CONDITIONS FOR WAVE PROPAGATION ALONG SOLAR MAGNETIC FLUX TUBES

by

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Presented to the Faculty of the Graduate School of

The University of Texas at Arlington in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE IN PHYSICS

THE UNIVERSITY OF TEXAS AT ARLINGTON

May 2006

ACKNOWLEDGEMENTS

I am indebted to my advisor Dr. Zdzislaw Musielak for his constant guidance and encouragement. I also sincerely thank my committee members Dr. Neil Fazleev and Dr. Manfred Cuntz for their patience and time in reviewing the first draft and suggesting improvements. I am immense in gratitude for Bodhisatwa Chakravarty for helping me with diagrams. I am also indebted to my husband Snehanshu Saha and our friends Arghya and Soma for their support throughout this effort. It is fortunate to be in such gracious company. I would like to sincerely thank my parents and my brother for their continuous support and encouragement.

April 18, 2006

ABSTRACT

CONDITIONS FOR WAVE PROPAGATION ALONG

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Publication No. ______

Swati Routh, M.S.

The University of Texas at Arlington, 2006

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There is an intimate relation between all aspects of the solar activity and the presence of solar magnetic fields. Observations showed that the magnetic field at the photospheric level consists of magnetic flux tubes, which are located at supergranule boundaries. The flux tubes interact with the solar granulation and convection and, as a result, longitudinal, transverse and torsional magnetic tube waves are generated. These waves are responsible for carrying the energy to the upper parts of the solar atmosphere and heating the atmosphere to temperatures much higher than the solar photosphere. The energy carried by transverse and torsional waves may also play a dominant role in acceleration of the solar wind. It is important to determine the propagation conditions

for all three types of magnetic tube waves. The main purpose of analytical studies performed in this thesis is to derive cutoff frequencies for the waves propagating along thin and vertically oriented magnetic flux tubes, and to determine the range of frequencies corresponding to propagating and evanescent waves under the solar conditions. Among new results obtained in this thesis is an alternative much simpler approach to derive the transverse wave equation, the derivation of the wave equation for torsional waves propagating along different parts of exponentially diverging flux tubes, and the cutoff frequencies for torsional waves.

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CHAPTER 1

INTRODUCTION

The Sun, like all stars, is such a massive ball of plasma that it is held together and compressed under its own gravitational attraction. It consists mainly of H (90%) and He (10%), mostly in an ionized state because of the high temperature; the remaining elements, such as C, N, O, comprise about 0.1% and are present in roughly the same proportions as on Earth.

Traditionally, solar phenomena have been divided into two classes, *quiet* and *active*; the *quiet* Sun is viewed as a static, spherically symmetric ball of plasma, whose properties depend to a first approximation on radial distance from the centre and whose magnetic field is negligible. The *active* Sun consists of transient phenomena, such as sunspots, prominences and flares, which are superimposed on the quiet atmosphere and most of which owe their existence to the magnetic field.

The overall structure of the Sun is sketched in figure 1.The interior is divided into three regions, namely the core, intermediate (or radiative) zone and convection zone, where different physical processes are dominant. The core contains only half the mass of the Sun in only about one-fiftieth of its volume, but generates 99% of the energy. This energy is slowly transferred outwards across the intermediate zone by radiative diffusion, as the photons are absorbed and emitted many times.

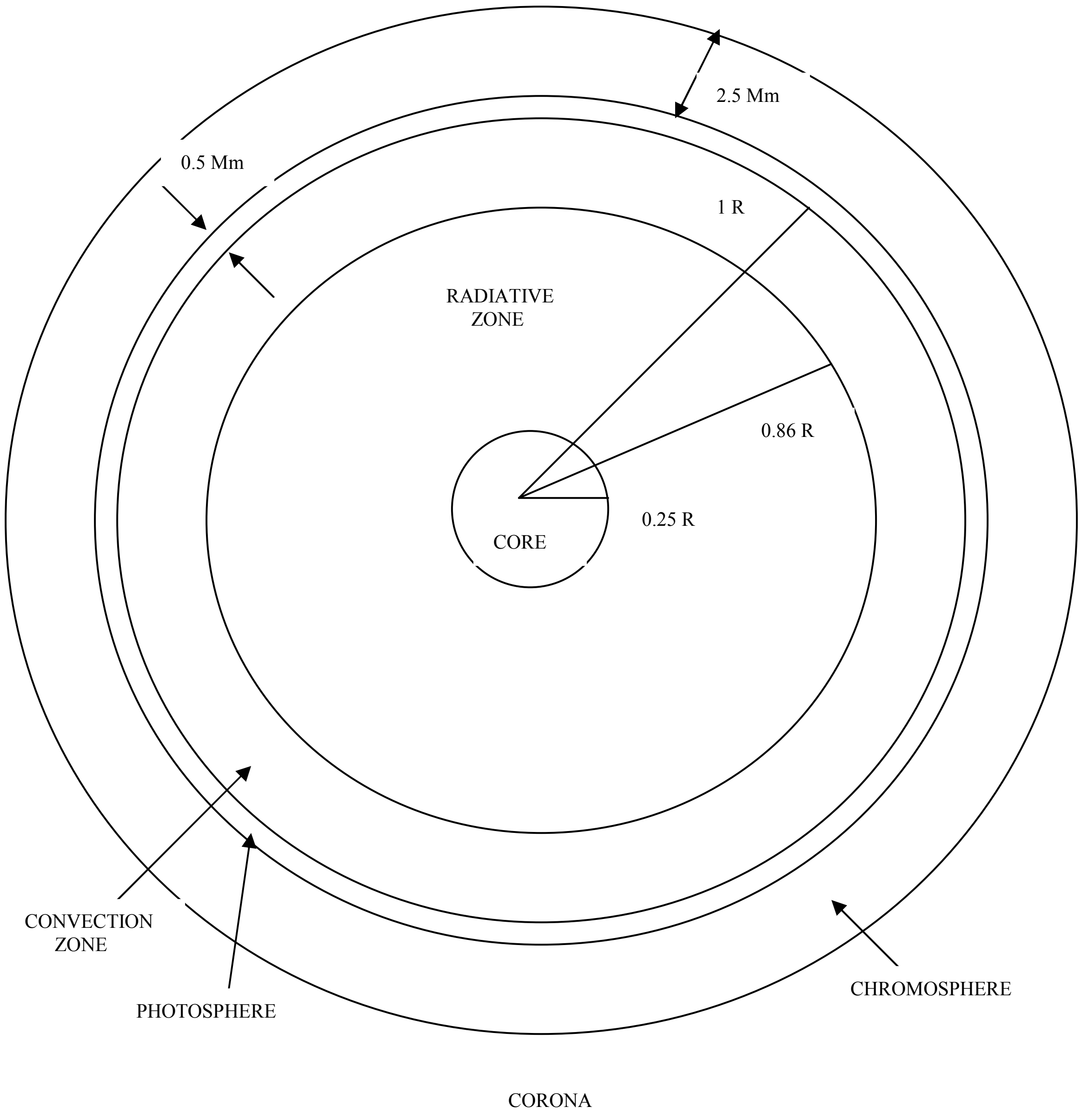

Fig. 1.1 The overall structure of the Sun

The solar interior is so opaque that, whereas an unimpeded photon would take 2 s to reach the surface from the centre, there are so many collisions (absorptions and re-emissions) that photons in practice take 10^7 years for the journey! The effect of these collisions is to increase the typical wavelength from that of high-energy gamma rays in the core to that of visible light at the solar surface.

Convective motions take place in the solar convection zone because the temperature gradient is too great for the material to remain in static equilibrium and so convective instability ensues. Convection transports energy because an individual blob of plasma carries heat as it rises and then gives up some of it before falling and picking up more. In fact, convection is the dominant means of energy transport in the convective zone. According to dynamo theory this zone (or its lower boundary) is also the region where the Sun's magnetic field is generated.

The visible solar outer atmosphere consists of three regions with different physical properties. The lowest is an extremely thin layer of plasma, called the photosphere, which relatively dense and opaque and emits most the solar radiation. Above it lies the rarer and more transparent chromosphere, while the corona extends from the top of a narrow transition region to the Earth and beyond.

Before 1940 it was thought, quite naturally, that the temperature decreases as one goes away from the solar surface. But, since then, it has been realized that, after falling from about 6600K (at the bottom of the photosphere) to a minimum value of about 4300K (at the top of the photosphere), the temperature rises slowly through the lower chromosphere and then dramatically through the transition region to a few million degrees in corona. Thereafter the temperature falls slowly in the outer corona, which is expanding outwards as the solar wind, to a value of 10^5 K at 1 AU. The reason for the temperature rise above the photosphere has been one of the major problems in solar physics and is not yet fully answered; the low chromosphere is probably heated by sound waves that are generated in the noisy convection zone, propagate outwards and then dissipate their energy after forming shocks; higher levels may well be heated by several magnetic mechanism (see Priest, 1982).

The principal goal of this thesis is to study analytically the propagation of the MHD waves which are important non-radiative sources of the chromospheric and coronal heating— longitudinal, transverse and torsional waves in vertical magnetic flux tubes. Before proceeding to study the tube waves we start from a simpler situation— magnetoacoustic waves in a uniform medium which could be present on the Sun (for example, a Moreton or flare induced coronal wave is sometimes emitted from the flare site and moves across the disc: it is probably a fast magnetoacoustic wave, Priest,1982).

CHAPTER 2

MHD WAVES IN A UNIFORM MEDIUM

This chapter focuses on the physical properties of linear magnetohydrodynamic (MHD) waves and new results concerning propagation of these waves in a uniform medium are presented. The behavior of linear MHD waves propagating in a homogeneous medium with a uniform magnetic field of arbitrary direction is presently well-understood. In this case, there are three types of MHD modes: fast, slow and Alfven waves. In general, fast and slow MHD waves (also called magnetoacoustic waves) have both longitudinal and transverse components, however, Alfven waves are associated only with purely transverse motions (e.g., Priest 1982). Analysis of the group velocity of magnetoacoustic waves shows that the energy propagation of fast MHD waves is almost independent of the direction (similar to the phase velocity) and that, in contrast, slow MHD waves have the striking property that the wave energy associated with their propagation is always carried within a small angle with respect to the background magnetic field. For Alfven waves the associated wave energy propagates only along the magnetic field line direction.

To describe the propagation of these waves, one must consider a set of basic MHD equations. In the following, the MHD equations are described and then used in all analytical studies presented in this thesis.

2.1 Governing MHD Equations

Magnetohydrodynamics (MHD) is the study of highly conducting fluids in the presence of magnetic fields. It has broad applications in laboratory plasmas, magnetospheric physics, space physics and astrophysics. The basic set of MHD equations is derived from conservation laws (i.e., conservation of mass, momentum and energy) in conjunction with Maxwell's equations. The main assumptions of MHD are well summarized by Priest (1982):

- The characteristic length scales are much greater than those of the plasma.
- The characteristic time scales are much greater than the particle collision time scales.
- Plasma properties are isotropic.
- The characteristic speeds are much smaller than the speed of light, so that relativistic effects can be neglected.

We further assume that the fluid is isentropic and isothermal, that the gas pressure is a scalar, and that the displacement currents and electrostatic forces may be neglected; the molecular viscosity and Ohmic diffusion is negligible as long as shock formation does not occur. Our assumption leads to the following perfect fluid, ideal MHD equations [see any text book on Magnetohydrodynamics]

Continuity equation:

$$\frac{d\rho}{dt} + \rho\nabla.u = 0 \qquad (2.1a)$$

Energy equation:

$$\frac{dp}{dt} - V_s^2 \frac{d\rho}{dt} = 0 \qquad (2.1b)$$

Momentum equation:

$$\frac{d\vec{u}}{dt} + \frac{\nabla p}{\rho} - \vec{g} - (1/4\pi\rho)(\nabla \times \vec{B}) \times \vec{B} = 0 \qquad (2.1c)$$

and

Induction equation:

$$\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{u} \times \vec{B}) = 0 \qquad (2.1d)$$

with the solenoidal condition:

$\nabla.\vec{B} = 0$

where we have used Ohm's law,

$$\vec{E} + \vec{u} \times \vec{B} = 0\,, \qquad (2.1e)$$

to simplify equation (1.1d); $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u}.\nabla$,and $V_s \left(\equiv \sqrt{\frac{\gamma RT}{\mu}} \right)$ is the sound velocity; the remaining symbols and notations are standard.. We further assume that the sound and Alfven velocities do not vary in space, so that a dispersion relation can be derived. In addition the background uniform magnetic field $\vec{B}_0$ is oriented at an arbitrary angle θ with respect to the vertical z axis and the wave vector $\vec{K}$ is oriented at an arbitrary angle ϕ with respect to z axis.

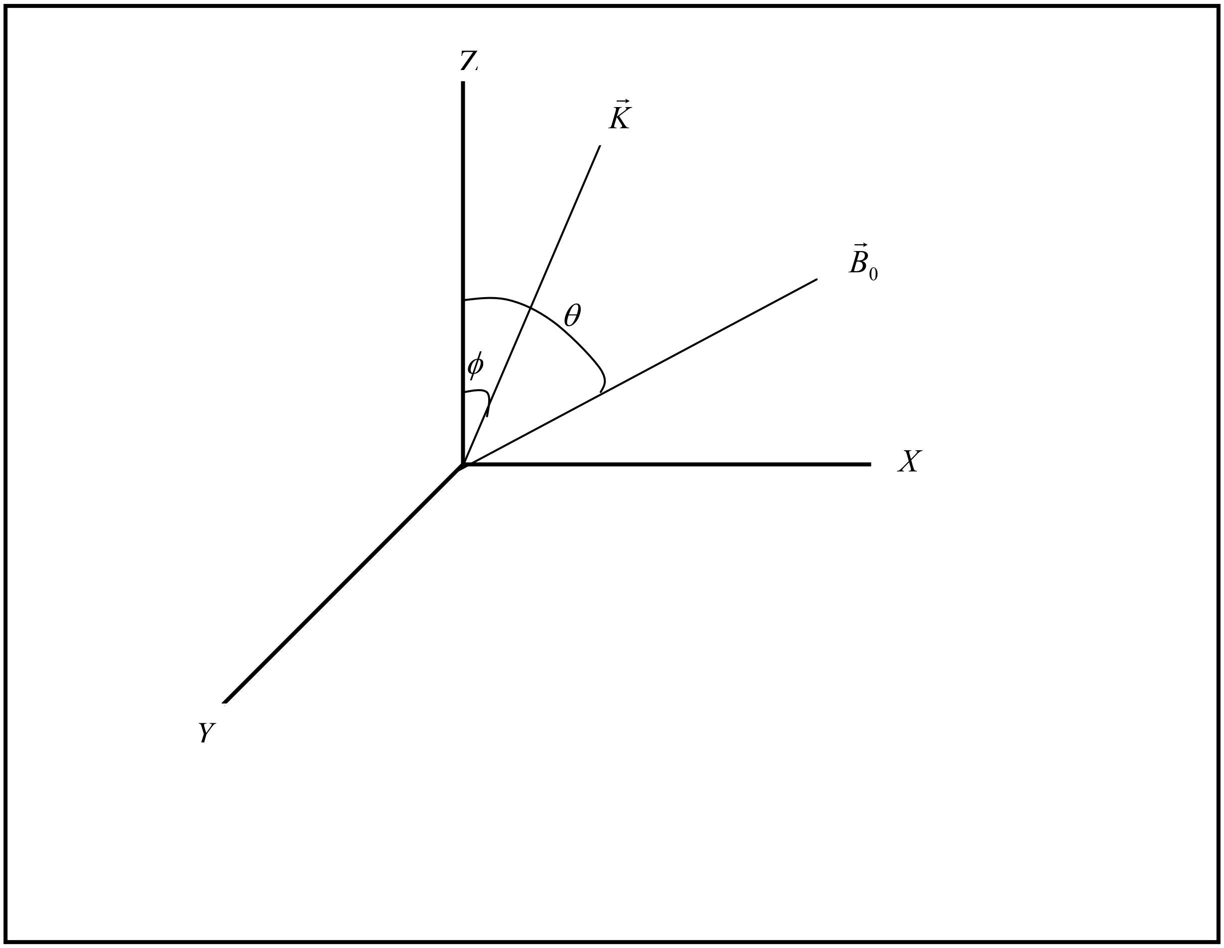

Fig. 2.1 Orientation of $\vec{K}$ and $\vec{B}_0$: $\vec{K}$ is oriented at an angle ϕ with Z axis and $\vec{B}_0$ is oriented at an angle θ with the vertical

2.2 Deriving Magnetoacoustic Wave Equation

We regard every variable as the sum of its equilibrium value (indicated by subscript zero) and the small perturbation due to the wave motion. We also assume that the magnetic field $\vec{B}_0$ and all the perturbations lie solely in the (x, z) plane and we are in the plasma's proper frame of reference (i.e., $\vec{u}_0 = 0$). Linearizing all the equations (2.1a)-(2.1d) in the homogeneous background medium and neglecting gravity we get the following equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho_0 u_x)}{\partial x} + \frac{\partial(\rho_0 u_z)}{\partial z} = 0 \qquad (2.2a)$$

$$\frac{\partial p}{\partial t} + {v_s}^2 \rho_0 (\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}) = 0 \qquad (2.2b)$$

$$\frac{\partial u_x}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} - \frac{1}{4\pi\rho_0} B_{0z} (\frac{\partial b_x}{\partial z} - \frac{\partial b_z}{\partial x}) = 0 \qquad (2.2c)$$

$$\frac{\partial u_z}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{1}{4\pi\rho_0} B_{0x} (\frac{\partial b_z}{\partial x} - \frac{\partial b_x}{\partial z}) = 0 \qquad (2.2d)$$

$$\frac{\partial b_x}{\partial t} + B_{0x}\frac{\partial u_z}{\partial z} - B_{0z}\frac{\partial u_x}{\partial z} = 0 \qquad (2.2e)$$

$$\frac{\partial b_z}{\partial t} + B_{0z}\frac{\partial u_x}{\partial x} - B_{0x}\frac{\partial u_z}{\partial x} = 0 \qquad (2.2f)$$

In equations (2.2c) and (2.2d) we used the vector identity:

$$(\nabla \times \vec{B}) \times \vec{B} = (\vec{B}.\nabla)\vec{B} - \frac{1}{2}\nabla \vec{B}^2$$

In equations (2.2e) and (2.2f) we used the vector identity:

$$\nabla \times (\vec{A} \times \vec{B}) = (\nabla.\vec{B})\vec{A} - (\nabla.\vec{A})\vec{B} + (\vec{B}.\nabla)\vec{A} - (\vec{A}.\nabla)\vec{B}$$

From equations (2.2a)—(2.2f) we get the following two totally symmetric coupled equations for u_x and u_z:

$$\frac{\partial^2 u_x}{\partial t^2} - (V_s^2 + \frac{B_{0z}^{\;2}}{4\pi\rho_0})\frac{\partial^2 u_x}{\partial x^2} - V_s^2 \frac{\partial^2 u_z}{\partial x \partial z} + \frac{B_{0x}B_{0z}}{4\pi\rho_0}(\frac{\partial^2 u_z}{\partial z^2} + \frac{\partial^2 u_z}{\partial x^2}) - \frac{B_{0z}^{\;2}}{4\pi\rho_0}\frac{\partial^2 u_x}{\partial z^2} = 0 \qquad (2.3a)$$

$$\frac{\partial^2 u_z}{\partial t^2} - (V_s^2 + \frac{B_{0x}^{\;2}}{4\pi\rho_0})\frac{\partial^2 u_z}{\partial z^2} - V_s^2 \frac{\partial^2 u_x}{\partial x \partial z} + \frac{B_{0x}B_{0z}}{4\pi\rho_0}(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial z^2}) - \frac{B_{0x}^{\;2}}{4\pi\rho_0}\frac{\partial^2 u_z}{\partial x^2} = 0 \qquad (2.3b)$$

Solving the above two equations we get a fourth order differential equation for u_z:

$$\partial^4 u_z / \partial t^4 + 2 v_s^{\,2} B_{0x} B_{0z} / 4\pi\rho_0 (\partial^4 u_z / \partial x \partial z^3 + \partial^4 u_z / \partial x^3 \partial z) - (v_s^{\,2} + v_a^{\,2})\ \partial^4 u_z / \partial z^2 \partial t^2$$

$$-(v_s^{\,2} + v_a^{\,2})\ \partial^4 u_z / \partial x^2 \partial t^2 + v_s^{\,2} v_a^{\,2} \partial^4 u_z / \partial x^2 \partial z^2 + v_s^{\,2} v_{az}^{\;2} \partial^4 u_z / \partial z^4 + v_s^{\,2} v_{ax}^{\;2} \partial^4 u_z / \partial x^4 = 0$$

$$\ldots. \qquad (2.4)$$

Equation for u_x is also the same.

2.3 The Dispersion Relation

In order to derive a dispersion relation, we look for wavelike solutions of the form

$$\vec{u}(x,z,t) = \vec{u}\exp i(wt - k_x x - k_z z)$$

Then we obtain the dispersion relation

$$w^4 - (V_s^{\,2} + V_A^{\,2}) w^2 k^2 + V_s^{\,2} V_A^{\,2} k^4 (\sin\theta \sin\Phi + \cos\theta\cos\Phi)^2 = 0 \qquad (2.5)$$

where $V_A^{\,2} (\equiv B_0^{\,2} / 4\pi\rho_0)$ is the square of the Alfven velocity.

2.4 The Phase Velocity

The characteristic phase velocity is given by

$$V_F^{\ 2} = \frac{1}{2}(V_s^{\ 2} + V_A^{\ 2}) + \frac{1}{2}[(V_s^{\ 2} + V_A^{\ 2})^2 - 4V_s^{\ 2}V_A^{\ 2}(\sin\theta\sin\Phi + \cos\theta\cos\Phi)^2]^{1/2} \qquad (2.6a)$$

and

$$V_S^{\ 2} = \frac{1}{2}(V_s^{\ 2} + V_A^{\ 2}) - \frac{1}{2}[(V_s^{\ 2} + V_A^{\ 2})^2 - 4V_s^{\ 2}V_A^{\ 2}(\sin\theta\sin\Phi + \cos\theta\cos\Phi)^2]^{1/2} \qquad (2.6b)$$

where V_F ,the high-frequency mode is known as the fast magnetoacoustic wave and V_S is known as the slow magnetoacoustic wave.

It is straightforward to show from equations (2.6a) and (2.6b) that the magnetoacoustic modes generally show the attributes of both purely longitudinal and transverse waves. If we consider magnetic field perturbation in the plane perpendicular to the background magnetic field then we would get Alfven mode as well. The Alfven wave phase speed lies between that of the slow and fast waves, and so the Alfven wave is sometimes referred as the intermediate mode. The dispersion relation for this mode would be

$$\omega^2 = k_z^{\ 2}V_A^{\ 2}$$

The properties of the three modes are summarized in the following table:

Table 1: Properties of linear fast, slow and Alfven waves in uniform medium

Fast magnetoacoustic wave	Driven by tension and pressure forces; involves pressure and density variations; gas and magnetic pressure are in phase. Roughly isotropic, propagating fastest across the field. Forms an orthogonal triad with the other two modes.
Slow magnetoacoustic mode	Driven by tension and pressure forces; involves pressure and density variations; gas and magnetic pressure variations are out of phase. Anisotropic, does not propagate across the field; energy flow is confined to the magnetic field direction.
Alfven wave	Driven by tension forces; no pressure or density variations; motions transverse to both the applied magnetic field and the direction of propagation. Anisotropic, does not propagate across the field; energy flows along the field at the Alfven wave speed.

2.5 Special Cases

We now consider three special cases:

i. $\theta = 0, \Phi = 0$: This is the simplest case for a vertical magnetic field with the purely one-dimensional vertical propagation. In this case, the fast mode is then a purely longitudinal acoustic mode when $V_s > V_A$, and is a purely transverse mode when $V_s < V_A$; and the slow mode has a purely transverse character when $V_s > V_A$, and is purely longitudinal when $V_s < V_A$.

ii. $\theta = 0, \Phi = \dfrac{\pi}{2}$ or $\theta = \dfrac{\pi}{2}, \Phi = 0$: The slow mode has zero phase velocity for its propagation perpendicular to the background magnetic field, but $\dfrac{\omega}{k_B} \equiv \dfrac{\omega}{k\cos\theta} \rightarrow c_T \equiv \dfrac{V_A V_s}{(V_{A^2} + V_s^{\,2})^{1/2}}$ where k_B is the wavenumber component along the field. Thus c_T is the component of the phase velocity along the field for propagation almost perpendicular to the field, so that the wavelength along the field is much longer than the wavelength across the field. Note that c_T represents the cusp speed for the group velocity of the slow wave.

iii. $\theta = \phi$: The phase velocity is decoupled; the fast mode is a purely longitudinal acoustic mode (V_s) and the slow mode is a purely transverse mode (V_A).

The two magnetoacoustic waves may be regarded as a sound wave, modified by the magnetic field, and a compressional Alfven wave, modified by the plasma pressure; the modification being most marked for propagation away from the magnetic field direction. In the case of vanishing magnetic field ($V_A = 0$), the slow wave disappears and the fast wave becomes a pure acoustic wave. When the plasma $\beta (\equiv \frac{2\mu p_0}{B_0^{\ 2}})$, the ratio of gas to magnetic pressure, where μ is the mean molecular weight) is much larger than unity so that $\frac{V_s^{\ 2}}{V_{A^2}}$ is also much larger than one, the plasma may be regarded as incompressible as far as the magnetic field effects are concerned, since sound wave propagates almost instantaneously (Priest, 1982)

<u>2.6 The Group Velocity</u>

The group velocity of a wave, $\vec{V}_g$, is defined by

$\vec{V}_g = \nabla_k \omega$.

Note that the direction and the magnitude of $\vec{V}_g$ are in general quite different from those of the phase velocity $V_{F,S}$.

If we express the gradient ∇_k in terms of its components parallel and perpendicular to the $\vec{B}_0$ direction and use the dispersion relation (2.5), we obtain for the magnetoacoustic modes (for $\Phi = 0$) (see Musielak and Rosner, 1987)

$$\vec{V}_g = \frac{V_{F,S}^4 (\vec{k}/k) - V_s^{\ 2} V_A^{\ 2} (k_{\parallel}/k)}{V_{F,S}(2V_{F,S}^2 - V_s^{\ 2} - V_A^{\ 2})} \qquad (2.7a)$$

$$V_g^{\ 2} = V_{F,S}^2 + \frac{V_s^2 V_A^2 \sin^2 2\theta}{16 V_{F,S}^2 (\beta_{sA}^2 - \cos^2\theta)} \qquad (2.7b)$$

and

$$\tan\theta_g = \frac{V_{F,S}^4}{V_{F,S}^4 - V_s^2 V_A^2}\tan\theta \qquad (2.7c)$$

where $\beta_{sA}^2 = \dfrac{(V_s^2 + V_A^2)^2}{4V_s^2 V_A^2}$ (2.7d)

One can see from equation (2.7a) that when $V_s > V_A$ or $V_A > V_s$, the group velocity of the fast mode has almost the same direction as V_F; in contrast, the slow mode has the striking property that the wave energy associated with its propagation is always carried within a small angle of B_0.

An important limitation on magnetohydrodynamic theory is that $\omega < \Omega_i$, where $\Omega_i = eB_0 m_i$ is the ion-gyration (or cyclotron) frequency with which ions gyrate about the magnetic field. When this is violated the slow wave ceases to exist and the fast and Alfven waves are modified. Thus, for example, with a magnetic field of 1 G, our analysis is valid provided the wave period exceeds 7×10^{-4} s (Priest,1982).

CHAPTER 3

MAGNETIC FLUX TUBES

3.1 Introduction

The magnetic fields in astrophysical situations are often quite inhomogeneous, due to the fact that they are embedded in highly turbulent fluids. In the case of the solar convection zone observations of the surface show fields ($B \geq 10^3 G$) which are essentially discrete, i.e. consisting of individual strands separated by field free fluid. Theoretical calculations of the behavior of fields in turbulent media support the idea that the field gets concentrated into a small fraction of the volume within a few turnover times of the turbulent eddies (Kraichnan ,1976, Spruit, 1981).This is especially the case if the back-reaction of the field on the flow is taken into account(e.g. Peckover and Weiss,1978, Spruit, 1981).

To study the behavior of such complicated field, one needs approximations. Weak inhomogeneities can be treated as perturbations of a homogeneous field. The very strong inhomogeneity often encountered suggests an opposite point of view to consider the field as existing of discrete structures separated by field free regions. We will call these structures the “magnetic flux tubes”.

3.2 Various flux tubes at different regions of Sun

Discrete flux tubes are found in many different forms on the Sun. Sunspots are present where a large 3000 Gauss flux tube breaks through the surface, typically in pairs with the tube coming up through one spot and going back down through the other. Also a spot (of typical diameter 20 Mm) is suggestive that it may itself consist of many smaller tubes. In the chromosphere above a sunspot pair one sees fibril structures joining one spot to the other and presumably outlining the magnetic field. In photospheric magnetic field maps one finds that, outside the active regions surrounding sunspot groups, the solar surface is covered with a fragmentary network structure consisting of many tiny flux tubes at the boundaries of large convection cells (the supergranulation).These tubes are a few hundred kilometers across and have a field strength of about 1500 Gauss.

The solar corona is seen in soft X-rays to consist of myriads of flux loops, both outside active regions and also within an active region. Hot flux loops ($\sim 10^7$ K) may also be created by large solar flares which subsequently cool to give an arcade of cool loops joining the two ribbons which make up the flare in its main phase in the chromosphere. Solar prominences are huge vertical sheets of plasma up in the corona but with a density a factor of a hundred higher and a temperature a factor of a hundred lower than the surrounding coronal plasma. Occasionally they loose equilibrium and erupt outwards when they undergo a metamorphosis and take on the appearance of a large twisted flux tube.

3.3 Vertical magnetic flux tubes

Parker (1955) suggested that an isolated horizontal flux tube in the solar interior would tend to rise by so-called magnetic buoyancy. The argument is very simple. If a tube is in lateral equilibrium with its field-free surroundings having a plasma pressure p_e, then its internal pressure (p_0) and magnetic field (B_0) satisfy

$$p_e = p_0 + \frac{B_0^2}{2\mu}, \tag{2.1}$$

or, if the temperature (T) is uniform,

$$RT\rho_e = RT\rho_0 + \frac{B_0^2}{2\mu} \tag{2.2}$$

Thus

$$\rho_e > \rho_o$$

and the plasma in the tube experiences a buoyancy force, which exceeds the magnetic tension if

$$(\rho_e - \rho_0)g > \frac{B_0^2}{\mu L}, \tag{2.3}$$

where L is the length of tube which is curved upwards. After substituting for $(\rho_e - \rho_0)$ from (2.2) this condition becomes

$$L > 2H$$

where $H = \frac{RT}{g}$ is the scale height.

If such a large flux tube in the interior rises and breaks through the surface it will form a pair of sunspots. In practice, the unbalanced force would make flux tubes rise much faster than a solar cycle period and so it is thought that the flux tubes are created by dynamo action not throughout the convection zone but only at its base. Most of the flux tubes which penetrate the solar surface are thought to be almost vertical due to magnetic buoyancy. A whole hierarchy of such tubes exists from the tiniest, only one to two hundred kilometers across, to enormous sunspots with a diameter of thirty Megameters (Zwaan, 1978; Roberts, 1989).

3.4 Structure of tube

A basic problem is to determine the structure of such a tube (its pressure p_0, density ρ_0, field B_0 and radius r) as a function of height (z).As the external pressure and density fall off in value with height, so the internal field strength tends to decrease and the tube spreads out(as the radius r increases).One needs to solve the equilibrium condition

$$-\nabla\left(p_0+\frac{B_0^2}{2\mu}\right)+(\vec{B}_0.\nabla)\frac{\vec{B}_0}{\mu}+\rho_0\vec{g}=0$$

inside the tube together with the hydrostatic equilibrium equation

$$\frac{dp_e}{dz}=\rho_e g$$

outside and a pressure matching condition

$$p_0 + \frac{B_0^2}{2\mu} = p_e$$

on the surface (S) of the tube. For a thin tube with

$$p_0 \sim p_e \sim e^{-z/H}$$

pressure balance gives

$$B^2 \sim e^{-z/H}$$

and so the tube radius expands exponentially like

$$r \sim e^{z/4H}\,.$$

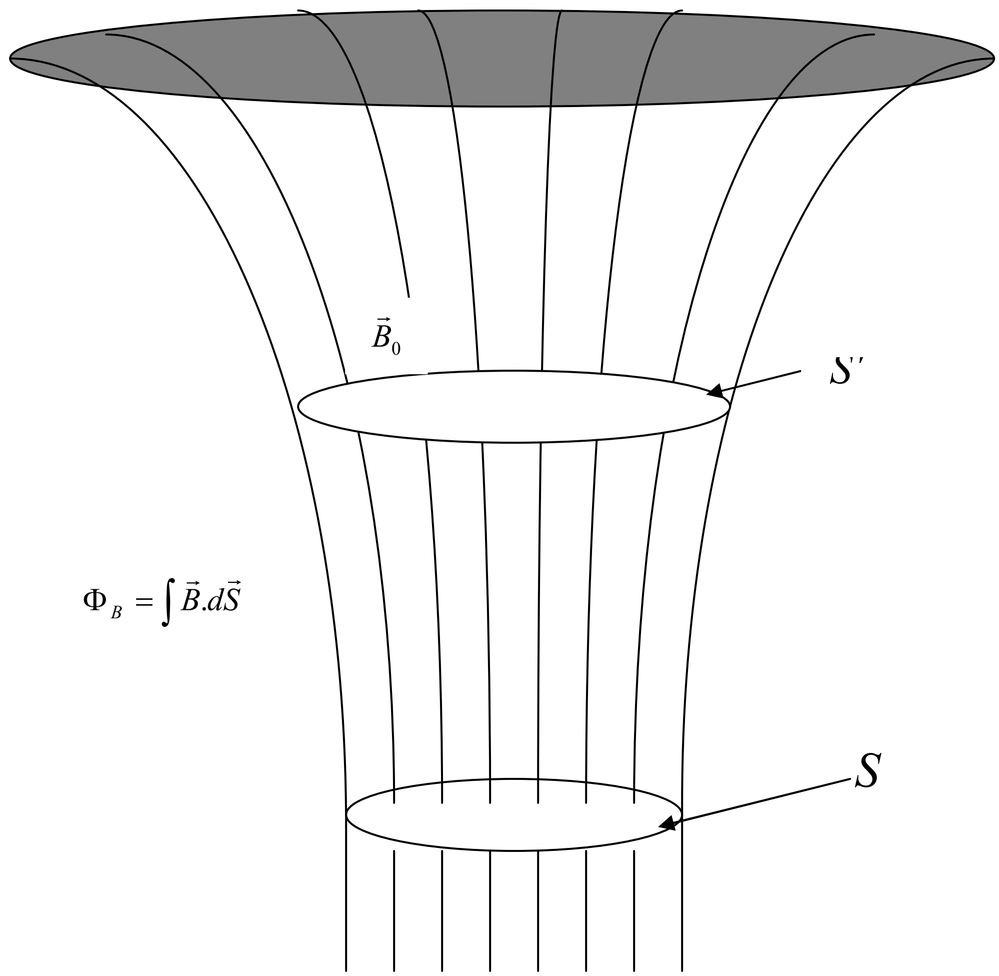

Fig 3.1 Divergent flux tube: The field lines that penetrate the surface are held such that as the surface stretches and shrinks the field lines are pushed and pulled together. This in effect leaves Φ_B unchanged.

3.5 Thin flux tubes

A very simplified model of the field is obtained, if one assumes that the flux tube is narrow enough. They are supposed to be so thin that they are always in pressure balance with their surroundings, and such that their diameter changes only slowly along their length. This approximation is certainly not valid for sunspots, but it is reasonable for the small elements of which the field outside of sunspots consists. Assuming the internal field is also nearly uniform across the cross section of the tube, the field can then be characterized by one value B, which depends on depth only. As we assume that there are no forces except gravity acting on the gas in the interior, the vertical balance of forces is then given by $\frac{dp}{dz} = \rho g$ and the horizontal balance by $\Delta p = p_e - p = \frac{B^2}{8\pi}$, where p_e and p are the external and internal gas pressures, respectively, and B is the field strength at the inside of the tube. It is also assumed that at each depth inside temperature T is equal to the external temperature T_e.We also assume that viscosity and resistivity are negligible and that the motion is adiabatic (ideal MHD limit).

The solar magnetic fields outside sunspots are concentrated into magnetic flux tube structures (e.g., Stenflo 1978; Solanki 1993) which support magnetic tube waves. In many theoretical considerations, the tubes are assumed to be so thin that their horizontal structure is neglected (see, however, Chapter IV in this thesis) and that no difference in physical parameters occurs on either side of the tubes. In this case, there are three different types of magnetic waves propagating along these thin magnetic flux tubes (e.g., Spruit 1987):

(i) *longitudinal (sausage) tube waves*, which are similar to slow MHD waves in the low-beta plasma limit;

(ii) *transverse (kink) tube waves*, which are similar to Alfven MHD waves;

(iii) *torsional tube waves*, which have no analogy to MHD waves.

If the thin flux tube approximation is released and the physical parameters change across the tube, then the tube boundary may support MHD surface waves. These waves exist whenever there are boundaries in a medium separating regions with different physical parameters (i.e., boundaries of loops and current sheets), and they are confined to the boundary as the wave amplitude decreases exponentially inwards and outwards from the boundary (see Roberts 1991). In the following, we consider the three types of magnetic flux tubes but do not discuss the surface waves.

CHAPTER 4

LONGITUDINAL TUBE WAVES

To derive the wave equation that describes the propagation of longitudinal tube waves along vertical flux tubes, we consider the following assumptions and model. This derivation follows Defouw (1976) and Musielak et al. (1989, 1995).

4.1 Assumptions and Model

(a) thin, isolated and vertical magnetic flux tube,

(b) no magnetic field outside the tube,

(c) parallel atmosphere with $\vec{g} = -g\hat{z}$ =constant,

(d) isothermal (T_0 =constant) and stratified atmosphere inside $[\rho_0 = \rho_0(z)]$ and outside the tube $[\rho_e = \rho_e(z)]$,

(e) gas pressure is a scalar,

(f) interior of the tube is force-free,

(g) no turbulence outside,

(h) linear and adiabatic waves,

(i) ideal MHD limit i.e. displacement currents and electrostatic forces may be neglected, molecular viscosity and Ohmic diffusion be negligible

The assumptions lead to the ideal MHD equations, which are written in the following linearized form:

$$\frac{\partial \rho}{\partial t} + \rho_0(\nabla.\vec{u}) + u_z \rho_0' = 0 \qquad \text{where} \quad \rho_0' = \frac{d\rho_0}{dz} \tag{4.1}$$

$$\rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla p - \rho \vec{g} - \frac{1}{4\pi}\left[\left(\nabla \times \vec{b}\right) \times \vec{B}_0 + (\nabla \times \vec{B}_0) \times \vec{b}\right] = 0 \tag{4.2}$$

$$\frac{\partial \vec{b}}{\partial t} - \nabla \times (\vec{u} \times \vec{B}_0) = 0 \tag{4.3}$$

$$\frac{\partial p}{\partial t} + \vec{u}.\nabla p_0 - V_s^2(\partial \rho / \partial t + \rho_0' u_z) = 0 \tag{4.4}$$

where ρ_0, p_0 and $\vec{B}_0$ refer to the unperturbed atmosphere inside the tube; and ρ, p, $\vec{u}$ and $\vec{b}$ are perturbations associated with the waves.

In writing equation (4.2) we used the hydrostatic equilibrium condition for the atmosphere inside the tube which requires

$$\nabla p_0 - \rho_0 \vec{g} - \frac{1}{4\pi}[(\nabla \times \vec{B}_0) \times \vec{B}_0] = 0 \tag{4.5}$$

We also need another equation which is horizontal pressure balance equation in the following form:

$$p_0 + p + \frac{1}{8\pi}(B_0 + 2B_0 b + b^2) = p_e$$

where p_e is the external pressure.

For linear waves b^2 is small and we have

$$p_0 + \frac{B_0^2}{8\pi} = p_e \text{ and } p + \frac{B_0 b}{4\pi} = 0 \tag{4.6}$$

4.2 The wave equation

The following wave equation is derived:

$$\frac{\partial^2 u}{\partial t^2} - V_t^2 \frac{\partial^2 u}{\partial z^2} + \frac{V_t^2}{2H}\frac{\partial u}{\partial z} + \frac{V_t^2}{H^2}\left[\frac{1}{2} - \frac{1}{2\gamma} + \frac{V_s^2}{V_A^2}\left(\frac{\gamma - 1}{\gamma^2}\right)\right]u = 0 \tag{4.7}$$

where the characteristic velocity for longitudinal tube waves is given by

$$V_t^2 = \frac{V_A^2 V_s^2}{V_A^2 + V_s^2}$$

4.3 Klein-Gordon Wave Equation and the cutoff frequency

To remove the 1st order term we use the following transformation

$$u = v\left(\frac{B_0}{\rho_0}\right)^{1/2}$$

Then the Klein-Gordon form of the wave equation derived by them is the following:

$$\left(\frac{\partial^2}{\partial t^2} - V_t^2 \frac{\partial^2}{\partial z^2} + \Omega_T^2\right)v = 0 \tag{4.8}$$

where the cut-off frequency is

$$\Omega_T^2 = \frac{V_t^2}{H^2}\left(\frac{9}{16} - \frac{1}{2\gamma} + \frac{V_s^2}{V_A^2}\frac{\gamma - 1}{\gamma^2}\right) \tag{4.9}$$

The constant V_s^2, V_A^2 and H implies that Ω_T is a constant; so it does not change along the tube axis.

The wave equation (4.7) and the K-G equation (4.8) have the same form for all wave variables (u, ρ, p).

4.4 Dispersion relation

Since $\Omega_T =$constant, one can make Fourier transforms in time and space and derive the global dispersion relation $(\omega^2 - \Omega_T^2) - V_t^2 k^2 = 0$, where ω is the wave frequency and $k = k_z$ is the wave vector. This shows that the waves are propagating when $\omega > \Omega_T$ and k is real, and they are non-propagating when either $\omega = \Omega_T$ with k =0 or $\omega < \Omega_T$ with k being imaginary; in the latter case, the waves are called evanescent waves.

CHAPTER 5

TRANSVERSE TUBE WAVES

We now derive the wave equation for transverse tube waves using first a method developed by Spruit (1981) and then a new method developed in this thesis. We also derive the cutoff frequency for these waves.

5.1 Previous Method of Deriving the Wave Equation

We consider a magnetic flux tube that is oriented vertically and is displaced from its equilibrium position in the horizontal direction. We denote this horizontal displacement by ξ and the resulting velocity by u and magnetic field

$\vec{B}_0 = B_0(r,\varphi,l)\hat{l}$ in the local coordinate system or

$$\vec{B}_0 = B_{0z}(z)\hat{z} + b_x(z,t)\hat{x}, \qquad (5.1.1)$$

in the xz-coordinate system.

We consider linear oscillations, so that $\vec{l}$ vector along the tube axis has the following components

$$l_x = \frac{b}{B_0} \text{ or } l_x = \frac{\partial \xi}{\partial z} = \left(\frac{\partial}{\partial t}\right)^{-1} \frac{\partial u}{\partial z} \qquad (5.1.2)$$

In addition,we have

$$l_z = 1, \qquad (5.1.3)$$

and

$$k_x = \frac{\partial^2 \xi}{\partial z^2} = \left(\frac{\partial}{\partial t}\right)^{-1} \frac{\partial^2 u}{\partial z^2} \tag{5.1.4}$$

where k_x is the x-component of the curvature vector.

We begin with the momentum equation for the tube and write it down in the following form

$$\rho_0 \frac{\partial \vec{u}}{\partial t} = -\nabla\left(p_0 + \frac{B_0^2}{8\pi}\right) + \frac{1}{4\pi}\left(\vec{B}_0.\nabla\right)\vec{B}_0 + \rho_0 \vec{g} \tag{5.1.5}$$

We assume that there are motions outside the tube and their velocity is $\vec{v}$. The momentum equation for these motions is

$$\rho_e \frac{\partial \vec{v}}{\partial t} = -\nabla p_e + \rho_e \vec{g} \tag{5.1.6}$$

The momentum equation for the tube and for the external motions are related to each other through the horizontal pressure balance given by

$$p_0 + \frac{B_0^2}{8\pi} = p_e \tag{5.1.7}$$

Using (5.1.7) and (5.1.6), we may write

$$\nabla\left(p_0 + \frac{B_0^2}{8\pi}\right) = \nabla p_e = \rho_e \vec{g} - \rho_e \frac{\partial \vec{v}}{\partial t}$$

which allows writing (5.1.5) as

$$\rho_0 \frac{\partial \vec{u}}{\partial t} = \frac{1}{4\pi}\left(\vec{B}_0.\nabla\right)\vec{B}_0 + (\rho_0 - \rho_e)\vec{g} + \rho_e \frac{\partial \vec{v}}{\partial t} \tag{5.1.8}$$

Since we are only interested in transverse motions of the tube, we consider the perpendicular component of equation (5.1.8). Thus we get,

$$(\rho_0+\rho_e)\left(\frac{\partial\vec{u}}{\partial t}\right)_\perp=\frac{1}{4\pi}\left[\hat{l}\times(\vec{B}_0.\nabla)\vec{B}_0\right]\times\hat{l}+\left(\rho_0-\rho_e\right)\left[(\hat{l}\times\vec{g})\times\hat{l}\right] \qquad (5.1.9)$$

as for the continuity of motions at the tube boundary

$(\vec{u})_\perp=-(\vec{v})_\perp$.

Since

$$(\vec{B}_0.\nabla)\vec{B}_0=(B_0\,\hat{l}.\nabla)B_0\,\hat{l}=B_0\,\hat{l}\frac{\partial B_0}{\partial l}+B_0{}^2\frac{\partial\hat{l}}{\partial l}=B_0\,\hat{l}\frac{\partial B_0}{\partial l}+B_0{}^2\vec{k}$$

where $\hat{k}=\dfrac{\partial\hat{l}}{\partial l}$

$$\left[\hat{l}\times(\vec{B}_0.\nabla)\vec{B}_0\right]\times\hat{l}=\left[B_0(\hat{l}\times\hat{l})\frac{\partial B_0}{\partial l}\right]\times\hat{l}+B_0^2(\hat{l}\times\hat{k})\times\hat{l}=B_0^2\vec{k}$$

as $\vec{k}\perp\hat{l}$

and

$$(\hat{l}\times\vec{g})\times\hat{l}=(\hat{l}.\hat{l})\vec{g}-(\vec{g}.\hat{l})\hat{l}=-g\,\hat{z}+gl_z\,\hat{l}=g(l_z\,\hat{l}-\hat{z})$$

This gives

$$(\rho_0+\rho_e)\left(\frac{\partial\vec{u}}{\partial t}\right)_\perp=\frac{B_0^2}{4\pi}\vec{k}+(\rho_0-\rho_e)g(l_z\,\hat{l}-\hat{z})$$

Defining

$\left(\dfrac{\partial\vec{u}}{\partial t}\right)_\perp\equiv\dfrac{\partial u}{\partial t}$ and $(\vec{k})_\perp\equiv k_x$

we have

$$(\rho_0+\rho_e)\frac{\partial u}{\partial t}=\frac{B_0^2}{4\pi}k_x+(\rho_0-\rho_e)g(l_z l_x)$$

Using (5.1.2)-(5.1.4) we can write the above equation as

$$\frac{\partial u}{\partial t}-\frac{B_0^2}{4\pi(\rho_0+\rho_e)}\left(\frac{\partial}{\partial t}\right)^{-1}\left(\frac{\partial^2 u}{\partial z^2}\right)-\frac{\rho_0-\rho_e}{\rho_0+\rho_e}g\left(\frac{\partial}{\partial t}\right)^{-1}\frac{\partial u}{\partial z}=0 \qquad (5.1.10)$$

Introducing

$V_K^2\equiv\dfrac{B_0{}^2}{4\pi(\rho_0+\rho_e)}$, we have

$$\frac{\partial^2 u}{\partial t^2}-V_K^2\frac{\partial^2 u}{\partial z^2}-\frac{\rho_0-\rho_e}{\rho_0+\rho_e}g\frac{\partial u}{\partial z}=0 \qquad (5.1.11)$$

From the horizontal pressure balance, we find

$$p_0-p_e=-\frac{B_0^2}{8\pi}$$

And, using the ideal gas law thermodynamics

$$\frac{RT_0\rho_0}{\mu}-\frac{RT_e\rho_e}{\mu}=-\frac{B_0{}^2}{8\pi} \qquad (5.1.12)$$

Here it is assumed that $T_0=T_e$=constant; the temperature is also constant throughout the tube length. Equation (5.1.12) gives

$$\rho_0-\rho_e=-\frac{B_0^2}{8\pi}\frac{\mu}{RT_0}$$

Therefore,

$$\frac{\rho_0 - \rho_e}{\rho_0 + \rho_e} g = -\frac{1}{2} \frac{B_0^2}{4\pi(\rho_0 + \rho_e)} \gamma \frac{g\mu}{\gamma R T_0} \tag{5.1.13}$$

Using equation (5.1.13), the definition $V_K^2 \equiv \frac{B_0{}^2}{4\pi(\rho_0 + \rho_e)}$, the pressure/density scale height $H = \frac{V_s{}^2}{\gamma g}$, and the acoustic wave speed V_s where $V_s^2 = \frac{\gamma R T_0}{\mu}$, it can be obtained from equation (5.1.12) that

$$\frac{\partial^2 u}{\partial t^2} - V_K^2 \frac{\partial^2 u}{\partial z^2} + \frac{V_K^2}{2H} \frac{\partial u}{\partial z} = 0 \tag{5.1.14}$$

which is the wave equation for transverse tube waves.

To obtain the Klein-Gordon equations for this wave, we use the transformation

$u = v \rho_0{}^{-1/4}$

which gives

$$\frac{\partial^2 v}{\partial t^2} - V_K^2 \frac{\partial^2 v}{\partial z^2} + \Omega_K{}^2 v = 0 \tag{5.1.15}$$

where $\Omega_K{}^2 \equiv \frac{C_t{}^2}{16H^2}$, is the cut-off frequency.

The constant Ω_K implies that the cutoff frequency is global, which means that it does not change along the tube axis. By making a Fourier transform in time and space, the global dispersion relation $(\omega^2 - \Omega_K{}^2) - k^2 V_K{}^2 = 0$ is obtained. According to this relation all the partial waves with frequencies higher than the cutoff frequency will keep on propagating unless damped by dissipation.

5.2 New Method to Derive the Wave Equation

Here we have developed an alternative much simpler method to derive the transverse wave equation.

The momentum equation of the oscillating tube is

$$\rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla p_0 - \rho_0 \vec{g} - \frac{1}{4\pi}[(\nabla \times \vec{B}) \times \vec{B}] = 0 \qquad (5.2.1)$$

where $\vec{B}(\equiv \vec{B}_0 + \vec{b} = B_0(z)\hat{z} + b_x(z,t)\hat{x})$, is the total magnetic field.

We can write equation (5.2.1) in the following way:

$$\rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla p_0 - \rho_0 \vec{g} - \frac{1}{4\pi}\left[(\nabla \times \vec{B}_0) \times \vec{B}_0 + (\nabla \times \vec{b}) \times \vec{B}_0 + (\nabla \times \vec{B}_0) \times \vec{b}\right] = 0$$

or

$$\rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla \left(p_0 + \frac{B_0^2}{8\pi} \right) - \rho_0 \vec{g} - \frac{1}{4\pi}\left[(\vec{B}_0.\nabla)\vec{B}_0 + (\vec{B}_0.\nabla)\vec{b}\right] = 0 \qquad (5.2.2)$$

We assume that there are motions outside the tube and their velocity is $\vec{v}$. The equation of motion of the fluid outside the tube is

$$\rho_e \frac{\partial \vec{v}}{\partial t} + \nabla p_e - \rho_e \vec{g} = 0 \qquad (5.2.3)$$

According to the horizontal pressure balance equation $p_0 + \frac{B_0^2}{8\pi} = p_e$, we have

$$\nabla \left(p_0 + \frac{B_0^2}{8\pi} \right) = \nabla p_e$$

with

$$\nabla p_e = \rho_e \vec{g} - \rho_e \frac{\partial \vec{v}}{\partial t} \qquad \text{(see equation (5.2.3))}$$

and

$$\nabla \left(p_0 + \frac{B_0^2}{8\pi} \right) = \rho_e \vec{g} - \rho_e \frac{\partial \vec{v}}{\partial t} \tag{5.2.4}$$

Then equation (5.2.2) can be written as

$$\rho_0 \frac{\partial \vec{u}}{\partial t} - \rho_e \frac{\partial \vec{v}}{\partial t} + (\rho_e - \rho_0)\vec{g} - \frac{1}{4\pi}\left[(\vec{B}_0.\nabla)\vec{B}_0 + (\vec{B}_0.\nabla)\vec{b} \right] = 0 \tag{5.2.5}$$

The perpendicular component (the x-component) of equation (5.2.5) becomes

$$(\rho_0 + \rho_e)\frac{\partial u_x}{\partial t} - \frac{1}{4\pi}(B_0 \frac{\partial b_x}{\partial z}) = 0 \tag{5.2.6}$$

since at the tube boundary $u_x = -v_x$.

In addition to the momentum equation (5.2.6), we need the induction equation:

$$\frac{\partial \vec{b}}{\partial t} - \nabla \times (\vec{u} \times \vec{B}) = 0 \tag{5.2.7}$$

Linearizing equation (5.2.7) and taking x- component we get

$$\frac{\partial b_x}{\partial t} - B_0 \frac{\partial u_x}{\partial z} = 0 \tag{5.2.8}$$

Taking time derivative of equation (5.2.6) we obtain

$$(\rho_0 + \rho_e)\frac{\partial^2 u_x}{\partial t^2} - \frac{1}{4\pi}(B_0 \frac{\partial^2 b_x}{\partial z \partial t}) = 0 \tag{5.2.9}$$

Next, we consider the derivative of equation (5.2.8) with respect to z:

$$\frac{\partial^2 b_x}{\partial z \partial t} = -\frac{B_0}{2H}\frac{\partial u_x}{\partial z} + B_0 \frac{\partial^2 u_x}{\partial z^2} \tag{5.2.10}$$

Substituting the above equation in (5.2.9) we get

$$\frac{\partial^2 u_x}{\partial t^2} - V_K^2 \frac{\partial^2 u_x}{\partial z^2} + \frac{V_K^2}{2H}\frac{\partial u_x}{\partial z} = 0 \qquad (5.2.11)$$

Where we define the characteristic velocity V_K for transverse tube waves

$$V_K = \frac{B_0}{\sqrt{4\pi(\rho_0 + \rho_e)}}$$

Similarly, to derive the wave equation for b_x we take time derivative of equation (5.2.8) and substitute equation (5.2.6) into it and thus obtain

$$\frac{\partial^2 b_x}{\partial t^2} - V_K^2 \frac{\partial^2 b_x}{\partial z^2} - \frac{V_K^2}{2H}\frac{\partial b_x}{\partial z} = 0 \qquad (5.2.12)$$

The sign of the last term is different in (5.2.11) than in (5.2.12).

Now, to transform (5.2.11) and (5.2.12) into the Klein-Gordon form we use the transformations

$$u_x = \rho_0^{-1/4} u$$

$$b_x = \rho_0^{1/4} b$$

and the resulting Klein-Gordon equations for transverse wave are

$$\left(\frac{\partial^2}{\partial t^2} - C_t^2 \frac{\partial^2}{\partial z^2} + \Omega_K^2\right)(u, b) = 0 \qquad (5.2.13)$$

where $\Omega_K = \dfrac{C_t}{4H}$,is the cut-off frequency.

This is the same cutoff frequency as that originally obtained by Spruit (1981). The physical meaning of this cutoff was already discussed in the previous section.

CHAPTER 6

TORSIONAL MAGNETIC TUBE WAVES

Propagation of torsional tube waves was considered in the literature by Hollweg (1982) and Noble et al.(2003) etc. Here we derive the wave equation for two different cases. First, we assume that the tube is thin enough so that the radial component of the magnetic field can be neglected. Second, we discuss the case when the radial component is considered.

6.1 Derivation of Torsional Wave Equation: First Case

To describe torsional tube waves we adopt a cylindrical coordinate system (r, ϕ, z), with z being the tube axis.

Using thin flux tube approximation the tube magnetic field can be written as

$$\vec{B}_0 = B_0(z)\hat{z} \qquad (6.1.1)$$

Where $B_0 = B_{0z}$

Torsional waves are described by

$$\vec{v} = v_\varphi(z,t)\hat{\varphi} \qquad (6.1.2)$$

and

$$\vec{b} = b_\varphi(z,t)\hat{z}$$

Let us begin with the linearized momentum equation:

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = \frac{1}{4\pi}[(\nabla \times \vec{B}_0) \times \vec{b} + (\nabla \times \vec{b}) \times \vec{B}_0] \qquad (6.1.3)$$

where we have neglected the source term.

Since the flux tube was initially untwisted i.e. $B_{0\varphi} = 0$,

$$\therefore\ \nabla \times \vec{B}_0 = 0$$

So we need to calculate the last term on the right hand side of (6.1.3)

only:

$$\nabla \times \vec{b} = -\frac{1}{r}\frac{\partial}{\partial z}(rb_\varphi)\hat{r} + \frac{1}{r}\frac{\partial}{\partial r}(rb_\varphi)\hat{z}$$

$$\therefore\ (\nabla \times \vec{b}) \times \vec{B}_0 = \frac{B_0}{r}\frac{\partial}{\partial z}(rb_\varphi)\hat{\varphi}$$

So, equation (6.1.3) becomes

$$\frac{\partial v_\varphi}{\partial t} - \frac{B_0}{4\pi\rho_0 r}\frac{\partial}{\partial z}(rb_\varphi) = 0 \qquad (6.1.4)$$

Now the second basic equation is the induction equation, which is

$$\frac{\partial \vec{b}}{\partial t} - \nabla \times (\vec{v} \times \vec{B}_0) = 0 \qquad (6.1.5)$$

Now, $\vec{v} \times \vec{B}_0 = v_\varphi B_0 \hat{r}$

$$\therefore\ \nabla \times (\vec{v} \times \vec{B}_0) = \frac{\partial}{\partial z}(v_\varphi B_0)\hat{\varphi}$$

So we can write equation (6.1.5) as

$$\frac{\partial b_\varphi}{\partial t} - \frac{\partial}{\partial z}(v_\varphi B_0) = 0 \qquad (6.1.6)$$

As magnetic flux is conserved (i.e. $B_0 r^2 =$ constant), we can write equation (6.1.6) as

$$\frac{\partial b_\varphi}{\partial t} - B_0 r^2 \frac{\partial}{\partial z}(\frac{v_\varphi}{r^2}) = 0$$

As r and z are independent coordinates, we can write the above equation as

$$\frac{\partial b_\varphi}{\partial t} - B_0 r \frac{\partial}{\partial z}(\frac{v_\varphi}{r}) = 0 \qquad (6.1.7)$$

Knowing that $r \neq r(t)$, we rewrite (6.1.4) and (6.1.7) as

$$\frac{\partial}{\partial t}(\frac{v_\varphi}{r}) - \frac{B_0}{4\pi\rho_0 r^2}\frac{\partial}{\partial z}(rb_\varphi) = 0 \qquad (6.1.8)$$

and

$$\frac{\partial}{\partial t}(rb_\varphi) - B_0 r^2 \frac{\partial}{\partial z}(\frac{v_\varphi}{r}) = 0 \qquad (6.1.9)$$

Defining $x \equiv \frac{v_\varphi}{r}$ and $y \equiv rb_\varphi$ we obtain Hollweg's equations from (6.1.8) and (6.1.9)

$$\frac{\partial x}{\partial t} = \frac{B_0}{4\pi\rho_0 r^2}\frac{\partial y}{\partial z} \qquad (6.1.10)$$

$$\frac{\partial y}{\partial t} = B_0 r^2 \frac{\partial x}{\partial z}$$

For Hollweg's tubes $z \equiv s$ where s is a spatial variable which is along a field line and B_0, ρ_0 are functions of s whereas for our thin tubes B_0, ρ_0 are functions of z.

Now let us consider v_φ and b_φ as the wave variables.

From equations (6.1.4) and (6.1.7) we get

$$\frac{\partial v_\varphi}{\partial t} = \frac{B_0}{4\pi\rho_0}\frac{\partial b_\varphi}{\partial z} \tag{6.1.11}$$

and

$$\frac{\partial b_\varphi}{\partial t} = B_0\frac{\partial v_\varphi}{\partial z} \tag{6.1.12}$$

In deriving equations (6.1.11) and (6.1.12) we again used the fact that r and z are independent coordinates.

Derivative of equation (6.1.11) with respect to time gives us

$$\frac{\partial^2 v_\varphi}{\partial t^2} = \frac{B_0}{4\pi\rho_0}\frac{\partial^2 b_\varphi}{\partial z \partial t}$$

Substituting (6.1.12) in the above equation we get

$$\frac{\partial^2 v_\varphi}{\partial t^2} - \frac{B_0 B_0'}{4\pi\rho_0}\frac{\partial v_\varphi}{\partial z} - \frac{B_0{}^2}{4\pi\rho_0}\frac{\partial^2 v_\varphi}{\partial z^2} = 0 \tag{6.1.13}$$

where $B_0' = \frac{\partial B_0}{\partial z}$.

From the horizontal pressure balance equation, $p_0 + \frac{B_0{}^2}{8\pi} = p_e$, we can write

$$B_0' = \frac{\partial B_0}{\partial z} = -\frac{B_0}{2H}$$

So (6.1.13) becomes

$$\frac{\partial^2 v_\varphi}{\partial t^2} + \frac{V_A{}^2}{2H}\frac{\partial v_\varphi}{\partial z} - V_A{}^2\frac{\partial^2 v_\varphi}{\partial z^2} = 0 \tag{6.1.14}$$

where $V_A{}^2 = \dfrac{B_0{}^2}{4\pi\rho_0}$,is the Alfven velocity.

Taking time derivative of (6.1.12) we get

$$\frac{\partial^2 b_\varphi}{\partial t^2} = B_0 \frac{\partial^2 v_\varphi}{\partial z \partial t}$$

Substituting (6.1.11) into the above equation we get

$$\frac{\partial^2 b_\varphi}{\partial t^2} - \frac{V_A{}^2}{2H}\frac{\partial b_\varphi}{\partial z} - V_A{}^2 \frac{\partial^2 b_\varphi}{\partial z^2} = 0 \qquad (6.1.15)$$

To transform (6.1.14) and (6.1.15) into Klein-Gordon form, we use

$$v_\varphi = \rho_0{}^{-1/4} v$$

and

$$b_\varphi = \rho_0{}^{1/4} b$$

Then we write (6.1.14) as

$$\frac{\partial^2 v}{\partial t^2} - V_A{}^2 \frac{\partial^2 v}{\partial z^2} + \frac{V_A{}^2}{16H^2} v = 0 \qquad (6.1.16)$$

and (6.15) as

$$\frac{\partial^2 b}{\partial t^2} - V_A{}^2 \frac{\partial^2 b}{\partial z^2} + \frac{V_A{}^2}{16H^2} b = 0 \qquad (6.1.17)$$

So both (6.1.16) and (6.1.17) have same form with same cut-off frequency

$$\Omega_A = \frac{V_A}{4H}.$$

So we can take V_A to be a constant.

Now we check the cut-off frequency for Hollweg's variables $x \equiv \frac{v_\varphi}{r}$ and $y \equiv r b_\varphi$.

Taking time derivative of (6.1.10) we get

$$\frac{\partial^2 x}{\partial t^2} = \frac{B_0}{4\pi\rho_0 r^2}\frac{\partial^2 y}{\partial z \partial t}$$

Substituting the other equation in (6.1.10) in to the above equation we finally obtain

$$\frac{\partial^2 x}{\partial t^2} + \frac{V_A^{\ 2}}{2H}\frac{\partial x}{\partial z} - V_A^{\ 2}\frac{\partial^2 x}{\partial z^2} = 0 \qquad (6.1.18)$$

and similarly we get from the first equation in (6.1.10)

$$\frac{\partial^2 y}{\partial t^2} - \frac{V_A^{\ 2}}{2H}\frac{\partial y}{\partial z} - V_A^{\ 2}\frac{\partial^2 y}{\partial z^2} = 0 \qquad (6.1.19)$$

The above two equations of x and y have same form as (6.1.14) and (6.1.15).

So when we transform them in K-G form they will also have same form of equation like (6.1.17) with same cut-off frequency

$$\Omega_A = \frac{V_A}{4H}.$$

So all the variables (v, b, x, y) have same cut-off frequency.

Since both the propagation velocity V_A of torsional waves and the scale height H are constant along the tube, the cutoff frequency Ω_A is a global cutoff as it is the same along the entire length of the tube. Torsional waves are propagating waves when their frequency ω satisfies the condition $\omega > \Omega_A$.

6.2 Deriving Wave Equation for Torsional Wave Equation: Second Case

To describe torsional tube waves we adopt a cylindrical coordinate system (r, ϕ, z), with z being the tube axis.

In general, the tube magnetic field is given by $\vec{B}_0 = B_{0z}(z)\hat{z} + B_{0r}(r)\hat{r}$, which is based on the assumption that initially the field is not twisted, $B_{0\phi} = 0$.

For a thin flux tube we may write

$$B_{0r}(r) = B_{0r}(0) + r\left(\frac{\partial B_{0r}}{\partial r}\right)_{r=0} + ..., \tag{6.2.1}$$

and take $B_{0r}(0) = 0$,which is the essence of the thin flux tube approximation.

Now, $\dfrac{dz}{dr} = \dfrac{B_{0z}}{B_{0r}} = \dfrac{B_{0z}}{r\dfrac{\partial B_{0r}}{\partial r}}$, from equation (6.2.1), (6.2.2)

Since $\nabla.\vec{B}_0 = 0$, we obtain

$$\nabla.\vec{B}_0 = \frac{\partial B_{0z}}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(rB_{0r}\right) = \frac{\partial B_{0z}}{\partial z} + 2\frac{\partial B_{0r}}{\partial r}, \text{ from equation (6.2.1)}$$

$$= \frac{dB_{0z}}{dz} + 2\frac{B_{0z}}{r}\frac{dr}{dz}, \text{ from equation(6.2.2)}$$

$$= \frac{dB_{0z}}{B_{0z}} + 2\frac{dr}{r} = 0$$

which shows that $r^2 B_{0z}$=constant for the thin flux tube approximation

To describe torsional wave, we introduce as before

$$\vec{v} = v_\varphi(z,t)\hat{\varphi}$$

and

$$\vec{b} = b_\varphi(z,t)\hat{\varphi}$$

The linearized momentum equation is

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = \frac{1}{4\pi}[(\nabla \times \vec{B}_0) \times \vec{b} + (\nabla \times \vec{b}) \times \vec{B}_0] \quad (6.2.3)$$

As the tube was initially untwisted i.e. $B_{0\varphi} = 0$,

$\therefore\ \nabla \times \vec{B}_0 = 0$

So we need to calculate the last term on the right hand side of (6.2.3) only:

$$\nabla \times \vec{b} = -\frac{1}{r}\frac{\partial}{\partial z}(rb_\varphi)\hat{r} + \frac{1}{r}\frac{\partial}{\partial r}(rb_\varphi)\hat{z}$$

$$\therefore\ (\nabla \times \vec{b}) \times \vec{B}_0 = \frac{B_{0z}}{r}\frac{\partial}{\partial z}(rb_\varphi)\hat{\varphi} + \frac{B_{0r}}{r}\frac{\partial}{\partial r}(rb_\varphi)\hat{\varphi}$$

So, equation (6.2.3) becomes

$$\frac{\partial v_\varphi}{\partial t} - \frac{1}{4\pi\rho_0}\left[\frac{B_{0r}}{r}\frac{\partial}{\partial r}(rb_\varphi) + \frac{B_{0z}}{r}\frac{\partial}{\partial z}(rb_\varphi)\right] = 0 \quad (6.2.4)$$

or, if we define $B_{0r}\frac{\partial}{\partial r} + B_{0z}\frac{\partial}{\partial z} \equiv B_0\frac{\partial}{\partial s}$, where s is a spatial variable along the field line

then we may write (6.2.4) as

$$\frac{\partial v_\varphi}{\partial t} - \frac{1}{4\pi\rho_0}\left[\frac{B_0}{r}\frac{\partial}{\partial s}(rb_\varphi)\right] = 0 \quad (6.2.5)$$

Now the second basic equation is the induction equation, which is

$$\frac{\partial \vec{b}}{\partial t} - \nabla \times (\vec{v} \times \vec{B}_0) = 0 \quad (6.2.6)$$

Now, $\vec{v}\times\vec{B}_0 = v_\varphi B_{0z}\hat{r} - v_\varphi B_{0r}\hat{z}$

$$\therefore\ \nabla\times(\vec{v}\times\vec{B}_0) = \left\{\frac{\partial}{\partial z}(v_\varphi B_{0z}) + \frac{\partial}{\partial r}(v_\varphi B_{0r})\right\}\hat{\varphi}$$

$$= \left\{v_\varphi \frac{\partial B_{0z}}{\partial z} + B_{0z}\frac{\partial v_\varphi}{\partial z} + \frac{\partial}{\partial r}\left(\frac{v_\varphi}{r} r B_{0r}\right)\right\}\hat{\varphi}$$

(as r and z are independent coordinates)

$$= \left[v_\varphi\left\{\frac{\partial B_{0Z}}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rB_{0r})\right\} + rB_{0z}\frac{\partial}{\partial z}\left(\frac{v_\varphi}{r}\right) + rB_{0r}\frac{\partial}{\partial r}\left(\frac{v_\varphi}{r}\right)\right]\hat{\varphi}$$

$$= \left[v_\varphi(\nabla.\vec{B}_0) + rB_{0r}\frac{\partial}{\partial r}\left(\frac{v_\varphi}{r}\right) + rB_{0z}\frac{\partial}{\partial z}\left(\frac{v_\varphi}{r}\right)\right]\hat{\varphi}$$

Using the solenoidal condition $\nabla.\vec{B}_0 = 0$, we finally write equation (6.2.6) as

$$\frac{\partial b_\varphi}{\partial t} - rB_0\frac{\partial}{\partial s}\left(\frac{v_\varphi}{r}\right) = 0 \qquad (6.2.7)$$

Knowing that $r \neq r(t)$,we rewrite (6.2.6) and (6.2.7) as

$$\frac{\partial}{\partial t}\left(\frac{v_\varphi}{r}\right) - \frac{B_0}{4\pi\rho_0 r^2}\frac{\partial}{\partial s}(rb_\varphi) = 0 \qquad (6.2.8)$$

and

$$\frac{\partial}{\partial t}(rb_\varphi) - B_0 r^2\frac{\partial}{\partial s}\left(\frac{v_\varphi}{r}\right) = 0 \qquad (6.2.9)$$

Defining $x \equiv \frac{v_\varphi}{r}$ and $y \equiv rb_\varphi$ we obtain Hollweg's equations from (6.2.8) and (6.2.9)

$$\frac{\partial x}{\partial t} = \frac{B_0}{4\pi\rho_0 r^2}\frac{\partial y}{\partial s} \tag{6.2.10}$$

$$\frac{\partial y}{\partial t} = B_0 r^2 \frac{\partial x}{\partial s} \tag{6.2.11}$$

Solving the above two equations we get

$$\frac{\partial^2 x}{\partial t^2} = V_A^2 \frac{\partial^2 x}{\partial s^2} \tag{6.2.12}$$

where $V_A(s) = \frac{B_0}{\sqrt{4\pi\rho_0}}$,is the Alfven speed.

Equation (6.2.12) is usually studied [see [6]]. It has no cutoff frequency.

But if we transform the above equation in Klein-Gordon form following the way Musielak et al. (2006) derived for acoustic waves in non-isothermal medium, we can show that there is indeed cutoff frequency.

We introduce new variable $d\tau = \frac{ds}{V_A}$ and obtain

$$\frac{\partial^2 x}{\partial t^2} - \frac{\partial^2 x}{\partial \tau^2} + \frac{V_A'}{V_A}\frac{\partial x}{\partial \tau} = 0 \tag{6.2.13}$$

where $V_A' = \frac{dV_A}{d\tau}$.

To remove the 1st order derivative we use the following transformation

$x(t,\tau) = \tilde{x}(t,\tau)e^{\zeta/2}$

where $\zeta = \int_0^{\tau} [V_A'(\tilde{\tau})/V_A(\tilde{\tau})]d\tilde{\tau}$

Thus equation (6.2.13) becomes

$$\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial \tau^2}+\Omega_x^2(\tau)\right]\tilde{x}(t,\tau)=0 \qquad (6.2.14)$$

where Ω_x is the critical frequency given by

$$\Omega_x^2(\tau)=\frac{3}{4}\left(\frac{V_A{}'}{V_A}\right)-\frac{1}{2}\left(\frac{V_A{}''}{V_A}\right) \qquad (6.2.15)$$

where $V_A{}''=\dfrac{d^2V_A}{d\tau^2}$.

If we express the cutoff frequency in terms of spatial variable s, then equation (6.2.15) becomes

$$\Omega_x^2(s)=\frac{3}{8}\frac{V_A}{H}\left(1-\frac{1}{2}\frac{V_A}{H}\right) \qquad (6.2.16)$$

where $V_A \sim \exp\left(\dfrac{s}{2H}\right)$

So from (6.2.16) we can show that as we go upward along the field line the cutoff frequency increases.

The physical meaning of this cutoff frequency is that torsional waves must have frequency ω higher than $\Omega_x(s)$ in order to reach a given height s and be propagating waves at this height.

CHAPTER 7

DERIVATION OF THE WAVE EQUATIONS FROM LAGRANGIAN

If we can come up with a Lagrangian $L(u,u',\dot{u})$ (where $u' \equiv \frac{\partial u}{\partial z}$ and $\dot{u} \equiv \frac{\partial u}{\partial t}$) of the following form

$$L = \frac{1}{2}\dot{u}^2 - \frac{1}{2}V_{t,K,A}^2 u'^2 - \frac{1}{2}\Omega_{T,K,A}^2 u^2 \tag{7.1}$$

then we can write the following Euler-Lagrangian equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right) + \frac{d}{dz}\left(\frac{\partial L}{\partial u'}\right) - \frac{\partial L}{\partial u} = 0 \tag{7.2}$$

From the above equation we get the following wave equations

$$\frac{\partial^2 u}{\partial t^2} - V_{t,K,A}{}^2 \frac{\partial^2 u}{\partial z^2} + \Omega_{T,K,A}^2 u = 0 \tag{7.3}$$

which are the Klein-Gordon wave equations for longitudinal, transverse and torsional waves where Ω_T , Ω_K and Ω_A are the critical frequencies for longitudinal, transverse and torsional respectively.

CHAPTER 8

RELATIONSHIP BETWEEN THE TUBE CHARACTERISTIC VELOCITIES AND THE CUTOFF FREQUENCIES

For longitudinal tube waves, we have

$$V_t^2 = \frac{V_s^2 V_A^2}{V_s^2 + V_A^2},$$

and

$$\Omega_T{}^2 = \frac{V_t^2}{H^2}\left(\frac{9}{16} - \frac{1}{2\gamma} + \frac{V_s^2}{V_A^2}\frac{\gamma - 1}{\gamma^2}\right)$$

For transverse tube waves, we have

$$V_K^2 = \frac{B_0{}^2}{4\pi(\rho_0 + \rho_e)} = V_A^2\left(\frac{\rho_0}{\rho_0 + \rho_e}\right)$$

and

$$\Omega_K^2 = \frac{V_K^2}{16H^2}$$

For torsional tube waves, we have

$$V_A^2 = \frac{B_0^2}{4\pi\rho_0},$$

$$\Omega_A^2 = \frac{V_A^2}{16H^2}$$

Now, longitudinal characteristic velocity V_t and sound speed $V_s \left(\equiv \left(\frac{\gamma p_0}{\rho_0} \right)^{1/2} \right)$ are related as

$$V_t = \frac{V_s}{\sqrt{1 + \frac{\gamma\beta}{2}}} \tag{8.1}$$

and transverse characteristic velocity V_K and sound speed V_s are related as

$$V_K = \frac{V_s}{\sqrt{\gamma\left(\beta + \frac{1}{2}\right)}} \tag{8.2}$$

Now the cutoff frequencies of the longitudinal wave and sound waves (Ω_s) are related as

$$\Omega_T = \frac{\Omega_s}{\sqrt{1 + \frac{\gamma\beta}{2}}} \left(\frac{9}{4} - \frac{2}{\gamma} + 2\beta \frac{\gamma - 1}{\gamma} \right)^{1/2} \tag{8.3}$$

where $\Omega_s^2 = \frac{V_s^2}{4H^2}$

The cutoff frequency of the transverse wave and that of sound wave are related as

$$\Omega_K = \frac{\Omega_s}{2\sqrt{\gamma\left(\beta + \frac{1}{2}\right)}} \tag{8.4}$$

(see appendix for derivations of equations (8.1) and (8.2),(8.3) and (8.4))

Finally, we may write

$$\frac{V_K}{V_t} = \sqrt{\frac{2+\gamma\beta}{\gamma(1+2\beta)}} \tag{8.5}$$

and

$$\Omega_K = \Omega_T \sqrt{\frac{2+\gamma\beta}{\gamma(1+2\beta)}} \left(9 - \frac{8}{\gamma} + 8\beta \frac{\gamma-1}{\gamma}\right)^{-1/2} \tag{8.6}$$

From equations (8.5) and (8.6) we can see that for our values of β (The surface value of β is near unity both in sunspots and in the small-scale fields. Fields with $\beta \geq 1.8$ are subject to an instability in the top of the convection zone that concentrates such fields to $\beta \approx 1$, the transverse wave has a faster velocity and a lower cutoff frequency. Both of these effects enhance the energy flux which can be carried by this wave. ,and the transverse wave is a good candidate for coronal and chromospheric heating.

Since torsional waves propagate with the characteristic speed V_A, we have

$$V_A = V_K \left(\frac{\rho_0 + \rho_e}{\rho_0}\right)^{1/2}$$

where $\frac{\rho_e}{\rho_0} = 1 + \frac{1}{\beta}$ (as from horizontal pressure balance equation $p_0 + \frac{B_0^2}{8\pi} = p_e$, we can write $1 + \frac{B_0^2}{8\pi p_0} = \frac{p_e}{p_0} = \frac{\rho_e}{\rho_0}$ since $T_e = T_0$).

Thus, we have $V_A = V_K \sqrt{\frac{2\beta+1}{\beta}}$ (8.7)

If $\beta \gg 1$, then $V_A \approx V_K \sqrt{2}$.

If $\beta \ll 1$, then $V_A \approx \frac{V_K}{\sqrt{\beta}}$.

The definitions of the cutoff frequency for torsional, Ω_A, and transverse (kink), Ω_K, tube waves are very similar in form, however, the ratio of the cutoffs is

$$\frac{\Omega_A}{\Omega_K} = \frac{V_A}{V_K} = \sqrt{1+\frac{\rho_0}{\rho_e}} = \sqrt{\frac{(2\beta+1)}{\beta}} \tag{8.8}$$

where $\beta = \frac{8\pi p_0}{B_0^2}$.

For $\beta \gg 1$, it is seen that $\frac{\Omega_A}{\Omega_K} \approx \sqrt{2}$, and for $\beta \ll 1$, one finds $\frac{\Omega_A}{\Omega_K} \approx \frac{1}{\sqrt{\beta}}$. This clearly shows that torsional tube wave always propagate faster than transverse tube waves, and also that the cutoff frequency for the former is higher than for the latter; the higher cutoff implies that the wave energy spectra for torsional tube waves are not as broad as those for transverse tube waves.

It is instructive to compare the torsional and longitudinal velocities and cutoff frequencies as well. Using (5) we may write

$$V_A = V_t\sqrt{\frac{2+\gamma\beta}{\gamma\beta}} \tag{8.9}$$

When $\beta \gg 1$, then $V_A \approx V_t$ and when $\beta \ll 1$, then $V_A \approx V_t\sqrt{\frac{2}{\gamma\beta}}$.

.The ratio of these two cutoffs is

$$\frac{\Omega_A}{\Omega_T} = \sqrt{\frac{2+\gamma\beta}{\gamma\beta}\left(9-\frac{8}{\gamma}+8\beta\frac{\gamma-1}{\gamma}\right)^{-1}} \tag{8.10}$$

with $\frac{\Omega_A}{\Omega_T} \approx \sqrt{\frac{\gamma}{8\beta(\gamma-1)}} \approx \frac{0.56}{\sqrt{\beta}}$ when $\beta \gg 1$ and $\frac{\Omega_A}{\Omega_T} \approx \sqrt{\frac{2}{(9\gamma-8)\beta}} \approx \frac{0.53}{\sqrt{\beta}}$ when $\beta \ll 1$.

This shows that the ratio is very sensitive to the value of plasma β; for typical values of β considered here Ω_A is always lower than Ω_T, which implies that the wave energy spectra for torsional tube waves are always broader than those obtained for longitudinal waves (Noble et al.(2003)). Finally, the comparison of the cutoff frequency for torsional and acoustic, $\Omega_s = \frac{V_s}{2H}$, waves show that Ω_A is always marginally lower than Ω_s and, therefore, the torsional wave energy spectra are broader (e.g., Ulmschneider et al. (1998)).

Analytical and numerical studies have revealed that the acoustic waves inside the flux tubes are being damped and dissipated more rapidly comparing to other MHD modes (torsional and transverse).Thus these waves play significant roles in the heating process at the base of the chromospheres.

For upper chromosphere heating, the prominent energy carriers from the convection zone are the transverse and torsional modes, and possible other non-wave sources. This is due to the fact that transverse and torsional wave modes are much more difficult to damp than the other two modes. A fraction of the energy carried by transverse tube waves becomes also available for the chromospheric-heating through nonlinear mode-coupling (see Ulmschneider and Musielak (1998)).

CHAPTER 9

APPLICATIONS OF MHD WAVES

9.1 MHD Fast, Slow and Alfven Waves

Since slow and Alfven MHD waves transfer energy primarily along the magnetic field lines (see Chapter 2), the waves may be used to explain the observed association between the enhanced local heating of the solar atmosphere and the enhanced magnetic field strength. Extensive discussions of the role played by these MHD waves in the solar atmosphere can be found in early papers written by Kulsrud (1955), Osterbrock (1961) and Parker (1964), as well as by Priest (1982) and Collins (1989a,b). However, the assumptions of uniform magnetic fields and uniform background media are inconsistent with the solar data (Stenflo 1978; Saar 1987; Solanki 1993) which show highly inhomogeneous structures in the observable part of the solar atmosphere. As a result, a simple wave treatment of MHD waves propagating in uniform media has very limited applications to the solar atmosphere (see Narain and Ulmschneider (1996)).

To study the propagation of MHD waves in a stratified and magnetized medium some authors have used a local dispersion relation, in which stratification is introduced via the use of local cutoff frequencies (see Thomas 1983, and Campos 1987, and references therein). In general, this approach can be justified either when the vertical wavelength is much smaller than the atmospheric characteristic scale heights (the WKB approximation) or when a very special distribution of the atmospheric parameters is assumed (Nye and Thomas, 1974). Thomas (1983) and Musielak (1990) discussed some restrictions on the validity of local dispersion relations, and showed that some results previously obtained were outside the range of validity of this approach; in particular, they demonstrated that some cutoff frequencies were incorrectly calculated.

A number of authors have considered the propagation of linear Alfven waves in an inhomogeneous solar atmosphere assuming specific forms of inhomogeneity for which full analytical solutions to the Alfven wave equations can be found (Ferraro and Plumpton 1958; Hollweg 1978; Leroy 1978; Heinemann and Olbert 1980; Rosner, Low and Holzer 1986). For an isothermal, hydrostatic, plane-parallel atmosphere with constant magnetic field, the obtained solutions have been used to demonstrate that Alfven waves are reflected and that the region where reflection is strong can be determined from the condition that the wave frequency is smaller than the local cutoff frequency for these waves (Rosner, Low and Holzer 1986; An et al. 1989, 1990). A new analytical approach for assessing the reflection of linear Alfven waves in smoothly nonuniform media has been presented by Musielak, Fontenla and Moore (1992); see also Musielak, Musielak and Mobashi (2006).

9.2 Longitudinal, Transverse and Torsional Tube Waves

After extensive discussion of various types of magnetic waves in Chapters 3, 4, 5 and 6, we now consider the role these waves may play in the heating of the solar atmosphere. To do so, it is now necessary to identify sources of these waves in the Sun.

The theory of generation of magnetic tube waves by turbulent motions in the solar convection zone is presently well-developed (e.g., Musielak et al. 1995, 2002) and it allows calculating the wave energy spectra and fluxes. There are also some other sources of magnetic tube waves in the solar atmosphere. One of them is the excitation of magnetic tube waves by purely transverse footpoint shaking (caused by photospheric granulation) of a vertical magnetic flux tube embedded in an otherwise non-magnetic gravitational atmosphere (Ulmschneider, Z\"ahringer and Musielak 1991; Hasan and Kalkofen 2002) who showed that point shaking of magnetic flux tubes results in transverse waves which have horizontal swaying velocities and amplitudes that increase rapidly with height. The swaying velocities are similar for waves of different period but result in larger horizontal tube excursions in long period waves. Due to stretching and compressive actions of the vertical components of the curvature forces, longitudinal (compressional) waves are also generated. Among other sources, the excitation of MHD waves in magnetic loops by the interaction of the loops with upward propagating acoustic waves (Chitre and Davila 1991) and the generation of Alfven waves by solar microflares (Parker 1991) seem to be important.

Among different types of MHD surface waves (see Roberts 1991), Alfvenic surface waves are probably the most promising candidates for the heating (e.g., Narain and Ulmschneider 1996).

To determine the role played by MHD waves in chromospheric and coronal heating, it is necessary to address the problem of the wave energy transfer from the region of wave generation to the region where the local heating takes place. The key issues are the efficiency of wave energy transfer and the rate of dissipation of this energy at different heights in the solar atmospheres. Calculations of this sort are rather difficult because the solar atmosphere shows both continuous changes in physical parameters with height and localized inhomogeneities. The problem has not yet been solved as it requires large scale 3D numerical simulations of the wave propagation in the solar atmosphere. Clearly, this is out of the scope of this thesis, which instead is concentrated on analytical studies of the conditions for the propagation of magnetic tube waves in stratified but otherwise isothermal solar atmosphere.

CHAPTER 10

SUMMARY

In this thesis we discussed the propagation of magnetoacoustic waves in a stratified medium with an embedded uniform magnetic field. Here the direction of magnetic field and the wave vector both are allowed to vary with respect to the vertical axis (the opposite direction of gravity is taken to be vertical axis). We focused on the special case of two-dimensional geometry (i.e., the magnetic field lines and the wave vector lie in the vertically oriented planes, and the coordinate perpendicular to these planes are not considered). When both the magnetic force and pressure gradient are important but the effect of gravity is neglected then for outward-propagating disturbances there are two distinct modes, namely, the fast and slow magnetoacoustic waves. Their properties are listed and specific results obtained for special magnetic field and wave vector geometries are discussed for these two modes.

The main goal of this thesis is to derive cutoff frequencies for the waves (longitudinal, transverse and torsional) propagating along thin and vertically oriented magnetic flux tubes. We briefly discussed the longitudinal wave and developed an alternative much simpler method to derive the transverse wave equation. We also derived the wave equations for torsional waves propagating along different parts of exponentially divergent tubes and determined their cutoff frequencies. The cutoff frequencies for very thin tubes (thin enough to consider magnetic field lines to be essentially axial) for all the three waves, longitudinal, transverse and torsional, are constant throughout the tube length. But for the tubes where the radial component of the magnetic field is considered though it is very small in amplitude so that we still do not loose the essence of thin tubes, the cutoff frequencies of the torsional waves are not constant; instead they are local quantities that vary in the media and their specific values at a given height determine the frequency that torsional waves must have in order to be propagating at this height.

Specific applications of the obtained results to the solar atmosphere are also briefly discussed. It is shown how the cutoffs may affect the spectra of magnetic tube waves generated in the solar convection zone and how they can be used to determine the range of frequencies corresponding to propagating waves in different regions of the solar atmosphere.

In future work, the obtained results will be extended to include the effects of temperature gradients on the cutoff frequencies of magnetic tubes waves. Since these cutoffs will be local quantities, they may be used to determine the propagation conditions for longitudinal, transverse and torsional tube waves in several known empirical models of the solar atmosphere. Recent solar observations show that there are regions of the solar atmosphere where tilted magnetic flux tubes exist. Therefore, one the main goals of our future work is to derive the wave equations and corresponding cutoff frequencies for magnetic waves propagating along these titled flux tubes. The obtained results will be used to predict periods of oscillations of these tubes, which will be compared to the current solar observations.

APPENDIX A

RELATIONSHIP OF TUBE CHARACTERISTIC VELOCITIES AND CUTOFF FREQUENCIES WITH SOUND SPEED AND SOUND CUTOFF FREQUENCY

For longitudinal tube waves, the characteristic velocity and the cutoff frequency are

$$V_t^2 = \frac{V_s^2 V_A^2}{V_s^2 + V_A^2},$$

and

$$\Omega_T{}^2 = \frac{V_t^2}{H^2}\left(\frac{9}{16} - \frac{1}{2\gamma} + \frac{V_s^2}{V_A^2}\frac{\gamma - 1}{\gamma^2}\right)$$

For transverse tube waves, we have

$$V_K^2 = \frac{B_0{}^2}{4\pi(\rho_0 + \rho_e)} = V_A^2\left(\frac{\rho_0}{\rho_0 + \rho_e}\right)$$

and

$$\Omega_K^2 = \frac{V_K^2}{16H^2}$$

Sound speed and the cutoff frequency are defined by

$$V_s{}^2 = \frac{\gamma p_0}{\rho_0}$$

and

$$\Omega_s^2 = \frac{V_s^2}{4H^2}$$

Since

$$\frac{1}{V_t^2} = \frac{1}{V_s^2} + \frac{1}{V_A^2}$$

we have

$$\frac{V_s^2}{V_t^2} = 1 + \frac{V_s^2}{V_A^2} = 1 + \frac{\gamma\beta}{2}$$

because

$$\frac{V_s^2}{V_A^2} = \frac{\gamma\, p_0}{\rho_0}\frac{4\pi\rho_0}{B_0^2} = \frac{\gamma}{2}\frac{8\pi\, p_0}{B_0^2} = \frac{\gamma}{2}\beta$$

Therefore

$$V_t = \frac{V_s}{\sqrt{1+\dfrac{\gamma\beta}{2}}}$$

which is equation (8.1).

Since $\Omega_T{}^2 = \dfrac{V_t^2}{H^2}\left(\dfrac{9}{16} - \dfrac{1}{2\gamma} + \dfrac{V_s^2}{V_A^2}\dfrac{\gamma-1}{\gamma^2}\right)$

$$= \frac{V_s^2}{H^2(1+\gamma\beta/2)}\left(\frac{9}{16} - \frac{1}{2\gamma} + \frac{\gamma}{2}\beta\frac{\gamma-1}{\gamma^2}\right)$$

$$= \Omega_T{}^2 = \frac{\Omega_s^2}{1+\gamma\beta/2}\left(\frac{9}{4} - \frac{2}{\gamma} + 2\beta\frac{\gamma-1}{\gamma}\right)$$

Therefore

$$\Omega_T = \frac{\Omega_s}{\sqrt{1+\gamma\beta/2}}\left(\frac{9}{4} - \frac{2}{\gamma} + 2\beta\frac{\gamma-1}{\gamma}\right)^{1/2}$$

which is equation (8.3).

Now,

$$\frac{V_s^2}{V_t^2} = \frac{\gamma\, p_0}{\rho_0}\frac{4\pi(\rho_0+\rho_e)}{B_0^2}$$

$$=\frac{\gamma}{2}\left(1+\frac{\rho_e}{\rho_0}\right)\frac{8\pi p_0}{B_0^2}$$

$$=\frac{\gamma}{2}\left(1+\frac{\rho_e}{\rho_0}\right)\beta$$

with $p_0+\frac{B_0^2}{8\pi}=p_e \Rightarrow 1+\frac{B_0^2}{8\pi\rho_0}=\frac{p_e}{p_0}$

we have

$$1+\frac{1}{\beta}=\frac{\gamma p_e}{\rho_e}\frac{\rho_0}{\gamma p_0}\frac{\rho_e}{\rho_0}=\frac{V_{se}^2}{V_{so}^2}\frac{\rho_e}{\rho_0}=\frac{\rho_e}{\rho_0} \qquad \text{(as for isothermal atmosphere } V_{se}=V_{so}\text{)}$$

where V_{se} and V_{so} are sound speeds in external and inside tube respectively.

Therefore

$$V_K=\frac{V_s}{\sqrt{\gamma(\beta+1/2)}}$$

which is equation (8.2).

Now $\frac{V_s^2}{V_T^2}=\frac{1}{4}\frac{V_s^2}{4H^2}\frac{16H^2}{V_K^2}=\frac{1}{4}\frac{\Omega_s^2}{\Omega_K^2}=\gamma\left(\beta+\frac{1}{2}\right)$

therefore

$$\Omega_K=\frac{\Omega_s}{2\sqrt{\gamma\left(\beta+\frac{1}{2}\right)}}$$

which is equation (8.4).

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BIOGRAPHICAL INFORMATION

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