathematics, the set housing this new number i is called the set of complex numbers, which is responsible for our ability to find the roots of *all* polynomials:

Definition: The set of all numbers of the form $a + bi$ where a and b are real numbers, expressed as

$$\{a + bi \mid a, b \in \mathbb{R}\},$$

is called the *set of complex numbers*, denoted by $\mathbb{C}$.

Elements of this set are typically denoted by $z = a + bi$ (Brown & Churchill, 2013). The set of real numbers is contained in the complex numbers as the special complex numbers where b is zero.

Complex numbers are capable of being expressed as $z = r(cos\theta + i\ sin\theta)$, referred to as polar coordinate form. We can see this graphically:

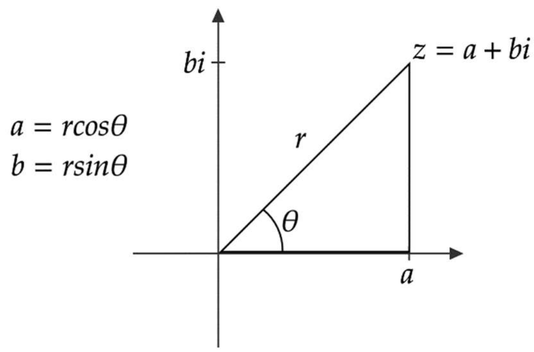


Figure 1.5: Graphical Demonstration of Polar Form

One can then derive Euler's formula using Taylor series identities:

$$re^{i\theta} = r(cos\theta + i\ sin\theta)$$

Thus, complex numbers can also be expressed as $z = re^{i\theta}$. Furthermore, r is called the *modulus* of z, expressed as $r = |z| = \sqrt{z\overline{z}}$, where $\overline{z}$ is the *complex conjugate* of z. The complex conjugate of a complex number $z = a + bi$ is $\overline{z} = a - bi$. The sign in the middle is switched out for the other. These are all the complex numbers with a modulus of 1:

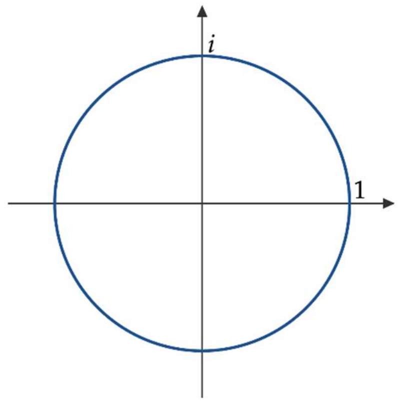


Figure 1.6: Complex Numbers with Modulus 1

Not all complex numbers lie on this circle, just the ones with modulus 1. Other complex numbers will have different moduli, and thus lie on different circles. The circle shape is justified when trying to find numbers with modulus 1:

$$|z| = \sqrt{z\bar{z}} = 1$$

$$\begin{aligned}\sqrt{z\bar{z}} &= \sqrt{(a+bi)(a-bi)} \\ &= \sqrt{a^2 - abi + abi + b^2} \\ &= \sqrt{a^2 + b^2} = 1\end{aligned}$$

Doing this again with $|z| = r$, we can say that r can be thought of as the radius of a circle.

The next theorem states how many roots a polynomial in the complex number system possesses.

1.2.3 The Fundamental Theorem of Algebra

<ins>Theorem – The Fundamental Theorem of Algebra (FTA)</ins>: Every nonzero polynomial of degree n with complex coefficients has exactly n roots, counting multiplicity.

Providing proof of this theorem is outside the scope of this paper. There are at least six, and they are all fairly complicated or generally difficult (Fine & Rosenberger, 1997)

The conclusion of the theorem is most important, as we will be comparing results from new number systems to results we expect from the FTA. There is one more implication of the FTA needing explanation.

From the FTA, it follows that we can uniquely factor a complex polynomial into a product of linear factors:

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0 = (x - b_1)(x - b_2)\cdots(x - b_{n-1})(x - b_n),$$

where each coefficient a_k of the polynomial is complex and each b_k for $1 \leq k \leq n$ is a complex number, not necessarily distinct from one another. Each linear factor represents a root of the polynomial. This factored form of a polynomial may look familiar to an algebra-based class. The first a_n real coefficient is omitted, but the next proof still works if we included it.

We can demonstrate that linear factorization is unique using the FTA. Suppose that we could factor a polynomial of degree n in two ways like this:

$$(x - b_1)(x - b_2)\cdots(x - b_{n-1})(x - b_n) = (x - c_1)(x - c_2)\cdots(x - c_{m-1})(x - c_m)$$

Assume that $m < n$, or that there are fewer linear factors on the right-hand side (RHS) than the left-hand side (LHS) of the equation. Then when all the factors multiply together on the RHS, the degree of the polynomial would be less than n which is not possible. Let us assume that $m > n$ instead. The same argument applies, and we now have a polynomial of degree greater than n which is also not possible. So, $m = n$. It follows that the root values and multiplicity of each root must be same, or else if one side had a root the other did not, then that would contradict the FTA, since its conclusion states that the number of roots (counting multiples) is equal to the degree of the polynomial. Therefore, linear factorization of polynomials over the complex numbers is unique.

The FTA allows us to predetermine how many solutions a given polynomial has, as well as find the completely factored form of a polynomial with complex coefficients. For example, we now know that the polynomial equation

$$x^3 - x = 0$$

possesses exactly 3 roots, 1, 0, and -1. The unique factorization of the polynomial looks like:

$$x^3 - x = (x-1)(x-0)(x+1) = x(x-1)(x+1).$$

For a polynomial with complex roots, let us return to $x^2 + 1$:

$$x^2 + 1 = (x - i)(x + i)$$

The simple inclusion of i not only enables us to find roots where we would otherwise be unable to with real numbers, but also creates an interesting phenomenon where the root count *exactly* matches the degree of the polynomial. This next example will demonstrate the concept of multiplicity:

$$\begin{aligned} x^4 - 4x^3 + 6x^2 - 4x + 1 &= (x-1)(x-1)(x-1)(x-1) \\ &= (x-1)^4 \end{aligned}$$

For the polynomial $x^4 - 4x^3 + 6x^2 - 4x + 1$, the root 1 has a multiplicity of 4. In other words, the root "shows up" four times after transforming the polynomial into its linear factors. This multiplicity value exactly matches the degree of the polynomial.

A forest with complex polynomial trees is an upgrade from one with real polynomial trees. All its trees, barring the zero constant polynomial tree, possess exactly as many roots as the degree of the polynomial they represent. Say one were to jump down the rabbit hole from the real polynomials to complex polynomials. They would probably see a tree like:

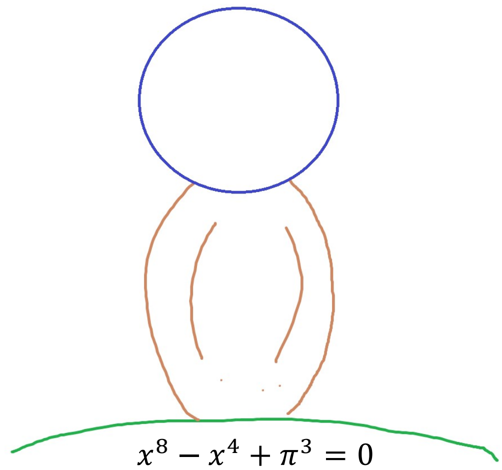


Figure 1.7: Spherical Complex Tree of $x^8 - x^4 + \pi^3 = 0$

It would be too much work to dig up all roots of this tree to see if there are indeed eight. Knowing, however, that there are eight because of the FTA is equivalent to finding a perfectly circular cave entrance in the ground, heading under, and spelunking up to the point below the tree.

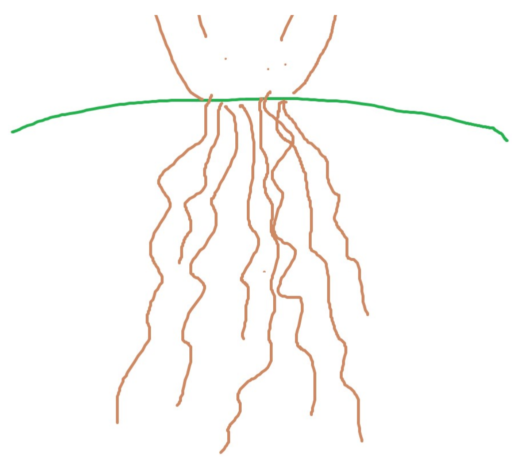


Figure 1.8: Roots of the Complex Tree of $x^8 - x^4 + \pi^3 = 0$

Indeed, there are eight roots.

So that is all, right? Everything is perfectly simple now that we have the complex numbers – if we have a polynomial with degree n, then it has exactly n roots (counting multiple of the same roots). It seems that we have nothing left to do. Except, if you wish to play around a little bit with the following complex number form:

$$a + bi$$

Squaring this form yields $a^2 - b^2 + 2abi$. We can obtain this result precisely due to the presence of i:

$$\begin{aligned}(a+bi)^2 &= (a+bi)(a+bi)\\ &= a^2 + abi + abi + b^2i^2\\ &= a^2 + 2abi - b^2\\ &= a^2 - b^2 + 2abi\end{aligned}$$

Squaring i leads to the real term becoming a difference of two squared real numbers, expressed as $a^2 - b^2$. Algebraically, we can use $a^2 - b^2 + 2abi$ to investigate the square roots of various complex numbers. Roots of the polynomial $x^2 - k$, for some k complex number, will be affected by i.

The mathematical influence of i on the roots of polynomials is noticeable and motivates a slew of questions. What if i squared did not equal -1? What if i were something else? Are there other nonreal numbers like i that we can inspect, numbers whose squares are not 1? Should we find such numbers, could we see how they fare against the FTA? In other words, how many roots would a polynomial with coefficients from a set possessing one of these numbers have? Are there other forests to explore? Indeed, there are other nonreal numbers like i. In fact, there are uncountably infinitely many more. However, we need only investigate two nonreal numbers. Infinitely many other nonreal numbers belong to a system that is like the complex numbers or one of the soon-to-be investigated nonreal number systems, up to scaling (Harkin & Harkin, 2004).

1.2.4 Hyperbolic Numbers

Let us introduce a new number, call it τ (tau), and give it the following property:

$$\tau^2 = 1.$$

This new number is neither 1 nor -1. It is simply τ until it is squared, exactly like i. To a skeptical reader, this may seem redundant. There already are two numbers whose square is 1, being 1 and -1. The polynomial $x^2 - 1$ has solutions. What purpose could there be for such a number? Observe that we can take the same form from the complex numbers and generate a new number set.

Definition: The set of all numbers of the form $a + b\tau$ where a and b are real numbers, expressed as

$$\{a + b\tau \mid a, b \in \mathbb{R}\},$$

is called the *set of hyperbolic numbers*, denoted by $\mathbb{L}$.

Elements of this set are typically denoted by $z = a + b\tau$. The reason we use a double-struck L instead of a double-struck H to denote the set is because $\mathbb{H}$ represents the hyperbolic plane, which means something different from the set of hyperbolic numbers. Hyperbolic numbers can alternatively be called Lorentz numbers, named after the Dutch physicist Hendrik Lorentz. Numbers from this set are utilized for spacetime models (Naber, 1989).

Like complex numbers, hyperbolic numbers can be expressed in exponential form. Unlike the complex exponential form, where one form works for all nonzero complex numbers, hyperbolic numbers fall under four different exponential forms, depending on where they lie in the plane. Consider the four open quadrants cut out by the lines $y = x$ and $y = -x$:

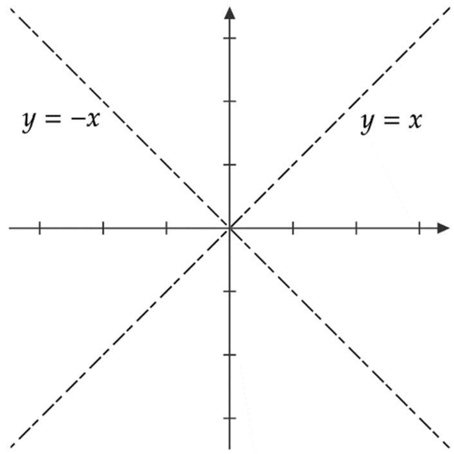


Figure 1.9: Four Open Quadrants Cut Out by $y = x$ and $y = -x$

Using similar derivations for Euler's formula (Atmai, Kizer-Pugh, & Shipman, 2022), we find that hyperbolic numbers in the open right quadrant have the form

$$re^{\tau\theta} = r(cosh\theta + \tau\ sinh\theta).$$

In the other three quadrants, the hyperbolic numbers are expressed as

$$\tau re^{\tau\theta} = \tau r(cosh\theta + \tau\ sinh\theta) \text{ in the top quadrant,}$$

$$-re^{\tau\theta} = -r(cosh\theta + \tau\ sinh\theta) \text{ in the left quadrant, and}$$

$$-\tau re^{\tau\theta} = -\tau r(cosh\theta + \tau\ sinh\theta) \text{ in the bottom quadrant.}$$

In hyperbolic numbers, r still signifies the modulus of a hyperbolic number z, expressed as $r = |z| = \sqrt{|z\overline{z}|}$, where $\overline{z}$ is the *hyperbolic conjugate* of z. The absolute value brackets within the radical are necessary, as will be shown. Like complex numbers, the hyperbolic conjugate of a hyperbolic number $z = a + b\tau$ is $\overline{z} = a - b\tau$. Below are hyperbolic numbers with modulus 1:

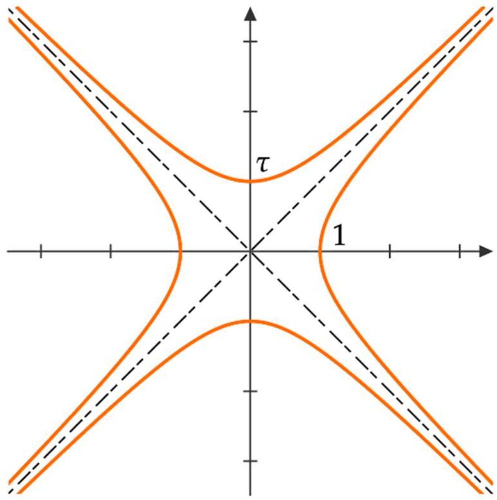


Figure 1.10: Hyperbolic Numbers with Modulus 1

Not all hyperbolic numbers lie on this hyperbolic shape, just the ones with modulus 1. Other hyperbolic numbers will have different moduli, and thus lie on different hyperbolae. This hyperbolic shape can again be justified when finding numbers with modulus 1. Suppose we write

$$|z| = \sqrt{z\overline{z}} = 1$$

$$\begin{aligned} \sqrt{z\overline{z}} &= \sqrt{(a+b\tau)(a-b\tau)} \\ &= \sqrt{a^2 - ab\tau + ab\tau - b^2} \\ &= \sqrt{a^2 - b^2} = 1 \end{aligned}$$

However, note that $a^2 - b^2$ can be negative. To accommodate this, mathematicians write the modulus like

$$|z| = \sqrt{|z\overline{z}|} = \sqrt{|a^2 - b^2|}\,,$$

hence the graph possessing four "branches."

Notice that in the set of complex numbers, the modulus of a complex number can be construed as a distance from the origin. For complex numbers on the unit circle, each possesses a modulus of 1, which makes sense visually speaking. Hyperbolic numbers are different. If we look at Figure 1.10, we can note that all points on the orange shape are hyperbolic numbers that possess a modulus of 1, shown in Figure 1.11:

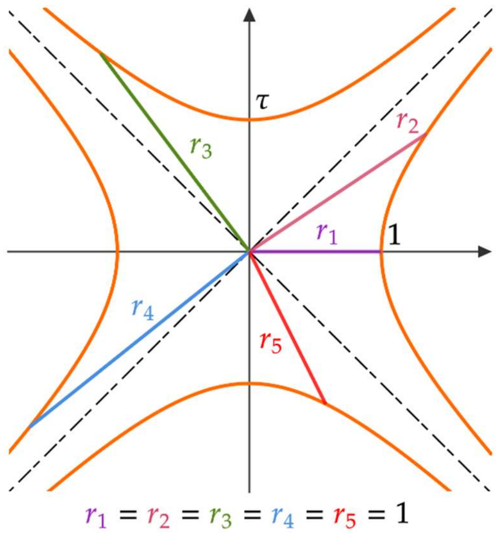


Figure 1.11: Points on the Unit Hyperbola Possess the Same Modulus Value

Finding the modulus of any point on this graph will yield 1. For hyperbolic numbers, the modulus cannot be so easily construed as "distance" as we are used to it.

1.2.5 Parabolic Numbers

Let us introduce another new number, call it ω (omega), and give it the following property:

$$\omega^2 = 0$$

Like before, ω is not 0. It is its own thing until it is squared. Again, this may seem redundant, but its purpose follows suit to the previously introduced nonreal numbers. With this number, we can generate a new number set:

<u>Definition</u>: The set of all numbers of the form $a + b\omega$ where a and b are real numbers, expressed as

$$\{a + b\omega \mid a, b \in \mathbb{R}\}$$

is called the *set of parabolic numbers*, denoted by $\mathbb{P}$.

Elements of this set are typically denoted by $z = a + b\omega$. Unlike the previous sets, we do not know whether these numbers have applications.

Like the previous sets, we can derive the form

$$re^{\omega\theta} = r(1 + \omega\theta)$$

for parabolic numbers. Again, r is the modulus of a parabolic number, expressed as $r = |z| = \sqrt{z\bar{z}}$, where $\bar{z}$ is the *parabolic conjugate* of z. The following is the graphical representation of parabolic numbers with modulus 1:

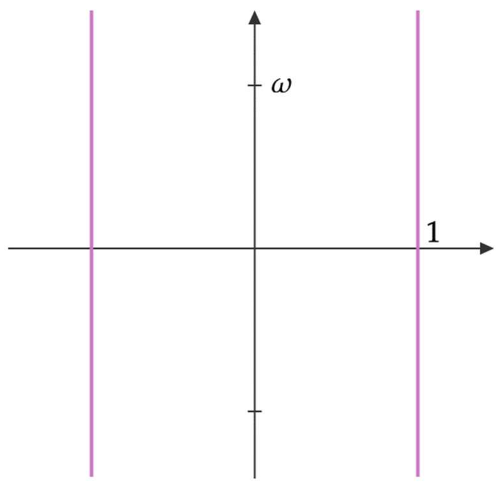


Figure 1.12: Parabolic Numbers with Modulus 1

Not all parabolic numbers lie on these two vertical lines, just the ones with modulus 1. Other parabolic numbers will have different moduli, and thus lie on different vertical lines. These vertical lines can be justified when investigating which parabolic numbers possess modulus 1:

$$|z| = \sqrt{z\bar{z}} = 1$$

$$\sqrt{z\bar{z}} = \sqrt{(a + b\omega)(a - b\omega)}$$

$$= \sqrt{a^2 - ab\omega + ab\omega + b^2(0)}$$

$$= \sqrt{a^2} = 1 \Rightarrow a = \pm 1$$

The result shows that parabolic numbers $a + b\omega$, where $a = 1$ or $a = -1$, possess modulus 1, hence the two vertical lines at 1 and -1. Note, again, that "distance" here is quite different:

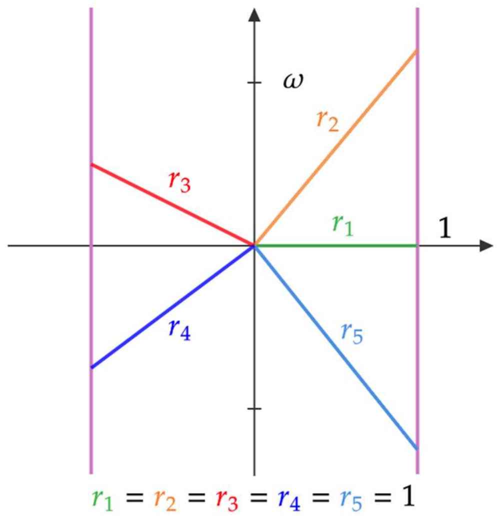


Figure 1.13: Points on the Unit Parabolic Shape Possess the Same Modulus Value

If we were to measure the Euclidean length of each of these distances with a ruler, we would see that they are different. In the parabolic numbers, however, the "lengths" are all equal – another bizarre result.

Now knowing the basics of hyperbolic and parabolic numbers, what would forests consisting of hyperbolic and parabolic polynomial trees look like?

1.2.6 Miscellaneous Mathematical Knowledge

1.2.6.1 Even and Odd Integers

Since this paper will be covering powers of numbers from number systems different from the real line, knowing what the form of a power looks like will streamline one's understanding of the concepts. Powers will be positive integers, and there are two types of integers: even and odd. Even integers have the form

$$2k,$$

where k is an integer. Examples:

$$k = 1 \Rightarrow 2(1) = 2$$

$$k = -2 \Rightarrow 2(-2) = -4$$

$$k = 5 \Rightarrow 2(5) = 10$$

$$k = -90 \Rightarrow 2(-90) = -180$$

These examples are even since they are divisible by 2 or are multiples of 2. 0 is also considered an even integer. Odd integers have the form

$$2k + 1$$

where k is an integer. Examples:

$$k = 1 \Rightarrow 2(1) + 1 = 3$$

$$k = -2 \Rightarrow 2(-2) + 1 = -3$$

$$k = 5 \Rightarrow 2(5) + 1 = 11$$

$$k = -90 \Rightarrow 2(-90) + 1 = -179$$

These examples are odd since they are not divisible by 2 or are not multiples of 2. Powers in this paper will be of either of these forms, and all powers will be non-negative.

1.2.6.2 The Quadratic Formula

For a polynomial equation of the form

$$ax^2 + bx + c = 0,$$

the solutions of polynomial equation will be

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This works for polynomial equations where the variable possesses a higher power. For example, let us say that $x = h^8$, where h is a variable. The above polynomial equation can be expressed as $ah^{16} + bh^8 + c = 0$, and solving for h, we have

$$h^8 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow h = \pm \sqrt[8]{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

This will be useful to know later when discussing square roots of hyperbolic numbers.

1.2.6.3 The Binomial Theorem

There is one more concept to introduce before we delve into the rest of the paper. Because we will be dealing with powers of numbers with the form $a + bt$, where t is either τ or ω, it would be enlightening to see how this form is expanded out when it is multiplied n times. The binomial theorem formula will help this become a lot easier, describing how one can expand expressions of the form $(x + y)^n$:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1y^{n-1} + \binom{n}{n}y^n$$

$$= x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1y^{n-1} + y^n$$

Every $\binom{n}{j} = \frac{n!}{j!(n-j)!}$ for $0 \leq j \leq n$. For this paper, all one really needs to know is that each $\binom{n}{j}$ is a real coefficient. One can think of x being a and y being bt, thus we will be seeing what $(a + bt)^n$ looks like.

CHAPTER 2

HYPERBOLIC ROOTS

2.1 Simple Hyperbolic Polynomial Equations

2.1.1 A Simple Hyperbolic Polynomial of Even Degree

We can begin with the following example:

$$x^2 - 1 = 0$$

Recall that $\tau^2 = 1$ in the hyperbolic number system. It follows that $(-\tau)^2 = 1$. Because the hyperbolic set also includes the real numbers, that means that the equation above has at least four solutions: 1, -1, τ, and -τ. The polynomial has more roots than its degree! We have, right off the bat, seen an example of a polynomial that contradicts what we are used to with real and complex numbers. We can also see the following:

$$\begin{aligned} x^2 - 1 &= (x-1)(x+1) \\ &= (x-\tau)(x+\tau) \end{aligned}$$

Factorization is not unique! Another contradiction of what we are used to. One may ask next how many square roots of 1 exist. In other words, if we let $(a + b\tau)^2 = 1$, how many solutions are there? We can calculate the following:

$$\begin{aligned} (a+b\tau)^2 &= (a+b\tau)(a+b\tau) \\ &= a^2 + ab\tau + ab\tau + b^2\tau^2 \\ &= a^2 + 2ab\tau + b^2 \\ &= a^2 + b^2 + 2ab\tau \end{aligned}$$

Should we set this new form to $a^2 + b^2 + 2ab\tau = 1$, we can immediately see that the term $2ab\tau$ must equal 0, since 1 is a real number. We also see that $a = b = 0$ is not a valid option, since if we allowed both a and b to be 0, then $a^2 + b^2 + 2ab\tau = 0$. So only exactly one of them must be 0. Let us say it is b. That leaves us with $a^2 = 1$. Because a is a real number, the only two options we have are $a = 1$ and $a = -1$. Likewise, if we set $a = 0$, then we have $b^2 = 1$. Again, because b is a real number, the only two options we have are $b = 1$ and $b = -1$. That leaves us with the following combinations:

$$a = 1, b = 0 \Rightarrow a + b\tau = 1$$

$$a = -1, b = 0 \Rightarrow a + b\tau = -1$$

$$a = 0, b = 1 \Rightarrow a + b\tau = \tau$$

$$a = 0, b = -1 \Rightarrow a + b\tau = -\tau$$

Thus, there are *exactly* four solutions to the equation $(a + b\tau)^2 = 1$: 1, -1, τ, and -τ. In other words, there are exactly four square roots of 1. The following figure demonstrates this:

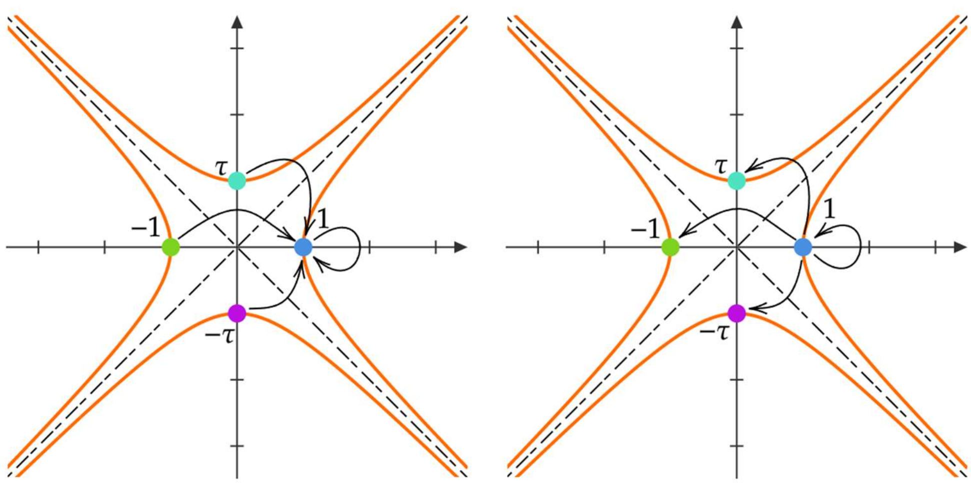


Figure 2.1: Squares Equal to 1 and Square Roots of 1

2.1.2 Square Roots of a Hyperbolic Number

Can we see that there are exactly four square roots of any hyperbolic number? To answer this, let us begin by considering the squaring function. Taking any hyperbolic number $a + b\tau$, its square is $(a + b\tau)^2 = a^2 + b^2 + 2ab\tau = p + q\tau$. Then we have that $a^2 + b^2 = p$ and $2ab = q$. Observe that $(a + b\tau)^2 = a^2 + b^2 + 2ab\tau \geq 0$, since the square of a hyperbolic number must be positive. Applying some simple algebra, we see that

$$a^2 + b^2 + 2ab\tau \geq 0$$

$$\Rightarrow a^2 + b^2 \geq -2ab\tau$$

$$\Rightarrow p \geq -q$$

But we can also take $(a - b\tau)^2 = a^2 + b^2 - 2ab\tau$ and yield a similar result:

$$a^2 + b^2 - 2ab\tau \geq 0$$

$$\Rightarrow a^2 + b^2 \geq 2ab\tau$$

$$\Rightarrow p \geq q$$

So $p \geq |q|$, since p is greater than or equal to positive and negative q. Hyperbolic numbers $p + q\tau$ that possess the trait $p \geq |q|$ are in the closed right quadrant, which includes hyperbolic numbers that lie on the lines $y = x$ and $y = -x$ for $x \geq 0$.

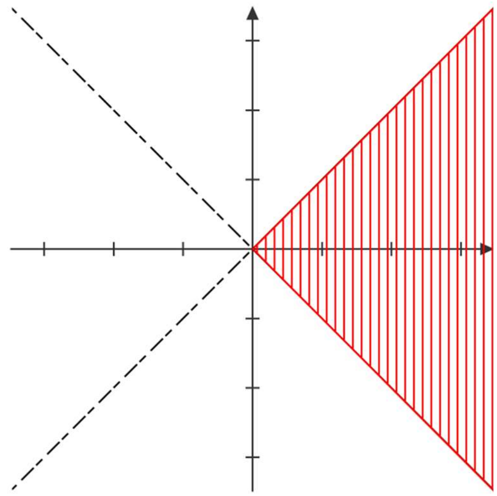


Figure 2.2: Where Squares of Hyperbolic Numbers Lie

For now, let us take a number $p + q\tau$ in the open right quadrant, where $p > |q|$, which excludes numbers lying on the lines $y = x$ and $y = -x$. Suppose $a + b\tau$ is a square root of $p + q\tau$. Then from our calculation above, $a^2 + b^2 = p$ and $2ab\tau = q$. Assume that $q = 0$. Then we have the following combinations:

$$a = \pm\sqrt{p}, b = 0$$

$$b = \pm\sqrt{p}, a = 0$$

Note that since $q = 0$ and $p > |q| = 0$, p is not zero. In this case, there are exactly 4 square roots of the hyperbolic number p where $p > 0$. They are

$$a = \sqrt{p}, b = 0 \Rightarrow a + b\tau = \sqrt{p}$$

$$a = -\sqrt{p}, b = 0 \Rightarrow a + b\tau = -\sqrt{p}$$

$$a = 0, b = \sqrt{p} \Rightarrow a + b\tau = \sqrt{p}\tau$$

$$a = 0, b = -\sqrt{p} \Rightarrow a + b\tau = -\sqrt{p}\tau$$

Now, assume that $q \neq 0$. Since $2ab = q$, we can say that $b = q/2a$. Plugging back into $a^2 + b^2 = p$, we have:

$$a^2 + b^2 = a^2 + \left(\frac{q}{2a}\right)^2 = a^2 + \frac{q^2}{4a^2} = p \Rightarrow a^2 + \frac{q^2}{4a^2} - p = 0$$

Multiplying both sides by a^2, just so we can cancel out the factor a^2 in the denominator of the $q^2/4a^2$ term, we get:

$$a^2a^2 + \frac{q^2}{4a^2}a^2 - pa^2 = 0(a^2)$$

$$a^4 + \frac{q^2}{4} - pa^2 = 0$$

$$a^4 - pa^2 + \frac{q^2}{4} = 0$$

This form enables usage of the quadratic formula to find a^2:

$$a^2 = \frac{p \pm \sqrt{(-p)^2 - 4(1)\left(\frac{q^2}{4}\right)}}{2(1)}$$

$$= \frac{p \pm \sqrt{p^2 - q^2}}{2}$$

The above result shows that a^2 has only two viable solutions, if $p^2 - q^2 > 0$. Solving further for a, we get

$$a = \pm\sqrt{\frac{p \pm \sqrt{p^2 - q^2}}{2}}$$

As can be seen, there are two possible numbers within the large radical, both of which are positive and real since $p^2 - q^2 > 0$. Therefore, there are four different a's. Recall that $b = q/2a$. It follows that there is a single b assigned to each a:

$$a = \sqrt{\frac{p + \sqrt{p^2 - q^2}}{2}}, b = \frac{q}{2\sqrt{\frac{p + \sqrt{p^2 - q^2}}{2}}}$$

$$a = \sqrt{\frac{p - \sqrt{p^2 - q^2}}{2}}, b = \frac{q}{2\sqrt{\frac{p - \sqrt{p^2 - q^2}}{2}}}$$

$$a = -\sqrt{\frac{p + \sqrt{p^2 - q^2}}{2}}, b = \frac{q}{-2\sqrt{\frac{p + \sqrt{p^2 - q^2}}{2}}}$$

$$a = -\sqrt{\frac{p - \sqrt{p^2 - q^2}}{2}}, b = \frac{q}{-2\sqrt{\frac{p - \sqrt{p^2 - q^2}}{2}}}$$

Therefore, there are exactly four square roots of a hyperbolic number $p + qt$.

2.1.3 Attempting to Solve for Even Index Roots Algebraically

Is it possible that there are only four roots of higher even power polynomials of the form $x^{2k} - z$, where z is a hyperbolic number in the open right quadrant? For example, if we found the roots of $x^8 - (9 + 8\tau)$, would there be exactly four roots? For even indices greater than 2, could we find roots of hyperbolic numbers outside of the closed right quadrant? If we are to assume that for parabolic number $p + q\tau$, there exists a $2n$-index root such that we can express that root like

$$(a + b\tau)^{2k} = p + q\tau,$$

then for what a and b would the expression be true? So far, we have attempted to solve these problems algebraically, so let us attempt to continue this trend for the general even case.

Let $c_j = \binom{n}{j}$ for $0 \leq j \leq n$ (keep this notation for the rest of this paper) and let $n = 2k$ for some positive integer k. If we utilize the binomial theorem, we can see $(a + b\tau)^{2n}$ expanded out is

$$\begin{aligned}(a + b\tau)^{2n} &= c_0 a^{2n} + c_1 a^{2n-1}(b\tau)^1 + c_2 a^{2n-2}(b\tau)^2 + \cdots + c_{2n-1}a^1(b\tau)^{2n-1} \\ &\quad + c_{2n}(b\tau)^{2n} \\ &= c_0 a^{2n} + c_2 a^{2n-2} b + \cdots + c_{2n} b^{2n} \\ &\quad + \tau(c_1 a^{2n-1} b^1 + c_3 c_2 a^{2n-3} b^3 + \cdots + c_{2n-1} a^1 (b\tau)^{2n-1}) = p + q\tau\end{aligned}$$

This shows that $c_0 a^{2n} + c_2 a^{2n-2} b + \cdots + c_{2n} b^{2n} = p$ and $c_1 a^{2n-1} b^1 + c_3 c_2 a^{2n-3} b^3 + \cdots + c_{2n-1} a^1 (b\tau)^{2n-1} = q$, which looks very complicated. For $k = 1$, however, we get $a^2 + b^2 + 2ab\tau = p + q\tau$, so $a^2 + b^2 = p$ and $2ab = q$, which is much simpler. This simplicity does not extend with higher multiples of 2. Take $(a + b\tau)^4 = p + q\tau$ for instance (coefficients have already been given)

$$\begin{aligned}(a + b\tau)^4 &= (a + b\tau)(a + b\tau)(a + b\tau)(a + b\tau) \\ &= a^4 + 4a^3 b\tau + 6a^2(b\tau)^2 + 4a(b\tau)^3 + (b\tau)^4 \\ &= a^4 + 6a^2 b^2 + b^4 + \tau(4a^3 b + 4ab^3) = p + q\tau\end{aligned}$$

This calculation shows that $a^4 + 6a^2 b^2 + b^4 = p$ and $4a^3 b + 4ab^3 = q$. How could one easily go about solving this? If we expand $(a + b\tau)^6$, we get

$$\begin{aligned}(a + b\tau)^6 &= (a + b\tau)(a + b\tau)(a + b\tau)(a + b\tau)(a + b\tau)(a + b\tau) \\ &= a^6 + 6a^5 b\tau + 15a^4(b\tau)^2 + 20a^3(b\tau)^3 + 15a^2(b\tau)^4 + 6a(b\tau)^5 + (b\tau)^6 \\ &= a^6 + 15a^4 b^2 + 15a^2 b^4 + b^6 + \tau(6a^5 b + 20a^3 b^3 + 6ab^5) = p + q\tau\end{aligned}$$

This new calculation shows that $a^6 + 15a^4 b^2 + 15a^2 b^4 + b^6 = p$ and $6a^5 b + 20a^3 b^3 + 6ab^5 = q$. This is even *more* arduous to solve algebraically. Is there an alternative non-algebraic method that obviates this issue? For now, let us move on to a simple hyperbolic polynomial of odd degree. As we work in the next section, perhaps a solution will manifest itself.

2.1.4 A Simple Hyperbolic Polynomial of Odd Degree

What about this polynomial equation:

$$x^3 - 1 = 0$$

We can use the same process when solving for $x^2 - 1 = 0$. Let us take $(a + b\tau)^3$ and multiply it out:

$$\begin{aligned}
(a + b\tau)^3 &= (a + b\tau)^2(a + b\tau) \\
&= (a^2 + b^2 + 2ab\tau)(a + b\tau) \\
&= a^3 + ab^2 + 2a^2b\tau + a^2b\tau + b^3\tau + 2ab^2\tau^2 \\
&= a^3 + ab^2 + 2a^2b\tau + a^2b\tau + b^3\tau + 2ab^2 \\
&= a^3 + 3ab^2 + 3a^2b\tau + b^3\tau \\
&= a^3 + 3ab^2 + \tau(3a^2b + b^3)
\end{aligned}$$

We attempt to set $a^3 + 3ab^2 + \tau(3a^2b + b^3) = 1$, assuming there are solutions. Since 1 is real, we must get rid of the τ term. To get rid of the τ term, we must have some a, b combination to make it become 0. After some deliberation, we can see that it is not possible for $3a^2b + b^3 = 0$ for any real numbers a and b unless $b = 0$. That leaves $a^3 = 1$, which means $a = 1$. Only one combination is available:

$$a = 1, b = 0 \Longrightarrow (a + b\tau) = 1,$$

and therefore, there is only one root in the hyperbolic numbers: 1. Indeed, you can see this when we attempt to factor $x^3 - 1$ across the hyperbolic numbers:

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

$x^2 + x + 1$ cannot be factored further into linear factors across the hyperbolic numbers. Further factorization is possible across complex numbers, but hyperbolic numbers do not have this luxury. To prove this is true, assume that $x = a + b\tau$ for some a, b. Then we have that

$$\begin{aligned}(x-1)(x^2+x+1) &= \big((a+b\tau)-1\big)((a+b\tau)^2+(a+b\tau)+1) \\ &= (a+b\tau)^3+(a+b\tau)^2+(a+b\tau)-(a+b\tau)^2-(a+b\tau)-1 \\ &= (a+b\tau)^3-1 \\ &= x^3-1\end{aligned}$$

Therefore, the LHS and RHS are equal for any hyperbolic number, which means that the factorization of $x^3 - 1$ is correct. Suppose that the hyperbolic number z is a root of $x^2 + x + 1$. Then we have that

$$\begin{aligned}z^3-1 &= (z-1)(z^2+z+1) \\ &= (z-1)(0) \\ &= 0 \Rightarrow z^3-1=0\end{aligned}$$

Note, however, that we have already shown that the only hyperbolic number that satisfies $z^3 - 1 = 0$ is 1. This number is not a root of $x^2 + x + 1$, since if $x = 1$, we have

$$x^2 + x + 1 = 1^2 + 1 + 1 = 3 \neq 0$$

Therefore, we have obtained a contradiction – our assumption is wrong. There are no hyperbolic roots of $x^2 + x + 1$. Therefore, it is not further factorable across the hyperbolic numbers.

The factor $x - 1$ occurs only once, so we have a root of multiplicity 1. Compare this to the degree of the polynomial – we have fewer roots, counting multiplicity, than the degree of the polynomial.

If we were to find the cubic roots of τ, -1, and $-\tau$, the same process shown above would be utilized. The only cubic roots for these values, respectively, are τ, -1, and $-\tau$.

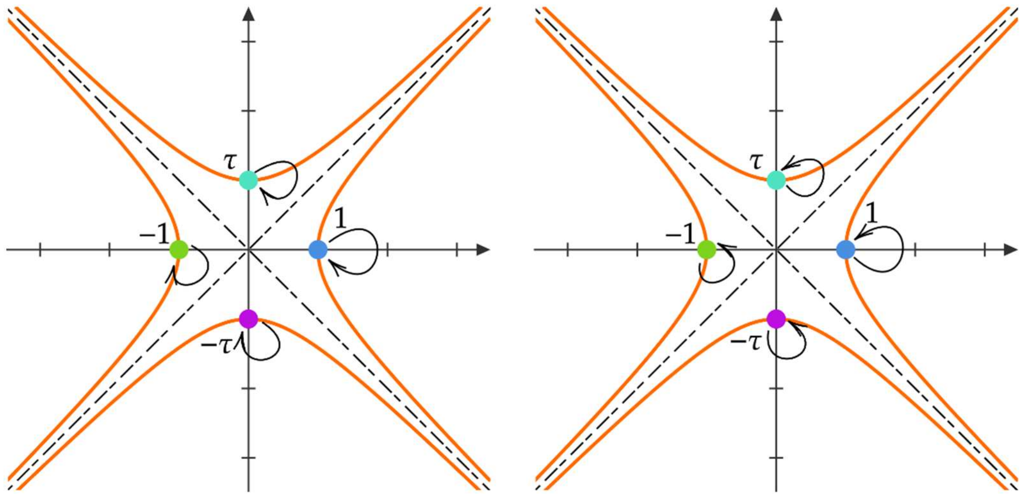


Figure 2.3: Cubes and Cubic Roots of 1, τ, -1, and τ

Figure 2.3 shows a different image of where these numbers “travel” as they are cubed and as their cubic roots are calculated.

2.1.5 Attempting to Solve for Odd Index Roots Algebraically

Finding results for the general odd power case is an algebraic mess. Should we start with a hyperbolic number $p + q\tau$ and assume there exists a, b such that

$$(a + b\tau)^{2k-1} = p + q\tau,$$

then we will produce complicated algebras like those in the even power section. Even attempting to work with $(a + b\tau)^3$ is problematic (coefficients have already been given):

$$\begin{aligned}(a + b\tau)^3 &= (a + b\tau)(a + b\tau)(a + b\tau)\\ &= a^3 + 3a^2b\tau + 3a(b\tau)^2 + (b\tau)^3\\ &= a^3 + 3ab^2 + \tau(3a^2b + b^3) = p + q\tau\end{aligned}$$

We have that $a^3 + 3ab^2 = p$ and $3a^2b + b^3 = q$. This complication does not transfer if we knew that $p = 0$ or $q = 0$, since we would have that $q = b^3$ and $p = a^3$, respectively, and thus $b = \sqrt[3]{q}$ and $a = \sqrt[3]{p}$ for those respective cases. Since p and q are real, and there is only one real cubic root for each real number, there exists only one a and one b such that $a^3 = p$ and $b^3 = q$. This shows that for 1, -1, τ, and $-\tau$, the cubic roots are 1, -1, τ, and $-\tau$, respectively. It can easily be shown that this result extends to odd higher-index roots using a similar process. However, if p and q are nonzero, finding solutions is not so simple. We have not found an algebraic method in this section that could have helped with generalizing the even case. Thus, a different method is needed.

2.2 The Simplicity of the Exponential Form and Generalizing Results for Hyperbolic Numbers in the Four Open Quadrants

Recall that a hyperbolic number $a + b\tau$, where $a \neq \pm b$, can be expressed exponentially. Let us quickly work with hyperbolic numbers that cannot be expressed exponentially, or hyperbolic numbers that are on the lines $y = x$ and $y = -x$.

Claim: For $a + a\tau$ on the line $y = x$, for any $n \in \mathbb{N}$,

$$(a + a\tau)^n = (2a)^{n-1}(a + a\tau).$$

Proof: The proof will be by induction. Suppose that $n = 1$. Then

$$\begin{aligned}(a + a\tau)^1 &= 1(a + a\tau)\\ &= (2a)^{1-1}(a + \tau)\end{aligned}$$

Now, suppose that for positive integer k, the relationship is true. Let us investigate $k + 1$:

$$\begin{aligned}(a + a\tau)^{k+1} &= (a + a\tau)^k(a + a\tau)\\ &= (2a)^{k-1}(a + a\tau)(a + a\tau)\\ &= (2a)^{k-1}(a + a\tau)^2\\ &= (2a)^{k-1}(2a)(a + a\tau)\\ &= (2a)^k(a + a\tau)\end{aligned}$$

$$= (2a)^{(k+1)-1}(a + a\tau)$$

Therefore, for $a + a\tau$ on the line $y = x$, for any $n \in \mathbb{N}$, $(a + a\tau)^n = (2a)^{n-1}(a + a\tau)$. A similar proof can be done for the line $y = -x$ with numbers of the form $a - a\tau$.

As the proof demonstrates, hyperbolic numbers on the lines $y = x$ and $y = -x$, when taken to an arbitrary power, stay on those lines. If we were to attempt to find all roots of a number not on the lines $y = x$ and $y = -x$, i.e., from the four open quadrants, no roots would be from those lines. We can freely work with exponential numbers in the four open quadrants without worrying about possible extraneous roots. Let us begin by working with the even power case.

<u>Claim</u>: For $n = 2k$, even, where $k \geq 0$, $f(z) = z^{2k}$ takes each open quadrant bijectively onto the right open quadrant.

<u>Proof</u>: It is important to demonstrate that numbers from the four open quadrants have the property that their even powers are in the right open quadrant. Suppose we have

$$z_1 = re^{\tau\theta}$$

$$z_2 = \tau re^{\tau\theta}$$

$$z_3 = -re^{\tau\theta}$$

$$z_4 = -\tau re^{\tau\theta}$$

Then when we input these values into the function $f(z) = z^{2k}$, we get

$$f(z_1) = \left(re^{\tau\theta}\right)^{2k} = r^{2k}e^{\tau 2k\theta}$$

$$f(z_2) = \left(\tau re^{\tau\theta}\right)^{2k} = r^{2k}e^{\tau 2k\theta}$$

$$f(z_3) = \left(-re^{\tau\theta}\right)^{2k} = r^{2k}e^{\tau 2k\theta}$$

$$f(z_4) = \left(-\tau re^{\tau\theta}\right)^{2k} = r^{2k}e^{\tau 2k\theta}$$

One can see that all outputs are the same and that the form of these powers is that of a hyperbolic number in the open right quadrant. Therefore, the codomain of the function $f(z) = z^{2k}$ is the open right quadrant.

We shall show the claim is true for hyperbolic numbers in the top quadrant. To show bijectivity of a function, it must be demonstrated that the function is one-to-one (injective) and onto (surjective). Let us first show the one-to-one case. Assume that for two hyperbolic numbers $\tau r_1 e^{\tau\theta_1}$ and $\tau r_2 e^{\tau\theta_2}$ we have that $\left(\tau r_1 e^{\tau\theta_1}\right)^{2k} = \left(\tau r_2 e^{\tau\theta_2}\right)^{2k}$. Then

$$\left(\tau r_1 e^{\tau\theta_1}\right)^{2k} = \left(\tau r_2 e^{\tau\theta_2}\right)^{2k}$$

$$\tau^{2k} r_1^{2k} e^{\tau 2k\theta_1} = \tau^{2k} r_2^{2k} e^{\tau 2k\theta_2}$$

$$r_1^{2k} e^{\tau 2k\theta_1} = r_2^{2k} e^{\tau 2k\theta_2}$$

Since both r_1 and r_2 are positive real numbers, it must be true that if $r_1^{2k} = r_2^{2k}$, then $r_1 = r_2$. Similarly, in the hyperbolic numbers, there are no coterminal angles. Therefore, since $e^{\tau 2k\theta_1} = e^{\tau 2k\theta_2}$, we have that $e^{\tau\theta_1} = e^{\tau\theta_2}$. Both hyperbolic numbers are the same number – each output has only one unique input. A similar process can be done for the other open quadrants to yield the same result.

Now, let us show the onto case. Assume we have the hyperbolic number $re^{\tau\theta}$ from the open right quadrant. Take $\tau r^{\frac{1}{2k}} e^{\tau\frac{\theta}{2k}}$ from the top quadrant. Taking this hyperbolic number to the $2k$'th power, we have

$$\left(\tau r^{\frac{1}{2k}} e^{\tau\frac{\theta}{2k}}\right)^{2k} = re^{\tau\theta}$$

Therefore, each hyperbolic number in the open right quadrant has a hyperbolic number from the top quadrant that maps to it. The mapping is onto. A similar proof can be done for the other open quadrants to get the same result. Therefore, for $n = 2k$, even, where $k \geq 0$, $f(z) = z^{2k}$ takes each open quadrant bijectively onto the right open quadrant.

The proof above shows that even index powers only exist in the open right quadrant. Recall from the proof that each open quadrant for the function $f(z) = z^{2k}$ has a bijective mapping to the open right quadrant. If we were to find the $2k$ index root of a number in the right open quadrant, we would see that there are exactly four roots, each from one of the four open quadrants. What about the odd power case?

<u>Claim</u>: For $n = 2k + 1$, odd, where $k \geq 0$, $f(z) = z^{2k+1}$ takes each open quadrant bijectively onto itself.

<u>Proof</u>: It is important to demonstrate that numbers from the four open quadrants have the property that their odd powers are in the same quadrant Suppose we have

$$z_1 = re^{\tau\theta}$$

$$z_2 = \tau re^{\tau\theta}$$

$$z_3 = -re^{\tau\theta}$$

$$z_4 = -\tau re^{\tau\theta}$$

Then when we input these values into the function $f(z) = z^{2k+1}$, we get

$$f(z_1) = \left(re^{\tau\theta}\right)^{2k+1} = r^{2k+1}e^{\tau(2k+1)\theta}$$

$$f(z_2) = \left(\tau re^{\tau\theta}\right)^{2k+1} = \tau r^{2k+1}e^{\tau(2k+1)\theta}$$

$$f(z_3) = \left(-re^{\tau\theta}\right)^{2k+1} = -r^{2k+1}e^{\tau(2k+1)\theta}$$

$$f(z_4) = \left(-\tau re^{\tau\theta}\right)^{2k+1} = -\tau r^{2k+1}e^{\tau(2k+1)\theta}$$

The odd powered number of each number above is in the same quadrant as the original number. Therefore, the codomain of each open quadrant for the function $f(z) = z^{2k+1}$ is the same open quadrant.

We shall show the claim is true for hyperbolic numbers in the top quadrant. Let us show the one-to-one case. Assume that for two hyperbolic numbers $\tau r_1 e^{\tau\theta_1}$ and $\tau r_2 e^{\tau\theta_2}$ we have that $\left(\tau r_1 e^{\tau\theta_1}\right)^{2k+1} = \left(\tau r_2 e^{\tau\theta_2}\right)^{2k+1}$. Then

$$\left(\tau r_1 e^{\tau\theta_1}\right)^{2k+1} = \left(\tau r_2 e^{\tau\theta_2}\right)^{2k+1}$$

$$\tau^{2k+1} r_1^{2k+1} e^{\tau(2k+1)\theta_1} = \tau^{2k+1} r_2^{2k+1} e^{\tau(2k+1)\theta_2}$$

$$\tau r_1^{2k+1} e^{\tau(2k+1)\theta_1} = \tau r_2^{2k+1} e^{\tau(2k+1)\theta_2}$$

Like the even power case, since both r_1 and r_2 are positive real numbers, it must be true that if $r_1^{2k+1} = r_2^{2k+1}$, then $r_1 = r_2$. In the hyperbolic numbers, there are no coterminal angles. Therefore, since $e^{\tau(2k+1)\theta_1} = e^{\tau(2k+1)\theta_2}$, we have that $e^{\tau\theta_1} = e^{\tau\theta_2}$. Both hyperbolic numbers are the same number, and thus, the function is injective for the top quadrant. A similar process can be done for the other open quadrants to yield the same result.

Next, let us show the onto case. Assume we have the hyperbolic number $\tau r e^{\tau\theta}$ from the open top quadrant. Take $\tau r^{\frac{1}{2k+1}} e^{\tau\frac{\theta}{2k+1}}$ from the open top quadrant. Taking this hyperbolic number to the $2k+1$ power, we have

$$\left(\tau r^{\frac{1}{2k+1}} e^{\tau\frac{\theta}{2k+1}}\right)^{2k+1} = \tau r e^{\tau\theta}$$

Therefore, each hyperbolic number in the open top quadrant has a hyperbolic number from the top quadrant that maps to it. The mapping is onto for the open top quadrant. For the other open quadrants, a similar proof can be performed to produce the same result.

Therefore, for $n = 2k + 1$, even, where $k \geq 0$, $f(z) = z^{2k+1}$ takes each open quadrant bijectively onto itself.

The proof shows that odd index powers exist in all four open quadrants. Recall from the proof that each open quadrant for the function $f(z) = z^{2k+1}$ has a bijective mapping to itself. If we were to find the $2k + 1$ index root of a hyperbolic number from one of the open quadrants, there would be exactly one root that is from the same quadrant.

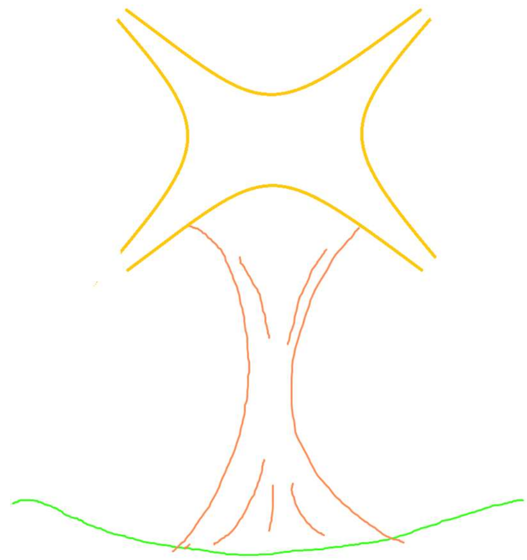
We now know that hyperbolic numbers in the open right quadrant possess exactly four even index roots. Resulting from this, polynomials with a factor of the form $x^n - a$, where n is an even, positive integer and a is a hyperbolic number from the open right quadrant, will have a root count either exceeding, meeting, or falling short of the degree, counting multiplicity. Here are some examples:

$x^3 - x = x(x^2 - 1)$; degree 3, but 5 total roots

$x^5 - x = x(x^4 - 1)$; degree 5 with 5 total roots

$x^7 - x = x(x^6 - 1)$; degree 7, but 5 total roots

After going through a copious amount of mathematics, one may wonder, if we are to continue with this tree analogy, what hyperbolic polynomial trees may look like. Suppose we tripped down another rabbit hole conveniently located in the forest of complex numbers, landing (safely) in the middle of an alien-looking forest. We may immediately see a tree that looks like this:



$x^2 - 92 = 0$

Figure 2.4: Curvy Hyperbolic Tree of $x^2 - 92 = 0$

Of course, our experience would deceive us, and we may think that there are only two roots. However, we spot what seems to be four roots peeking out at the base of the tree. That is impossible, we may think. A spade is not enough to verify this, we need something more resolute. Luckily, we see a conveniently formed cave hole in the ground nearby, head in, and stop right below the tree we saw above. Clear as day, we witness four roots:

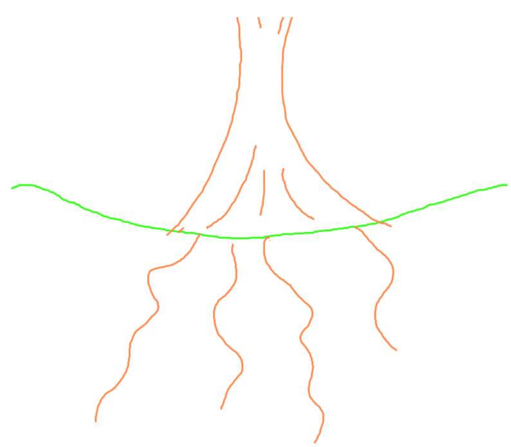


Figure 2.5: Roots of Hyperbolic Tree $x^2 - 92 = 0$

Completely unlike anything we are used to in common mathematics. This tree “should not” have more roots than what we expect from its form, but indeed, it does.

CHAPTER 3

PARABOLIC ROOTS

3.1 Simple Parabolic Polynomial Equations

3.1.1 A Simple Parabolic Polynomial of Even Degree

We can begin again with the following example:

$$x^2 - 1 = 0$$

Immediately, we can see that 1 and -1 are solutions to the equation. Unlike the hyperbolic numbers, we do not seem to have any other number whose square is 1. To be safe, let us investigate what happens when we square an arbitrary parabolic number, just to see how a solution to the equation might look:

$$\begin{aligned}(a + b\omega)^2 &= (a + b\omega)(a + b\omega)\\ &= a^2 + ab\omega + ab\omega + b^2\omega^2\\ &= a^2 + ab\omega + ab\omega + 0\\ &= a^2 + 2ab\omega\end{aligned}$$

Since we are trying to find squares that equal 1 to solve the equation, it must be that either a or b is 0 to get rid of the $2ab\omega$ term. But a cannot be zero, since a^2 is the only real term. Hence $b = 0$ and a is 1 or -1. We have confirmed that there are exactly two solutions to this equation, matching the results of the real and complex numbers.

3.1.2 A Simple Parabolic Polynomial of Odd Degree

Just like with hyperbolic numbers, let us again look at the following polynomial equation:

$$x^3 - 1 = 0$$

Multiplying out $(a + b\omega)^3$, we get

$$\begin{aligned}(a + b\omega)^3 &= (a + b\omega)^2(a + b\omega) \\ &= (a^2 + 2ab\omega)(a + b\omega) \\ &= a^3 + a^2b\omega + 2a^2b\omega + 2ab^2\omega^2 \\ &= a^3 + a^2b\omega + 2a^2b\omega + 0 \\ &= a^3 + 3a^2b\omega\end{aligned}$$

After setting this equal to 0, we can easily see that $b = 0$ and that $a = 1$, and just like the hyperbolic numbers, we have a single root (being 1) of multiplicity 1. So far, the results for the parabolic numbers seem to follow suit to the real numbers. However, if we pay attention to the forms of a parabolic number when it is squared and cubed, a pattern seems to emerge – the following conjecture may be true: $(a + b\omega)^n = a^n + na^{n-1}b\omega$.

3.1.3 The General Form of a Powered Parabolic Number

I will use proof by induction to show that for all positive integers n:

$$(a + b\omega)^n = a^n + na^{n-1}b\omega$$

Let $n = 1$. Then:

$$\begin{aligned}(a + b\omega)^1 &= a + b\omega \\ &= a^1 + 1a^{1-1}b\omega\end{aligned}$$

Assume that the proposition works for some positive integer k and investigate $k + 1$:

$$\begin{aligned}(a + b\omega)^{k+1} &= (a + b\omega)^k(a + b\omega)^1 \\ &= (a^k + ka^{k-1}b\omega)(a + b\omega) \\ &= a^{k+1} + a^kb\omega + ka^kb\omega + ka^{k-1}b^2\omega^2 \\ &= a^{k+1} + a^kb\omega + ka^kb\omega + 0\end{aligned}$$

$$= a^{k+1} + (k+1)a^k b\omega$$

$$= a^{k+1} + (k+1)a^{k+1} b\omega$$

Therefore, $(a + b\omega)^n = a^n + na^{n-1}b\omega$ for every positive integer n. Through the lens of the binomial theorem, we can see that all terms with a power of ω greater than 1 would be zero. This would give the same result. To demonstrate this, assume that $n \geq 2$. Then

$$\begin{aligned}
(a+\omega)^n &= c_0 a^n + c_1 a^{n-1}(b\omega)^1 + c_2 a^{n-2}(b\omega)^2 + \cdots + c_{n-1}a^1(b\omega)^{n-1} + c_n(b\omega)^n \\
&= c_0 a^n + c_1 a^{n-1}b^1\omega^1 + c_2 a^{n-2}b^2\omega^2 + \cdots + c_{n-1}a^1 b^{n-1}\omega^{n-1} + c_n b^n \omega^n \\
&= c_0 a^n + c_1 a^{n-1}b^1\omega^1 + c_2 a^{n-2}b^2(0) + \cdots + c_{n-1}a^1 b^{n-1}(0) + c_n b^n(0) \\
&= c_0 a^n + c_1 a^{n-1}b\omega \\
&= \frac{n!}{0!\,n!}a^n + \frac{n!}{1!\,(n-1)!}a^n \\
&= a^n + na^{n-1}b\omega
\end{aligned}$$

We can use this result to our advantage when determining how many solutions there are to the general equation $x^n = 0$ in the set of parabolic numbers. This equation can be alternatively seen as attempting to find a, b such that

$$a^n + na^{n-1}b\omega = 0$$

Since a is a real number, it must be true that $a = 0$, since it is the only real term. For $n \geq 2$, this causes the term $na^{n-1}b\omega$ to be 0. Note, however, that b need not be any specific real number. Since b is a real number, and there are uncountably infinitely many real numbers, there are uncountably infinitely many roots of 0! It follows that any polynomial equation of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + \omega x = 0,$$

where $a_1 = \omega$ and $a_0 = 0$ from the general polynomial form, also has uncountably infinitely many solutions. Let us look at an example:

$$8x^4 + 3x^3 + 9x^2 + \omega x = 0$$

A polynomial of degree 4 over the complex numbers would possess exactly 4 roots. Over the parabolic numbers, we can see that $0 + 8\omega$, $0 + \pi\omega$, $0 + \frac{1}{99999999}\omega$, and any number of the form $0 + b\omega$ are valid solutions to the above polynomial equation. Let us look at another example:

$$65937x^{100} - 9404x^{86} + \sqrt[3]{3}x^{53} - x^{37} + 90x^{30} - 1.3x^{11} + \omega x = 0$$

Like the previous example, $0 + 8\omega$, $0 + \pi\omega$, $0 + \frac{1}{99999999}\omega$, and any number of the form $0 + b\omega$ are included in the set of solutions of this parabolic polynomial equation. This is an amazing result – it is unlike anything that can be produced in the real and complex number sets.

The following illustrates where these solutions lie:

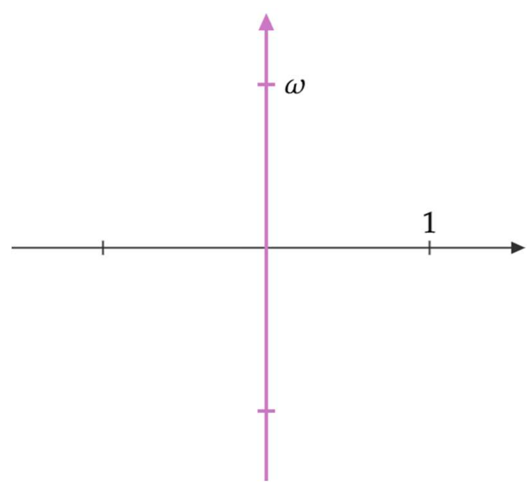
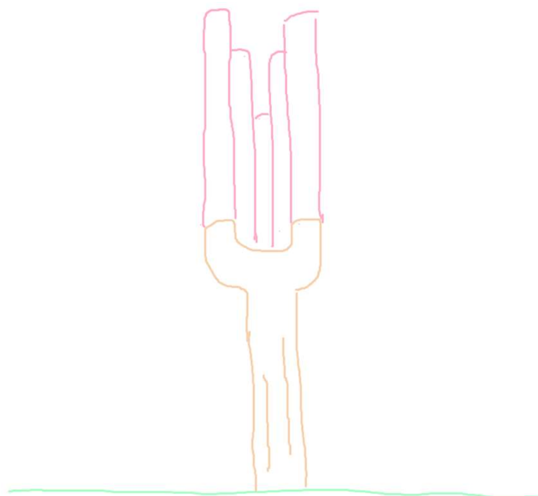


Figure 3.1: Solutions to $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + \omega x = 0$

Solutions lie along the $\omega\mathbb{R}$ axis. Numbers along this axis have a real component equal to 0, hence each of these numbers has a modulus of 0.

Exploring a forest of hyperbolic polynomials, compared to a parabolic forest, takes much, much longer. If we happened to find ourselves in a parabolic forest, perhaps again by some tripping accident, we would pick our heads up to see an unusual tree like this:



$x^2 - \omega x = 0$

Figure 3.2: Pole-like Hyperbolic Tree of $x^2 - \omega x = 0$

We have already encountered a parabolic polynomial tree of degree two with four roots. How much more bizarre could it get? After finding and entering another convenient cave entrance nearby, we trot along until we find ourselves directly below the tree:

Figure 3.3: Roots of Hyperbolic Tree $x^2 - \omega x = 0$

There are too many roots to count. Each time we attempt to keep count of the roots, they shift in and out of each other and you lose track of the roots you counted. All one can do is stare in awe.

CHAPTER 4

CONCLUSION

4.1 Some Future Prospects

The general results of the hyperbolic numbers do not include numbers that lie on the $y = x$ and $y = -x$ lines. Since these numbers cannot be converted to exponential form, the ease with which we can extract meaningful patterns from them is still ambiguous. In the future, perhaps, further research will be performed on this exact topic. For now, we can make conjectures about the number of roots these numbers possess.

Generally, square roots of hyperbolic numbers in the open quadrants look like this geometrically:

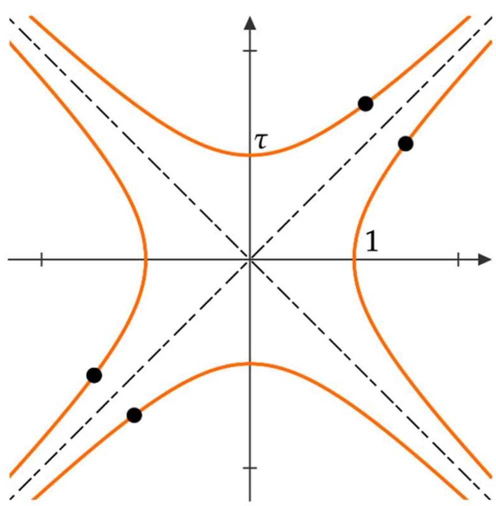


Figure 4.1: Arbitrary Square Roots of a Hyperbolic Number in the Right Open Quadrant

Note that these numbers are reflections across the lines $y = x$ and $y = -x$. This phenomenon is best understood when multiplying the arbitrary square root in the open right quadrant by τ, -1, and $-\tau$ to get the square roots in the top, left, and bottom open quadrants, respectively. This pattern is kept for all square roots of a number in the open right quadrant. Let us look at different square roots that possess the same angle but are located on an inner hyperbola:

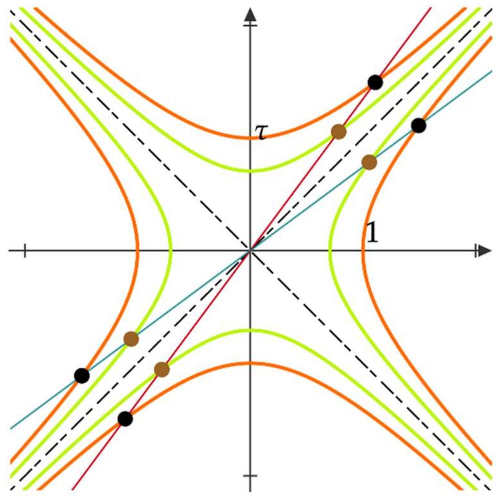


Figure 4.2: Where Same Angle Square Roots Tend to on Inner Hyperbolas

The trend makes sense considering that as we keep moving through hyperbolas closer and closer to the lines $y = x$ and $y = -x$, r decreases to 0. Hence the trend for same angle square roots is to go to zero. So, as r decreases to zero, the four roots on each set of hyperbolas coalesce into one. Thus, one might say that the polynomial $x^2 = 0$ has the root zero with multiplicity four. More work would be needed to define multiplicity properly and prove that this is the case.

To predict what square roots on the $y = x$ and $y = -x$ lines look like. Let us instead look at square roots that are colinear to the square roots on the unit hyperbola and perpendicular to either the $y = x$ or $y = -x$ lines:

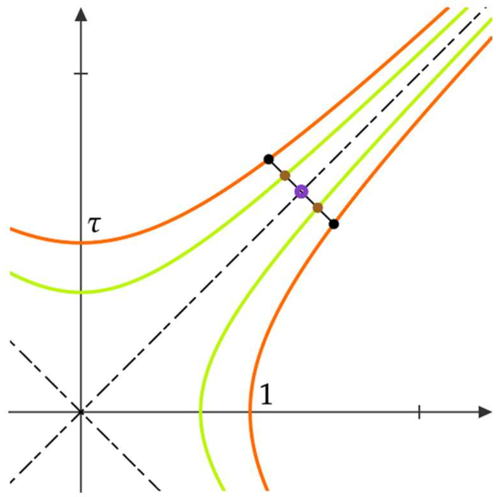


Figure 4.3: Points that Tend Perpendicularly

This trend is even more interesting and pertinent. Square roots along the black perpendicular line are converging to one point. We know that the original square roots and any along the black line, excluding the purple point, come in pairs. This same trend can be observed for square roots on the other side of the $y = -x$ line. Would a hyperbolic number on the $y = x$ and $y = -x$ lines then have two roots, each with multiplicity two? We save this question for future investigation!

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BIOGRAPHICAL INFORMATION

Nathan Easley was born in Jerusalem, Israel in 2001. Not very particular to anything in the academic world prior to college, he enjoyed learning for its own sake. Initially, he did not have the intent to go to college and acquire a degree after high school. A fateful encounter with his former world history teacher (who insisted he go and set up the circumstances for this path) changed his plans and spurned motivation. He began attending The University of Texas at Arlington in the fall of 2019, deciding to become a mathematics major after careful deliberation. His reasons for pursuing a mathematics degree spawned from his viewpoint on mathematics – that it is fundamental, versatile, and enhances critical and analytical thinking.

Nathan had hoped that his college experience would gradually enlighten him on his own desires, wants, and goals. He did not expect there to be strong mental hurdles to overcome while simply acquiring the degree, distracting him from gaining anything meaningful in terms of self-understanding. Despite this, Nathan chose to stick to the end (even joining the UTA Honors College) and has recently begun thinking, with a clearer mind, about his future. He is considering graduate school and is excited to gain a deeper comprehension of himself and what lies ahead.

Outside of academia, Nathan enjoys playing video games, listening to music, reading books (from various genres), exercising, and life itself. He owns a piano which he intends to further master.