OBSERVER SYNTHESIS FOR LINEAR/NONLINEAR DYNAMICAL SYSTEMS SUBJECT TO MEASUREMENT DELAYS

by

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To my Parents.

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ABSTRACT

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In this research, problems associated with communication delays on the stability of formation of a group of unmanned vehicles is examined and solution to this problem is proposed. Ideally, it is required that the information transfer between the vehicles participating in a cooperative task be available immediately. But, in reality, there is always a delay between the instants at which the information is transmitted by one vehicle and the instant at which it is received by the other vehicles in the formation.

A state observer is used to estimate the time delayed signal. The observer consists of two dynamic systems connected in cascade. The delayed output signal is used to estimate the current states using this cascade observer. The reconstruction of the states is done at several sub-intervals within the delay window.

Two types of time delays profiles are considered for analysis. Initially, the delays are assumed to be known constants. Next, the assumption is relaxed to study the effects of known time varying delays are examined. In both the cases, the structure of the state observer remains unchanged. The analysis is done for both continuous-time and discrete-time linear time invariant systems and continuous-time nonlinear system. The

state observer, in each of these cases is an guarantees either asymptotic or exponential while convergence to the true states.

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CHAPTER 1

INTRODUCTION

Measurement delays in dynamical systems are ubiquitous. Multiple systems performing a common task often experience a delay in receiving the measurement due to the basic fact that they are physically separated from each other and the measurement from one system has to travel this distance to reach the second system. However if the delay was only due to the distance, it would be of very small magnitude. But owing to the uncertainties in the communication channel, there are several factors which contribute to a significant increase in delay magnitude.

This scenario can be experienced in a various biological, ecological and engineering systems. In this research the effects of such delay on engineering systems is analyzed. Specifically, in the cases of multiple dynamical systems performing a common task. A state observer to estimate the current states from the delayed measurements is proposed.

In the succeeding sections, a brief mathematical descriptions of state observers and time delay systems are presented.

1.1 State Observers

State observer is a device that estimates the unknown states of a dynamical system. It utilizes the system model and measurements of the system inputs and outputs for the estimation process. The system model may be represented either by a differential equation (continuous time systems) or by a difference equation (discrete time systems). Three main quantitative state observers are: Luenberger observer[1], adaptive observer[2] and Kalman [3] filter.

In the deterministic case, when no random noise is present, the Luenberger observer and its extensions are used for time-invariant systems with known parameters. When the parameters of the system are unknown or time varying, an adaptive observer is used. The adaptive observer, in addition to estimating the system states, is also used to estimate the unknown system parameters. The corresponding observer for a stochastic system containing additive noise processes, with known/unknown parameters, is a stochastic observer with a structure attributed to Kalman. Kalman filter is a recursive estimator. If the system parameters are unknown, the filter can be used to estimate these along with system states.

In this research, the dynamical systems under consideration are of the deterministic type.

1.1.1 Observability and Observers

Consider a deterministic linear system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \quad (1.1)$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) \quad (1.2)$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state vector. $\boldsymbol{u}(t) \in \mathbb{R}^p$ is the system input. $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are the constant matrices with appropriate dimensions. $\boldsymbol{y}(t) \in \mathbb{R}^q$ is the system output. The initial conditions for the systems is specified as

$$\boldsymbol{x}(0) = \boldsymbol{x}_0 \quad (1.3)$$

System (1.1)-(1.2) is said to be *observable* if at any time t, the state $\boldsymbol{x}(t)$ can be determined from the input sequence $\boldsymbol{u}(s)$ and the output $\boldsymbol{y}(s)$, $0 < s < t$. $\boldsymbol{x}(t)$ can be determined if the initial state $\boldsymbol{x}_0$ and the inputs are known. Hence, observability can equivalently be defined as the problem of finding the initial states from the given input-output measurements. If the initial states cannot be determined from the given sequence of inputs and output measurements, then the system is not observable.

Any linear system of the form (1.1)-(1.2) can be tested for observability by constructing the *observability matrix*

$$\boldsymbol{O} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \vdots \\ \boldsymbol{CA}^{n-1} \end{bmatrix} \tag{1.4}$$

If the matrix O is of full rank, then the pair $(\boldsymbol{C}, \boldsymbol{A})$ is observable. However, if $Rank(\boldsymbol{O}) < n$, then the system is not observable. This implies that some of the states cannot be determined from the input-output sequences. In this research we consider systems that are completely observable, i.e, $Rank(\boldsymbol{O}) = n$.

Several unstable systems can generally be stabilized by using stabilizing control laws to place the system poles at desired locations. The inputs to the controller are the state measurements. But, in many systems, all the state may not be measurable. In this case, with the knowledge of the initial state, the state trajectory, can in principle be determined if $(\boldsymbol{C}, \boldsymbol{A})$ is observable. However, the procedure involves integration and inversion of a matrix which is ill-conditioned. An alternative, more robust and practical approach to estimate the states. The states can be estimated using a *state observer*.

1.2 Luenberger Observers for Linear and Nonlinear Dynamical Systems

The state observer for estimating the unknown states of a deterministic linear system was first proposed by Luenberger [1] in 1971.

The equation for the Luenberger observer, in addition to the system dynamics, contains a term that corrects the current state estimates by an amount proportional to the prediction error: the estimation of the current output minus the actual measurement. Inclusion of this correction ensures stability and convergence of the observer even when the system being observed is unstable.

For the system (1.1)-(1.2) the observer is of the form

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}(\boldsymbol{y} - \hat{\boldsymbol{y}}), \quad \hat{\boldsymbol{x}}(0) = \hat{\boldsymbol{x}}_0 \tag{1.5}$$

$$\hat{\boldsymbol{y}} = \boldsymbol{C}\hat{\boldsymbol{x}} \tag{1.6}$$

where $\hat{\boldsymbol{x}}_0$ is arbitrarily chosen. The error in estimation is defined as

$$\boldsymbol{e}(t) = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t) \tag{1.7}$$

The error dynamics is given by

$$\dot{\boldsymbol{e}} = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}(t), \quad \boldsymbol{e}(0) = \boldsymbol{x}_0 - \hat{\boldsymbol{x}}_0 \tag{1.8}$$

If all the eigenvalues of matrix $\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}$ is chosen to lie in the left half $s - plane$, then regardless of $\boldsymbol{e}(0)$, $\boldsymbol{e}(t) \to 0$ as $t \to \infty$ exponentially and accurate state estimates are obtained.

Similarly, the concept for state observers can be extended to nonlinear deterministic continuous time systems. But the design process is far more complicated than the linear systems. Several algorithms have been proposed to realize a stable nonlinear observer.

The earliest nonlinear observer designs by Krener and Isidori (Ref. [4]-[5]) were based on a set of conditions to linearize the observation error.

One of the most complete nonlinear observer design was proposed by Gauthier et al. [6]. In this observer design, the concept of global nonlinear coordinate changes is used to realize the observer and it guarantees global convergence for uniformly observable inputs. The existence of a global nonlinear coordinate change implies uniform observability of the system. Later, in 1993, Ciccarella et. al. [7] proposed several improvements to the observer design by Gauthier et. al. to achieve global asymptotic stability of the observer system.

The observer design based on the work by Ciccarella et al. [7] is as follows

For a nonlinear systems of type,

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t)) && (1.9) \\ \boldsymbol{y}(t) &= \boldsymbol{h}(\boldsymbol{x}(t)) && (1.10) \end{aligned}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{u}(t) \in \mathbb{R}^p$ in the system input, $\boldsymbol{y} \in \mathbb{R}^m$ is the system output and the vector functions $\boldsymbol{f},\ \boldsymbol{g},\ \boldsymbol{h}$ are C^∞, the dynamics of the state observer is

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}}(t) &= \boldsymbol{f}(\hat{\boldsymbol{x}}(t)) + \boldsymbol{g}(\hat{\boldsymbol{x}}(t), \boldsymbol{u}(t)) + \boldsymbol{Q}^{-1}(\hat{\boldsymbol{x}}(t))\boldsymbol{K}(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)) && (1.11) \\ \hat{\boldsymbol{y}} &= \boldsymbol{h}(\hat{\boldsymbol{x}}(t))) && (1.12) \end{aligned}$$

where $\boldsymbol{Q}(\hat{\boldsymbol{x}}(t))$ is the observability matrix defined as

$$\boldsymbol{Q}(\hat{\boldsymbol{x}}(t)) = \frac{d\boldsymbol{\phi}(\boldsymbol{x})}{d\boldsymbol{x}} \qquad (1.13)$$

$$\boldsymbol{\phi}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{h}(\boldsymbol{x}) \\ L_f\boldsymbol{h}(\boldsymbol{x}) \\ \vdots \\ L_f^{n-1}\boldsymbol{h}(\boldsymbol{x}) \end{bmatrix} \qquad (1.14)$$

$\boldsymbol{\phi}(\boldsymbol{x})$ defines the global change of coordinates. $L_f\boldsymbol{h}(\boldsymbol{x})$ denotes the Lie derivative of the function $\boldsymbol{h}$ along $\boldsymbol{f}$ and $L_f^k\boldsymbol{h}(\boldsymbol{x})$ denotes the k^{th} order repeated Lie derivative of the function $\boldsymbol{h}$ along $\boldsymbol{f}$.

In the linear case, $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{C}\boldsymbol{x}$ and $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ and matrix (1.14) reduces to the observability matrix defined in (1.4).

The theory of state observers can similarly be extended to discrete linear and nonlinear systems. The first step in discrete observer realization is the discretization of the system dynamics followed by the observer design in the discrete domain.

1.3 Time Delay Systems

The field of time-delay systems had its origin in the 18th century [8] and has received substantial attention since then. Investigation into the stability of delay differential equations and hence time-delay systems, was pioneered by the likes of Bellman [9], Nyquist, Chebotarev and Pontryagin and more recently by Hale [10]. In the last two decades, the advances in numerical methods and control theory, especially the robust and adaptive control theories have had a considerable impact on the field. Efficient numerical algorithms to solve linear matrix inequalities (LMI) and robust stability analysis techniques of uncertain polynomials have enabled researchers to solve the stability problems associated with time-delay systems.

The concept of time-delay is used in biological, ecological and engineering systems. The time-delay systems are also referred to as hereditary systems, systems with after-effects, systems with time-lag and infinite dimensional systems.

Delay can occur in the system dynamics, the control input or the system output. Irrespective of which part of the system is affected by delays, the evolution of a time-delay system depends both on the present state and also its history. In general, this dependence can be represented by a functional differential equations. In particular, the differential-difference equations are best suited to describe a time-delay system.

1.3.1 Functional Differential Equations

Consider a simple linear functional differential equation of the form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{x}(t-\Delta)) \tag{1.15}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ are the system states, $\boldsymbol{f} : \mathbb{R}^n \times C \rightarrow \mathbb{R}^n$ and Δ is the delay magnitude. Eq. (1.15) suggests and that the the derivative of the states at any instant t, is a function of both the states at the current time t and the previous values of the $\boldsymbol{x}(t)$.

To propagate the system states beyond the time $t = 0$, $\dot{\boldsymbol{x}}(0)$ has to calculated. This implies that the values of $\boldsymbol{x}(t)$ at $t = 0$ and $t = -\Delta$ are needed. This is because, the solution of (1.15) at any instant t is

$$\boldsymbol{x}(t) = \exp\left(A(t-\Delta)\right)\boldsymbol{x}(0) \tag{1.16}$$

Similarly, to calculate $\dot{\boldsymbol{x}}$ at $t = \xi,\ 0 \leq \xi \leq \Delta$, both $\boldsymbol{x}(\xi)$ and $\boldsymbol{x}(\xi - \Delta)$ are needed. But $\boldsymbol{x}(\xi - \Delta)$ cannot be generated as a solution of Eq. (1.15) since $-\Delta \leq \xi - \Delta < 0$. Hence for the solution to be uniquely defined, the value of $\boldsymbol{x}(t)$ has to be completely defined in the interval in the interval $-\Delta \leq t \leq 0$. This is the initial condition of the system, specified as

$$\boldsymbol{x}(0) = \boldsymbol{\phi}(t), \quad t \in [-\Delta, 0] \tag{1.17}$$

Once the initial conditions are well defined, the system can be propagated from $t = 0$ using the differential equation (1.15).

As stated earlier, the evolution of time delay systems depends both on the present state and also its history, similar to the functional differential equations. Hence, any time delay systems can be aptly described by a functional differential equation.

Equations of the type (1.15) are referred to as *retarded functional differential equation* (RFDE). Ordinary differential equations are a special class of the RFDE. In an RFDE the delay variable does not appear in the highest order derivative term. If it does appear, then it is a functional differential equation of the neutral type. For example,

$$\dot{\boldsymbol{x}}(t) + 3\boldsymbol{x}(t - \Delta) + \boldsymbol{x}(t) = 0 \tag{1.18}$$

is an RFDE. While,

$$\dot{\boldsymbol{x}} + 3\dot{\boldsymbol{x}}(t - \Delta) + \boldsymbol{x}(t) - \boldsymbol{x}(t - \Delta) = 0 \tag{1.19}$$

is a *neutral functional differential equation* (NFDE).

1.3.2 Stability of Time-Delay Systems

Stability is an important factor for time-delay systems, since delays can be a major source of instability in an otherwise stable process. Hence, it is necessary to compensate for time delays. The most common method of compensation is the realization of a controller to stabilize the system. Some of the common methods for the compensation of fixed time-delay include the recursive response method, state-augmented compensation method, controllability based stabilization method, the Smith predictor method and the Padé approximation method. All the techniques are applicable to any control algorithm to be used for controlled design.

In the case of linear systems, two important methods are: *Padé* approximation [11] of the delay and Smith predictor [12] based control. Both the methods are transfer function based approach. The two methods are briefly explained below:

Padé approximation: Control systems with time-delays are difficult to analyze and simulate. One of the reasons is that a closed-loop control system with delays is in fact an infinite dimensional system, i.e. it has infinite number of poles. It is also difficult to determine all the system poles. One of the most widely recommended remedies to overcome this difficulty is the Padé approximation method to compensate for the delay.

Consider the following model-matching problem for a transfer function

$$G(s) = e^{-t_d s} G_0(s) \tag{1.20}$$

where $G_0(s)$ is the proper stable rational transfer function and $e^{-t_d s}$ is the pure delay term. $G(s)$ is approximated by a transfer function

$$\hat{G}(s) = e^{-t_d s} G_0(s) \tag{1.21}$$

The same input is applied to both the systems as shown in Fig. (1.1). Then by comparing the two outputs, the unknown transfer function $\hat{G}(s)$ can be determined.

This implies that the term $P_d(s)$ has to be matched with the delay term $e^{-t_d s}$. This is accomplished by expanding $e^{-t_d s}$ a matching series expansion of a rational function whose numerator is a polynomial of degree p and denominator is a polynomial of degree q as follows

$$e^{-t_d s} = P_d(s) = \frac{N_d(s)}{D_d(s)} = \frac{\sum_{k=0}^{n} (-1)^k c_k t_d^k s^k}{\sum_{k=0}^{n} c_k t_d^k s^k} \tag{1.22}$$

The coefficients are

$$c_k = \frac{(2n-k)!n!}{2n!k!(n-k)!}, \quad k = 0, 1, \ldots, n \tag{1.23}$$

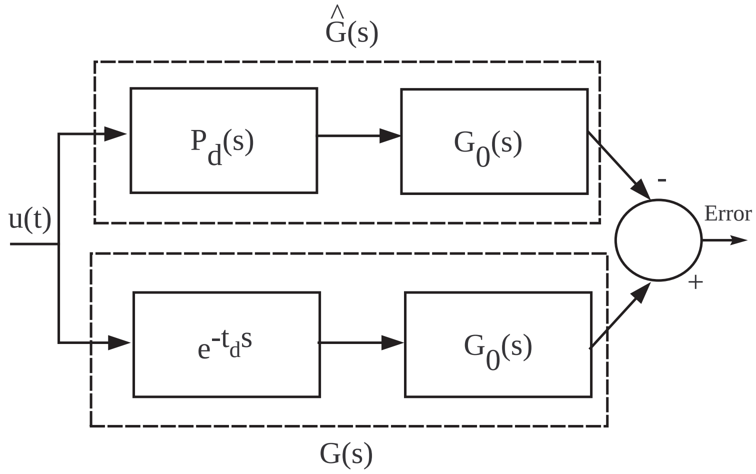

Figure 1.1 Model-matching Problem using Padé approximation method.

After obtaining the approximated transfer function of the system, $\hat{G}(s)$, it can be used for designing the controller for time delay systems.

The number of terms in the expansion of the numerator and denominator polynomials depends on the accuracy of the systems. If the term $P_d(s)$ has to be very close to $e^{-t_d s}$, then n is very large. Also, as time delay t_d increases, n should be increased to keep the level of the approximation error fixed.

Smith Predictor: Smith predictor is a predictive model-based control scheme that requires state prediction. It was suggested to design controllers to stabilize factory processes with long transport delays, for example catalytic crackers and steel mills, but the idea can be generalized to all control processes that have long loop delays.

Fig. (1.2) shows the block diagram of a process with delay and Fig. (1.3) the Smith predictor.

The development of Smith predictor is based on the knowledge of the process model. The first step in designing the controller for the delayed process is to design a suitable predictive controller when the system is free of delay.

The process is composed of the delay-free stable rational transfer function $G_0(s)$ and a pure time delay term $e^{-t_d s}$. $w(t)$ is a reference input for the system. The Smith

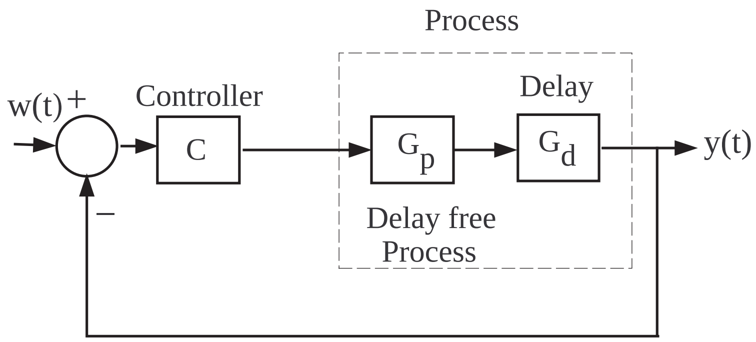

Figure 1.2 Block Diagram of a Time Delayed Process.

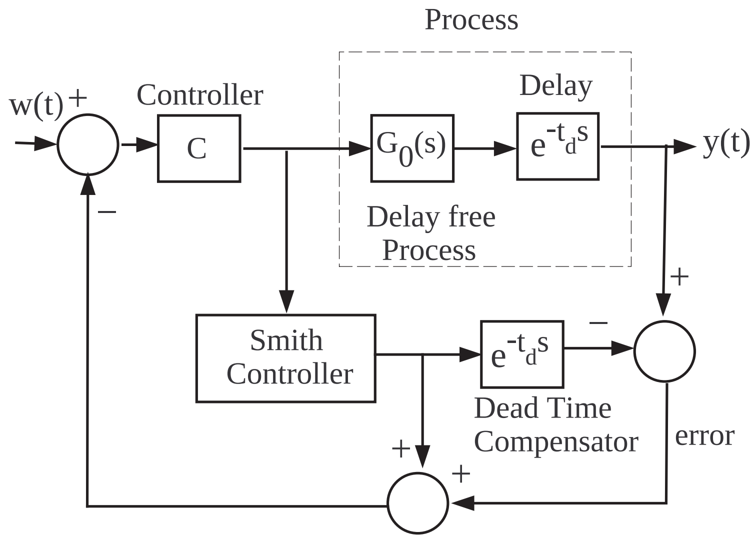

Figure 1.3 Block Diagram of the Smith Predictor.

controller incorporates the system model, thus allowing for the prediction of the system variables, and then the controller is designed as though the system is delay free.

The predictor consists of two internal feedback loops, the Smith controller and a dead time compensator. The Smith controller includes the dynamic model of the process, $G_0(s)$ but excludes all transport delays; the other includes both $G_0(s)$ and the delays. Since the transport delays are excluded from the first model, it can be a high-gain, low-delay, negative feedback loop. If this model is accurate, and the plant performance reliable, this loop can provide near optimal control of the plant.

The Smith predictor is very popular and widely used in the process control systems. The main advantage is that the delay time is removed from the closed loop system.

But, the disadvantages outweigh the advantages. Primary ones include sensitivity to process model mismatch, difficulty in coping with disturbances, and too simplified process model. Also, the implementation of an analog predictor is complicated. However, its discrete equivalent can be easily implemented in practice.

Several modifications to the original Smith predictor have been proposed, including a predictor for nonlinear systems and adaptive versions.

1.4 Stability Analysis of Time Delay System

Stability analysis of time delay system often reveals the range of delay magnitudes that could be tolerated by the dynamic system without becoming unstable. For delays associated with system states and inputs, a controller can be implemented to stabilize the system.

As noted in the previous section, an open loop system cannot be stabilized using a Smith predictor. But, with the advances in robust and adaptive control methodologies, several elegant stability analysis were introduced. Controllers based on backstepping and sliding modes have been used to stabilize time-delay systems.

The widely used stability analyses are the Lyapunov theorems and extensions namely, Lyapunov-Krasovskii (LK) theorem and Razumikhin theorem. The numerical method techniques like the LMI coupled with the Lyapunov based theorems are also useful for stability analysis of these systems. The following two theorems are quoted verbatim from [8]

Lyapunov-Krasovskii Theorem: *Suppose* $f : \mathbb{R} \times \mathcal{C} \rightarrow \mathbb{R}^n$ *in (1.15) maps* $\mathbb{R}\times$ *(bounded sets of* $\mathcal{C}$*) into a bounded sets in* $\mathbb{R}^n$ *and that* $u, v, w : \bar{\mathbb{R}}_+ \rightarrow \bar{\mathbb{R}}_+$ *are continuous nondecreasing functions , where additionally u(s) and v(s) are positive for* $s > 0$*, and*

$u(0) = v(0) = 0$. *If there exists a continuous differentiable functional* $V : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ *such that*

$$u(\|\phi(0)\|) \leq V(t, \phi) \leq v(\|\phi\|_c) \tag{1.24}$$

and

$$\dot{V}(t, \phi) \leq -w(\|\phi(0)\|) \tag{1.25}$$

then trivial solution of (1.15) is uniformly stable. If $w(s) > 0$ *for* $s > 0$, *then it is uniformly asymptotically stable. If in addition,* $\lim_{s\to\infty} u(s) = \infty$, *then it is globally asymptotically stable.*

Razumikhin Theorem: *Suppose* $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ *in (1.15) maps* $\mathbb{R}\times$ *(bounded sets of* $\mathcal{C}$*) into a bounded sets in* $\mathbb{R}^n$ *and that* $u, v, w : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ *are continuous non-decreasing functions , where additionally u(s) and v(s) are positive for* $s > 0$, *and* $u(0) = v(0) = 0$, *v strictly increasing. If there exists a continuous differentiable functional* $V : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ *such that*

$$u(\|x\|) \leq V(t, \phi) \leq v(\|x\|_c), \quad \text{for } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n \tag{1.26}$$

and

$$\dot{V}(t, \phi) \leq -w(\|\phi(0)\|), whenever V(t + \theta, x(t + \theta)) \leq V(t, x(t)) \tag{1.27}$$

for $\theta \in [-\Delta,\ 0]$, *then system is uniformly stable.*

If, in addition, $w(s) > 0$ *for* $s > 0$, *and there exits a continuous nondecreasing function* $p(s) > s$ *for* $s > 0$ *such that condition (1.28) is strengthened to*

$$\dot{V}(t, \phi) \leq -w(\|\phi(0)\|), whenever V(t + \theta, x(t + \theta)) \leq pV(t, x(t)) \tag{1.28}$$

for $\theta \in [-\Delta,\ 0]$, *then system (1.15) is uniformly asymptotically stable.*

f in addition, $\lim_{s\to\infty} u(s) = \infty$, *then the system (1.15) is globally asymptotically stable.*

Ref. [8] cites detailed proofs of the two theorems.

The two theorems are widely used in the stability analysis of state and input delays. [13],[14],[15],[16].

1.5 Thesis Organization

The remainder of the thesis is organized as follows. Chapter 2 describes in detail the Motivation behind the development the time delay observer and a description of the observer proposed be Germani et al. [17]. In chapter 3 the time delay observer analysis for linear systems is considered. First the results of the work by Germani [18] for constant delays is extended to a discrete linear system with constant output delays. Then, the effects of time varying delays on both continuous time and discrete time systems is considered. Simulation results for all the cases are presented. Chapter 4 introduces the concept of time delay observer for nonlinear continuous time systems. An alternate result for constant output delay problem is first derived. This is an improvement of the result proposed by Germani, since it account for the observer gain as a function of delay. Next, the effects of time varying delay on nonlinear systems is analyzed and the simulation results are presented. Finally Chapter 5 briefly states the conclusions from this research work and possible avenues for future work.

CHAPTER 2

RESEARCH MOTIVATION

The fundamental motivation for this research are the problems associated with co-operative control of multiple unmanned aerial vehicles (UAV) due to delays in information flow.

Over the last two decades, cooperative control of multiple vehicles has received substantial amount of attention from the research community. Multiple vehicles performing a common task greatly improves the results, since the task can be equally shared amongst the participating vehicles and it can be completed in a shorter duration as compared to a single vehicle performing the same task. The application areas of cooperative control of multiple vehicles are several ranging from robot teams, micro-robot swarming, unmanned ground vehicles, unmanned aerial vehicles as well as micro-satellite clusters (Refs. [19], [20], [21], [22]). Significant developments in control techniques for single vehicles, computation methods, communication capabilities and miniaturization of technologies have further boosted the efforts in exploiting the features of multiple vehicle performing common tasks. The environments in which these vehicles operate can be highly unstructured.

In scenarios with multiple vehicles performing a common task, they may be required to maintain a certain formation. The reason for this is to optimize the performance of the vehicles, subject to several constraints including limited fuel, low bandwidth communication, limited time to complete the task etc. Also, moving in a fixed formation will solve the problems of collisions of the vehicles within the groups. Two of the widely used formation control approaches are the "leader-follower" techniques and the "virtual leader" techniques. In the former method, one out of all the participating vehicles is assigned as the leader of the group. While, in the latter approach there is only a fictitious leader. It can be recognized that the formation control of cooperating vehicles depends heavily on information flow between the participating vehicles. This information flow may be subject to uncertainties and transmission delays. In an interconnected dynamical system the behavior of the participating systems depends not only on the individual vehicle dynamics, but also on the nature of the interconnections. Therefore, it is important to study the stability and performance of such systems [23] and the effect of influences that alter the nature of these interconnections.

The unmanned aerial vehicle (UAV) cooperative control with information flow constraints problems was studied by Luo[24] and Fax and Murray[25]. In [24], three different situations of inter-vehicular communication were investigated. The communication between vehicles was represented as a sequence of impulses, a bandwidth-limited signal, and a range-limited signal respectively. The information flow constraints between the vehicles was posed as a series of generalized optimal control problems, by considering the communicated information as one of the control inputs to the UAV. In the work by Fax and Murray, ideas from graph theory and system theory were used to develop information exchange strategies to improve formation stability and consequently achieve better performance that is robust to changes in communication topology. The authors also a consider a wide range of inter-vehicular connection possibilities.

In this succeeding section, we look at a cooperative control in an AHS environment and how communication delays between the vehicles can cause serious problems.

2.1 Problem Description

An example of multiple vehicle control is the study on the effects of communication delays on string stability on automated highway systems (AHS)[26]. AHS was proposed to improve the efficiency on highways by reducing severe traffic congestion. In this scheme, the string of vehicles plying on the highways are automated and controlled to

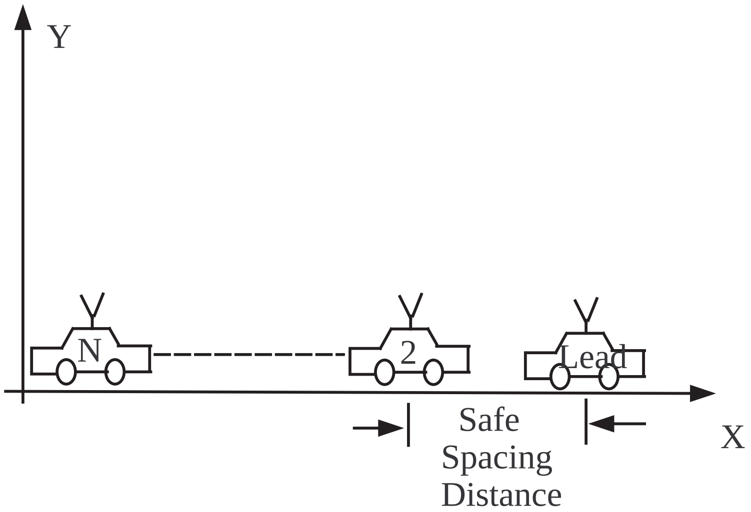

Figure 2.1 String of automated vehicles moving at a safe intervehicular spacing.

move at a spacing that is much closer than is safe for human drivers as shown in Fig. (2.1). To maintain the fixed constant spacing, each vehicle in the string has to transmit its position information to other vehicles. It was observed that when delays [26] crept into the communication, the string system became unstable with vehicles moving much closer to each other than the prescribed spacing. The ultimate effect was collision of the vehicles.

In most of these cases, the problem due to delays were tried to overcome by using a stabilizing controller within each vehicle. The instability was fixed by prescribing the maximum and minimum tolerable delay values and tuning the controller to cater to the problems created by the delay.

However in this thesis, instead of addressing the issue using a controller, the delay problems are addressed using the concept of state reconstruction of the present state using the delayed information.

2.2 Time Delay Observer

The concept of using a state observer to estimate the current states form a delayed outputs is fairly new. One of the earliest contributions in this direction include observers for nonlinear systems with delays in the output which are linearizable by additive output injection [27]. Another significant work is the observer design proposed by Germani et al. [17]-[18] The attractive feature of the observer design proposed by Germani et al. is that the states are reconstructed at different time-delay instants within the delay window. Hence, theoretically, as the delay magnitude increases the current states can be estimated with zero observation error just by increasing the number of systems in the observer chain. For a linear system they have proved that an observer with just two systems in the chain is sufficient to estimate the current states for any delay magnitude, assuming it to be known.

The concept of chain-observers [28] was first proposed by the same authors for the design of state observers for nonlinear system without any delays. In this work, Germani and others have presented a state observer design algorithm based on the drift-observability property. This property refers to the observability of the system for zero input. The observer constructed based on this property has an interesting chain-like structure. This concept has been extended by them for estimating the states from delayed output measurements.

One of the main problems in [17]-[18] is that the observer gain is arbitrarily selected to stabilize the system, without any relation to the delay magnitude. In [29], Kazantzis and Wright have proposed a nonlinear observer with a state-dependent gain. The gain is computed from the solution of a system of first-order singular partial differential equations (PDEs). The observer retains the same chain structure as proposed by Germani.

Since we have used the same concept as proposed by Germani et al. [17], a brief overview of their work from Ref. [17] is presented, for nonlinear systems.

The single-input single-output (SISO) nonlinear systems considered for observation are of the type

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{g}(\boldsymbol{x}(t))u(t) \quad t \geq \Delta, \quad \boldsymbol{x}(-\Delta) = \bar{\boldsymbol{x}} \tag{2.1}$$

$$\bar{\boldsymbol{y}}(t) = \boldsymbol{h}(\boldsymbol{x}(t-\Delta)) \tag{2.2}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$ the vector functions $\boldsymbol{f},\ \boldsymbol{g},\ \boldsymbol{h}$ are C^∞. The undelayed output is represented as $\boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t))$. and $\Delta > 0$ is a known constant measurement delay.

For system (2.1)-(2.2) a square map, $\boldsymbol{z} = \boldsymbol{\Phi}(\boldsymbol{x})$, is defined as

$$\boldsymbol{\Phi}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{h}(\boldsymbol{x}) \\ \boldsymbol{L}_f\boldsymbol{h}(\boldsymbol{x}) \\ \vdots \\ \boldsymbol{L}_f^{n-1}\boldsymbol{h}(\boldsymbol{x}) \end{bmatrix} \tag{2.3}$$

where $\boldsymbol{L}_f\boldsymbol{h}(\boldsymbol{x})$ denotes the Lie derivative of the function $\boldsymbol{h}$ along $\boldsymbol{f}$ and $\boldsymbol{L}_f^k\boldsymbol{h}(\boldsymbol{x})$ denotes the k^{th} order repeated Lie derivative of the function $\boldsymbol{h}$ along $\boldsymbol{f}$.

The Jacobian, $\boldsymbol{Q}(\boldsymbol{x})$, of the map $\boldsymbol{\Phi}(\boldsymbol{x})$ and the Jacobian, $\boldsymbol{Q}^{-1}(\boldsymbol{x})$, of the the inverse map $\boldsymbol{\Phi}^{-1}(\boldsymbol{x})$ are defined as

$$\boldsymbol{Q}(\boldsymbol{x}) = \frac{\partial \boldsymbol{\Phi}(\boldsymbol{x})}{\partial \boldsymbol{x}} \tag{2.4}$$

$$\boldsymbol{Q}^{-1}(\boldsymbol{x}) = \left.\frac{\partial \boldsymbol{\Phi}^{-1}(\boldsymbol{z})}{\partial \boldsymbol{z}}\right|_{\boldsymbol{z}=\boldsymbol{\Phi}(\boldsymbol{x})} \tag{2.5}$$

The non-linear system (2.1)-(2.2) can be represented in the new co-ordinate system as:

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}_n\boldsymbol{z}(t) + \widetilde{\boldsymbol{H}}(\boldsymbol{z}(t), u(t)), \quad t \geq \Delta, \quad \boldsymbol{z}(-\Delta_1) = \boldsymbol{\Phi}(\bar{x}) \tag{2.6}$$

$$\bar{\boldsymbol{y}}(t) = \boldsymbol{C}_n\boldsymbol{z}(t-\Delta) \quad t \geq 0 \tag{2.7}$$

where

$$\widetilde{\boldsymbol{H}}(\boldsymbol{z}(t), u(t)) = \widetilde{\boldsymbol{H}}(\boldsymbol{x}(t), u(t))|_{\boldsymbol{x}=\boldsymbol{\Phi}^{-1}(\boldsymbol{z})} \tag{2.8}$$

$$\widetilde{\boldsymbol{H}}(\boldsymbol{x}(t), u(t)) = \boldsymbol{B}_n\boldsymbol{L}_f^n\boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{Q}(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x})u(t) \tag{2.9}$$

Matrices $\boldsymbol{A}_n,\ \boldsymbol{B}_n,\ \boldsymbol{C}_n$ are the Brunowski triple of order n.

The following assumptions are made about system (2.1)-(2.2).

$\mathbf{H}_1$: *The system Eqs.(2.1)-(2.2) has a uniform observation relative degree equal to n (n is the dimension of vector x), i.e,*

$$\begin{aligned} \forall \boldsymbol{x} \in \mathbb{R}^n \quad \boldsymbol{L}_g \boldsymbol{L}_f^k \boldsymbol{h}(\boldsymbol{x}) &= 0, \quad k = 0, 1, \ldots, n-2 \\ \exists \boldsymbol{x} \in \mathbb{R}^n \quad \boldsymbol{L}_g \boldsymbol{L}_f^{n-1} \boldsymbol{h}(\boldsymbol{x}) &\neq 0 \end{aligned} \tag{2.10}$$

$\mathbf{H}_2$: *System Eqs.(2.1)-(2.2) is globally drift observable and the diffeomorphism* $\boldsymbol{z} = \boldsymbol{\Phi}(x)$ *and its inverse* $\boldsymbol{x} = \boldsymbol{\Phi}^{-1}(z)$ *are globally Lipschitz in* $\mathbb{R}^n$*, i.e.,*

$$\|\boldsymbol{\Phi}(\boldsymbol{x}_1) - \boldsymbol{\Phi}(\boldsymbol{x}_2)\| \leq \boldsymbol{\gamma}_\Phi \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \quad \forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n \tag{2.11}$$

$$\|\boldsymbol{\Phi}^{-1}(\boldsymbol{z}_1) - \boldsymbol{\Phi}^{-1}(\boldsymbol{z}_2)\| \leq \boldsymbol{\gamma}_{\Phi^{-1}} \|\boldsymbol{z}_1 - \boldsymbol{z}_2\|, \quad \forall \boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{R}^n \tag{2.12}$$

Under assumption 2, the Jacobian matrices $\boldsymbol{Q}(\boldsymbol{x})$ *and* $\boldsymbol{Q}^{-1}(\boldsymbol{x})$ *are non-singular in* $\mathbb{R}^n$*.*

$\mathbf{H}_3$: *The vector function* $\widetilde{\boldsymbol{H}}(\boldsymbol{z}(t), u(t))$ *is globally uniformly Lipschitz with respect to* $\boldsymbol{z}$*, and the Lipschitz coefficient* $\boldsymbol{\gamma}_{\widetilde{H}}$ *is a non- decreasing function of* $|u|$*, i.e,*

$$\|\widetilde{\boldsymbol{H}}(\boldsymbol{z}_1, u) - \widetilde{\boldsymbol{H}}(\boldsymbol{z}_2, u)\| \leq \gamma_{\widetilde{H}}(|u|) \|\boldsymbol{z}_1 - \boldsymbol{z}_2\| \tag{2.13}$$

Lemma X[17]: Consider a function $s(t) \leq 0$, t $\in [-\delta, +\infty)$, with $\delta \leq 0$, such that

$$\int_{-\delta}^{0} s(\tau) d\tau < +\infty, \tag{2.14}$$

$$s(t) \leq \mu \exp(\bar{\alpha} t) + \gamma \int_{t-\delta}^{t} s(\tau) d\tau, \quad t \geq 0 \tag{2.15}$$

where $\bar{\alpha}$,γ, μ are positive real.

If $\gamma\delta < 1$ then there exists a positive $\alpha < \bar{\alpha}$ such that

$$s(t) < \bar{\mu} \exp(-\alpha t), \quad t \geq 0 \tag{2.16}$$

where

$$\bar{\mu} = \frac{\exp(\alpha\delta)}{1-c}\left(\mu + \gamma\int_{t-\delta}^{t} s(\tau)d\tau\right) \quad (2.17)$$

$$c = \frac{\gamma}{\alpha}(\exp(\alpha\delta) - 1) < 1 \quad (2.18)$$

The proposed Chain Observer for system (2.1)-(2.2) has the following structure:

$$\dot{\hat{\boldsymbol{x}}}_0(t) = \boldsymbol{f}(\hat{\boldsymbol{x}}_0(t)) + \boldsymbol{g}(\hat{\boldsymbol{x}}_0(t))u_0(t) + \boldsymbol{Q}^{-1}(\hat{\boldsymbol{x}}_0(t))\boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{h}(\hat{\boldsymbol{x}}_0(t))) \quad (2.19)$$

$$\begin{aligned}\dot{\hat{\boldsymbol{x}}}_j(t) &= \boldsymbol{f}(\hat{\boldsymbol{x}}_j(t)) + \boldsymbol{g}(\hat{\boldsymbol{x}}_j(t))u_j(t) + \boldsymbol{Q}^{-1}(\hat{\boldsymbol{x}}_j(t))\left\{\exp(\boldsymbol{A}_n\frac{\Delta}{m}j)\boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{h}(\hat{\boldsymbol{x}}_0(t)))\right.\\ &\left.+\sum_{i=0}^{j-1}\exp(\boldsymbol{A}_n\frac{\Delta}{m}(i-j))\left(\boldsymbol{H}(\hat{\boldsymbol{x}}_i(t), u_i(t)) - \boldsymbol{H}(\hat{\boldsymbol{x}}_{i+1}\left(t-\frac{\Delta}{m}\right), u_i(t))\right)\right\},\\ &j = 1, 2, \ldots, m \end{aligned} \quad (2.20)$$

The initial conditions are:

$$\dot{\hat{\boldsymbol{x}}}_0(0) = \hat{\boldsymbol{x}}(-\Delta) \quad (2.21)$$

$$\dot{\hat{\boldsymbol{x}}}_j(\tau) = \hat{\boldsymbol{x}}\left(\tau - \Delta + \frac{j}{m}\Delta\right) \quad \tau \in \left[-\frac{\Delta}{m}, 0\right], \quad j = 1, 2, ..., m \quad (2.22)$$

where m is the length of the chain observer i.e, the chain consists of $m+1$ observer systems. $\hat{\boldsymbol{x}}(\tau) \in [-\Delta, 0]$ is any *a priori* estimate of the state. The variable $\hat{\boldsymbol{x}}_j(\tau)$ is an estimate of the delayed state $\boldsymbol{x}(t - \Delta + \frac{j}{m}\Delta)$, denoted also as $\boldsymbol{x}_j(t)$. These 'm+1' cascaded systems form the links in the "chain observer".

2.2.1 Chain Observer Stability

Theorem 7: For systems (2.1)-(2.2), assume that hypothesis H_2, H_3 are satisfied. Take a positive real, $\tilde{\boldsymbol{u}}_M$ and an integer m such that the Lipschitz coefficient of function $\widetilde{\boldsymbol{H}}(\boldsymbol{z}, u)$ defined in Eq (2.13) and the delay Δ are such that

$$\gamma_{\widetilde{H}}(\tilde{u}_M)\|\exp(\boldsymbol{A}_n\frac{\Delta}{m})\|\frac{\Delta}{m} < 1 \quad (2.23)$$

then there exists a positive α, a positive $u_M \leq \tilde{u}_M$, and a gain vector $\boldsymbol{K}$ for the observer such that if $|u(t)| \leq u_M$ for $t \geq -\Delta$, then

$$\|\boldsymbol{x}(t) - \hat{\boldsymbol{x}}_m(t)\| \leq \nu\exp(-\alpha t) \quad (2.24)$$

where $\boldsymbol{\nu}$ depends on the estimation error in $[-\Delta, 0]$ as:

$$\nu = \nu_1 \parallel \boldsymbol{x}(-\Delta) - \hat{\boldsymbol{x}}_m(-\Delta) \parallel + \nu_2 \int_{-\Delta}^{0} \parallel \boldsymbol{x}(\tau) - \hat{\boldsymbol{x}}_m(\tau) \parallel d\tau \tag{2.25}$$

where ν_1 and ν_2 are suitable positive constants.

If assumption H_1 also holds, then bound u_M on $|u(t)|$ can be equal to $\tilde{u}_M$, given by Eq. (2.23).

The detailed proof of the theorem is reported in Ref. [17]. But, some of the important steps in the stability analysis are as follows:

- The stability of the observer is proved first in the z-coordinates. Hence an expression for the observer equations in the z-coordinates is

$$\begin{aligned} \dot{\hat{\boldsymbol{z}}}_0(t) &= \boldsymbol{A}_n \hat{\boldsymbol{z}}_0(t) + \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_0(t), u_0(t)) + \boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{C}_n \hat{\boldsymbol{z}}_0(t)) \quad t \geq 0 && (2.26) \\ \hat{\boldsymbol{z}}_j(t) &= \exp(\boldsymbol{A}_n \frac{\Delta}{m}) \hat{\boldsymbol{z}}_{j-1}(t) + \int_{t-\frac{\Delta}{m}}^{t} e^{\boldsymbol{A}_n(t-\tau)} \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_j(\tau), u_j(\tau)) d\tau, \quad j = 1, \ldots, m && (2.27) \end{aligned}$$

$$\begin{aligned} \hat{\boldsymbol{z}}_0(0) &= \boldsymbol{\Phi}(\hat{\boldsymbol{x}}(-\Delta)) && (2.28) \\ \hat{\boldsymbol{z}}_j(\tau) &= \boldsymbol{\Phi}\left(\hat{\boldsymbol{x}}\left(\tau - \Delta + \frac{j}{m}\Delta\right)\right) \quad \tau \in \left[-\frac{\Delta}{m}, 0\right], \quad j = 1, 2, \ldots, m && (2.29) \end{aligned}$$

- The observation error in z-coordinates is

$$\boldsymbol{e}_{z,j}(t) = \boldsymbol{z}_j(t) - \hat{\boldsymbol{z}}_j(t) \tag{2.30}$$

Using Lemma XS and Eq. (2.23) the observation error at $j = m$ is shown to be

$$\|\boldsymbol{e}_{z,m}(t)\| \leq \widetilde{\mu}_m \exp(-\alpha t) \tag{2.31}$$

This proves exponential convergence of the observer in the z-coordinates. Using assumption H_2 it follows:

$$\|\boldsymbol{x}(t) - \hat{\boldsymbol{x}}_m(t)\| \leq \tilde{\boldsymbol{\nu}} e^{-\alpha t} \tag{2.32}$$

with

$$\tilde{\nu} = \gamma_\Phi \gamma_{\Phi^{-1}} \left(\lambda^m \mu_0 \|\boldsymbol{x}(-\Delta) - \hat{\boldsymbol{x}}_0(t)\| + \gamma_{\widetilde{H}}(u_M) \sum_{j=1}^{m} \lambda^{m-j} \int_{-\frac{\Delta}{m}}^{0} \|\boldsymbol{x}_j(\tau) - \hat{\boldsymbol{x}}_j(\tau)\| d\tau \right) \tag{2.33}$$

and

$$\nu_1 = \gamma_\Phi \gamma_{\Phi^{-1}} \mu_0 \tag{2.34}$$

$$\nu_2 = \gamma_\Phi \gamma_{\Phi^{-1}} \gamma_{\widetilde{H}}(u_M) \tag{2.35}$$

μ_0 defines the rate of error decay at the zeroth observer and hence the proof.

2.2.2 Advantages and Disadvantages of the Chain Observer

The main advantage of the chain observer is that, if the three assumptions $H_1,\ H_2,\ H_3$, are satisfied, an exponentially stable observer can be realized. The length, m, of the chain depends on the delay magnitude as given by Eq. (2.23). It can be easily simplified to observe states of a linear system with delayed outputs[18].

However, the main problem in this design is the assumption that delay is a known constant. If the system output is time stamped, the clock information can be extracted to ascertain the exact value of the delay. But, given the uncertainties in the communication channel, it can almost be guaranteed that the delay will not be a constant at all instants. Hence it is feasible to assume that the delay is a time-varying quantity.

Second, for a nonlinear system, theoretically infinite number of observers can be used in the chain to estimate the states for any delay magnitude. In this case, the estimation process itself would take considerable amount of time to estimate the current states. But in the case of cooperative control or string stability problems, where the estimated states have to be used for control, the amount of time available for the observer to estimate the states will be limited. Hence, a long chain of observer is not a practical situation. Hence, in this thesis we assume that the length of the chain is limited to two: the zeroth and the first observer.

Third, in the observer stability analysis, the maximum delay that can be tolerated by the observer does not seem to have any relation to the chosen observer gain. We rederive the stability conditions for the nonlinear observer establishing a relation between the delay magnitude and the chosen observer gain $\boldsymbol{K}$.

Finally, in all their works, Germani et al. have assumed only continuous time systems. The concept of the time delay observer is extended to estimate delayed outputs from a discrete-time linear systems.

Summarizing, the following are the contributions of our research:

- Observer analysis for both continuous and discrete time linear systems for both constant known delays and time varying known delays.
- Observer analysis for continuous time nonlinear systems for both constant known delays and time varying known delays and an equation establishing a relation between the delay magnitude and the observer gain.

CHAPTER 3

STATE OBSERVERS FOR LINEAR TIME INVARIANT SYSTEMS WITH DELAYED OUTPUTS

The state space model of the general class of continuous-time single-input single-output (SISO) linear time invariant (LTI) systems considered for observation is

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \quad t \geq \Delta, \quad \boldsymbol{x}(-\Delta) = \bar{\boldsymbol{x}} \tag{3.1}$$

$$\bar{\boldsymbol{y}}(t) = \boldsymbol{C}\boldsymbol{x}(t-\Delta) \tag{3.2}$$

where $\Delta > 0$ is the measurement delay. $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state vector. $u(t) \in \mathbb{R}$ is the system input. $\boldsymbol{A}$ and $\boldsymbol{B}$ are the constant matrices with appropriate dimensions.The un-delayed output is represented as $\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t)$. Though the observer theory is developed for SISO systems, it can easily be extended to multiple-input multiple-output (MIMO) systems by using system matrices of appropriate dimensions.

Similarly, for discrete-time systems, a parallel theory can be extended to estimate the delayed outputs using a discrete chain-observer. The discrete-time equivalent of the continuous-time system is

$$\boldsymbol{x}(k+1) = \boldsymbol{\Phi}\boldsymbol{x}(k) + \boldsymbol{\Gamma}u(k) \tag{3.3}$$

$$\bar{\boldsymbol{y}}(k) = \boldsymbol{C}\boldsymbol{x}(k-n) \tag{3.4}$$

where $n > 0$ is the delay interval. $\boldsymbol{\Phi}$ and $\boldsymbol{\Gamma}$ are constant matrices, $\boldsymbol{\Phi} = \exp(\boldsymbol{A}T)$ and $\boldsymbol{\Gamma} = \int_{t_0}^{t} \exp(A\tau)\boldsymbol{B}u(\tau)dt$. T is the sampling interval.

The general structure of the chain observer, as explained in Chapter 2, consists of two systems referred to as: the zeroth observer and the first observer. The zeroth observer estimates the states at the delayed instant i.e, at $t-\Delta$ (or $k-n$ for discrete system) and the first observer estimates the states at the current instant. The inputs to the zeroth observer are the delayed measurements from the actual system and the delayed control inputs. Using these measurements the zeroth observer estimates the states at the delayed time instant. The inputs to the first observer are the observed states from the zeroth observer along with the delayed measurements and current control inputs. The estimated states from the zeroth observer are then propagated forward by the first observer to obtain the states at the current instant. Thus, current state estimates are obtained from the delayed measurements.

3.1 Observer Stability Analysis

The stability analysis is a two-step process. First, the stability of the zeroth observer is proven and then the stability of first observer is analyzed. The stability analysis reveals accuracy of the estimated states to the actual states. This accuracy is specified by the observation error vector. If the magnitude of error vector is close to zero, it implies that the estimated states follow the true states closely. If the error magnitude is significant, then it implies that the observer states have not converged to the true states.

The observer stability is analyzed for two cases:

- Constant Delay, i.e Δ (n for discrete systems) is a known constant.
- Time-varying Delays. In this case, the delay profile is also assumed to be known, owing to the fact that the system outputs are time stamped.

3.2 Case 1: Constant Delays

3.2.1 Continuous-time LTI systems

The theory of chain observers for continuous-time LTI systems was proposed by Germani et al. [18]. In this section, the design process is explained in detail.

The chain observer dynamics for estimating the delayed measurement is

$$\dot{\hat{\boldsymbol{x}}}(t-\Delta) = \boldsymbol{A}\hat{\boldsymbol{x}}(t-\Delta) + \boldsymbol{B}u(t-\Delta) + \boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{C}\hat{\boldsymbol{x}}(t-\Delta)) \quad (3.5)$$

$$\dot{\hat{\boldsymbol{x}}}_1(t) = \boldsymbol{A}\hat{\boldsymbol{x}}_1(t) + \boldsymbol{B}u(t) + \exp(\boldsymbol{A}\Delta)\boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{C}\hat{\boldsymbol{x}}_1(t)) \quad (3.6)$$

The variable $\hat{\boldsymbol{x}}(t-\Delta)$ denotes the estimate of the states at a time $t-\Delta$ and variable $\hat{\boldsymbol{x}}_1$, the estimate of the states at the current time t. The observer states are initialized as follows

$$\hat{\boldsymbol{x}}(t-\Delta)|_{t=0} = \hat{\boldsymbol{x}}(-\Delta) \quad (3.7)$$

$$\hat{\boldsymbol{x}}_1(0) = \hat{\boldsymbol{x}}(0) \quad (3.8)$$

The gain vector $\boldsymbol{K}$ is chosen such that stable eigen values are assigned to the matrix $\boldsymbol{A} - \boldsymbol{KC}$.

3.2.1.1 Stability analysis of the zeroth observer

The observation error is defined as

$$\boldsymbol{\eta}_0(t) = \boldsymbol{x}(t-\Delta) - \hat{\boldsymbol{x}}(t-\Delta) \quad (3.9)$$

where $\boldsymbol{x}(t-\Delta)$ are the system states at $t-\Delta$. The time derivative of $\boldsymbol{\eta}_0(t)$ is

$$\begin{aligned}
\dot{\boldsymbol{\eta}}_0(t) &= \dot{\boldsymbol{x}}(t-\Delta) - \dot{\hat{\boldsymbol{x}}}(t-\Delta) \\
\dot{\boldsymbol{\eta}}_0(t) &= \{\boldsymbol{A}\boldsymbol{x}(t-\Delta) + \boldsymbol{B}u(t-\Delta)\} - \{\boldsymbol{A}\hat{\boldsymbol{x}}(t-\Delta) + \boldsymbol{B}u(t-\Delta) + \boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{C}\hat{\boldsymbol{x}}(t-\Delta))\} \\
\dot{\boldsymbol{\eta}}_0(t) &= (\boldsymbol{A} - \boldsymbol{KC})(\boldsymbol{x}(t-\Delta) - \hat{\boldsymbol{x}}(t-\Delta)) \\
\dot{\boldsymbol{\eta}}_0(t) &= \boldsymbol{A}_m\boldsymbol{\eta}_0(t) \quad (3.10)
\end{aligned}$$

where $\boldsymbol{A}_m = \boldsymbol{A} - \boldsymbol{KC}$ is Hurwitz. The solution of Eq. (3.10) is

$$\boldsymbol{\eta}_0(t) = \exp(\boldsymbol{A}_m t)\boldsymbol{\eta}_0(0) \quad (3.11)$$

This proves exponential convergence of the delayed state estimates.

3.2.1.2 Stability analysis of the first observer

The observation error is defined as

$$\boldsymbol{\eta}_1(t) = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}_1(t) \tag{3.12}$$

The expression for $\hat{\boldsymbol{x}}_1(t)$ can be given as

$$\hat{\boldsymbol{x}}_1(t) = \exp(\boldsymbol{A}\Delta))\hat{\boldsymbol{x}}(t-\Delta) + \int_{t-\Delta}^{t} \exp(\boldsymbol{A}(t-\tau))\boldsymbol{B}u(\tau)\mathrm{d}\tau \tag{3.13}$$

Eq. (3.13) can be verified by differentiating $\hat{\boldsymbol{x}}_1(t)$,

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}}_1(t) &= \exp(\boldsymbol{A}\Delta)\dot{\hat{\boldsymbol{x}}}(t-\Delta) \\ &\quad + \boldsymbol{A}\int_{t-\Delta}^{t} \exp(\boldsymbol{A}(t-\tau))\boldsymbol{B}u(\tau)\mathrm{d}\tau + \boldsymbol{B}u(t) - \exp(\boldsymbol{A}(t-\tau))\boldsymbol{B}u(t-\Delta) \end{aligned} \tag{3.14}$$

From Eq. (3.13), $\int_{t-\Delta}^{t} \exp(\boldsymbol{A}(t-\tau))\boldsymbol{B}u(\tau)\mathrm{d}\tau = \hat{\boldsymbol{x}}_1(t) - \exp(\boldsymbol{A}\Delta)\hat{\boldsymbol{x}}(t-\Delta)$. Substituting for the integral,

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}}_1(t) &= \exp(\boldsymbol{A}\Delta)\dot{\hat{\boldsymbol{x}}}(t-\Delta) + \boldsymbol{A}\left\{\hat{\boldsymbol{x}}_1(t) - \exp(\boldsymbol{A}\Delta)\hat{\boldsymbol{x}}(t-\Delta)\right\} \\ &\quad + \boldsymbol{B}u(t) - \exp(\boldsymbol{A}(t-\tau)\}\boldsymbol{B}u(t-\Delta) \\ &= \boldsymbol{A}\hat{\boldsymbol{x}}_1(t) + \boldsymbol{B}u(t) + \exp(\boldsymbol{A}\Delta))\{\dot{\hat{\boldsymbol{x}}}(t-\Delta) - \boldsymbol{A}\hat{\boldsymbol{x}}(t-\Delta) + \boldsymbol{B}u(t-\Delta)\} \end{aligned} \tag{3.15}$$

From Eq. (3.5), $\dot{\hat{\boldsymbol{x}}}(t-\Delta) - \boldsymbol{A}\hat{\boldsymbol{x}}(t-\Delta) + \boldsymbol{B}u(t-\Delta) = \boldsymbol{K}(\bar{\boldsymbol{y}} - \boldsymbol{C}\hat{\boldsymbol{x}}(t-\Delta))$. Using this result, (3.15) can be re-written as

$$\dot{\hat{\boldsymbol{x}}}_1(t) = \boldsymbol{A}\hat{\boldsymbol{x}}_1(t) + \boldsymbol{B}u(t) + \boldsymbol{K}(\bar{\boldsymbol{y}} - \boldsymbol{C}\hat{\boldsymbol{x}}(t-\Delta)) \tag{3.16}$$

which is the expression for observer 1. Similarly, state $\boldsymbol{x}(t)$ can be written as

$$\boldsymbol{x}(t) = \exp(\boldsymbol{A}\Delta))\boldsymbol{x}(t-\Delta) + \int_{t-\Delta}^{t} \exp(\boldsymbol{A}(t-\tau))\boldsymbol{B}u(\tau)\mathrm{d}\tau \tag{3.17}$$

Subtracting Eq. (3.13) from (3.17)

$$\boldsymbol{\eta}_1(t) = \exp(\boldsymbol{A}\Delta)\boldsymbol{\eta}_0(t) \tag{3.18}$$

This proves exponential convergence to zero of the observation error. It is a very interesting result, since convergence is guaranteed for any delay Δ, as long as it is known and constant. The convergence results change significantly when the delay is not a constant. The implications of a time varying Δ on the observer stability is discussed in later sections.

3.2.1.3 Simulation Results

The performance of the observer is examined for a string stability problem[26]. The system consists of a string of automated vehicles traveling at highway speeds with very small intervehicular spacings. Currently, with human drivers, the throughput on highways is 2000 vehicles per lane per hour on a given highway. But, studies have shown that [30] that maintaining such a formation increases the throughput to more than 6000 vehicles per lane per hour on a given highway.

The safety of the platoon depends heavily on the timely communication between the vehicles. Fig. (2.1) shows a typical scenario with intervehicular communication. Any delays in information exchange could potentially lead to disastrous situations, like collisions between the vehicles.

For simulation, three vehicles with identical dynamics moving along a straight line is considered. The control strategy is to maintain a constant spacing with respect to the lead vehicle in the formation. It is assumed that two vehicles which are following the lead vehicle are within the communication range R_{lead} of the lead vehicle. The position information from the lead vehicle in the platoon is transmitted to all the preceding vehicles. The control law for each of the the preceding vehicles is designed using the position of the lead vehicle and its own position. This enables the vehicles to align themselves and hence maintain a constant distance between each other and also from the lead vehicle.

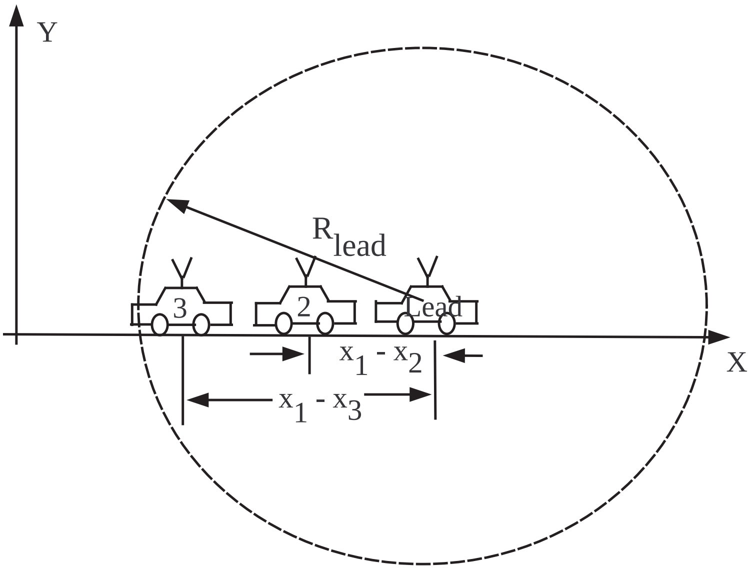

Figure 3.1 Constant spacing strategy using the lead vehicle's position information

Due to physical separation between the vehicles and low bandwidth communication channels, the output of the lead vehicle is transmitted to other vehicles after a delay. The further away the vehicle in the platoon, greater is the delay magnitude. Fig. (3.1) represents the constant spacing strategy based on the position information from the lead vehicle.

In the Fig. (3.1), it can be seen that both vehicles 2 and 3 are within the communication range of the lead vehicle. Hence, they can receive the information from the lead vehicle. However, one of the issues that is not addressed here is the possibility of the vehicles going out of this communication range, in which case, a different kind of spacing strategy needs to be developed to maintain constant separation.

The dynamics of the vehicles used for simulation is

$$\dot{x}_i(t) = v_i(t) \tag{3.19}$$

$$\dot{v}_i(t) = u_i(t) \tag{3.20}$$

where $\boldsymbol{x}_i(t)$ is the position of the $i-th$ vehicle. $\boldsymbol{v}_i(t)$ is the velocity and $\boldsymbol{u}_i(t)$ is the control input.

Defining the states of the system as

$$\boldsymbol{x}(t) = [x_1 \; x_2]^T = [x_i \; v_i]^T \tag{3.21}$$

The state space representation is

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{3.22}$$

$$\bar{\boldsymbol{y}}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}(t - \Delta) \tag{3.23}$$

$\bar{\boldsymbol{y}}(t)$ is the delayed output from the lead vehicle. Since vehicles 2 and 3 track the position of the lead vehicle. The control law for each of the vehicles is based on the position error calculated with respect to the lead vehicle. The error vector is defined as

$$e = x_1 - x_i, \quad i = 2, 3 \tag{3.24}$$

A desired tracking error dynamics is prescribed as

$$\ddot{e} + 2\zeta\omega_n \dot{e} + \omega_n^2 e = 0 \tag{3.25}$$

Substituting the values for $\ddot{e}$, $\dot{e}$ and e and simplifying for the input u_i,

$$u_i = u_1 - 2\zeta\omega_n \dot{e} - \omega_n^2 e, \quad i = 2, 3 \tag{3.26}$$

At vehicles 2 and 3, only delayed position information from the lead vehicle is available. The current position information is estimated using the chain observer. The initial conditions for the vehicles and the observer are

$$\boldsymbol{x}_{lead}(0) = \begin{bmatrix} 600 \\ 10 \end{bmatrix}, \; \boldsymbol{x}_{vehicle2}(0) = \begin{bmatrix} 500 \\ 10 \end{bmatrix}, \; \boldsymbol{x}_{vehicle3}(0) = \begin{bmatrix} 400 \\ 10 \end{bmatrix} \tag{3.27}$$

$$\hat{\boldsymbol{x}}_0(\tau) = \begin{bmatrix} 600 \\ 10 \end{bmatrix}, \; \hat{\boldsymbol{x}}_1(0) = \begin{bmatrix} 600 \\ 10 \end{bmatrix}, \; \tau \in [-\Delta, 0] \tag{3.28}$$

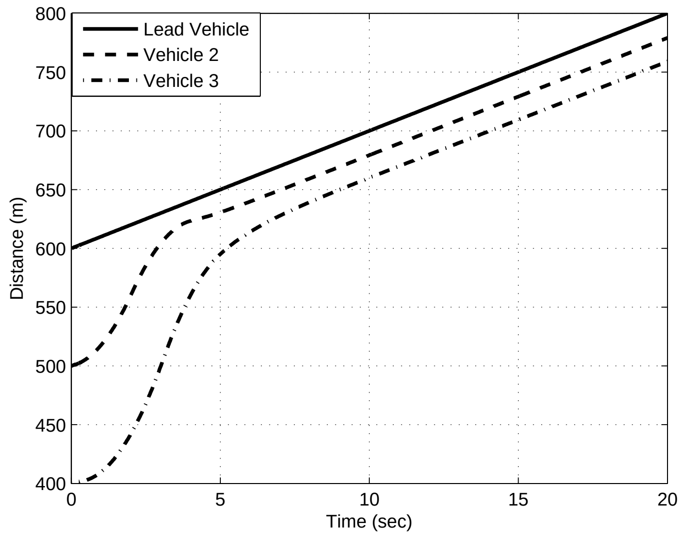

Figure 3.2 Positions of the three vehicles

The gain matrix $\boldsymbol{K}$ is chosen such that the eigen values of the matrix $\boldsymbol{A} - \boldsymbol{KC}$ are at -10 and -20 respectively. The position from lead vehicle arrives at vehicle 2 after a delay of $\Delta_1 = 0.1$ *sec* and at vehicle 3 after a delay of $\Delta_2 = 0.3$ *sec*.

The control law is implemented to maintain the vehicle 2 at a distance of $20m$ from the lead vehicle and vehicle 3 at $40m$ from the lead vehicle. The simulation results are shown below

Fig. (3.2) shows the evolution of the position vector with time of the three vehicles. Fig. (3.3) is the plot of the error in position between the lead vehicles and the preceding vehicles. From the two plots it can be observed that the vehicles have successfully implemented the control laws. In a very short time, they have reached the steady state positions ie., 20 m and 30 m respectively for vehicles 2 and 3 from the lead vehicles. This is possible due to the estimation of current position information from the delayed outputs by the chain observer.

The observation error vectors for the two vehicles are as shown in Figs. (3.4) and (3.4) respectively. Similar observer dynamics are used in vehicles 2 and 3. From the two plots it can be seen that the observer performance is expected. In spite of the large initial

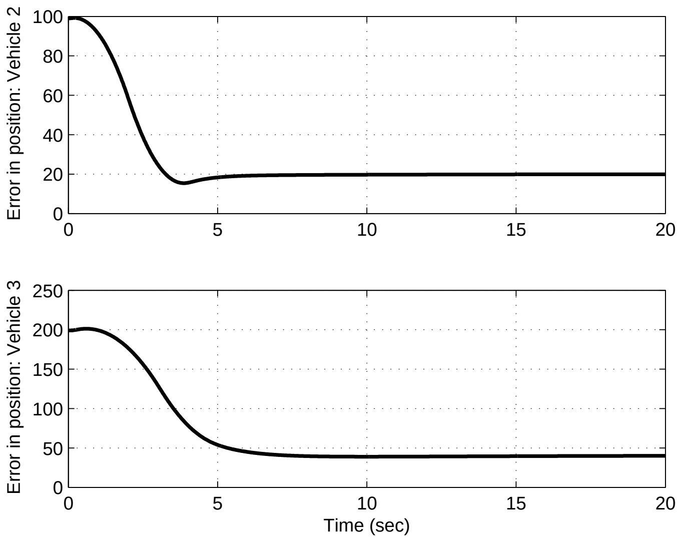

Figure 3.3 Error in position

condition errors, the observer is able to converge to the steady state values quickly and hence the control law using the estimated states is accurate.

Fig. (3.6) shows the control inputs to vehicle 2 and 3 respectively. The control inputs to the two vehicles is the acceleration profiles of the two vehicles. The acceleration profile of the two vehicles is similar. Starting at very large initial position errors, the two vehicles accelerate to move closer to the lead vehicles. But, overshoots in the dynamic response of the system cause the vehicles to move closer than the prescribed spacing. Hence, the control law is triggered again to decelerate the vehicles and reach the prescribed steady state conditions.

3.2.2 Discrete-time LTI systems

In this section, the chain observer theory is extended to discrete LTI system with delayed outputs.

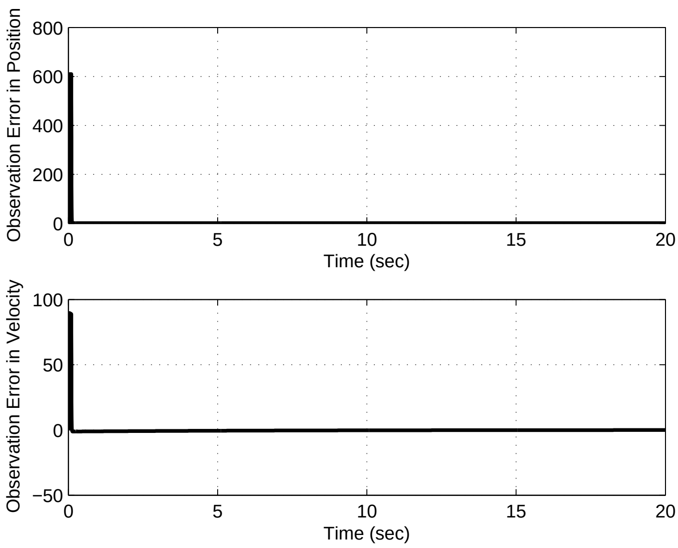

Figure 3.4 Observation Error in states of the Chain Observer at System 2

The structure of the time-delay observer is similar to that used for continuous-time systems. The observer equations are

$$\hat{\boldsymbol{x}}(k-n+1) = \boldsymbol{\Phi}\hat{\boldsymbol{x}}(k-n) + \boldsymbol{\Gamma}u(k-n) + \boldsymbol{\kappa}(\bar{\boldsymbol{y}}(k) - \boldsymbol{C}\hat{\boldsymbol{x}}(k-n)) \tag{3.29}$$

$$\hat{\boldsymbol{x}}_1(k+1) = \boldsymbol{\Phi}\hat{\boldsymbol{x}}_1(k) + \boldsymbol{\Gamma}u(k) + \boldsymbol{\Phi}^n\boldsymbol{\kappa}(\bar{\boldsymbol{y}}(k) - \boldsymbol{C}\hat{\boldsymbol{x}}(k-n)) \tag{3.30}$$

where $\hat{\boldsymbol{x}}(k-n)$ is an estimate of the state at the instant $k-n$ and $\hat{\boldsymbol{x}}_1(k)$ is an estimate of the state at k. $\boldsymbol{\kappa}$ is the gain matrix, which is chosen such that the eigen values of $\boldsymbol{\Phi} - \boldsymbol{\kappa}\boldsymbol{C}$ lie within the unit circle. The observer states are initialized as follows

$$\hat{\boldsymbol{x}}(k-n)|_{k=0} = \hat{\boldsymbol{x}}(-n) \tag{3.31}$$

$$\hat{\boldsymbol{x}}_1(0) = \hat{\boldsymbol{x}}(0) \tag{3.32}$$

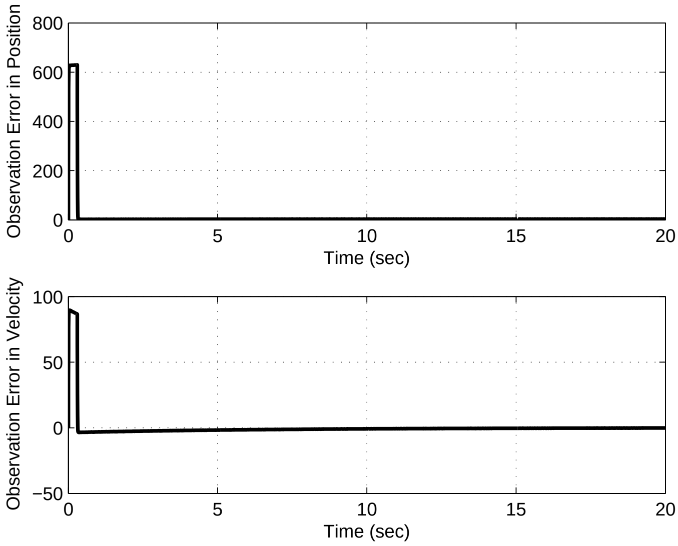

Figure 3.5 Observation Error in states of the Chain Observer at System 3

3.2.2.1 Stability analysis of the zeroth observer

The estimation errors are defined as:

$$\begin{aligned}
\boldsymbol{\eta}_0(k+1) &= \boldsymbol{x}(k-n+1) - \hat{\boldsymbol{x}}(k-n+1) \\
\boldsymbol{\eta}_0(k+1) &= \boldsymbol{\Phi}\boldsymbol{x}(k-n) + \boldsymbol{\Gamma}u(k-n) - \{\boldsymbol{\Phi}\hat{\boldsymbol{x}}(k-n) + \boldsymbol{\Gamma}u(k-n) + \boldsymbol{\kappa}(\bar{\boldsymbol{y}}(k) - \boldsymbol{C}\hat{\boldsymbol{x}}(k-n))\} \\
\boldsymbol{\eta}_0(k+1) &= (\boldsymbol{\Phi} - \boldsymbol{\kappa}\boldsymbol{C})(\boldsymbol{x}(k-n) - \hat{\boldsymbol{x}}(k-n)) \\
\boldsymbol{\eta}_0(k+1) &= \boldsymbol{\Phi}_m\boldsymbol{\eta}_0(k) \qquad (3.33)
\end{aligned}$$

Since the matrix $\boldsymbol{\Phi}_m = \boldsymbol{\Phi} - \boldsymbol{\kappa}\boldsymbol{C}$ is Schur stable, the error dynamics are exponentially stable.

3.2.2.2 Stability analysis of the first observer

The observation error for the first observer is defined as

$$\begin{aligned}
\boldsymbol{\eta}_1(k+1) &= \boldsymbol{x}(k+1) - \hat{\boldsymbol{x}}(k+1) \\
\boldsymbol{\eta}_1(k+1) &= \boldsymbol{\Phi}\boldsymbol{x}(k) + \boldsymbol{\Gamma}u(k) - \{\boldsymbol{\Phi}\hat{\boldsymbol{x}}_1(k) + \boldsymbol{\Gamma}u(k) + \boldsymbol{\Phi}^n\boldsymbol{\kappa}(\bar{\boldsymbol{y}}(k) - \boldsymbol{C}\hat{\boldsymbol{x}}(k-n))\} \\
\boldsymbol{\eta}_1(k+1) &= \boldsymbol{\Phi}\boldsymbol{\eta}_1(k) - \boldsymbol{\Phi}^n\boldsymbol{\kappa}\boldsymbol{C}\boldsymbol{\eta}_0(k) \qquad (3.34)
\end{aligned}$$

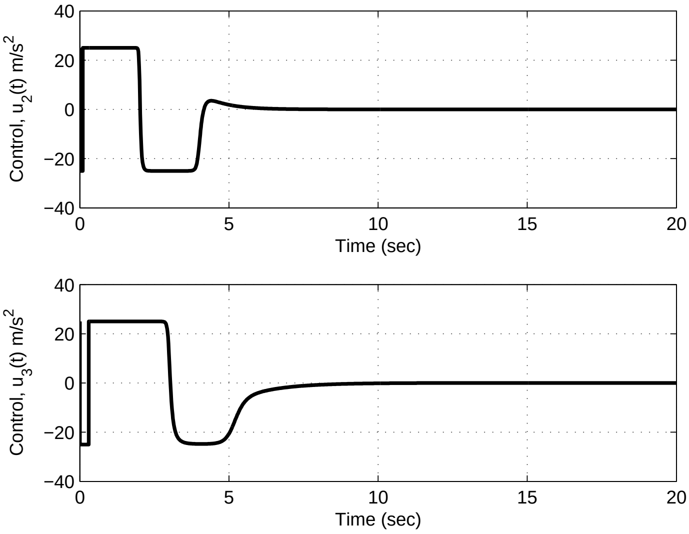

Figure 3.6 Control inputs of Vehicles 2 and 3 respectively.

Form the definition of state transition matrix,

$$\boldsymbol{\eta}_1(k) = \boldsymbol{\Phi}^n \boldsymbol{\eta}_0(k) \tag{3.35}$$

Using this property and substituting for $\boldsymbol{\eta}_1(k)$ in Eq.(3.89),

$$\begin{aligned} \boldsymbol{\eta}_1(k+1) &= \boldsymbol{\Phi}.\boldsymbol{\Phi}^n \boldsymbol{\eta}_0(k) - \boldsymbol{\Phi}^n \boldsymbol{\kappa} \boldsymbol{C} \boldsymbol{\eta}_0(k) \\ \boldsymbol{\eta}_1(k+1) &= \boldsymbol{\Phi}^n [\boldsymbol{\Phi} - \boldsymbol{\kappa} \boldsymbol{C}] \boldsymbol{\eta}_0(k) \end{aligned} \tag{3.36}$$

Using Eq.(3.33),

$$\boldsymbol{\eta}_1(k+1) = \boldsymbol{\Phi}^n \boldsymbol{\eta}_0(k+1) \tag{3.37}$$

The error dynamics of the zeroth observer is shown to be exponentially stable. Additionally, if Φ is also stable, i.e, its eigen values lie within the unit circle, then the term $\boldsymbol{\Phi}^n$, serves as a simple scaling factor to the exponentially decaying $\boldsymbol{\eta}_0(k+1)$. This proves the exponential stability of observer 1.

3.2.2.3 Simulation Results

The performance of the observer is again examined for a string stability problem [26] for discrete-time systems. The situation used for simulation is similar to that explained in Section (3.2.1.3). The string consists of three vehicles, a lead and two predecessors as shown in Fir. (3.1). It is also assumed that the two followers always remain within the communication range of the lead vehicle.

The discrete equivalent of the continuous-time state space model of the vehicles is obtained by using the *c2d* function in *MATLAB*. The sampling interval is 0.25 *sec.*

$$\boldsymbol{x}(k) = \begin{bmatrix} 1 & 0.25 \\ 0 & 1 \end{bmatrix} \boldsymbol{x}(k-1) + \begin{bmatrix} 0.03125 \\ 0.25 \end{bmatrix} \boldsymbol{u}(k) \tag{3.38}$$

$$\bar{\boldsymbol{y}}(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}(k-n) \tag{3.39}$$

$\bar{\boldsymbol{y}}(k)$ is the delayed output of the lead vehicle. As in the previous case, three vehicles with identical dynamics traveling along a straight line with constant spacing are simulated. Vehicles 2 and 3 track the position of the lead vehicle. The error vector is defined as

$$\boldsymbol{e}_x(k) = \boldsymbol{x}_1(k) - \boldsymbol{x}_i(k), \quad i = 2, 3 \tag{3.40}$$

The desired tracking error dynamics valent controller Eq. (3.26). The prescribed control law is

$$\boldsymbol{u}_i(k) = \boldsymbol{u}_{lead}(k) - k_1(\boldsymbol{x}_{2,i}(k) - \boldsymbol{x}_{2,lead})(k) - k_2\boldsymbol{e}(k), \quad i = 2, 3 \tag{3.41}$$

The initial conditions for the vehicles and the observer are

$$\boldsymbol{x}_{lead}(0) = \begin{bmatrix} 600 \\ 10 \end{bmatrix}, \ \boldsymbol{x}_{vehicle2}(0) = \begin{bmatrix} 500 \\ 10 \end{bmatrix}, \ \boldsymbol{x}_{vehicle3}(0) = \begin{bmatrix} 400 \\ 10 \end{bmatrix} \tag{3.42}$$

$$\hat{\boldsymbol{x}}_0(\tau) = \begin{bmatrix} 600 \\ 10 \end{bmatrix}, \ \hat{\boldsymbol{x}}_1(0) = \begin{bmatrix} 600 \\ 10 \end{bmatrix}, \ \tau \in [-\Delta, 0] \tag{3.43}$$

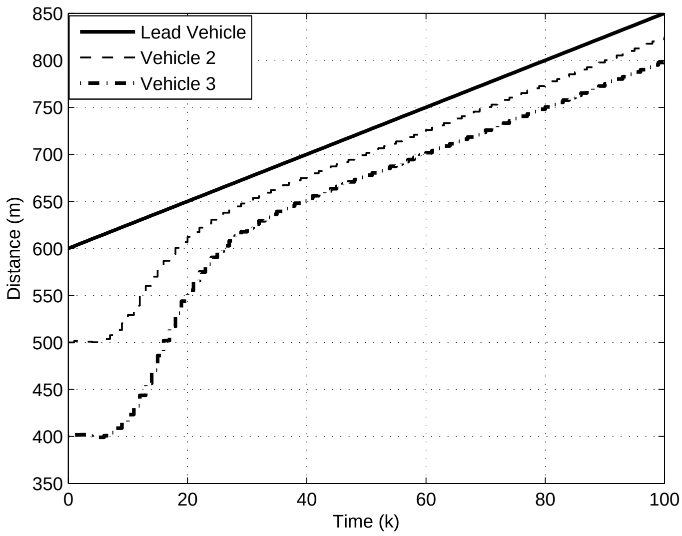

Figure 3.7 Positions of the three vehicles

The gain matrix $\boldsymbol{\kappa}$ is chosen such that the eigen values of the matrix $\boldsymbol{\Phi} - \boldsymbol{\kappa C}$ are at 0.9 and 0.5 respectively. The position information from the lead vehicle arrives at vehicle 2 after a delay of $n = 2$ *sample* and at vehicle 3 after a delay of $n = 3$ *samples*.

The control law is implemented to maintain the vehicle 2 at a distance of 25 m from the lead vehicle and vehicle 3 at 50 m from the lead vehicle. The simulation results are shown below

Fig. (3.7) shows the evolution of the positions of the three vehicles with time. Fig. (3.8) shows the error in position. Initial error in positions of the two vehicles is 100 m and 200 m respectively. But with application of the control input, the vehicles reach the steady state conditions of 25 m and 50 m respectively.

The observation errors in the two states of both the vehicles is plotted in Fig. (3.9) and (3.9) respectively. The performance of the discrete chain observer is similar to its continuous-time counterpart. The chain observer is able to track the actual states accurately and hence the observation error decays to zero quickly from the initial condition errors, as shown in the two plots.

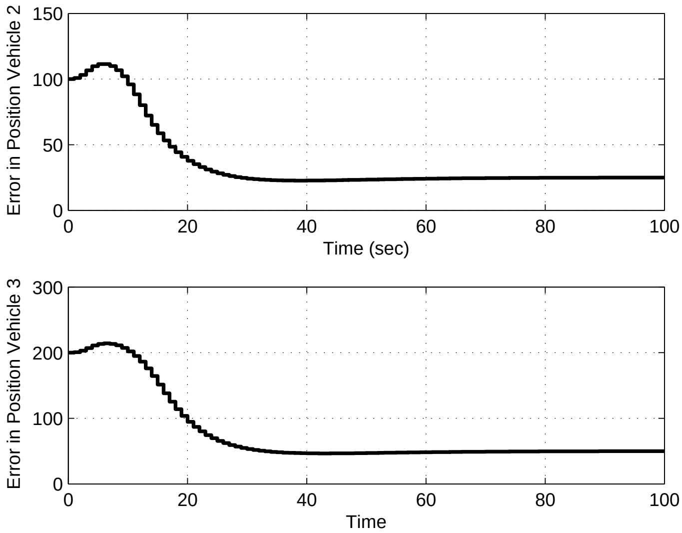

Figure 3.8 Error in position

Fig. (3.11) is the plot the control inputs to vehicle 2 and 3 respectively. The performance of the string is similar to that in the continuous case. Both the vehicles converge and maintain the prescribed spacing from the lead vehicle.

3.3 Case 2: Time-varying Delays

3.3.1 Continuous-time LTI systems

In this section, the observer theory is proposed for the case when the measurement delay, Δ is a function of time, i.e, its magnitude varies randomly with time. The range of delays is $0 < \Delta(t) \leq \tilde{\Delta}$, $\tilde{\Delta}$ is the upper bound on all the delays. The output delay is piecewise constant, i.e, the value Δ_i is a constant in the interval $(t_{i-1},\ t_i)$. For the next interval $(t_i,\ t_{i+1})$, it assumes a different value. Δt_i is the width of the interval over which Δ_i is a constant i.e, $\Delta t_i = t_i - t_{i-1}$. The delay profile is assumed to be known at the observer.

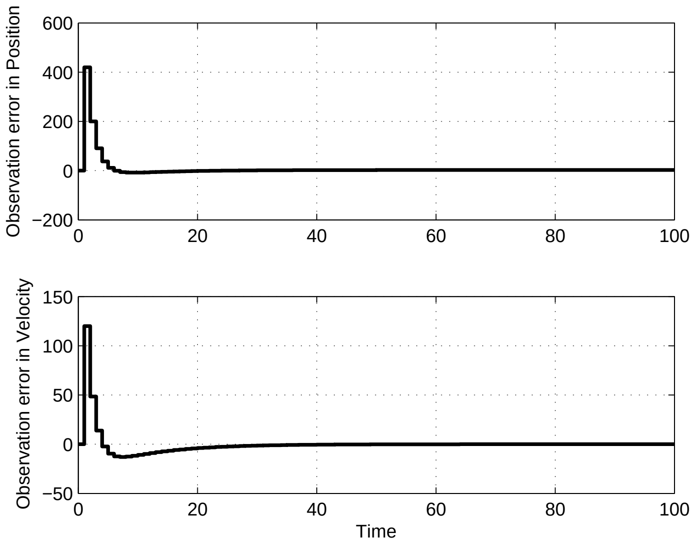

Figure 3.9 Observation Error in states of the Chain Observer at System 2

3.3.1.1 Chain observer of state estimation of output delayed LTI systems

Motivated by a construction similar to the one in Ref. [18], the following structure for the observer dynamics is proposed for estimating the states using the delayed output measurements

$$\dot{\hat{\boldsymbol{x}}}(t-\Delta) = \boldsymbol{A}\hat{\boldsymbol{x}}(t-\Delta) + \boldsymbol{B}u(t-\Delta) + \boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{C}\hat{\boldsymbol{x}}(t-\Delta)) \tag{3.44}$$

$$\dot{\hat{\boldsymbol{x}}}_1(t) = \boldsymbol{A}\hat{\boldsymbol{x}}_1(t) + \boldsymbol{B}u(t) + \exp(\boldsymbol{A}\Delta(t))\boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{C}\hat{\boldsymbol{x}}(t-\Delta)) \tag{3.45}$$

The variable $\hat{\boldsymbol{x}}(t-\Delta)$ denotes the estimate of the states at a time $t-\Delta$ and variable $\hat{\boldsymbol{x}}$, the estimate of the states at the current time t. The observer states are initialized as follows

$$\hat{\boldsymbol{x}}(t-\Delta)|_{t=0} = \hat{\boldsymbol{x}}(-\Delta_1) \tag{3.46}$$

$$\hat{\boldsymbol{x}}_1(0) = \hat{\boldsymbol{x}}(0) \tag{3.47}$$

Δ_1 is the output delay magnitude in the interval $t_0 \leq t < t_1$.

The observer dynamics in Eq. (3.44) is denoted as the zeroth observer and the dynamics in Eq. (3.45) as the first observer.

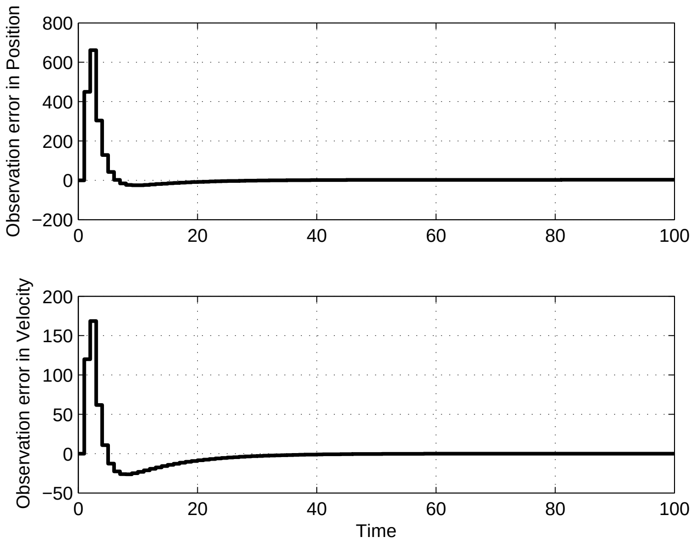

Figure 3.10 Observation Error in states of the Chain Observer at System 3

3.3.1.2 Stability analysis of the zeroth observer

In this section, the effects of the change in the delay at specific instants of time on the observer error dynamics is analyzed.

First, the observation error is defined as

$$\boldsymbol{\eta}_0(t) = \boldsymbol{x}(t-\Delta) - \hat{\boldsymbol{x}}(t-\Delta) \tag{3.48}$$

The time derivative of $\boldsymbol{\eta}_0(t)$ in the interval over which the delay is a constant is

$$\begin{aligned}
\dot{\boldsymbol{\eta}}_0(t) &= \dot{\boldsymbol{x}}(t-\Delta) - \dot{\hat{\boldsymbol{x}}}(t-\Delta) && (3.49)\\
\dot{\boldsymbol{\eta}}_0(t) &= \boldsymbol{A}\boldsymbol{x}(t-\Delta) + \boldsymbol{B}u(t-\Delta) - \{\boldsymbol{A}\hat{\boldsymbol{x}}(t-\Delta) + \boldsymbol{B}u(t-\Delta) + \boldsymbol{K}\boldsymbol{C}\boldsymbol{\eta}_0(t)\}\\
\dot{\boldsymbol{\eta}}_0(t) &= \boldsymbol{A}_m\boldsymbol{\eta}_0(t) && (3.50)
\end{aligned}$$

Note, $\boldsymbol{A}_m = \boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$ is Hurwitz.

Next, the error in observation due to changes in the delay magnitude is analyzed.

- At $t = t_1$ the zeroth observer estimates the states at the instant $t_1 - \Delta_1$, while the first observer estimates the states at t_1.
- At the same instant, the output delay changes to a new value, say Δ_2.

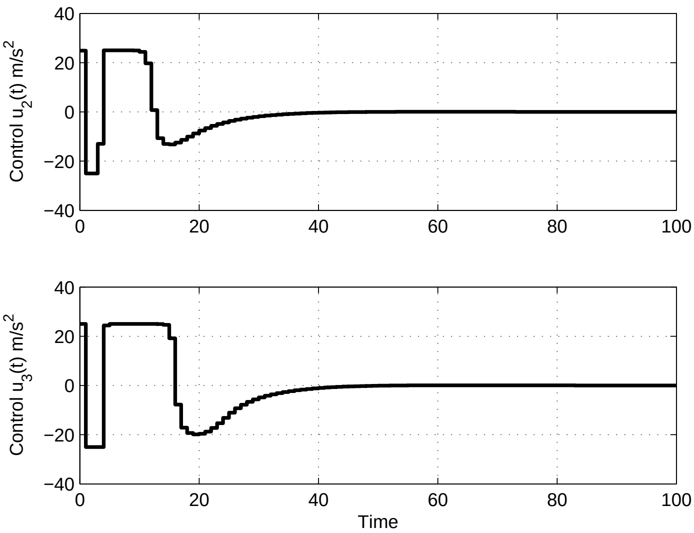

Figure 3.11 Control inputs of Vehicles 2 and 3 respectively.

- This implies that the zeroth observer would now have to start estimating the states from $t_1 - \Delta_2$. This is due to the fact that, with the change in Δ, the initial condition for the zeroth observer changes correspondingly for the interval over which this new delay value remains a constant i.e, at $t = 0$, the initial condition for the zeroth observer is $\hat{\boldsymbol{x}}(-\Delta_1)$ in the interval $[0, t_1]$ and with the change in delay at $t = t_1$, the initial condition for the zeroth observer is $\hat{\boldsymbol{x}}(t_1 - \Delta_2)$ in the interval $[t_1,\ t_2]$.
- The estimated values for this interval $[t_1,\ t_2]$ need not be specified explicitly because it is available from the output of first observer.
- The zeroth observer now starts estimating the states as though the they are propagating from t_1 with an initial condition starting from $t - \Delta_2$.
- Hence there is a finite "jump" in the observer states from $t_1 - \Delta_1$ to $t_1 - \Delta_2$. This contributes to the error in the observation which is equal to $\hat{\boldsymbol{x}}(t_1 - \Delta_2) - \hat{\boldsymbol{x}}(t_1 - \Delta_1)$.

Differentiating $\hat{\boldsymbol{x}}(t_1-\Delta_2)-\hat{\boldsymbol{x}}(t_1-\Delta_1)$ and substituting the observer dynamics,

$$\begin{aligned}
\dot{\hat{\boldsymbol{x}}}(t_1-\Delta_2)-\dot{\hat{\boldsymbol{x}}}(t_1-\Delta_1) &= \boldsymbol{A}\hat{\boldsymbol{x}}(t_1-\Delta_2)+\boldsymbol{B}u(t_1-\Delta_2)+\boldsymbol{K}(\bar{\boldsymbol{y}}(t_1)-\boldsymbol{C}\hat{\boldsymbol{x}}(t_1-\Delta_2)) \\
&\quad -\{\boldsymbol{A}\hat{\boldsymbol{x}}(t_1-\Delta_1)+\boldsymbol{B}u(t_1-\Delta_1)+\boldsymbol{K}(\bar{\boldsymbol{y}}(t_1)-\boldsymbol{C}\hat{\boldsymbol{x}}(t_1-\Delta_1))\} \\
\dot{\hat{\boldsymbol{x}}}(t_1-\Delta_2)-\dot{\hat{\boldsymbol{x}}}(t_1-\Delta_1) &= (\boldsymbol{A}-\boldsymbol{KC})(\hat{\boldsymbol{x}}(t_1-\Delta_2)-\hat{\boldsymbol{x}}(t_1-\Delta_1))+\boldsymbol{B}(u(t_1-\Delta_2)-u(t_1-\Delta_1)) \\
\dot{\hat{\boldsymbol{x}}}(t_1-\Delta_2)-\dot{\hat{\boldsymbol{x}}}(t_1-\Delta_1) &= \boldsymbol{A}_m\boldsymbol{\delta}_{\hat{\boldsymbol{x}}1}(t)+\boldsymbol{B}\boldsymbol{\delta}_{\boldsymbol{u}1}(t) \qquad (3.51)
\end{aligned}$$

Generalizing Eq.(3.51), at any instant $t=t_i$, when the magnitude of the output delay changes, the error in observer due to the "jump" in initial conditions can be expressed as

$$\dot{\hat{\boldsymbol{x}}}(t_i-\Delta_{i+1})-\dot{\hat{\boldsymbol{x}}}(t_i-\Delta_i)=\boldsymbol{A}_m\boldsymbol{\delta}_{\hat{\boldsymbol{x}}i}(t)+\boldsymbol{B}\boldsymbol{\delta}_{\boldsymbol{u}i}(t) \qquad (3.52)$$

where Δ_i is the delay in the interval $t_{i-1}\leq t<t_i$ and Δ_{i+1} is the delay in the next interval $t_i\leq t<t_{i+1}$. $\boldsymbol{\delta}_{\hat{\boldsymbol{x}}i}(t)=\hat{\boldsymbol{x}}(t_i-\Delta_{i+1})-\hat{\boldsymbol{x}}(t_i-\Delta_i)$ and $\boldsymbol{\delta}_{ui}(t)=\boldsymbol{u}(t_i-\Delta_{i+1})-\boldsymbol{u}(t_i-\Delta_i)$. However, this error exists assuming that the observer 1 has not yet converged to the true states at $t=t_i-\Delta_{i+1}$.

Substituting Eq. (3.52) in Eq.(3.49), for any $t=t_i$ and simplifying,

$$\dot{\boldsymbol{\eta}}_0(t_i)=\boldsymbol{A}_m\boldsymbol{\eta}_0(t_i)+\boldsymbol{A}_m\boldsymbol{\delta}_{\hat{\boldsymbol{x}}i}(t)+\boldsymbol{B}\boldsymbol{\delta}_{ui}(t) \qquad (3.53)$$

Therefore, the observation error dynamics can be written as

$$\dot{\boldsymbol{\eta}}_0(t)=\begin{cases}\boldsymbol{A}_m\boldsymbol{\eta}_0(t)+\boldsymbol{A}_m\boldsymbol{\delta}_{\hat{\boldsymbol{x}}i}(t)+\boldsymbol{B}\boldsymbol{\delta}_{ui}(t), & \text{at } t>t_i \\ \\ \boldsymbol{A}_m\boldsymbol{\eta}_0(t) & \text{for } t\in[t_{i-1},\ t_i)\end{cases} \qquad (3.54)$$

The delay profile assumed at the observer is shown in Fig.(3.12).

Next, a general expression for $\boldsymbol{\eta}_0(t)$ for any instant $t=t_N$, $N=0,1,2,\ldots,\infty$ at which the delay magnitude changes is derived. The procedure is as follows:

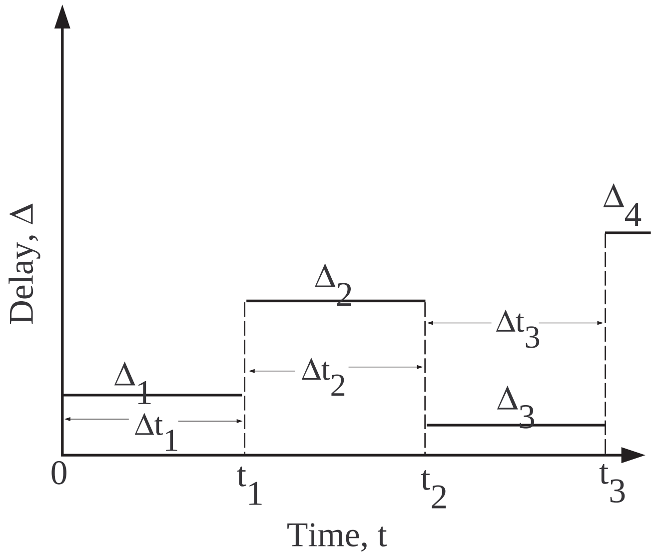

Figure 3.12 Profile of time-varying delay.

The solution of the dynamic equation (3.54) at $t = t_1$ is

$$\begin{aligned}
\boldsymbol{\eta}_0(t_1) &= \exp(\boldsymbol{A}_m(t_1 - 0))\boldsymbol{\eta}_0(0) + \int_0^{t_1} \exp(\boldsymbol{A}_m(t_1 - s)) \left\{\boldsymbol{A}_m \boldsymbol{\delta}_{\hat{\boldsymbol{x}}1}(s) + \boldsymbol{B}\boldsymbol{\delta}_{u1}(s)\right\} ds \\
\|\boldsymbol{\eta}_0(t_1)\| &\leq \|\exp(\boldsymbol{A}_m \Delta t_1)\boldsymbol{\eta}_0(0)\| + \left\| \int_0^{t_1} \exp(\boldsymbol{A}_m(t_1 - s)) \left\{\boldsymbol{A}_m \boldsymbol{\delta}_{\hat{\boldsymbol{x}}1}(s) + \boldsymbol{B}\boldsymbol{\delta}_{u1}(s)\right\} ds \right\| \\
\|\boldsymbol{\eta}_0(t_1)\| &\leq \|\exp(\boldsymbol{A}_m \Delta t_1)\|.\|\boldsymbol{\eta}_0(0)\| + \left\| \int_0^{t_1} \exp(\boldsymbol{A}_m(t_1 - s)) \left\{\boldsymbol{A}_m \boldsymbol{\delta}_{\hat{\boldsymbol{x}}1}(s) + \boldsymbol{B}\boldsymbol{\delta}_{u1}(s)\right\} ds \right\|
\end{aligned} \tag{3.55}$$

Since $\boldsymbol{A}_m$ is Hurwitz, we have $\|\exp(\boldsymbol{A}_m t)\| = \exp(-mt)$.

Denoting $\bar{\boldsymbol{\delta}}_1 = \left\| \int_0^{t_1} \exp(\boldsymbol{A}_m(t_1 - s)) \left\{\boldsymbol{A}_m \boldsymbol{\delta}_{\hat{\boldsymbol{x}}1}(s) + \boldsymbol{B}\boldsymbol{\delta}_{u1}(s)\right\} ds \right\|$, (3.55) can be re-written as

$$\|\boldsymbol{\eta}_0(t_1)\| \leq \exp\left(-m\Delta t_1\right) \|\boldsymbol{\eta}_0(0)\| + \bar{\boldsymbol{\delta}}_1 \tag{3.56}$$

Next, at $t = t_2$, the error vector is given as

$$\begin{aligned}
\boldsymbol{\eta}_0(t_2) &= \exp(\boldsymbol{A}_m \Delta t_2)\boldsymbol{\eta}_0(t_1) + \int_{t_1}^{t_2} \exp(\boldsymbol{A}_m(t_2 - s)) \left\{\boldsymbol{A}_m \boldsymbol{\delta}_{\hat{\boldsymbol{x}}2}(s) + \boldsymbol{B}\boldsymbol{\delta}_{u2}(s)\right\} ds \\
\|\boldsymbol{\eta}_0(t_2)\| &\leq \|\exp(\boldsymbol{A}_m \Delta t_2)\boldsymbol{\eta}_0(t_1)\| + \left\| \int_{t_1}^{t_2} \exp(\boldsymbol{A}_m(t_2 - s)) \left\{\boldsymbol{A}_m \boldsymbol{\delta}_{\hat{\boldsymbol{x}}2}(s) + \boldsymbol{B}\boldsymbol{\delta}_{u2}(s)\right\} ds \right\| \\
\|\boldsymbol{\eta}_0(t_2)\| &\leq \exp(-m\Delta t_2)\|\boldsymbol{\eta}_0(t_1)\| + \bar{\boldsymbol{\delta}}_2
\end{aligned} \tag{3.57}$$

Substituting for $\|\eta_0(t_1)\|$ from (3.56), we get

$$\begin{aligned}\|\boldsymbol{\eta}_0(t_2)\| &\leq \exp(-m\Delta t_2)\exp\{-m\Delta t_1\}\|\boldsymbol{\eta}_0(0)\| + \exp(-m\Delta t_2)\bar{\boldsymbol{\delta}}_1 + \bar{\boldsymbol{\delta}}_2 \\ \|\boldsymbol{\eta}_0(t_2)\| &\leq \exp\left(-m(\Delta t_1 + \Delta t_2)\right)\|\boldsymbol{\eta}_0(0)\| + \exp(-m\Delta t_2)\bar{\boldsymbol{\delta}}_1 + \bar{\boldsymbol{\delta}}_2 \end{aligned} \tag{3.58}$$

where $\bar{\boldsymbol{\delta}}_2 = \left\| \int_{t_1}^{t_2} \exp(\boldsymbol{A}_m(t_2 - s))\left\{\boldsymbol{A}_m \boldsymbol{\delta}_{\hat{\boldsymbol{x}}2}(s) + \boldsymbol{B}\boldsymbol{\delta}_{u2}(s)\right\} ds \right\|$

Proceeding further, the error vector at $t = t_3$ is

$$\begin{aligned}\boldsymbol{\eta}_0(t_3) &= \exp\left(A_m\Delta t_3\right)\boldsymbol{\eta}_0(t_2) + \int_{t_2}^{t_3} \exp(\boldsymbol{A}_m(t_3 - s))\left\{\boldsymbol{A}_m\boldsymbol{\delta}_{\hat{\boldsymbol{x}}3}(s) + \boldsymbol{B}\boldsymbol{\delta}_{u3}(s)\right\} ds \\ \|\boldsymbol{\eta}_0(t_3)\| &\leq \exp\left(-m\Delta t_3\right)\|\boldsymbol{\eta}_0(t_2)\| + \bar{\boldsymbol{\delta}}_3 \\ \|\boldsymbol{\eta}_0(t_3)\| &\leq \exp\left(-m\Delta t_3\right)\left\{\exp\left(-m(\Delta t_1 + \Delta t_2)\right)\|\boldsymbol{\eta}_0(0)\| + \exp\left(-m\Delta t_2\right)\bar{\boldsymbol{\delta}}_1 + \bar{\boldsymbol{\delta}}_2\right\} + \bar{\boldsymbol{\delta}}_3 \\ \|\boldsymbol{\eta}_0(t_3)\| &\leq \exp\left(-m\left(\Delta t_1 + \Delta t_2 + \Delta t_3\right)\right)\|\boldsymbol{\eta}_0(0)\| + \exp\left(-m(\Delta t_2 + \Delta t_3)\right)\bar{\boldsymbol{\delta}}_1 \\ &\quad + \exp\left(-m\Delta t_3\right)\bar{\boldsymbol{\delta}}_2 + \bar{\boldsymbol{\delta}}_3 \end{aligned} \tag{3.59}$$

Continuing as above, the error vector at any instant $t = t_N$ can be written as

$$\|\boldsymbol{\eta}_0(t_N)\| \leq \exp\left(-m\sum_{i=1}^{N}\Delta t_i\right)\|\boldsymbol{\eta}_0(0)\| + \sum_{j=1}^{N-1}\exp\left(-m\sum_{i=j+1}^{N}\Delta t_i\right)\bar{\boldsymbol{\delta}}_j + \bar{\boldsymbol{\delta}}_N \tag{3.60}$$

where $\bar{\boldsymbol{\delta}}_i = \left\| \int_{t_{i-1}}^{t_i} \exp(\boldsymbol{A}_m(t_i - s))\left\{\boldsymbol{A}_m\boldsymbol{\delta}_{\hat{\boldsymbol{x}}i}(s) + \boldsymbol{B}\boldsymbol{\delta}_{ui}(s)\right\} ds \right\|, \quad i = 1, 2, \ldots, N.$

Remark 1: In the Eq. (3.60), the term $\bar{\boldsymbol{\delta}}_i$ represents the error in observation due to the delay variations. This error is multiplied by an exponential term, which decay to zero and hence as $t \to \infty$, these cumulative delays also decay to zero. If the delay changes very fast, i.e, Δt_i is very small, then the cumulative error due the delay changes will be larger, since the time available for the multiplying exponentials will be smaller. Similarly, if the delay changes at a very slow rate, then the then the cumulative error as time increases will be of smaller magnitude and the observer will be able to track the system states much better.

Remark 2: From (3.60), it can be seen that at any $t = t_N$, there will be a residual non-decaying error term $\bar{\boldsymbol{\delta}}_N$. This implies that the observation will not be equal to zero, but a finite non-zero value given by the magnitude of the term $\bar{\boldsymbol{\delta}}_N$. It also implies that the observation error can be expected to lie in a residual set of magnitude defined by the error term $\bar{\boldsymbol{\delta}}_N$. The residuals at $t = t_N$ due to earlier delay changes may have decayed to zero due to the multiplying exponential term.

Remark 3: In (3.60), if there were no delay changes, i.e the measurement delay was a constant over the entire time horizon, then there would be no residual errors, ie. $\bar{\boldsymbol{\delta}}_i = 0$. and (3.60) reduces to

$$\|\boldsymbol{\eta}_0(t_N)\| \leq \exp(-mt_N)\|\boldsymbol{\eta}_0(0)\| \tag{3.61}$$

which is the expression for observation error with constant delays.

Remark 4: If all the delays intervals were of equal width, i.e $\Delta t_i = \Delta t_j, \quad i \neq j$, then the observation error at $t = t_N$ is given as

$$\|\boldsymbol{\eta}_0(t_N)\| \leq \exp(-mN\Delta t)\|\boldsymbol{\eta}_0(0)\| + \sum_{j=1}^{N} \exp(-m(N-j)\Delta t)\,\bar{\boldsymbol{\delta}}_j \tag{3.62}$$

3.3.1.3 Stability analysis of the first observer

Using the state transition matrix, the system state at t i.e, $\boldsymbol{x}_1(t)$ can be written as

$$\boldsymbol{x}_1(t) = \exp(\boldsymbol{A}\Delta(t))\boldsymbol{x}_0(t) + \int_{t-\Delta(t)}^{t} \exp(\boldsymbol{A}(t-s))\boldsymbol{B}u(s)ds \tag{3.63}$$

Similarly, the observer states estimating the system states at t can be expressed as:

$$\hat{\boldsymbol{x}}_1(t) = \exp(\boldsymbol{A}\Delta(t))\hat{\boldsymbol{x}}_0(t) + \int_{t-\Delta(t)}^{t} \exp(\boldsymbol{A}(t-s))\boldsymbol{B}u(s)ds \tag{3.64}$$

Subtracting Eq. (3.64) from Eq. (3.63) we obtain,

$$\boldsymbol{\eta}_1(t) = \exp(\boldsymbol{A}\Delta(t))\boldsymbol{\eta}_0(t) \tag{3.65}$$

Next, a general expression for the observation error $\boldsymbol{\eta}_1(t)$ at any $t = t_i, i = 1, 2, \ldots, N$ is derived as follows:

At $t = t_1$, Eq. (3.65) can be written as

$$\|\boldsymbol{\eta}_1(t_1)\| \leq \|\exp(\boldsymbol{A}\Delta(t))\|.\|\boldsymbol{\eta}_0(t)\| \tag{3.66}$$

Substituting for $\|\boldsymbol{\eta}_0(t)\|$,

$$\begin{aligned}\|\boldsymbol{\eta}_1(t_1)\| &\leq \alpha_1\{\exp\left(-m\Delta t_1\right)\|\boldsymbol{\eta}_0(0)\| + \bar{\boldsymbol{\delta}}_1\} \\ \|\boldsymbol{\eta}_1(t_1)\| &\leq \alpha_1 \exp\left\{-m\Delta t_1\right)\|\boldsymbol{\eta}_0(0)\| + \alpha_1\bar{\boldsymbol{\delta}}_1\end{aligned} \tag{3.67}$$

where $\|\exp(\boldsymbol{A}\Delta_1)\| \leq \alpha_1$. Similarly, at $t = t_2$, the observation error is

$$\begin{aligned}\|\boldsymbol{\eta}_1(t_2)\| &\leq \alpha_2\{\exp\left(-m(\Delta t_1 + \Delta t_2)\right)\|\boldsymbol{\eta}_0(0)\| + \exp\left(-m\Delta t_2\right)\bar{\boldsymbol{\delta}}_1 + \bar{\boldsymbol{\delta}}_2\} \\ \|\boldsymbol{\eta}_1(t_2)\| &\leq \alpha_2 \exp\left(-m(\Delta t_1 + \Delta t_2)\right)\|\boldsymbol{\eta}_0(0)\| + \alpha_2\exp(m_1\Delta_1)\bar{\boldsymbol{\delta}}_1 + \alpha_2\bar{\boldsymbol{\delta}}_2\end{aligned} \tag{3.68}$$

where $\|\exp(\boldsymbol{A}\Delta_2)\| \leq \alpha_2$. Hence a general expression for the observation error at any instant $t = t_N$ is

$$\|\boldsymbol{\eta}_1(t_N)\| \leq \alpha_N \exp\left(-m\sum_{i=1}^{N}\Delta t_i\right)\|\boldsymbol{\eta}_0(0)\| + \alpha_N\sum_{j=1}^{N-1}\exp\left(-m\sum_{i=j+1}^{N}\Delta t_i\right)\bar{\boldsymbol{\delta}}_j + \alpha_N\bar{\boldsymbol{\delta}}_N \tag{3.69}$$

where $\|\exp(\boldsymbol{A}\Delta_N)\| \leq \alpha_N$. (3.69) implies that there will be a finite non-zero observation error at any instant $t = t_N$. This is the error that occurs in the zeroth observer due to the delay changes. Additionally, in observer 1, each of these residual error is multiplied by a scaling constant $\|\exp(\boldsymbol{A}\Delta_i)\| \leq \alpha_i, \quad i = 1, 2, \ldots, N$. For a non-Hurwitz matrix A, each $\boldsymbol{\alpha}_i$ will amplify the residual error further.

If matrix A is Hurwitz, $\|\exp(\boldsymbol{A}\Delta_N)\| = \|\exp(-m_1\Delta_N)\|$, which is an exponentially decaying quantity. Thus, in this case the estimated states will be closer to the true states. Note, m_1 is the smallest eigen value of $\boldsymbol{A}$,

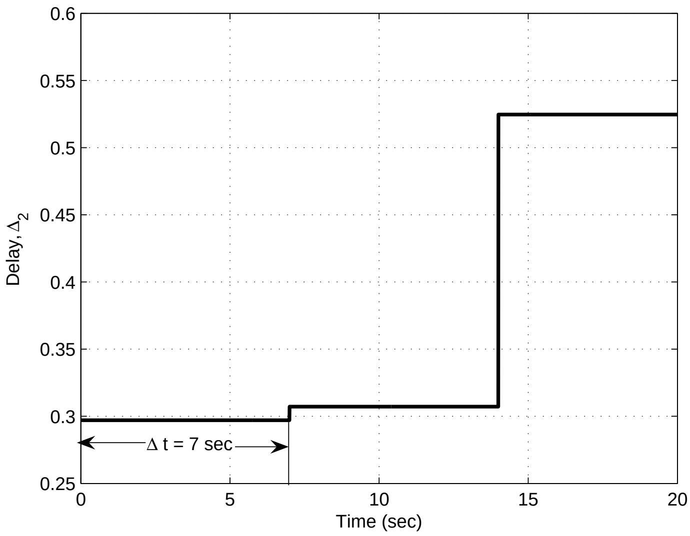

Figure 3.13 Delay profile of the measurement delay available at vehicle 2.

3.3.1.4 Simulation Results

The string system with a lead vehicle transmitting its position information to the two follower vehicles is the scenario considered for simulation. This is similar to the example considered in Section 3.2.1.3.

The main difference is that delay is no longer a constant but time-varying. It is assumed that delay of the output available at vehicle 2 changes at every 7 *sec* and the delay of the output at vehicle 3 changes at every 5 *sec*. The delay profiles are as shown in Figs. (3.13) and (3.14). In both the cases, $\Delta \in [0.1,\ 1]$.

The error control law and the initial condition for the vehicles and the observer is similar to that used in Sec. 3.2.1.3. The control law aims to position the two following vehicles at a distance of 20 m and 50 m from the lead vehicle. The simulation results are shown below

Fig. (3.15) shows the evolution of the positions of the three vehicles with time. Fig. (3.16) is the plot of error in position. There is a significant change in the steady state value of the position error vector. Due to constant changes in delays, the observer dynamics

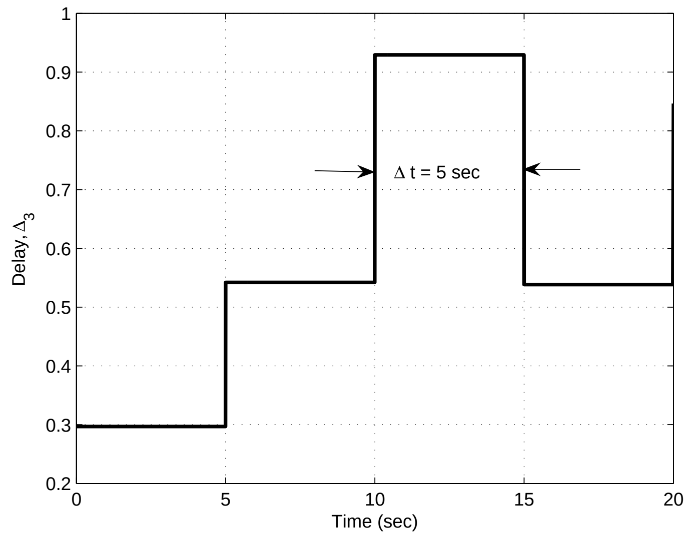

Figure 3.14 Delay profile of the measurement delay available at vehicle 3.

is always perturbed, which in turn causes the change in the control law. Supporting the theory developed for the observer estimating the output with time-varying delay, there is always a residual in the observation error. This fact can also be observed from the observation error plots shown in Figs. (3.17) and (3.18).

Fig. (3.19) is the plot of control input to the two vehicles. In the constant delay case, once the steady state was reached by the control, it remained a constant for the rest of the time horizon. But, the constant change in delay causes the chain observer to update its response based on the delay magnitude. This is reflected as a change in position of the vehicles which are perturbed from their steady state conditions. Since the control law aims to maintain the vehicles at the prescribed spacing, it causes the vehicle to accelerate or decelerate, thus trying to maintain the constant spacing. But due to the residual errors in observation, it can never meet its goal and there is constant error. From Fig. (3.16) it can be seen that vehicle 2 is at a distance of 30 m from the lead vehicle and vehicle 3 is at a distance slightly greater than the prescribed 50 m separation.

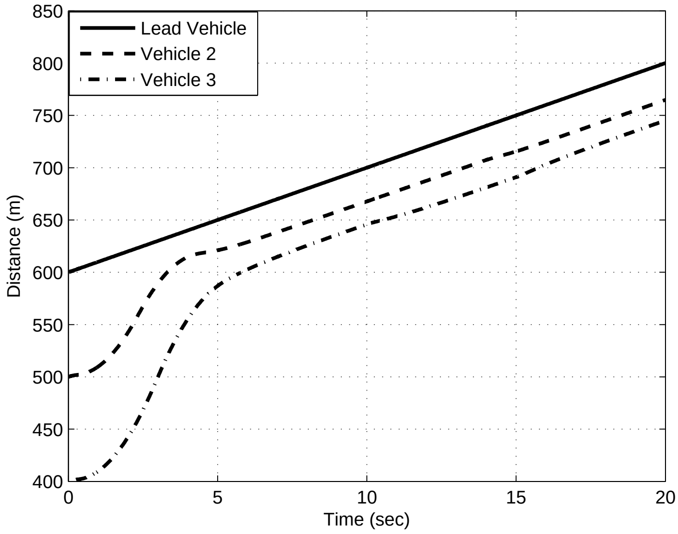

Figure 3.15 Positions of the three vehicles

Thus, the simulation suggests that this control scheme of constant spacing is not very effective in the time varying case. Hence, an a more robust control law is needed to maintain the constant spacing in the presence of time varying delays.

3.3.2 Discrete LTI systems

In this section, the discrete chain observer theory is extended to cases when the delay n, is a function of time, i.e, its magnitude varies randomly with time and $0 < n \leq N$. N is the upper bound on all the delays. n is a piecewise constant, i.e, the value n_i is a constant in the interval $(k_{i-1},\ k_i)$. For the next interval $(k_i,\ k_{i+1})$, it assumes a different value. The width of each time interval $(k_i - k_{i-1}), \quad \forall i = 1 \dots L$, is assumed to of size p_i. The output measurements are assumed to be time stamped. This timing information can be extracted at the observer, which helps in ascertaining the value of the delay.

The dynamics of the chain observer is

$$\hat{\boldsymbol{x}}(k-n+1) = \boldsymbol{\Phi}\hat{\boldsymbol{x}}(k-n) + \boldsymbol{\Gamma}u(k-n) + \boldsymbol{\kappa}(\bar{\boldsymbol{y}}(k) - \boldsymbol{C}\hat{\boldsymbol{x}}(k-n)) \qquad (3.70)$$

$$\hat{\boldsymbol{x}}_1(k+1) = \boldsymbol{\Phi}\hat{\boldsymbol{x}}_1(k) + \boldsymbol{\Gamma}u(k) + \boldsymbol{\Phi}^{n(k)}\boldsymbol{\kappa}(\bar{\boldsymbol{y}}(k) - \boldsymbol{C}\hat{\boldsymbol{x}}(k-n)) \qquad (3.71)$$

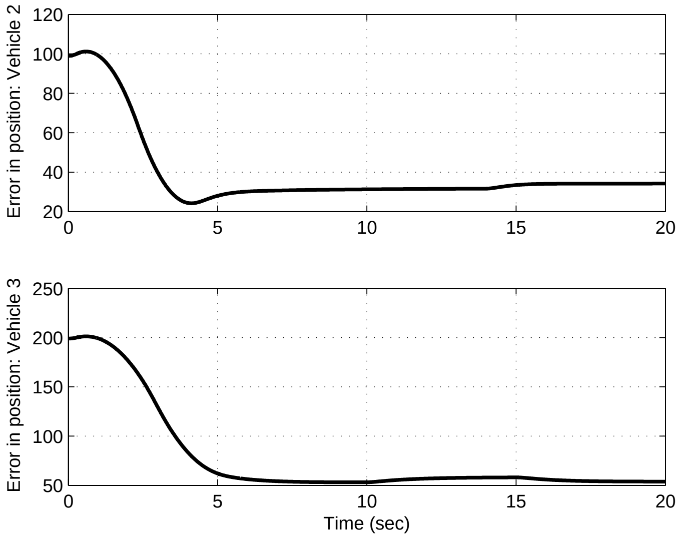

Figure 3.16 Error in position

where $\hat{\boldsymbol{x}}(k-n)$ is an estimate of the state at the instant $k-n$ and $\hat{\boldsymbol{x}}_1(k)$ is an estimate of the state at k. $\boldsymbol{\kappa}$ is the gain matrix, which is chosen such that the eigen values of $\boldsymbol{\Phi} - \boldsymbol{\kappa C}$ lie within the unit circle.

The observer states are initialized as follows

$$\hat{\boldsymbol{x}}(k-n)|_{k=0} = \hat{\boldsymbol{x}}(-n_1) \tag{3.72}$$

$$\hat{\boldsymbol{x}}_1(0) = \hat{\boldsymbol{x}}(0) \tag{3.73}$$

n_1 is the output delay magnitude in the interval $0 \leq k < k_1$.

3.3.2.1 Stability analysis of the zeroth observer

In this section, the effects of the change in the delay at specific instants of time on the observer error dynamics is analyzed.

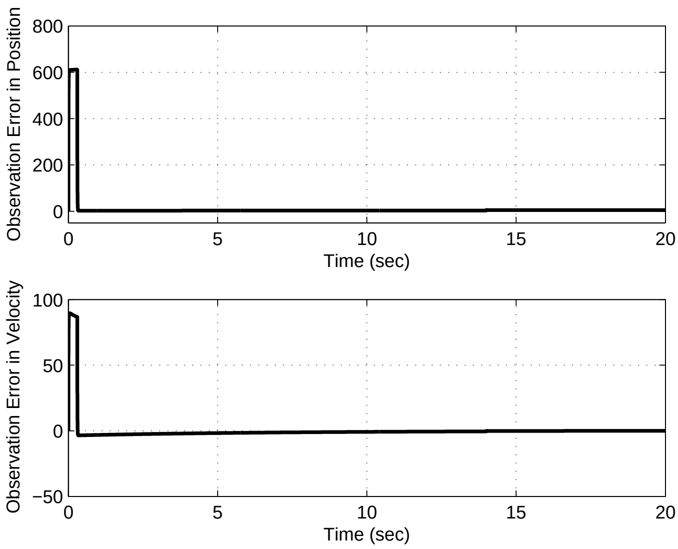

Figure 3.17 Observation Error in states of the Chain Observer at System 2

The observation error is defined as

$$\boldsymbol{\eta}_0(k+1) = \boldsymbol{x}(k-n+1) - \hat{\boldsymbol{x}}(k-n+1) \tag{3.74}$$

$$\boldsymbol{\eta}_0(k+1) = \boldsymbol{\Phi}\boldsymbol{x}(k-n) + \boldsymbol{\Gamma}u(k-n) - \boldsymbol{\Phi}\hat{\boldsymbol{x}}(k-n) - \boldsymbol{\Gamma}u(k-n) - \boldsymbol{\kappa}(\bar{\boldsymbol{y}}(k) - \boldsymbol{C}\hat{\boldsymbol{x}}(k-n))$$

$$\boldsymbol{\eta}_0(k+1) = (\boldsymbol{\Phi} - \boldsymbol{\kappa}\boldsymbol{C})(\boldsymbol{x}(k-n) - \hat{\boldsymbol{x}}(k-n))$$

$$\boldsymbol{\eta}_0(k+1) = \boldsymbol{\Phi}_m\boldsymbol{\eta}_0(k) \tag{3.75}$$

Next, the error in estimation due to change in delay is examined. As explained in the Section (3.3.1.2) the error in observation is due to the changing delay. At each $k = k_1$, the delay changes from n_i to a random value n_{i+1}. Hence the zeroth observer which started estimating the states with an initial condition $-n_1$, has to estimate states as though the initial condition is $t_i - n_{i+1}$. Thus, a change in delay translates to a shift in the initial condition for the zeroth observer.

However, the state values at the "new" initial condition can be obtained from the output of the first observer. For example, at $k = k_i$, the zeroth observer which was previously estimating states at the delayed instant $k_1 - n_1$ has to "jump" to the instant $k_1 - n_2$ and propagate its dynamics forward from this new time instant. The difference

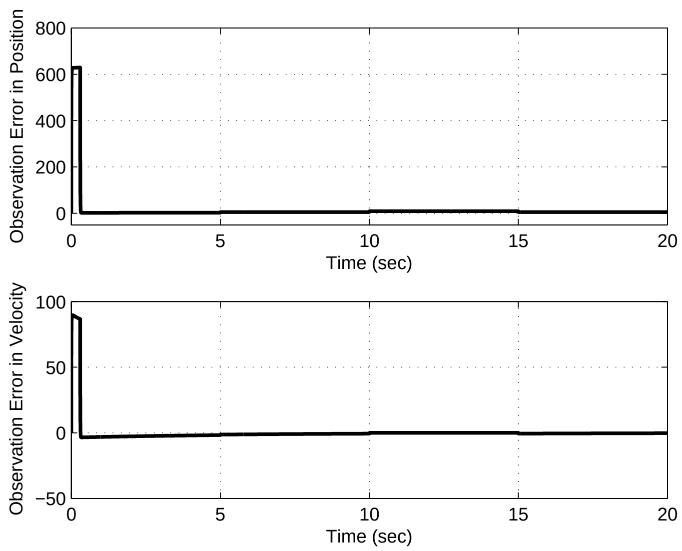

Figure 3.18 Observation Error in states of the Chain Observer at System 3

in magnitude at $k = k_1$ is $\hat{\boldsymbol{x}}(k_1 - n_2 + 1) - \hat{\boldsymbol{x}}(k_1 - n_1 + 1)$. Substituting the observer dynamics and simplifying,

$$\begin{aligned}\hat{\boldsymbol{x}}(k_1 - n_2 + 1) - \hat{\boldsymbol{x}}(k_1 - n_1 + 1) &= \boldsymbol{\Phi}\hat{\boldsymbol{x}}(k_1 - n_2) + \boldsymbol{\Gamma}u(k_1 - n_2) + \boldsymbol{\kappa}(\bar{\boldsymbol{y}}(k) - \boldsymbol{C}\hat{\boldsymbol{x}}(k_1 - n_2)) \\ &\quad - (\boldsymbol{\Phi}\hat{\boldsymbol{x}}(k_1 - n_1) + \boldsymbol{\Gamma}u(k_1 - n_1) + \boldsymbol{\kappa}(\bar{\boldsymbol{y}}(k) - \boldsymbol{C}\hat{\boldsymbol{x}}(k_1 - n_1))) \\ \hat{\boldsymbol{x}}(k_1 - n_1 + 1) &= \hat{\boldsymbol{x}}(k_1 - n_2 + 1) - \boldsymbol{\Phi}_m\boldsymbol{\delta}_{\hat{\boldsymbol{x}}1} - \boldsymbol{\Gamma}\boldsymbol{\delta}_{u1} \qquad (3.76)\end{aligned}$$

where $\boldsymbol{\Phi}_m = \boldsymbol{\Phi} - \boldsymbol{\kappa}\boldsymbol{C}$ is Schur stable. $\boldsymbol{\delta}_{\hat{\boldsymbol{x}}1} = \hat{\boldsymbol{x}}(k_1 - n_2) - \hat{\boldsymbol{x}}(k_1 - n_1)$ and $\boldsymbol{\delta}_{\boldsymbol{u}1} = u(k_1 - n_2) - u(k_1 - n_1)$.

Generalizing Eq. (3.76), the error in observation due the "jump" in initial condition at any instant $k = k_i$ is

$$\hat{\boldsymbol{x}}(k_i - n_i + 1) = \hat{\boldsymbol{x}}(k_i - n_{i+1} + 1) - \boldsymbol{\Phi}_m\boldsymbol{\delta}_{\hat{\boldsymbol{x}}i} - \boldsymbol{\Gamma}\boldsymbol{\delta}_{ui} \qquad (3.77)$$

Substituting Eq. (3.77) in Eq. (3.74), the observation error dynamics at $k = k_i$ is given as

$$\boldsymbol{\eta}_0(k_i + 1) = \boldsymbol{\Phi}_m\boldsymbol{\eta}_0(k_i) + \boldsymbol{\Phi}_m\boldsymbol{\delta}_{\hat{\boldsymbol{x}}i} + \boldsymbol{\Gamma}\boldsymbol{\delta}_{ui} \qquad (3.78)$$

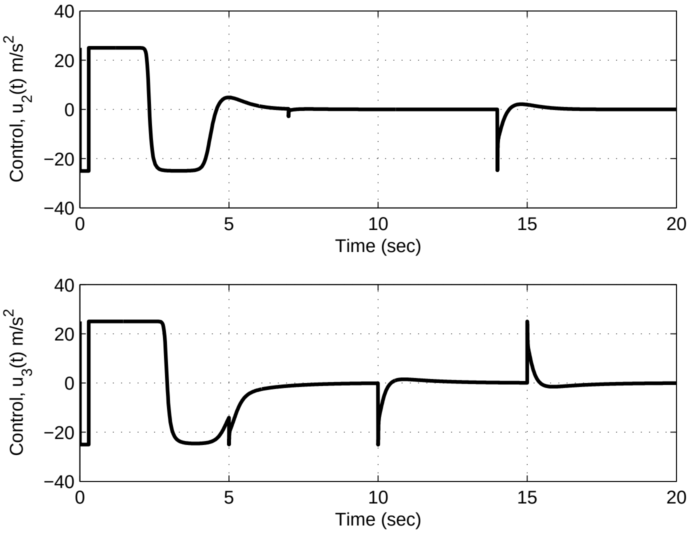

Figure 3.19 Control inputs of Vehicles 2 and 3 respectively.

Defining $\bar{\boldsymbol{\delta}}_i = \boldsymbol{\Phi}_m \boldsymbol{\delta}_{\hat{\boldsymbol{x}}i} + \boldsymbol{\Gamma}\boldsymbol{\delta}_{ui}$, Eq. (3.78) can be rewritten as

$$\boldsymbol{\eta}_0(k_i + 1) = \boldsymbol{\Phi}_m \boldsymbol{\eta}_0(k_i) + \bar{\boldsymbol{\delta}}_i \tag{3.79}$$

Hence, the observation error dynamics can be written as

$$\boldsymbol{\eta}_0(k+1) = \begin{cases} \boldsymbol{\Phi}_m \boldsymbol{\eta}_0(k) + \bar{\boldsymbol{\delta}}_i, & \text{at } k > k_i \\ \\ \boldsymbol{\Phi}_m \boldsymbol{\eta}_0(k) & \text{for } k \in [k_{i-1},\ k_i) \end{cases} \tag{3.80}$$

The delay profile for a discrete LTI system with delayed outputs is shown in Fig.(3.20).

Next, a general expression for the observation error dynamics at any instant $k_L, \quad L = 0, 1, \ldots, \infty$ is derived.

At $k = k_1$, the error equation is

$$\boldsymbol{\eta}_0(k_1 + 1) = \boldsymbol{\Phi}_m \boldsymbol{\eta}_0(k_1) + \bar{\boldsymbol{\delta}}_1 \tag{3.81}$$

Using the method of back substitution, the equation for error at k_1 in terms of the initial error vector is

$$\boldsymbol{\eta}_0(k_1 + 1) = \boldsymbol{\Phi}_m^{(p_1+1)} \boldsymbol{\eta}_0(0) + \bar{\boldsymbol{\delta}}_1 \tag{3.82}$$

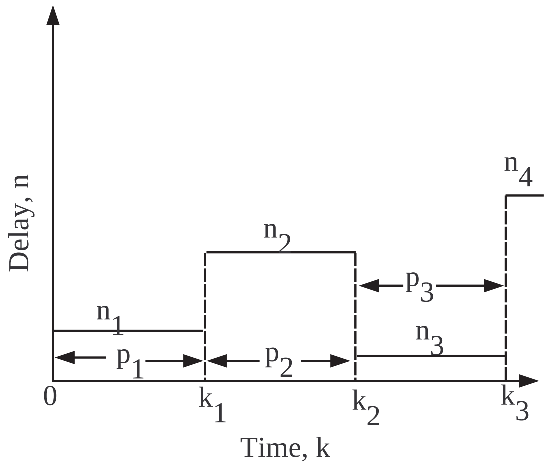

Figure 3.20 Variation of delay with time , for a system with delayed output.

Propagating the error vector in the interval $k_1 \leq k < k_2$, the observation error at $k = k_2$ is

$$\begin{aligned} \boldsymbol{\eta}_0(k_2+1) &= \boldsymbol{\Phi}_m \boldsymbol{\eta}_0(k_2) + \bar{\boldsymbol{\delta}}_2 \\ \boldsymbol{\eta}_0(k_2+1) &= \boldsymbol{\Phi}_m^{p_2} \boldsymbol{\eta}_0(k_1) + \bar{\boldsymbol{\delta}}_2 \end{aligned} \tag{3.83}$$

Substituting for $\boldsymbol{\eta}_0(k_1+1)$ from Eq. (3.84),

$$\boldsymbol{\eta}_0(k_2+1) = \boldsymbol{\Phi}_m^{(p_1+p_2+1)} \boldsymbol{\eta}_0(0) + \boldsymbol{\Phi}_m^{p_2} \bar{\boldsymbol{\delta}}_1 + \bar{\boldsymbol{\delta}}_2 \tag{3.84}$$

Continuing further, the equation for observation error at $k = k_3$ is

$$\begin{aligned} \boldsymbol{\eta}_0(k_3+1) &= \boldsymbol{\Phi}_m \boldsymbol{\eta}_0(k_3) + \bar{\boldsymbol{\delta}}_3 \\ \boldsymbol{\eta}_0(k_3+1) &= \boldsymbol{\Phi}_m^{(p_1+p_2+p_3+1)} \boldsymbol{\eta}_0(0) + \boldsymbol{\Phi}_m^{(p_2+p_3)} \bar{\boldsymbol{\delta}}_1 + \boldsymbol{\Phi}_m^{p_3} \bar{\boldsymbol{\delta}}_2 + \bar{\boldsymbol{\delta}}_3 \end{aligned} \tag{3.85}$$

Therefore, a general equation for the error vector at any instant $k = k_L, \quad L = 1, 2, \ldots, \infty$ is given as

$$\boldsymbol{\eta}_0(k_L+1) = \boldsymbol{\Phi}_m^{\left(\sum_{i=1}^{L} p_i + 1\right)} \boldsymbol{\eta}_0(0) + \sum_{i=1}^{L-1} \boldsymbol{\Phi}_m^{\left(\sum_{j=i+1}^{L} p_i\right)} \bar{\boldsymbol{\delta}}_i + \bar{\boldsymbol{\delta}}_L \tag{3.86}$$

Remark 1: Eq. (3.86) is similar to its continuous-time counterpart. It implies that at any instant $k = K_L$, there is a finite residual error due to the shift in initial condition. The magnitude of the error depends on the magnitude of the change in delay.

Remark 2: If there was no change in delay, then each $\bar{\delta}_i = 0$ and Eq. (3.86) reduces to

$$\boldsymbol{\eta}_0(k_L + 1) = \boldsymbol{\Phi}_m^{(p+1)} \boldsymbol{\eta}_0(0) \tag{3.87}$$

which is the error equation due to constant delay.

Remark 3: If all the p's were of equal intervals, i.e, $p_i = p_j$, $\forall i \neq j$, the observation error equation is given as

$$\boldsymbol{\eta}_0(k_L + 1) = \boldsymbol{\Phi}_m^{(L.p+1)} \boldsymbol{\eta}_0(0) + \sum_{i=1}^{L-1} \boldsymbol{\Phi}_m^{((L-j)p)\}} \bar{\boldsymbol{\delta}}_i \tag{3.88}$$

3.3.2.2 Stability analysis of the first observer

The observation errors for the first observer is defined as:

$$\begin{aligned}
\boldsymbol{\eta}_1(k+1) &= \boldsymbol{x}(k+1) - \hat{\boldsymbol{x}}(k+1) \\
\boldsymbol{\eta}_1(k+1) &= \boldsymbol{\Phi x}(k) + \boldsymbol{\Gamma} u(k) - \boldsymbol{\Phi} \hat{\boldsymbol{x}}(k) - \boldsymbol{\Gamma} u(k) - \boldsymbol{\Phi}^n \boldsymbol{\kappa}(\bar{y}(k) - \boldsymbol{C} \hat{\boldsymbol{x}}) \\
\boldsymbol{\eta}_1(k+1) &= \boldsymbol{\Phi} \boldsymbol{\eta}_1(k) - \boldsymbol{\Phi}^n \boldsymbol{\kappa} \boldsymbol{C} \boldsymbol{\eta}_0(k) \qquad (3.89)
\end{aligned}$$

Form the definition of state transition matrix, we have

$$\boldsymbol{\eta}_1(k) = \boldsymbol{\Phi}^n \boldsymbol{\eta}_0(k) \tag{3.90}$$

Using this property and substituting for $\boldsymbol{\eta}_1(k)$ in Eq.(3.89),

$$\begin{aligned}
\boldsymbol{\eta}_1(k+1) &= \boldsymbol{\Phi} \boldsymbol{\Phi}^n \boldsymbol{\eta}_0(k) - \boldsymbol{\Phi}^n \boldsymbol{\kappa} \boldsymbol{C} \boldsymbol{\eta}_0(k) \\
\boldsymbol{\eta}_1(k+1) &= \boldsymbol{\Phi}^n [\boldsymbol{\Phi} - \boldsymbol{\kappa} \boldsymbol{C}] \boldsymbol{\eta}_0(k) \\
\boldsymbol{\eta}_1(k+1) &= \boldsymbol{\Phi}^n \boldsymbol{\eta}_0(k+1) \qquad (3.91)
\end{aligned}$$

Next, a general expression for the observation error at the first observer for any $k = k_L$, when the delay changes, is derived.

At $k = k_1$, $\boldsymbol{\eta}_1(k+1)$ is given as

$$\boldsymbol{\eta}_1(k_1 + 1) = \boldsymbol{\Phi}^{n_1} \boldsymbol{\eta}_0(k_1 + 1) \tag{3.92}$$

Substituting for $\boldsymbol{\eta}_0(k_1+1)$ and simplifying,

$$\boldsymbol{\eta}_1(k_1+1) = \boldsymbol{\Phi}^{n_1}\boldsymbol{\Phi}_m^{(p_1+1)}\boldsymbol{\eta}_0(0) + \boldsymbol{\Phi}^{n_1}\bar{\boldsymbol{\delta}}_1 \tag{3.93}$$

Similarly the error at $k = k_2$ is

$$\boldsymbol{\eta}_1(k_1+1) = \boldsymbol{\Phi}^{n_2}\boldsymbol{\Phi}_m^{(p_1+p_2+1)}\boldsymbol{\eta}_0(0) + \boldsymbol{\Phi}^{n_2}\boldsymbol{\Phi}_m^{p_2}\bar{\boldsymbol{\delta}}_1 + \boldsymbol{\Phi}^{n_2}\bar{\boldsymbol{\delta}}_2 \tag{3.94}$$

Hence, a general expression for $\boldsymbol{\eta}_1$ at $k = k_L$ is

$$\boldsymbol{\eta}_0(k_L+1) = \boldsymbol{\Phi}^{n_L}\boldsymbol{\Phi}_m^{\left(\sum_{i=1}^{L} p_i+1\right)}\boldsymbol{\eta}_0(0) + \boldsymbol{\Phi}^{n_L}\sum_{i=1}^{L-1}\boldsymbol{\Phi}_m^{\left(\sum_{j=i+1}^{L} p_i\right)}\bar{\boldsymbol{\delta}}_i + \boldsymbol{\Phi}^{n_L}\bar{\boldsymbol{\delta}}_L \tag{3.95}$$

In Eq. (3.95), the term $\boldsymbol{\Phi}^{n_L}$ serves as an amplification factor for the error vector if $\boldsymbol{\Phi}$ is not Hurwitz. Alternately, a Hurwitz $\boldsymbol{\Phi}$ decays exponentially to zero. This implies that the residual error will also decay to zero and hence the observation error in this case is smaller compared to the case when $\boldsymbol{\Phi}$ is not Hurwitz.

3.3.2.3 Simulation Results

The example used for simulation is the same string stability problem in the presence of communication delays explained in the previous sections. The discrete version of the string stability problem for the time-varying delays is simulated. The sampling interval used for conversion of continuous-time system to its discrete equivalent is 0.25 *sec*. The system state space representation is converted from continuous-time to discrete-time representation using the *c*2*d* function in *MATLAB*.

The measurement delays are assumed to vary over constant intervals of time. The intervals p used for simulation is $p = 10$ *samples* for vehicle 2 and $p = 15$ *samples* at vehicle 3. In both the cases, the minimum delay is 1 *sample* and the maximum delay interval is 4 *samples*. The delay profiles are as shown in Figs. (3.21) and (3.22).

The control law used is the sampled version of Eq. (3.26) and the initial condition for the vehicles and the observer is similar to that used in Sec. 3.2.1.3. The control law

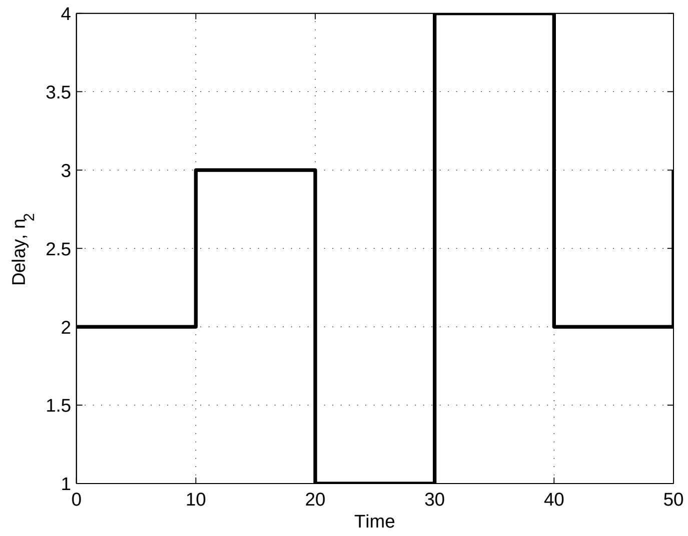

Figure 3.21 Delay profile of the measurement delay available at vehicle 2.

aims to position the two following vehicles at a distance of 20 m and 50 m from the lead vehicle. The simulation results are shown below

Fig. (3.23) shows the evolution of the positions of the three vehicles with time. Fig. (3.24) is the plot of error in position. There is a significant change in the steady state value of the position error vector. The position errors do not settle down at a constant value. From Fig. (3.24) it can be seen that vehicle 2 is at a distance of 30 m from the lead vehicle and vehicle 3 is at a distance slightly greater than the prescribed 50 m separation.

This behavior is due to constant changes in delays. The observer dynamics is always perturbed, which in turn causes the change in the control law. Supporting the theory developed, it can be noticed from Figs. (3.25) and (3.26) that there is always a residual error in the observation.

Fig. (3.27) is the plot of control input to the two vehicles. In the constant delay case, once the steady state was reached, no control effort was required further to steer the vehicles. But, the constant change in delay causes the chain observer to update its response based on the delay magnitude. This is reflected as a change in position of the

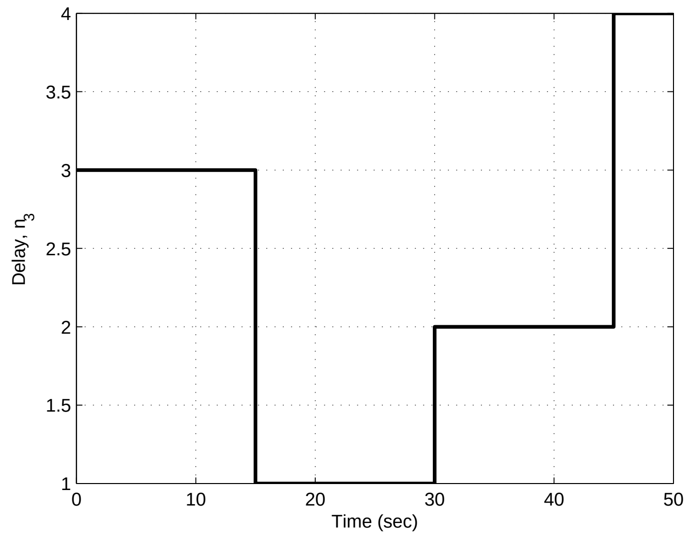

Figure 3.22 Delay profile of the measurement delay available at vehicle 3.

vehicles which are perturbed from their steady state conditions. Since the control law aims to maintain the vehicles at the prescribed spacing, it causes the vehicle to accelerate or decelerate, thus trying to maintain the constant spacing. But due to the residual errors in observation, it can never meet its goal and there is constant error.

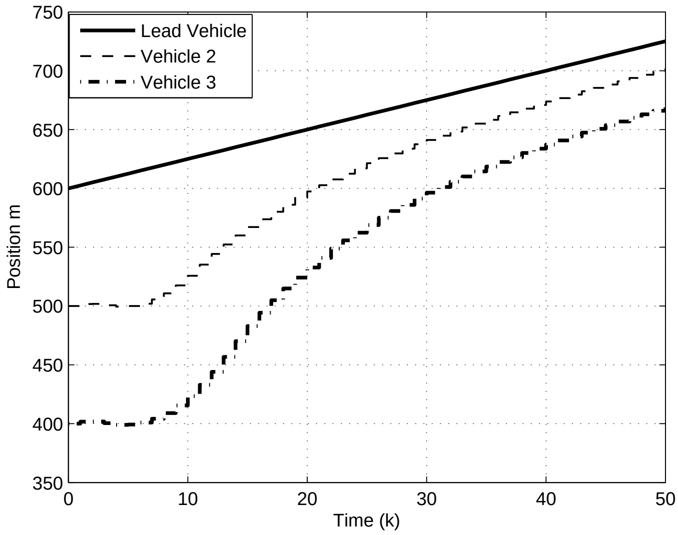

Figure 3.23 Positions of the three vehicles

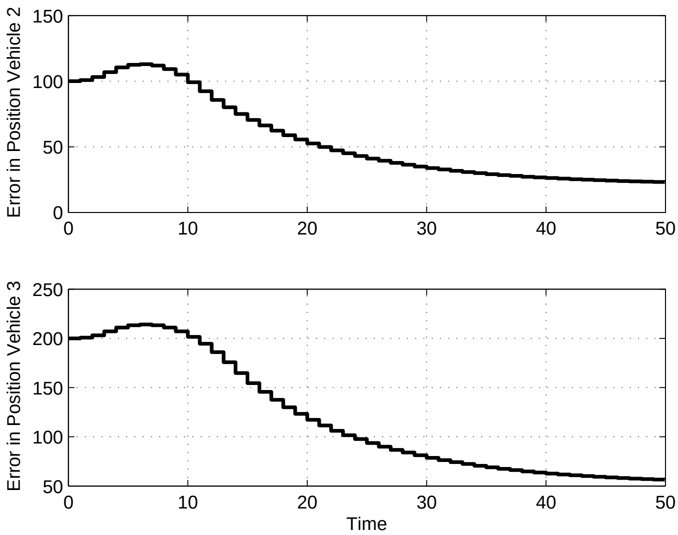

Figure 3.24 Error in position

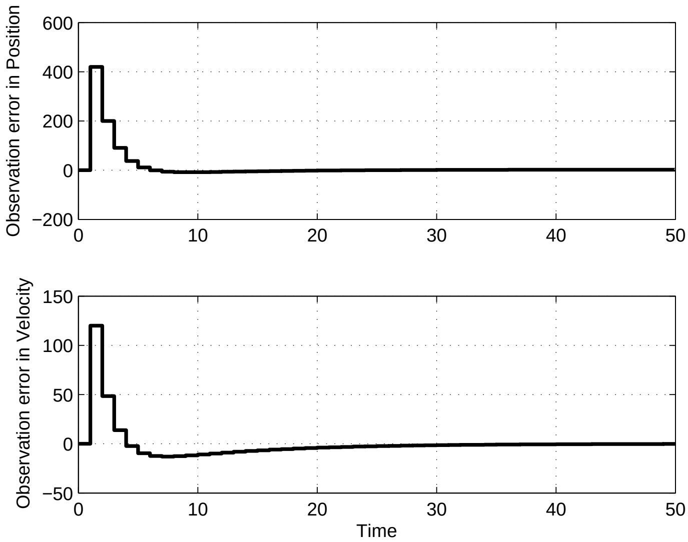

Figure 3.25 Observation Error in states of the Chain Observer at System 2

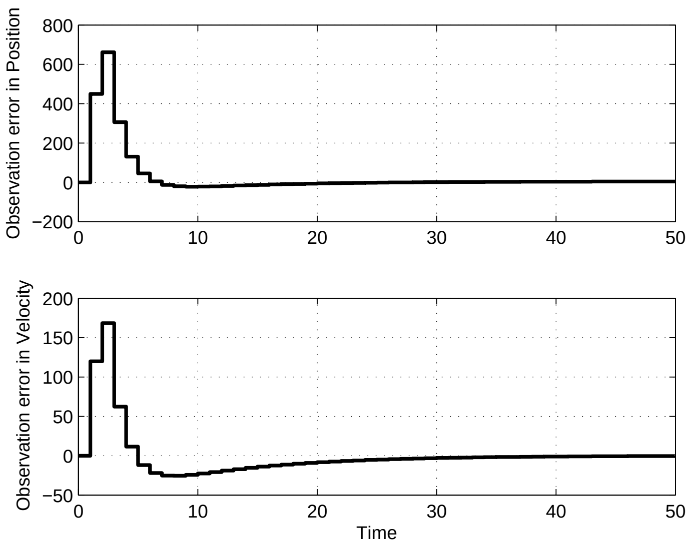

Figure 3.26 Observation Error in states of the Chain Observer at System 3

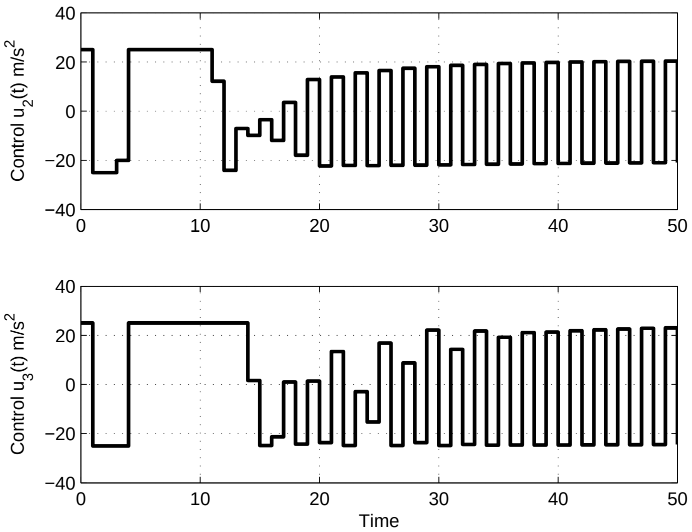

Figure 3.27 Control inputs of Vehicles 2 and 3 respectively.

CHAPTER 4

STATE OBSERVERS FOR NONLINEAR SYSTEMS WITH DELAYED OUTPUTS

The state space model of the general class of continuous-time single-input single-output (SISO) nonlinear systems considered for observation is

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{g}(\boldsymbol{x}(t))u(t) \quad t \geq \Delta, \quad \boldsymbol{x}(-\Delta) = \bar{\boldsymbol{x}} \tag{4.1}$$

$$\bar{\boldsymbol{y}}(t) = \boldsymbol{h}(\boldsymbol{x}(t-\Delta)) \tag{4.2}$$

where $\Delta > 0$ is the measurement delay. where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ the input and the vector functions $\boldsymbol{f},\ \boldsymbol{g},\ \boldsymbol{h}$ are $\boldsymbol{C}^{\infty}$. The undelayed output is represented as $\boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t))$.

The chain observer is implemented in the same reference coordinate in which the nonlinear system is defined. However, for stability analysis a nonlinear coordinate transformation (diffeomorphism) [17] is applied to the system model and the observer equation and the resulting set of equations are in the linear perturbed form. The stability results obtained in the transformed coordinates are then translated back to the original coordinate system. The process of coordinate change is as follows:

For system (4.1)-(4.2) a square map, $\boldsymbol{z} = \boldsymbol{\Phi}(\boldsymbol{x})$, is defined as

$$\boldsymbol{z} = \boldsymbol{\Phi}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{h}(\boldsymbol{x}) \\ \boldsymbol{L}_f\boldsymbol{h}(\boldsymbol{x}) \\ \vdots \\ \boldsymbol{L}_f^{n-1}\boldsymbol{h}(\boldsymbol{x}) \end{bmatrix} \tag{4.3}$$

where $\boldsymbol{L}_f\boldsymbol{h}(\boldsymbol{x})$ denotes the Lie derivative of the function $\boldsymbol{h}$ along $\boldsymbol{f}$ and $\boldsymbol{L}_f^k\boldsymbol{h}(\boldsymbol{x})$ denotes the k^{th} order repeated Lie derivative of the function $\boldsymbol{h}$ along $\boldsymbol{f}$.

The Jacobian, $\boldsymbol{Q}(\boldsymbol{x})$, of the map $\boldsymbol{\Phi}(\boldsymbol{x})$ and the Jacobian, $\boldsymbol{Q}^{-1}(\boldsymbol{x})$, of the the inverse map $\boldsymbol{\Phi}^{-1}(\boldsymbol{x})$ are defined as

$$\boldsymbol{Q}(\boldsymbol{x}) = \frac{\partial \boldsymbol{\Phi}(\boldsymbol{x})}{\partial \boldsymbol{x}} \tag{4.4}$$

$$\boldsymbol{Q}^{-1}(\boldsymbol{x}) = \frac{\partial \boldsymbol{\Phi}^{-1}(\boldsymbol{z})}{\partial \boldsymbol{z}}\Big|_{\boldsymbol{z}=\Phi(\boldsymbol{x})} \tag{4.5}$$

Using definition (4.3), the non-linear system (4.1)-(4.2) can be represented in the new co-ordinate system as

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}_n \boldsymbol{z}(t) + \widetilde{\boldsymbol{H}}(\boldsymbol{z}(t), u(t)), \quad t \geq \Delta, \quad \boldsymbol{z}(-\Delta) = \boldsymbol{\Phi}(\bar{\boldsymbol{x}}) \tag{4.6}$$

$$\bar{\boldsymbol{y}}(t) = \boldsymbol{C}_n \boldsymbol{z}(t-\Delta) \quad t \geq 0 \tag{4.7}$$

where

$$\widetilde{\boldsymbol{H}}(\boldsymbol{z}(t), u(t)) = \widetilde{\boldsymbol{H}}(\boldsymbol{x}(t), u(t))\Big|_{\boldsymbol{x}=\Phi^{-1}(\boldsymbol{z})} \tag{4.8}$$

$$\widetilde{\boldsymbol{H}}(\boldsymbol{x}(t), u(t)) = \boldsymbol{B}_n \boldsymbol{L}_f^n \boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{Q}(\boldsymbol{x}) \boldsymbol{g}(\boldsymbol{x}) u(t) \tag{4.9}$$

Matrices $\boldsymbol{A}_n$, $\boldsymbol{B}_n$ and $\boldsymbol{C}_n$ are the Brunowski triple of order n defined as

$$\boldsymbol{A}_n = \begin{bmatrix} 0_{(n-1)\times 1} & \boldsymbol{I}_{n-1} \\ 0 & 0_{1\times(n-1)} \end{bmatrix} \quad \boldsymbol{B}_n = \begin{bmatrix} 0_{(n-1)\times 1} \\ 1 \end{bmatrix} \quad \boldsymbol{C}_n = \begin{bmatrix} 1 & 0_{1\times(n-1)} \end{bmatrix} \tag{4.10}$$

$\boldsymbol{I}_{n-1}$ is the identity matrix of dimension $n-1 \times n-1$.

Eq. (4.6)-(4.7) is the linear perturbed form of Eq. (4.1)-(4.2). In Eq. (4.6), the term $\boldsymbol{A}_n \boldsymbol{z}(t)$ is the linear part and the term $\widetilde{\boldsymbol{H}}(\boldsymbol{z}(t), u(t))$ is the nonlinear perturbation.

The following assumptions are made about system (4.1)-(4.2).

Assumption 1 *The system (4.1)-(4.2) has a uniform observation relative degree equal to n (n is the dimension of vector $\boldsymbol{x}$), i.e,*

$$\begin{aligned} \forall \boldsymbol{x} \in \mathbb{R}^n \quad \boldsymbol{L}_g \boldsymbol{L}_f^k \boldsymbol{h}(\boldsymbol{x}) &= 0, \quad k = 0, 1, \ldots, n-2 \\ \exists \boldsymbol{x} \in \mathbb{R}^n \quad \boldsymbol{L}_g \boldsymbol{L}_f^{n-1} \boldsymbol{h}(\boldsymbol{x}) &\neq 0 \end{aligned} \tag{4.11}$$

Assumption 2 *System (4.1)-(4.2) is globally drift observable and the diffeomorphism* $\boldsymbol{z} = \boldsymbol{\Phi}(\boldsymbol{x})$ *and its inverse* $\boldsymbol{x} = \boldsymbol{\Phi}^{-1}(\boldsymbol{z})$ *are globally Lipschitz in* $\mathbb{R}^n$*, i.e.,*

$$\|\boldsymbol{\Phi}(\boldsymbol{x}_1) - \boldsymbol{\Phi}(\boldsymbol{x}_2)\| \leq \gamma_{\Phi}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \quad \forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n \tag{4.12}$$

$$\|\boldsymbol{\Phi}^{-1}(\boldsymbol{z}_1) - \boldsymbol{\Phi}^{-1}(\boldsymbol{z}_2)\| \leq \gamma_{\Phi^{-1}}\|\boldsymbol{z}_1 - \boldsymbol{z}_2\|, \quad \forall \boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{R}^n \tag{4.13}$$

Under assumption 2, the Jacobian matrices $\boldsymbol{Q}(\boldsymbol{x})$ *and* $\boldsymbol{Q}^{-1}(\boldsymbol{x})$ *are non-singular in* $\mathbb{R}^n$*.*

Assumption 3 *The vector function* $\widetilde{\boldsymbol{H}}(\boldsymbol{z}(t), u(t))$ *is globally uniformly Lipschitz with respect to* $\boldsymbol{z}$*, and the Lipschitz coefficient* $\boldsymbol{\gamma}_{\widetilde{H}}$ *is a non-decreasing function of* $|u|$*, i.e,*

$$\|\widetilde{\boldsymbol{H}}(\boldsymbol{z}_1, u) - \widetilde{\boldsymbol{H}}(\boldsymbol{z}_2, \boldsymbol{u})\| \leq \boldsymbol{\gamma}_{\widetilde{H}}(|u|)\|\boldsymbol{z}_1 - \boldsymbol{z}_2\| \tag{4.14}$$

4.1 Multiple-input Multiple-output (MIMO) Systems

Eq. (4.3) defines the diffeomorphism (nonlinear coordinate change) for SISO systems. However, for a MIMO systems the equations are modified as explained below [31]

The general state-space representation of MIMO nonlinear systems is

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{G}(\boldsymbol{x}(t))u(t) \quad t \geq \Delta, \quad \boldsymbol{x}(-\Delta) = \bar{\boldsymbol{x}} \tag{4.15}$$

$$\bar{\boldsymbol{y}}(t) = \boldsymbol{H}(\boldsymbol{x}(t - \Delta)), \quad t \geq 0 \tag{4.16}$$

where $\Delta > 0$ is the measurement delay, $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{U}(t) \in \mathbb{R}^p$, $\boldsymbol{y}(t) \in \mathbb{R}^q$, ($p \geq q$), the vector functions $\boldsymbol{f}$, $\boldsymbol{G}$, $\boldsymbol{H}$ are $\boldsymbol{C}^\infty$, the term $\boldsymbol{H}(t)$ can be written as $[\boldsymbol{h}_1(\boldsymbol{x})\ \boldsymbol{h}_2(\boldsymbol{x}) \ldots \boldsymbol{h}_q(\boldsymbol{x})]$ The undelayed output is represented as $\boldsymbol{y}(t) = \boldsymbol{H}(\boldsymbol{x}(t))$. The three assumptions mentioned in the case of SISO systems are modified as follows for the MIMO systems

Assumption 1: The system (4.15)-(4.16) has a global observation relative degree equal to $\sum_{k=1}^{p} r_k = n$, (n is the dimension of vector $\boldsymbol{x}$, $r = [r_1 \ldots r_q]$), i.e,

$$\begin{aligned} \forall \boldsymbol{x} \in \Omega_r, \quad \boldsymbol{L}_G \boldsymbol{L}_f^j \boldsymbol{H}_i(\boldsymbol{x}) &= 0 \quad j = 0, 1, \ldots, r_i - 2 \qquad (4.17) \\ \exists \boldsymbol{x} \in \Omega_r : \quad \boldsymbol{L}_G \boldsymbol{L}_f^{r_i - 1} \boldsymbol{H}_i(\boldsymbol{x}) &\neq 0 \qquad (4.18) \end{aligned}$$

Under this assumption, in an open set $\Omega_r \in \mathbb{R}^n$, Eq.(4.3) is accordingly modified as

$$\boldsymbol{\Phi}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{h}_1(\boldsymbol{x}) \\ \vdots \\ \boldsymbol{L}_f^{r_1 - 1} \boldsymbol{h}_1(\boldsymbol{x}) \\ \boldsymbol{h}_2(\boldsymbol{x}) \\ \vdots \\ \boldsymbol{L}_f^{r_2 - 1} \boldsymbol{h}_2(\boldsymbol{x}) \\ \vdots \\ \boldsymbol{h}_r(\boldsymbol{x}) \\ \vdots \\ \boldsymbol{L}_f^{r_q - 1} \boldsymbol{h}_q(\boldsymbol{x}) \end{bmatrix} \qquad (4.19)$$

where $\boldsymbol{L}_f \boldsymbol{h}_i(\boldsymbol{x})$ denotes the Lie derivative of the function $\boldsymbol{h}_i(\boldsymbol{x})$ along $\boldsymbol{f}$ and $\boldsymbol{L}_f^k \boldsymbol{h}_i(\boldsymbol{x})$ denotes the k-th order repeated Lie derivative of the function $\boldsymbol{h}_i(\boldsymbol{x})$ along $\boldsymbol{f}$.

Assumption 2: System (4.15)-(4.16) is globally drift observable and the diffeomorphism $\boldsymbol{z} = \boldsymbol{\Phi}(\boldsymbol{x})$ and its inverse $\boldsymbol{x} = \boldsymbol{\Phi}^{-1}(\boldsymbol{z})$ are globally Lipschitz in $\mathbb{R}^n$, i.e.,

$$\|\boldsymbol{\Phi}(\boldsymbol{x}_1) - \boldsymbol{\Phi}(\boldsymbol{x}_2)\| \leq \boldsymbol{\gamma}_{\Phi} \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \quad \forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n \qquad (4.20)$$

$$\|\boldsymbol{\Phi}^{-1}(\boldsymbol{z}_1) - \boldsymbol{\Phi}^{-1}(\boldsymbol{z}_2)\| \leq \boldsymbol{\gamma}_{\Phi^{-1}} \|\boldsymbol{z}_1 - \boldsymbol{z}_2\|, \quad \forall \boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{R}^n \qquad (4.21)$$

Under the Assumption 2, the Jacobian of the map $\boldsymbol{\Phi}(\boldsymbol{x})$, denoted as $\boldsymbol{Q}(\boldsymbol{x})$, and the Jacobian of the inverse map $\boldsymbol{\Phi}^{-1}(\boldsymbol{x})$, denoted as $\boldsymbol{Q}^{-1}(\boldsymbol{x})$ are non-singular in $\mathbb{R}^n$.

From the definition of $\boldsymbol{\Phi}(\boldsymbol{x})$, the following properties are obtained:

$$\boldsymbol{Q}(\boldsymbol{x}).\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}_n\boldsymbol{\Phi}(\boldsymbol{x}) + \boldsymbol{B}_n\boldsymbol{L}_f^n\boldsymbol{h}(\boldsymbol{x}), \quad \boldsymbol{H}(\boldsymbol{x}(t)) = \boldsymbol{C}_n\boldsymbol{\Phi}(\boldsymbol{x}) \tag{4.22}$$

where $(\boldsymbol{A}_n, \boldsymbol{B}_n, \boldsymbol{C}_n)$ are block diagonal matrices defined as

$$\boldsymbol{A}_n = diag_{i=1}^n\boldsymbol{A}_i, \quad \boldsymbol{B}_n = diag_{i=1}^n\boldsymbol{B}_i, \quad \boldsymbol{C}_n = diag_{i=1}^n\boldsymbol{C}_i \tag{4.23}$$

The n triples $(\boldsymbol{A}_j, \boldsymbol{B}_j, \boldsymbol{C}_j)$ are the Brunowski matrices.

$$\boldsymbol{A}_j = \begin{bmatrix} 0_{(j-1)\times 1} & \boldsymbol{I}_{j-1} \\ 0 & 0_{1\times(j-1)} \end{bmatrix}, \quad \boldsymbol{B}_j = \begin{bmatrix} 0_{(j-1)\times 1} \\ 1 \end{bmatrix}, \quad \boldsymbol{C}_j = \begin{bmatrix} 1 & 0_{1\times(j-1)} \end{bmatrix} \tag{4.24}$$

Remark 1: The pair $\boldsymbol{A}_n$,$\boldsymbol{C}_n$ is observable and the pair $\boldsymbol{A}_n$,$\boldsymbol{B}_n$ is controllable.

Under assumption 1, $\boldsymbol{z} = \boldsymbol{\Phi}(\boldsymbol{x})$ defines a global change of coordinates. Differentiating $\boldsymbol{z} = \boldsymbol{\Phi}(\boldsymbol{x})$ w.r.t t, and using Eq.(4.22), the nonlinear system equation in terms of the $\boldsymbol{z}$-coordinates is

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}_n\boldsymbol{z}(t) + \widetilde{\boldsymbol{H}}(\boldsymbol{z}(t), \boldsymbol{U}(t)), \quad t \geq -\Delta, \quad \boldsymbol{z}(-\Delta) = \boldsymbol{\Phi}(\bar{\boldsymbol{x}}) \tag{4.25}$$

$$\bar{\boldsymbol{y}}(t) = \boldsymbol{C}_n\boldsymbol{z}(t-\Delta), \quad t \geq 0 \tag{4.26}$$

where,

$$\widetilde{\boldsymbol{H}}(\boldsymbol{z}(t), \boldsymbol{U}(t)) = \boldsymbol{H}(\boldsymbol{x}(t), \boldsymbol{U}(t))|_{\boldsymbol{x}=\boldsymbol{\Phi}^{-1}(\boldsymbol{z})} \tag{4.27}$$

$$\widetilde{\boldsymbol{H}}(\boldsymbol{x}(t), \boldsymbol{U}(t)) = \boldsymbol{B}_n\left(\begin{bmatrix} \boldsymbol{L}_f^{r_1}\boldsymbol{h}_1(\boldsymbol{x}) \\ \boldsymbol{L}_f^{r_2}\boldsymbol{h}_2(\boldsymbol{x}) \\ \vdots \\ \boldsymbol{L}_f^{r_q}\boldsymbol{h}_q(\boldsymbol{x}) \end{bmatrix} + \begin{bmatrix} \boldsymbol{L}_G L_f^{r_1}\boldsymbol{h}_1(\boldsymbol{x}) \\ \boldsymbol{L}_G\boldsymbol{L}_f^{r_2}\boldsymbol{h}_2(\boldsymbol{x}) \\ \vdots \\ \boldsymbol{L}_G L_f^{r_q}\boldsymbol{h}_q(\boldsymbol{x}) \end{bmatrix}\boldsymbol{U}(t)\right) \tag{4.28}$$

Assumption 3: The vector function $\widetilde{\boldsymbol{H}}(\boldsymbol{z}(t), \boldsymbol{U}(t))$ defined in Eq.(4.27) is globally uniformly Lipschitz with respect to $\boldsymbol{z}$, and the Lipschitz coefficient $\boldsymbol{\gamma}_{\widetilde{H}}$ is a non decreasing function of $|u(t)|$, i.e,

$$\|\widetilde{\boldsymbol{H}}(\boldsymbol{z}_1, u) - \widetilde{\boldsymbol{H}}(\boldsymbol{z}_2, u)\| \leq \boldsymbol{\gamma}_{\widetilde{H}}(|u|)|\boldsymbol{z}_1 - \boldsymbol{z}_2\|, \quad \forall \boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{R}^n \tag{4.29}$$

$$\boldsymbol{\gamma}_{\widetilde{H}}(|u|) = \max\{\boldsymbol{\gamma}_{\widetilde{H}}(|u_1|), \boldsymbol{\gamma}_{\widetilde{H}}(|u_2|), \ldots, \boldsymbol{\gamma}_{\widetilde{H}}(|u_q|)\} \tag{4.30}$$

The stability analysis will be presented for a SISO nonlinear system. The same proof can be extended to MIMO systems, by using the appropriate matrices as explained above.

4.2 State Observer for Nonlinear SISO systems

A state observer for undelayed nonlinear system is[7]

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{f}(\hat{\boldsymbol{x}}(t)) + \boldsymbol{g}(\hat{\boldsymbol{x}}(t))u(t) + \boldsymbol{Q}^{-1}(\hat{\boldsymbol{x}}(t))\boldsymbol{K}(\boldsymbol{y}(t) - \boldsymbol{h}(\hat{\boldsymbol{x}}(t))) \tag{4.31}$$

The exponential convergence to zero of the observation error is expressed as

$$\|\boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)\| \leq \mu \exp(-\alpha t)\|\boldsymbol{x}(0) - \hat{\boldsymbol{x}}(0)\| \tag{4.32}$$

and is guaranteed by the following theorems:

Theorem 1: Consider system (4.1)-(4.2) with undelayed output $\boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t))$ under assumptions 2 and 3. Then, for any positive α there exist a gain vector $\boldsymbol{K}$ for the observer (4.31) and positive constants μ and u_M such that if $|u(t)| \leq u_M$ for $t \geq 0$, then (4.32) holds.

Theorem 2: Consider system (4.1)-(4.2) with undelayed output $\boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t))$, under assumptions 1, 2 and 3. Assume that there exists u_M such that $|u(t)| \leq u_M$, for t ≥ 0. Then, for any positive α there exists a gain vector $\boldsymbol{K}$ for (4.32) holds.

Remark 2: In Theorems 1 and 2 the convergence of the observer (4.31) is implied without the assumption of uniform observability [6], which is a much stronger assumption than drift-observability (in uniformly observable systems all states are distinguishable independently of input). On the other hand some *a priori* limitation on the input is required. Theorem 2 states that if assumptions 1, 2 and 3 hold, then, for any given *a priori* bound u_M, on the inputs, an observer with any prescribed exponential convergence rate μ can be found. If only assumptions 2 and 3 hold, as in Theorem 1, then the existence of an exponential observer is guaranteed if the input amplitudes satisfies an upper bound (depending on the prescribed convergence rate μ). This happens because assumption 1 (global observation relative degree equal to n) implies uniform observability. For systems that do not satisfy assumption 1 a condition excluding bad inputs (those that destroy observability) is needed. The bound on the inputs amplitude given by Theorem 1 is sufficient to exclude such bad inputs.

The time-delay observer theory is developed based on the theory of nonlinear observers for undelayed systems. In the subsequent sections, the observer dynamics and stability analysis are presented for the following two cases:

- Output Measurements with Constant Known Delays.
- Output Measurements with Time-varying known delays

4.3 Case 1: Constant Delays

The theory of chain observers for nonlinear systems was proposed by Germani et al. [17]. According to them, theoretically, an infinite number of systems in the chain observer can be used to estimate the output with any delay magnitude. However, for systems which requires faster response, this scheme would fail to give desired results in a shorter time interval. Hence, in this thesis an observer chain with two systems is considered. The systems are: zeroth observer and the first.

Unlike the linear case, the two observer system is not sufficient to estimate delayed signal of any delay magnitude. The relation between the delay magnitude and the length of observer chain was derived by Germani et al.[17]. As stated in Chapter 2, this relation seems to suggest that the gain matrix $\boldsymbol{K}$, chosen to stabilize the observer is completely unrelated to the delay magnitude. But, intuitive thinking would suggest otherwise. Hence, the stability proof of the observer is reformulated in this section and thereby obtaining an explicit relation between the delay magnitude and the observer gain.

The observer dynamics for systems with delayed output has the following structure

$$\dot{\hat{\boldsymbol{x}}}(t-\Delta) = \boldsymbol{f}(\hat{\boldsymbol{x}}(t-\Delta)) + \boldsymbol{g}(\hat{\boldsymbol{x}}(t-\Delta))u(t-\Delta) + \boldsymbol{Q}^{-1}(\hat{\boldsymbol{x}}(t-\Delta))\boldsymbol{K}\left(\bar{\boldsymbol{y}}(t) - \boldsymbol{h}(\hat{\boldsymbol{x}}(t-\Delta))\right) \tag{4.33}$$

$$\begin{aligned}\dot{\hat{\boldsymbol{x}}}_1(t) &= \boldsymbol{f}(\hat{\boldsymbol{x}}_1(t)) + \boldsymbol{g}(\hat{\boldsymbol{x}}_1(t))u_1(t) + \boldsymbol{Q}^{-1}(\hat{\boldsymbol{x}}_1(t))\{\exp(\boldsymbol{A}_n\Delta)\boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{h}(\hat{\boldsymbol{x}}(t-\Delta)))\\ &\quad + \exp(\boldsymbol{A}_n\Delta)(\boldsymbol{H}(\hat{\boldsymbol{x}}(t-\Delta), u(t-\Delta)) - \boldsymbol{H}(\hat{\boldsymbol{x}}_1(t-\Delta), u_1(t-\Delta)))\}\end{aligned} \tag{4.34}$$

The variable $\hat{\boldsymbol{x}}(t-\Delta)$ is an estimate of the states at time $t-\Delta$ and variable $\hat{\boldsymbol{x}}_1$ is the estimate of the states at time t with the following initial conditions

$$\hat{\boldsymbol{x}}(t-\Delta)|_{t=0} = \hat{\boldsymbol{x}}(-\Delta) \tag{4.35}$$

$$\hat{\boldsymbol{x}}_1(\tau) = \hat{\boldsymbol{x}}(\tau), \quad \tau \in [-\Delta, 0] \tag{4.36}$$

where $\hat{\boldsymbol{x}}(\tau) \in [-\Delta, 0]$ is any *a priori* estimate of the state.

4.3.1 Stability analysis of the zeroth observer

The observer (4.33) after applying the nonlinear change of coordinates (4.3) can be expressed as

$$\dot{\hat{\boldsymbol{z}}}(t-\Delta) = \boldsymbol{A}_n\hat{\boldsymbol{z}}(t-\Delta) + \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(t-\Delta), u(t-\Delta)) + \boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{C}_n\hat{\boldsymbol{z}}(t-\Delta)) \quad t \geq 0 \tag{4.37}$$

with initial condition

$$\hat{\boldsymbol{z}}(t-\Delta)|_{t=0} = \boldsymbol{\Phi}(\hat{\boldsymbol{x}}(-\Delta)) \tag{4.38}$$

Proof: The time derivative of $\hat{\boldsymbol{z}}(t-\Delta) = \boldsymbol{\Phi}(\hat{\boldsymbol{x}}(t-\Delta))$ is

$$\begin{aligned}\dot{\hat{\boldsymbol{z}}}(t-\Delta) &= \frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{x}(t-\Delta)}\dot{\hat{\boldsymbol{x}}}(t-\Delta)\\ &= \boldsymbol{Q}(\hat{\boldsymbol{x}}(t-\Delta))(\boldsymbol{f}(\hat{\boldsymbol{x}}(t-\Delta)) + \boldsymbol{g}(\hat{\boldsymbol{x}}(t-\Delta))u(t-\Delta)\\ &\quad + \boldsymbol{Q}^{-1}(\hat{\boldsymbol{x}}(t-\Delta))\boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{h}(\hat{\boldsymbol{x}}(t-\Delta))))\\ &= \boldsymbol{A}_n\boldsymbol{\Phi}(\hat{\boldsymbol{x}}(t-\Delta)) + \boldsymbol{B}_n\boldsymbol{L}_f^n\boldsymbol{h}(\hat{\boldsymbol{x}}(t-\Delta))\\ &\quad + \boldsymbol{Q}(\hat{\boldsymbol{x}}(t-\Delta))\boldsymbol{g}(\hat{\boldsymbol{x}}(t-\Delta))u(t-\Delta) + \boldsymbol{K}\left(\bar{\boldsymbol{y}}(t) - \boldsymbol{h}(\hat{\boldsymbol{x}}(t-\Delta))\right)\\ &= \boldsymbol{A}_n\hat{\boldsymbol{z}}(t-\Delta)(t) + \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(t-\Delta), u(t-\Delta)) + \boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{C}_n\hat{\boldsymbol{z}}(t-\Delta))\end{aligned}$$

The observer stability is proved using the Lyapunov theorem.

The observation error is defined as:

$$\boldsymbol{\eta}_{z_0}(t) = \boldsymbol{z}(t-\Delta) - \hat{\boldsymbol{z}}(t-\Delta) \tag{4.39}$$

The time derivative of Eq.(4.39) is

$$\begin{aligned}
\dot{\boldsymbol{\eta}}_{z_0}(t) &= \dot{\boldsymbol{z}}(t-\Delta) - \dot{\hat{\boldsymbol{z}}}(t-\Delta) \\
\dot{\boldsymbol{\eta}}_{z_0}(t) &= \boldsymbol{A}_n\boldsymbol{z}(t-\Delta) + \widetilde{\boldsymbol{H}}(\boldsymbol{z}(t-\Delta), u(t-\Delta)) \\
&\quad - \Big(\boldsymbol{A}_n\hat{\boldsymbol{z}}(t-\Delta) + \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(t-\Delta), u(t-\Delta)) + \boldsymbol{K}\boldsymbol{C}_n(\boldsymbol{z}(t-\Delta) - \hat{\boldsymbol{z}}(t-\Delta))\Big) \\
\dot{\boldsymbol{\eta}}_{z_0}(t) &= \boldsymbol{A}_n(\boldsymbol{z}(t-\Delta) - \hat{\boldsymbol{z}}(t-\Delta)) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(t-\Delta), u(t-\Delta)) + \widetilde{\boldsymbol{H}}(\boldsymbol{z}(t-\Delta), u(t-\Delta)) \\
&\quad - \boldsymbol{K}\boldsymbol{C}_n(\boldsymbol{z}(t-\Delta) - \hat{\boldsymbol{z}}(t-\Delta)) \\
\dot{\boldsymbol{\eta}}_{z_0}(t) &= \boldsymbol{A}_n\boldsymbol{\eta}_{z_0}(t) + \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_0, \hat{\boldsymbol{z}}_0, u_0) - \boldsymbol{K}\boldsymbol{C}_n\boldsymbol{\eta}_{z_0}(t) \\
\dot{\boldsymbol{\eta}}_{z_0}(t) &= \boldsymbol{A}_m\boldsymbol{\eta}_{z_0}(t) + \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_0, \hat{\boldsymbol{z}}_0, u_0) \qquad (4.40)
\end{aligned}$$

where,

$$\boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_0, \hat{\boldsymbol{z}}_0, u_0) = \widetilde{\boldsymbol{H}}(\boldsymbol{z}(t-\Delta), u(t-\Delta)) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(t-\Delta), u(t-\Delta)), \quad \boldsymbol{A}_m = \boldsymbol{A}_n - \boldsymbol{K}\boldsymbol{C}_n$$

Next, consider a Lyapunov function of the form,

$$\boldsymbol{V}_{\eta_{z_0}} = \frac{1}{2}\boldsymbol{\eta}_{z_0}^T\boldsymbol{P}\boldsymbol{\eta}_{z_0} \tag{4.41}$$

where $\boldsymbol{P}$ is a positive definite symmetric matrix that satisfies the Lyapunov equation:

$$\boldsymbol{A}_m^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_m = -\boldsymbol{Q} \tag{4.42}$$

for $\boldsymbol{Q} = \boldsymbol{Q}^T > 0$.

The time derivative of $\boldsymbol{V}_{\eta_{z_0}}$ along error trajectory is

$$\begin{aligned}
\dot{\boldsymbol{V}}_{\eta_{z_0}} &= \frac{1}{2}\boldsymbol{\eta}_{z_0}^T\left(\boldsymbol{P}\boldsymbol{A}_m + \boldsymbol{A}_m^T\boldsymbol{P}\right)\boldsymbol{\eta}_{z_0} + \boldsymbol{\eta}_{z_0}^T\boldsymbol{P}\boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_0, \hat{\boldsymbol{z}}_0, u_0) \\
\|\dot{\boldsymbol{V}}_{\eta_{z_0}}\| &\leq -\frac{1}{2}\|\boldsymbol{\eta}_{z_0}^T\boldsymbol{Q}\boldsymbol{\eta}_{z_0}\| + \|\boldsymbol{\eta}_{z_0}^T\|.\|\boldsymbol{P}\|.\gamma_{\widetilde{H}}(|u(t)|)\|\boldsymbol{z}(t-\Delta) - \hat{\boldsymbol{z}}(t-\Delta)\| \\
&\leq -\left(\frac{1}{2}\lambda_{\min}(\boldsymbol{Q}) - \lambda_{\max}(\boldsymbol{P})\boldsymbol{\gamma}_{\widetilde{H}}(|u(t)|)\right)\|\boldsymbol{\eta}_{z_0}\|^2 \qquad (4.43)
\end{aligned}$$

Hence, the zeroth observer is exponentially stable for $\left(\frac{1}{2}\lambda_{\min}(\boldsymbol{Q}) - \lambda_{\max}(\boldsymbol{P})\boldsymbol{\gamma}_{\widetilde{H}}(|u(t)|)\right) >$ 0

$$\Rightarrow \|\boldsymbol{\eta}_{z_0}\| = \mu_0 \exp(-\alpha_0 t) \tag{4.44}$$

4.3.2 Stability analysis of the first observer

For notational convenience, the following convention is used: $\boldsymbol{z}_{10} = \boldsymbol{z}_1(t-\Delta)$, $\hat{\boldsymbol{z}}_{10} = \hat{\boldsymbol{z}}_1(t-\Delta)$ and $u_{10} = u(t-\Delta)$.

Lemma 1: Using the nonlinear change of coordinates, observer 1 can be rewritten as

$$\hat{\boldsymbol{z}}_1(t) = \exp(\boldsymbol{A}_n\Delta)\hat{\boldsymbol{z}}(t-\Delta) + \int_{t-\Delta}^{t} \exp(\boldsymbol{A}_n(t-\tau))\widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(\tau), u_1(\tau))d\tau \tag{4.45}$$

Proof of Lemma 1 is reported in the appendix.

Similarly, the nonlinear states can be represented as

$$\boldsymbol{z}_1(t) = \exp(\boldsymbol{A}_n\Delta)\boldsymbol{z}(t-\Delta) + \int_{t-\Delta}^{t} \exp(\boldsymbol{A}_n(t-\tau))\widetilde{\boldsymbol{H}}(\boldsymbol{z}_1(\tau), u_1(\tau))d\tau \tag{4.46}$$

The observation error for is defined as

$$\boldsymbol{\eta}_{z_1}(t) = \boldsymbol{z}_1(t) - \hat{\boldsymbol{z}}_1(t) \tag{4.47}$$

Substituting Eqs. (4.45) and (4.46),

$$\boldsymbol{\eta}_{z_1}(t) = \exp(\boldsymbol{A}_n\Delta)\boldsymbol{\eta}_{z_0}(t) + \int_{t-\Delta}^{t} \exp(\boldsymbol{A}_n(t-\tau))\left(\widetilde{\boldsymbol{H}}(\boldsymbol{z}_1(\tau), u_1(\tau)) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(\tau), u_1(\tau))\right)d\tau \tag{4.48}$$

The time derivative of $\boldsymbol{\eta}_{z_1}(t)$ is

$$\begin{aligned}\dot{\boldsymbol{\eta}}_{z_1}(t) &= \exp(\boldsymbol{A}_n\Delta)\dot{\boldsymbol{\eta}}_{z_0}(t) + \boldsymbol{A}_n\int_{t-\Delta}^{t} \exp(\boldsymbol{A}_n(t-\tau))\left(\widetilde{\boldsymbol{H}}(\boldsymbol{z}_1(\tau), u_1(\tau)) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(\tau), u_1(\tau))\right)d\tau \\ &\quad + \widetilde{\boldsymbol{H}}(\boldsymbol{z}_1(t), u_1(t)) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(t), u_1(t)) \\ &\quad - \exp(\boldsymbol{A}_n\Delta)\left(\widetilde{\boldsymbol{H}}(\boldsymbol{z}_1(t-\Delta), u_1(t-\Delta)) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(t-\Delta), u_1(t-\Delta))\right)\end{aligned} \tag{4.49}$$

From Eq.(4.48),

$$\int_{t-\Delta}^{t} \exp(\boldsymbol{A}_n(t-\tau))\left(\widetilde{\boldsymbol{H}}(\boldsymbol{z}_1(\tau), u_1(\tau)) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(\tau), u_1(\tau))\right)d\tau = \boldsymbol{\eta}_{z_1}(t) - \exp(\boldsymbol{A}_n\Delta)\boldsymbol{\eta}_{z_0}(t)$$

Substituting this result in Eq.(4.49),

$$\begin{aligned}\dot{\boldsymbol{\eta}}_{z_1}(t) &= \exp(\boldsymbol{A}_n\Delta)\dot{\boldsymbol{\eta}}_{z_0}(t) + \boldsymbol{A}_n\left(\boldsymbol{\eta}_{z_1}(t) - \exp(\boldsymbol{A}_n\Delta)\boldsymbol{\eta}_{z_0}(t)\right) + \widetilde{\boldsymbol{H}}(\boldsymbol{z}_1(t), u_1(t)) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(t), u_1(t)) \\ &\quad + \exp(\boldsymbol{A}_n\Delta)\left(\widetilde{\boldsymbol{H}}(\boldsymbol{z}_1(t-\Delta), u_1(t-\Delta)) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(t-\Delta), u_1(t-\Delta))\right) \end{aligned} \tag{4.50}$$

Substituting $\dot{\boldsymbol{\eta}}_{z_0}(t)$ from Eq.(4.40) and rearranging,

$$\begin{aligned}\dot{\boldsymbol{\eta}}_{z_1}(t) &= \boldsymbol{A}_n\boldsymbol{\eta}_{z_1}(t) + \exp(\boldsymbol{A}_n\Delta)\left(\boldsymbol{A}_m\boldsymbol{\eta}_{z_0}(t) + \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_0, \hat{\boldsymbol{z}}_0, u_0) - \boldsymbol{A}_n\boldsymbol{\eta}_{z_0}(t)\right) \\ &\quad + \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_1(t), \hat{\boldsymbol{z}}_1(t), u_1(t)) + \exp(\boldsymbol{A}_n\Delta)\boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_{10}, \hat{\boldsymbol{z}}_{10}, u_{10}) \\ &= \boldsymbol{A}_n\boldsymbol{\eta}_{z_1}(t) + \exp(\boldsymbol{A}_n\Delta)\left(-\boldsymbol{K}\boldsymbol{C}_n\boldsymbol{\eta}_{z_0}(t) + \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_0(t), \hat{\boldsymbol{z}}_0(t), u_0(t))\right) \\ &\quad + \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_1(t), \hat{\boldsymbol{z}}_1(t), u_1(t)) + \exp(\boldsymbol{A}_n\Delta)\boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_{10}, \hat{\boldsymbol{z}}_{10}, u_{10}) \end{aligned} \tag{4.51}$$

Adding and subtracting $\boldsymbol{K}\boldsymbol{C}_n\boldsymbol{\eta}_{z_1}(t)$,

$$\begin{aligned}\dot{\boldsymbol{\eta}}_{z_1}(t) &= \boldsymbol{A}_m\boldsymbol{\eta}_{z_1}(t) + \boldsymbol{K}\boldsymbol{C}_n(\boldsymbol{\eta}_{z_1} - \exp(\boldsymbol{A}_n\Delta)\boldsymbol{\eta}_{z_0}(t)) + \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_1(t), \hat{\boldsymbol{z}}_1(t), u_1(t)) \\ &\quad + \exp(\boldsymbol{A}_n\Delta)\left\{\boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_0(t), \hat{\boldsymbol{z}}_0(t), u_0(t)) + \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_{10}, \hat{\boldsymbol{z}}_{10}, u_{10})\right\} \end{aligned} \tag{4.52}$$

Substituting the term $\boldsymbol{\eta}_{z_1} - \exp(\boldsymbol{A}_n\Delta)\boldsymbol{\eta}_{z_0}(t)$ with the integral from (4.48),

$$\begin{aligned}\dot{\boldsymbol{\eta}}_{z_1}(t) &= \boldsymbol{A}_m\boldsymbol{\eta}_{z_1}(t) + \boldsymbol{K}\boldsymbol{C}_n\left(\int_{t-\Delta}^{t} \exp(\boldsymbol{A}_n(t-\tau))\boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_1(\tau), \hat{\boldsymbol{z}}_1(\tau), u_1(\tau))d\tau\right) \\ &\quad + \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_1(t), \hat{\boldsymbol{z}}_1(t), u_1(t)) + \exp(\boldsymbol{A}_n\Delta)\left\{\boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_0(t), \hat{\boldsymbol{z}}_0(t), u_0(t)) + \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_{10}, \hat{\boldsymbol{z}}_{10}, u_{10})\right\} \end{aligned} \tag{4.53}$$

The solution for $\boldsymbol{\eta}_{z_1}(t)$ from Eq.(4.53) for $t \geq \Delta$ is given as

$$\begin{aligned}\boldsymbol{\eta}_{z_1}(t) &= \exp(A_m(t-\Delta))\boldsymbol{\eta}_{z_1}(\Delta) \\ &\quad + \int_{\Delta}^{t} \exp(\boldsymbol{A}_m(t-s))\left(\int_{s-\Delta}^{s} \boldsymbol{K}\boldsymbol{C}_n \exp(\boldsymbol{A}_n(s-\theta))\boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_1(\theta), \hat{\boldsymbol{z}}_1(\theta), u_1(\theta))d\theta\right) ds \\ &\quad + \int_{\Delta}^{t} \exp(\boldsymbol{A}_m(t-s))\boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_1(s), \hat{\boldsymbol{z}}_1(s), u_1(s))ds \\ &\quad + \int_{\Delta}^{t} \exp(\boldsymbol{A}_m(t-s))\exp(\boldsymbol{A}_n\Delta)\left\{\boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_0(s), \hat{\boldsymbol{z}}_0(s), u_0(s)) + \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_{10}(s), \hat{\boldsymbol{z}}_{10}(s), u_{10}(s))\right\} ds \end{aligned}$$

It follows that,

$$
\begin{aligned}
\|\boldsymbol{\eta}_{z_1}(t)\| \;\le\; & \|\exp(\boldsymbol{A}_m(t-\Delta))\boldsymbol{\eta}_{z_1}(\Delta)\| \\
& + \left\| \int_{\Delta}^{t} \exp(\boldsymbol{A}_m(t-s)) \left\{ \int_{s-\Delta}^{s} \boldsymbol{K}\boldsymbol{C}_n \exp(\boldsymbol{A}_n(s-\theta)) \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_1(\theta), \hat{\boldsymbol{z}}_1(\theta), u_1(\theta)) d\theta \right\} ds \right\| \\
& + \left\| \int_{\Delta}^{t} \exp(\boldsymbol{A}_m(t-s)) \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_1(s), \hat{\boldsymbol{z}}_1(s), u_1(s)) ds \right\| \\
& + \left\| \int_{\Delta}^{t} \exp(\boldsymbol{A}_m(t-s)) \exp(\boldsymbol{A}_n\Delta) \left\{ \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_0(s), \hat{\boldsymbol{z}}_0(s), u_0(s)) \right. \right. \\
& \left. \left. + \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_{10}(s), \hat{\boldsymbol{z}}_{10}(s), u_{10}(s)) \right\} ds \right\|
\end{aligned}
$$

Since matrix $\boldsymbol{A}_m$ is Hurwitz, $\|\exp(\boldsymbol{A}_m(t-\Delta))\| \le \exp(-m(t-\Delta)), \quad t \ge \Delta$. Using this result,

$$
\begin{aligned}
\|\boldsymbol{\eta}_{z_1}(t)\| \;\le\; & \boldsymbol{\epsilon} \exp(-m(t-\Delta)) \\
& + \int_{\Delta}^{t} \exp(-m(t-s)) \left\{ \int_{s-\Delta}^{s} \|\boldsymbol{K}\boldsymbol{C}_n \exp(\boldsymbol{A}_n(s-\theta))\| \, \boldsymbol{\gamma}_{\widetilde{H}}(|u|) \|\boldsymbol{\eta}_{z_1}(\theta)\| d\theta \right\} ds \\
& + \int_{\Delta}^{t} \exp(-m(t-s)) \boldsymbol{\gamma}_{\widetilde{H}}(|u|) \|\boldsymbol{\eta}_{z_1}(s)\| ds \\
& + \int_{\Delta}^{t} \exp(-m(t-s)) \|\exp(\boldsymbol{A}_n\Delta)\| \boldsymbol{\gamma}_{\widetilde{H}}(|u|) \|\boldsymbol{\eta}_{z_0}(s)\| ds \\
& + \int_{\Delta}^{t} \exp(-m(t-s)) \|\exp(\boldsymbol{A}_n\Delta)\| \boldsymbol{\gamma}_{\widetilde{H}}(|u|) \|\boldsymbol{\eta}_{z_1}(s-\Delta)\| ds
\end{aligned}
\tag{4.54}
$$

where, $\boldsymbol{\epsilon} = \sup_{-\Delta \le t \le \Delta} \|\boldsymbol{\eta}_{z_1}(t)\|$

$$
\begin{aligned}
\|\boldsymbol{\eta}_{z_1}(t)\| \;\le\; & \boldsymbol{\epsilon} \exp(-m(t-\Delta)) \\
& + \int_{\Delta}^{t} \exp(-m(t-s)) \boldsymbol{\gamma}_{\widetilde{H}}(|u|) \; \Delta \, \|\boldsymbol{K}\boldsymbol{C}_n \exp(\boldsymbol{A}_n\Delta)\| \sup_{s-\Delta \le \xi \le s} \|\boldsymbol{\eta}_{z_1}(\xi)\| ds \\
& + \int_{\Delta}^{t} \exp(-m(t-s)) \boldsymbol{\gamma}_{\widetilde{H}}(|u|) \; \|\exp(\boldsymbol{A}_n\Delta)\| \mu_0 \exp(-\alpha_0 s) ds \\
& + \int_{\Delta}^{t} \exp(-m(t-s)) \gamma_{\widetilde{H}}(|u|) \; \left(\|\boldsymbol{\eta}_{z_1}(s)\| + \|\exp(\boldsymbol{A}_n\Delta)\| . \|\boldsymbol{\eta}_{z_1}(s-\Delta)\| \right) ds
\end{aligned}
$$

Evaluating the integral associated with the term $\mu_0 \exp(-\alpha_0 s)$, the above can further be simplified as,

$$\begin{aligned}
\|\boldsymbol{\eta}_{z_1}(t)\| &\leq \boldsymbol{\epsilon} \exp(-m(t-\Delta)) \\
&+ \frac{\boldsymbol{\gamma}_{\widetilde{H}}(|u|)\ \|\exp(\boldsymbol{A}_n\Delta)\|\mu_0}{m-\alpha_0} \left(\exp(-\alpha_0 t) - \exp(-\alpha_0\Delta)\exp(-m(t-\Delta))\right) \\
&+ \int_{\Delta}^{t} \exp(-m(t-s))\boldsymbol{\gamma}_{\widetilde{H}}(|u|)\ \Delta\, \|\boldsymbol{K}\boldsymbol{C}_n \exp(\boldsymbol{A}_n\Delta)\| \sup_{s-\Delta\leq\xi\leq s} \|\boldsymbol{\eta}_{z_1}(\xi)\| ds \\
&+ \int_{\Delta}^{t} \exp(-m(t-s))\boldsymbol{\gamma}_{\widetilde{H}}(|u|)\ \left(\|\boldsymbol{\eta}_{z_1}(s)\| + \|\exp(\boldsymbol{A}_n\Delta)\|.\|\boldsymbol{\eta}_{z_1}(s-\Delta)\|\right) ds \\
\Rightarrow \|\boldsymbol{\eta}_{z_1}(t)\| &\leq \left(\boldsymbol{\epsilon} - \frac{\gamma_{\widetilde{H}}(|u|)\ \|\exp(\boldsymbol{A}_n\Delta)\|\boldsymbol{\mu}_0}{m-\alpha_0}\exp(-\alpha_0\Delta)\right)\exp(-m(t-\Delta)) \\
&+ \frac{\boldsymbol{\gamma}_{\widetilde{H}}(|u|)\ \|\exp(\boldsymbol{A}_n\Delta)\|\mu_0}{m-\boldsymbol{\alpha}_0}\exp(-\alpha_0 t) \\
&+ \int_{\Delta}^{t} \exp(-m(t-s))\gamma_{\widetilde{H}}(|u|)\ \Delta\, \|\boldsymbol{K}\boldsymbol{C}_n \exp(A_n\Delta)\| \sup_{s-\Delta\leq\xi\leq s} \|\boldsymbol{\eta}_{z_1}(\xi)\| ds \\
&+ \int_{\Delta}^{t} \exp(-m(t-s))\boldsymbol{\gamma}_{\widetilde{H}}(|u|)\ \left(\|\boldsymbol{\eta}_{z_1}(s)\| + \|\exp(\boldsymbol{A}_n\Delta)\|.\|\boldsymbol{\eta}_{z_1}(s-\Delta)\|\right) ds
\end{aligned}$$

Defining $\bar{\mu}_0 = \frac{\boldsymbol{\gamma}_{\widetilde{H}}(|u|)\ \|\exp(\boldsymbol{A}_n\Delta)\|\mu_0}{m-\alpha_0}\exp(-\alpha_0\Delta)$, $\Xi = \gamma_{\widetilde{H}}(|u|)\, \|\boldsymbol{K}\boldsymbol{C}_n \exp(\boldsymbol{A}_n\Delta)\|$, $\alpha = \gamma_{\widetilde{H}}(|u|)$, $\beta = \gamma_{\widetilde{H}}(|u|)\|\exp(\boldsymbol{A}_n\Delta)\|$ and $\epsilon_0 = \epsilon + \bar{\mu}_0$, the above inequality can be re-written as,

$$\begin{aligned}
\|\boldsymbol{\eta}_{z_1}(t)\| &\leq \boldsymbol{\epsilon}_0 \exp(-m(t-\Delta)) + \bar{\mu}_0\exp(-\alpha_0 t) + \int_{\Delta}^{t} \exp(-m(t-s))\Delta\ \Xi \sup_{s-\Delta\leq\xi\leq s} \|\boldsymbol{\eta}_{z_1}(\xi)\| ds \\
&+ \int_{\Delta}^{t} \exp(-m(t-s))\left(\alpha\|\boldsymbol{\eta}_{z_1}(s)\| + \beta\|\boldsymbol{\eta}_{z_1}(s-\Delta)\|\right) ds
\end{aligned}$$

Let

$$\begin{aligned}
\boldsymbol{r}(t) &= \boldsymbol{\epsilon}_0 \exp(-m(t-\Delta)) + \bar{\mu}_0\exp(-\alpha_0 t) + \int_{\Delta}^{t} \exp(-m(t-s))\Delta\ \Xi \sup_{s-\Delta\leq\xi\leq s} \|\boldsymbol{\eta}_{z_1}(\xi)\| ds \\
&+ \int_{\Delta}^{t} \exp(-m(t-s))\left(\alpha\|\boldsymbol{\eta}_{z_1}(s)\| + \beta\|\boldsymbol{\eta}_{z_1}(s-\Delta)\|\right) ds
\end{aligned}$$

Taking the time derivative of $\boldsymbol{r}(t)$ and adding and subtracting the term $m\bar{\mu}_0\exp(-\alpha_0 t)$,

$$\dot{\boldsymbol{r}}(t) = -m\boldsymbol{r}(t) + \Delta\Xi \sup_{t-\Delta\leq\xi\leq t} \|\boldsymbol{\eta}_{z_1}(\xi)\| + \alpha\|\boldsymbol{\eta}_{z_1}(s)\| + \beta\|\boldsymbol{\eta}_{z_1}(s-\Delta)\| + m\bar{\mu}_0\exp(-\alpha_0 t) - \alpha_0\bar{\mu}_0\exp(-\alpha_0 t) \tag{4.55}$$

Using the comparison theorem [32], one obtains

$$\|\boldsymbol{\eta}_{z_1}(t)\| \leq \|\boldsymbol{r}(t)\|$$

Hence, the asymptotic stability of $\boldsymbol{r}(t)$ implies that of $\boldsymbol{\eta}_{z_1}(t)$ and

$$\text{sup}_{t-\Delta\leq\omega\leq t}\boldsymbol{\eta}_{z_1}(\omega) \leq \text{sup}_{t-\Delta\leq\omega\leq t}\boldsymbol{r}(\omega)$$

and

$$\|\boldsymbol{\eta}_{z_1}(t-\Delta)\| \leq \boldsymbol{r}(t-\Delta) \leq \text{sup}_{t-\Delta\leq\omega\leq t}\boldsymbol{r}(\omega)$$

$$\dot{\boldsymbol{r}}(t) \leq -(m-\alpha)\boldsymbol{r}(t) + (\Delta\Xi+\beta)\text{sup}_{t-\Delta\leq\omega\leq t}\boldsymbol{r}(\omega) + (m-\alpha_0)\bar{\mu}_0\exp(-\alpha_0 t) \tag{4.56}$$

The asymptotic stability of the observer in z-coordinates implies the asymptotic stability of the observer in normal coordinate system.

Lemma 2 (Halanay [33], Kolmanovskii and Nosov [34]). *Let* $\dot{\boldsymbol{r}}(t) \leq -\alpha\boldsymbol{r}(t)+\beta\text{sup}_{t-\Delta\leq\omega\leq t}\boldsymbol{r}(\omega)$ *for* $t \geq t_0$ *and if* $\alpha > \beta > 0$, *then there exists a* $\xi > 0$ *and a* $k > 0$ *such that* $\boldsymbol{r}(t) \leq k\exp(-\xi(t-t_0))$ *for* $t \geq t_0$.

A modification of **Lemma 2** for the equations of type (4.56) is stated as follows:

Lemma 3: *Let* $\dot{\boldsymbol{r}}(t) \leq -\alpha\boldsymbol{r}(t) + \beta\text{sup}_{t-\Delta\leq\omega\leq t}\boldsymbol{r}(\omega) + \vartheta\exp(-\varpi t)$ *for* $t \geq t_0$ *and if* $\alpha > \beta > 0$, $\varpi > 0$, *then there exists a* $\xi > 0$, $k > 0$ *and a* $k_0 > 0$ *such that* $\boldsymbol{r}(t) \leq k\exp(-\xi(t-t_0)) + k_0\{\exp(-\varpi(t-t_0)) - \exp(-\alpha(t-t_0))\}$ *for* $t \geq t_0$.

Applying **Lemma 3** to (4.56),

$$m-\alpha > \Delta\Xi+\beta > 0, \quad (m-\alpha_0)\bar{\mu}_0 > 0 \tag{4.57}$$

which is the condition in Lemma 3, then there exists a $\xi > 0$, $k > 0$ and a $k_0 > 0$ such that $\boldsymbol{r}(t) \leq k\exp(-\xi(t-\Delta)) + k_0\{\exp(-\alpha_0(t-\Delta)) - \exp(-\alpha(t-\Delta))\}$ and the inequality holds for $\boldsymbol{\eta}_{z_1}(t)$ similarly. In this inequality, $k_0\exp(-\alpha_0(t-\Delta))$ is the quantity associated with the zeroth observer which is already proved to be exponentially stable. $k_0\exp(-\alpha(t-\Delta))$ is also stable, since $\alpha = \gamma_{\widetilde{H}}(|u|)$, is the Lipschitz constant of the

system. By definition, the lowest value possible for a Lipschitz constant is zero, which implies that $k_0 \exp(-\alpha(t-\Delta))$ is also an exponentially decaying term.
ξ is the unique solution to the equation,

$$-\xi = -m + \alpha + (\beta + \Delta\Xi)\exp(\Delta\xi) \tag{4.58}$$

Consequently the observer error dynamics is asymptotically stable so long as the following equation is satisfied.

$$-m + \alpha + (\beta + \Delta\Xi) < 0 \tag{4.59}$$

Substituting the respective values of the variables above,

$$-m + \gamma_{\widetilde{H}}(|u|) + \gamma_{\widetilde{H}}(|u|)\|\exp(\boldsymbol{A}_n\Delta)\| + \Delta\|\boldsymbol{K}\boldsymbol{C}_n\|\gamma_{\widetilde{H}}(|u|)\|\exp(\boldsymbol{A}_n\Delta)\| < 0$$

$$\|\exp(\boldsymbol{A}_n\Delta)\|\,(1 + \Delta\|\boldsymbol{K}\boldsymbol{C}_n\|) < \frac{m}{\gamma_{\widetilde{H}}(|u|)} - 1$$

Using, $\max|u| = U_M, \Rightarrow \gamma_{\widetilde{H}}(|u|) \leq \gamma_{U_M}$, thus the maximum amount of time-delay that can be accommodated for a given γ_{U_M} is given by

$$\|\exp(\boldsymbol{A}_n\Delta)\|\,(1 + \Delta\|\boldsymbol{K}\boldsymbol{C}_n\|) < \frac{m}{\gamma_{U_M}} - 1 \tag{4.60}$$

For a solution to exist for the above nonlinear inequality, i.e. to accommodate finite time delays in the output, $\frac{m}{\gamma_{U_M}} > 1$ and this in turn dictates the size of the minimum eigenvalue of $\boldsymbol{A}_m$. Since $\boldsymbol{A}_m = \boldsymbol{A}_n - \boldsymbol{K}\boldsymbol{C}_n$. This in turn can be used to compute the observer gain matrix, $\boldsymbol{K}$.

4.3.3 Simulation Results

The formation control of spacecrafts is used as a simulation example for nonlinear systems. The formation consists of two low-thrust spacecrafts, initially separated by an angle of 90^0 as shown in Fig. (4.1) in the geostationary orbit. The formation is described in earth centered coordinate frame. Both the spacecrafts are moving in the anti-clockwise

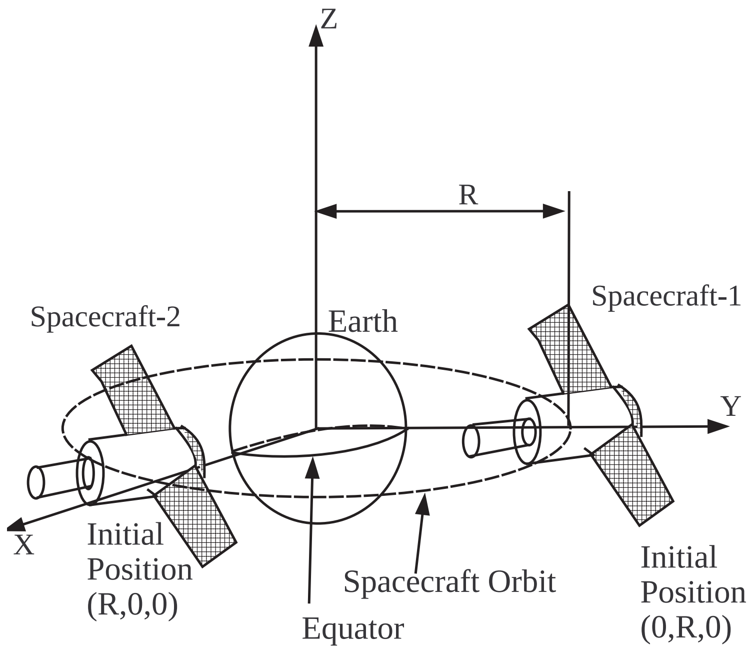

Figure 4.1 Initial Positions of the two spacecrafts.

direction. The control is applied on the second spacecraft such that it aligns itself at a 45^0 angle from the spacecraft-1 at all times. The final formation to be achieved is shown in Fig. (4.2).

The two spacecrafts are communicating with each other. To reach the required formation, it is assumed that the spacecraft-1 transmits its position information. The second satellite, which lies within the communication range of the first, uses this information to orient its position accordingly. However, the output from the first spacecraft is received after a delay. The time delay observer is used to estimate the current states from the delayed measurements. The estimated states are then used to compute the control effort required to maneuver the second spacecraft.

The spacecraft model used for observation is:

$$\ddot{\overrightarrow{\boldsymbol{R}}} = -\frac{\mu}{\boldsymbol{R}^3}\overrightarrow{\boldsymbol{R}} + u(t) \tag{4.61}$$

where $\mu = 3.986 \times 10^5$ and $R = |\overrightarrow{\boldsymbol{R}}|$. $\overrightarrow{\boldsymbol{R}}$ is the position vector.

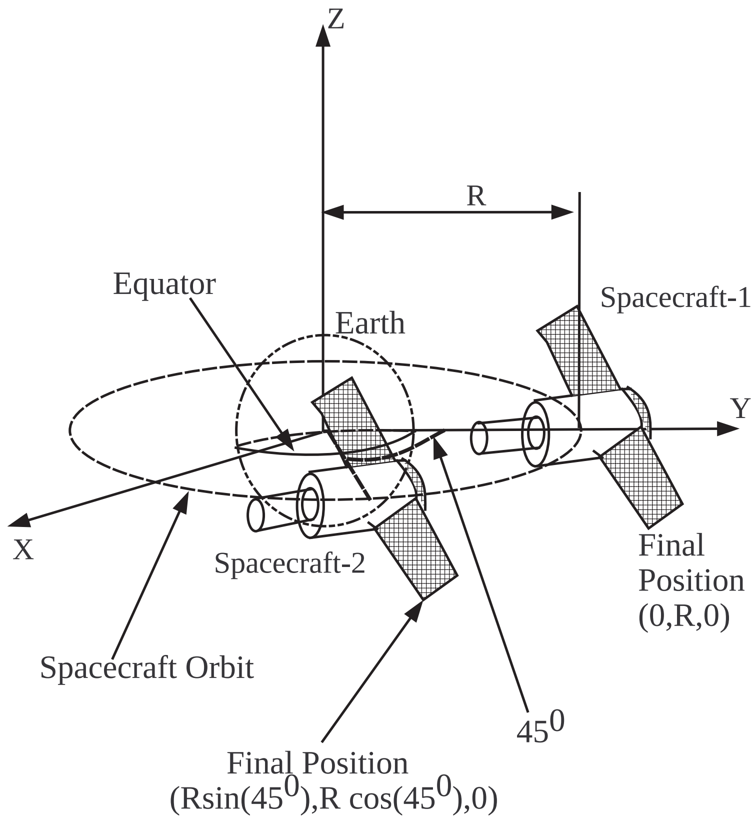

Figure 4.2 Final formation to be achieved by controlling the position of the second spacecraft.

The system states are

$$\boldsymbol{x} = [\boldsymbol{R_x} \quad \boldsymbol{R}_y \quad \boldsymbol{R}_z \quad \dot{\boldsymbol{R}}_{\boldsymbol{x}} \quad \dot{\boldsymbol{R}}_y \quad \dot{\boldsymbol{R}}_z]^T \tag{4.62}$$

State space representation of the spacecraft system is

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} \boldsymbol{x}_4 \\ \boldsymbol{x}_5 \\ \boldsymbol{x}_6 \\ \frac{-\mu \boldsymbol{x}_1}{\boldsymbol{R}^3} \\ \frac{-\mu \boldsymbol{x}_2}{\boldsymbol{R}^3} \\ \frac{-\mu x_3}{\boldsymbol{R}^3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ u_x \\ u_y \\ u_z \end{bmatrix} \tag{4.63}$$

where $\boldsymbol{R} = (\boldsymbol{x}_1^2 + \boldsymbol{x}_2^2 + \boldsymbol{x}_3^2)^{1/2}$

The delayed output of the system is:

$$\bar{\boldsymbol{y}} = \overrightarrow{\boldsymbol{R}}(t-\Delta) \tag{4.64}$$

$$\bar{\boldsymbol{y}} = \begin{bmatrix} \boldsymbol{x}_1(t-\Delta) \\ \boldsymbol{x}_2(t-\Delta) \\ \boldsymbol{x}_3(t-\Delta) \end{bmatrix} \tag{4.65}$$

The delay magnitude for all the outputs are assumed to be 0.5 *sec.* To construct the chain observer, square map is defined as

$$\boldsymbol{z} = \boldsymbol{\Phi}(\boldsymbol{x}) = [\boldsymbol{x}_1 \quad \boldsymbol{x}_4 \quad \boldsymbol{x}_2 \quad \boldsymbol{x}_5 \quad \boldsymbol{x}_3 \quad \boldsymbol{x}_6]^T \tag{4.66}$$

The observation matrix $\boldsymbol{Q}(\boldsymbol{x})$ and the gain matrix K are

$$\boldsymbol{Q}(\boldsymbol{x}) = \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{x}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad K = \begin{bmatrix} 7 & 0 & 0 \\ 10 & 0 & 0 \\ 0 & 0.11 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 10 \end{bmatrix} \tag{4.67}$$

The initial conditions are:

$$Spacecraft-1 = \begin{bmatrix} 0 \\ 7000 \\ 0 \\ -\sqrt{\frac{\mu}{\boldsymbol{R}}} \\ 0 \\ 0 \end{bmatrix} \quad Spacecraft-2 = \begin{bmatrix} 7000 \\ 0 \\ 0 \\ \sqrt{\frac{\mu}{\boldsymbol{R}}} \\ 0 \\ 0 \end{bmatrix} \quad Observer = \begin{bmatrix} 0 \\ 7000 \\ 0 \\ -\sqrt{\frac{\mu}{\boldsymbol{R}}} \\ 0 \\ 0 \end{bmatrix}$$

To compute the control input for spacecraft-2, the error vector is defined as:

$$\boldsymbol{e}(t) = [\boldsymbol{x}_2 - \boldsymbol{x}_1 \quad \boldsymbol{y}_2 - \boldsymbol{y}_1 \quad \boldsymbol{z}_2 - \boldsymbol{z}_1]^T \tag{4.68}$$

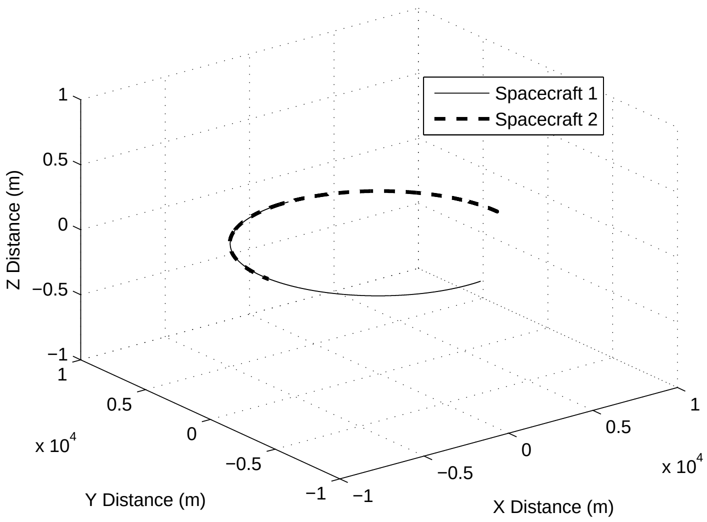

Figure 4.3 Trajectories of the two spacecrafts.

where, $\boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{z}_1$ are the position vectors of spacecraft-1 and $\boldsymbol{x}_2, \boldsymbol{y}_2, \boldsymbol{z}_2$ are the position vectors of spacecraft-2. Also,

$$\ddot{\boldsymbol{e}} = \frac{-\mu\overrightarrow{\boldsymbol{R}_2}}{\boldsymbol{R}_2} + u_2 - \left\{ \frac{-\mu\overrightarrow{\boldsymbol{R}_1}}{\boldsymbol{R}_1} + u_1 \right\} \tag{4.69}$$

The tracking error equation used for control is

$$\ddot{\boldsymbol{e}} + 2\zeta\omega_n\dot{\boldsymbol{e}} + \omega_n^2\boldsymbol{e} = 0 \tag{4.70}$$

Substituting the values for $\ddot{\boldsymbol{e}}$, $\dot{\boldsymbol{e}}$ and $\boldsymbol{e}$ and simplifying,

$$u_2 = \frac{\mu\overrightarrow{\boldsymbol{R}_2}}{\boldsymbol{R}_2} - \frac{\mu\overrightarrow{\boldsymbol{R}_1}}{\boldsymbol{R}_1} + u_1 - 2\zeta\omega_n\dot{\boldsymbol{e}} - \omega_n^2\boldsymbol{e} \tag{4.71}$$

The simulation results are shown below:

Fig. (4.3) is the plot of the trajectories of the two spacecrafts in the geostationary orbit. The two spacecraft which started at an initial angular separation of 90^0 with respect to the center of the earth, converge to maintain angular separation of 45^0 with respect to the center of the earth. This is possible because the estimates obtained from the chain observer is accurate and current state estimates of the spacecraft 1 is used

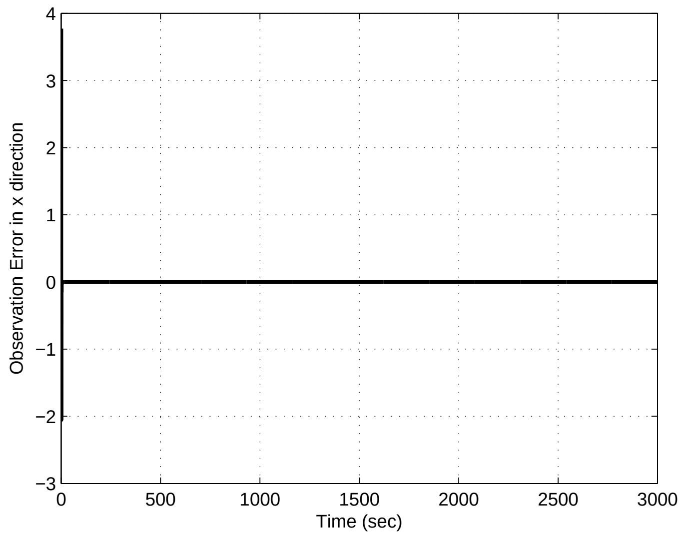

Figure 4.4 Observation Error in Position along X-axis

to compute the control inputs for the second spacecrafts and not the delayed position information from the first spacecraft.

Fig. (4.4), (4.5), (4.6) are the plots of observation error between the true states of the chief spacecraft and the estimated states at the deputy spacecraft. The plots show that the observer performance is good and the observation error decays to zero quickly. In fact, the observation error in the z direction is zero throughout, since there spacecraft movement is confined to the $X - Y$ plane.

Fig. (4.7), (4.8), (4.9) are the plots of the three control inputs computed using the current states of the deputy spacecraft and estimated states of the chief spacecraft from the chain observer. Since, both are low thrust spacecrafts, the maximum thrust that can be handled is assumed to be 1×10^{-5} N. Hence from the Figs. (4.7), (4.8) it can be noted that the control input is saturated at this maximum value, to maintain the second spacecraft at the desired position. Since, there is no movement in the z directions, the control input is zero.

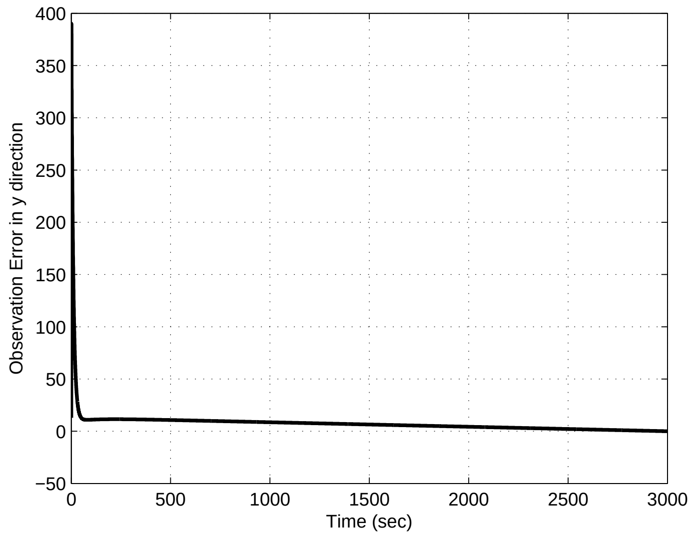

Figure 4.5 Observation Error in Position along Y-axis

4.4 Case 2: Time-varying Delays

In this section, the effect of time-varying delays on the observer stability is analyzed. The delay, as before, is modeled as a piecewise constant, varying at intervals of width Δt_i. $\Delta t_i = t_{i+1} - t_i$.

The observer dynamics for systems with delayed output has the following structure

$$\dot{\hat{\boldsymbol{x}}}(t-\Delta) = \boldsymbol{f}(\hat{\boldsymbol{x}}(t-\Delta)) + \boldsymbol{g}(\hat{\boldsymbol{x}}(t-\Delta))u(t-\Delta) + \boldsymbol{Q}^{-1}(\hat{\boldsymbol{x}}(t-\Delta))\boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{h}(\hat{\boldsymbol{x}}(t-\Delta))) \tag{4.72}$$

$$\begin{aligned}\dot{\hat{\boldsymbol{x}}}_1(t) &= \boldsymbol{f}(\hat{\boldsymbol{x}}_1(t)) + \boldsymbol{g}(\hat{\boldsymbol{x}}_1(t))u_1(t) + \boldsymbol{Q}^{-1}(\hat{\boldsymbol{x}}_1(t))\left\{\exp(\boldsymbol{A}_n\Delta(t))\boldsymbol{K}\left(\bar{\boldsymbol{y}}(t) - \boldsymbol{h}(\hat{\boldsymbol{x}}_0(t))\right)\right. \\ &\quad \left. + \exp(\boldsymbol{A}_n\Delta(t))\left(\boldsymbol{H}(\hat{\boldsymbol{x}}(t-\Delta), u(t-\Delta)) - \boldsymbol{H}(\hat{\boldsymbol{x}}_1(t-\Delta(t)), u_1(t-\Delta))\right)\right\}\end{aligned} \tag{4.73}$$

The variable $\hat{\boldsymbol{x}}(t-\Delta)$ is an estimate of the states at time $t-\Delta(t)$ and variable $\hat{\boldsymbol{x}}_1$ is the estimate of the states at time t with the following initial conditions

$$\hat{\boldsymbol{x}}(t-\Delta)|_{t=0} = \hat{\boldsymbol{x}}(-\Delta_1) \tag{4.74}$$

$$\hat{\boldsymbol{x}}_1(\tau) = \hat{\boldsymbol{x}}(\tau), \quad \tau \in [-\Delta_1, 0] \tag{4.75}$$

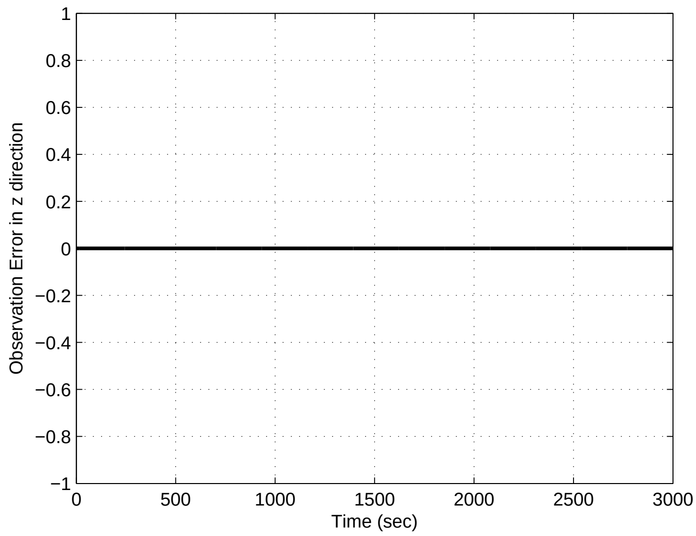

Figure 4.6 Observation Error in Position along Z-axis

where $\hat{\boldsymbol{x}}(\tau) \in [-\Delta_1, 0]$ is any *a priori* estimate of the state. $-\Delta_1$ is the delay magnitude in $t \in [0, t_1]$.

4.4.1 Stability analysis of the zeroth observer

As explained earlier, the observer stability is proved in the z-coordinate system. The observation error is defined as

$$\boldsymbol{\eta}_{z_0}(t) = \boldsymbol{z}(t-\Delta) - \hat{\boldsymbol{z}}(t-\Delta) \tag{4.76}$$

Differentiating Eq. (4.76),

$$\begin{aligned} \dot{\boldsymbol{\eta}}_{z_0}(t) &= \dot{\boldsymbol{z}}(t-\Delta) - \dot{\hat{\boldsymbol{z}}}(t-\Delta) \qquad (4.77) \\ \dot{\boldsymbol{\eta}}_{z_0}(t) &= \boldsymbol{A}_n \boldsymbol{z}(t-\Delta) + \widetilde{\boldsymbol{H}}(\boldsymbol{z}(t-\Delta), u(t-\Delta)) \\ &\quad -(\boldsymbol{A}_n \hat{\boldsymbol{z}}(t-\Delta) + \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(t-\Delta), u(t-\Delta)) + \boldsymbol{K}\boldsymbol{C}_n(\boldsymbol{z}(t-\Delta) - \hat{\boldsymbol{z}}(t-\Delta))\} \\ \dot{\boldsymbol{\eta}}_{z_0}(t) &= \boldsymbol{A}_m \boldsymbol{\eta}_{z_0}(t) + \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_0, \hat{\boldsymbol{z}}_0, u_0) \qquad (4.78) \end{aligned}$$

$\boldsymbol{A}_m = \boldsymbol{A}_n - \boldsymbol{K}\boldsymbol{C}_n$ is Hurwitz and $\boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_0, \hat{\boldsymbol{z}}_0, u_0) = \widetilde{\boldsymbol{H}}(\boldsymbol{z}(t-\Delta), u(t-\Delta)) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(t-\Delta), u(t-\Delta))$.

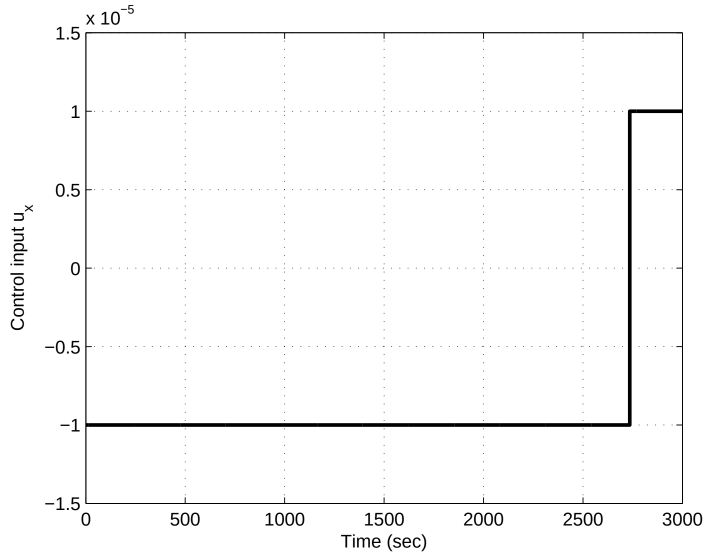

Figure 4.7 Control input u_x

As seen in the case of linear observer, the change in delay translates to a "jump" in the initial condition for zeroth observer at the instant when the delay changes. This "jump" at $t = t_1$ is expressed as $\hat{\boldsymbol{z}}(t_1 - \Delta_2) - \hat{\boldsymbol{z}}(t_1 - \Delta_1)$. Δ_2 is the new value of the delay at t_1 and Δ_1 is the delay magnitude before the change occurs. Differentiating $\hat{\boldsymbol{z}}(t_1 - \Delta_2) - \hat{\boldsymbol{z}}(t_1 - \Delta_1)$ and substituting the observer dynamics,

$$\begin{aligned}
\dot{\hat{\boldsymbol{z}}}(t_1-\Delta_2) - \dot{\hat{\boldsymbol{z}}}(t_1-\Delta_1) &= \boldsymbol{A}_n\hat{\boldsymbol{z}}(t-\Delta_2) + \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(t_1-\Delta_2), u(t_1-\Delta_2)) + \boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{C}_n\hat{\boldsymbol{z}}(t-\Delta_2)) \\
&\quad - \Big\{\boldsymbol{A}_n\hat{\boldsymbol{z}}(t-\Delta_1) + \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(t_1-\Delta_1), u(t_1-\Delta_1)) \\
&\quad + \boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{C}_n\hat{\boldsymbol{z}}(t-\Delta_1))\} \\
\dot{\hat{\boldsymbol{z}}}(t_1-\Delta_2) - \dot{\hat{\boldsymbol{z}}}(t_1-\Delta_1) &= (\boldsymbol{A}_n - \boldsymbol{K}\boldsymbol{C}_n)(\hat{\boldsymbol{z}}(t-\Delta_2) - \hat{\boldsymbol{z}}(t-\Delta_1)) \\
&\quad + \Big\{\widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(t_1-\Delta_2), u(t_1-\Delta_2)) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(t_1-\Delta_1), u(t_1-\Delta_1))\Big\} \\
\dot{\hat{\boldsymbol{z}}}(t_1-\Delta_1) &= \dot{\hat{\boldsymbol{z}}}(t_1-\Delta_2) - \boldsymbol{A}_m\boldsymbol{\delta}_{\hat{\boldsymbol{z}}1}(t) - \boldsymbol{\delta}_{\widetilde{\boldsymbol{H}}1}(t)
\end{aligned} \tag{4.79}$$

Generalizing Eq.(4.79), at any instant $t = t_i$, when the magnitude of the output delay changes, the error in observer due to the "jump" in initial conditions can be expressed as

$$\dot{\hat{\boldsymbol{z}}}(t_i-\Delta_i) = \dot{\hat{\boldsymbol{z}}}(t_i-\Delta_{i+1}) - \boldsymbol{A}_m\boldsymbol{\delta}_{\hat{\boldsymbol{z}}i}(t) - \boldsymbol{\delta}_{\widetilde{\boldsymbol{H}}i}(t) \tag{4.80}$$

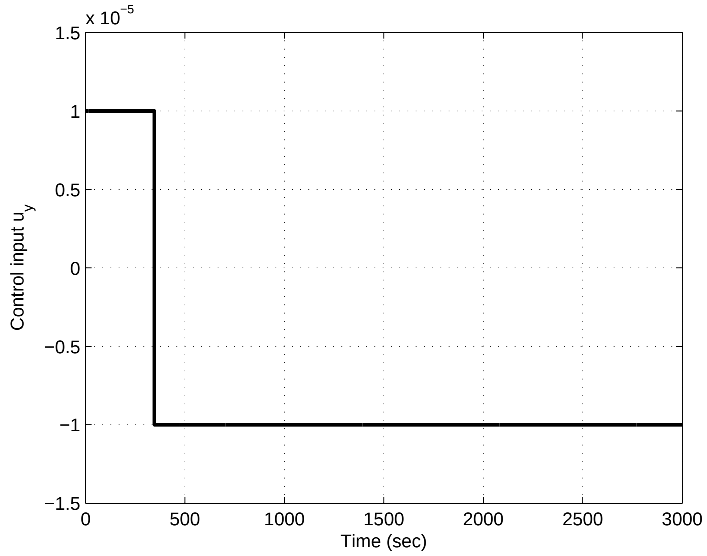

Figure 4.8 Control input u_y

where Δ_i is the delay in the interval $t_{i-1} \leq t < t_i$ and Δ_{i+1} is the delay in the next interval $t_i \leq t < t_{i+1}$. $\boldsymbol{\delta}_{\hat{\boldsymbol{z}}i}(t) = \hat{\boldsymbol{z}}(t_i - \Delta_{i+1}) - \hat{\boldsymbol{z}}(t_i - \Delta_i)$ and $\boldsymbol{\delta}_{\widetilde{\boldsymbol{H}}i}(t) = \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(t_i - \Delta_{i+1}), u(t_i - \Delta_{i+1})) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(t_i - \Delta_i), u(t_i - \Delta_i))$.

Substituting Eq. (4.79) in Eq.(4.77), for any $t = t_i$ and simplifying,

$$\dot{\boldsymbol{\eta}}_{z_0}(t_i) = \boldsymbol{A}_m \boldsymbol{\eta}_{z_0}(t_i) + \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_0, \hat{\boldsymbol{z}}_0, u_0) + \boldsymbol{A}_m \boldsymbol{\delta}_{\hat{\boldsymbol{z}}i}(t) + \boldsymbol{\delta}_{\widetilde{\boldsymbol{H}}i}(t) \tag{4.81}$$

Therefore, the observation error dynamics can be written as

$$\dot{\boldsymbol{\eta}}_{z_0}(t) = \begin{cases} \boldsymbol{A}_m \boldsymbol{\eta}_{z_0}(t_i) + \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_0, \hat{\boldsymbol{z}}_0, u_0) + \boldsymbol{A}_m \boldsymbol{\delta}_{\hat{\boldsymbol{z}}i}(t) + \boldsymbol{\delta}_{\widetilde{\boldsymbol{H}}i}(t), & \text{at } t > t_i \\ \\ \boldsymbol{A}_m \boldsymbol{\eta}_{z_0}(t) + \boldsymbol{\eta}_{\widetilde{H}}(\boldsymbol{z}_0, \hat{\boldsymbol{z}}_0, u_0) & \text{for } t \in [t_{i-1},\ t_i) \end{cases} \tag{4.82}$$

The delay profile assumed at the observer is shown in Fig.(4.10).

Next, a general expression for $\boldsymbol{\eta}_{z_0}(t)$ for any instant $t = t_N$, $N = 0, 1, 2, \ldots, \infty$ at which the delay magnitude changes, is derived. The procedure is as follows:

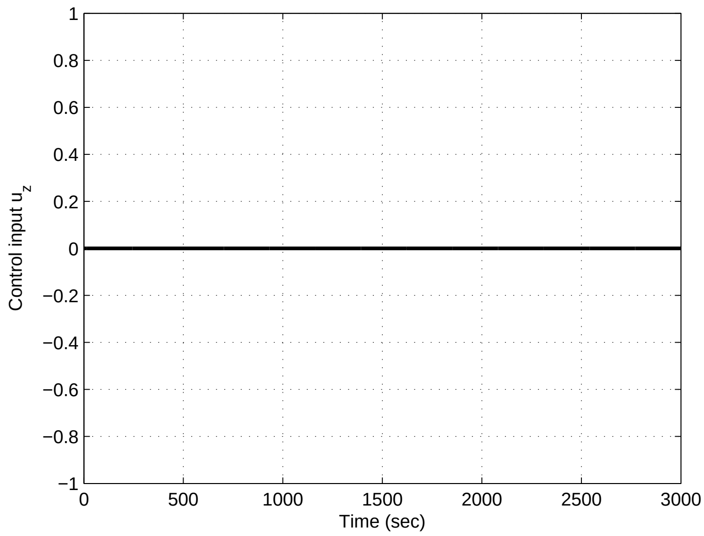

Figure 4.9 Control input u_z

The solution of the dynamic equation (4.82) at $t = t_1$ is

$$\begin{aligned}
\boldsymbol{\eta}_{z_0}(t_1) &= \exp\left(\boldsymbol{A}_m\left(t_1 - 0\right)\right)\boldsymbol{\eta}_{z_0}(0) + \int_0^{t_1} \exp\left(\boldsymbol{A}_m\left(t_1 - s\right)\right)\left\{\boldsymbol{\eta}_{\widetilde{H}}(s) + \boldsymbol{A}_m\boldsymbol{\delta}_{\hat{\boldsymbol{z}}1}(s) + \boldsymbol{\delta}_{\widetilde{\boldsymbol{H}}1}(s)\right\} ds \\
\|\boldsymbol{\eta}_{z_0}(t_1)\| &\leq \exp\left(-m\Delta t_1\right)\|\boldsymbol{\eta}_{z_0}(0)\| \\
&\quad + \gamma_{\widetilde{H}}(|u|)M_1 \int_0^{t_1} \|\boldsymbol{\eta}_{z_0}(s)\| ds + \left\|\int_0^{t_1} \exp\left(\boldsymbol{A}_m\left(t_1 - s\right)\right)\left\{\boldsymbol{A}_m\boldsymbol{\delta}_{\hat{\boldsymbol{z}}1}(s) + \boldsymbol{\delta}_{\widetilde{\boldsymbol{H}}1}(s)\right\} ds\right\| \\
\|\boldsymbol{\eta}_{z_0}(t_1)\| &\leq \exp\left(-m\Delta t_1\right)\|\boldsymbol{\eta}_{z_0}(0)\| + \gamma_{\widetilde{H}}(|u|)M_1 \sup_{0\leq s\leq t_1} \|\boldsymbol{\eta}_{z_0}(s)\| + \bar{\boldsymbol{\delta}}_1 \qquad (4.83)
\end{aligned}$$

where $\int_0^{t_1} \|\exp\left(\boldsymbol{A}_m(t_1 - s)\right)\| ds \leq M_1$ and
$\bar{\boldsymbol{\delta}}_1 = \left\|\int_0^{t_1} \exp\left(\boldsymbol{A}_m\left(t_1 - s\right)\right)\left\{\boldsymbol{A}_m\boldsymbol{\delta}_{\hat{\boldsymbol{z}}1}(s) + \boldsymbol{\delta}_{\widetilde{\boldsymbol{H}}1}(s)\right\} ds\right\|$.

From assumption 4, $\gamma_{\widetilde{H}}(|u|) < \frac{1}{M_1}$. Hence, from (4.83),

$$\sup_{0\leq s\leq t_1} \|\boldsymbol{\eta}_{z_0}(s)\| \leq \frac{1}{(1 - \gamma_{\widetilde{H}}(|u|)M_1)}\{\exp\left(-m\Delta t_1\right)\|\boldsymbol{\eta}_{z_0}(0)\| + \bar{\boldsymbol{\delta}}_1\}$$

Therefore, (4.83) can be rewritten as

$$\|\boldsymbol{\eta}_{z_0}(t_1)\| \leq \frac{1}{(1 - \gamma_{\widetilde{H}}M_1)}\{\exp\left(-m\Delta t_1\right)\|\boldsymbol{\eta}_{z_0}(0)\| + \bar{\boldsymbol{\delta}}_1\} \qquad (4.84)$$

Defining, $\frac{1}{(1-\gamma_{\widetilde{H}}M_1)} = \beta_1$, the observation error at $t = t_1$ is

$$\|\boldsymbol{\eta}_{z_0}(t_1)\| \leq \beta_1 \exp\left(-m\Delta t_1\right)\|\boldsymbol{\eta}_{z_0}(0)\| + \beta_1\bar{\boldsymbol{\delta}}_1 \qquad (4.85)$$

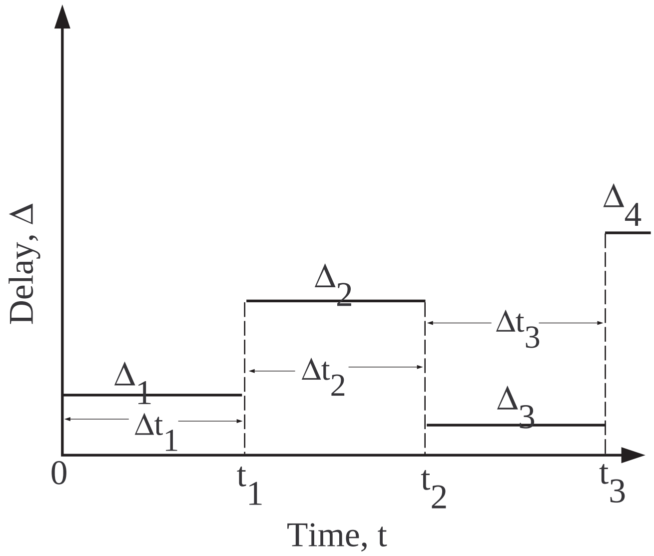

Figure 4.10 Profile of time-varying delay.

Next, at $t = t_2$, the error vector is given as

$$\begin{aligned}\boldsymbol{\eta}_{z_0}(t_2) &= \exp\left(\boldsymbol{A}_m\left(t_2 - t_1\right)\right)\boldsymbol{\eta}_{z_0}(t_1) + \int_{t_1}^{t_2} \exp\left(\boldsymbol{A}_m\left(t_2 - s\right)\right)\left\{\boldsymbol{\eta}_{\widetilde{H}}(s) + \boldsymbol{A}_m\boldsymbol{\delta}_{\hat{\boldsymbol{z}}2}(s) + \boldsymbol{\delta}_{\widetilde{\boldsymbol{H}}2}(s)\right\} ds \\ \Rightarrow \|\boldsymbol{\eta}_{z_0}(t_2)\| &\leq \beta_2 \exp\left(-m\Delta t_2\right)\|\boldsymbol{\eta}_{z_0}(t_1)\| + \beta_2\bar{\boldsymbol{\delta}}_2 \end{aligned} \quad (4.86)$$

Substituting for $\|\boldsymbol{\eta}_{z_0}(t_1)\|$ and simplifying,

$$\|\boldsymbol{\eta}_{z_0}(t_2)\| \leq \beta_1\beta_2 \exp\left(-m(\Delta t_1 + \Delta t_2)\right)\|\boldsymbol{\eta}_{z_0}(0)\| + \beta_1\beta_2 \exp\left(-m\Delta t_2\right)\bar{\boldsymbol{\delta}}_1 + \beta_2\bar{\boldsymbol{\delta}}_2 \quad (4.87)$$

Continuing as above, the error vector at any instant $t = t_N$ can be written as

$$\|\boldsymbol{\eta}_{z_0}(t_N)\| \leq \beta \exp\left(-m\sum_{i=1}^{N}\Delta t_i\right)\|\boldsymbol{\eta}_{z_0}(0)\| + \sum_{j=1}^{N-1}\exp\left(-m\sum_{i=j+1}^{N}\Delta t_i\right)\beta_j\bar{\boldsymbol{\delta}}_j + \beta_N\bar{\boldsymbol{\delta}}_N \quad (4.88)$$

where $\beta = \beta_1\beta_2\ldots\beta_N$. $\beta_i = \frac{1}{(1-\gamma_{\widetilde{H}}M_i)}$. $\int_{t_{i-1}}^{t_i}\|\exp\left(A_m(t_i - s)\right)\|ds \leq M_i$.
$\bar{\boldsymbol{\delta}}_i = \left\|\int_{t_{i-1}}^{t_i}\exp(\boldsymbol{A}_m(t_i - s))\left\{\boldsymbol{A}_m\boldsymbol{\delta}_{\hat{\boldsymbol{z}}i}(s) + \boldsymbol{\delta}_{\widetilde{\boldsymbol{H}}i}(s)\right\}ds\right\|, \quad i = 1, 2, \ldots, N.$

The form of Eq.(4.88) is similar to the expression obtained in the linear case with time-varying delays. This also suggests that due to time-varying delays, there will always be a residual error in observation.

Remark 1: Evaluating each M_i,

$$M_i = \frac{1}{m}(\exp(-m(t_i - t_{i-1})) - 1) \tag{4.89}$$

But $t_i - t_{i-1} = \Delta t_i$. Therefore

$$M_i = \frac{1}{m}(\exp(-m\Delta t_i) - 1) \tag{4.90}$$

The corresponding β_i is

$$\beta_i = \frac{1}{1 - \frac{\gamma_{\widetilde{H}}}{m}(\exp(-m\Delta t_i) - 1)} \tag{4.91}$$

Three possible values of Δt_i:

- $\Delta t_i = 0$ implies that the system is undelayed.
- $\Delta t_i < 1$ implies that the delay is changing at a very fast rate. In this case, $\exp(-m\Delta t_i) > 1$. Assuming that $m = 1$, $\gamma_{\widetilde{H}} = 1$, the denominator term in β_i will be less than 1. Hence $\beta_i > 1$. Since this scales the $\bar{\delta}_i$, it would amplify the residual error term, which in turn increases the magnitude of observation error.
- $\Delta t_i > 1$ implies that the delay is changing at a relatively slower rate. In this case, $\exp(-m\Delta t_i) < 1$. Assuming that $m = 1$, $\gamma_{\widetilde{H}} = 1$, the denominator term in β_i will be greater than 1. Hence $\beta_i < 1$ and therefore in this case, the scaling factor β_i attenuates the residual error which in turn reduces the observation error.

Remark 2: If each $\bar{\delta}_i = 0$, (4.88) reduces to

$$\|\boldsymbol{\eta}_{z_0}(t_N)\| \leq \beta \exp(-mt_N) \|\boldsymbol{\eta}_{z_0}(0)\| \tag{4.92}$$

which is the expression for observation error for no changes in delay, i.e, the constant delay case.

Remark 3: For equal intervals of time delay, (4.88) simplifies to

$$\|\boldsymbol{\eta}_{z_0}(t_N)\| \leq \beta \exp(-mN\Delta t) \|\boldsymbol{\eta}_{z_0}(0)\| + \sum_{j=1}^{N} \exp(-m(N-j)\Delta t) \beta_j \bar{\boldsymbol{\delta}}_j \tag{4.93}$$

In this case, each $M_i = \frac{1}{m}(\exp(-m\Delta t) - 1)$ and $\beta_i = \beta_j = \tilde{\beta}, \quad i \neq j$. Hence the term $\beta = \tilde{\beta}.\tilde{\beta}\ldots\tilde{\beta} = \tilde{\beta}^N$. (4.93) can be rewritten as

$$\|\boldsymbol{\eta}_{z_0}(t_N)\| \leq \tilde{\beta}^N \exp\left(-mN\Delta t\right)\|\boldsymbol{\eta}_{z_0}(0)\| + \sum_{j=1}^{N} \exp\left(-m(N-j)\Delta t\right)\tilde{\beta}^{N-j}\bar{\boldsymbol{\delta}}_j \tag{4.94}$$

4.4.2 Stability analysis of the first observer

The proof of stability of the first observer is similar to that described in the constant delay case. If $\tilde{\Delta}$ is the maximum measurement delay that can occur in the system, then the observer gain can be computed for this value of delay using Eq. (4.60). However, the observer response does change at the instants when the delay changes. This is can noted from the following development:

From Eq. (4.48)

$$\begin{aligned}
\|\boldsymbol{\eta}_{z_1}(t)\| &= \|\exp(\boldsymbol{A}_n\Delta(t))\|.\|\boldsymbol{\eta}_{z_0}(t)\| \\
&\quad + \left\|\int_{t-\Delta}^{t} \exp(\boldsymbol{A}_n(t-\tau))\left(\widetilde{\boldsymbol{H}}(\boldsymbol{z}_1(\tau), u_1(\tau)) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(\tau), u_1(\tau))\right)d\tau\right\| \\
\|\boldsymbol{\eta}_{z_1}(t)\| &= \beta_{z_1}\|\boldsymbol{\eta}_{z_0}(t)\| \qquad (4.95)
\end{aligned}$$

where $\beta_{z_1} = \frac{\alpha}{1-\gamma_{\widetilde{H}}(|u|)M}$, $\int_{t-\Delta}^{t} \|\exp(\boldsymbol{A}_n(t-\tau))\|ds \leq M$ and $\alpha = \|\exp(\boldsymbol{A}_n\Delta(t))\|$. Substituting the general expression for $\|\boldsymbol{\eta}_{z_0}(t)\|$ from (4.94),

$$\|\boldsymbol{\eta}_{z_1}(t)\| = \beta_{z_1}\left(\tilde{\beta}^N \exp\left(-mN\Delta t\right)\|\boldsymbol{\eta}_{z_0}(0)\| + \sum_{j=1}^{N} \exp\left(-m(N-j)\Delta t\right)\tilde{\beta}^{N-j}\bar{\boldsymbol{\delta}}_j\right) \tag{4.96}$$

It can be noted that value of α does not change with the change in the delay magnitude because the eigen values of $\alpha = \|\exp(\boldsymbol{A}_n\Delta(t))\|$ are always at 1, irrespective of the delay magnitude. Hence the observer 1, faithfully propagates the states from zeroth observer forward to the current instant and is scaled by the constant $\frac{1}{1-\gamma_{\widetilde{H}}(|u|)M}$.

4.4.3 Simulation Results

The example used for simulation is the formation control of spacecrafts explained in Section (4.3.3). As explained earlier, the cooperative system consists of two spacecrafts

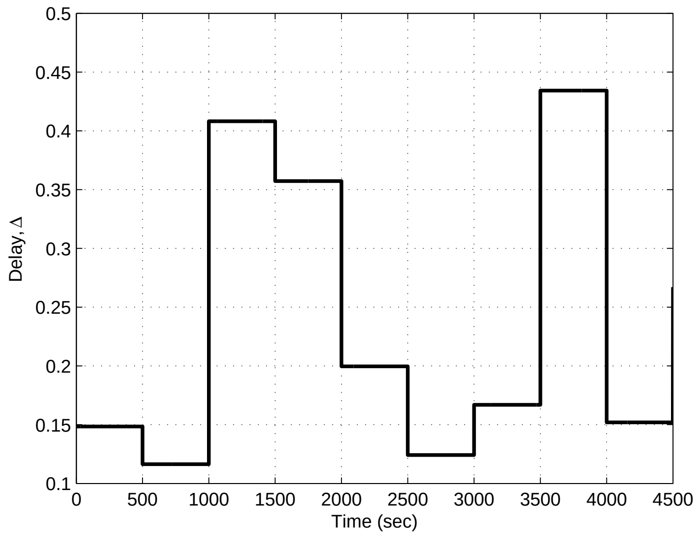

Figure 4.11 Profile of the measurement Delay.

orbiting the earth in the geostationary orbit. The cooperative control is applied to maintain the two spacecrafts at an angular separation of 45^0. The initial positions of the two spacecrafts and desired final formation geometry are as shown in Fig. (4.1) and (4.2) respectively.

The information from the first spacecraft to the second is assumed to available after a finite delay. However, the delay in this case, is a function of time. The delay profile of the output from the first spacecraft is as shown in Fig. (4.11). It is assumed that all the outputs are delayed by the same interval. The delay varies in the interval $(0.05,\ 0.5)$ and its magnitude changes randomly for every 500 *seconds*.

The spacecraft model and the initial conditions are as mentioned in Sec (4.3.3). The simulation plots are as shown in the figures below.

Fig. (4.12) is the plot of the trajectories of the two spacecrafts. A closer look at the trajectory of the second spacecraft's trajectory reveals the effects of the time-varying delay. It can be noted that it is not a perfectly circular orbit as compared to that of the first spacecraft. Due to the variations in delay magnitude, it is constantly perturbed

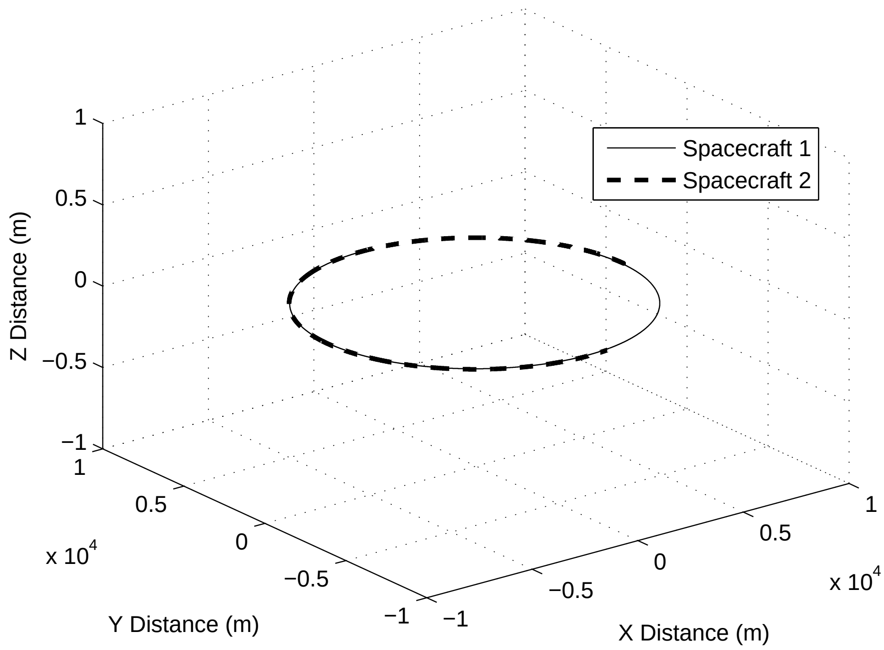

Figure 4.12 Trajectories of the two spacecraft

from its orbit. Hence, the second spacecraft does not exactly follow the path of first spacecraft.

This effect can also be observed from Fig. (4.13) and (4.14), which are the plots of observation error between the true states of the first spacecraft and the estimated states at the second spacecraft. In both the plots, it can be seen that the observer dynamics is constantly excited at every 500 *sec*. After accommodating this change in the delay in its dynamics, the observer tries to catch up with the true states. Again after 500 *sec*, another change in the delay magnitude cause the chain observer to account for the change in delay. Since, the spacecrafts are in an equatorial orbit, there is no movement along the z-direction and hence no error in observation.

Fig. (4.16), (4.17), (4.18) are the plots of the three control inputs computed using the current states of the second spacecraft and estimated states of the first spacecraft

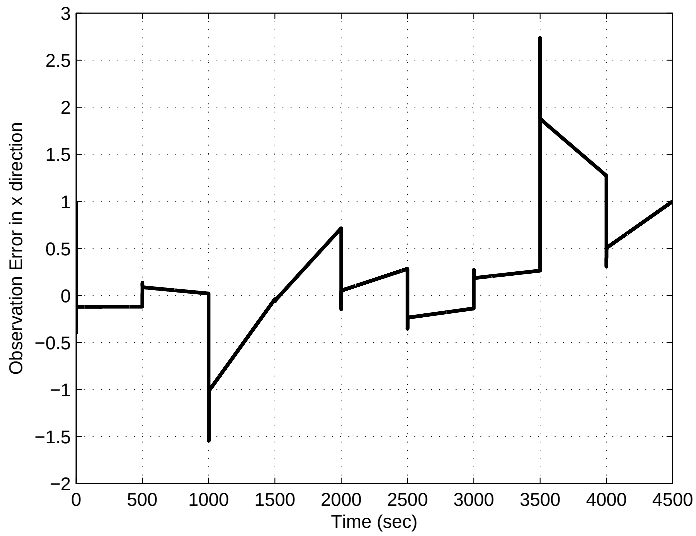

Figure 4.13 Observation Error in Position along X-axis

from the chain observer. Since the maximum achievable control in limited to $\pm 1 \times 10^{-5}$, the control inputs remain saturated at this value in the X and Y directions.

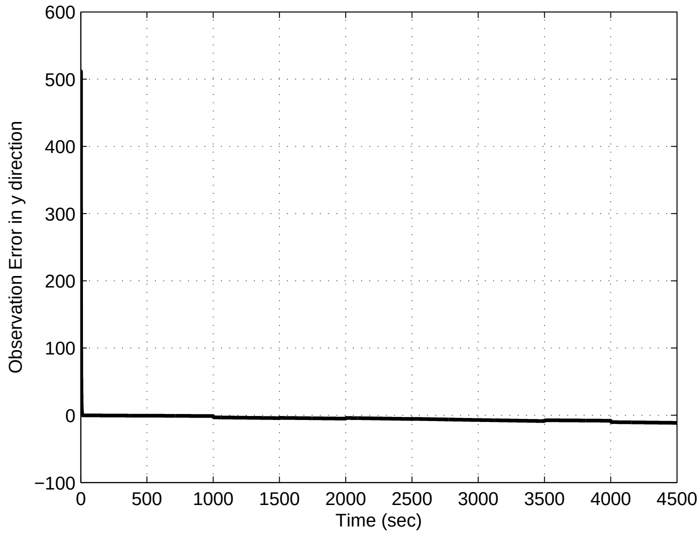

Figure 4.14 Observation Error in Position along Y-axis

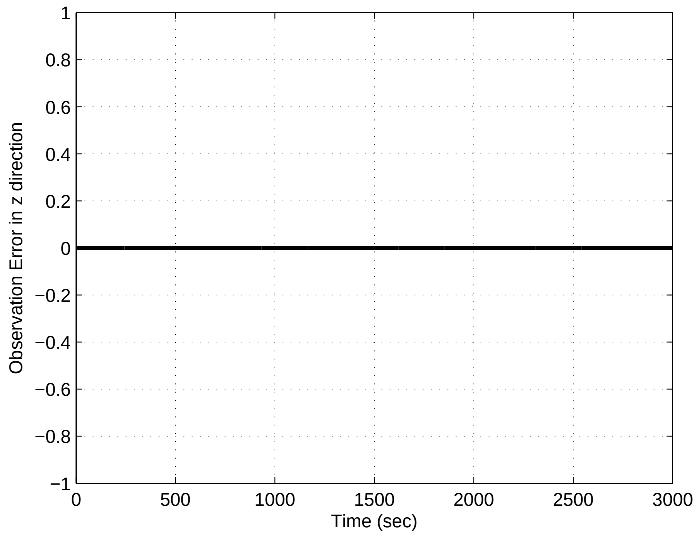

Figure 4.15 Observation Error in Position along Z-axis

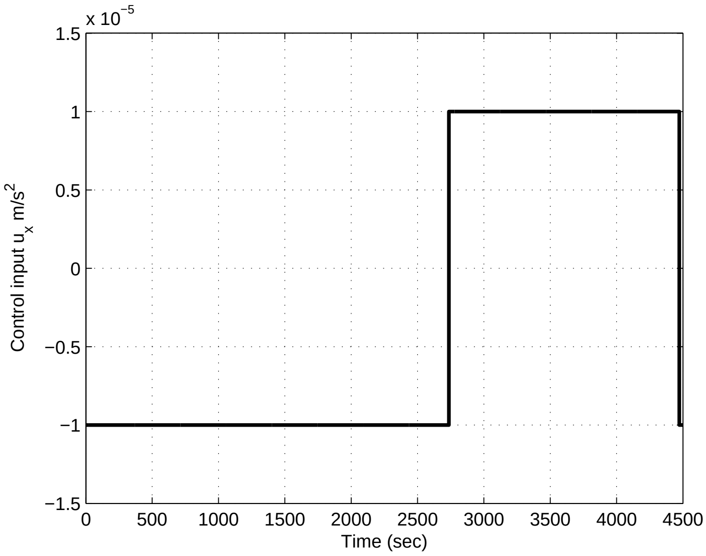

Figure 4.16 Control input u_x

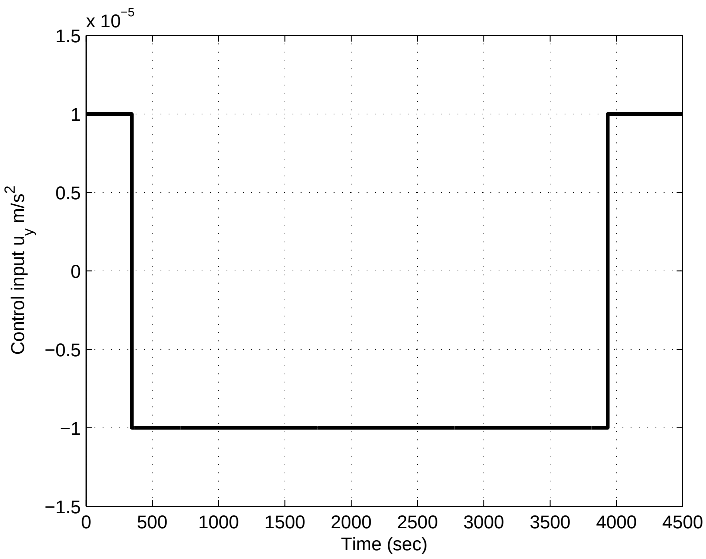

Figure 4.17 Control input u_y

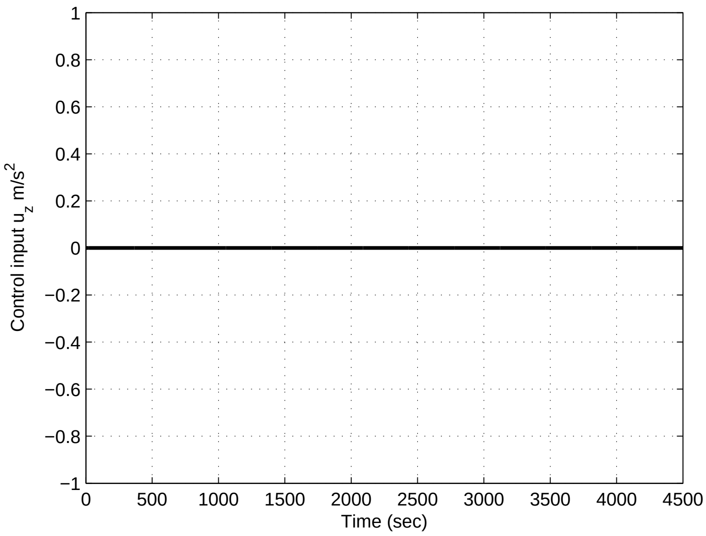

Figure 4.18 Control input u_z

CHAPTER 5

CONCLUDING REMARKS

Studies involving multiple unmanned vehicles performing a common task have shown that the results obtained are much better than using a single vehicle to perform this task. But, the multi-vehicle control is plagued by several problems and one of the critical issues is the delays in communication between the participating vehicles. Studies have further shown that this is a serious problem and delay is a potential source of instability.

In this research one of the methods to overcome the instability problem is addressed. The method suggested is to use a state observer to construct the current state information from the delayed measurements. The observer has an interesting chain-like structure with multiple systems in the chain performing the estimation process.

The stability analysis showed that for linear systems, the chain observer can provide an exponential stability for the case when delay is known constant. However from a practical point of view, the delays are generally not of constant magnitude for the entire operation time. Owing to channel uncertainties, the delay magnitude does change with time. Stability analysis in this case showed that the chain observer can guarantee only asymptotic stability with a finite non-zero observation error. Based on the error tolerance levels in the system, it was found that the residual observation error magnitude was fairly small and did not pose serious threat to the system stability. The cooperative control laws were implemented using the estimated states from the observer. The simulation results showed that the control laws thus derived were stable and the performance of the cooperative system was satisfactory.

The results of the observer were then extended to nonlinear systems with delayed measurements. Due to the complicated nature of the system in itself, it was found that several assumptions were needed to implement a stable observer. These assumptions also restricted the observer implementation to a certain class of nonlinear systems which satisfied these assumptions. For nonlinear systems, the observer was able to provide asymptotic stability for both constant and time-varying delays. As expected, in the time-varying case, the observation error decays to a finite residual set.

APPENDIX A

PROOF OF LEMMA 1

Lemma 1: Using the nonlinear change of coordinates, observer 1 can be rewritten as

$$\hat{\boldsymbol{z}}_1(t) = \exp(\boldsymbol{A}_n\Delta)\hat{\boldsymbol{z}}(t-\Delta) + \int_{t-\Delta}^{t} \exp(\boldsymbol{A}_n(t-\tau))\widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(\tau), u_1(\tau))d\tau \tag{A.1}$$

To prove this lemma, it is required to show that by differentiating Eq. (A.1) gives back the expression obtained by transforming Equation for observer 1

Proof: Differentiating $\hat{\boldsymbol{z}}_1(t) = \boldsymbol{\Phi}(\hat{\boldsymbol{x}}_1(t))$ w.r.t time and substituting for $\boldsymbol{\Phi}(\hat{\boldsymbol{x}}_1(t))$,

$$\begin{aligned}
\dot{\hat{\boldsymbol{z}}}_1(t) &= \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{x}_1}\dot{\hat{\boldsymbol{x}}}_1(t) \\
&= \boldsymbol{Q}(\hat{\boldsymbol{x}}_1(t))\left\{\boldsymbol{f}(\hat{\boldsymbol{x}}_1(t)) + \boldsymbol{g}(\hat{\boldsymbol{x}}_1(t))u_1(t) + \boldsymbol{Q}^{-1}(\hat{\boldsymbol{x}}_1(t))\left\{\exp(\boldsymbol{A}_n\Delta)\boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{h}(\hat{\boldsymbol{x}}(t-\Delta)))\right.\right. \\
&\quad \left.\left.\exp(\boldsymbol{A}_n\Delta)(\boldsymbol{H}(\hat{\boldsymbol{x}}(t-\Delta), u(t-\Delta)) - \boldsymbol{H}(\hat{\boldsymbol{x}}_1(t-\Delta), u_1(t-\Delta)))\right\}\right\} \\
\Rightarrow \dot{\hat{\boldsymbol{z}}}_1(t) &= \boldsymbol{A}_n\boldsymbol{z}_1(t) + \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(t), u_1(t)) + \exp(\boldsymbol{A}_n\Delta)\boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{h}(\hat{\boldsymbol{x}}(t-\Delta))) \\
&\quad + \exp(\boldsymbol{A}_n\Delta)(\widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(t-\Delta), u(t-\Delta)) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(t-\Delta), u_1(t-\Delta))) \qquad (A.2)
\end{aligned}$$

Now, it is sufficient to prove that by differentiating Eq. (A.1), Eq.(A.2) is obtained. Differentiation of Eq. (A.1) gives,

$$\begin{aligned}
\dot{\hat{\boldsymbol{z}}}_1(t) &= \exp(\boldsymbol{A}_n\Delta)\dot{\hat{\boldsymbol{z}}}(t-\Delta) + \boldsymbol{A}_n\int_{t-\Delta}^{t} \exp(\boldsymbol{A}_n(t-\tau))\widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(\tau-\Delta), u(\tau-\Delta))d\tau \\
&\quad + \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(t), u_1(t)) - \exp(\boldsymbol{A}_n\Delta)\widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(t-\Delta), u_1(t-\Delta))
\end{aligned}$$

Substituting for the integral in the above equation with $\hat{\boldsymbol{z}}_1(t) - \exp(\boldsymbol{A}_n\Delta)\hat{\boldsymbol{z}}(t-\Delta)$ from Eq. (A.1) and rearranging the terms,

$$\begin{aligned}
\dot{\hat{\boldsymbol{z}}}_1(t) &= \boldsymbol{A}_n\hat{\boldsymbol{z}}_1(t) + \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(t), u_1(t)) + \exp(\boldsymbol{A}_n\Delta)(\dot{\hat{\boldsymbol{z}}}(t-\Delta) - \boldsymbol{A}_n\hat{\boldsymbol{z}}(t-\Delta)) \\
&\quad - \exp(\boldsymbol{A}_n\Delta)\widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(t-\Delta), u_1(t-\Delta))
\end{aligned}$$

Adding and subtracting $\exp(\boldsymbol{A}_n\Delta)\widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(t-\Delta), u(t-\Delta))$ to the above equation and rearranging,

$$\begin{aligned}
\dot{\hat{\boldsymbol{z}}}_1(t) &= \boldsymbol{A}_n\hat{\boldsymbol{z}}_1(t) + \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(t), u_1(t)) \\
&\quad + \exp(\boldsymbol{A}_n\Delta)\left\{\dot{\hat{\boldsymbol{z}}}(t-\Delta) - \boldsymbol{A}_n\hat{\boldsymbol{z}}(t-\Delta) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(t-\Delta), u(t-\Delta))\right\} \\
&\quad + \exp(\boldsymbol{A}_n\Delta)\left\{\widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(t-\Delta), u(t-\Delta) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(t-\Delta), u_1(t-\Delta)))\right\} \qquad (A.3)
\end{aligned}$$

Now, defining the variable $\boldsymbol{s}_j(t)$ for $j = 0, 1$ as

$$\boldsymbol{s}_j(t) = \dot{\hat{\boldsymbol{z}}}_j(t) - \boldsymbol{A}_n \hat{\boldsymbol{z}}_j(t) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_j(t), u_j(t)) \tag{A.4}$$

Eq. (A.3) can be written in the difference equation form, for $j = 1$, as

$$\boldsymbol{s}_j(t) = \exp(\boldsymbol{A}_n\Delta)\boldsymbol{s}_{j-1}(t) + \exp(\boldsymbol{A}_n\Delta)(\widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_{j-1}(t), u_{j-1}(t)) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_j(t-\Delta), u_j(t-\Delta)))$$

The zeroth observer in difference equation form can be written as

$$\boldsymbol{s}_0(t) = \boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{C}_n \hat{\boldsymbol{z}}(t-\Delta)) \tag{A.5}$$

Using standard equation for discrete time systems, the following equation is obtained

$$\boldsymbol{s}_j(t) = \exp(\boldsymbol{A}_n\Delta)\boldsymbol{s}_0(t) + \exp(\boldsymbol{A}_n\Delta)\left(\widetilde{H}(\hat{\boldsymbol{z}}(t-\Delta), u(t-\Delta)) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(t-\Delta), u_1(t-\Delta))\right)$$

Substituting the expressions for $\boldsymbol{s}_j(t), \quad j = 1$ and $\boldsymbol{s}_0(t)$ in the above equation,

$$\begin{aligned} \dot{\hat{\boldsymbol{z}}}_1(t) \;\; = \;\; & \boldsymbol{A}_n \hat{\boldsymbol{z}}_1(t) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(t), u_1(t)) + \exp(\boldsymbol{A}_n\Delta)\boldsymbol{K}(\bar{\boldsymbol{y}}(t) - \boldsymbol{C}_n \hat{\boldsymbol{z}}(t-\Delta)) \\ & + \exp(\boldsymbol{A}_n\Delta)\left(\widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}(t-\Delta), u_i(t)) - \widetilde{\boldsymbol{H}}(\hat{\boldsymbol{z}}_1(t-\Delta), u_1(t-\Delta))\right) \end{aligned} \tag{A.6}$$

Hence equality between expressions Eq. (A.6) and Eq. (A.1) is proved

APPENDIX B

PROOF OF LEMMA 3

Consider an equation of the form,

$$\dot{\boldsymbol{r}}(t) = -\alpha\boldsymbol{r}(t) + \beta\text{sup}_{t-\Delta\leq\omega\leq t}\boldsymbol{r}(\omega) + \vartheta\exp(-\varpi t) \tag{B.1}$$

where, $\vartheta\exp(-\varpi t)$ is an exponentially decaying signal.

Denoting $\text{sup}_{t-\Delta\leq\omega\leq t}r(\omega)$ as a constant λ, Eq. (B.1) can be rewritten as,

$$\dot{\boldsymbol{r}}(t) = -\alpha\boldsymbol{r}(t) + \beta\lambda + \vartheta\exp(-\varpi t) \tag{B.2}$$

The solution of $\boldsymbol{r}(t)$ is given as

$$\begin{aligned}
\boldsymbol{r}(t) &= \exp(-\alpha(t-t_0))\boldsymbol{r}(t_0) + \int_{t_0}^{t}\exp(-\alpha(t-\tau))\left(\beta\lambda + \vartheta\exp(-\varpi\tau)\right)d\tau \\
\boldsymbol{r}(t) &= \exp(-\alpha(t-t_0))\boldsymbol{r}(t_0) + \beta\lambda\exp(-\alpha t)\int_{t_0}^{t}\exp(\alpha\tau)d\tau + \vartheta\exp(-\alpha t)\int_{t_0}^{t}\exp((\alpha-\varpi)\tau)d\tau
\end{aligned} \tag{B.3}$$

Evaluating the two integrals and simplifying,

$$\begin{aligned}
\boldsymbol{r}(t) &= \exp(-\alpha(t-t_0))\boldsymbol{r}(t_0) + \frac{\beta\lambda}{\alpha}\exp(-\alpha t)\{\exp(\alpha t) - \exp(\alpha t_0)\} \\
&\quad + \frac{\vartheta\exp(-\varpi t)}{\alpha-\varpi}\{\exp(-\varpi(t-t_0)) - \exp(-\alpha(t-t_0))\}
\end{aligned}$$

Using Lemma 2 (Halanay [33],Kolmanovskii and Nosov [34]), for any $k > 0$ and $\xi > 0$ the above equation can be rewritten as,

$$\boldsymbol{r}(t) = k\exp(-\xi(t-t_0)) + k_0\{\exp(-\varpi(t-t_0)) - \exp(-\alpha(t-t_0))\} \tag{B.4}$$

where $k_0 = \frac{\vartheta\exp(-\varpi t)}{\alpha-\varpi}$

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BIOGRAPHICAL STATEMENT

Praveen was born in Marandahalli, Tamil Nadu, India in 1981. He grew up in Bangalore, India where he completed his schooling. He attended the M.S. Ramaiah Institute of Technology affiliated to the Visweswariah Technological University in Bangalore from 1999 to 2003. He graduated with his bachelors degree in Electronics and Communication.

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