FUNDAMENTAL DYNAMICAL EQUATIONS FOR TWO AND FOUR-COMPONENT SPINORS IN GALILEAN AND MINKOWSKI SPACE-TIME

by

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To my wife Karin for her loving support and boundless patience

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ABSTRACT

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A search for new fundamental (Galilean and Poincaré invariant) dynamical equations for free elementary particles represented by spinor state functions is conducted in Galilean and Minkowski space-time. A dynamical equation is considered as fundamental if it is invariant under the symmetry operations of the group of the space-time metric and if its state functions transform like the irreducible representations of the group of the metric. It is shown that there are no Galilean invariant equations for two-component spinor wave functions thus the Pauli equation is not fundamental. It is formally proved that the Lévy-Leblond and Schrödinger equations are the only Galilean invariant 4-component spinor equations for the Schrödinger phase factor. New fundamental dynamical equations for four-component spinors are found using generalized phase factors. For the extended Galilei group a generalized Lévy-Leblond equation is found to be the only first order Galilean invariant four-component spinor equation.

For the Poincaré group a generalized Dirac equation is found to be the only first order Poincaré invariant four-component spinor equation. In the non-relativistic limit the generalized Dirac equation is shown to reduce to the generalized Lévy-Leblond equation. A new momentum-energy relation is derived from the analysis of stationary states of the generalized Dirac equation. The new energy-momentum relationship is used to show that the behavior of a particle obeying the generalized Dirac equation is different from that of a particle governed by the standard Dirac equation because of the existence of additional momentum and energy terms. Since this new energy-momentum relationship differs from the well-known energy-momentum relationship of Special Theory of Relativity, it cannot describe ordinary matter. Hence, it is suggested that the new energy-momentum relationship represents a different form of matter that may be identified as Dark Matter.

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CHAPTER 1

INTRODUCTION

1.1 History and Motivation

Astronomical observations show that our Universe is dominated by Dark Matter (DM) and Dark Energy (DE), and that ordinary matter makes up only a very small fraction of all matter in the Universe. The realization that some unknown form of matter and mysterious form of energy dominate the structure and evolution of the entire Universe is one of the most profound discoveries in science. Numerous ideas have been put forward to explain DM and DE but so far explanations of their origin and nature remains inconclusive.

The existence of DM was first suggested by Fritz Zwicky in the early 1930's. Zwicky determined the radial velocities of eight galaxies and found out that they were 400 times greater than that expected by the shared gravity of luminous matter in those galaxies. The explanation given by Zwicky for his extraordinary finding was to suggest the existence of what he called "missing matter", or matter which cannot be directly observed but can be inferred indirectly by its gravitational influence on visible matter. There is a large body of literature devoted to the subject with detailed descriptions of both observational and theoretical research given in numerous research papers (e.g. [1, 2, 3, 4, 5]), many review papers (e.g., [6, 7, 8, 9, 10]) and in some books (e.g., [11, 12]). Since the main focus of this dissertation is on theoretical research of dark matter, theoretical ideas relevant to the research presented in this dissertation will now be briefly described.

It is now believed that DM may be composed of many different constituents. A small portion may be baryonic DM that falls into the category of Massive Compact Halo Objects (MACHO'S). Most DM must be non-baryonic DM that is usually classified by its kinetic energy as either Cold Dark Matter (CDM): objects at classical velocities, Warm Dark Matter (WDM): particles at relativistic velocities ($< .1c$), or Hot Dark Matter (HDM): particles moving at ultra-relativistic velocities ($> .95c$). The observation of galaxy sized clustering of the DM mass distribution in space supports CDM as the most common constituent of non-baryonic DM. Other observations support this finding. For example, Cosmic Microwave Background Radiation (CMBR) and studies of big bang nucleosynthesis require CDM to explain the present day structure of the universe. There are at least two noteworthy discrepancies between the model and observation. The first is the cuspy halo problem and the second is the missing satellites problem (e.g., [3, 13, 4]).

Standard proposals to explain the non-baryonic DM are based upon unification of the fundamental forces, new forces, new particles in the standard model, string theory and the so-called 'fuzzy' DM. There are some papers postulating that the axion is a dark matter particle and that its mass is about 10-23 eV (e.g., [14] and others are devoted to the idea of cold DM (CDM) particles with mass around 10-23 eV (e.g., [2, 3, 5]); often such particles are called Extremely Light Bosonic Dark Matter (or ELBDM) to distinguish it from CDM axion. An interesting result is that the uncertainty principle must operate on galactic scales because the considered masses of these particles are so small. If the particles are bosons, then they form a Bose-Einstein Condensate (BEC). A detailed description of formation and properties of such astronomical BEC is given by [4] for non-relativistic scalar fields and by [15] for relativistic scalar fields.

A different approach to the DM problem has been taken by [16, 17, 18, 19] who searched for new invariant dynamical equations. These equations are required to describe state functions that transform like irreducible representations (irreps) of the group of all transformations that leave the space-time metric invariant. This approach follows Wigner [20] and others (e.g., [21, 22]) in their assessment that an elementary particle must transform as one of the irreducible representations of the group of the metric in a Hilbert space. The developed method is based on the Principle of Analyticity and the Principle of Relativity, and it has been used to formulate fundamental physical theories of waves and particles in the space-time of a given metric [16, 17, 18, 19, 23].

The main focus of this dissertation is to establish new physics that will explain the origin and nature of DM. To achieve this challenging goal, a search for new invariant dynamical equations describing free elementary particles in both Galilean and Poincaré space-time will be conducted. It is anticipated that one of the new invariant dynamical equations to be obtained in this project can be used to formulate a fundamental physical theory that correctly describes the nature and behavior of DM, and that predictions of this theory can be verified in laboratory experiments and by astronomical observations. The main goals of the research presented in this dissertation will now be described.

1.2 Main Goals of this Dissertation

The subject of this work is to search for new fundamental dynamical equations that describe free elementary particles. These equations may predict new forms of matter that may be good candidates for DM. The work of this dissertation focuses on searching for spinor equations that may have been overlooked in the two most fundamental (Galilean and Minkowski) metrics of physics. The presented approach is different than that used by [16, 17] and [18] whose method was based on the eigenvalue equations derived from the properties of the extended Galilei group and the Poincaré group (e.g., [21]). Here, a general form of first order partial differential equations is considered and their invariance with respect to all transformations that leave the Galilean and Minkowski metrics invariant is investigated. Specific goals include searches for invariant dynamical equations for

(i) two-component spinor wave functions in Galilean space-time;

(ii) four-component spinor wave functions in Galilean space-time;

(iii) four-component spinor wave functions in Minkowski space-time.

The searches have been completed and the obtained results are presented in this dissertation. The obtained results can be divided into two groups, namely, the previously known Lévy-Leblond, Schrödinger, and Dirac equations, which have been formally derived, and new invariant dynamical equations, which have also been found.

The material of this dissertation is organized in the following chapters. A search for new dynamical spinor equations in Galilean space-time with the Schrödinger phase function is presented in Chapter 2. The most notable results of this investigation are the proofs of uniqueness of the Schrödinger and Lévy-Leblond equations as the only fundamental dynamical equations for 4-component spinors in Galilean space-time with the Schrödinger phase function. Another result is a proof that there are no Galilei invariant dynamical equations for 2-component spinors. The search for dynamical equations in Galilean space-time is continued with alternative phase functions in Chapter 3. A search for new dynamical equations in Minkowski space-time is presented in Chapter 4. Physical consequences of the obtained results are discussed in Chapter 5, which also contains conclusions and describes possible future directions of this research.

CHAPTER 2

DYNAMICAL EQUATIONS FOR TWO AND FOUR-COMPONENT SPINOR STATE FUNCTIONS IN GALILEAN SPACE-TIME

2.1 Background

A physical theory of free particles in Galilei space-time must be based on a fundamental dynamical equation that has the same form in all isometric frames of reference. All coordinate transformations that do not change the Galilei metric form a representation of the group of the metric. In order for two observers with the same metric to identify the same particle, the state functions describing this particle must transform like one of the irreducible representations (irreps) of the Galilei group. This definition was first formally introduced by Wigner [20], who determined all unitary irreps of the Poincaré group [21] and used them to classify the elementary particles in Minkowski space-time.

Vector irreps of the Galilei group have no physical interpretation [24] but there is an infinite number of projective (ray) irreps, which are different from the vector irreps of the group [25]. Typically, the projective irreps are determined by the method of induced representations [26, 27], and they are characterized by a constant that enters a phase factor in defining the projective irreps [21]. The process of introducing the constant is the central extension of the Galilei Lie algebra and the corresponding group is called the extended Galilei group [21, 28, 22].

In the previously obtained results, Lévy-Leblond [29] used the method of Bargmann and Wigner [26] and derived a Galilei invariant dynamical equation for free particles with arbitrary spin. The state function describing these particles is a four-component spinor. Similar studies were performed by Fushchich and Nikitin [22], however, they used a different method. Lévy-Leblond [30] also derived the Pauli-Schrödinger (PS) equation [31, 32] by adding the electromagnetic field to his Galilei invariant equation. The fields were made Galilei invariant by dropping the Maxwell term from Maxwell's equations. He obtained the PS equation, a two-component spinor equation but did not evaluate the Galilei invariance of the PS equation.

Four-component spinors were introduced to Quantum Field Theory (QFT) by Dirac when he formulated his relativistic theory of electrons and positrons and obtained Poincaré invariant dynamical equations for these particles [33, 34]. On the other hand, two-component spinors are now widely used in General Relativity (GR) primarily through the work of Penrose [35]. Extensive discussions of the role played by four-components spinors in quantum mechanics and QFT, and two-component spinors, in GR can be found in [36] and [37], respectively.

In a more recent work on free and spinless particles described by scalar and analytic state functions in Galilei space-time, it was established that Schrödinger's equation [38] is Galilei invariant [39, 40, 41] and, therefore, the equation plays the central role in Galilei relativity [16]. An important result is that Schrödinger-like equations are the only Galilei invariant dynamical equations [17]. The fact that the Dirac, Pauli-Schrödinger and Schrödinger equations are intimately related is well-known [42, 43, 44]. The PS equation is an approximation to the Dirac equation for small electron velocities and the Schrödinger equation can be obtained from the PS equation by neglecting magnetic interaction of the spin.

One of the main goals of this dissertation is to search for new fundamental (Galilei invariant) dynamical equations for two and four-component spinors or to eliminate the possibility of their existence. The obtained results are based on the extended Galilei group [21, 29, 40, 41] and therefore relies upon the Schrödinger phase factor [22]. We define a dynamical equation to be fundamental if it has the following properties: (i) invariance under the symmetry operators of the group of the metric; (ii) no mixed time and space partial derivatives; and (iii) state functions that transform like the irreducible representations of the group of the metric. One such fundamental dynamical equation in Galilei relativity is the first-order partial differential equation for a four-component spinor wave function known as the Lévy-Leblond equation [29, 30]. Here, this equation is derived by using a different method and it is proved that the Lévy-Leblond equation is the only first order Galilei invariant equation that can be obtained by using the Schrödinger phase factor.

Another interesting result is that there are no Galilei invariant equations for two-component spinors. It is also proved that the Schrödinger equation which is first order in time and second order in space is the only second order fundamental equation and that there are no higher order fundamental equations for two and four-component spinors. It is important to note that these results are obtained for the Schrödinger phase factor of the extended Galilei group. Other phase factors are possible and they are investigated in Chapter 3.

2.2 Scalar State Functions in Galilean Relativity

2.2.1 Group of the Galilei metric

Galilei space time is defined by the Galilei metric: $ds_1^2 = dx^2 + dy^2 + dz^2$ and $ds_2^2 = dt^2$, where x, y, z, and t are spatial and time coordinates. The metric is invariant under a set of transformations that forms the Galilei group. The group may be decomposed into subgroups such that

$$\mathcal{G} = [T(1) \otimes R(3)] \otimes_s [T(3) \otimes B(3)] \ , \tag{2.1}$$

where $T(1)$, $R(3)$, $T(3)$, and $B(3)$ are the subgroups of translation in time, rotations in space, translations in space, and boosts respectively. The direct product and semi-direct product are denoted $\otimes$ and $\otimes_s$.

The Galilei transformations can be used to relate the coordinate systems of two observers that are spatially rotated, translated, and boosted relative to one another. A Galilei transformation can be defined by

$$\mathbf{x} \to \mathbf{x}' = R\mathbf{x} + \mathbf{v}t + \mathbf{a} \quad \text{and} \quad t \to t' = t + b \;, \tag{2.2}$$

where R is a rotation matrix, $\mathbf{v}$ is the velocity vector of a boost relating the two coordinate systems, and $\mathbf{a}$ is a spacial translation relating the two coordinate systems. The inverse Galilei transformation is

$$\mathbf{x} = R^{-1}\mathbf{x}' - R^{-1}\mathbf{v}t' - R^{-1}\left(\mathbf{a} - \mathbf{v}b\right) \quad \text{and} \quad t = t' - b \;. \tag{2.3}$$

The chain rule can be used to determine how the differential operators transform under the Galilei transformation

$$\frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial x^i}{\partial t'}\frac{\partial}{\partial x^i} = \frac{\partial}{\partial t} - R^{-1}_{ij} v_j \frac{\partial}{\partial x^i}, \tag{2.4}$$

and

$$\frac{\partial}{\partial x'^i} = \frac{\partial t}{\partial x'^i}\frac{\partial}{\partial t} + \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j} = R^{-1}_{ij}\frac{\partial}{\partial x^j} \;. \tag{2.5}$$

It has been demonstrated for scalar wave functions that the Galilei group does not lead to any dynamical equations that satisfy the principles of analyticity and relativity [16]. The first principle requires that state functions are analytic and the second principle demands that dynamical equations governing the state function are Galilei invariant. Therefore an additional symmetry $|\psi^*\psi| = |\psi'^*\psi'|$ must be added to the group of the metric.

The expanded symmetry group called the extended Galilei group is the universal covering group of the Galilei group [27, 28]. The extended Galilei group exhibits structure that is similar to the Poincare group [21, 29, 17]. The arguments used in [16] for scalar wave functions apply equally well to n-component functions such as spinors and vectors. Consequently, we begin this work with the extended Galilei group, which has the structure

$$\mathcal{G}_e = [R(3) \otimes_s B(3)] \otimes_s [T(3+1) \otimes U(1)] \ , \tag{2.6}$$

where $U(1)$ is a one-parameter unitary group [21]. We consider only the proper isochronal subgroup $\mathcal{G}_+^\uparrow$ of $\mathcal{G}_e$ which omits the space and time inversions that can be treated separately.

Measurements of the norm of the scalar state function must produce the same results for all observers related by the Galilei transformations. Hence, the resulting transformation of the wave function ψ is

$$\psi(x,t) \rightarrow \psi'(x',t') = e^{i\phi(x,t)}\psi(x,t) \ , \tag{2.7}$$

with $\phi(x,t)$ being a phase function to be determined.

2.2.2 Galilei Invariant Equations for Scalar State Functions

In the previous work, Musielak and Fry [16] used the Galilei group of the metric and the principles of analyticity and relativity to formally derive Schrödinger-like equations. They concluded that the Galilei group was incomplete for forming a fundamental theory of free particles and that the necessary modifications of the group led to the extended Galilei group. The derived Schrödinger-like equation can be written in the following form

$$i\frac{\partial\psi}{\partial t} + \frac{\omega}{k^2}\nabla^2\psi = 0 \ , \tag{2.8}$$

where ω and k are the eigenvalues of the translation operators in time and space. Properties of the eigenvalue equations ensure that $\omega/k^2 = 1/2M$ is a constant in all inertial frames of reference. M is referred to as the "wave mass" and is related to the classical mass through the Planck constant $m = \hbar M$ [39, 45, 41, 16]. The phase function

$$\phi(\mathbf{x}, t) = m\mathbf{v} \cdot \mathbf{x} + \frac{1}{2} m v^2 t + mc \tag{2.9}$$

makes equation (2.8) Galilei invariant.

The state function ψ in Eq. (2.8) is a scalar function. According to Musielak and Fry [17], the Schrödinger-like equation is the only Galilei invariant equations for scalar functions.

When the state function is a spinor it has two or more components and each component must satisfy the Schrödinger-like equation [32, 46, 47]. Invariance of the Schrödinger-like equation under transformations of the extended Galilei group requires a phase factor $e^{i\phi(x,t)}$ with a phase function (2.9) that is of the same form as for scalars [22]. We will call this the Schrödinger phase factor and we will use this condition when searching for fundamental (Galilei invariant) equations for two and four-component spinor state functions.

2.3 Dynamical Equation for Spinor State Functions

We now search for fundamental dynamical equations for two and four-component spinors. We consider a dynamical equation fundamental if it has the properties of (i) invariance under the symmetry operators of the group of the metric, (ii) state functions that transform like the irreducible representations of the group of the metric, and (iii) no mixed time and space partial derivatives.

We begin the search with first order differential equations acting on the N-component spinor state functions ψ. The most general form of such a first order equation is

$$\left[B_1 \frac{\partial}{\partial t} + B_{2j} \frac{\partial}{\partial x_j} + B_3\right] \psi(r,t) = 0 \ , \tag{2.10}$$

where B_1, B_{2j}, and B_3 are arbitrary $N \times N$ matrices that are assumed to be free of any dependence on the space and time coordinates.

Dynamical equations must be invariant under Galilei transformations, so we will require Galilei invariance of the first order differential equation in order to derive a set of restrictions on the matrices B. In the next subsection, the restrictions will be used to find a set of matrices that will allow the equation to be invariant under Galilei transformations.

Applying a Galilei transformation to Eq. (2.10) and regrouping the terms, we obtain

$$\left[B_1' \frac{\partial}{\partial t} + (B_1' R_{jk} v_k + B_{2k}' R_{kj}) \frac{\partial}{\partial x_j}\right] \psi(\mathbf{r},t) \\ + \left[-\frac{i}{2} m v^2 B_1' - i m R_{jl} v_k R_{kl} B_{2j}' + B_3'\right] \psi(\mathbf{r},t) = 0 \ . \tag{2.11}$$

For Eq. (2.10) to be invariant under Galilei transformation, Eq. (2.11) must be of the same form. This requirement leads to the following conditions on the set of matrices

$$B_1 = G B_1 G^{-1} \ , \tag{2.12}$$

$$B_{2j} = G B_1 G^{-1} R_{ij} v_j + G B_{2i} G^{-1} R_{ij} \ , \tag{2.13}$$

and

$$B_3 = -\frac{i}{2} m v^2 G B_1 G^{-1} - i m v_i G B_{2i} G^{-1} + G B_3 G^{-1} \ . \tag{2.14}$$

These conditions will be used in our search for fundamental dynamical equations.

2.3.1 First-Order Equations for Two-Component Spinors

Our main result obtained for two-component spinor state functions is given by the following proposition.

Proposition 1. *There are no Galilei invariant equations for two-component spinors.*

Proof:

Applying rotations only (no boosts) in the conditions given by Eq. (2.12) through Eq. (2.14) constrains the matrices to the following forms: $B_1 = c_1 I$, $B_{2j} = c_2 \sigma_j$ and $B_3 = c_3 I$, where c_1, c_2, and c_3 are arbitrary constants. This demonstrates that there are first-order equations that are rotationally invariant. To be Galilei invariant the equation must also be boost invariant.

It turns out that it is not possible to construct Galilei boost operators for two-component spinors. In general, one may construct a boost matrix from the velocity parameters v_j and boost generators X_j as the exponential expression given by $B(v) = e^{iX_j v_j}$. The generators of Galilei boosts must obey the following commutation relations of the Galilei group: $\left[X_{\theta_i}, X_{\theta_j}\right] = iX_{\theta_k}\epsilon_{ijk}$, $\left[X_{v_i}, X_{v_j}\right] = 0$ and $\left[X_{v_i}, X_{\theta_j}\right] = iX_{v_k}\epsilon_{ijk}$.

Because Galilei boosts commute, they form an Abelian subgroup and a one dimensional irreducible representation exists. However, the composition of boosts and rotations is the result of a semi-direct product and requires boosts and rotations to obey the group composition law that can be written as

$$G(a, b, v, R(\theta)) = T(b)S(a)B(v)R(\theta)$$

$$= G(a_2, b_2, v_2, R_2)G(a_1, b_1, v_1, R_1)$$

$$= G(R_2 a_1 + a_2 - v_2 b_1, b_1 + b_2, R_2 v_1 + v_2, R_2 R_1)\ , \tag{2.15}$$

which is composed of translations in time $T(b)$, translations in space $S(a)$, boosts $B(v)$, and rotations $R(\theta)$. For the 2×2 generators of rotations $X_{\theta_i} = \sigma_i/2$, there are no 2×2 matrices that are able to satisfy the commutation relations as boost generators.

An interesting result is that this problem does not exit in the Minkowski space-time [35, 36]. Hence, one may try to take the limit $c \to \infty$ of the Lorentz boost for two-component spinors. The result is [44]

$$B_{v_x} = \begin{pmatrix} \cosh v_x/c & \sinh v_x/c \\ \sinh v_x/c & \cosh v_x/c \end{pmatrix} \tag{2.16}$$

$$B_{v_y} = \begin{pmatrix} \cosh v_y/c & i \sinh v_y/c \\ -i \sinh v_y/c & \cosh v_y/c \end{pmatrix} \tag{2.17}$$

and

$$B_{v_z} = \begin{pmatrix} e^{v_z/c} & 0 \\ 0 & e^{v_z/c} \end{pmatrix} \tag{2.18}$$

and it is seen that the diverging matrix elements are obtained. This is not surprising since the Galilei spinor boosts cannot be represented with 2×2 matrices. Lorentz boosts do not commute as their Galilei counterparts. As such the Lorentz group has a different universal covering group, SL(2,C) and it can be represented with 2×2 matrices [35]. The physical implications of this result are that one cannot perform a Galilei boost of two component spinors, which concludes the proof of Proposition 1.

2.3.2 First-Order Equations for Four-Component Spinors

After showing that there are no Galilei invariant dynamical equations for two-component spinor wave functions, we searched for fundamental dynamical equations describing evolution of four-component spinor wave functions in time and space. The obtained results are summarized by the following proposition.

Proposition 2. *The only Galilei invariant first-order differential equation for four-component spinors is the Lévy-Leblond equation [29, 30]*

$$\left[\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}\frac{\partial}{\partial t} + \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}\frac{\partial}{\partial x_j} + \begin{pmatrix} 0 & 2imI \\ 0 & 0 \end{pmatrix}\right]\psi(r,t) = 0 \tag{2.19}$$

where I is the 2×2 identity matrix.

Proof: As in the case for two-component spinors, we seek a set of matrices that satisfy the conditions for invariance given by Eq. (2.12) through Eq. (2.14). Let B_1 be an arbitrary 4×4 matrix, then

$$B_1 = \begin{pmatrix} P & Q \\ S & T \end{pmatrix} \tag{2.20}$$

where P, Q, S, and T are arbitrary 2×2 matrices. Applying an arbitrary rotation in the condition given by Eq. (2.12) results in four conditions on the 2×2 matrices P, Q, S, and T. The conditions are:

$$\begin{aligned} B_1 = RB_1R^{-1} &= \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}\begin{pmatrix} P & Q \\ S & T \end{pmatrix}\begin{pmatrix} U^{-1} & 0 \\ 0 & U^{-1} \end{pmatrix} \\ &= \begin{pmatrix} UPU^{-1} & UQU^{-1} \\ USU^{-1} & UTU^{-1} \end{pmatrix} \end{aligned} \tag{2.21}$$

Individually these four conditions are identical in form to that given by Eq. (2.12) for two-component spinors and the results are the same. Therefore the 2×2 matrices must be diagonal with arbitrary constant coefficients p, q, s, and t and

$$B_1 = \begin{pmatrix} pI & qI \\ sI & tI \end{pmatrix}. \tag{2.22}$$

For rotations only, the condition given by Eq. (2.14) has the same form as that given by Eq. (2.12). Therefore, the matrix B_3 is similarly constrained to

$$B_3 = \begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix}. \tag{2.23}$$

For rotations only, the condition given by Eq. (2.13) produces four 2×2 conditions with the results as those found for two-component spinors. The matrices are constrained to

$$B_{2j} = \begin{pmatrix} e\sigma_j & f\sigma_j \\ g\sigma_j & h\sigma_j \end{pmatrix} \tag{2.24}$$

where σ_j are the 2×2 Pauli matrices.

Applying boosts (without rotations), then the condition given by Eq. (2.12) leads $q = 0$ and $t = p$, so that

$$B_1 = \begin{pmatrix} pI & 0 \\ sI & pI \end{pmatrix}. \tag{2.25}$$

In the case of boosts again without rotations, the condition given by Eq. (2.13) leads to $e = -h = s$ and $f = 0$, so that

$$B_{2j} = \begin{pmatrix} s\sigma_j & 0 \\ g\sigma_j & -s\sigma_j \end{pmatrix}. \tag{2.26}$$

Applying boosts without rotations in the condition given by Eq. (2.14) leads to $b = 2ims$, $p = 0$, $a = d$, and $g = 0$, and the group of matrices become

$$B_1 = \begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix}, \tag{2.27}$$

$$B_{2j} = \begin{pmatrix} s\sigma_j & 0 \\ 0 & -s\sigma_j \end{pmatrix}, \tag{2.28}$$

and

$$B_3 = \begin{pmatrix} aI & 2imsI \\ cI & aI \end{pmatrix}. \tag{2.29}$$

Applying together rotations and boosts in the conditions leads to $a = 0$ and $c = 0$, so that

$$B_1 = \begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix}, \tag{2.30}$$

$$B_{2j} = \begin{pmatrix} s\sigma_j & 0 \\ 0 & -s\sigma_j \end{pmatrix}, \tag{2.31}$$

and

$$B_3 = \begin{pmatrix} 0 & 2imsI \\ 0 & 0 \end{pmatrix}. \tag{2.32}$$

The constant s can now be factored out of the equation leaving the following first-order differential equation

$$\left[\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \frac{\partial}{\partial t} + \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix} \frac{\partial}{\partial x_j} + \begin{pmatrix} 0 & 2imI \\ 0 & 0 \end{pmatrix} \right] \psi(r,t) = 0 \tag{2.33}$$

which is known to be the the only Galilei invariant first-order dynamical equation for four-component spinors. This concludes the proof of Proposition 2.

Although the obtained equation is unique, it can be cast into several different but equivalent forms. This can be done by similarity transformations, which corresponds to a change of basis. The equation can also be transformed into other representations such as momentum representation [29, 30].

2.3.3 Higher-Order Equations for Four-Component Spinors

We also search for higher-order dynamical equations for four-component spinor wave functions. The following proposition summarizes the obtained results.

Proposition 3. *The Lévy-Leblond equation with the operator raised to N-th power*

$$\left[\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}\frac{\partial}{\partial t} + \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}\frac{\partial}{\partial x_j} + \begin{pmatrix} 0 & 2imI \\ 0 & 0 \end{pmatrix}\right]^N \psi(\mathbf{r}, t) = 0 \qquad (2.34)$$

is Galilei invariant.

Proof: Let

$$\mathcal{L} = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}\frac{\partial}{\partial t} + \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}\frac{\partial}{\partial x_j} + \begin{pmatrix} 0 & 2imI \\ 0 & 0 \end{pmatrix} \qquad (2.35)$$

be the Lévy-Leblond operator. It has already been proven that the first-order equation is invariant, therefore we have

$$G\mathcal{L}G^{-1}G\psi(\vec{x}, t) = G\mathcal{L}G^{-1}e^{i\phi}\psi(\vec{x}, t) = e^{i\phi}\mathcal{L}\psi(\vec{x}, t) = 0. \qquad (2.36)$$

This process can be repeated for each power of $\mathcal{L}$ until the phase factor has been commuted fully to the left.

$$G\mathcal{L}^N G^{-1}T\psi(\vec{x}, t) = G\mathcal{L}^{N-1}G^{-1}e^{i\phi}\mathcal{L}\psi(\vec{x}, t) = e^{i\phi}\mathcal{L}^N\psi(\vec{x}, t) = 0. \qquad (2.37)$$

Corollary: The case of $N = 2$ produces the Schrödinger equation [38]

$$\mathcal{L}^2\psi(\vec{x}, t) = \left[\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}\frac{\partial}{\partial t} + \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}\frac{\partial}{\partial x_j} + \begin{pmatrix} 0 & 2imI \\ 0 & 0 \end{pmatrix}\right]^2 \psi(\vec{x}, t) = 0$$

$$= \left[2im\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\frac{\partial}{\partial t} + \begin{pmatrix} \{\sigma_j, \sigma_k\} & 0 \\ 0 & \{\sigma_j, \sigma_k\} \end{pmatrix}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\right]\psi(\vec{x}, t) = 0$$

$$= \left[2im\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\frac{\partial}{\partial t} + 2\delta_{jk}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\right]\psi(\vec{x}, t) = 0$$

$$= \left[imI\partial_t + I\partial_j^2\right]\psi(\vec{x}, t) = 0 \ . \qquad (2.38)$$

2.3.4 Fundamental Dynamical Equations for 4-component Spinors

We have already shown that the only Galilei invariant first-order dynamical equation is the Lévy-Leblond equation (see Proposition 2). Furthermore there is an entire class of higher-order Galilei invariant equations that result from taking the N^{th} power of the Lévy-Leblond operator (see Proposition 3). Among this class of Galilei invariant equations, there are the Lévy-Leblond [29, 30], the Schrödinger equation [38], and Schrödinger-like equations [16, 17].

In addition to being Galilei invariant, the fundamental dynamical equations must also have no mixed partial differentials. The Lévy-Leblond equation, the Schrödinger and Schrödinger-like equations are fundamental because they have no mixed partial differentials.

Proposition 4. *The only fundamental equations for four-component spinors are the Lévy-Leblond (see Eq. 2.19) and Schrödinger-like equations (see Eq. 2.8).*

Proof: Let us first consider equations among the Lévy-Leblond class. The Nth order equation is obtained by raising the Lévy-Leblond operator to the Nth power. The even and odd powers can be examined separately such that

$$\mathcal{L}^N = \begin{cases} \mathcal{L}^{2M} & N even \\ \mathcal{L}^{2Q+1} & N odd \end{cases} \tag{2.39}$$

where $M = N/2 \geq 1$ and $Q = (N-1)/2 \geq 1$. Using the result given by Eq. (2.38) for $N = 2$ and expanding the binomial to power M produces

$$\mathcal{L}^{2M} = \left[2imI\partial_t + 2I\partial_j^2\right]^M = \sum_{p,q}^{p+q=M} \frac{M!}{p!q!}(2im\partial_t)^p(2\partial_j)^q \tag{2.40}$$

Mixed partial differentials are produced for every term where $p > 0$ and $q > 0$, so there are no fundamental equations for $M > 1$. When N is odd, then

$$\mathcal{L}^{2Q+1} = \sum_{p,q}^{p+q=Q} \frac{Q!}{p!q!}(2im\partial_t)^p(2\partial_j)^q \tag{2.41}$$

In this case, the mixed partial differentials result for every $Q > 0$. There is no fortuitous canceling of terms as it was found for $N = 2$, which gave the Schrödinger equation, so there are no other fundamental equations.

Knowing that there are no other fundamental equations among the Lévy-Leblond class does not rule out other 2nd and higher-order fundamental equations. To prove that there are no other fundamental equations for four-component spinor state functions, we must consider the most arbitrary N^{th} order differential equation

$$\left[\sum_{a,b}^{a+b\leq N} D_{abj}\partial_t^a\partial_j^b\right]\psi(\vec{x},t) = 0 \ , \tag{2.42}$$

where D_{abj} are assumed to be 4×4 constant matrices and there is an implied summation over the index $j = 1, 2, 3$. Eliminating mixed partial differential terms, Eq. (2.42) becomes

$$\left[\sum_{a}^{0<a\leq N} D_{a0}\partial_t^a + \sum_{b}^{0<b\leq N} D_{0bj}\partial_j^b + D_{00}\right]\psi(\vec{x},t) = 0 \ . \tag{2.43}$$

The Galilei transformation rule (see Eqs 2.4 and 2.5)

$$G{\partial_t}^p{\partial_i}^q G^{-1} e^{i\varphi(\vec{x},t)} \ \psi\left(\vec{x},t\right)$$

$$= e^{i\varphi(\vec{x},t)}[k_1 + \partial_t + k_{2i}\partial_i]^p[k_{3i} + k_{4ji}\partial_j]^q \ \psi\left(\vec{x},t\right) \tag{2.44}$$

where the constants introduced are defined

$$k_1 = -\frac{i}{2}mv^2$$

$$k_{2i} = R_{ji}v_j$$

$${k_{2i}}^2 = \left(R_{ji}v_j\right)^2 = v^2$$

$$k_{3i} = -imR_{ji}v_kR_{jk}$$

$${k_{3i}}^2 = \left(-imR_{ji}v_kR_{jk}\right)^2 = -m^2v^2$$

$$k_{4ji} = R_{ji}.$$

transforms (Eq. 2.43) into

$$e^{i\varphi(\vec{x},t)} \left[\sum_{a}^{0<a\leq N} D'_{a0} \left(k_1 + \partial_t + k_{2i}\partial_i\right)^a \right.$$

$$\left. + \sum_{b}^{0<b\leq N} D'_{0bj} \left(k_{3j} + k_{4ij}\partial_i\right)^b + D'_{00} \right] \psi(\vec{x},t) = 0 \tag{2.45}$$

The trinomial and binomials can be expanded by their respective powers

$$e^{i\varphi(\vec{x},t)} \left[\sum_{a}^{0<a\leq N} D'_{a0} \left(\sum_{p,q,r}^{p+q+r=a} P\left(k_1^p, 1^q, k_{2i}^r\right) \partial_t^q \partial_i^r \right) \right.$$

$$\left. + \sum_{b}^{0<b\leq N} D'_{0bj} \left(\sum_{u,v}^{u+v=b} P\left(k_{3j}^u, k_{4ij}^v\right) \partial_i^v \right) + D'_{00} \right] \psi(\vec{x},t) = 0 \tag{2.46}$$

where the function $P()$ produces the permutation of its arguments. To be Galilei invariant this equation must be equal to the untransformed equation (see Eq. 2.43) and terms with mixed partial differentials must vanish. The condition that must be met for the sum of mixed partial differential terms of like powers e and f to vanish is

$$\sum_{a}^{0<a\leq N} D'_{a0} \left(\sum_{p}^{p+e+f=a} P\left(k_1^p, 1^e, k_{2i}^f\right) \right) = 0 \tag{2.47}$$

with $e > 0$ and $f > 0$.

The permuted terms do not vanish so the matrices must sum together to equal zero. The remaining conditions for invariance are

$$D_{e0} = \sum_{a}^{0<a\leq N} D'_{a0} \left(\sum_{p}^{p+e=a} P\left(k_1^p, 1^e, k_{2i}^0\right) \right) \tag{2.48}$$

$$D_{0fi} = \sum_{a}^{0<a\leq N} D'_{a0} \left(\sum_{p}^{p+f=a} P\left(k_1^p, 1^0, k_{2i}^f\right) \right)$$

$$+ \sum_{b}^{0<b\leq N} D'_{0bj} \left(\sum_{u}^{u+f=b} P\left(k_{3j}^u, k_{4ij}^f\right) \right) \tag{2.49}$$

$$D_{00} = \sum_{a}^{0<a\leq N} D'_{a0}\left(P\left(k_1^a, 1^0, k_{2i}^0\right)\right) + \sum_{b}^{0<b\leq N} D'_{0bj}\left(P\left(k_{3j}^b, k_{4ij}^0\right)\right) + D'_{00} \tag{2.50}$$

Setting $f = 0$ in Eq. (2.47) and combining with Eq. (2.48) proves that $D_{a0} = 0$. So there are no other Galilei invariant dynamical equations that are free of mixed partial differentials for 4-component spinors. This concludes the proof Proposition 4.

2.4 Discussion

There are three new and important results presented in this Chapter. First, we demonstrated that there are no fundamental dynamical equations for two-component spinor wave functions in Galilei space-time. Second, we derived the Lévy-Leblond equation for a four-component spinor wave function by using a different method than the original Lévy-Leblond approach [29, 30]. Finally, the most important result is a formal proof that the Lévy-Leblond and Schrödinger equations are the only fundamental dynamical equations for four-component spinor wave functions in Galilei space-time that can be derived with the Schrödinger phase factor. The obtained results have far reaching consequences that will be now discussed.

Quantum mechanics textbooks [41, 47] frequently present the Schrödinger equation for two-component spinors as a way of introducing the concept of spin. Rotational invariance is expected of the equation and rotations are applied to the spinor state functions. However it is not possible to do the same for Galilei boosts. Consequently, in non-relativistic quantum mechanics there is no way to relate the spinor state function of one observer to that of another observer moving with a constant relative velocity. Furthermore even the Schrödinger equation with its identity matrix coefficients cannot be shown to be Galilei invariant for two-component spinors because there are no 2×2 Galilei boost matrices.

Now, the Pauli-Schrödinger (PS) equation can be derived from the Lévy-Leblond equation [30]. Using phase invariance of the second kind the fields V and A are introduced by the substitutions

$$i\partial_t \to i\partial_t - V(x,t) \; , \tag{2.51}$$

and

$$-i\partial_j \to -i\partial_j - A(x,t) \; , \tag{2.52}$$

where (V, A) is the 4-potential of the electro-magnetic field. Maxwell's equations break Galilei invariance but may be cast into a Galilei invariant form by the elimination of Maxwell's term in the non-relativistic limit ($c \to \infty$). Performing the above substitution on the Lévy-Leblond equation produces a pair of coupled equations of the form

$$\sigma_j(i\partial_j + A_j)\phi - 2m\chi = 0 \tag{2.53}$$

and

$$(i\partial_t - v)\phi + \sigma_j(i\partial_j + A_j)\chi = 0 \tag{2.54}$$

Then the Pauli-Schrödinger equation [31, 38, 30] is obtained by eliminating χ from the above pair of equations, and we have

$$i\partial_t\phi = V\phi + \frac{1}{2m}\left[(i\partial_j + A_j)^2 + i\sigma \cdot (i\partial_j + A_j) \times (i\partial_j + A_j)\right]\phi \tag{2.55}$$

The PS equation is a second order differential equation governing the space and time evolution of a two-component spinor. As a two-component spinor equation it cannot be proven to be Galilei invariant as demonstrated by Proposition 1. This means that the PS equation is not fundamental in Galilei space-time. It is interesting that the 2-component PS equation can be derived from the four-component Lévy-Leblond equation, which is fundamental. Thus the validity of the two-component PS equation follows from the existence of the fundamental (Galilei invariant) Lévy-Leblond equation.

It has been suggested that the PS equation is covariant in the low velocity limit [44]. However that proof uses the four-component boost matrix to determine the effect of a boost on the separable two-components of the spinor. Normally the four-component boost matrix mixes the components of the spinor but in the limit of low velocity the four-component spinor boost matrix becomes an identity matrix and no mixing occurs. As such it effectively applies no boost at all. Consequently, this approach does not show Galilei boost invariance in a low velocity limit.

Based on the above results, some confusion may arise from the unusual fact that the four-component Schrödinger equation is fundamental while the two-component Schrödinger equation is not. Since the matrices of the Schrödinger equation are diagonal there is no mixing of the pair of two-component spinors in the four-component equation. This makes it possible to separate the equations into two two-component Schrödinger equations [47], which indicates that the validity of the (non-fundamental) two-component Schrödinger equation is a consequence of the fundamental four-component Schrödinger equation.

The above discussion of the PS equation clearly shows that the Lévy-Leblond and Schrödinger equations for four-component spinor wave functions are the only fundamental dynamical equation in Galilei space-time with the Schrödinger phase factor. This is an important result as it shows that only the Lévy-Leblond and Schrödinger equations are available to formulate field theories of elementary particles described by four-component spinors in Galilei space-time.

2.5 Summary

A search for fundamental dynamical equations for two and four-component spinor wave functions was conducted in Galilei space-time represented by the extended Galilei group. For a dynamical equation to be considered as fundamental, it was required that the equation was invariant under the symmetry operators of the group of the Galilei metric, that the state functions transformed like the irreducible representations of the group of the metric, and that the equation did not have mixed time and space partial derivatives.

The main results obtained are: (i) there are no fundamental dynamical equations for two-component spinor wave functions in Galilei space-time; (ii) the Lévy-Leblond equation for a four-component spinor wave function can be derived by using a different method than the one originally used by Lévy-Leblond; (iii) a formal proof that the Lévy-Leblond and Schrödinger equations are the only fundamental dynamical equations for four-component spinor wave functions in Galilei space-time.

Among important physical implications of the obtained results is that the Pauli-Schrödinger equation is not a fundamental (Galilei invariant) equation in Galilei space-time. This remains true despite the fact that the original Lévy-Leblond equation for four-component spinor wave functions, from which the Pauli-Schrödinger equation can be obtained, is a Galilei invariant dynamical equation.

CHAPTER 3

NEW INVARIANT DYNAMICAL EQUATIONS IN GALILEAN SPACE-TIME

3.1 Background

To find new fundamental physics equations that describe Dark Matter in Galilean space-time, we must go beyond the work presented in the previous chapter. The reason is that in Chapter 2 [48] all equations were derived using the Schrödinger phase factor exclusively. It was shown that under this constraint, the Levy-Leblond and Schrödinger equations are the only Galilei invariant dynamical equations for four-component spinors. It is well-known that both equations describe ordinary matter in Galilean space-time.

In this chapter, we conduct a search for new Galilei invariant dynamical equations by removing the requirement of a Schrödinger phase factor. This leads to new invariant equations that are called here the generalized Lévy-Leblond and generalized Schrödinger equations. Using these new equations, we obtain a new energy-momentum relation, which generalizes the well-known non-relativistic energy-momentum relationship. It is also demonstrated that these generalized equations reduce to the standard Lévy-Leblond and Schrödinger equations as a special case when the Schrödinger phase factor is applied. An important and new result is that the choice of phase factor determines the form of the energy-momentum relation and that the standard non-relativistic momentum-energy relation is obtained only when the Schrödinger phase function is used. A physical interpretation of the of the new equations is also provided.

3.2 Deriving New Fundamental Dynamical Equations

3.2.1 Conditions for Invariance of the First Order Equation

For convenience we will use the following notation throughout Chapter 3 when dealing with Galilean space and time coordinates. Unless states otherwise in the text there is an implied summation over repeated indices. For greek indices the sum will run from 0 to 3 while latin indices will run from 1 to 3. The Galilei metric tensor $g_{\mu\nu} = diag1, 1, 1, 1$ implies for all quantities X_μ that $X_\mu = X^\mu = (X_0, X_j)$ and $X_\mu X^\mu = X_0^2 + X_j^2$. The partial differential ∂_μ is with respect to the coordinate defined by the index such that $\partial^\mu = \partial_\mu = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$.

In order to search for Galilei invariant equations we begin with the most arbitrary first order differential equation

$$[B_0\partial_t + B_j\partial_j + B_c]\,\psi(x,t) = [B^\mu\partial_\mu + B_c]\,\psi(x,t) = 0 \tag{3.1}$$

where B_0, B_j, and B_c are constant $n \times n$ matrices and ψ is an n-component state function and we use the notation $B^\mu = B_\mu = (B_0, B_j)$.

A space time point in the coordinates of one observer $x_\nu = (t, x, y, z)$ is related to the same point in the coordinates of another observer $x'_\nu = (t, x, y, z)$ by a linear transformation

$$x'^\nu = \Lambda^\nu_\mu x^\mu + b^\nu \tag{3.2}$$

where there is an implied sum over repeated indices Under the linear transformation a differential operator transforms like

$$\partial'_\mu = \Lambda^\rho_\mu \partial_\rho \tag{3.3}$$

Applying the linear transformations to the first order equation produces

$$\left[B'^\mu\partial'_\mu + B'_c\right] e^{i\phi(x,t)}\psi(x,t) = 0 \tag{3.4}$$

Substitute the transformed differential operators to restore the equation to its original variables

$$\left[B'^{\mu}\Lambda^{\rho}_{\mu}\partial_{\rho} + B'_{c}\right] e^{i\phi(x,t)}\psi(x,t) = 0 \tag{3.5}$$

The phase factor can be commuted through the differential operator

$$\Lambda^{\rho}_{\mu}\partial_{\rho}e^{i\phi(x,t)} = e^{i\phi(x,t)}\Lambda^{\rho}_{\mu}\left(\partial_{\rho} + i\partial_{\rho}\phi\right) \tag{3.6}$$

Commuting the phase factor through to the left side of the transformed first order equation and dividing it out produces

$$\left[B'^{\mu}\Lambda^{\rho}_{\mu}\left(\partial_{\rho} + i\partial_{\rho}\phi\right) + B'_{c}\right]\psi(x,t) = 0 \tag{3.7}$$

For the first order equation to be invariant under the transformation, the transformed first order equation must be equal to the original first order equation. Equating terms of like differential powers then generates a set of conditions on the matrices B that must be met for the dynamical equation to be invariant.

$$B^{\mu} = B'^{\beta}\Lambda^{\mu}_{\beta} \tag{3.8}$$

$$B_{c} = B'_{c} + iB'^{\beta}\Lambda^{\mu}_{\beta}\partial_{\mu}\phi \tag{3.9}$$

The equation is invariant for phase functions of the form

$$\phi(x,t) = \zeta_{\mu}x^{\mu} + \zeta_{c} \tag{3.10}$$

where ζ_{μ}, ζ_{c} are scalar functions of the transformation parameters a^{μ}, v^{i}, and θ^{i} and potentially any number of other parameters that have yet to be introduced. The extended Galilei group relies upon the introduction of one parameter in ζ_{c}. Using the replacement $\partial_{\mu}\phi = \zeta_{\mu}$ the condition for invariance (3.9) becomes

$$B_{c} = B'_{c} + iB'^{\beta}\Lambda^{\mu}_{\beta}\zeta_{\mu} \tag{3.11}$$

which can be further simplified by substitution of (4.10)

$$B_{c} = B'_{c} + iB^{\mu}\zeta_{\mu} \tag{3.12}$$

3.2.2 Matrices that Satisfy the Conditions for Galilean Invariance

The linear transformation used here is the Galilean transformation

$$\hat{\Lambda} = [\Lambda^{\mu}_{\nu}] = \begin{pmatrix} I & 0 \\ Rv & R \end{pmatrix} \tag{3.13}$$

$$\hat{\Lambda}^{-1} = \begin{pmatrix} I & 0 \\ -v & R^{-1} \end{pmatrix} \tag{3.14}$$

The matrices B^{μ} and B_c must satisfy the conditions in order to form a Galilean invariant first order equation.

Applying rotations only to the first condition constrains the matrices to the form

$$B^t = \begin{pmatrix} pI & qI \\ sI & tI \end{pmatrix} \quad B^j = \begin{pmatrix} e\sigma_j & f\sigma_j \\ g\sigma_j & h\sigma_j \end{pmatrix} \tag{3.15}$$

where $a, b, c, d, e, f, g, h, p, q, s,$ and t are as yet undetermined constants. Applying boosts only to the first condition further constrains the matrices to

$$B^t = \begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix} \quad B^j = \begin{pmatrix} s\sigma_j & 0 \\ g\sigma_j & -s\sigma_j \end{pmatrix} \tag{3.16}$$

Combining boosts and rotations in the conditions creates no further constraints.

The second condition contains the unknown function $\zeta(v^i, \theta^j)$ that must be determined in addition to the matrix B_c. With the Galilean transformation substituted in the second condition reads

$$B_c = B'_c + iB'^t\zeta_t + iB'^i R^j_i \zeta_j \tag{3.17}$$

Written explicitly in matrix form

$$B_c - B'_c$$

$$= \begin{pmatrix} A & B \\ C & D \end{pmatrix} - \begin{pmatrix} U_R & 0 \\ -\frac{\sigma\cdot v}{2}U_R & U_R \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} U_R^{-1} & 0 \\ U_R^{-1}\frac{\sigma\cdot v}{2} & U_R^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} A - UAU^{-1} - UBU^{-1}\frac{\sigma\cdot v}{2} & B - UBU^{-1} \\ C - UCU^{-1} + \frac{\sigma\cdot v}{2}UAU^{-1} + \frac{\sigma\cdot v}{2}UBU^{-1}\frac{\sigma\cdot v}{2} + UDU^{-1}\frac{\sigma\cdot v}{2} & D - UDU^{-1} + \frac{\sigma\cdot v}{2}UBU^{-1} \end{pmatrix}$$

$$= iB'^t\zeta_t + iB'^i R_i^j \zeta_j$$

$$= iB^t\zeta_t + iB^j\zeta_j$$

$$= i\begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix}\zeta_t + i\begin{pmatrix} s\sigma_j & 0 \\ g\sigma_j & -s\sigma_j \end{pmatrix}\zeta_j$$

$$= \begin{pmatrix} is\sigma_j\zeta_j & 0 \\ isI\zeta_t + ig\sigma_j\zeta_j & -is\sigma_j\zeta_j \end{pmatrix} \tag{3.18}$$

The second component of the matrix equation (3.18) requires

$$B = bI \tag{3.19}$$

where b is a scalar constant and I is the 2×2 identity matrix.

The first and fourth components of (3.18) add together to produce a relationship between A and D

$$A + D = UAU^{-1} + UDU^{-1} = U(A + D)U^{-1}. \tag{3.20}$$

From (3.20) we can conclude that

$$A + D = qI \tag{3.21}$$

where q is an as yet undetermined scalar constant.

The first component of (3.18) combined with (3.19) is

$$A - UAU^{-1} - \frac{b\sigma_j v_j}{2} = is\sigma_j\zeta_j \tag{3.22}$$

Solving for ζ_j

$$\zeta_j(\theta_i, v_j) = \frac{-ib}{2s}\sigma_j\sigma_i v_i - \frac{i\sigma_j}{s}(A - UAU^{-1}) \tag{3.23}$$

we find that the functional dependence of ζ_j on θ_i is seperate from its dependence on v_i. Furthermore there exists a choice of constant terms $A = aI$ that will remove the rotational dependence from ζ_j leaving it simply proportional to velocity.

The third component of (3.18) provides the last piece of information that can be extracted from the matrix equation. By substituting (3.19) and (3.23) into the expression to eliminate ζ_j the functional dependence of ζ_t is found to be

$$\zeta_t = \frac{1}{is}\left[C - UCU^{-1} + \frac{\sigma \cdot v}{2}UAU^{-1} + \frac{1}{4}bv^2 - UDU^{-1}\frac{\sigma \cdot v}{2}\right.$$
$$\left. + \frac{bg}{2s}\sigma \cdot v + \frac{g}{s}\left(A - UAU^{-1}\right)\right] \tag{3.24}$$

and the first order equation is

$$\left[\begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix}\partial_0 + \begin{pmatrix} s\sigma_j & 0 \\ g\sigma_j & -s\sigma_j \end{pmatrix}\partial_j + \begin{pmatrix} A & bI \\ C & qI - A \end{pmatrix}\right]\psi(x,t) = 0. \tag{3.25}$$

3.2.3 Requiring Hermiticity of the First Order Equation

Requiring hermiticity of the Hamiltonian can result in further constraint on the matrices. First hermiticity of the B_i matrix requires $g = 0$. Second, hermiticity of the sum $\kappa B_t + B_c$ where κ is an arbitrary scalar constant requires $C = (b - \kappa s)I$ and $A^\dagger = A$. Applying these constraints to (3.23) and (3.24) lead to far more constrained conditions on the phase function. Redefining the constants to absorb s (i.e. $b/s \to b$, $q/s \to q$, and $A/s \to A$).

$$\zeta_j(\theta_i, v_j) = \frac{-ib}{2}\sigma_j\sigma_i v_i - i\sigma_j(A - UAU^{-1}) \tag{3.26}$$

$$\zeta_t = -i\left[\frac{\sigma \cdot v}{2}UAU^{-1} + \frac{1}{4}bv^2 - UDU^{-1}\frac{\sigma \cdot v}{2}\right] \tag{3.27}$$

$$= -\frac{i}{4}bv^2 - i\frac{\sigma \cdot v}{2}UAU^{-1} + iUAU^{-1}\frac{\sigma \cdot v}{2} - q\frac{\sigma \cdot v}{2} \tag{3.28}$$

Dividing through by s and redefining constants the set of matrices is constrained to

$$\left[\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}\partial_0 + \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}\partial_j + \begin{pmatrix} A & bI \\ (b-\kappa)I & qI - A \end{pmatrix}\right]\psi(x,t) = 0 \tag{3.29}$$

where A must be Hermitian $A^\dagger = A$ and q, b, and κ must be real. Equation (3.29) is the generalized Lévy-Leblond equation.

3.2.4 Deriving a Second Order Equation: the Generalized Schrödinger Equation

A second order equation can be created by squaring the first order operator on the state function. This equation may also be true and it might be used to limit the constants of the first order equation. The second order equation should not contain any mixed partial differentials. The second order equation produced by squaring the operator of the first order equation is

$$\left[\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}\partial_0 + \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}\partial_j + \begin{pmatrix} A & bI \\ (b-\kappa)I & qI - A \end{pmatrix}\right]^2\psi(x,t)$$

$$= \left[\begin{pmatrix} bI & 0 \\ qI & bI \end{pmatrix}\partial_t + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\partial_j^2 + \begin{pmatrix} \sigma_j A + A\sigma_j & 0 \\ 0 & -2q\sigma_j + A\sigma_j + \sigma_j A \end{pmatrix}\partial_j\right.$$

$$\left. + \begin{pmatrix} A^2 + b(b-\kappa)I & bqI \\ q(b-\kappa)I & b(b-\kappa)I + q^2I - 2qA + A^2 \end{pmatrix}\right]\psi(x,t) = 0 \tag{3.30}$$

If the second order equation is required to be a dynamical equation of the state function then the Hamiltonian of the second order equation must also be Hermitian. We have already established that A is Hermitian and q and b are real. Under these restrictions the second and third terms of (3.30) are Hermitian. The sum of the first and fourth terms of (3.30) is also Hermitian. This second order equation (3.30) is the generalized Schrödinger equation for four-component spinors.

3.2.5 Deriving a Momentum-Energy Relation

A momentum energy relation can be derived by looking for stationary states. The first order dynamical equation can be transformed into a time-independent equation assuming that the space and time parts of the state function are separable i.e.

$$\psi(x,t) = \psi(x)e^{-i\epsilon t}. \tag{3.31}$$

The bispinor notation is used to break the four component spinor into a pair of two component spinors ϕ and χ

$$\psi(x) = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}. \tag{3.32}$$

States with a definite momentum p_j are also seperable

$$\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} \varphi_0 \\ \chi_0 \end{pmatrix} e^{ip_j x_j}. \tag{3.33}$$

Furthermore we can substitute the constant 2×2 matrix A with a sum of the linearly combination of the Pauli matrices and identity matrix. Substituting these expression into the generalized Lévy-Leblond equation and replacing the momentum operator $\hat{p}_j$ with its eigenvalues p_j produces the pair of linked equations

$$\varepsilon\varphi_0 + \tilde{\varepsilon}\varphi_0 - \sigma_j(p_j + \tilde{p}_j)\chi_0 = 0 \tag{3.34}$$

$$-\sigma_j(p_j + \tilde{p}_j)\varphi_0 + 2m_0\chi_0 = 0. \tag{3.35}$$

As a linear homogeneous system of equations for φ_0 and χ_0 non-trivial solutions exist only when the determinant of the coefficients vanishes, i.e.

$$\begin{vmatrix} \varepsilon + \tilde{\varepsilon} & \sigma_j(p_j + \tilde{p}_j) \\ \sigma_j(p_j + \tilde{p}_j) & 2m_0 \end{vmatrix} = 0. \tag{3.36}$$

So we have

$$2m_0(\varepsilon + \tilde{\varepsilon}) - \sigma_j(p_j + \tilde{p}_j)\sigma_k(p_k + w_k) \tag{3.37}$$

$$= 2m_0\varepsilon - (p_j + \tilde{p}_j)^2 = 0 \tag{3.38}$$

and the momentum-energy relation for the generalized Lévy-Leblond equation is

$$\epsilon = -\tilde{\varepsilon} + (p_j + \tilde{p}_j)^2/2m. \tag{3.39}$$

3.3 Discussion

The main result of this chapter is a new Galilean invariant equation for four component spinors that is called here the generalized Lévy-Leblond equation. It must be pointed out that the state functions of this equation do not obey the standard Schrödinger equation, which can be seen by squaring the operators of the generalized Lévy-Leblond equation to produce a new second order equation that becomes a generalized Schrödinger equation; the latter has more terms and even more important it mixes the spinor components. This is an important result as it shows that other fundamental equations do exist in Galilean space-time. As expected, the generalized Lévy-Leblond equation reduces to the standard Lévy-Leblond equation obtained in Chapter 2 for the special case when the phase factor is set to the Schrödinger phase factor. Based on equations (3.26-3.27), the Schrödinger phase factor is obtained when $A = 0$. The free parameter b remains an arbitrary constant but can be set to $b = 2m$.

Using the generalized Lévy-Leblond equation, we obtain a new energy-momentum relation, which generalizes the well-known non-relativistic energy-momentum relationship; note that the latter is obtained only when the Schrödinger phase factor is used and the standard Lévy-Leblond equation is derived. This dependence of the non-relativistic energy-momentum relationships on the phase factors is a new phenomenon discovered here.

The discovery of the new invariant dynamical equation and the resulting new non-relativistic energy-momentum relationship may have important physical consequences because the conditions under which this equation and the relationship were derived do not apply to ordinary matter whose atomic structure is described by the Schrödinger equation. The new equation is independent from the existence of the Schrödinger equation because its Galilean invariance is guaranteed by another (non-Schrödinger) phase function. This clearly indicates that the equation may describe an extraordinary matter such as Dark Matter whose nature and origin remains currently unknown. This interesting proposition will be further considered in Chapter 5 after a search for fundamental dynamical equations in Minkowski space-time is presented (see Chapter 4).

3.4 Summary

A new fundamental (Galilean invariant) dynamical equation for four-component spinor state functions was found. The equation describes free particles represented by such state functions. Since in a special case this new equation reduces to the standard Lévy-Leblond equation derived in Chapter 2, it is called here the generalized Lévy-Leblond equation. The main difference between these equations is that they are obtained with different phase factors and that energy-momentum relationships resulting from the equations are also different. The dependence of the non-relativistic energy-momentum relationship on the constant terms is a new phenomenon discovered here. Moreover, the obtained results clearly show that the energy-momentum resulting from the generalized Lévy-Leblond equation is not the same as the well-known non-relativistic energy-momentum relationship. Hence, it is suggested that such new energy-momentum relationship may describe an extraordinary matter such as Dark Matter whose nature and origin still remains unknown.

CHAPTER 4

DIRAC EQUATION AND OTHER NEW FUNDAMENTAL EQUATIONS IN MINKOWSKI SPACE-TIME

4.1 Background

In this dissertation, a physical theory is called fundamental in Minkowski space-time when its dynamical equations are invariant with respect to all transformations of coordinates that leave the Minkowski metric invariant. The transformations form a representation of the inhomogeneous Lorentz group, which is also known as the Poincaré group (e.g., [21, 22]). We refer to an invariant dynamical equation as a Poincaré invariant equation. In 1939 Wigner [20] classified all irreducible representations (irreps) of the Poincaré group and used them to determine classes of elementary particles that exist in Minkowski space-time (e.g., [21]).

The Klein-Gordon (KG) equation [49, 50] is one of the fundamental (Poincaré invariant) equations of relativistic field theories. Particles described by the KG equation are bosons with spin zero and they are represented by scalar wave functions. The difficulties with negative energy states and the probabilistic interpretation of this equation are eliminated in quantum field theories (QFT) by the procedure of field quantization (e.g., [47]). Other fundamental equations are the Dirac equation, which describes fermions with spin 1/2, and the Proca equation, which describe bosons with spin 1. The state functions that represent fermions are spinors while vector wave functions represent bosons. Typically, it is required that different components of these state functions satisfy the KG equation. The non-relativistic limits of the KG, Dirac and Proca equations have also been investigated and the resulting Galilean invariant equations were obtained [29, 30, 47, 22].

To obtain the KG equation, the relativistic energy-momentum relationship is typically used and differential operators are substituted for the energy and momentum. On the other hand, the Dirac equation is usually derived by using the transformation properties of spinors under the Lorentz group. A similar procedure is also used to obtain the Proca equation. All these equations can also be formally obtained from the corresponding Lagrangians (e.g., [47]). Another method that is based on group theory was introduced by Wigner and Bargmann (1948) [26], and used by these authors to obtain the equations of relativistic field theories.

A different approach was used by Fry, Musielak & Chen (2011) [18], who used a method based on the Poincaré group, which is the group of the Minkowski metric, and formally derived the KG equation for free spin-zero particles. Using the same approach, Musielak & Fry (2011) [18] demonstrated that the KG equation is the only second-order differential equation, which is Poincaré invariant, and that there is also an infinite set of higher-order KG-like equations, all of which are Poincaré invariant.

As already shown in the previous chapters, the approach developed in this dissertation differs from those adopted in the previous works. Specific differences were already discussed in Chapter 2 for Galilean space-time and they are also relevant for Minkowski space-time. Now, the approach will be extended to the Minkowski metric and free elementary particles represented by four-component spinor state functions will be considered. Before the formal derivation of the results is given, we first briefly summarize the basic properties of the Minkowski metric and the Poincaré group.

The Minkowski metric can be written as $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$, where x, y and z are the spatial coordinates and t is the time coordinate given in natural units where the speed of light is $c = 1$ so that space and time have the same units.

The group of this metric is the Poincaré group P, whose structure is given by the following semi-direct product: $P = H_p \otimes_s T(3+1)$, where $T(3+1)$ is an invariant subgroup of space-time translations and H_p is a non-invariant subgroup consisting of the remaining transformations and the identity transformation. In this dissertation, we consider the so-called proper orthochronous group $P_+^\uparrow$ that is a subgroup of P.

We now derive Poincaré invariant dynamical equations for 4-component spinor wave functions in Minkowski space-time.

4.2 Poincaré Invariant Dynamical Equations

4.2.1 Deriving Conditions for Invariance of the First Order Equation

To find a fundamental dynamical equation we consider the most arbitrary first order differential equation of the form

$$[B^\mu \partial_\mu + B_c]\,\psi(x,t) = 0 \tag{4.1}$$

where B^μ and B_c are constant $n \times n$ matrices and ψ is an n-component state function.

A space time point in the coordinates of one observer x^ν is related to the same point in the coordinates of another observer x'^ν by a linear transformation

$$x'^\nu = \Lambda^\nu_\mu x^\mu + b^\nu. \tag{4.2}$$

Under the linear transformation a differential operator transforms like

$$\partial'_\mu = \Lambda^\rho_\mu \partial_\rho. \tag{4.3}$$

The state function transforms like

$$T\psi(x,t) = \psi'(x',t'). \tag{4.4}$$

In general the state functions are related by a phase factor

$$e^{i\phi(x,t)}\psi(x,t) = \psi'(x',t'). \tag{4.5}$$

The principle of relativity requires that if a dynamical equation such as equation (4.1) is true for one observer that there exist an equations of the same form for all other observers who have a coordinate system that is translated, rotated, and boosted relative to the first observer. An arbitrary second observer would have an equation that features objects such as B'^{μ}, B'_c, and ∂'_μ that may have been changed by the coordinate transformation such as

$$\left[B'^{\mu}\partial'_\mu + B'_c\right]\psi'(x', t') = 0 \tag{4.6}$$

Substituting the transformation rules of the differential operator (4.3) and state function (4.4) into equation (4.5) produces a first order equation that is now written in terms of the original coordinate system x^μ

$$\left[B'^{\mu}\Lambda^{\rho}_{\mu}\partial_\rho + B'_c\right]e^{i\phi(x,t)}\psi(x, t) = 0 \tag{4.7}$$

The phase factor can be commuted through the differential operator

$$\Lambda^{\rho}_{\mu}\partial_\rho e^{i\phi(x,t)} = e^{i\phi(x,t)}\Lambda^{\rho}_{\mu}\left(\partial_\rho + i\partial_\rho\phi\right) \tag{4.8}$$

Commuting the phase factor through to the left side of the transformed first order equation and dividing it out produces

$$\left[B'^{\mu}\Lambda^{\rho}_{\mu}\left(\partial_\rho + i\partial_\rho\phi\right) + B'_c\right]\psi(x, t) = 0. \tag{4.9}$$

For the first order equation to be invariant under the transformation the transformed first order equation must be equal to the original first order equation. Equating terms of like differential powers then generates a set of conditions on the matrices B^μ and B_c that must be met for the dynamical equation to be invariant.

$$B^\mu = B'^{\beta}\Lambda^{\mu}_{\beta} \tag{4.10}$$

$$B_c = B'_c + iB'^{\beta}\Lambda^{\mu}_{\beta}\partial_\mu\phi \tag{4.11}$$

The equation is invariant for phase functions of the form

$$\phi(x,t) = \zeta_\mu x^\mu + \zeta_c \tag{4.12}$$

where ζ_μ, ζ_c are scalar functions of the transformation parameters a^μ, v^i, and θ^i and potentially any number of other parameters that have yet to be introduced. The extended Galilei group relies upon the introduction of one parameter in ζ_c. Using the replacement $\partial_\mu \phi = \zeta_\mu$ the condition for invariance (4.11) becomes

$$B_c = B'_c + iB'^\beta \Lambda^\mu_\beta \zeta_\mu. \tag{4.13}$$

4.2.2 Finding Matrices that Satisfy the Conditions for Invariance

The matrices B^μ and B_c must satisfy the conditions (4.10) and (4.13) in order to form a first order differential equation that is invariant under the linear transformation $\hat{\Lambda}$. From this point on the linear transformation will be a Poincare or inhomogeneous Lorentz transformation. This transformation can be separated into rotations and boosts to ease calculations. The rotations matrices for 4-component spinors are

$$R_{\theta_j} = \cos\frac{\theta}{2} + \epsilon_{jkl}\gamma_k\gamma_l \sin\frac{\theta}{2} \tag{4.14}$$

where γ_j and γ_0 form a basis for the 4×4 spinor matrices and θ_j is the rotation about the j-axis and in the $k-l$-plane. The boost matrices for 4-component spinors are

$$S_{v_j} = \cosh\frac{\eta}{2} + i\gamma_j\gamma_0 \sinh\frac{\eta}{2} \tag{4.15}$$

where η is the boost angle. The boost angle is related to the velocity by $\tanh\eta = \beta = v/c$ where $c = 1$ is the speed of light in natural units. The Dirac representation is chosen for this work whenever explicit representation of the gamma matrices γ^μ is required

$$\gamma^j = \begin{pmatrix} 0 & i\sigma^j \\ -i\sigma^j & 0 \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{4.16}$$

where σ^j are the Pauli matrices.

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.17}$$

In this representation the Minkowski metric $g^{\mu\nu}$ has signature $(+---)$. The covariant gamma matrices are related to the contravariant form by $\gamma_\mu = g_{\mu\nu}\gamma^\nu = \{\gamma^0, -\gamma^j\}$. The gamma matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu} I. \tag{4.18}$$

4.2.3 Invariant Equations for the Standard Phase Function $\phi = 0$

Before dealing with the more general case we consider the consequences of assuming the phase function $\phi(x,t) = 0$. This case requires $\zeta_\mu = \zeta_c = 0$ and simplifies the condition (4.13) to

$$B_c = B'_c. \tag{4.19}$$

Applying rotations only to the conditions and (4.10) constrains the matrices to the form

$$B^t = \begin{pmatrix} pI & qI \\ sI & tI \end{pmatrix} \quad B^j = \begin{pmatrix} e\sigma^j & f\sigma^j \\ g\sigma^j & h\sigma^j \end{pmatrix} \tag{4.20}$$

where e, f, g, h, p, q, s, and t are unknown scalar constants.

We re-write condition (4.10) with the boost transformation S

$$\Lambda^\beta_\mu B^\mu = B'^\beta = S B^\beta S^{-1} \tag{4.21}$$

Applying a Lorentz boost along the z-direction in condition (4.21) the right hand side becomes

$$S_z B^\beta S_z^{-1} = \left(\cosh\frac{\eta}{2} - i\gamma_z\gamma_0 \sinh\frac{\eta}{2}\right) B^\beta \left(\cosh\frac{\eta}{2} + i\gamma_z\gamma_0 \sinh\frac{\eta}{2}\right) \tag{4.22}$$

$$= B^\beta \cosh^2 \frac{\eta}{2} + \gamma_z\gamma_0 B^\beta \gamma_z\gamma_0 \sinh^2 \frac{\eta}{2} \tag{4.23}$$

$$+i\left(B^\beta\gamma_z\gamma_0 - \gamma_z\gamma_0 B^\beta\right)\sinh\frac{\eta}{2}\cosh\frac{\eta}{2} \tag{4.24}$$

$$= \frac{1}{2}\left(B^\beta + \gamma_z\gamma_0 B^\beta\gamma_z\gamma_0\right)\cosh\eta + \frac{1}{2}\left(-B^\beta + \gamma_z\gamma_0 B^\beta\gamma_z\gamma_0\right) \tag{4.25}$$

$$+\frac{i}{2}\left(B^\beta\gamma_z\gamma_0 - \gamma_z\gamma_0 B^\beta\right)\sinh\eta \tag{4.26}$$

While the left hand side of the equation (4.21) is

$$\Lambda^t_\mu B^\mu = B^t \cosh\eta - iB^z \sinh\eta \tag{4.27}$$

for $\beta = t$ and a boost in the z-direction. Alternately the left hand side is

$$\Lambda^z_\mu B^\mu = B^z \cosh\eta + iB^t \sinh\eta \tag{4.28}$$

for $\beta = z$ while $\Lambda^x_\mu B^\mu = B^x$ and $\Lambda^y_\mu B^\mu = B^y$. Here we used the identities

$$\sinh^2\frac{\eta}{2} = \frac{1}{2}\cosh\eta + \frac{1}{2} \tag{4.29}$$

and

$$\sinh\frac{\eta}{2}\cosh\frac{\eta}{2} = \frac{1}{2}\sinh\eta \tag{4.30}$$

and

$$\cosh^2\frac{\eta}{2} = \frac{1}{2}\cosh\eta - \frac{1}{2} \tag{4.31}$$

The right hand side of (4.21) is equal the left hand side when

$$\left\{B^t, \gamma_z\gamma_0\right\} = 0 \tag{4.32}$$

$$\left\{B^z, \gamma_z\gamma_0\right\} = 0 \tag{4.33}$$

$$\left[B^x, \gamma_z\gamma_0\right] = 0 \tag{4.34}$$

$$\left[B^y, \gamma_z\gamma_0\right] = 0 \tag{4.35}$$

The above procedure may be repeated for boosts in the x and y directions to produce a set of constraints similar to (4.32)-(4.35). Taken together the constraints are equivalent to requiring the matrices B^μ obey the Clifford algebra

$$\{B^\mu, B^\nu\} = 2\delta^{\mu\nu} I. \tag{4.36}$$

Returning to the condition on B_c, equation (4.19), application of a Lorentz boost produces

$$B_c = S_z B_c S_z^{-1} \tag{4.37}$$

$$= \left(\cosh\frac{\eta}{2} - i\gamma_z\gamma_0 \sinh\frac{\eta}{2}\right) B_c \left(\cosh\frac{\eta}{2} + i\gamma_z\gamma_0 \sinh\frac{\eta}{2}\right) \tag{4.38}$$

$$= B_c \cosh^2\frac{\eta}{2} + \gamma_z\gamma_0 B_c \gamma_z\gamma_0 \sinh^2\frac{\eta}{2} \tag{4.39}$$

$$+ i\left(B_c\gamma_z\gamma_0 - \gamma_z\gamma_0 B_c\right)\sinh\frac{\eta}{2}\cosh\frac{\eta}{2} \tag{4.40}$$

This leads to

$$[B_c, \gamma_z\gamma_0] = 0 \tag{4.41}$$

Applying rotations in equation (4.19)

$$B_c = R_z B_c R_z^{-1} = \left(\cos\frac{\theta}{2} - \gamma_x\gamma_y \sin\frac{\theta}{2}\right) B_c \left(\cos\frac{\theta}{2} + \gamma_x\gamma_y \sin\frac{\theta}{2}\right) \tag{4.42}$$

$$= B_c \cos^2\frac{\theta}{2} - \gamma_x\gamma_y B_c \gamma_x\gamma_y \sin^2\frac{\theta}{2} + \left(B_c\gamma_x\gamma_y - \gamma_x\gamma_y B_c\right)\sin\frac{\theta}{2}\cos\frac{\theta}{2} \tag{4.43}$$

Which leads to the constraint

$$[B_c, \gamma_x\gamma_y] = 0 \tag{4.44}$$

This reasoning can be repeated for x- and y-directions as well to produce similar constraints. Under these conditions the matrix B_c is constrained to

$$B_c = a \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{4.45}$$

where a remains a free parameter. Combining these results produces the equation

$$\left[\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\partial_j + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\partial_t + a\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right]\psi(x,t) = 0. \tag{4.46}$$

We have found a Poincaré invariant equation for 4-component spinor state functions. This equation satisfies the principles of relativity and analyticity but we must also insure that the state functions transform like irreducible representations of the Poincaré group. This means that the state functions must obey the eigen equations $i\partial_t\psi = \omega\psi$ and $-i\partial_j\psi = k_j\psi$. This requirement can be used to remove the free parameter a. Multiplying equation (4.46) by i and moving the time derivative to the left hand side produces the equation

$$i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\partial_t\psi(x,t) = -i\left[\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\partial_j + a\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right]\psi(x,t). \tag{4.47}$$

The operator on the right hand side is the Hamiltonian and the principle of relativity requires that the operator be Hermitian in order for the eigenvalues of this operator to be real. The terms are all Hermitian under the condition that the constant a is imaginary or that ia is real. Now we substitute the eigen-values for their operators

$$\omega\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\psi(x,t) = k_j\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\psi(x,t) - ia\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\psi(x,t). \tag{4.48}$$

Substituting the bispinor

$$\psi(x,t) = \begin{pmatrix} \phi \\ \eta \end{pmatrix} \tag{4.49}$$

for the state function in equation (4.48) produces a pair of linked equations

$$\omega\phi = k_j\sigma_j\eta - ia\phi \tag{4.50}$$

and

$$-\omega\eta = -k_j\sigma_j\phi - ia\eta. \tag{4.51}$$

The four momentum p_μ contains the eigen-values of energy and momentum

$$p_\mu = (\omega, k_j) \,. \tag{4.52}$$

There will always exist some frame of reference where the momentum is zero ($k_j = 0$) and the energy equals the rest mass, so that $\omega = \omega_0$ where ω_0 is called the invariant frequency. When these values are substituted into the pair of equations the only remaining free parameter a can be determined in terms of the invariant frequency $ia = \pm\omega_0$ and the equation (4.48) can be written

$$i \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \partial_t \psi(x,t) = \left[-i \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \partial_j \pm \omega_0 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \right] \psi(x,t).. \tag{4.53}$$

By identifying the wave mass with the rest mass we are able to identify equation (4.53) as the Dirac equation [33, 34].

We began with the most arbitrary first order differential equation for 4-component spinors. The constant matrices were constrained to insure that the dynamical equation would be Poincare invariant for the phase function $\phi = 0$. Consequently this is the only Poincare invariant first order differential equation for $\phi = 0$.

4.2.4 Other Phase Functions

We now address the more general case where the phase function can be other than $\phi = 0$. Since ϕ is not present in the condition (4.10) there is no impact on the results obtained for B^μ in the previous section. The matrix B_c however is constrained by the condition (4.13) where the presence of ϕ will introduce the function $\zeta_\mu(\theta_i, v_j)$. Now the problem is to determine the value of the functions $\zeta_\mu(\theta_i, v_j)$ in addition to the matrix B_c.

First notice that the condition (4.13) can be simplified by the substitution of condition (4.10) and the use of the result $B^\mu = \gamma^\mu$

$$B_c = B_c' + iB'^\beta \Lambda^\mu_\beta \zeta_\mu = B_c' + iB^\mu \zeta_\mu = B_c' + i\gamma^\mu \zeta_\mu. \tag{4.54}$$

By setting $\theta_j = 0$ and $v_j = 0$ it is clear that $\zeta_\mu(0,0) = 0$. To find the rotational dependence we move the terms containing B_c to the left hand side and apply a z-rotation

$$B_c - B_c' = (B_c + \gamma_1\gamma_2 B_c \gamma_1\gamma_2)\sin^2\frac{\theta_3}{2} - (B_c\gamma_1\gamma_2 - \gamma_1\gamma_2 B_c)\sin\frac{\theta_3}{2}\cos\frac{\theta_3}{2} \tag{4.55}$$

where we have used the identities

$$\sin\theta_3 = 2\sin\frac{\theta_3}{2}\cos\frac{\theta_3}{2}, \qquad \cos\theta_3 = \cos^2\frac{\theta_3}{2} - \sin^2\frac{\theta_3}{2}, \tag{4.56}$$

and

$$\frac{1-\cos\theta_3}{2} = \sin^2\frac{\theta_3}{2}. \tag{4.57}$$

Any 4×4 matrix can be represented using the matrices I, γ^μ, $\sigma^{\mu\nu}$, $\gamma^5\gamma^\mu$, and γ^5 as a basis where

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{4.58}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{4.59}$$

$$\gamma^5\gamma^j = \begin{pmatrix} -\sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} \qquad \gamma^5\gamma^0 = i\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{4.60}$$

are all given for γ in the Dirac basis. If a matrix is required to be hermitian then the basis can be restricted to I, γ^μ, $\sigma^{\mu\nu}$, and $\gamma^5\gamma^\mu$ excluding the anti-hermitian γ^5. Since the matrix B_c must be hermitian it can be written in a basis composed of the hermitian matrices

$$B_c = aI + b_{\mu\nu}\sigma^{\mu\nu} + c_\mu\gamma^\mu + d_\mu\gamma^5\gamma^\mu \tag{4.61}$$

where

$$c_\mu \gamma^\mu = c_0\gamma_0 - c_1\gamma_1 - c_2\gamma_2 - c_3\gamma_3 \tag{4.62}$$

and where a, b_i, c_μ, and d_μ are undetermined constants. Using this basis we can calculate $B_c + \gamma_x\gamma_y B_c \gamma_x\gamma_y$ and $B_c\gamma_x\gamma_y - \gamma_x\gamma_y B_c$. We do this for the aI term first

$$aI + \gamma_1\gamma_2 aI\gamma_1\gamma_2 = 0 \tag{4.63}$$

$$aI\gamma_1\gamma_2 - \gamma_1\gamma_2 aI = 0. \tag{4.64}$$

Since the quantities vanish a is allowed to remain a free parameter which it must in order to have the Dirac equation as a possible result of the more general ϕ. For the $b_{\mu\nu}\sigma^{\mu\nu}$ term

$$b_{\mu\nu}\sigma^{\mu\nu} + \gamma_1\gamma_2 b_{\mu\nu}\sigma^{\mu\nu}\gamma_1\gamma_2 \tag{4.65}$$

$$= i\left(-2b_{10}\gamma_0\gamma_1 - b_{20}\gamma_0\gamma_2 - b_{12} + b_{12}\gamma_1\gamma_2 + 2b_{23}\gamma_2\gamma_3 - 2b_{31}\gamma_1\gamma_3\right) \tag{4.66}$$

$$b_{\mu\nu}\sigma^{\mu\nu}\gamma_1\gamma_2 - \gamma_1\gamma_2 b_{\mu\nu}\sigma^{\mu\nu} = -2i\left(b_{10}\gamma_0\gamma_2 - b_{23}\gamma_1\gamma_3 - b_{31}\gamma_2\gamma_3\right) \tag{4.67}$$

It is apparent that all but the b_{30} term must equal zero. For the $c_\mu\gamma^\mu$ term

$$c_\mu\gamma^\mu + \gamma_x\gamma_y c_\mu\gamma^\mu\gamma_x\gamma_y = -2c_1\gamma_1 - 2c_2\gamma_2 \tag{4.68}$$

and

$$c_\mu\gamma^\mu\gamma_x\gamma_y - \gamma_x\gamma_y c_\mu\gamma^\mu = -2c_1\gamma_2 + 2c_2\gamma_1. \tag{4.69}$$

These terms will survive since they are coefficients of individual γ-matrices which appear in $\gamma^m u\zeta_m u$. The $d^\mu\gamma_5\gamma_\mu$ term produces

$$d^\mu\gamma_5\gamma_\mu + \gamma_1\gamma_2 d^\mu\gamma_5\gamma_\mu\gamma_1\gamma_2 = -2id_1\gamma_0\gamma_2\gamma_3 + 2id_2\gamma_0\gamma_1\gamma_3 \tag{4.70}$$

$$d^\mu\gamma_5\gamma_\mu\gamma_1\gamma_2 - \gamma_1\gamma_2 d^\mu\gamma_5\gamma_\mu = 2id_1\gamma_0\gamma_1\gamma_3 + 2id_2\gamma_0\gamma_3 \tag{4.71}$$

These terms must vanish to satisfy invariance of the first order equation so we conclude $d_1 = 0$ and $d_2 = 0$.

Putting these results together we have

$$\gamma_x\gamma_y B_c \gamma_x\gamma_y = -c_0\gamma_0 - c_1\gamma_1 - c_2\gamma_2 + c_3\gamma_3 \tag{4.72}$$

and

$$B_c\gamma_x\gamma_y - \gamma_x\gamma_y B_c = -2c_1\gamma_2 + 2c_2\gamma_1. \tag{4.73}$$

The remaining terms must vanish and thus provide a means of eliminating some of the free parameters of equation (4.61). Inserting these results into the left hand side of (4.55) produces

$$i\gamma^\mu\zeta_\mu = B_c - B_c' = (-2c_1\gamma_1 - 2c_2\gamma_2)\sin^2\frac{\theta_3}{2} - (-2c_1\gamma_2 + 2c_2\gamma_1)\sin\frac{\theta_3}{2}\cos\frac{\theta_3}{2} \tag{4.74}$$

$$= (-c_1\gamma_1 - c_2\gamma_2)(1-\cos\theta_3) + (c_1\gamma_2 - c_2\gamma_1)\sin\theta_3 \tag{4.75}$$

The γ-matrices are linearly independent so the coefficients of each matrix must vanish independently. For a z-rotation we conclude that $\zeta_3(\theta_3) = 0$ and $\zeta_0(\theta_3) = 0$ and

$$\zeta_1 = ic_1(1-\cos\theta_3) + ic_2\sin\theta_3 \tag{4.76}$$

$$\zeta_2 = ic_2(1-\cos\theta_3) - ic_1\sin\theta_3. \tag{4.77}$$

Similar expressions can be produced for the dependence of ζ_1 and ζ_2 on the angles θ_1 and θ_2. All the free parameters of B_c in equation (4.61) will vanish except for the c_μ terms and a.

The phase function ϕ will also depend on the boost parameters v_j. To find the functional dependence of ϕ on v_j we apply boosts in equation (4.13) and attempt to solve for ϕ. Equation (4.13) is simplified by substituting in (4.10) to get

$$B_c = B_c' + iB'^\beta\Lambda^\mu_\beta\zeta_\mu = B_c' + iB^\beta\zeta_\beta. \tag{4.78}$$

Applying a Lorentz boost along the z-direction produces

$$B_c - S_z B_c S_z^{-1} = B_c - \left(\cosh\frac{\eta}{2} - i\gamma_z\gamma_0\sinh\frac{\eta}{2}\right) B_c \left(\cosh\frac{\eta}{2} + i\gamma_z\gamma_0\sinh\frac{\eta}{2}\right) \tag{4.79}$$

$$= B_c - B_c \cosh^2 \frac{\eta}{2} - \gamma_z \gamma_0 B_c \gamma_z \gamma_0 \sinh^2 \frac{\eta}{2} \tag{4.80}$$

$$-i \left(B_c \gamma_z \gamma_0 - \gamma_z \gamma_0 B_c \right) \sinh \frac{\eta}{2} \cosh \frac{\eta}{2}. \tag{4.81}$$

With the remaining free parameters and $B_c = aI + c^\nu \gamma_\nu$ we calculate

$$B_c \gamma_3 \gamma_0 - \gamma_3 \gamma_0 B_c = -2(c_0 \gamma_3 + c_3 \gamma_0) \tag{4.82}$$

and

$$\gamma_3 \gamma_0 B_c \gamma_3 \gamma_0 = c_0 \gamma_0 + c_1 \gamma + c_2 \gamma_2 - c_3 \gamma_3. \tag{4.83}$$

Then equation (4.81) is

$$i\gamma^\mu \zeta_m u = i \left[\gamma_0 \zeta_0 - \gamma_1 \zeta_1 - \gamma_2 \zeta_2 - \gamma_3 \zeta_3 \right] \tag{4.84}$$

$$= \gamma_0 \left[-2c_0 \sinh^2 \frac{\eta}{2} + 2ic_3 \sinh \frac{\eta}{2} \cosh \frac{\eta}{2} \right] + \gamma_3 \left[2c_3 \sinh^2 \frac{\eta}{2} + 2ic_0 \sinh \frac{\eta}{2} \cosh \frac{\eta}{2} \right] \tag{4.85}$$

and the z-boost dependence of ζ is $\zeta_1 = 0$, $\zeta_2 = 0$,

$$\zeta_0 = 2ic_0 \sinh^2 \frac{\eta}{2} + 2c_3 \sinh \frac{\eta}{2} \cosh \frac{\eta}{2}, \tag{4.86}$$

and

$$\zeta_3 = 2ic_3 \sinh^2 \frac{\eta}{2} - 2c_0 \sinh \frac{\eta}{2} \cosh \frac{\eta}{2}. \tag{4.87}$$

Equations (4.86-4.87) show how the phase function ϕ depends upon a boost in the z-direction. This procedure can be repeated for boosts in the x- and y-directions with similar results.

After performing boosts and rotations in condition (4.54) the only remaining free parameters for B_c from equation (4.61) are a and c_μ. No other restriction is available to eliminate these free parameters so they will appear in the first order equation. These free parameters are allowed because of the added flexibility in the state function that is afforded by the freedom to select any phase function ϕ to offset non-invariant terms generated by the transformation of the constant matrix B_c. The free parameters c_μ and a will appear in the first order equation making it more general than the Dirac equation thus we have a generalized Dirac equation

$$[\gamma^\mu \partial_\mu + aI + c_\mu \gamma^\mu]\, \psi(x,t) = 0. \tag{4.88}$$

Other generalized Dirac equations have appeared in the literature [51, 52]. These equations are of a completely different nature from equation (4.88). The generalized Dirac equation of [51] refers to a Dirac equation that was generalized to account for a particle with two mass states. The context, purpose, and method of that equations construction are not provided however the equation is a second order differential equation. As a second order equation it resembles the Klein-Gordon equation but with the additional first order parameter found in the Dirac equation. Thus it resembles the sum of a Dirac and Klein-Gordon equation. The equation described in [52] follows from a generalized uncertainty principle and exists in the context of spacetime with an assumed minimal distance on the order of the Planck length. This system is not Poincaré invariant and therefore in no way related to the work of this dissertation.

Equation (4.88) satisfies the principles of relativity and analyticity but we have not yet required that the state functions transform like irreducible representation of the Poincaré group. This will be done in the course of producing an energy momentum relation in the next section.

4.3 Analysis of the Generalized Dirac Equation

4.3.1 Alternate Form of the Generalized Dirac Equation

The first order equation can be transformed by multiplying by $i\gamma_0$, using the momentum operator $\hat{p}_i = -i\partial_i$, energy operator $\hat{\varepsilon} = i\partial_t$, and separating the time derivative from the space derivative resulting in

$$\hat{\varepsilon}\psi = \mathcal{H} = \left[\alpha_i \hat{p}_i + \tilde{a}\beta + \tilde{c_0} I + \alpha_i \tilde{c_i}\right]\psi \tag{4.89}$$

where $\alpha_i = \gamma_0\gamma_i$, $\beta = \gamma_0$, $\tilde{a} = -ia$, $\tilde{c_0} = -ic_0$, and $\tilde{c_i} = -ic_i$. In this form the right hand side can be identified as the Hamiltonian $\mathcal{H}$ acting on the state function ψ. In order to satisfy the principle of relativity the eigenvalues of $\mathcal{H}$ must be real and thus $\mathcal{H}$ must be hermitian. Since α_i and β are hermitian the hermiticity of the Hamiltonian is satisfied if $\tilde{a}$ and $\tilde{c_\mu}$ are real.

4.3.2 Non-relativistic Limit: Derivation of the Generalized Pauli Equation

Its easier to interpret the non-relativistic limit by introducing the speed of light c. The generalized Dirac equation is then

$$i\partial_t\psi = \mathcal{H} = \left[c\alpha_i \hat{p}_i + c\tilde{a}\beta + c\tilde{c_0} I + c\alpha_i \tilde{c_i}\right]\psi. \tag{4.90}$$

We now introduce the electromagnetic four potential

$$A^\mu = (A_0(x), A_j(x)) \tag{4.91}$$

which can be incorporated into the generalized Dirac equation via the minimal coupling

$$p^\mu \to p^\mu - \frac{e}{c}A^\mu \equiv \Pi^\mu \tag{4.92}$$

where Π^μ is the kinetic momentum and e is the electric charge. The generalized Dirac equation with electromagnetic potentials is

$$i\partial_t\psi = \left[c\alpha_i\left(\hat{p}_i - \frac{e}{c}A_i\right) + eA_0 + c\tilde{a}\beta + c\tilde{c_0} I + c\alpha_i \tilde{c_i}\right]\psi. \tag{4.93}$$

The four-component spinor ψ can be decomposed into a pair of two-component spinors $\tilde{\phi}$ and $\tilde{\chi}$

$$\psi = \begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} \tag{4.94}$$

The generalized Dirac equation then becomes

$$i\partial_t \begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} = c \begin{pmatrix} \sigma_i \Pi_i \tilde{\chi} \\ \sigma_i \Pi_i \tilde{\phi} \end{pmatrix} + eA_0 \begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} + c\tilde{a} \begin{pmatrix} \tilde{\phi} \\ -\tilde{\chi} \end{pmatrix} + c\tilde{c_0} \begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} + c\tilde{c_i} \begin{pmatrix} \sigma_i \tilde{\chi} \\ \sigma_i \tilde{\phi} \end{pmatrix} \tag{4.95}$$

after inserting explicit representations for the matrices α_i, β, and I.

Producing the non-relativistic limit of the generalized Dirac equation requires making some assumptions about the value of the free parameters $\tilde{a}$ and $\tilde{c_\mu}$. In the standard Dirac equation $\tilde{a} = \bar{m}c$ where m is the mass of the particle and c is the speed of light. We will assume this value for $\tilde{a}$ in order to interpret the third term on the right hand side as a rest mass energy. The similarity of the fifth term to the first term is suggestive of a momentum interpretation while the similarity between the fourth term and left hand side term suggests an energy interpretation. We will thus set $c\tilde{c_0} = -\tilde{\varepsilon}$ where the negative sign is chosen to match the sign of ε as it will appear on the left. Now if we assume $\bar{m}c^2$ is the largest energy, the spinor state function may be further split into two parts

$$\begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \exp\left[-i\bar{m}c^2 t\right]. \tag{4.96}$$

The generalized Dirac equation is now

$$i\partial_t \begin{pmatrix} \phi \\ \chi \end{pmatrix} = c \begin{pmatrix} \sigma_i \Pi_i \chi \\ \sigma_i \Pi_i \phi \end{pmatrix} + eA_0 \begin{pmatrix} \phi \\ \chi \end{pmatrix} - 2\bar{m}c^2 \begin{pmatrix} 0 \\ \chi \end{pmatrix} - \tilde{\varepsilon} \begin{pmatrix} \phi \\ \chi \end{pmatrix} + c\tilde{c_i} \begin{pmatrix} \sigma_i \chi \\ \sigma_i \phi \end{pmatrix}. \tag{4.97}$$

If we consider the second component of the equation when kinetic and potential energies are small compared to the rest mass energy i.e. when $|i\partial\chi/\partial t| \ll |\bar{m}c^2\chi|$, $|\varepsilon\tilde{\chi}| \ll |\bar{m}c^2\chi|$, and $|eA_0\chi| \ll |\bar{m}c^2\chi|$ then we find that

$$\chi = \frac{\sigma_j\left(\Pi_j + \tilde{c_j}\right)}{2\bar{m}c}\phi. \tag{4.98}$$

This relation can be substituted into the first component equation to produce the generalized Pauli equation

$$i\partial_t\phi = \left[\frac{1}{2\bar{m}}\left(p_j - \frac{e}{c}A_j + \tilde{c_j}\right)^2 - \frac{e\hbar}{2\bar{m}c}\sigma_j B_j + \frac{}{2\bar{m}}\sigma_j\epsilon_{jkl}\partial_k\tilde{c_l} + eA_0 - \tilde{\varepsilon}\right]\phi. \tag{4.99}$$

For a constant $\tilde{c_j}$ the curl term vanishes leaving one difference from the original Pauli equation. Here the $\tilde{c_j}$ term contributes to the time rate of change of ϕ in the same way as momentum.

4.3.3 Non-relativistic Limit: Derivation of the Generalized Lévi-Leblond Equation

Returning to the first component equation of (4.97), using the energy eigen-operator $\hat{\varepsilon} = i\partial_t$ and the eigen-equation $\hat{\varepsilon}\psi = \varepsilon\psi$, and setting the electromagnetic potential and charge to zero we obtain the first half of the generalized Lévy-Leblond equation. Doing the same with equation (4.98) produces the second half of the generalized Lévy-Leblond equation.

$$(\varepsilon + \tilde{\varepsilon})\phi = \sigma_j\left(p_j + \tilde{c_j}\right)\chi \tag{4.100}$$

$$2\bar{m}\chi = \sigma_j\left(p_j + \tilde{c_j}\right)\phi. \tag{4.101}$$

4.3.4 Deriving the Generalized Klein-Gordon Equation

A second order equation such as the Klein-Gordon equation [49, 50] can be constructed by applying the operators of the generalized Dirac equation to the state function a second time. Starting with equation (4.90) and making substitutions $\tilde{a} = \bar{m}$, $\tilde{c_0} = -\tilde{\varepsilon}$, $\tilde{c_j} = \tilde{p}_j$, and $\hat{p}_j = -i\partial_j$ produces the first order equation

$$i\partial_t\psi = \left[-i\alpha_j\partial_j + \bar{m}\beta - \tilde{\varepsilon} + \alpha_j\tilde{p}_j\right]\psi \tag{4.102}$$

After apply the operators to the state function a second time the equation is produced

$$-\partial_t^2\psi = -\partial_j^2\psi + \bar{m}^2\psi + \tilde{p}_j^2\psi + \tilde{\varepsilon}^2\psi - 2i\tilde{p}_j\partial_j\psi - 2\bar{m}\tilde{\varepsilon}\beta\psi$$
$$+2i\tilde{\varepsilon}\alpha_j\partial_j\psi - 2\tilde{\varepsilon}\tilde{p}_j\alpha_j\psi \tag{4.103}$$

This is the generalized Klein-Gordon equation. The terms are all hermitian and the equation is Poincaré invariant.

4.3.5 Free Motion of the Dirac Particle and Momentum Energy Relation

To examine the free motion of a particle described by the generalized Dirac equation (4.102) we look for stationary states $\psi(x)$ that have no time dependence. The time dependent part of the wave function is assumed to be separable from the stationary state such that

$$\psi(x,t) = \psi(x)exp[-i\varepsilon t]. \tag{4.104}$$

Under this condition the generalized Dirac equation is a time independent equation of the stationary state

$$\varepsilon\psi(x) = \left[-i\alpha_j\partial_j + \bar{m}\beta - \tilde{\varepsilon} + \alpha_j\tilde{p}_j\right]\psi(x). \tag{4.105}$$

Splitting the four-component spinor into a pair of two-component spinors and using explicit representations for the matrices α and β produces from equation (4.105) the pair of equations

$$\varepsilon\phi = \sigma_j p_j\chi + \bar{m}\phi - \tilde{\varepsilon}\phi + \sigma_j\tilde{p}_j\chi \tag{4.106}$$

and

$$\varepsilon\chi = \sigma_j p_j \phi - \bar{m}\chi - \tilde{\varepsilon}\chi + \sigma_j \tilde{p}_j \phi. \tag{4.107}$$

States with a defined momentum p_j

$$\begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \phi_0 \\ \chi_0 \end{pmatrix} exp\left[i\hat{p}_j x_j\right] \tag{4.108}$$

can be substituted into the equations (4.106)-(4.107) while replacing the operator $\hat{p}_j$ with its eigenvalues p_j.

$$\left(\varepsilon + \tilde{\varepsilon} - \bar{m}\right)\phi_0 - \sigma_j\left(p_j + \tilde{p}_j\right)\chi_0 = 0 \tag{4.109}$$

$$-\sigma_j\left(p_j + \tilde{p}_j\right)\phi_0 + \left(\varepsilon + \tilde{\varepsilon} + \bar{m}\right)\chi_0 = 0 \tag{4.110}$$

This system of equations admits non-trivial solution only when

$$\left(\varepsilon + \tilde{\varepsilon} - \bar{m}\right)\left(\varepsilon + \tilde{\varepsilon} + \bar{m}\right) - \left[\sigma_j\left(p_j + \tilde{p}_j\right)\right]^2 = 0. \tag{4.111}$$

Simplifying the equation to

$$\left(\varepsilon + \tilde{\varepsilon}\right)^2 - \bar{m}^2 - \left(p_j + \tilde{p}_j\right)^2 = 0 \tag{4.112}$$

leads to the solution

$$\varepsilon = -\tilde{\varepsilon} \pm \sqrt{\bar{m}^2 + (p_j + \tilde{p}_j)^2}. \tag{4.113}$$

This is an expression for the energy of a stable state system of the generalized Dirac equation.

4.4 Discussion

In this chapter, we searched for fundamental, Poincaré invariant, dynamical equations describing free elementary particles that were represented by four component spinor state functions. The original Dirac equation was obtained and shown to be the only Poincaré invariant dynamical equation for 4-component spinor state functions when the phase function is $\phi = 0$. A more important result was the discovery of a new fundamental Poincaré invariant equation, which was called the generalized Dirac equation (4.88-4.89) because the original Dirac equation is obtained from it as a special case when the phase function ϕ is set to zero. To be more specific, the standard results are obtained when the phase function $\phi = 0$ requires $\zeta_\mu = 0$ which in turn requires the free parameters to vanish $c_\mu = 0$. Without these free parameters the generalized Dirac equation becomes the standard Dirac equation.

To check the validity of this generalized Dirac equation, we evaluated it in the non-relativistic limit and showed that the generalized Lévy-Leblond equation (4.100-4.101) and the generalized Pauli-Schrödinger equation (4.99) were obtained. Moreover, we also derived a second order equation that was called the generalized Klein-Gordon (KG) equation (4.103). It was also found that the generalized KG equation reduced to the standard KG equation when the phase function ϕ is set to zero.

A new and important result is that the generalized Dirac equation gives an energy-momentum relationship that is different than that required by the Special Theory of Relativity (STR). The well-known STR energy-momentum relation is obtained when ϕ is set to zero. Differences between the new and STR energy-momentum relationships indicate that the new relation does not describe ordinary matter, which means that the generalized Dirac equation provides new physics that goes beyond the standard model.

There are several ways to interpret the generalized Dirac equation. One way is to consider the free parameter $\tilde{c}_0$ is a new energy term that we can call $\tilde{\varepsilon}$ while the free parameter $\tilde{c}_j$ is a new momentum term that can be called $\tilde{p}_j$ and the free paramter a is a mass term that can be called $\bar{m}$. While the first two terms are energy-like and momentum-like they are different from the standard energy and momentum terms because the free parameters do not correspond to the existing energy $\hat{\varepsilon}$ and momentum $\hat{p}_j$ operators. It has already been established by the eigen-equations that measurement of the energy produces the eigen-value ε and measurement of the momentum produces the eigen-value p_j. These quantities can be combined into the four-momentum $p_\mu = (\varepsilon, p_j)$. The contraction of the four-momentum on itself produces an invariant quantity that can be assigned a label m that may be called the rest mass

$$p^\mu p_\mu = \varepsilon^2 - p_j^2 = m^2. \tag{4.114}$$

The four-momentum of the alternative energy and momentum can be defined as $\tilde{p}_\mu = (\tilde{\varepsilon}, \tilde{p}_j)$, which we will call the alternative four-momentum. The contraction of the alternative four-momentum on itself then produces a constant of the motion that can be assigned a label $\tilde{m}$ that may be called the alternative rest mass

$$\tilde{p}_\mu \tilde{p}^\mu = \tilde{\varepsilon}^2 - \tilde{p}_j^2 = \tilde{m}^2. \tag{4.115}$$

Another four-momentum $\bar{p}_\mu$ can be determined from the momentum-energy relation by solving for $\bar{m}^2$ to get

$$\bar{p}^\mu \bar{p}_\mu = (\varepsilon + \tilde{\varepsilon})^2 - (p_j + \tilde{p}_j)^2 = \bar{m}^2. \tag{4.116}$$

From this the total four-momentum of the particle is

$$\bar{p}_\mu = (\varepsilon', \bar{p}_j) = (\varepsilon + \tilde{\varepsilon}, p_j + \tilde{p}_j)\,. \tag{4.117}$$

where we have defined the total energy $\varepsilon' = \varepsilon + \tilde{\varepsilon}$ and total momentum $\bar{p}_j = p_j + \tilde{p}_j$. Furthermore, from the sum of the four-momenta

$$\bar{p}_\mu = p_\mu + \tilde{p}_\mu \tag{4.118}$$

it follows that the total mass is $\bar{m} = m + \tilde{m}$.

A consistent treatment requires that all measurable quantities of a quantum theory must correspond to eigen-values that have associated operators. Thus we could introduce an operator $\hat{\tilde{\varepsilon}} = i\partial_{\tilde{t}}$ associated with the new alternative energy and an operator $\hat{\tilde{p}}_j = i\partial_{\tilde{j}}$ associated with the new alternative momentum values. For each operator there is an eigen-equation and using these eigen-equations the generalized Dirac equation can be transformed into an equation with two energy operators and two momentum operators.

$$\left[\hat{\varepsilon} + \hat{\tilde{\varepsilon}}\right]\psi = \left[\alpha_i \hat{p}_i + (m + \tilde{m})\beta + \alpha_i \hat{\tilde{p}}_i\right]\psi \tag{4.119}$$

In short it looks like the sum of two Dirac equations or a Dirac equation governing the evolution of two distinct particles. Since the energy and momentum terms may be split as many times as you like the equation can be interpretted as governing a multi-particle system. In this interpretation, the only thing that makes this equation different from the standard Dirac equation is that the existence of the alternative particle depends upon a non-zero phase function. It is perhaps an interesting result that non-zero phase functions are possible but yield the same results as could be found with the standard Dirac equation.

A simpler way to interpret the generalized Dirac equation is to insist that the eigen-values ε and p_j represent the total energy and momentum of the particle. In this case there is no escaping the conclusion that $\varepsilon = \varepsilon + \tilde{\varepsilon}$ and $p_j = p_j + \tilde{p}_j$ thus the new energy and momentum terms vanish $\tilde{\varepsilon} = 0$, $\tilde{p}_j = 0$ and the generalized Dirac equation reduces to the standard Dirac equation.

A more radical interpretation is to suppose that the alternative energy and alternative momentum values are not directly measurable. Measurement of the momentum would result in the standard eigenvalues of momentum p_j while measurement of the energy would result in the standard eigenvalue of energy ε. But if one then used the standard momentum-energy relation $\varepsilon^2 = m^2 + p_j^2$ to find the mass m of the particle the result would be wrong. The particle must obey the new momentum-energy relation so it would behave as if it had additional mass $\tilde{m}$. In this interpretation the new momentum-energy relation (4.113) requires a generalized Dirac particle to behave in a new way.

The fact that the new energy-momentum relationship does not describe ordinary matter is an interesting result that is used here the generalized Dirac equation may govern the evolution of a new form of matter that might fit the characteristics of DM. A more detailed discussion of this idea is given in Chapter 5.

4.5 Summary

The search for new Poincaré invariant equations described in this Chapter allowed us to formally derive the original Dirac equation and a generalized Dirac equation. The latter was used to derive a second order equation that was identified as a generalized KG equation. The non-relativistic limit of these equations was evaluated and as expected the generalized Lévy-Leblond equation and the original KG equations were obtained. It was also demonstrated how to reduce the generalized Dirac equation to the generalized Pauli-Schrödinger equation.

A new energy-momentum relation was derived from the analysis of stationary states of the generalized Dirac equation. The new energy-momentum relationship was used to show that the behavior of a particle obeying the generalized Dirac equation would be different from that of a particle governed by the standard Dirac equation because of the existence of additional momentum and energy terms. Since this new energy-momentum relationship differs from the well-known energy-momentum relationship of the Special Theory of Relativity, it cannot describe ordinary matter.

Hence, it was suggested that the new energy-momentum relationship represents a different form of matter that may be identified as DM.

CHAPTER 5

CONCLUSION

5.1 Theoretical Predictions Based on the New Fundamental Equations

The main purpose of the research described in this dissertation was to find new fundamental Poincaré and Galilei invariant dynamical equations for free particles, which are represented by spinor state functions. The motivation for finding new equations was to predict new particles that could explain DM. To explain DM requires new physics that must go beyond the Standard Model. In the previous proposals, DM was explained by suggesting the existence of new kinds of fields or particles (see Sec. 1.1), however, the new results presented in this dissertation are unique because they clearly identified new fundamental equations that have been overlooked in the previous studies (see Sec. 2.1 and 4.1).

Particles governed by the generalized fundamental equations obtained in this research could make good candidates for DM. The standard Dirac equation describes the behavior of many known particles within the standard model while the generalized Dirac equation encompasses these known particles and may also describe new particles that are beyond the Standard Model. A particle with low or zero rest mass similar to the neutrino is one such possibility.

Neutrinos are spin half particles with no charge and a small mass. While it is currently believed that neutrinos have a small mass they were originally proposed as zero mass particles. Neutrinos have no electric charge and are detected through weak force interactions. But even though neutrinos are hard to see they do not make good DM candidates because the quantity of neutrinos predicted to exist in the universe can only account for a small fraction of the observed DM. Another problem is that neutrinos have very low mass and thus move at high velocities. This prevents them from forming galactic halo distributions such as have been determined by combining astronomical observations with many body numerical simulations.

A particle governed by the generalized Dirac equation may provide a better DM candidate. The new constants present in the generalized Dirac equation produce new terms in the the resulting energy-momentum relation. The presence of these constants in the momentum-energy relation allow us to infer several things about a potential DM particle. First, the constant $\tilde{c}_0$ is a new energy term while $\tilde{c}_j$ is a new momentum term. The source of these alternative momentum and energy terms is unknown but they create the possibility of a particle with new or hidden energy and momentum.

A particle governed by the generalized Dirac equation may make a better DM candidate because it can have alternative mass. A particle of this kind could resemble a neutrino (having spin half, zero electric charge, and a negligible standard mass) but have a high alternative mass. With characteristics like this the particle would be difficult to detect since it only interacts gravitationally but unlike the neutrino the additional alternative mass would allow for a slower moving particle that could form galactic mass distributions like those that are observed.

5.2 Summary

A search for fundamental (Galilean and Poincaré invariant) dynamical equations in space-time with the Galilei and Minkowski metrics was conducted for free elementary particles described by spinor state functions. The primary novel products of this research are four new Galilei and Poincaré invariant spinor equations. The work produced the generalized Lévy-Leblond and generalized Schrödinger equations for Galilean space-time. The work also produced the generalized Dirac and generalized Klein-Gordon equations for Minkowski space-time. The work of this dissertation was also unique because it focused on finding dynamical spinor equations by a different approach than the previous research [16, 17] that focused on scalar fields and used eigen equations to derive dynamical laws.

The main results obtained for Galilean space-time with the Schrödinger phase factor are: (i) there are no fundamental dynamical equations for two-component spinor wave functions; (ii) the Lévy-Leblond equation for a four-component spinor wave function can be derived by using a different method than the one originally used by Lévy-Leblond; (iii) a formal proof that the Lévy-Leblond and Schrödinger equations are the only fundamental dynamical equations for four-component spinor wave functions in Galilei space-time. Among important physical implications of the obtained results is that the Pauli-Schrödinger equation is not a fundamental equation in Galilei space-time.

A new fundamental (Galilean invariant) dynamical equation for state functions being four component spinors was found. The equation describes free particles represented by such state functions. Since in a special case this new equation reduces to the standard Lévy-Leblond equation, it is called here the generalized Lévy-Leblond equation. The main difference between these equations is that they are obtained with different phase factors and that energy-momentum relationships resulting from the equations are also different. This dependence of the non-relativistic energy-momentum relationships on the phase factors is a new phenomenon discovered here. Moreover, the obtained results clearly show that the energy-momentum resulting from the generalized Lévy-Leblond equation is not the same as the well-known non-relativistic energy-momentum relationship.

The search for new Poincaré invariant equations described in this dissertation allowed us to formally derive the original Dirac equation and also the so-called generalized Dirac equation. The latter was used to derive a second order equation that was identified as the generalized KG equation. The non-relativistic limit of these equations was considered and as expected the generalized Lévy-Leblond equation and the original KG equations were obtained. It was also demonstrated how to reduce the generalized Dirac equation to the generalized Pauli-Schrödinger equation.

A new energy-momentum relation was derived from the analysis of stationary states of the generalized Dirac equation. The new energy-momentum relationship was used to show that the behavior of a particle obeying the generalized Dirac equation would be different from that of a particle governed by the standard Dirac equation because of the existence of additional momentum and energy terms. Since such new energy-momentum relationship differs from the well-known energy-momentum relationship of Special Theory of Relativity, it cannot describe ordinary matter. Hence, it was suggested that the new energy-momentum relationship represents a different form of matter that may be identified as DM.

5.3 Future Research

Further work is required to fully explore the consequences and possibilities of the new dynamical equations. The first step would be to quantize the obtained fundamental equations and determine physical properties of free particles resulting from quantization of the fields. Since the particle's properties will depend on some constants that are present in the theory, experiments must be suggested that could measure the values of these constants. Once the physical properties of the free particles are determined, the form of interaction between the particles could be established and then incorporated into the new theory. If the particles do account for DM, then gravity will be the only force acting between them. This will require formulating quantum theories on galactic scales.

Hopefully, the results obtained in this dissertation may be used in other areas of physics, such as solid state physics, where the method of deriving new dynamical equations may be applied to finding dynamical laws governing quasiparticles. Quasiparticles are an emergent phenomena that result when a many-body system is in a low-level excited state. The excitation can be treated as a discrete particle interacting with its surroundings in place of treating the more complicated many-body problem [53]. Some examples of quasiparticles are electron holes (missing electrons), phonons (quantized vibrations of repetitive structures), magnons (a collective excitations of electron spin structures in a crystal lattice), as well as excitons, plasmons, polaritons, politons, rotons, trions, etc. Composite fermions may be of special relevance to this work. These are quasiparticles that may have less than an electron charge for example, fractional charges like e/3, e/4, e/5, and e/7 are possible. Composite fermions can also be anyons, quasiparticles that are neither bosons nor fermions. Future work will determine whether the generalized Dirac equation derived in this dissertation will be useful for explaining these kinds of quasiparticles.

APPENDIX A

DERIVING GALILEI INVARIANT SPINOR EQUATIONS

In appendix A concepts relevant to the Galilei group are developed.

A.1 Galilei Transformations

Classical and non-relativistic quantum mechanics take place in Galilean space-time. Mathematically space-time is a manifold upon which is defined a metric that provides infinitesimal measures of space and time intervals. Galilean space-time is defined by the Galilei metric

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{A.1}$$

$$d\tau = dt \tag{A.2}$$

which displays the familiar dicotomy of space and time.

Galilean space-time also displays several symmetries that can be identified by the transformations that leave the metric invariant. These Galilei transformations

$$\vec{x} \to \vec{x}' = R\vec{x} + \vec{v}t + \vec{a} \tag{A.3}$$

$$t \to t' = t + b \tag{A.4}$$

relate the coordinates $\vec{x}, t$ of one observer to those of another observer $\vec{x}', t'$ by a velocity boost vector $\vec{v}$, a spatial translation vector $\vec{a}$, a time translation scalar b, and a rotation matrix $R(\vec{\theta})$ that is a function of three rotation angles $\vec{\theta}$.

A.2 The Galilei Group

The Galilei transformations form a group called the Galilei group $\mathcal{G}$ with transformation operator $G(\vec{a}, b, \vec{v}, R(\theta))$. The group composition law can be worked out by inserting one transformation into another thus

$$\vec{x}'' = R_2\vec{x}' + \vec{v}_2 t' + \vec{a}_2 \tag{A.5}$$

$$= R_2(R_1\vec{x} + \vec{v}_1 t + \vec{a}_1) + \vec{v}_2(t + b_1) + \vec{a}_2 \tag{A.6}$$

and after regrouping terms

$$= (R_2R_1)\vec{x} + (R_2\vec{v}_1 + \vec{v}_2)t + (R_2\vec{a}_1 + \vec{v}_2 b_1 + \vec{a}_2). \tag{A.7}$$

The combination can be identified as a single transformation from which the group composition law is apparent

$$G(\vec{a}_2, b_2, \vec{v}_2, R_2)G(\vec{a}_1, b_1, \vec{v}_1, R_1) \tag{A.8}$$

$$= G(R_2\vec{a}_1 + \vec{v}_2 b_1 + \vec{a}_2, b_1 + b_2, R_2\vec{v}_1 + \vec{v}_2, R_2R_1). \tag{A.9}$$

This group composition law is an expression of the group multiplication operation and shows how elements of the group behave when applied in conjunction.

It can now be proven that the Galilei transformations form a group by showing that the elements satisfy all the properties of a group. The group composition law already demonstrates the property of closure. It can also be used to demonstrate the property of associativity

$$G(\vec{a}_3, b_3, \vec{v}_3, R_3)G(R_2\vec{a}_1 + \vec{v}_2 b_1 + \vec{a}_2, b_1 + b_2, R_2\vec{v}_1 + \vec{v}_2, R_2R_1) \tag{A.10}$$

$$= G(R_3\vec{a}_2 + \vec{v}_3 b_2 + \vec{a}_3, b_2 + b_3, R_3\vec{v}_2 + \vec{v}_3, R_2R_2)G(\vec{a}_1, b_1, \vec{v}_1, R_1). \tag{A.11}$$

Additionally the identity property requires the existence of the identity element

$$E = G(0, 0, 0, I) \tag{A.12}$$

where the identity matrix I is used for rotations with a zero angle. Lastly the inverse property is satisfied by the existence of a unique inverse

$$G^{-1} = G(-R^{-1}(\vec{a} - \vec{v}b), -b, -R^{-1}\vec{v}, R^{-1}) \tag{A.13}$$

which corresponds to the inverse Galilei transformation

$$\vec{x} = R^{-1}\vec{x}' - R^{-1}\vec{v}t' - R^{-1}\vec{a} + R^{-1}\vec{v}b \tag{A.14}$$

$$t = t' - b. \tag{A.15}$$

A.3 The Extended Galilei Group

The Galilei metric can be extended by adding an additional measure to the metric. This measure is the norm of the state function $\psi(\vec{x}, t)$. The norm of the state function is invariant under Galilei transformations

$$|\psi^* \psi| = |\psi'^* \psi'| \,. \tag{A.16}$$

This means that measurements of the norm of the state function must produce the same result for all observers related by the Galilei transformations. From the invariance of the norm the transformation of the state function is found to be

$$\psi(\vec{x}, t) \rightarrow \psi'(\vec{x}', t') = e^{i\phi(\vec{x},t)}\psi(\vec{x}, t). \tag{A.17}$$

where the gauge function $e^{i\phi(\vec{x},t)}$ contains the phase function $\phi(\vec{x}, t)$ that is an unknown function of space and time. Adding this transformation to the Galilei group forms the extended Galilei group $\mathcal{G}_e$ which will provide some advantages when worked with as we shall see later.

A.4 Group Decompositions

The Galilei group may be decomposed into subgroups such that

$$\mathcal{G} = [T(1) \otimes R(3)] \otimes_s [T(3) \otimes B(3)] \ \ , \tag{A.18}$$

where $T(1)$, $R(3)$, $T(3)$, and $B(3)$ are the subgroups of translation in time, rotations in space, translations in space, and boosts respectively. The direct product and semi-direct product are denoted $\otimes$ and $\otimes_s$.

It has been demonstrated for scalar wave functions that the Galilei group does not lead to any dynamical equations that satisfy the principles of analyticity and relativity [16]. Therefore an additional symmetry $|\psi^*\psi| = |\psi'^*\psi'|$ must be added to the group of the metric.

The expanded symmetry group called the extended Galilei group is the universal covering group of the Galilei group [27, 28]. The extended Galilei group exhibits structure that is similar to the Poincare group [21, 29, 17]. The arguments used in [16] for scalar wave functions apply equally well to n-component functions such as spinors and vectors. Consequently, we use the extended Galilei group, which has the structure

$$\mathcal{G}_e = [R(3) \otimes_s B(3)] \otimes_s [T(3+1) \otimes U(1)] \ , \tag{A.19}$$

where $U(1)$ is a one-parameter unitary group [21]. We consider only the proper isochronal subgroup $\mathcal{G}$ of $\mathcal{G}_e$ which omits the space and time inversions which can be treated separately.

A.5 Transforming Scalar Functions

A coordinate transformation is defined as a set of linear functions g_α that perform a mapping from one set of coordinates x_α to another set of coordinates x'_α according to a set of transformation parameters Π_j such that

$$x'_\alpha = g_\alpha(x_\alpha, \Pi_j). \tag{A.20}$$

The inverse coordinate transformation performs the inverse of the mapping and can be written as the inverse of the transformation function or as the transformation function of the inverse parameters

$$x_\alpha = g_\alpha^{-1}(x'_\alpha, \Pi_j) = g_\alpha(x'_\alpha, \Pi_j^{-1}). \tag{A.21}$$

A scalar function $F(x_\alpha)$ that is transformed and written in transformed coordinates is identical to the original function in the original coordinates

$$F'(x'_\alpha) = F'\left(g(x_\alpha)\right) = F(x_\alpha). \tag{A.22}$$

A transformation operator T_Π that is a linear operator and function of the parameters Π_j can be defined by its effect on a function

$$F'(x_\alpha) = T_\Pi F(x_\alpha). \tag{A.23}$$

A consequence of the two previous definitions is that the effect of a transformation operator on a function is equal to the function given in its inversely transformed coordinates

$$T_\Pi F(x_\alpha) = F(g^{-1}(x_\alpha, \Pi_j)). \tag{A.24}$$

For consistency notice that the transformation of the coordinates is itself a function and when the transformation operator is applied it inverts the coordinate transformation

$$T_{\Pi_j} x'_\alpha = T_{\Pi_j} g_\alpha(x_\alpha) = g\left(g_\alpha^{-1}(x_\alpha)\right) = x_\alpha. \tag{A.25}$$

A.6 Transforming Multi-Component Functions

The transformation operator defined in the previous section was for scalar quantities. When dealing with multi-component objects such as spinors or vectors each component may transform differently. Since we are assuming the transformation operators are linear they can be represented by a matrix. The effect of the transformation operator T on a multi-component object $F_\beta(x_\alpha)$ then looks like a combination of a matrix multiplication by matrix $M_{\beta\gamma}$ on the components of F and an inverse coordinate transformation of the functions in each component

$$F'(x_\alpha) = TF_\beta(x_\alpha) = M_{\beta\gamma} F_\beta(g^{-1}(x_\alpha)). \tag{A.26}$$

This is commonly written in terms of the transformed coordinates

$$F'(x'_\alpha) = TF_\beta(x'_\alpha) = M_{\beta\gamma}F_\beta(x_\alpha). \tag{A.27}$$

A.7 Transforming Operators

An operator O is transformed by a combination of covariant and contravariant transformations

$$O'(x_\alpha) = TO(x_\alpha)T^{-1} = MO\left(g^{-1}(x_\alpha)\right)M^{-1}. \tag{A.28}$$

The transformation operator transforms by this rule as well but remains invariant under its own action

$$T' = TTT^{-1} = T. \tag{A.29}$$

When a covariant and contravariant transformation is applied to a scalar function $f(x)$ the function can be treated as an operator by introducing the identity operator I

$$Tf(x_\alpha)T^{-1} = T\left[If(x_\alpha)\right]T^{-1} = MIf(g^{-1}(x_\alpha))M^{-1} = f\left(g^{-1}(x_\alpha)\right). \tag{A.30}$$

A.8 Galilei Transformation of Differential Operators

The chain rule can be used to determine how differential operators $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x_i}$ transform. For a Galilean transformation G operating on the differential operators

$$G\frac{\partial}{\partial t}G^{-1} = \frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial x^i}{\partial t'}\frac{\partial}{\partial x^i} = \frac{\partial}{\partial t} + R_{ji}v_j\frac{\partial}{\partial x^i} = \frac{\partial}{\partial t} + Rv\cdot\nabla \tag{A.31}$$

$$G\frac{\partial}{\partial x^i}G^{-1} = \frac{\partial}{\partial x'^i} = \frac{\partial t}{\partial x'^i}\frac{\partial}{\partial t} + \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j} = R_{ji}\frac{\partial}{\partial x^j} \tag{A.32}$$

where $\frac{\partial}{\partial t'}$ and $\frac{\partial}{\partial x'^i}$ are the partial differentials in the primed frame. The same equation can be expressed in vector notation

$$G\nabla G^{-1} = R\nabla. \tag{A.33}$$

Noting that $R\vec{v} \cdot R\vec{v} = \vec{v} \cdot \vec{v}$ for any vector $\vec{v}$ we conclude that

$$G\nabla^2 G^{-1} = G\frac{\partial}{\partial x^i} G^{-1} G\frac{\partial}{\partial x^i} G^{-1} \tag{A.34}$$

$$= \frac{\partial}{\partial x'^i}\frac{\partial}{\partial x'^i} = R_{ki}\frac{\partial}{\partial x^k} R_{ji}\frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i} = \nabla^2. \tag{A.35}$$

Extending this argument to higher powers we find that for even powers of m the gradient operator transforms like

$$G\nabla^m G^{-1} = \nabla^m \tag{A.36}$$

while odd powers of m produce

$$G\nabla^m G^{-1} = \nabla^{m-1} R\nabla. \tag{A.37}$$

Transformation of the time differential to a power n is

$$G\left(\frac{\partial}{\partial t}\right)^n G^{-1} = \left(\frac{\partial}{\partial t} + R\vec{v} \cdot \nabla\right)^n. \tag{A.38}$$

A.9 Deriving Differential Generators

A generator X_Π is defined through its operation on a function $g(x)$

$$X_\Pi g(x) = i \lim_{\Pi \to 0} \left[\frac{g(x') - g(x)}{\Pi}\right]. \tag{A.39}$$

The differential generator can be calculated from the coordinate transformation $x'_j = f_j(x_i, \Pi_k)$ as follows

$$X_{\Pi_k} = -i \left.\frac{\partial f'_j(x_i, \Pi_l)}{\partial \Pi_k}\right|_{\Pi_k=0} \frac{\partial}{\partial x'^j} = -i \left.\frac{\partial f'_j(x_i, \Pi_l)}{\partial \Pi_k}\right|_{\Pi_k=0} \frac{\partial}{\partial x^j}. \tag{A.40}$$

Note that the transformation near the origin is equal to the inverse transformation near the origin so the prime and unprimed coordinates can be exchanged

$$X_{\Pi_k} = -i \left[\left.\frac{\partial x'}{\partial \Pi_k}\right|_{\Pi_k=0} \frac{\partial}{\partial x'} + \left.\frac{\partial y'}{\partial \Pi_k}\right|_{\Pi_k=0} \frac{\partial}{\partial y'} + \left.\frac{\partial z'}{\partial \Pi_k}\right|_{\Pi_k=0} \frac{\partial}{\partial z'}\right]. \tag{A.41}$$

A.10 Differential Generators of the Galilei and Extended Galilei Groups

The differential generators of the Galilei group can be calculated from the Galilei transformation functions. The differential generators for translations in space, translations in time, boosts, and rotations are

$$X_{a_k} = -i\frac{\partial}{\partial x_k}, \tag{A.42}$$

$$X_{a_0} = X_b = -i\frac{\partial}{\partial t}, \tag{A.43}$$

$$X_{v_k} = -it\frac{\partial}{\partial x_k}, \tag{A.44}$$

and

$$X_{\theta_l} = -i\left[x_j\frac{\partial}{\partial x_k}\right]\epsilon_{jkl}. \tag{A.45}$$

A.11 Commutation Relations of the Galilei and Extended Galilei Groups

The differential generators can be used to determine the commutation relations of the group. For this particular operation its easier to calculate the commutators by writing the generators in terms of momentum p_i. The differential generators in momentum representation are

$$X_{a_k} = p_k = -i\frac{\partial}{\partial x_k}, \tag{A.46}$$

$$X_{a_0} = p_0 = -i\frac{\partial}{\partial t}, \tag{A.47}$$

$$X_{v_k} = B_k = tp_k, \tag{A.48}$$

and

$$X_{\theta_l} = J_{\theta_l} = x_j p_k \epsilon_{jkl}. \tag{A.49}$$

From these generators we determine the commutation relations are

$$[p_j, p_k] = 0, \tag{A.50}$$

$$[p_j, p_0] = 0, \tag{A.51}$$

$$[p_0, p_0] = 0, \tag{A.52}$$

$$[B_j, p_k] = 0, \tag{A.53}$$

$$[B_j, B_k] = 0, \tag{A.54}$$

$$[t, p_0] = \left[t, -i\frac{\partial}{\partial t}\right] = -i\left[t, \frac{\partial}{\partial t}\right] = -i\left(t\frac{\partial}{\partial t} - 1 - t\frac{\partial}{\partial t}\right) = i, \tag{A.55}$$

$$[t, p_j] = 0, \tag{A.56}$$

$$[x_k, p_j] = \left[x_k, -i\frac{\partial}{\partial x_j}\right] = -i\left[x_k, \frac{\partial}{\partial x_j}\right] = -i\left(x_k\frac{\partial}{\partial x_j} - \delta_{kj} - x_k\frac{\partial}{\partial x_j}\right) = i\delta_{kj}, \tag{A.57}$$

$$[x, p_0] = 0, \tag{A.58}$$

$$[B_k, p_0] = [tp_k, p_0] = t\,[p_k, p_0] + [t, p_0]\,p_k = ip_k, \tag{A.59}$$

$$[J_{\theta_l}, J_{\theta_o}] = [x_j p_k \epsilon_{jkl}, x_m p_n \epsilon_{mno}] \tag{A.60}$$

$$= (x_j[p_k, x_m p_n] + [x_j, x_m p_n]p_k)\,\epsilon_{jkl}\epsilon_{mno} \tag{A.61}$$

$$= (x_j x_m[p_k, p_n] + x_j[p_k, x_m]p_n + x_m[x_j, p_n]p_k + [x_j, x_m]p_n p_k)\,\epsilon_{jkl}\epsilon_{mno} \tag{A.62}$$

$$= (-ix_j p_n + ix_m p_k)\,\epsilon_{jkl}\epsilon_{mno} \tag{A.63}$$

$$= iJ_{\theta_p}\epsilon_{lop}, \tag{A.64}$$

and

$$[B_k, J_{\theta_o}] = [tp_k, x_m p_n \epsilon_{mno}] = t\,[p_k, x_m]\,p_n\epsilon_{mno} = -i\delta_{mk}tp_n\epsilon_{mno} = iB_n\epsilon_{kon}. \tag{A.65}$$

For every set of commutation relations there is one universal covering group that is simply connected. The extended Galilei group is the universal covering group for the Galilei group.

A.12 Spinor Representations of the Extended Galilei Group

A representation is a set of matrices or linear transformations on a vector space that is homomorphic to the group multiplication law. One way to find a representation is to construct a set of generators that obey the commutation relations of the group. The generators are chosen to have a number of dimensions equal to that of the desired representation. However, the generators may form a representation of a different group from the one that was started with because several groups may share the same set of commutation relations (or Lie algebra). To insure the representation functions properly the group composition law should be examined and compared to that of the original group.

For the extended Galilei group we shall attempt to construct 2- and 4-dimensional spinor representations starting with 2-component spinors. The Pauli and identity matrices σ_i, I form a basis for 2×2 unitary matrices. It is evident that $\frac{\sigma_i}{2}$ are generators of rotation because they obey the commutation relations of the rotation operators

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\sigma_k}{2}. \tag{A.66}$$

Boosts form an abelian subgroup of the Galilei group so that by themselves boosts can be represented by diagonal matrices. However while boosts form an invariant subgroup within the Galilei group the same cannot be said of rotations. Therefore when boosts combine with rotations it is via a semi-direct product $R(3) \otimes_s B(3)$ and a 2×2 boost generator matrix is required.

Finding a set of three 2×2 unitary matrices that satisfy all the commutation relations of boosts and rotations is impossible. Therefore there is no 2×2 matrix generators for boosts and rotations. Without boost generators for 2-component spinors Galilean transformations cannot be constructed for 2-component spinors and Galilean invariance cannot exist for 2-component spinor equations.

We now consider 4-component spinors. Rotation matrices can be constructed from the Pauli matrices

$$X_{\theta_j} = \frac{1}{2} \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}. \tag{A.67}$$

The boost generators can then be constructed from lower triangular matrices

$$X_{v_j} = \frac{-i}{2} \begin{pmatrix} 0 & 0 \\ \sigma_j & 0 \end{pmatrix}. \tag{A.68}$$

These matrices satisfy the commutation relations for rotations and boosts. Translations don't need to be combined here because they split into subgroups differently.

A.13 Constructing Transformation Operators from Group Generators

Now that we have Galilean generators for 2- and 4-component spinors we can use them to construct transformation operators that form the spinor representation of the extended Galilei group. In general transformation operators can be constructed from the generators X_j and parameters Π_j of a group by the exponential function or infinite series

$$T = e^{-i\Pi_j X_j} = \sum_{k=0}^{\infty} \frac{(-i\Pi_j)^k}{k!} X_j^k. \tag{A.69}$$

Generators of a given dimension will result in operators of the same number of dimensions. The 2×2 and 4×4 generator matrices found in the last section will produce same sized transformation operators.

One-dimensional transformations are also important. A 1-dimensional generator will produce a scalar transformation. Remember that we started our analysis with scalar transformations for the Galilei group. The scalar transformations were used to construct the differential generators and these generators were then used to discover the commutation relations of the group. The commutation relations were then used to find generators for various representations. It turns out that the scalar transformations are an essential part of constructing higher dimensional transformations as well. While a matrix is needed to transform a multi-component object (a tensor) its also necessary to simultaneously transform functions within the components that depend upon the coordinates. In constructing every operator we must include a scalar as well as matrix transformation.

We begin with 2-component spinors. Using the generators of rotation $\frac{\sigma_j}{2}$ and rotation parameters $\theta_i = \{\gamma, \beta, \alpha\}$ the rotation operators can be found by

$$U_R^{(2)}(\theta_j) = e^{-i\sigma_j\theta_j/2} = \sum_{k=0}^{\infty} \frac{(-i\theta_j)^k}{k!}\sigma_j^k. \tag{A.70}$$

Working out the sum for each angle parameter and identifying the sum as an analytic function produces the rotation matrices for 2-component spinors

$$U_R(\alpha) = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}, \tag{A.71}$$

$$U_R(\beta) = \begin{pmatrix} \cos\beta/2 & \sin\beta/2 \\ -\sin\beta/2 & \cos\beta/2 \end{pmatrix}, \tag{A.72}$$

and

$$U_R(\gamma) = \begin{pmatrix} \cos\gamma/2 & i\sin\gamma/2 \\ i\sin\gamma/2 & \cos\gamma/2 \end{pmatrix}. \tag{A.73}$$

Since there isn't a generator for 2×2 boosts there is no corresponding 2×2 boost operator. As an abelian subgroup the boosts can be represented by a diagonal matrix or identity matrix combined with the scalar transformation. This would be fine for representing boosts alone but not useful for the combination of boosts with rotations. The combination must obey the group composition law (derived earlier) which reflects the nature of the semi-direct product $R(3) \otimes_s B(3)$.

Turning to transformation operators for 4-component spinors it is easy enough to produce the rotation matrix using the previous result for 2-component spinors

$$U_R^{(4)} = \sum_{k=0}^{\infty} \frac{(-i\theta_j)^k}{k!} \frac{1}{2} \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}^k = \begin{pmatrix} U_R^{(2)} & 0 \\ 0 & U_R^{(2)} \end{pmatrix} \tag{A.74}$$

where $U_R^{(2)}$ is the rotation matrix for 2×2 rotations. The boost matrix can be obtained by similar calculation

$$U_B^{(4)} = \sum_{k=0}^{\infty} \frac{(-iv_j)^k}{k!} \frac{-i}{2} \begin{pmatrix} 0 & 0 \\ \sigma_j & 0 \end{pmatrix}^k = \begin{pmatrix} I & 0 \\ -\frac{1}{2}\sigma_j v_j & I \end{pmatrix}. \tag{A.75}$$

Constructing a combined boost and rotation matrix is accomplished through the product

$$U_{RB}^{(4)} = \begin{pmatrix} U_R^{(2)} & 0 \\ 0 & U_R^{(2)} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\frac{1}{2}\sigma_j v_j & 0 \end{pmatrix} = \begin{pmatrix} U_R^{(2)} & 0 \\ -\frac{1}{2}\sigma_j v_j U_R^{(2)} & U_R^{(2)} \end{pmatrix}. \tag{A.76}$$

The rotation-boost transformation operator on a 4-component state function is

$$T\psi(x,t) = \begin{pmatrix} U_R^{(2)} & 0 \\ -\frac{1}{2}\sigma_j v_j U_R^{(2)} & U_R^{(2)} \end{pmatrix} \psi\left(R^{-1}(x - vt), t\right) \tag{A.77}$$

and it obeys the group composition law.

A.14 Transforming Equations

The phase function that makes the Schrodinger equation Galilei invariant is

$$\varphi\left(\vec{x}, t\right) = -m\vec{v} \cdot R\vec{x} + \frac{1}{2}mv^2 t + C. \tag{A.78}$$

Calculating the derivatives of the phase function produces

$$\frac{\partial \varphi\left(\vec{x}, t\right)}{\partial t} = \frac{1}{2}mv^2 \tag{A.79}$$

and

$$\frac{\partial \varphi\left(\vec{x},t\right)}{\partial x^{j}} = -mv_{i}R_{ki}\frac{\partial}{\partial x^{j}}x_{k} = -mv_{i}R_{ji}. \tag{A.80}$$

Combining equations derived earlier we see that application of a Galilei transformation to any combination of space and time differentials of arbitrary powers p and q yield polynomial terms with like powers

$$T_{G}\frac{\partial^{p}}{\partial t^{p}}\partial_{i}{}^{q}T_{G}{}^{-1} = \left(\frac{\partial}{\partial t} + R_{ji}v_{j}\frac{\partial}{\partial x^{i}}\right)^{p}\left(R_{ji}\frac{\partial}{\partial x^{j}}\right)^{q}. \tag{A.81}$$

When this transformation is applied to the state function a phase factor results that can be commuted through to the other side of the equation such that

$$T_{G}\frac{\partial^{p}}{\partial t^{p}}\partial_{i}{}^{q}T_{G}{}^{-1}e^{i\varphi(\vec{x},t)}\ \psi\left(\vec{x},t\right) = \left(\frac{\partial}{\partial t} + R_{ji}v_{j}\frac{\partial}{\partial x^{i}}\right)^{p}\left(R_{ji}\frac{\partial}{\partial x^{j}}\right)^{q}e^{i\varphi(\vec{x},t)}\ \psi\left(\vec{x},t\right) \tag{A.82}$$

$$= e^{i\varphi(\vec{x},t)}\left[\left(i\frac{\partial\varphi}{\partial t} + \frac{\partial}{\partial t}\right) + R_{ji}v_{j}\left(i\frac{\partial\varphi}{\partial x^{i}} + \frac{\partial}{\partial x^{i}}\right)\right]^{p}\left[R_{ji}\left(i\frac{\partial\varphi}{\partial x^{j}} + \frac{\partial}{\partial x^{j}}\right)\right]^{q}\ \psi\left(\vec{x},t\right) \tag{A.83}$$

$$= e^{i\varphi(\vec{x},t)}\left[\left(\frac{i}{2}mv^{2} + \frac{\partial}{\partial t}\right) + R_{ji}v_{j}\left(-imv_{k}R_{ik} + \frac{\partial}{\partial x^{i}}\right)\right]^{p} \tag{A.84}$$

$$\times\left[R_{ji}\left(-imv_{k}R_{jk} + \frac{\partial}{\partial x^{j}}\right)\right]^{q}\ \psi\left(\vec{x},t\right) \tag{A.85}$$

$$= e^{i\varphi(\vec{x},t)}\left[\frac{i}{2}mv^{2} - imR_{ji}v_{j}v_{k}R_{ik} + \frac{\partial}{\partial t} + R_{ji}v_{j}\frac{\partial}{\partial x^{i}}\right]^{p} \tag{A.86}$$

$$\times\left[-imR_{ji}v_{k}R_{jk} + R_{ji}\frac{\partial}{\partial x^{j}}\right]^{q}\ \psi\left(\vec{x},t\right) \tag{A.87}$$

$$= e^{i\varphi(\vec{x},t)}\left[k_{1} + \frac{\partial}{\partial t} + k_{2i}\frac{\partial}{\partial x^{i}}\right]^{p}\left[k_{3i} + k_{4ji}\frac{\partial}{\partial x^{j}}\right]^{q}\ \psi\left(\vec{x},t\right) \tag{A.88}$$

where the constants k_1, k_{2i}, k_{3i}, and k_{4ij} introduced here are defined

$$k_{1} = \frac{i}{2}mv^{2} - imR_{ji}v_{j}v_{k}R_{ik}, \tag{A.89}$$

$$k_{2i} = R_{ji}v_{j}, \tag{A.90}$$

$$k_{2i}{}^{2} = \left(R_{ji}v_{j}\right)^{2} = v^{2}, \tag{A.91}$$

$$k_{3i} = -imR_{ji}v_{k}R_{jk}, \tag{A.92}$$

$${k_{3i}}^2 = \left(-imR_{ji}v_{k}R_{jk}\right)^2 = -m^2v^2, \tag{A.93}$$

and

$$k_{4ji} = R_{ji}. \tag{A.94}$$

A.15 Deriving the Conditions for Galilei Invariant Spinor Equations

A.15.1 Conditions for Invariance of First Order Differential Equations

The most arbitrary first order dynamical equation for an n-component state function ψ is

$$\left[B_1\frac{\partial}{\partial t} + B_{2i}\frac{\partial}{\partial x^i} + B_3\right]\psi\left(\vec{r},t\right) = 0. \tag{A.95}$$

After transformation it becomes

$$\left[{B'}_1\left(k_1 + \frac{\partial}{\partial t} + k_{2i}\frac{\partial}{\partial x^i}\right) + {B'}_{2i}\left(k_{3i} + k_{4ji}\frac{\partial}{\partial x^j}\right) + {B'}_3\right]\psi\left(\vec{r},t\right) = 0. \tag{A.96}$$

The implied sums can be written explicitly

$$\begin{aligned}&\left[{B'}_1\left(k_1 + \frac{\partial}{\partial t} + k_{2x}\frac{\partial}{\partial x} + k_{2y}\frac{\partial}{\partial y} + k_{2z}\frac{\partial}{\partial z}\right)\right.\\ &\left.+{B'}_{2i}\left(k_{3i} + k_{4xi}\frac{\partial}{\partial x} + k_{4yi}\frac{\partial}{\partial y} + k_{4zi}\frac{\partial}{\partial z}\right) + {B'}_3\right]\psi\left(\vec{r},t\right) = 0.\end{aligned} \tag{A.97}$$

The terms can be grouped by differential powers

$$\left[{B'}_1\frac{\partial}{\partial t} + \left({B'}_1k_{2j} + {B'}_{2i}k_{4ji}\right)\frac{\partial}{\partial x^j} + \left({B'}_1k_1 + {B'}_{2i}k_{3i} + {B'}_3\right)\right]\psi\left(\vec{r},t\right) = 0. \tag{A.98}$$

Requiring invariance of the equation yields a set of conditions on the matrices B

$$B_1 = TB_1T^{-1}, \tag{A.99}$$

$$B_{2j} = TB_1T^{-1}k_{2j} + TB_{2i}T^{-1}k_{4ji}, \tag{A.100}$$

and

$$B_3 = TB_1T^{-1}k_1 + TB_{2i}T^{-1}k_{3i} + TB_3T^{-1}. \quad \text{(A.101)}$$

With the definition for k substituted in the conditions are

$$B_1 = TB_1T^{-1}, \quad \text{(A.102)}$$

$$B_{2j} = TB_1T^{-1}R_{ij}v_i + TB_{2i}T^{-1}R_{ji}, \quad \text{(A.103)}$$

and

$$B_3 = TB_1T^{-1}\left(\frac{i}{2}mv^2 - imR_{ji}v_jv_kR_{ik}\right) - TB_{2i}T^{-1}imR_{ji}v_kR_{jk} + TB_3T^{-1}. \quad \text{(A.104)}$$

A.15.2 Conditions for Invariance of Second Order Differential Equations

The starting equation for second order differentials is

$$\left[C_1\left(\frac{\partial}{\partial t}\right)^2 + C_{2i}\frac{\partial}{\partial t}\frac{\partial}{\partial x^i} + C_3\frac{\partial}{\partial t} + C_{4ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} + C_{5i}\frac{\partial}{\partial x^i} + C_6\right]\psi\left(\vec{r},t\right) = 0. \quad \text{(A.105)}$$

After transformation it becomes

$$\begin{aligned}\Bigg[C'_1\left(k_1 + \frac{\partial}{\partial t} + k_{2i}\frac{\partial}{\partial x^i}\right)^2 + C'_{2i}\left(k_1 + \frac{\partial}{\partial t} + k_{2q}\frac{\partial}{\partial x^q}\right)\left(k_{3i} + k_{4ji}\frac{\partial}{\partial x^j}\right) \\ +C'_3\left(k_1 + \frac{\partial}{\partial t} + k_{2i}\frac{\partial}{\partial x^i}\right) + C'_{4ij}\left(k_{3i} + k_{4qi}\frac{\partial}{\partial x^q}\right)\left(k_{3j} + k_{4qj}\frac{\partial}{\partial x^q}\right) \\ +C'_{5i}\left(k_{3i} + k_{4ji}\frac{\partial}{\partial x^j}\right) + C'_6\Bigg]\psi\left(\vec{r},t\right) = 0. \quad \text{(A.106)}\end{aligned}$$

After multiplying out the polynomials and grouping the terms by differentials of like powers the equation becomes

$$\Bigg[C'_1\left(\frac{\partial}{\partial t}\right)^2 + \left(C'_1 2k_{2i} + C'_{2j}k_{4ij}\right)\frac{\partial}{\partial t}\frac{\partial}{\partial x^i} + \left(C'_1 2k_1 + C'_{2i}k_{3i} + C'_3\right)\frac{\partial}{\partial t}$$

$$+C'_1\left(k_{2q}\frac{\partial}{\partial x^q}\right)\left(k_{2r}\frac{\partial}{\partial x^r}\right) + C'_{2i}k_{2r}k_{4qi}\frac{\partial}{\partial x^q}\frac{\partial}{\partial x^r} + C'_{4ij}\left(k_{4qi}\frac{\partial}{\partial x^q}\right)\left(k_{4rj}\frac{\partial}{\partial x^r}\right)$$

$$+\left(C'_1\left(k_1 k_{2i}+k_{2i}k_1\right)+C'_{2q}k_{2i}k_{3q}+C'_{2q}k_1 k_{4iq}+C'_3 k_{2i}\right.$$

$$\left.+C'_{4qr}\left(k_{3q}k_{4ir}+k_{4iq}k_{3r}\right)+C'_{5q}k_{4iq}\right)\frac{\partial}{\partial x^i}$$

$$+\left(C'_1 {k_1}^2+C'_{2i}k_1 k_{3i}+C'_3 k_1+C'_{4ij}k_{3i}k_{3j}+C'_{5i}k_{3i}+C'_6\right)\Big]\,\psi\left(\vec{r},t\right)=0. \quad \text{(A.107)}$$

Requiring invariance of the second order equation therefore leads to the following six conditions

$$C_1 = TC_1T^{-1}, \quad \text{(A.108)}$$

$$C_{2j} = 2TC_1T^{-1}k_{2j} + TC_{2i}T^{-1}k_{4ji}, \quad \text{(A.109)}$$

$$C_3 = 2TC_1T^{-1}k_1 + TC_{2i}T^{-1}k_{3i} + TC_3T^{-1}, \quad \text{(A.110)}$$

$$C_{4qr} = TC_1T^{-1}k_{2q}k_{2r} + TC_{2i}T^{-1}k_{2r}k_{4qi} + TC_{4ij}T^{-1}k_{4qi}k_{4rj}, \quad \text{(A.111)}$$

$$C_{5i} = TC_1T^{-1}\left(k_1k_{2i}+k_{2i}k_1\right) + TC_{2q}T^{-1}k_{2i}k_{3q} + TC_{2q}T^{-1}k_1k_{4iq}$$

$$+TC_3T^{-1}k_{2i} + TC_{4qr}T^{-1}\left(k_{3q}k_{4ir}+k_{4iq}k_{3r}\right) + TC_{5q}T^{-1}k_{4iq}, \quad \text{(A.112)}$$

and

$$C_6 = TC_1T^{-1}{k_1}^2 + TC_{2i}T^{-1}k_1k_{3i} + TC_3T^{-1}k_1 + TC_{4ij}T^{-1}k_{3i}k_{3j}$$

$$+TC_{5i}T^{-1}k_{3i} + TC_6T^{-1}. \quad \text{(A.113)}$$

Notice that substituting $\overline{C}_1 = 2C_1$ into the conditions would make the first three conditions identical to the conditions on B (the conditions for the first order equation).

A.15.3 Conditions for Invariance of Higher Order Equations

The first two differential equations demonstrate a trend that will continue with higher order differential equations. For every matrix introduced in the differential equation there will be an equation that must be satisfied for the differential equation to be invariant. As the number of matrices and conditions grows so does the number and complexity of the conditions. Furthermore the set of conditions for a given order differential equation will always contain as a subset the set of conditions of the next lower order differential equation.

A.16 Finding 2×2 Matrices that Satisfy the Conditions for Invariance

A.16.1 Finding a 2×2 Matrix to Satisfies the First Condition

Before working with 4×4 matrices we attempt to find a set of 2×2 matrices that satisfy the conditions for invariance of a first order dynamical equation. We begin by applying rotations to the first condition for invariance to see how the matrix B_1 is constrained. Let B_1 be an arbitrary 2×2 matrix

$$B_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{A.114}$$

where a, b, c, and d are unknown scalar constants. A rotation about the z-axis for 2-component spinors can be represented by the 2×2 matrix

$$R_z = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}. \tag{A.115}$$

Applying a z-rotation to the matrix B_1 in the first condition produces the following result

$$\begin{aligned} B_1 = {R_z}^{-1} B_1 R_z &= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \\ &= \begin{pmatrix} ae^{-i\alpha/2} & be^{-i\alpha/2} \\ ce^{i\alpha/2} & de^{i\alpha/2} \end{pmatrix} \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} = \begin{pmatrix} a & be^{-i\alpha} \\ ce^{i\alpha} & d \end{pmatrix}. \end{aligned} \tag{A.116}$$

From this we conclude that invariance of a 2×2 matrix under z-axis rotation constrains the matrix to

$$B_1 = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}. \tag{A.117}$$

A rotation about the y-axis for 2-component spinors can be represented

$$R_y = \begin{pmatrix} \cos\beta/2 & \sin\beta/2 \\ -\sin\beta/2 & \cos\beta/2 \end{pmatrix}. \tag{A.118}$$

Applying this rotation to the matrix B_1 in the first condition produces the following result

$$R_y{}^{-1} B_1 R_y = \begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos\beta/2 & \sin\beta/2 \\ -\sin\beta/2 & \cos\beta/2 \end{pmatrix}$$

$$= \begin{pmatrix} a\cos\beta/2 \; - c\sin\beta/2 & b\cos\beta/2 \; - d\sin\beta/2 \\ a\sin\beta/2 \; + c\cos\beta/2 & b\sin\beta/2 \; + d\cos\beta/2 \end{pmatrix} \begin{pmatrix} \cos\beta/2 & \sin\beta/2 \\ -\sin\beta/2 & \cos\beta/2 \end{pmatrix}$$

$$= \begin{pmatrix} a\cos^2\beta/2 \; - (c+b)\sin\frac{\beta}{2}\cos\frac{\beta}{2} \; + d\sin^2\frac{\beta}{2} & b\cos^2\frac{\beta}{2} \; + (a-d)\sin\frac{\beta}{2}\cos\frac{\beta}{2} \; - c\sin^2\frac{\beta}{2} \\ c\cos^2\frac{\beta}{2} \; + (a-d)\sin\frac{\beta}{2}\cos\frac{\beta}{2} \; - b\sin^2\frac{\beta}{2} & d\cos^2\frac{\beta}{2} \; + (c+b)\sin\frac{\beta}{2}\cos\frac{\beta}{2} \; + a\sin^2\frac{\beta}{2} \end{pmatrix}. \tag{A.119}$$

When a y-axis rotation must leave a matrix invariant notice that the squared sin and cos functions are symmetric but the product of sin and cos functions is anti-symmetric. To make it equal to a constant requires the asymmetric part to vanish so $c + b = 0$ and $a - d = 0$. Also note that sin and cos squared terms sum to a constant only when their coefficients are equal so $a = d$ and $b = -c$. From this we conclude the matrix B_1 is constrained to

$$B_1 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \tag{A.120}$$

A rotation about the x-axis for 2-component spinors can be represented as

$$R_x = \begin{pmatrix} \cos\gamma/2 & i\sin\gamma/2 \\ i\sin\gamma/2 & \cos\gamma/2 \end{pmatrix}. \tag{A.121}$$

Applying this rotation to the matrix B_1 in the first condition produces the following result

$$R_x{}^{-1} B_1 R_x = \begin{pmatrix} \cos\gamma/2 & -i\sin\gamma/2 \\ -i\sin\gamma/2 & \cos\gamma/2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos\gamma/2 & i\sin\gamma/2 \\ i\sin\gamma/2 & \cos\gamma/2 \end{pmatrix}$$

$$= \begin{pmatrix} a\cos\gamma/2 \ - ic\sin\gamma/2 & b\cos\gamma/2 \ - id\sin\gamma/2 \\ c\cos\gamma/2 \ - ia\sin\gamma/2 & d\cos\gamma/2 \ - ib\sin\gamma/2 \end{pmatrix} \begin{pmatrix} \cos\gamma/2 & i\sin\gamma/2 \\ i\sin\gamma/2 & \cos\gamma/2 \end{pmatrix}$$

$$= \begin{pmatrix} a\cos^2\gamma/2 \ + d\sin^2\frac{\gamma}{2} \ + i\,(b-c)\sin\gamma/2\,\cos\gamma/2 \\ c\cos^2\gamma/2 \ + b\sin^2\frac{\gamma}{2} \ + i\,(d-a)\sin\gamma/2\,\cos\gamma/2 \end{pmatrix.}$$

$$\begin{matrix} b\cos^2\gamma/2 \ + c\sin^2\frac{\gamma}{2} \ + i\,(a-d)\sin\gamma/2\,\cos\gamma/2 \\ d\cos^2\gamma/2 \ + a\sin^2\frac{\gamma}{2} \ + i\,(c-b)\sin\gamma/2\,\cos\gamma/2 \end{matrix}\Bigg). \tag{A.122}$$

From this result we see that invariance under a y-axis rotation constrains the matrix to

$$B_1 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}. \tag{A.123}$$

We have taken a close look at the effect of rotations about each of the three axes. Now these results can be combined. Requiring rotational invariance about all three axes (or any two for that matter) forces the matrix B_1 to be diagonal (i.e., a constant multiple of the identity matrix)

$$B_1 = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A.124}$$

A.16.2 Finding 2×2 Matrices to Satisfy the Second Condition

We now wish to find a set of $2{\times}2$ matrices B_{2j} that satisfies the second condition for invariance. We begin with a set of 3 arbitrary matrices for j=1, 2, 3. Each matrix has a unique set of constants e, f, g, and h where we have dropped the subscript from the individual constants and moved it outside the matrix such that

$$B_{2j} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}_j . \tag{A.125}$$

Applying roations only simplifies the second condition for invariance. Notice that this condition is almost identical to the first condition when the boost is zero. Applying only a rotation about the z-axis produces the condition

$$B_{2j} = R_z{}^{-1} B_1 R_z R_{ij} v_i + R_z{}^{-1} B_{2i} R_z R_{ji}. \tag{A.126}$$

Substituting the matrix (A.125) and rotation matrices used earlier produces the following result

$$\begin{aligned}\begin{pmatrix} e & f \\ g & h \end{pmatrix}_j &= a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} R_{ij} v_i \\ &+ \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}_i \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} R_{ji} \\ &= a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} R_{ij} v_i + \begin{pmatrix} e & f e^{-i\alpha} \\ g e^{i\alpha} & h \end{pmatrix}_i R_{ji}. \end{aligned} \tag{A.127}$$

Explicit representation of the matrices R_{ij} then yields the equation

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix}_1 \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}_2 \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}_3 \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} e & f e^{-i\alpha} \\ g e^{i\alpha} & h \end{pmatrix}_1 \times \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix} + \begin{pmatrix} e & f e^{-i\alpha} \\ g e^{i\alpha} & h \end{pmatrix}_2 \times \begin{pmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix}$$

$$+\begin{pmatrix} e & fe^{-i\alpha} \\ ge^{i\alpha} & h \end{pmatrix}_3 \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{A.128}$$

From the third component of this equation it is clear that $f_3 = g_3 = 0$ so

$$B_{23} = \begin{pmatrix} e_3 & 0 \\ 0 & h_3 \end{pmatrix}. \tag{A.129}$$

The first and second components of the equation (A.128) are equivalent to the restrictions

$$e_1\,(1 - \cos\alpha\,) = -e_2 \sin\alpha\ , \tag{A.130}$$

$$e_2\,(1 - \cos\alpha\,) = e_1 \sin\alpha\ , \tag{A.131}$$

$$h_1\,(1 - \cos\alpha\,) = -h_2 \sin\alpha\ , \tag{A.132}$$

$$h_2\,(1 - \cos\alpha\,) = h_1 \sin\alpha\ , \tag{A.133}$$

$$f_1\left(1 - e^{-i\alpha}\cos\alpha\,\right) = -f_2 e^{-i\alpha} \sin\alpha\ , \tag{A.134}$$

$$f_2\left(1 - e^{-i\alpha}\cos\alpha\,\right) = f_1 e^{-i\alpha} \sin\alpha\ , \tag{A.135}$$

$$g_1\left(1 - e^{-i\alpha}\cos\alpha\,\right) = -g_2 e^{-i\alpha} \sin\alpha\ , \tag{A.136}$$

and

$$g_2\left(1 - e^{-i\alpha}\cos\alpha\,\right) = g_1 e^{-i\alpha} \sin\alpha\ . \tag{A.137}$$

From these restrictions we deduce $e_1 = \pm ie_2$ and $h_1 = \pm ih_2$ and $f_1 = \pm if_2$ and $g_1 = \pm ig_2$ and we conclude that invariance under z-rotations restricts the matrices B_{2j} to

$$B_{21} = \begin{pmatrix} e_1 & f_1 \\ g_1 & h_1 \end{pmatrix}, \qquad B_{22} = \begin{pmatrix} \pm ie_1 & \pm if_1 \\ \pm ig_1 & \pm ih_1 \end{pmatrix},$$

and

$$B_{23} = \begin{pmatrix} e_3 & 0 \\ 0 & h_3 \end{pmatrix}. \tag{A.138}$$

Rotations about the y-axis may be applied by the same method to produce the equation

$$R_y{}^{-1} B_{2j} R_y = \begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}_j \begin{pmatrix} \cos\beta/2 & \sin\beta/2 \\ -\sin\beta/2 & \cos\beta/2 \end{pmatrix}$$

$$= \begin{pmatrix} e\cos\beta/2 \; - g\sin\beta/2 & f\cos\beta/2 \; - h\sin\beta/2 \\ e\sin\beta/2 \; + g\cos\beta/2 & f\sin\beta/2 \; + h\cos\beta/2 \end{pmatrix} \begin{pmatrix} \cos\beta/2 & \sin\beta/2 \\ -\sin\beta/2 & \cos\beta/2 \end{pmatrix}$$

$$= \begin{pmatrix} e\cos^2\beta/2 \; - (g+f)\sin\frac{\beta}{2}\cos\frac{\beta}{2} \; + h\sin^2\frac{\beta}{2} & f\cos^2\frac{\beta}{2} \; + (e-h)\sin\frac{\beta}{2}\cos\frac{\beta}{2} \; - g\sin^2\frac{\beta}{2} \\ g\cos^2\frac{\beta}{2} \; + (e-h)\sin\frac{\beta}{2}\cos\frac{\beta}{2} \; - f\sin^2\frac{\beta}{2} & h\cos^2\frac{\beta}{2} \; + (g+f)\sin\frac{\beta}{2}\cos\frac{\beta}{2} \; + e\sin^2\frac{\beta}{2} \end{pmatrix}. \tag{A.139}$$

The same can be done for x-rotations and when the results of all three rotations are combined the matrices B_{2j} are restricted to a multiple of the Pauli matrices

$$B_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B_{22} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

and

$$B_{23} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.140}$$

A.16.3 Finding a 2×2 Matrix that Satisfies the Third Condition

With this result the third condition (without a boost) has an identical form to the first condition and therefore leads to a similar result

$$B_3 = c \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A.141}$$

where p is an unknown constant.

To summarize the results so far, rotational invariance constrains the matrix coefficients of a first order dynamical equation for 2-component spinors to $B_1 = aI$, $B_{2j} = b\sigma_j$, and $B_3 = cI$.

A.17 Finding 4×4 Matrices that Satisfy the Conditions for Rotational Invariance

A.17.1 Finding a 4×4 Matrix that Satisfies the First Condition for Rotational Invariance

Let B_1 be an arbitrary 4×4 matrix

$$B_1 = \begin{pmatrix} P & Q \\ S & T \end{pmatrix} \tag{A.142}$$

where P, Q, S, and T are unknown constant 2x2 matrices. The rotation matrix about the z-axis on the four component spinor is

$$\overline{R}_z = \begin{pmatrix} R_z & 0 \\ 0 & R_z \end{pmatrix} \tag{A.143}$$

where R_z is a 2×2 matrix for rotating a 2-component spinor. With these matrices the first condition for a z=rotation is

$$B_1 = \overline{R}_z^{\,-1} B_1 \overline{R}_z = \begin{pmatrix} R_z^{\,-1} & 0 \\ 0 & R_z^{\,-1} \end{pmatrix} \begin{pmatrix} P & Q \\ S & T \end{pmatrix} \begin{pmatrix} R_z & 0 \\ 0 & R_z \end{pmatrix} \tag{A.144}$$

$$= \begin{pmatrix} R_z{}^{-1}P & R_z{}^{-1}Q \\ R_z{}^{-1}S & R_z{}^{-1}T \end{pmatrix} \begin{pmatrix} R_z & 0 \\ 0 & R_z \end{pmatrix} \tag{A.145}$$

$$= \begin{pmatrix} R_z{}^{-1}PR_z & R_z{}^{-1}QR_z \\ R_z{}^{-1}SR_z & R_z{}^{-1}TR_z \end{pmatrix}. \tag{A.146}$$

The same can be done for rotations about the y-axis and x-axis with similar results

$$B_1 = \overline{R}_y{}^{-1} B_1 \overline{R}_y = \begin{pmatrix} R_y{}^{-1} & 0 \\ 0 & R_y{}^{-1} \end{pmatrix} \begin{pmatrix} P & Q \\ S & T \end{pmatrix} \begin{pmatrix} R_y & 0 \\ 0 & R_y \end{pmatrix} \tag{A.147}$$

$$= \begin{pmatrix} R_y{}^{-1}PR_y & R_y{}^{-1}QR_y \\ R_y{}^{-1}SR_y & R_y{}^{-1}TR_y \end{pmatrix}. \tag{A.148}$$

These equations produce 4 conditions on the 2×2 matrices. The conditions are of the same form as the conditions found for 2-component spinors, e.g. $P = R_z^{-1}PR_z$. Since we have already found the constraints on the matrices P, Q, S, and T that result from these conditions on 2x2 matrices we know that the 4x4 matrix must have the form

$$B_1 = \begin{pmatrix} pI & qI \\ sI & tI \end{pmatrix} \tag{A.149}$$

where p, q, s, and t are arbitrary scalar constants and I is the 2x2 identity matrix.

A.17.2 Finding 4×4 Matrices that Satisfy the Second Condition for Rotational Invariance

The second condition for invariance is

$$B_{2j} = TB_1T^{-1}R_{ij}v_i + TB_{2i}T^{-1}R_{ji}. \tag{A.150}$$

Let B_{2j} be an arbitrary 4x4 matrix for each j=1, 2, 3 such that

$$B_{2j} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}_j \tag{A.151}$$

A rotation about the z-axis is applied to the second condition for invariance by substitution of the matrices into the equation for invariance

$$\begin{aligned} &\begin{pmatrix} E & F \\ G & H \end{pmatrix}_1 \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} E & F \\ G & H \end{pmatrix}_2 \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} E & F \\ G & H \end{pmatrix}_3 \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} R_z{}^{-1}ER_z & R_z{}^{-1}FR_z \\ R_z{}^{-1}GR_z & R_z{}^{-1}HR_z \end{pmatrix}_1 \times \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} R_z{}^{-1}ER_z & R_z{}^{-1}FR_z \\ R_z{}^{-1}GR_z & R_z{}^{-1}HR_z \end{pmatrix}_2 \times \begin{pmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} R_z{}^{-1}ER_z & R_z{}^{-1}FR_z \\ R_z{}^{-1}GR_z & R_z{}^{-1}HR_z \end{pmatrix}_3 \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{A.152}$$

Examining the components of this equation reveals the same condition on the 2x2 sub-matrices as the first condition on B for a rotation about z for 2x2 matrices. From this we know that B_{23} is composed of diagonal matrices

$$E_3 = \begin{pmatrix} e_{31} & 0 \\ 0 & e_{34} \end{pmatrix}, \quad F_3 = \begin{pmatrix} f_{31} & 0 \\ 0 & f_{34} \end{pmatrix}, \quad G_3 = \begin{pmatrix} g_{31} & 0 \\ 0 & g_{34} \end{pmatrix}, \tag{A.153}$$

and

$$H_3 = \begin{pmatrix} h_{31} & 0 \\ 0 & h_{34} \end{pmatrix}. \tag{A.154}$$

A rotation about the y-axis produces similar results

$$\begin{pmatrix} E & F \\ G & H \end{pmatrix}_1 \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} E & F \\ G & H \end{pmatrix}_2 \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} E & F \\ G & H \end{pmatrix}_3 \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} {R_y}^{-1}ER_y & {R_y}^{-1}FR_y \\ {R_y}^{-1}GR_y & {R_y}^{-1}HR_y \end{pmatrix}_1 \times \begin{pmatrix} \cos\beta \\ 0 \\ -\sin\beta \end{pmatrix}$$

$$+ \begin{pmatrix} {R_y}^{-1}ER_y & {R_y}^{-1}FR_y \\ {R_y}^{-1}GR_y & {R_y}^{-1}HR_y \end{pmatrix}_2 \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} {R_y}^{-1}ER_y & {R_y}^{-1}FR_y \\ {R_y}^{-1}GR_y & {R_y}^{-1}HR_y \end{pmatrix}_{31} \times \begin{pmatrix} \sin\beta \\ 0 \\ \cos\beta \end{pmatrix}. \tag{A.155}$$

The components of this equation contain the same conditions as was found for 2x2 matrices in the first condition under a rotation about the y-axis. Thus we may conclude that the components of B_{22} are restricted to

$$E_2 = \begin{pmatrix} e_{21} & e_{22} \\ -e_{22} & e_{21} \end{pmatrix}, \quad F_2 = \begin{pmatrix} f_{21} & f_{22} \\ -f_{22} & f_{21} \end{pmatrix}, \quad G_2 = \begin{pmatrix} g_{21} & g_{22} \\ -g_{22} & g_{21} \end{pmatrix}, \tag{A.156}$$

and

$$H_2 = \begin{pmatrix} h_{21} & h_{22} \\ -h_{22} & h_{21} \end{pmatrix}. \tag{A.157}$$

A rotation about the x-axis is applied to second condition

$$\begin{pmatrix} E & F \\ G & H \end{pmatrix}_1 \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} E & F \\ G & H \end{pmatrix}_2 \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} E & F \\ G & H \end{pmatrix}_3 \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} {R_x}^{-1}ER_x & {R_x}^{-1}FR_x \\ {R_x}^{-1}GR_x & {R_x}^{-1}HR_x \end{pmatrix}_1 \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} {R_x}^{-1}ER_x & {R_x}^{-1}FR_x \\ {R_x}^{-1}GR_x & {R_x}^{-1}HR_x \end{pmatrix}_2 \times \begin{pmatrix} 0 \\ \cos\gamma \\ \sin\gamma \end{pmatrix}$$

$$+ \begin{pmatrix} {R_x}^{-1}ER_x & {R_x}^{-1}FR_x \\ {R_x}^{-1}GR_x & {R_x}^{-1}HR_x \end{pmatrix}_{31} \times \begin{pmatrix} 0 \\ -\sin\gamma \\ \cos\gamma \end{pmatrix} . \tag{A.158}$$

The first component requires that B_{21} be rotationally invariant about the x-axis such that it is constrained to

$$E_1 = \begin{pmatrix} e_{11} & e_{12} \\ -e_{12} & e_{11} \end{pmatrix}, \quad F_1 = \begin{pmatrix} f_{11} & f_{12} \\ -f_{12} & f_{11} \end{pmatrix}, \quad G_1 = \begin{pmatrix} g_{11} & g_{12} \\ -g_{12} & g_{11} \end{pmatrix}, \tag{A.159}$$

and

$$H_1 = \begin{pmatrix} h_{11} & h_{12} \\ -h_{12} & h_{11} \end{pmatrix} . \tag{A.160}$$

Further restrictions can be found by using the first and second components of the z-rotation equation

$$E_1 = {R_z}^{-1}E_1R_z\cos\alpha \ - {R_z}^{-1}E_2R_z\sin\alpha \tag{A.161}$$

$$E_2 = {R_z}^{-1}E_1R_z\sin\alpha \ + {R_z}^{-1}E_2R_z\cos\alpha \ . \tag{A.162}$$

Putting the 2×2 matrices into these equations produces

$$E_1 = \begin{pmatrix} e_{11} & e_{12} \\ e_{12} & e_{11} \end{pmatrix}$$

$$= \begin{pmatrix} e_{11} & e_{12}e^{-i\alpha/2} \\ e_{12}e^{i\alpha/2} & e_{11} \end{pmatrix} \cos\alpha \; - \begin{pmatrix} e_{21} & e_{22}e^{-i\alpha/2} \\ -e_{22}e^{i\alpha/2} & e_{21} \end{pmatrix} \sin\alpha \tag{A.163}$$

and

$$E_2 = \begin{pmatrix} e_{21} & e_{22} \\ -e_{22} & e_{21} \end{pmatrix}$$

$$= \begin{pmatrix} e_{11} & e_{12}e^{-i\alpha/2} \\ e_{12}e^{i\alpha/2} & e_{11} \end{pmatrix} \sin\alpha \; + \begin{pmatrix} e_{21} & e_{22}e^{-i\alpha/2} \\ -e_{22}e^{i\alpha/2} & e_{21} \end{pmatrix} \cos\alpha \; . \tag{A.164}$$

Combining these equations produces

$$e_{11}\left(1 - \cos\alpha\ \right) = -e_{21}\sin\alpha\ , \tag{A.165}$$

$$e_{21}\left(1 - \cos\alpha\ \right) = e_{11}\sin\alpha\ , \tag{A.166}$$

$$e_{12}\left(1 - e^{-i\alpha/2}\cos\alpha\ \right) = -e_{22}e^{-i\alpha/2}\sin\alpha\ , \tag{A.167}$$

$$e_{22}\left(1 - e^{-i\alpha/2}\cos\alpha\ \right) = e_{12}e^{-i\alpha/2}\sin\alpha\ , \tag{A.168}$$

$$e_{12}\left(1 - e^{i\alpha/2}\cos\alpha\ \right) = e_{22}e^{i\alpha/2}\sin\alpha\ , \tag{A.169}$$

and

$$-e_{22}\left(1 - e^{\frac{i\alpha}{2}}\cos\alpha\right) = e_{12}e^{i\alpha/2}\sin\alpha \tag{A.170}$$

which requires $e_{21} = \pm ie_{11}$ and $e_{22} = \pm ie_{12}$. The matrix E_2 is then restricted to

$$E_2 = \begin{pmatrix} \pm ie_{11} & \pm ie_{12} \\ \mp ie_{12} & \pm ie_{11} \end{pmatrix}. \tag{A.171}$$

Using the first and third components of the y-rotation equation we know that

$$E_1 = {R_y}^{-1}E_1R_y\cos\beta\ + {R_y}^{-1}E_3R_y\sin\beta \tag{A.172}$$

and

$$E_3 = -{R_y}^{-1}E_1R_y\sin\beta\ + {R_y}^{-1}E_3R_y\cos\beta\ . \tag{A.173}$$

Explicit substitution produces

$$E_1 = \begin{pmatrix} e_{11} & e_{12} \\ e_{12} & e_{11} \end{pmatrix} = \begin{pmatrix} e_{11} - 2e_{12}\sin\frac{\beta}{2}\cos\frac{\beta}{2} & e_{12}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right) \\ e_{12}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right) & e_{11} + 2\, e_{12}\sin\frac{\beta}{2}\cos\frac{\beta}{2} \end{pmatrix} \cos\beta$$

$$+ \begin{pmatrix} e_{31}\cos^2\beta/2 + e_{34}\sin^2\frac{\beta}{2} & (e_{31} - e_{34})\sin\frac{\beta}{2}\cos\frac{\beta}{2} \\ (e_{31} - e_{34})\sin\frac{\beta}{2}\cos\frac{\beta}{2} & e_{34}\cos^2\frac{\beta}{2} + e_{31}\sin^2\frac{\beta}{2} \end{pmatrix} \sin\beta \qquad \text{(A.174)}$$

and

$$E_3 = \begin{pmatrix} e_{31} & 0 \\ 0 & e_{34} \end{pmatrix} = - \begin{pmatrix} e_{11} - 2e_{12}\sin\frac{\beta}{2}\cos\frac{\beta}{2} & e_{12}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right) \\ e_{12}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right) & e_{11} + 2\, e_{12}\sin\frac{\beta}{2}\cos\frac{\beta}{2} \end{pmatrix} \sin\beta$$

$$+ \begin{pmatrix} e_{31}\cos^2\beta/2 + e_{34}\sin^2\frac{\beta}{2} & (e_{31} - e_{34})\sin\frac{\beta}{2}\cos\frac{\beta}{2} \\ (e_{31} - e_{34})\sin\frac{\beta}{2}\cos\frac{\beta}{2} & e_{34}\cos^2\frac{\beta}{2} + e_{31}\sin^2\frac{\beta}{2} \end{pmatrix} \cos\beta\ . \qquad \text{(A.175)}$$

Adding the first and fourth components of the first equation produces

$$2e_{11} = 2e_{11}\cos\beta + (e_{31} + e_{34})\sin\beta\ . \qquad \text{(A.176)}$$

Adding the first and fourth components of the second equation produces

$$(e_{31} + e_{34}) = -2e_{11}\sin\beta + (e_{31} + e_{34})\cos\beta\ . \qquad \text{(A.177)}$$

These equations may be combined to give

$$2e_{11} = \pm i\,(e_{31} + e_{34})\,, \qquad \text{(A.178)}$$

$$e_{12} = e_{12}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right)\cos\beta + (e_{31} - e_{34})\sin\frac{\beta}{2}\cos\frac{\beta}{2}\sin\beta\ , \qquad \text{(A.179)}$$

and

$$0 = -e_{12}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right)\sin\beta + (e_{31} - e_{34})\sin\frac{\beta}{2}\cos\frac{\beta}{2}\cos\beta\ . \qquad \text{(A.180)}$$

From the second and third components of the x-rotation equation we know

$$E_2 = {R_x}^{-1}E_2R_x\cos\gamma + {R_x}^{-1}E_3R_x\sin\gamma \qquad \text{(A.181)}$$

and

$$E_3 = -{R_x}^{-1}E_2R_x\sin\gamma\ + {R_x}^{-1}E_3R_x\cos\gamma\ . \tag{A.182}$$

Explicit substitution produces

$$\begin{aligned}
E_2 &= \begin{pmatrix} e_{21} & e_{22} \\ -e_{22} & e_{21} \end{pmatrix} \\
&= \begin{pmatrix} e_{21} + i2e_{22}\sin\gamma/2\ \cos\gamma/2 & e_{22}\left(\cos^2\gamma/2\ - \sin^2\frac{\gamma}{2}\ \right) \\ e_{22}\left(-\cos^2\gamma/2\ + \sin^2\frac{\gamma}{2}\ \right) & e_{21} - i2e_{22}\sin\gamma/2\ \cos\gamma/2 \end{pmatrix}\cos\gamma \\
&+ \begin{pmatrix} e_{31}\cos^2\gamma/2\ + e_{34}\sin^2\frac{\gamma}{2} & i\left(e_{31} - e_{34}\right)\sin\gamma/2\ \cos\gamma/2 \\ i\left(e_{34} - e_{31}\right)\sin\gamma/2\ \cos\gamma/2 & e_{34}\cos^2\gamma/2\ + e_{31}\sin^2\frac{\gamma}{2} \end{pmatrix}\sin\gamma
\end{aligned} \tag{A.183}$$

and

$$\begin{aligned}
E_3 &= \begin{pmatrix} e_{31} & 0 \\ 0 & e_{34} \end{pmatrix} \\
&= -\begin{pmatrix} e_{21} + i2e_{22}\sin\gamma/2\ \cos\gamma/2 & e_{22}\left(\cos^2\gamma/2\ - \sin^2\frac{\gamma}{2}\ \right) \\ e_{22}\left(-\cos^2\gamma/2\ + \sin^2\frac{\gamma}{2}\ \right) & e_{21} - i2e_{22}\sin\gamma/2\ \cos\gamma/2 \end{pmatrix}\sin\gamma \\
&+ \begin{pmatrix} e_{31}\cos^2\gamma/2\ + e_{34}\sin^2\frac{\gamma}{2} & i\left(e_{31} - e_{34}\right)\sin\gamma/2\ \cos\gamma/2 \\ i\left(e_{34} - e_{31}\right)\sin\gamma/2\ \cos\gamma/2 & e_{34}\cos^2\gamma/2\ + e_{31}\sin^2\frac{\gamma}{2} \end{pmatrix}\cos\gamma\ .
\end{aligned} \tag{A.184}$$

Adding the first and fourth terms of the first and second equations produces

$$2e_{21} = 2e_{21}\cos\gamma\ + \left(e_{31} + e_{34}\right)\sin\gamma \tag{A.185}$$

and

$$\left(e_{31} + e_{34}\right) = -2e_{21}\sin\gamma\ + \left(e_{31} + e_{34}\right)\cos\gamma \tag{A.186}$$

which may be combined to reveal

$$2e_{21} = \pm i\left(e_{31} + e_{34}\right). \tag{A.187}$$

Similarly, the combination of

$$e_{22} = e_{22}\left(\cos^2\gamma/2 \ -\sin^2\frac{\gamma}{2}\ \right)\cos\gamma \ +i\left(e_{31}-e_{34}\right)\sin\gamma/2\ \cos\gamma/2\ \sin\gamma \tag{A.188}$$

and

$$0 = -e_{22}\left(\cos^2\gamma/2 \ -\sin^2\frac{\gamma}{2}\ \right)\sin\gamma \ +i\left(e_{31}-e_{34}\right)\sin\gamma/2\ \cos\gamma/2\ \cos\gamma \tag{A.189}$$

yields

$$2e_{22} = i\left(e_{31}-e_{34}\right). \tag{A.190}$$

The components of the y-rotation equation requires

$$2e_{11} = \pm i\left(e_{31}+e_{34}\right), \tag{A.191}$$

$$e_{12} = e_{12}\left(\cos^2\frac{\beta}{2} \ -\sin^2\frac{\beta}{2}\ \right)\cos\beta \ +\left(e_{31}-e_{34}\right)\sin\frac{\beta}{2}\ \cos\frac{\beta}{2}\ \sin\beta\ , \tag{A.192}$$

and

$$0 = -e_{12}\left(\cos^2\frac{\beta}{2} \ -\sin^2\frac{\beta}{2}\ \right)\sin\beta \ +\left(e_{31}-e_{34}\right)\sin\frac{\beta}{2}\ \cos\frac{\beta}{2}\ \cos\beta \tag{A.193}$$

which yields the result

$$2e_{12} = \left(e_{31}-e_{34}\right). \tag{A.194}$$

From these conditions we may conclude

$$e_{21} = e_{11}, \tag{A.195}$$

$$e_{22} = ie_{12}, \tag{A.196}$$

and

$$E_1 = \begin{pmatrix} e_{11} & e_{12} \\ e_{12} & e_{11} \end{pmatrix}, \quad E_2 = \begin{pmatrix} e_{11} & ie_{12} \\ -ie_{12} & e_{11} \end{pmatrix}, \tag{A.197}$$

and

$$E_3 = \begin{pmatrix} e_{31} & 0 \\ 0 & e_{34} \end{pmatrix}. \tag{A.198}$$

Comparing this result to the earlier result

$$E_2 = \begin{pmatrix} \pm ie_{11} & \pm ie_{12} \\ \mp ie_{12} & \pm ie_{11} \end{pmatrix} = \begin{pmatrix} e_{11} & ie_{12} \\ -ie_{12} & e_{11} \end{pmatrix} \tag{A.199}$$

allows us to conclude that $e_{11} = 0$ and thus

$$E_1 = \begin{pmatrix} 0 & e_{12} \\ e_{12} & 0 \end{pmatrix}, \quad E_2 = i \begin{pmatrix} 0 & -e_{12} \\ e_{12} & 0 \end{pmatrix}, \tag{A.200}$$

and

$$E_3 = \begin{pmatrix} e_{12} & 0 \\ 0 & -e_{12} \end{pmatrix}. \tag{A.201}$$

The above arguments apply to the other sub-matrices $F,\ G,\ H$ as well.

At this point we are able to combine the results of all the restrictions obtained from applying rotations to the second condition for invariance.

$$\begin{aligned} B_{21} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}_1 &= \begin{pmatrix} 0 & e_{12} & 0 & f_{12} \\ e_{12} & 0 & f_{12} & 0 \\ 0 & g_{12} & 0 & h_{12} \\ g_{12} & 0 & h_{12} & 0 \end{pmatrix} \\ &= \begin{pmatrix} e_{12} & f_{12} \\ g_{12} & h_{12} \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{A.202}$$

$$\begin{aligned} B_{22} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}_2 &= i \begin{pmatrix} 0 & -e_{12} & 0 & -f_{12} \\ e_{12} & 0 & f_{12} & 0 \\ 0 & -g_{12} & 0 & -h_{12} \\ g_{12} & 0 & h_{12} & 0 \end{pmatrix} \\ &= \begin{pmatrix} e_{12} & f_{12} \\ g_{12} & h_{12} \end{pmatrix} \times \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \end{aligned} \tag{A.203}$$

$$B_{23} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}_3 = \begin{pmatrix} e_{12} & 0 & f_{12} & 0 \\ 0 & -e_{12} & 0 & -f_{12} \\ g_{12} & 0 & h_{12} & 0 \\ 0 & -g_{12} & 0 & -h_{12} \end{pmatrix}$$

$$= \begin{pmatrix} e_{12} & f_{12} \\ g_{12} & h_{12} \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.204}$$

This final result can be written more compactly as

$$B_{2j} = \begin{pmatrix} e_{12}\sigma_j & f_{12}\sigma_j \\ g_{12}\sigma_j & h_{12}\sigma_j \end{pmatrix} = \begin{pmatrix} e\sigma_j & f\sigma_j \\ g\sigma_j & h\sigma_j \end{pmatrix} \tag{A.205}$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{A.206}$$

and

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.207}$$

are the Pauli matrices.

A.17.3 Finding a 4 × 4 Matrix that Satisfies the Third Condition for Rotational Invariance

When rotations only are applied to the third condition it has an identical form to the first condition. The constraints on the matrix are therefore identical and we conclude

$$B_3 = \begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix}. \tag{A.208}$$

A.18 Finding 4×4 Matrices that Satisfy the Conditions for Boost Invariance

A.18.1 Finding a 4×4 Matrix that Satisfies the First Condition for Boost Invariance

The 5 matrices still have 12 of the original 80 free parameters remaining after requiring rotational invariance. Boosts are applied now to further restrict the matrices from their current form

$$B_1 = \begin{pmatrix} pI & qI \\ sI & tI \end{pmatrix}, B_{2j} = \begin{pmatrix} e\sigma_j & f\sigma_j \\ g\sigma_j & h\sigma_j \end{pmatrix}, \tag{A.209}$$

and

$$B_3 = \begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix}. \tag{A.210}$$

The boost and inverse boost transformations are

$$T = \begin{pmatrix} I & 0 \\ -\sigma \cdot v/2 & I \end{pmatrix} \tag{A.211}$$

and

$$T^{-1} = \begin{pmatrix} I & 0 \\ \sigma \cdot v/2 & I \end{pmatrix}. \tag{A.212}$$

Plugging these matrices into the first condition produces the following equation

$$B_1 = \begin{pmatrix} pI & qI \\ sI & tI \end{pmatrix}$$

$$= TB_1T^{-1} = \begin{pmatrix} I & 0 \\ -\sigma \cdot v/2 & I \end{pmatrix} \begin{pmatrix} pI & qI \\ sI & tI \end{pmatrix} \begin{pmatrix} I & 0 \\ \sigma \cdot v/2 & I \end{pmatrix}$$

$$= \begin{pmatrix} pI & qI \\ -p\sigma \cdot \frac{v}{2} + sI & -q\sigma \cdot \frac{v}{2} + tI \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sigma \cdot v/2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} pI + q\sigma\cdot v/2 & qI \\ -p\sigma\cdot\frac{v}{2} + sI - q\left(\sigma\cdot\frac{v}{2}\right)^2 + t\sigma\cdot v/2 & -q\sigma\cdot\frac{v}{2} + tI \end{pmatrix}. \tag{A.213}$$

From the first and third components of this equation we can see that $q = 0$ and $t = p$ so we conclude that the matrix B_1 is restricted to

$$B_1 = \begin{pmatrix} pI & 0 \\ sI & pI \end{pmatrix}. \tag{A.214}$$

A.18.2 Finding 4×4 Matrices that Satisfy the Second Condition for Boost Invariance

Further constraints can be derived from applying boosts only to the second condition for invariance. Let B_{2j} be an unknown 4×4 matrix

$$B_{2j} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}_j. \tag{A.215}$$

Inserting this matrix and the boost only transformation into the second condition produces the following equation

$$\begin{aligned} B_{2j} &= TB_1T^{-1}\delta_{ij}v_i + TB_{2i}T^{-1}\delta_{ji} \\ &= \begin{pmatrix} pI & 0 \\ sI & pI \end{pmatrix} v_j + \begin{pmatrix} I & 0 \\ -\sigma\cdot v/2 & I \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix}_j \begin{pmatrix} I & 0 \\ \sigma\cdot v/2 & I \end{pmatrix} \\ &= \begin{pmatrix} pI & 0 \\ sI & pI \end{pmatrix} v_j + \begin{pmatrix} E & F \\ -\sigma\cdot\frac{vE}{2} + G & -\sigma\cdot\frac{vF}{2} + H \end{pmatrix}_j \begin{pmatrix} I & 0 \\ \sigma\cdot v/2 & I \end{pmatrix} \\ &= \begin{pmatrix} pI & 0 \\ sI & pI \end{pmatrix} v_j + \begin{pmatrix} E + F\sigma\cdot v/2 & F \\ -\sigma\cdot\frac{vE}{2} + G - \sigma\cdot\frac{vF}{2}\sigma\cdot v/2 + H\sigma\cdot v/2 & -\sigma\cdot\frac{vF}{2} + H \end{pmatrix}_j. \end{aligned} \tag{A.216}$$

From the first comonent of this equation we know that

$$-pIv_j = \frac{F_j\sigma_i v_i}{2}. \tag{A.217}$$

Explicit representations of the 2×2 matrices in this equation may be substituted in to produce the equation

$$-pIv_x = \begin{pmatrix} -pv_x & 0 \\ 0 & -pv_x \end{pmatrix} = \frac{1}{2}f_{12}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{pmatrix}$$

$$= \frac{1}{2}f_{12}\begin{pmatrix} v_x + iv_y & -v_z \\ v_z & v_x - iv_y \end{pmatrix}. \tag{A.218}$$

From this we must conclude $f_{12} = p = 0$ and $F_j = 0$.

The second and fourth terms of equation add nothing useful while the third term is

$$0 = sIv_j + -\sigma \cdot \frac{vE_j}{2} + H_j\sigma \cdot v/2. \tag{A.219}$$

Rearranging the equation to seperate terms containing σ produces the equation

$$\begin{aligned} -2sIv_j &= -\sigma \cdot vE_j + H_j\sigma \cdot v \\ &= -\sigma_i v_i E_j + H_j \sigma_i v_i \\ &= -\sigma_i v_i e \sigma_j + h\sigma_j \sigma_i v_i \\ &= (h\sigma_j\sigma_i - e\sigma_i\sigma_j)\, v_i. \end{aligned} \tag{A.220}$$

This can be seen explicitly for j=1

$$\begin{pmatrix} -2sv_x & 0 \\ 0 & -2sv_x \end{pmatrix}$$

$$= -\begin{pmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{pmatrix} e \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + h\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{pmatrix}$$

$$= -e\begin{pmatrix} v_x - iv_y & v_z \\ -v_z & v_x + iv_y \end{pmatrix} + h\begin{pmatrix} v_x + iv_y & -v_z \\ v_z & v_x - iv_y \end{pmatrix}. \tag{A.221}$$

Notice that from the second and third components we may conclude that $e = -h$. From this we conclude the free parameters are restricted to $h = -e = -s$.

To summarize the results up to this point we have determined the matrices of the first order equation are constrained to the form

$$B_1 = \begin{pmatrix} pI & 0 \\ sI & pI \end{pmatrix}, \quad B_{2j} = \begin{pmatrix} s\sigma_j & 0 \\ g\sigma_j & -s\sigma_j \end{pmatrix} \tag{A.222}$$

and

$$B_3 = \begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix}. \tag{A.223}$$

A.18.3 Finding a 4×4 Matrix that Satisfies the Third Condition for Boost Invariance

Further constraints may be obtained by applying boosts without rotations to the third and final condition

$$B_3 = TB_1T^{-1}\left(\frac{i}{2}mv^2 - imR_{ji}v_jv_kR_{ik}\right) - TB_{2i}T^{-1}imR_{ji}v_kR_{jk} + TB_3T^{-1}. \tag{A.224}$$

We begin by letting the B_3 take the form of an unknown 4×4 matrix

$$B_3 = \begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix}. \tag{A.225}$$

Putting the known matrices into the third condition produces the equation

$$\begin{aligned} = &\begin{pmatrix} I & 0 \\ -\sigma\cdot v/2 & I \end{pmatrix}\begin{pmatrix} pI & 0 \\ sI & pI \end{pmatrix}\begin{pmatrix} I & 0 \\ \sigma\cdot v/2 & I \end{pmatrix}\left(-\frac{i}{2}mv^2\right) \\ &- \begin{pmatrix} I & 0 \\ -\sigma\cdot v/2 & I \end{pmatrix}\begin{pmatrix} s\sigma_j & 0 \\ g\sigma_j & -s\sigma_j \end{pmatrix}\begin{pmatrix} I & 0 \\ \sigma\cdot v/2 & I \end{pmatrix} imv_j \\ &+ \begin{pmatrix} I & 0 \\ -\sigma\cdot v/2 & I \end{pmatrix}\begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix}\begin{pmatrix} I & 0 \\ \sigma\cdot v/2 & I \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} pI & 0 \\ sI & pI \end{pmatrix} \left(-\frac{i}{2}mv^2\right) - \begin{pmatrix} s\sigma_j & 0 \\ -s\sigma\cdot\frac{v}{2}\sigma_j + g\sigma_j - s\sigma_j\sigma\cdot v/2 & -s\sigma_j \end{pmatrix} imv_j$$

$$+ \begin{pmatrix} aI + b\sigma\cdot v/2 & bI \\ -a\sigma\cdot\frac{v}{2} + cI - b\sigma\cdot\frac{v}{2}\sigma\cdot\frac{v}{2} + d\sigma\cdot v/2 & -b\sigma\cdot\frac{v}{2} + dI \end{pmatrix}. \tag{A.226}$$

The second component of this equation provides no information. The first or fourth components can be extracted for closer examination

$$0 = -\frac{i}{2}mv^2pI - imsv_j\sigma_j + bv_j\sigma_j/2,$$

$$= -\frac{i}{2}mv^2p \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(-ims + \frac{b}{2}\right) \begin{pmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{pmatrix}. \tag{A.227}$$

This equation requires $b = 2ims$ and thus $p = 0$. The third component can also be examined more closely

$$0 = -\frac{i}{2}mv^2sI + \left(s\sigma\cdot\frac{v}{2}\sigma_j - g\sigma_j + s\sigma_j\sigma\cdot v/2\right) imv_j$$

$$-a\sigma\cdot\frac{v}{2} - b\sigma\cdot\frac{v}{2}\sigma\cdot\frac{v}{2} + d\sigma\cdot\frac{v}{2}$$

$$= -\frac{i}{2}mv^2sI + \left(s\sigma\cdot\frac{v}{2}\sigma_j + s\sigma_j\sigma\cdot v/2\right) imv_j - b\sigma\cdot\frac{v}{2}\sigma\cdot\frac{v}{2} + \frac{(d-a-2img)}{2}(\sigma\cdot v)$$

$$= -\frac{ims}{2}v^2I + \left(\frac{ims}{2}v^2 + \frac{ims}{2}v^2\right) I - \frac{b}{4}v^2I + \frac{(d-a-2img)}{2}(\sigma\cdot v)$$

$$= -\frac{ims}{2}v^2I + \frac{ims}{2}v^2I + \frac{ims}{2}v^2I - \frac{ims}{2}v^2I + \frac{(d-a-2img)}{2}(\sigma\cdot v)$$

$$= \frac{(d-a-2img)}{2}(\sigma\cdot v)$$

$$= \frac{(d-a-2img)}{2} \begin{pmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{pmatrix}. \tag{A.228}$$

This equation requires

$$img = \frac{(d-a)}{2} \tag{A.229}$$

So now we conclude that the matrices are constrained to the form

$$B_1 = \begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix}, \quad B_{2j} = \begin{pmatrix} s\sigma_j & 0 \\ \frac{-i(d-a)}{2m} & -s\sigma_j \end{pmatrix}, \tag{A.230}$$

and

$$B_3 = \begin{pmatrix} aI & 2misI \\ cI & dI \end{pmatrix}. \tag{A.231}$$

A.19 Finding 4×4 Matrices that Satisfy the Conditions for Rotational and Boost Invariance

A.19.1 Finding a 4×4 Matrix that Satisfies the First Condition for Rotational and Boost Invariance

Note the transformations can be defined with boosts before rotations or after rotations. Each choice results in the condition equations and transformation matrices having a different form.

Rotations and boosts can be applied to the first condition for invariance to produce the result

$$\begin{aligned} B_1 = TB_1T^{-1} &= \begin{pmatrix} R & 0 \\ -\sigma \cdot vR/2 & R \end{pmatrix} \begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix} \begin{pmatrix} R^{-1} & 0 \\ R^{-1}\sigma \cdot v/2 & R^{-1}R \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ sR & 0 \end{pmatrix} \begin{pmatrix} R^{-1} & 0 \\ R^{-1}\sigma \cdot v/2 & R^{-1}R \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix}. \end{aligned} \tag{A.232}$$

This result provides no new restriction on the matrices.

A.19.2 Finding 4×4 Matrices that Satisfy the Second Condition for Rotational and Boost Invariance

Rotations and boosts can be applied to the second condition for invariance leading to the following equation

$$\begin{aligned}
B_{2j} &= TB_1T^{-1}R_{ij}v_i + TB_{2i}T^{-1}R_{ji} \\
&= \begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix} R_{ij}v_i \\
&+ \begin{pmatrix} R & 0 \\ -\sigma\cdot vR/2 & R \end{pmatrix} \begin{pmatrix} s\sigma_i & 0 \\ \frac{-i(d-a)}{2m} & -s\sigma_i \end{pmatrix} \begin{pmatrix} R^{-1} & 0 \\ R^{-1}\sigma\cdot v/2 & R^{-1}R \end{pmatrix} R_{ji} \\
&= \begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix} R_{ij}v_i \\
&+ \begin{pmatrix} sR\sigma_i & 0 \\ -\sigma\cdot\frac{vR}{2}s\sigma_i + \frac{-i(d-a)}{2m}R & -sR\sigma_i \end{pmatrix} \begin{pmatrix} R^{-1} & 0 \\ R^{-1}\sigma\cdot v/2 & R^{-1}R \end{pmatrix} R_{ji} \\
&= \begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix} R_{ij}v_i + \\
&\begin{pmatrix} sR\sigma_iR^{-1} & 0 \\ -\sigma\cdot\frac{vR}{2}s\sigma_iR^{-1} + \frac{-i(d-a)}{2m} - sR\sigma_iR^{-1}\sigma\cdot v/2 & -sR\sigma_iR^{-1} \end{pmatrix} R_{ji}.
\end{aligned} \tag{A.233}$$

We already know the first, second, and fourth components work out from the earlier rotational invariance calculations. That is we know

$$R\sigma_iR^{-1}R_{ji} = \sigma_j. \tag{A.234}$$

It yet remains to work out the third component equation.

$$sIR_{ij}v_i + \left[-\sigma\cdot\frac{vR}{2}s\sigma_iR^{-1} + \frac{-i\,(d-a)}{2m} - sR\sigma_iR^{-1}\sigma\cdot v/2\right]R_{ji} = \frac{-i\,(d-a)}{2m}. \tag{A.235}$$

Which produces a new condition $a = d$ in order for the remainder of the equation to work out as follows

$$-2sIR_{ij}v_i = s\left[-\sigma\cdot vR\sigma_i R^{-1} - R\sigma_i R^{-1}\sigma\cdot v\right]R_{ji}$$

$$= s\left[-\sigma\cdot vR\sigma_i R^{-1}R_{ji} - R\sigma_i R^{-1}R_{ji}\sigma\cdot v\right]$$

$$= s\left[-\sigma\cdot v\sigma_j - \sigma_j\sigma\cdot v\right]$$

$$= s\left[-\sigma_k v_k\sigma_j - \sigma_j\sigma_k v_k\right]$$

$$= sv_k\left[-\sigma_k\sigma_j - \sigma_j\sigma_k\right]$$

$$= -sv_k\left\{\sigma_k, \sigma_j\right\}$$

$$= -sv_k 2\delta_{kj} I$$

$$= -sv_j 2I. \tag{A.236}$$

No additional restrictions can be found from this equation and s remains a free parameter. To summarize the results at this stage the matrices are

$$B_1 = \begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix}, \quad B_{2j} = \begin{pmatrix} s\sigma_j & 0 \\ 0 & -s\sigma_j \end{pmatrix}, \tag{A.237}$$

and

$$B_3 = \begin{pmatrix} aI & 2misI \\ cI & aI \end{pmatrix}. \tag{A.238}$$

A.19.3 Finding a 4×4 Matrix that Satisfies the Third Condition for Rotational and Boost Invariance

Rotations and boosts can be applied to the third condition for invariance to produce an equation

$$B_3 = TB_1T^{-1}\left(\frac{i}{2}mv^2 - imR_{ji}v_jv_kR_{ik}\right) - TB_{2i}T^{-1}imR_{ji}v_kR_{jk} + TB_3T^{-1}$$

$$= \begin{pmatrix} aI & 2misI \\ cI & aI \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix} \left(-\frac{i}{2}mv^2\right)$$

$$- \begin{pmatrix} sR\sigma_i R^{-1} & 0 \\ -\sigma \cdot \frac{vR}{2} s\sigma_i R^{-1} - sR\sigma_i R^{-1}\sigma \cdot v/2 & -sR\sigma_i R^{-1} \end{pmatrix} imR_{ji}v_k R_{jk}$$

$$+ \begin{pmatrix} R & 0 \\ -\sigma \cdot vR/2 & R \end{pmatrix} \begin{pmatrix} aI & 2misI \\ cI & aI \end{pmatrix} \begin{pmatrix} R^{-1} & 0 \\ R^{-1}\sigma \cdot v/2 & R^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix} \left(-\frac{i}{2}mv^2\right)$$

$$- \begin{pmatrix} sR\sigma_i R^{-1} & 0 \\ -\sigma \cdot \frac{vR}{2} s\sigma_i R^{-1} - sR\sigma_i R^{-1}\sigma \cdot \frac{v}{2} & -sR\sigma_i R^{-1} \end{pmatrix} imR_{ji}v_k R_{jk}$$

$$+ \begin{pmatrix} aI + 2ims\sigma \cdot v/2 & 2imsI \\ cI - ismv^2/2 & -ism\sigma \cdot v + aI \end{pmatrix}. \quad (A.239)$$

All the components of this equation check out without providing new restrictions. In conclusion, the only 4×4 matrices that can satisfy the conditions for rotational and boost invariance are

$$B_1 = \begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix}, \quad B_{2j} = \begin{pmatrix} s\sigma_j & 0 \\ 0 & -s\sigma_j \end{pmatrix}, \quad (A.240)$$

and

$$B_3 = \begin{pmatrix} aI & 2misI \\ cI & aI \end{pmatrix} \quad (A.241)$$

where a, c, and s are scalar free parameters.

A.20 Requiring Hermiticity of Observables

Observables in a quantum theory are represented by operators that act on the state functions to produce eigenvalues that are measureable quantities. In order for the eigenvalues to be real the observables (operators) must be Hermitian. The dynamical equation is restructured to help us impose this restriction. After multiplying through by i and moving the time differential to the left hand side the dynamical equation is

$$i\begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix}\partial_t\psi(x,t) = \left[-i\begin{pmatrix} s\sigma_j & 0 \\ 0 & -s\sigma_j \end{pmatrix}\partial_j - i\begin{pmatrix} aI & 2misI \\ cI & aI \end{pmatrix}\right]\psi(x,t). \tag{A.242}$$

The momentum operator $\hat{p_j} = -i\partial_j$ and energy operator $\hat{\varepsilon} = i\partial_t$ may be substituted into the equation

$$\begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix}\hat{\varepsilon}\psi(x,t) = \left[\begin{pmatrix} s\sigma_j & 0 \\ 0 & -s\sigma_j \end{pmatrix}\hat{p_j} - i\begin{pmatrix} aI & 2misI \\ cI & aI \end{pmatrix}\right]\psi(x,t). \tag{A.243}$$

The matrix B_j is already Hermitian if s is real. The triangular matrix B_t by itself cannot be Hermitian but in a linear combination $\kappa B_t + B_c$ it can be made Hermitian under the conditions that a is imaginary and $(2ms)^\dagger = -ic + \kappa s$ meaning that c is imaginary and κ is real.

A.21 Finding State Functions that Transform like Irreducible Representations

A.21.1 Deriving the Eigen-Equations

A set of functions ψ_j forms a basis for an irreducible representation if the transformation operations T of the group acting on the function can be written as a linear combination of the functions

$$T\psi_j = \sum_k A_{jk}\psi_k \tag{A.244}$$

where A_{jk} is a matrix. When the elements of the sum are infinite and the functions are analytic the series may be summed to an exponential. Therefore the state function $\psi(x,t)$ transformed by a space translation $T(a_j)$ can be written as

$$T(a_j)\psi(x,t) = \psi(x_j + a_j, t) = e^{ik_j a_j}\psi(x,t) \tag{A.245}$$

and a time translations $T(a_t)$ of the state function can be written as

$$T(a_t)\psi(x,t) = \psi(x, t + a_t) = e^{i\omega a_t}\psi(x,t). \tag{A.246}$$

Since we require the state functions to be analytic the transformation operators can be reformed by summing the Taylor series expansions of the transformed functions into exponential functions

$$T(a_j)\psi(x,t) = \psi(x_j + a_j, t) = e^{i(-ia_j\partial_j)}\psi(x,t) \tag{A.247}$$

and

$$T(a_t)\psi(x,t) = \psi(x_j, t + a_t) = e^{i(-ia_t\partial_t)}\psi(x,t) \tag{A.248}$$

where we define the generators of space translations $\hat{p}_j = -i\partial_j$ and time translations $\hat{E} = i\partial_t$ and where a_j and a_t are the parameters of the space and time translations respectively. Combining these results produces the eigen-equations for space translations

$$-i\partial_j\psi(x,t) = k_j\psi(x,t) \tag{A.249}$$

and time translations

$$i\partial_t\psi(x,t) = \omega\psi(x,t). \tag{A.250}$$

The eigen-equations can be written in terms of the operators $\hat{k}_j = -i\partial_j$ and $\hat{\omega} = i\partial_t$

$$\hat{k}_j\psi(x,t) = k_j\psi(x,t) \tag{A.251}$$

and

$$\hat{\omega}\psi(x,t) = \omega\psi(x,t). \tag{A.252}$$

Furthermore, operators of momentum $\hat{p}_j$ and energy $\hat{\varepsilon}$ may be defined in terms of the wave quantities $\hat{p}_j = \hbar\hat{k}_j$ and $\hat{\varepsilon} = \hbar\hat{\omega}$ where $\hbar$ relates the quantities by a choice of units. Then the eigen-equations may be expressed in term of these energy and momentum operators

$$\hat{p}_j\psi(x,t) = p_j\psi(x,t) \tag{A.253}$$

and

$$\hat{\varepsilon}\psi(x,t) = \varepsilon\psi(x,t). \tag{A.254}$$

In constructing irreducible representations we wish to accomplish several objectives. First, we want a set of variables that are invariant under the transformations of the group. This is accomplished when the operators correspond to the variables form a commuting set. The benefit of this is that when the variables are invariant the eigenvalues provide convenient labels for the state. Second, we want the variables to be interpreted as measurable values so they must be real. If the generators χ_j of the group are Hermitian, representations of the group $e^{i\chi_j\Pi_j}$ will be unitary. When operators are applied to the state function, eigenvalues are produced and the eigenvalues will be real if the operators are unitary. Finally, in addition to finding operator representations for the generators of the group we want to find any other unique operators which can commute with them. The additional operators are called Casimir operators. By Schur's lemma this maximal or complete set of commuting operators will form an irreducible representation if and only if the Casimir operators are multiples of unity. In this case the eigenvalues of the Casimir operators can be said to specify the representation.

A.21.2 Restricting the Dynamical Equations to State Functions that Transform like Irreducible Representations

The eigen-equations can be applied to further constrain the free parameters of the matrices in the dynamical equation (A.243). Replacing the operators in equation (A.243) with their eigen-values

$$\begin{pmatrix} 0 & 0 \\ sI & 0 \end{pmatrix} \varepsilon\psi(x,t) = \left[\begin{pmatrix} s\sigma_j & 0 \\ 0 & -s\sigma_j \end{pmatrix} p_j - i \begin{pmatrix} aI & 2misI \\ cI & aI \end{pmatrix} \right] \psi(x,t). \quad \text{(A.255)}$$

Writing the state functions in terms of a bispinor

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \quad \text{(A.256)}$$

allows the equation to be split into a pair of linked equations

$$s\sigma_j p_j \phi - ia\phi + 2ms\chi = 0 \quad \text{(A.257)}$$

and

$$s\varepsilon\phi + ic\phi + s\sigma_j p_j \chi + ia\chi = 0. \quad \text{(A.258)}$$

These equations will admit plane wave solutions only when the determinant vanishes

$$\begin{vmatrix} s\sigma_j p_j - ia & 2ms \\ s\varepsilon + ic & s\sigma_j p_j + ia \end{vmatrix} = 0. \quad \text{(A.259)}$$

From equation (A.259) it follows that

$$(s\sigma_j p_j - ia)(s\sigma_j p_j + ia) - 2ms(s\varepsilon + ic) = 0. \quad \text{(A.260)}$$

Solving for the energy ε produces a momentum-energy relation

$$\varepsilon = p^2/2m + a^2/2ms^2 - ic/s. \quad \text{(A.261)}$$

There always exists some frame of reference where momentum is zero $p_j = 0$ and energy $\varepsilon = \varepsilon_0$ is the rest energy ε_0. In this frame of reference the equation (A.261) is

$$\varepsilon_0 = a^2/2ms^2 - ic/s. \tag{A.262}$$

Employing this result we conclude that the momentum energy relation is

$$\varepsilon = p^2/2m + \varepsilon_0. \tag{A.263}$$

Furthermore, the remaining free parameters can be eliminated from the first order equation by using the Hermiticity conditions derived earlier in combination with equation (A.262). After dividing out the constant s the equation first order equation is

$$i \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \partial_t \psi(x,t) = -i \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix} \partial_j \psi(x,t)$$
$$-i \begin{pmatrix} (2m\varepsilon_0 - 4m^2 + 2m\kappa)^{\frac{1}{2}} I & 2imI \\ i(2m-\kappa)I & (2m\varepsilon_0 - 4m^2 + 2m\kappa)^{\frac{1}{2}} I \end{pmatrix} \psi(x,t). \tag{A.264}$$

Setting $\kappa = 2m$ and $\varepsilon_0 = 0$ produces the Lévy-Leblond equation.

APPENDIX B

DERIVING POINCARE INVARIANT SPINOR EQUATIONS

In appendix B concepts relevant to the Poincaré group are developed.

B.1 Poincaré Transformations

Minkowski space-time is defined by the Minkowski metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \tag{B.1}$$

Minkowski space-time is characterized by a set of transformations that leave the Minkowski metric invariant. These are the Poincaré transformations

$$x'^{\nu} = \Lambda^{\nu}_{\mu} x^{\mu} + b^{\nu} \tag{B.2}$$

and they relate the coordinates x^{ν} of one observer to those of another observer x'^{ν} by a rotation and Lorentz boost Λ and a space-time translation vector b^{ν}. The rotation matrices about the angles θ_j take the form

$$\Lambda(\theta_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix}. \tag{B.3}$$

$$\Lambda(\theta_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}, \tag{B.4}$$

and

$$\Lambda(\theta_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_3 & \sin\theta_3 & 0 \\ 0 & -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{B.5}$$

The matrices of the Lorentz boost are

$$\Lambda(\eta_1) = \begin{pmatrix} \cosh\eta_1 & \sinh\eta_1 & 0 & 0 \\ \sinh\eta_1 & \cosh\eta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{B.6}$$

$$\Lambda(\eta_2) = \begin{pmatrix} \cosh\eta_2 & 0 & \sinh\eta_2 & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\eta_2 & 0 & \cosh\eta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{B.7}$$

and

$$\Lambda(\eta_3) = \begin{pmatrix} \cosh\eta_3 & 0 & 0 & \sinh\eta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta_3 & 0 & 0 & \cosh\eta_3 \end{pmatrix} \tag{B.8}$$

where $\cosh\eta_j = (1 - v_j^2)^{-1/2}$ and $\sinh\eta_j = v_j(1 - v_j^2)^{-1/2}$.

B.2 The Poincaré Group

The Poincaré transformations form a group called the Poincaré group $\mathcal{P}$ with transformation operator $P(\Lambda, b^\mu)$ where Λ is the rotation and boost transformation matrix. The group composition law can be worked out by inserting one transformation into another thus

$$\begin{aligned} x''^\nu &= \Lambda^\nu_{2\mu} x'^\mu + b^\nu_2 \\ &= \Lambda^\nu_{2\mu}(\Lambda^\mu_{1\rho} x^\rho + b^\mu_1) + b^\nu_2 \\ &= \Lambda^\nu_{2\mu}\Lambda^\mu_{1\rho} x^\rho + \Lambda^\nu_{2\mu} b^\mu_1 + b^\nu_2 \end{aligned} \tag{B.9}$$

$$P(\Lambda_2\Lambda_1, \Lambda_2 b^\mu_1 + b^\mu_2) = P(\Lambda_2, b^\mu_2) P(\Lambda_1, b^\mu_1) \tag{B.10}$$

This group composition law is an expression of the group multiplication operation and shows how elements of the group behave when applied in conjunction.

It can now be proven that the Poincaré transformations form a group by showing that the elements satisfy all the properties of a group. The group composition law already demonstrates the property of closure. It can also be used to demonstrate the property of associativity

$$P(\Lambda_3\Lambda_2, \Lambda_3 b_2^\mu + b_3^\mu)P(\Lambda_1, b_1^\mu)$$

$$= P(\Lambda_3, b_3^\mu)P(\Lambda_2\Lambda_1, \Lambda_2 b_1^\mu + b_2^\mu). \tag{B.11}$$

Additionally the identity property requires the existence of the identity element

$$E = P(I, 0) \tag{B.12}$$

where the identity matrix I is used for rotations with a zero angle and boosts with zero velocity. Lastly the inverse property is satisfied by the existence of a unique inverse

$$P^{-1}(\Lambda, b^\mu) = P(\Lambda^{-1}, -\Lambda^{-1}b^\mu) \tag{B.13}$$

such that

$$P(\Lambda^{-1}, -\Lambda^{-1}b^\mu)P(\Lambda, b^\mu) = P(I, 0) = E. \tag{B.14}$$

B.3 Poincaré Transformation of Differential Operators

Under the Lorentz transformation a differential operator transforms like

$$\partial'_\mu = \Lambda_\mu^\rho \partial_\rho. \tag{B.15}$$

B.4 Differential Generators of the Poincaré Group

The differential generators of the Poincaré group can be calculated from the Poincaré transformation functions. The differential generators for translations in space-time, boosts, and rotations are

$$X_{b_\mu} = -i\frac{\partial}{\partial x_\mu}, \tag{B.16}$$

$$X_{v_k} = -i\left[t\frac{\partial}{\partial x_k} + x_k\frac{\partial}{\partial_t}\right], \tag{B.17}$$

and

$$X_{\theta_l} = -i\left[x_j\frac{\partial}{\partial x_k}\right]\epsilon_{jkl}. \tag{B.18}$$

B.5 Commutation Relations of the Poincaré Group

The differential generators can be used to determine the commutation relations of the group.

$$\left[X_{\theta_i}, X_{\theta_j}\right] = i\epsilon_{ijk}X_{\theta_k}, \tag{B.19}$$

$$\left[X_{\theta_i}, X_{v_j}\right] = i\epsilon_{ijk}X_{v_k}, \tag{B.20}$$

$$\left[X_{v_i}, X_{v_j}\right] = -i\epsilon_{ijk}X_{\theta_k}, \tag{B.21}$$

B.6 Spinor Representations of the Extended Poincaré Group

A representation is a set of matrices or linear transformations on a vector space that is homomorphic to the group multiplication law. The rotations matrices for 4-component spinors are

$$R_{\theta_j} = \cos\frac{\theta}{2} + \epsilon_{jkl}\gamma_k\gamma_l\sin\frac{\theta}{2} \tag{B.22}$$

where γ_j and γ_0 form a basis for the 4×4 spinor matrices and θ_j is the rotation about the j-axis and in the $k-l$-plane. The boost matrices for 4-component spinors are

$$S_{v_j} = \cosh\frac{\eta}{2} + i\gamma_j\gamma_0\sinh\frac{\eta}{2} \tag{B.23}$$

where η is the boost angle. The boost angle is related to the velocity by $\tanh\eta = \beta = v/c$ where $c = 1$ is the speed of light in natural units. The Dirac representation is chosen for this work whenever explicit representation of the gamma matrices γ^μ is required

$$\gamma^j = \begin{pmatrix} 0 & i\sigma^j \\ -i\sigma^j & 0 \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{B.24}$$

where σ^j are the Pauli matrices.

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.25}$$

In this representation the Minkowski metric $g^{\mu\nu}$ has signature $(+---)$. The covariant gamma matrices are related to the contravariant form by $\gamma_\mu = g_{\mu\nu}\gamma^\nu = \{\gamma^0, -\gamma^j\}$. The gamma matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu} I. \tag{B.26}$$

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BIOGRAPHICAL STATEMENT

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