COMPLETE STRESS ANALYSIS FOR TWO DIMENSIONAL INCLUSION PROBLEM USING COMPLEX VARIABLES

by

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ABSTRACT

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This thesis addresses the problem of finding stresses and displacements in an infinite plate with a two-dimensional circular inclusion with different material properties using the complex variable method with biaxial loading acting on it. The complex variable method is used so that the continuity equations of traction force and displacement field are satisfied at the interface. All the complex equations are solved using software *Mathematica.*

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

Figure Page

CHAPTER 1

INTRODUCTION TO ELASTICITY AND COMPLEX VARIABLE METHOD

1.1 Background

Mechanical engineering uses elasticity in numerous problems in analysis and design of machine elements. Such applications include general stress analysis, contact stresses, thermal stress analysis, fracture mechanics and fatigue. Elasticity is an elegant and fascinating subject that deals with determination of the stress, strain and distribution in an elastic solid under the influence of external forces. This subject also provides the basis for more advanced work in inelastic material behavior including plasticity and viscoelasticity.

Hooke's Law (Proposed in 1678): - When stress is applied to a solid within the elastic limit the strain produced is proportional to the loads producing them. Elasticity theory establishes a mathematical model of the deformation problem and this requires mathematical knowledge to understand the formulation and solution procedures. This is where complex variable theory comes into picture.

The complex variable method provides a very powerful tool for solution of many problems. This theory provided solutions for torsion problem and most importantly the plane problem. This technique is also useful for cases involving anisotropic and thermo elastic materials [1]. This method is based on the reduction of the elasticity boundary value problem to a formulation in the complex domain. This formulation then allows many powerful mathematical techniques available from the complex variable theory to be applied to the elasticity problem. The purpose of this theory is to introduce the basics of this method and to investigate its application to particular problems of engineering interest.

All realistic structures are three-dimensional. But complexities of elastic field equations in analytical closed form are very difficult to accomplish. Most of the solutions developed for elasticity problems include axisymmetry or two- dimensionality which simplify particular aspects of formulation and solution. Thus the theories set forth will be approximate models. The nature and accuracy of the approximation depend on problem and loading geometry.

The two basic theories of plane stress and plane strain represent the fundamental problem in elasticity. These two theories apply to significantly different types of two-dimensional bodies; however, their formulations yield very similar field equations. These two theories can be reduced to one governing equation in terms of a single unknown stress function. However the resulting strains and displacements calculated from these common stresses would not be the same for each plane theory. This occurs because plane strain and plane stress have different forms for Hooke's Law and strain-displacement relations.

Numerous solutions to plane stress and plane strain problems can be obtained through the use of a particular stress technique. The method employs the Airy stress function and will reduce the general formulation to a single governing equation in terms of a single unknown. The resulting governing equation is then solvable by several methods of applied mathematics and thus many analytical solutions to problems of interest can be generated. The stress function formulation is based on the general idea of developing a representation for the stress field that satisfies equilibrium and yields a single governing equation from the compatibility statement. The Airy stress function is represented in terms of functions of a complex variable and transforms the plane problem into one involving complex variable theory. This thesis addresses the problem of Infinite plate with circular inclusion with stresses acting on the plate .In order to solve the complex equations involved algebraic software *Mathematica* is used [1].

1.2 Overview of *Mathematica*

1.2.1 Introduction

Mathematica is a general application organizing many algorithmic, visualization and user interface capabilities within a document-like user interface paradigm. *Mathematica* addresses nearly every field of mathematics, it provides a cross platform support for a wide range of task such as giving computationally interactive presentations, a multifaceted language for data integration, graphics editing and symbolic user interface construction. The *Mathematica* system is very broad, and provides a systematic interface to all sorts of computations, from traditional numeric and symbolic computation, to visualization, to data format conversion, and the creation of user interfaces.

Mathematica is split into two parts, the "kernel" and the "front end"[2]. The kernel is the algorithmic engine for performing computations. The front end provides a convenient human interface for creating and manipulating programmatic structures, allowing graphics, mathematics, programs, text, and user interfaces can be freely edited and intermingled. The two communicate via the MathLink protocol. It is possible to use the kernel on one computer and the front end on another, although this is not common.

A distinguishing characteristic of *Mathematica*, compared to similar systems, is its attempt to uniformly capture all aspects of mathematics and computation, rather than just specialized areas. The main innovation that makes this possible is the idea of symbolic programming captured in the *Mathematica* programming language, emphasing the use of simple tree-like expressions to represent knowledge from a large number of domains.

1.2.2 Features of Mathematica

1. Vast web of mathematical, visualization, graphics, and general programming functions, typically with state of the art implementations.

2. Ability to instantly create user interfaces to arbitrary computations by just specifying parameters.

3. Highly general interface that allows the uniform manipulation and intermingling of graphics, programs, and user interfaces etc.

4. Support for efficient data structures such as sparse arrays, piecewise functions.

5. Support for emerging fields such as graph plotting and analysis, alternate input devices, new data formats.

6. Ability to create and publish programs that run on the free Mathematica Player.

1.2.3 Advantages

The standard *Mathematica* front end makes laying out computations very simple. Users may re-evaluate hierarchically-nested blocks of code by clicking on a set of braces and hitting shift-enter. Additionally, Mathematica is able to handle arbitrary-precision numbers and rational numbers, as compared to other mathematics programs such as Matlab, Excel, and most standard programming languages.

Mathematica also has very generalized functions as well as a great variety of functions. As a higher-level multi-paradigm programming language, it requires much less code than most programming languages in order to write the same thing.

1.2.4 Disadvantages

Mathematica code has been criticized as hard to debug, the most common way of debugging programs written for *Mathematica* was the use of print statements.

Mathematica's user interface and graphics support was not nearly as intuitive and advanced as its competitors, but version 6 introduced a large variety of new features and integration that may assuage these concerns [3].

CHAPTER 2

COMPLEX VARIABLE THEORY

2.1 Review of complex variable theory

A complex variable *z* is defined by two variables *x* and *y* in the form

$$z = x + i\ y \qquad (2.1.1)$$

where $i = \sqrt{-1}$ is called the imaginary unit, *x* is known as real part of *z*, that is, $x = Re(z)$, while *y* is called the imaginary part, $y = Im(z)$

It can be expressed in polar form as

$$z = r\,(\cos\theta + i \sin\theta) = re^{i\theta} \qquad (2.1.2)$$

where $r = \sqrt{x^2 + y^2}$ is known as the modulus of z and $\theta = \tan^{-1}(y / x)$ is the argument. Complex variables includes two quantities (real and imaginary parts) and they can represented by a two-dimensional vector with x and y components.

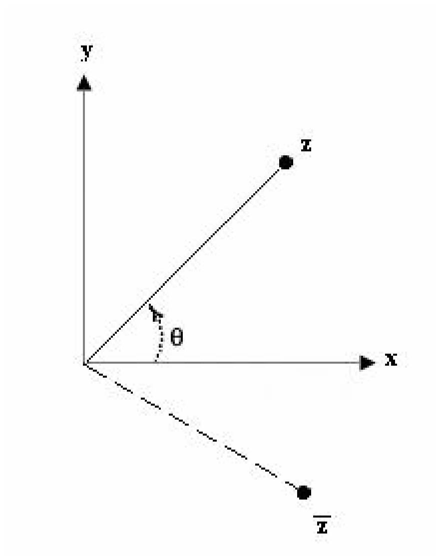

Figure 2.1 Complex Plane

The complex conjugate $\overline{z}$ of the variable z is defined as

$$\overline{z} = x - i\,y = r\,e^{-i\,\theta} \qquad (2.1.3)$$

It is apparent that this quantity is simply a result of changing the sign of the imaginary part of z, and in complex plane (Figure 2-1) is a reflection of z about the real axis. Note that $r = \sqrt{z\,z}$

Using equations (2.1.1) and (2.1.3) following relationships are held

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}$$

$$\frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right) \tag{2.1 .4}$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Addition, subtraction, multiplication, and division of complex numbers z_1 and z_2 are defined as

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2) \tag{2.1 .5}$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2)$$

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\frac{(y_1 x_2 - x_1 y_2)}{x_2^2 + y_2^2}$$

2.2 Formulation of Airy Stress Function

One of the hurdles associated with real world elasticity problems is that we need to calculate tensor fields for stresses .For this solution, a minimum of three partial differential equations are needed. A better approach is to reduce the partial differential equations to a single differential equation with solvable unknowns. This method is known as Airy stress function method which can be used successfully for plane stress and plane strain problem.

The stress function formulation is based on the general idea of developing a representation for the stress field that satisfies equilibrium and yields a single governing equation from the compatibility statement.

For plane strain, the equilibrium equations are

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y = 0 \end{aligned} \qquad (2.2.1)$$

where F_x and F_y are body forces but are assumed that they are derivable from a potential function V such that

$$\begin{aligned} F_x = \frac{-\partial V}{\partial x} \\ F_y = \frac{-\partial V}{\partial y} \end{aligned} \qquad (2.2.2)$$

This assumption is not very restrictive because many body forces found in applications (e.g., gravity loading) fall in this category. In this case the plane equilibrium equations can be written as [1]

$$\frac{\partial(\sigma_x - V)}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0$$
$$\frac{\partial(\sigma_y - V)}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y = 0 \qquad (2.2.3)$$

It is observed that these equations will be identically satisfied by choosing a representation

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} + V$$
$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} + V \qquad (2.2.4)$$
$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

where $\Phi = \Phi(x,y)$ is an arbitrary form called the Airy stress function.

Neglecting body forces equation (2.2.4) reduces to

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} \qquad (2.2.5)$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \, \partial y}$$

Now that the equilibrium condition is satisfied, we can shift our attention to compatibility relations in terms of stress.

For plane strain, the compatibility relations is given by

$$\frac{\partial^4 \phi}{\partial x^4} + 2\,\frac{\partial^4 \phi}{\partial x^2\, \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = -\frac{1-2\nu}{1-\nu}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \qquad (2.2.6)$$

For plane stress, the compatibility relations is given by

$$\frac{\partial^4 \phi}{\partial x^4} + 2\,\frac{\partial^4 \phi}{\partial x^2\, \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = -(1-\nu)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \qquad (2.2.7)$$

Equations (2.2.6) and (2.2.7) can also be written as

$$\nabla^4 \phi = -\left(\frac{1-2\nu}{1-\nu}\right)\nabla^2 V \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . \text{Plane Strain} \qquad (2.2.8)$$

$$\nabla^4 \phi = -(1-\nu)\,\nabla^2 V \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . \text{Plane Stress} \qquad (2.2.9)$$

The form

$$\nabla^4 = \nabla^2 \nabla^2 \qquad (2.2.10)$$

is called the biharmonic operator. Assuming no body forces, then, both plane strain and plane stress form reduces to

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = \nabla^4 \phi = 0 \qquad (2.2.11)$$

This relation is called the biharmonic equation [1] and its solutions are known as biharmonic functions. Thus the problem of plane elasticity has been reduced to single equation in terms of the Airy stress function Φ. This function is to be determined in two dimensional region R bounded by the boundary S as shown in Figure (2.2)

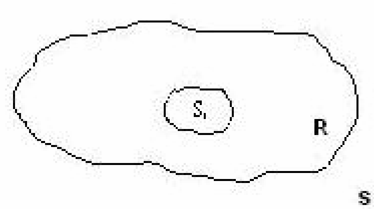

Figure 2.2 Typical Domain for the plane elasticity problem

Using appropriate boundary conditions over S, the solution can be obtained. The resulting strains and displacements calculated from these common stresses would not be the same for each plane theory. This occurs because plane strain and plane stress have different forms for Hooke's Law and strain-displacement relations.

2.3 Polar Coordinate Formulation

We will make use of polar coordinates in the solution of many plane problems in elasticity, thus the previous governing equations will now be developed in this curvilinear system. For such a coordinate system , the solution to plane stress and plane strain problems involves the determination of the in-plane displacements, strains and stresses in R{u_r, u_θ, e_r, e_θ , $e_{r\theta}$,σ_r, σ_θ, $T_{r\theta}$} subject to prescribed boundary conditions on S (Refer to Figure 2.2) .

Following is the stress transformation to polar coordinates

$$\begin{aligned}
\sigma_r &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\,\tau_{xy}\ \sin\theta\cos\theta \\
\sigma_\theta &= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\,\tau_{xy}\ \sin\theta\cos\theta \qquad (2.3.1) \\
\tau_{r\theta} &= -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau_{xy}\,(\cos^2\theta - \sin^2\theta)
\end{aligned}$$

Equation (2.2.5) which is a relation between the stress components and the Airy stress function can be easily transformed to polar form. These equations in polar form are

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} \qquad (2.3.2)$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

Also the biharmonic equation in polar form is

$$\nabla^4\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi = 0 \qquad (2.3.3)$$

The plane problem is then formulated in terms of Airy function $\Phi(r, \theta)$ with a single governing biharmonic equation [1]. Appropriate boundary conditions are necessary to complete a solution.

2.4 Complex Formulation of the Plane Elasticity Problem

Plane stress and plane strain are two different state of stresses .Although each case is related to a completely different two dimensional model, the basic formulations are quite similar, and by simple changes in elastic constants, solutions to each case can be shown. The equation for plane strain include expressions for stresses

$$\sigma_x = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\,\mu\,\frac{\partial u}{\partial x}$$

$$\sigma_y = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\,\mu\,\frac{\partial v}{\partial y} \qquad (2.4.1)$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

where λ is Lames constant and μ is the modulus of rigidity

The Navier equation is reduced to

$$\mu\,\nabla^2 u + (\lambda + \mu)\,\nabla\,(\nabla\,.\,u) = 0 \qquad (2.4.2)$$

where the Laplacian is defined as

$$\nabla^2 = ()_{xx} + ()_{yy} \qquad (2.4.3)$$

with the subscripts representing partial differentiation. For both plane strain and plane stress with zero body forces the stresses are expressed in a self-equilibrated form using the Airy stress function Φ as

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} \qquad (2.4.4)$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x\,\partial y}$$

and from the compatibility relations, Φ satisfies the biharmonic equation

$$\nabla^4 \phi = \phi_{xxxx} + 2\,\phi_{xxyy} + \phi_{yyyy} \qquad (2.4.5)$$

Thus, the stress formulation to the plane problem is reduced to solving the biharmonic equation.

Now we wish to represent the Airy stress function in terms of functions of a complex variable and transform the plane problem into one involving complex variable theory. Using relations (2.1.1) and (2.1.3) the variables x and y can be expressed in terms of z and $\bar{z}$. Repeated use of differential operators defined in equation (2.1.4) allows the following representation of harmonic and biharmonic operators

$$\nabla^2() = 4\frac{\partial^2()}{\partial z\,\partial \bar{z}} \tag{2.4.6}$$

$$\nabla^4() = 16\frac{\partial^4()}{\partial z^2\,\partial \bar{z}^2} \tag{2.4.7}$$

Therefore the governing biharmonic elasticity equation (2.4.5) can be expressed as

$$\frac{\partial^4()}{\partial z^2\,\partial \bar{z}^2} = 0 \tag{2.4.8}$$

Integrating this result yields

$$\begin{aligned}\phi(z,\bar{z}) &= \frac{1}{2}\left(\overline{z\,\gamma(z)} + \bar{z}\,\gamma(z) + \chi(z) + \overline{\chi(z)}\right)\\ &= \mathrm{Re}\left(\bar{z}\,\gamma(z) + \chi(z)\right)\end{aligned} \tag{2.4.9}$$

where γ and χ are arbitrary functions of the indicated variables considering Φ that is real [1]. This result demonstrates that the Airy stress function can be formulated in terms of two functions of a complex variable. Also considering Navier equation (2.4.2) and introducing a complex displacement U=u+i v we get,

$$2\mu U = \kappa\,\gamma(z) - \overline{\gamma'(z)} - \overline{\psi(z)} \tag{2.4.10}$$

where again $\chi(z)$ and $\psi(z) = \chi'(z)$ are arbitrary functions of a complex variable and the parameter κ depends only on Poisson's ratio υ.

$$\begin{aligned} &\kappa = 3 - 4\nu && \text{Plane Strain} \\ &\kappa = \frac{3-\nu}{1+\nu} && \text{Plane Stress} \end{aligned} \qquad (2.4.11)$$

Equation (2.4.10) is the complex variable formulation for the displacement field and is written in terms of two arbitrary functions of a complex variable.

Using equations (2.4.4) and (2.4.9) yields the fundamental stress combinations

$$\begin{aligned} &\sigma_x + \sigma_y = 2\,(\gamma'(z) - \overline{\gamma'(z)}) \\ &\sigma_y - \sigma_x + 2\,i\,\tau_{xy} = 2\,(\overline{z}\;\gamma''(z) + \psi' z)) \end{aligned} \qquad (2.4.12)$$

By adding and subtracting and equating real and imaginary parts, relation (2.4.12) can be easily solved for the individual stresses. Using standard transformation laws, the stresses and displacements in polar coordinates can be written as

$$\begin{aligned} &\sigma_r + \sigma_\theta = \sigma_x + \sigma_y \\ &\sigma_\theta - \sigma_r + 2\,i\,\tau_{r\theta} = (\sigma_y - \sigma_x + 2\,i\,\tau_{xy})\,e^{2i\theta} \\ &u_r + i\,u_\theta = (u + i\,v)\,e^{-i\theta} \end{aligned} \qquad (2.4.13)$$

Thus, we have demonstrated that all the basic variables in plane elasticity are expressible in terms of two arbitrary functions of a complex variable [1]. The solution to particular problems is then reduced to finding the appropriate potentials that satisfy the boundary conditions .This solution technique is greatly aided by mathematical methods of complex variable theory.

2.5 General Structure of Complex Potentials

We now have established that the solution to plane elasticity problems involves determination of two complex potential functions γ (z) and ψ (z) .These potentials have some general properties and structures as well as the relations for stresses and displacements that a particular indeterminacy or arbitrariness of the potentials can be found. Particular general forms of these potentials exist for regions of different topology. Most problems of interest involve finite simply-connected, finite multiply-connected and infinite multiply-connected domains.

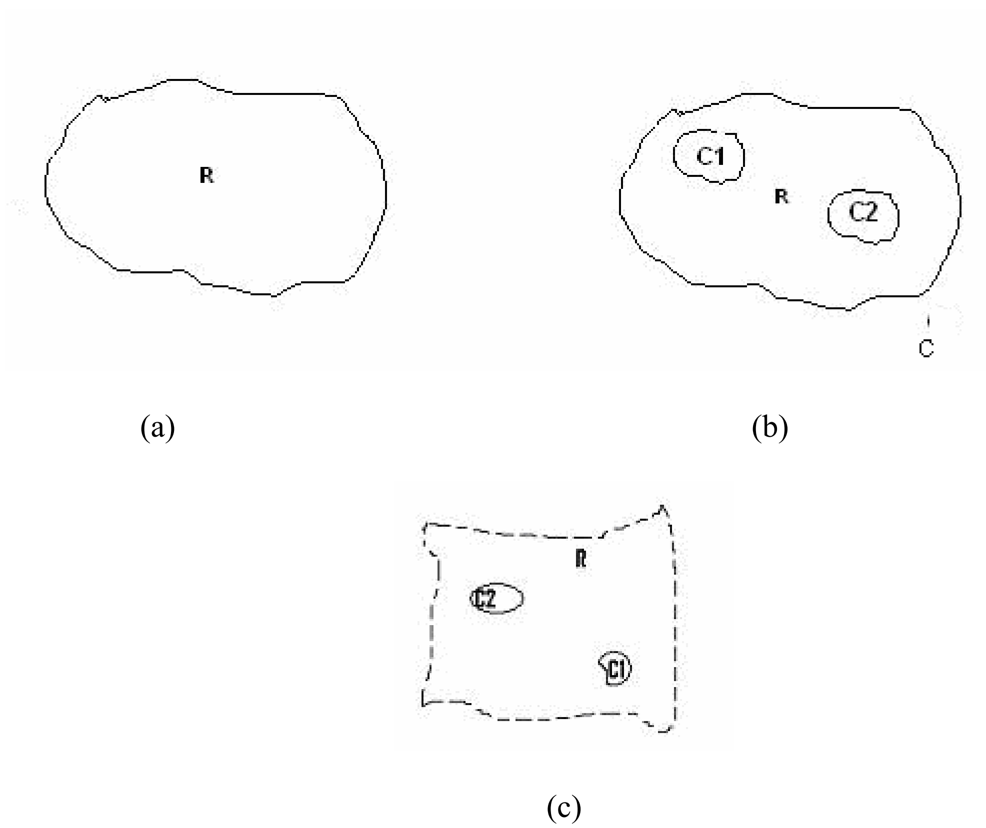

Figure 2.5 Typical Domains of Interest: (a) Finite Simply Connected Domain, (b) Finite Multiply Connected Domain, (c) Infinite Multiply Connected Domain.

2.5.1 Finite Simply Connected Domains

Consider a finite simply-connected domain as shown in Figure 2.5(a). For this case the potential functions, γ (z) and ψ (z) are analytic and single-valued in the domain and this allows the power series representation

$$\gamma = \sum_{n=0}^{\infty} a_n z^n$$
$$\psi = \sum_{n=0}^{\infty} b_n z^n \qquad (2.5.1)$$

where a_n and b_n are constants to be determined by the boundary conditions of the problem under study.

2.5.2 Finite Multiply Connected Domains

For a region shown in Figure 2.5 (b) it is assumed that the domain has k internal boundaries as shown. For this topology, the potential functions need not be single- valued demonstrated by considering the behavior of stresses and displacements around each of the n contours C_k in region R. Since multi-valued displacements lead to the theory of dislocations, for this case the problem is limited to be single- valued. The general potential form for this case is given as

$$\gamma(z) = -\sum_{k=1}^{n} \frac{F_k}{2\pi(1+\kappa)} \log(z - z_k) + \gamma^*(z)$$
$$\psi(z) = \sum_{k=1}^{n} \frac{\kappa \overline{F}_k}{2\pi(1+\kappa)} \log(z - z_k) + \psi^*(z) \qquad (2.5.2)$$

where F_k is the resultant force on each contour C_k, $\gamma^*(z)$ and $\psi^*(z)$ are arbitrary analytic function functions in R, z_k is a point within the contour C_k, κ is the material constant which depends on Poisson's ratio defined by equation (2.4.11) .

2.5.3 Infinite Domains

For a region shown in Figure 2.5(c) the general form of potential functions is given by

$$\gamma(z) = -\frac{\sum_{k=1}^{m} F_k}{2\pi(1+\kappa)} \log z + \frac{\sigma_x^\infty + \sigma_y^\infty}{4} + \gamma^{**}(z)$$

$$\psi(z) = \frac{\sum_{k=1}^{m} \kappa \overline{F_k}}{2\pi(1+\kappa)} \log z + \frac{\sigma_y^\infty - \sigma_x^\infty + 2i\tau_{xy}^\infty}{2} + \psi^{**}(z) \qquad (2.5.3)$$

where σ_x^∞ , σ_y^∞,τ_{xy}^∞ are the stresses at infinity and $\gamma^{**}(z)$ and $\psi^{**}(z)$ are arbitrary analytic functions outside the region enclosing all m contours .Using power series theory, these analytic functions can be expressed as

$$\gamma^{**}(z) = \sum_{n=1}^{\infty} a_n z^{-n}$$

$$\psi^{**}(z) = \sum_{n=1}^{\infty} b_n z^{-n} \qquad (2.5.4)$$

The displacements at infinity would indicate unbounded behavior unless all stresses at infinity vanish and $\Sigma F_k = \Sigma \overline{F_k} = 0$.This happens due to the fact that even a bounded strain over an infinite length will produce infinite displacements [1].The case of simply-connected domain and infinite domain is obtained by dropping summation terms in equation (2.5.3)

CHAPTER 3

APPLICATION OF AIRY STRESS FUNCTION FOR PLANE PROBLEMS

3.1 Infinite Plate with a Hole subjected to Biaxial Loading

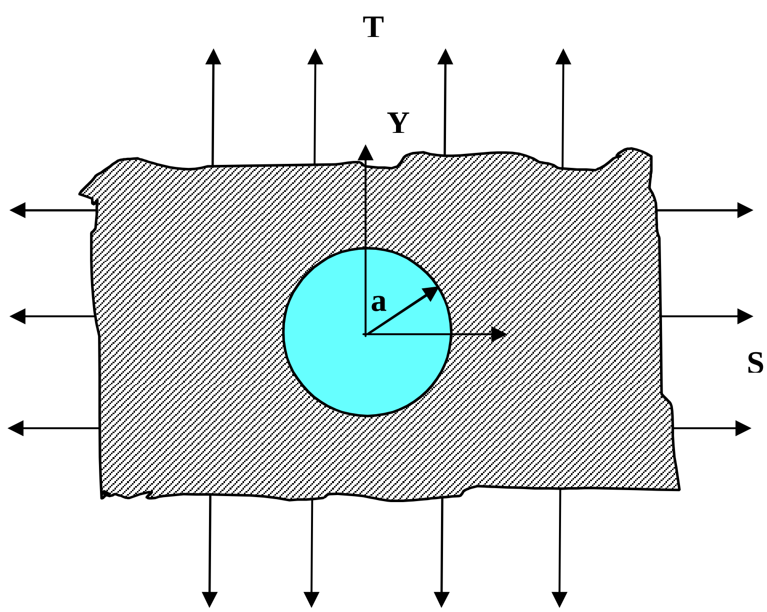

Figure 3.1 Infinite Plate with a Hole Subjected to Biaxial Loading

Let us consider an infinite plate containing a stress free circular hole with a radius ‘a’ subjected to biaxial stresses at infinity as shown in Figure 3.1 .Here we have taken

$$\sigma_x^\infty = S$$

$$\sigma_y^\infty = T \qquad (3.1.1)$$

$$\tau_{xy}^\infty = 0$$

We have complex potentials from Equation (2.5.3) as

$$\gamma(z) = -\frac{\sum_{k=1}^{m} F_k}{2\pi(1+\kappa)} \log z + \frac{\sigma_x^\infty + \sigma_y^\infty}{4} + \gamma^{**}(z)$$

$$\psi(z) = \frac{\sum_{k=1}^{m} \kappa \overline{F}_k}{2\pi(1+\kappa)} \log z + \frac{\sigma_y^\infty - \sigma_x^\infty + 2\,i\,\tau_{xy}^\infty}{2} + \psi^{**}(z)$$

However in this case, the logarithmic terms are dropped because the hole is stress free.

The complex potentials now are written as

$$\gamma(z) = \frac{\sigma_x^\infty + \sigma_y^\infty}{4} z + \sum_{n=1}^{\infty} a_n z^{-n}$$

$$(3.1.2)$$

$$\psi(z) = \frac{\sigma_y^\infty - \sigma_x^\infty + 2\,i\,\tau_{xy}^\infty}{4} z + \sum_{n=1}^{\infty} a_n z^{-n}$$

Substituting values of equation (3.1.1) in equation (3.1.2) and considering n =3 gives us

$$\gamma(z) = \frac{1}{4}(S+T)z + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3}$$

$$(3.1.3)$$

$$\psi(z) = \frac{1}{2}(-S+T)z + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3}$$

Integrating ψ (z) to get second potential function χ (z) as

$$\chi(z) = \frac{1}{4}(-S+T)z^2 + \mathrm{Log}[z]\,b_1 - \frac{b_2}{z} - \frac{b_3}{2z^2} \qquad (3.1.4)$$

Incorporating polar coordinates for the problem

$$z = r\, e^{i\theta} \qquad (3.1.5)$$

gives us the Airy stress function as

$$\phi = \frac{1}{4\, r^2}\, (r^4\, (S + T + (-S + T)\, \mathrm{Cos}[2\,\theta]) + 4\, \mathrm{Cos}[4\,\theta]\, a_3 +$$
$$2\, r\, (2\, r\, \mathrm{Cos}[2\,\theta]\, a_1 + 2\, \mathrm{Cos}[3\,\theta]\, a_2 + r\, \mathrm{Log}[r^2]\, b_1 - 2\, \mathrm{Cos}[\theta]\, b_2) \qquad (3.1.6)$$
$$- 2\, \mathrm{Cos}[2\,\theta]\, b_3)$$

Note that this φ = f(r,θ) since the equations is converted to polar form.

Now using the Airy stress function we can derive stresses using equation (2.3.2)

$$\sigma_{rr} = \frac{1}{2\, a^4}\, (a^4\, S + a^4\, T + a^4\, S\, \mathrm{Cos}[2\,\theta] -$$
$$a^4\, T\, \mathrm{Cos}[2\,\theta] - 8\, a^2\, \mathrm{Cos}[2\,\theta]\, a_1 - 20\, a\, \mathrm{Cos}[3\,\theta]\, a_2 - \qquad (3.1.7)$$
$$36\, \mathrm{Cos}[4\,\theta]\, a_3 + 2\, a^2\, b_1 + 4\, a\, \mathrm{Cos}[\theta]\, b_2 + 6\, \mathrm{Cos}[2\,\theta]\, b_3)$$

$$\sigma_{\theta\theta} = \frac{1}{2\, a^4}$$
$$(a^4\, S + a^4\, T - a^4\, S\, \mathrm{Cos}[2\,\theta] + a^4\, T\, \mathrm{Cos}[2\,\theta] + 4\, a\, \mathrm{Cos}[3\,\theta]\, a_2 +$$
$$12\, \mathrm{Cos}[4\,\theta]\, a_3 - 2\, a^2\, b_1 - 4\, a\, \mathrm{Cos}[\theta]\, b_2 - 6\, \mathrm{Cos}[2\,\theta]\, b_3) \qquad (3.1.8)$$

$$\tau_{r\theta} = \frac{1}{2\, a^4}\, (-a^4\, S\, \mathrm{Sin}[2\,\theta] + a^4\, T\, \mathrm{Sin}[2\,\theta] - 4\, a^2\, \mathrm{Sin}[2\,\theta]\, a_1 -$$
$$12\, a\, \mathrm{Sin}[3\,\theta]\, a_2 - 24\, \mathrm{Sin}[4\,\theta]\, a_3 + 4\, a\, \mathrm{Sin}[\theta]\, b_2 + \qquad (3.1.9)$$
$$6\, \mathrm{Sin}[2\,\theta]\, b_3)$$

Displacements can be calculated by using equation (2.4.10) to yield

$$u_r = \frac{1}{8 r^3 \mu} (r^4 ((S+T)\kappa + 2 (S-T) \mathrm{Cos}[2\theta] - r (S+T) \mathrm{Sin}[\theta]) +$$
$$4 (r^2 (\kappa \mathrm{Cos}[2\theta] + r \mathrm{Sin}[3\theta]) a_1 + r (\kappa \mathrm{Cos}[3\theta] + 2 r \mathrm{Sin}[4\theta]) a_2 +$$
$$\kappa \mathrm{Cos}[4\theta] a_3 + 3 r \mathrm{Sin}[5\theta] a_3 - r^2 b_1 - r \mathrm{Cos}[\theta] b_2 - \mathrm{Cos}[2\theta] b_3)) \tag{3.1.10}$$

$$u_\theta = -\frac{1}{8 r^3 \mu} (r^4 \mathrm{Cos}[\theta] (r (S+T) + 4 (S-T) \mathrm{Sin}[\theta]) +$$
$$4 (r^2 (-r \mathrm{Cos}[3\theta] + \kappa \mathrm{Sin}[2\theta]) a_1 +$$
$$r (-2 r \mathrm{Cos}[4\theta] + \kappa \mathrm{Sin}[3\theta]) a_2 - 3 r \mathrm{Cos}[5\theta] a_3 +$$
$$\kappa \mathrm{Sin}[4\theta] a_3 + r \mathrm{Sin}[\theta] b_2 + \mathrm{Sin}[2\theta] b_3)) \tag{3.1.11}$$

The real form of equations (3.1.7), (3.1.8), (3.1.9), (3.1.10), and (3.1.11) can be found out by solving the constants. The only way to find constants is by using boundary conditions. Since we know that it is a stress free condition on the interior of the hole we can use at r = a

$$(\sigma_{rr} - i \tau_{r\theta})_{r=a} = 0 \tag{3.1.12}$$

Using equation (3.1.12) to separate out the terms to give at r =a

$$\sigma_{rr} = 0$$
$$\tau_{r\theta} = 0 \tag{3.1.13}$$

Substituting values of σ_{rr} and $\tau_{r\theta}$ in equation (3.1.12) to give

$$
\begin{aligned}
&\frac{S}{2}+\frac{1}{4}E^{-2I\theta}S+\frac{1}{4}E^{2I\theta}S+\frac{T}{2}-\frac{1}{4}E^{-2I\theta}T-\\
&\frac{1}{4}E^{2I\theta}T-\frac{2E^{-2I\theta}a_1}{a^2}-\frac{2E^{2I\theta}a_1}{a^2}-\frac{5E^{-3I\theta}a_2}{a^3}-\frac{5E^{3I\theta}a_2}{a^3}-\\
&\frac{9E^{-4I\theta}a_3}{a^4}-\frac{9E^{4I\theta}a_3}{a^4}+\frac{b_1}{a^2}+\frac{E^{-I\theta}b_2}{a^3}+\frac{E^{I\theta}b_2}{a^3}+\frac{3E^{-2I\theta}b_3}{2a^4}+\\
&\frac{3E^{2I\theta}b_3}{2a^4}-\frac{1}{4}E^{-2I\theta}S-\frac{1}{4}E^{2I\theta}S-\frac{1}{4}E^{-2I\theta}T+\frac{1}{4}E^{2I\theta}T+\\
&\frac{E^{-2I\theta}a_1}{a^2}-\frac{E^{2I\theta}a_1}{a^2}+\frac{3E^{-3I\theta}a_2}{a^3}-\frac{3E^{3I\theta}a_2}{a^3}+\frac{6E^{-4I\theta}a_3}{a^4}-\\
&\frac{6E^{4I\theta}a_3}{a^4}-\frac{E^{-I\theta}b_2}{a^3}+\frac{E^{I\theta}b_2}{a^3}-\frac{3E^{-2I\theta}b_3}{2a^4}+\frac{3E^{2I\theta}b_3}{2a^4}
\end{aligned}
\tag{3.1.14}
$$

Equating like powers of $e^{in\theta}$ gives relations for the coefficients of a_n and b_n

$$
\begin{aligned}
&a_1\to\frac{1}{2}a^2(S-T),\ a_2\to 0,\ a_3\to 0\\
&b_1\to-\frac{1}{2}a^2(S+T),\ b_2\to 0,\ b_3\to\frac{1}{2}a^4(S-T)
\end{aligned}
\tag{3.1.15}
$$

Using equation (3.1.15) we can solve for Airy stress function Φ developed in (3.1.6)

$$
\begin{aligned}
\phi_{\pi}=&\frac{r^2 S}{4}+\frac{r^2 T}{4}-\frac{1}{4}r^2 S\,\mathrm{Cos}[2\theta]+\frac{1}{2}a^2(S-T)\,\mathrm{Cos}[2\theta]-\\
&\frac{a^4(S-T)\,\mathrm{Cos}[2\theta]}{4r^2}+\frac{1}{4}r^2 T\,\mathrm{Cos}[2\theta]-\frac{1}{4}a^2(S+T)\,\mathrm{Log}[r^2]
\end{aligned}
\tag{3.1.16}
$$

Using equation (3.1.16) we can obtain stresses and displacements as

$$\sigma_{rr} = \frac{(a - r)(a + r)(-r^2 (S + T) + (3 a^2 - r^2)(S - T) \mathrm{Cos}[2\theta])}{2 r^4}$$

$$\sigma_{\theta\theta} = \frac{r^2 (a^2 + r^2)(S + T) - (3 a^4 + r^4)(S - T) \mathrm{Cos}[2\theta]}{2 r^4} \tag{3.1 .17}$$

$$\tau_{r\theta\mathrm{final}} = \frac{1}{2 r^4} (3 a^4\ S\ \mathrm{Sin}[2\theta] - 2 a^2 r^2\ S\ \mathrm{Sin}[2\theta] -$$

$$r^4\ S\ \mathrm{Sin}[2\theta] - 3 a^4\ T\ \mathrm{Sin}[2\theta] + 2 a^2 r^2\ T\ \mathrm{Sin}[2\theta]$$

$$+ r^4\ T\ \mathrm{Sin}[2\theta])$$

$$u_{rr} = \frac{1}{8 r^3 \mu} (-2 (S - T)(a^4 - r^4 - a^2 r^2 \kappa) \mathrm{Cos}[2\theta] +$$

$$r^2 ((S + T)(2 a^2 + r^2 \kappa - r^3\ \mathrm{Sin}[\theta]) +$$

$$2 a^2 r (S - T) \mathrm{Sin}[3\theta]))$$

$$(3.1 .18)$$

$$u_{\theta\theta} = \frac{1}{8 r^3 \mu} (-r^5 (S + T) \mathrm{Cos}[\theta] + 2 (S - T)(a^2 r^3 \mathrm{Cos}[3\theta] -$$

$$(a^4 + r^4 + a^2 r^2 \kappa) \mathrm{Sin}[2\theta]))$$

3.2 Infinite Plate with Two-Dimensional Circular Inclusion Subjected to Biaxial Loading

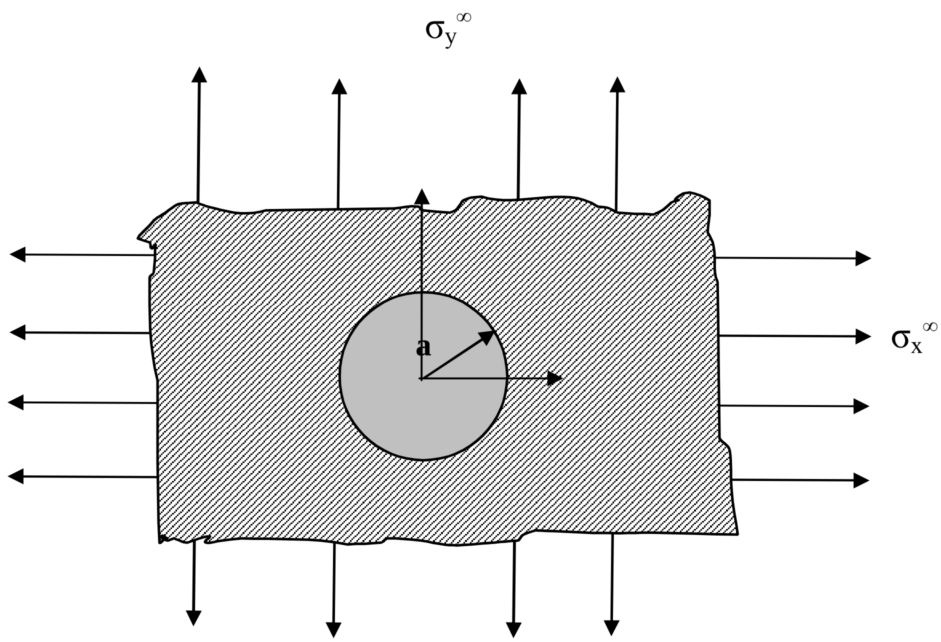

Figure 3.2 Infinite Plate with Two-Dimensional Circular Inclusion

Consider an infinite plate with a two dimensional circular inclusion as shown in Figure 3.2.The circular inclusion has radius 'a' with material properties κ and μ. To differentiate between the material properties of the infinite plate and the circular inclusion we shall assume the material properties of the infinite plate as $\kappa 1$ and $\mu 1$ where κ and $\kappa 1$ are the parameters depending on the Poisson's ratio and μ and $\mu 1$ are the shear moduli of the circular inclusion and the infinite plate respectively. In order to evaluate the stresses and displacements, we have to club the results of the infinite plate with a hole subjected to biaxial loading and the finite simply-connected domain. The results can be only be validated by maintaining the equilibrium and continuity of stresses and displacements at the boundary of two phases. In this case, we assume

$$\sigma_x = S$$
$$\sigma_y = T \qquad (3.2.1)$$
$$\tau_{xy} = 0$$

3.2.1 Stresses and Displacements in two dimensional circular inclusion

The complex potentials for the two-dimensional circular inclusion are given by

$$\gamma(z) = \sum_{n=0}^{2} a_n z^n$$
$$\qquad (3.2.2)$$
$$\psi(z) = \sum_{n=0}^{2} b_n z^n$$

Thus

$$\gamma(z) = a_0 + z a_1 + z^2 a_2$$
$$\qquad (3.2.3)$$
$$\psi(z) = b_0 + z b_1 + z^2 b_2$$

Integrating ψ (z) to get the second potential function χ (z) as

$$\chi(z) = z b_0 + \frac{z^2 b_1}{2} + \frac{z^3 b_2}{3} \qquad (3.2.4)$$

Incorporating the polar coordinates for the problem

$$z = r e^{i\theta} \qquad (3.2.5)$$

gives the Airy stress function as

$$\phi_{\text{cirinc}} = \frac{1}{6} r (6 r a_1 + 6 \text{Cos}[\theta] (a_0 + r^2 a_2 + b_0)$$
$$+ 3 r \text{Cos}[2\theta] b_1 + 2 r^2 \text{Cos}[3\theta] b_2) \tag{3.2.6}$$

Note that Φ = f(r,θ) since the equation is converted to polar form.

Now using the Airy stress function we can derive stresses using equation (2.3.2).

Also these stresses are taken at condition r = a

$$\sigma_{\text{rrcirinc}} = 2 a_1 + 2 a \text{Cos}[\theta] a_2 - \text{Cos}[2\theta] b_1 - 2 a \text{Cos}[3\theta] b_2$$
$$\sigma_{\theta\theta\text{cirinc}} = 2 a_1 + 6 a \text{Cos}[\theta] a_2 + \text{Cos}[2\theta] b_1 + 2 a \text{Cos}[3\theta] b_2 \tag{3.2.7}$$
$$\tau_{\text{r}\theta\text{cirinc}} = 2 a \text{Sin}[\theta] a_2 + \text{Sin}[2\theta] b_1 + 2 a \text{Sin}[3\theta] b_2$$

The displacements can be calculated by using equation (2.4.10) at r =a to yield

$$u_{\text{rrcirinc}} = \frac{1}{2\mu} (\kappa \text{Cos}[\theta] a_0 - a a_1 + a \kappa a_1 - 2 a^2 \text{Cos}[\theta] a_2$$
$$+ a^2 \kappa \text{Cos}[\theta] a_2 - \text{Cos}[\theta] b_0 - a \text{Cos}[2\theta] b_1 - a^2 \text{Cos}[3\theta] b_2)$$
$$\tag{3.2.8}$$
$$u_{\theta\theta\text{cirinc}} = \frac{1}{2\mu} (-\kappa \text{Sin}[\theta] a_0 + 2 a^2 \text{Sin}[\theta] a_2 + a^2 \kappa \text{Sin}[\theta] a_2 +$$
$$\text{Sin}[\theta] b_0 + a \text{Sin}[2\theta] b_1 + a^2 \text{Sin}[3\theta] b_2)$$

3.2.2 Stresses and Displacements in Infinite Plate with a Circular Hole

The complex potentials for infinite plate with a hole are given by

$$\gamma(z) = \frac{S+T}{4} z + \sum_{n=1}^{1} m_n z^{-n}$$
$$\tag{3.2.9}$$
$$\psi(z) = \frac{T-S}{2} z + \sum_{n=1}^{3} f_n z^{-n}$$

Thus

$$\gamma(z) = \frac{S+T}{4} z + \frac{m_1}{z}$$
$$\psi(z) = \frac{T-S}{2} z + \frac{f_1}{z} + \frac{f_2}{z^2} + \frac{f_3}{z^3} \quad (3.2.10)$$

Integrating ψ (z) to get the second potential function χ (z) as

$$\chi(z) = \frac{1}{4}(-S+T) z^2 + \mathrm{Log}[z] f_1 - \frac{f_2}{z} - \frac{f_3}{2 z^2} \quad (3.2.11)$$

Incorporating the polar coordinates for the problem

$$z = r e^{i\theta} \quad (3.2.12)$$

gives the Airy stress function as

$$\phi_{\mathrm{plate}} = \frac{1}{4 r^2}\left(r^4 (S + T + (-S+T)\,\mathrm{Cos}[2\theta]) + 2 r^2\,\mathrm{Log}[r^2]\, f_1 - 4 r\,\mathrm{Cos}[\theta]\, f_2 - 2\,\mathrm{Cos}[2\theta]\,(f_3 - 2 r^2 m_1)\right) \quad (3.2.12)$$

Note that $\Phi = f(r,\theta)$ since the equations is converted to polar form.

Now using the Airy stress function we can derive the stresses using equation

(2.3.2). Also these stresses are taken at condition r = a

$$\sigma_{\text{rrplate}} = \frac{1}{2\,a^4}\left(a^4\,S + a^4\,T + a^4\,S\,\text{Cos}[2\,\theta] - a^4\,T\,\text{Cos}[2\,\theta] + 2\,a^2\,f_1 + \right.$$

$$\left. 4\,a\,\text{Cos}[\theta]\,f_2 + 6\,\text{Cos}[2\,\theta]\,f_3 - 8\,a^2\,\text{Cos}[2\,\theta]\,m_1\right)$$

$$\sigma_{\theta\theta\text{plate}}\ \frac{1}{2\,a^4}\left(a^4\,S + a^4\,T - a^4\,S\,\text{Cos}[2\,\theta] + a^4\,T\,\text{Cos}[2\,\theta] - 2\,a^2\,f_1 - \right. \tag{3.2.13}$$

$$\left. 4\,a\,\text{Cos}[\theta]\,f_2 - 6\,\text{Cos}[2\,\theta]\,f_3\right)$$

$$\tau_{r\theta\text{plate}} = \frac{1}{2\,a^4}\left(-a^4\,S\,\text{Sin}[2\,\theta] + a^4\,T\,\text{Sin}[2\,\theta] + 4\,a\,\text{Sin}[\theta]\,f_2 + \right.$$

$$\left. 6\,\text{Sin}[2\,\theta]\,f_3 - 4\,a^2\,\text{Sin}[2\,\theta]\,m_1\right)$$

The displacements can be calculated by using equation (2.4.10) to yield

$$u_{\text{rrplate}} = \frac{1}{8\,a^3\,\mu 1}\left(a^4\,S - a^4\,T - a^4\,S\,\kappa 1 + a^4\,T\,\kappa 1 - 2\,a^4\,S\,\text{Cos}[2\,\theta]\right.$$

$$+ 2\,a^4\,T\,\text{Cos}[2\,\theta] - 4\,a^2\,f_1 - 4\,a\,\text{Cos}[\theta]\,f_2 - 4\,\text{Cos}[2\,\theta]\,f_3 +$$

$$\left. 4\,a^2\,\text{Cos}[2\,\theta]\,m_1 + 4\,a^2\,\kappa 1\,\text{Cos}[2\,\theta]\,m_1\right) \tag{3.2.14}$$

$$u_{\theta\theta\text{plate}} = \frac{1}{4\,a^3\,\mu 1}\left(a^4\,S\,\text{Sin}[2\,\theta] - a^4\,T\,\text{Sin}[2\,\theta] - 2\,a\,\text{Sin}[\theta]\,f_2 - \right.$$

$$\left. 2\,\text{Sin}[2\,\theta]\,f_3 + 2\,a^2\,\text{Sin}[2\,\theta]\,m_1 - 2\,a^2\,\kappa 1\,\text{Sin}[2\,\theta]\,m_1\right)$$

The continuity condition can be satisfied if the traction force and displacements are the same at the interface.

Equating equations (3.2.7) to (3.2.13) and equations (3.2.7) to (3.2.14), we have

$$\frac{1}{2a^4}(a^4 S + a^4 T + a^4 S \cos[2\theta] - a^4 T \cos[2\theta] + 2a^2 f_1 + 4a \cos[\theta] f_2 + 6 \cos[2\theta] f_3 - 8a^2 \cos[2\theta] m_1) - (2a_1 + 2a \cos[\theta] a_2 - \cos[2\theta] b_1 - 2a \cos[3\theta] b_2) = 0 \tag{3.2.15}$$

$$\frac{1}{2a^4}(a^4 S + a^4 T - a^4 S \cos[2\theta] + a^4 T \cos[2\theta] - 2a^2 f_1 - 4a \cos[\theta] f_2 - 6 \cos[2\theta] f_3) - (2a_1 + 6a \cos[\theta] a_2 + \cos[2\theta] b_1 + 2a \cos[3\theta] b_2) = 0 \tag{3.2.16}$$

$$\frac{1}{2a^4}(-a^4 S \sin[2\theta] + a^4 T \sin[2\theta] + 4a \sin[\theta] f_2 + 6 \sin[2\theta] f_3 - 4a^2 \sin[2\theta] m_1) - (2a \sin[\theta] a_2 + \sin[2\theta] b_1 + 2a \sin[3\theta] b_2) = 0 \tag{3.2.17}$$

$$\frac{1}{8a^3 \mu 1}(a^4 S - a^4 T - a^4 S \kappa 1 + a^4 T \kappa 1 - 2a^4 S \cos[2\theta] + 2a^4 T \cos[2\theta] - 4a^2 f_1 - 4a \cos[\theta] f_2 - 4 \cos[2\theta] f_3 + 4a^2 \cos[2\theta] m_1 + 4a^2 \kappa 1 \cos[2\theta] m_1) - \left(\frac{1}{2\mu}(\kappa \cos[\theta] a_0 - a a_1 + a \kappa a_1 - 2a^2 \cos[\theta] a_2 + a^2 \kappa \cos[\theta] a_2 - \cos[\theta] b_0 - a \cos[2\theta] b_1 - a^2 \cos[3\theta] b_2)\right) = 0 \tag{3.2.18}$$

$$\frac{1}{4\,a^3\,\mu 1}\,(a^4\,S\,\mathrm{Sin}[2\,\theta] - a^4\,T\,\mathrm{Sin}[2\,\theta] -$$

$$2\,a\,\mathrm{Sin}[\theta]\,f_2 - 2\,\mathrm{Sin}[2\,\theta]\,f_3 + 2\,a^2\,\mathrm{Sin}[2\,\theta]\,m_1 - 2\,a^2\,\kappa 1\,\mathrm{Sin}[2\,\theta]\,m_1) -$$

$$\left(\frac{1}{2\,\mu}\,(-\kappa\,\mathrm{Sin}[\theta]\,a_0 + 2\,a^2\,\mathrm{Sin}[\theta]\,a_2 + a^2\,\kappa\,\mathrm{Sin}[\theta]\,a_2 + \right. \qquad (3.2\,.19)$$

$$\left.\mathrm{Sin}[\theta]\,b_0 + a\,\mathrm{Sin}[2\,\theta]\,b_1 + a^2\,\mathrm{Sin}[3\,\theta]\,b_2)\right) = 0$$

Equating coefficients of cos [θ],cos [2 θ],cos [3 θ],cos [4 θ],sin [θ],sin [2 θ],sin[3 θ], sin[4 θ] and then equating them to zero would give us the following equations

$$2\,a\,a_2 - \frac{2\,f_2}{a^3} \tag{3.2.20}$$

$$\frac{1}{2}\left(-S + T - 2\,b_1 - \frac{6\,f_3}{a^4} + \frac{8\,m_1}{a^2}\right) \tag{3.2.21}$$

$$-\,2\,a\,b_2 \tag{3.2.22}$$

$$0 \tag{3.2.23}$$

$$2\,a\,a_2 - \frac{2\,f_2}{a^3} \tag{3.2.24}$$

$$b_1 + \frac{1}{2}\left(S - T - \frac{6\,f_3}{a^4} + \frac{4\,m_1}{a^2}\right) \tag{3.2.25}$$

$$2\,a\,b_2 \tag{3.2.26}$$

$$0 \tag{3.2.27}$$

$$\frac{\kappa\,a_0}{2\,\mu} - \frac{a^2\,a_2}{\mu} + \frac{a^2\,\kappa\,a_2}{2\,\mu} - \frac{b_0}{2\,\mu} + \frac{f_2}{2\,a^2\,\mu 1} \tag{3.2.28}$$

$$\frac{a\,S}{4\,\mu 1} - \frac{a\,T}{4\,\mu 1} - \frac{a\,b_1}{2\,\mu} + \frac{f_3}{2\,a^3\,\mu 1} - \frac{m_1}{2\,a\,\mu 1} - \frac{\kappa 1\,m_1}{2\,a\,\mu 1} \tag{3.2.29}$$

$$-\frac{a^2 b_2}{2\mu} \tag{3.2.30}$$

$$0 \tag{3.2.31}$$

$$-\frac{\kappa a_0}{2\mu}+\frac{a^2 a_2}{\mu}+\frac{a^2 \kappa a_2}{2\mu}+\frac{b_0}{2\mu}+\frac{f_2}{2a^2 \mu 1} \tag{3.2.32}$$

$$-\frac{a S}{4\mu 1}+\frac{a T}{4\mu 1}+\frac{a b_1}{2\mu}+\frac{f_3}{2a^3 \mu 1}-\frac{m_1}{2a\mu 1}+\frac{\kappa 1\, m_1}{2a\mu 1} \tag{3.2.33}$$

$$\frac{a^2 b_2}{2\mu} \tag{3.2.34}$$

$$0 \tag{3.2.35}$$

Since the above equations are insufficient to solve for the number of constants we get two more equations

$$-\frac{a S}{8\mu 1}+\frac{a T}{8\mu 1}+\frac{a S \kappa 1}{8\mu 1}-\frac{a T \kappa 1}{8\mu 1}-\frac{a a_1}{2\mu}+\frac{a \kappa a_1}{2\mu}+\frac{f_1}{2a\mu 1} \tag{3.2.36}$$

$$-\frac{S}{2}-\frac{T}{2}+2a_1-\frac{f_1}{a^2} \tag{3.2.37}$$

Solving equations (3.2.20) to (3.2.37) we get constants as

$$a_0 \to \frac{b_0}{\kappa}$$

$$a_1 \to -\frac{(-3\,S - T + S\,\kappa 1 - T\,\kappa 1)\,\mu}{4\,(2\,\mu - \mu 1 + \kappa\,\mu 1)}$$

$$a_2 \to 0$$

$$b_1 \to -\frac{-S\,\mu + T\,\mu + S\,\kappa 1\,\mu - T\,\kappa 1\,\mu}{2\,(\kappa 1\,\mu + \mu 1)}$$

$$b_2 \to 0$$

$$f_1 \to a^2\left(-\frac{S}{2} - \frac{T}{2}\right) - \frac{a^2\,(-3\,S - T + S\,\kappa 1 - T\,\kappa 1)\,\mu}{2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)}$$

$$f_2 \to 0$$

$$f_3 \to -\frac{-a^4\,S\,\mu + a^4\,T\,\mu - a^4\,S\,\mu 1 + a^4\,T\,\mu 1}{2\,(\kappa 1\,\mu + \mu 1)}$$

$$m_1 \to \frac{a^2\,(S\,\mu - T\,\mu + S\,\mu 1 - T\,\mu 1)}{2\,(\kappa 1\,\mu + \mu 1)}$$

Using the above values of the constants to get revised equation (3.2.7) and equation (3.2.8) to get

$$\sigma_{\text{rcirinc}} = -\frac{(-3\,S - T + S\,\kappa 1 - T\,\kappa 1)\,\mu}{2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)} + \frac{(-S\,\mu + T\,\mu + S\,\kappa 1\,\mu - T\,\kappa 1\,\mu)\,\text{Cos}[2\,\theta]}{2\,(\kappa 1\,\mu + \mu 1)}$$

$$\sigma_{\theta\text{cirinc}} = -\frac{(-3\,S - T + S\,\kappa 1 - T\,\kappa 1)\,\mu}{2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)} - \frac{(-S\,\mu + T\,\mu + S\,\kappa 1\,\mu - T\,\kappa 1\,\mu)\,\text{Cos}[2\,\theta]}{2\,(\kappa 1\,\mu + \mu 1)} \quad (3.2.38)$$

$$\tau_{r\theta\text{cirinc}} = -\frac{(-S\,\mu + T\,\mu + S\,\kappa 1\,\mu - T\,\kappa 1\,\mu)\,\text{Sin}[2\,\theta]}{2\,(\kappa 1\,\mu + \mu 1)}$$

$$u_{rcirinc} = \frac{1}{8} r \left(-\frac{(-1+\kappa)(S(-3+\kappa 1) - T(1+\kappa 1))}{2\mu + (-1+\kappa)\mu 1} + \frac{2(S-T)(-1+\kappa 1)\,\mathrm{Cos}[2\theta]}{\kappa 1\,\mu + \mu 1} \right)$$

$$u_{\theta cirinc} = -\frac{r(S-T)(-1+\kappa 1)\,\mathrm{Sin}[2\theta]}{4(\kappa 1\,\mu + \mu 1)} \qquad (3.2.39)$$

Again using the values of the constants to get revised equation (3.2.13) and equation (3.2.14)

$$\sigma_{rplate} =$$

$$\frac{S}{2} - \frac{a^2 S}{2 r^2} + \frac{T}{2} - \frac{a^2 T}{2 r^2} + \frac{3 a^2 S \mu}{2 r^2 (2\mu - \mu 1 + \kappa \mu 1)} + \frac{a^2 T \mu}{2 r^2 (2\mu - \mu 1 + \kappa \mu 1)} -$$

$$\frac{a^2 S \kappa 1\,\mu}{2 r^2 (2\mu - \mu 1 + \kappa \mu 1)} + \frac{a^2 T \kappa 1\,\mu}{2 r^2 (2\mu - \mu 1 + \kappa \mu 1)} + \frac{1}{2} S\,\mathrm{Cos}[2\theta] -$$

$$\frac{1}{2} T\,\mathrm{Cos}[2\theta] + \frac{3 a^4 S \mu\,\mathrm{Cos}[2\theta]}{2 r^4 (\kappa 1\,\mu + \mu 1)} - \frac{2 a^2 S \mu\,\mathrm{Cos}[2\theta]}{r^2 (\kappa 1\,\mu + \mu 1)} - \qquad (3.2.40)$$

$$\frac{3 a^4 T \mu\,\mathrm{Cos}[2\theta]}{2 r^4 (\kappa 1\,\mu + \mu 1)} + \frac{2 a^2 T \mu\,\mathrm{Cos}[2\theta]}{r^2 (\kappa 1\,\mu + \mu 1)} + \frac{3 a^4 S \mu 1\,\mathrm{Cos}[2\theta]}{2 r^4 (\kappa 1\,\mu + \mu 1)} -$$

$$\frac{2 a^2 S \mu 1\,\mathrm{Cos}[2\theta]}{r^2 (\kappa 1\,\mu + \mu 1)} - \frac{3 a^4 T \mu 1\,\mathrm{Cos}[2\theta]}{2 r^4 (\kappa 1\,\mu + \mu 1)} + \frac{2 a^2 T \mu 1\,\mathrm{Cos}[2\theta]}{r^2 (\kappa 1\,\mu + \mu 1)}$$

$$\sigma_{\theta plate} = \frac{S}{2} + \frac{a^2 S}{2 r^2} + \frac{T}{2} + \frac{a^2 T}{2 r^2} - \frac{3 a^2 S \mu}{2 r^2 (2 \mu - \mu 1 + \kappa \mu 1)} - \frac{a^2 T \mu}{2 r^2 (2 \mu - \mu 1 + \kappa \mu 1)} + \frac{a^2 S \kappa 1 \mu}{2 r^2 (2 \mu - \mu 1 + \kappa \mu 1)} - \frac{a^2 T \kappa 1 \mu}{2 r^2 (2 \mu - \mu 1 + \kappa \mu 1)} - \frac{1}{2} S \operatorname{Cos}[2\theta] + \frac{1}{2} T \operatorname{Cos}[2\theta] - \frac{3 a^4 S \mu \operatorname{Cos}[2\theta]}{2 r^4 (\kappa 1 \mu + \mu 1)} + \frac{3 a^4 T \mu \operatorname{Cos}[2\theta]}{2 r^4 (\kappa 1 \mu + \mu 1)} - \frac{3 a^4 S \mu 1 \operatorname{Cos}[2\theta]}{2 r^4 (\kappa 1 \mu + \mu 1)} + \frac{3 a^4 T \mu 1 \operatorname{Cos}[2\theta]}{2 r^4 (\kappa 1 \mu + \mu 1)} \qquad (3.2.41)$$

$$\tau_{r\theta plate} = -\frac{1}{2} S \operatorname{Sin}[2\theta] + \frac{1}{2} T \operatorname{Sin}[2\theta] + \frac{3 a^4 S \mu \operatorname{Sin}[2\theta]}{2 r^4 (\kappa 1 \mu + \mu 1)} - \frac{a^2 S \mu \operatorname{Sin}[2\theta]}{r^2 (\kappa 1 \mu + \mu 1)} - \frac{3 a^4 T \mu \operatorname{Sin}[2\theta]}{2 r^4 (\kappa 1 \mu + \mu 1)} + \frac{a^2 T \mu \operatorname{Sin}[2\theta]}{r^2 (\kappa 1 \mu + \mu 1)} + \frac{3 a^4 S \mu 1 \operatorname{Sin}[2\theta]}{2 r^4 (\kappa 1 \mu + \mu 1)} - \frac{a^2 S \mu 1 \operatorname{Sin}[2\theta]}{r^2 (\kappa 1 \mu + \mu 1)} - \frac{3 a^4 T \mu 1 \operatorname{Sin}[2\theta]}{2 r^4 (\kappa 1 \mu + \mu 1)} + \frac{a^2 T \mu 1 \operatorname{Sin}[2\theta]}{r^2 (\kappa 1 \mu + \mu 1)} \qquad (3.2.42)$$

$$u_{r plate} = -\frac{1}{8 r^3 \mu 1} \left(\frac{2 a^2 r^2 (-(S - T)(-1 + \kappa 1) \mu - (S + T)(-1 + \kappa) \mu 1)}{2 \mu + (-1 + \kappa) \mu 1} + \frac{2 a^2 (S - T)(a^2 - r^2 (1 + \kappa 1))(\mu + \mu 1) \operatorname{Cos}[2\theta]}{\kappa 1 \mu + \mu 1} + r^4 (S - T)(-1 + \kappa 1 + 2 \operatorname{Cos}[2\theta]) \right) \qquad (3.2.43)$$

$$u_{\theta plate} = -\frac{(S - T)(a^4 (\mu + \mu 1) + a^2 r^2 (-1 + \kappa 1)(\mu + \mu 1) - r^4 (\kappa 1 \mu + \mu 1)) \operatorname{Sin}[2\theta]}{4 r^3 \mu 1 (\kappa 1 \mu + \mu 1)} \qquad (3.2.44)$$

3.3 Example

Consider an infinite plate with a two-dimensional circular inclusion .The material of the infinite plate is iron while the two dimensional circular inclusion with a radius of 50 mm is made of chromium. There is a biaxial force acting on the plate with $1000N/mm^2$ in the X-direction while a biaxial force of $500N/mm^2$ in the Y direction.

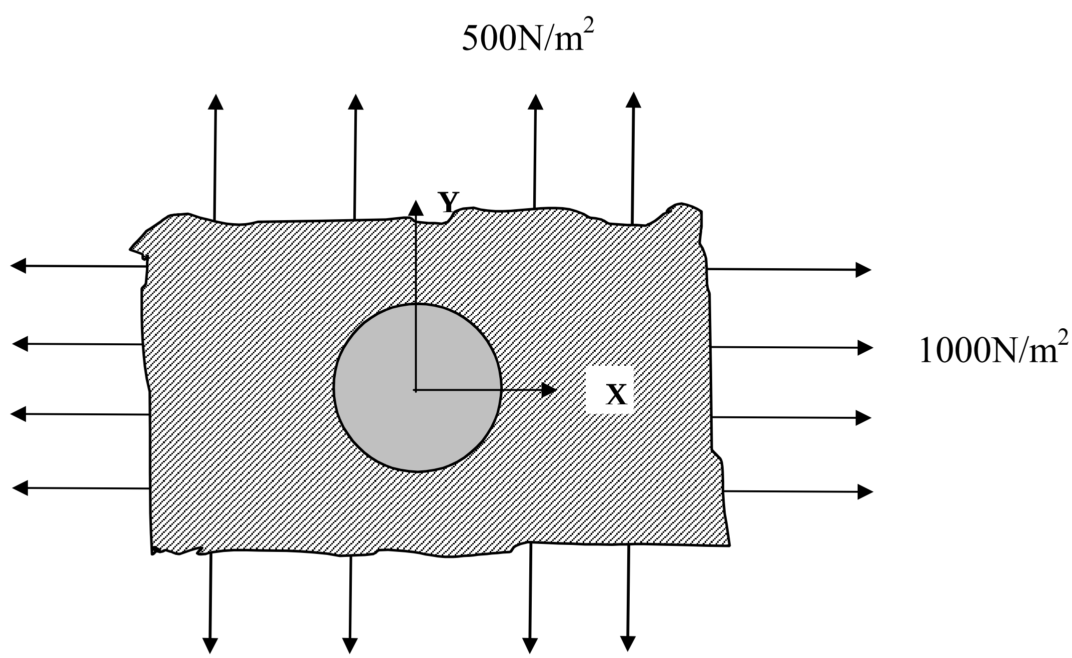

Figure 3.3 Infinite iron plate with chromium inclusion

We have the material properties of chromium [5] as

Poisson's ratio, $\upsilon = 0.21$

Shear modulus, μ= 115GPa

We have the material properties of iron [6] as

Poisson's ratio, $\upsilon = 0.28$

Shear modulus, μ1= 82GPa

Also

$$\kappa = 3\text{-}4\,\upsilon$$

Thus

κ =2.16……………………for chromium

κ1 =1.88……………………..…for iron

Substituting these values in equations (3.2.38), (3.2.40), (3.2.41), and (3.2.42) we get

For the chromium circular inclusion

$$\begin{aligned}
\sigma_{\text{rcirinc}} &= 452.756 + 84.842\,\text{Cos}[2\theta] \\
\sigma_{\theta\text{cirinc}} &= 452.755 - 84.842\,\text{Cos}[2\theta] \qquad (3.4\ .1) \\
\tau_{r\theta\text{cirinc}} &= -84.842\,\text{Sin}[2\theta]
\end{aligned}$$

For the infinite iron plate

$$\begin{aligned}
\sigma_{\text{rplate}} &= 750 - \frac{743110.230}{r^2} + 250\,\text{Cos}[2\theta] + \frac{3.096\,\text{Cos}[2\theta]}{r^4} - \frac{1.651\,\text{Cos}[2\theta]}{r^2} \\
\sigma_{\theta\text{plate}} &= 750 + \frac{743110.236}{r^2} - 250\,\text{Cos}[2\theta] - \frac{3.096\,\text{Cos}[2\theta]}{r^4} \qquad (3.4\ .2) \\
\tau_{r\theta\text{plate}} &= -250\,\text{Sin}[2\theta] + \frac{3.096\,\text{Sin}[2\theta]}{r^4} - \frac{825788.061\,\text{Sin}[2\theta]}{r^2}
\end{aligned}$$

Converting equations (3.4.1) and (3.4.2) to the Cartesian coordinate system we obtain the following result

For the chromium inclusion

$$\sigma_{xcirinc} = 537.598$$

$$\sigma_{ycirinc} = 367.914 \qquad (3.4.3)$$

$$\tau_{xycirinc} = 0$$

For the iron plate

$$\sigma_{xplate} = \frac{1}{2\,(x^2+y^2)^2}\Big(-4.768*10^{-7} + 3.492\;*10^{-10}\,(x^2+\;y^2) + 2000\,(x^2+y^2)^2 +$$
$$(4.7683*\;10^{-7} - 3.137*10^{6}\,(x^2+\;y^2))\,\mathrm{Cos}\Big[2\tan^{-1}\Big(\frac{y}{x}\Big)\Big] + \qquad (3.4.4)$$
$$(6.193*10^{9} - 1.651*10^{6}\;(x^2+y^2))\,\mathrm{Cos}\Big[4\tan^{-1}\Big(\frac{y}{x}\Big)\Big]\Big)$$

$$\sigma_{yplate} = \frac{1}{2\,(x^2+y^2)^2}\Big(4.768*10^{-7} + 3.492*10^{-10}\,(x^2+y^2) +$$
$$1000.\;(x^2+y^2)^2 + (4.768*10^{-7} - 165355.650\,(x^2+y^2)) \qquad (3.4.5)$$
$$\mathrm{Cos}\Big[2\tan^{-1}\Big(\frac{y}{x}\Big)\Big] \;+\; (-6.193*10^{9} + 1.651*10^{6}\,(x^2+y^2))\,\mathrm{Cos}\Big[4\tan^{-1}\Big(\frac{y}{x}\Big)\Big]\Big)$$

$$\tau_{xyplate} = -\frac{743110.23\,\mathrm{Sin}\big[2\tan^{-1}\big(\frac{y}{x}\big)\big]}{(x^2+y^2)} +$$
$$\qquad (3.4.6)$$
$$\frac{\big(-2.384*10^{-7} + 250\,(x^2+y^2)^2\big)\,\mathrm{Sin}\big[4\tan^{-1}\big(\frac{y}{x}\big)\big]}{(x^2+y^2)^2}$$

CHAPTER 4

CONCLUSIONS AND FUTURE WORK

4.1 Conclusions

All the work presented in Chapters 1 to Chapter 3 shows an estimated idea for solving elasticity problems using the complex variable theory method. Using the Airy stress function method we can arrive at results for stresses and displacements. Although the work is mostly restricted to two-dimensionality, it gives a fair idea of the behavior of an infinite plate with a hole and an infinite plate with a two-dimensional circular inclusion under biaxial loading. The value of stresses and displacements are dependent on the material properties of both the plate and two dimensional circular inclusion. For circular inclusion the stresses are constant while the shear stress is zero subjected to a far field stress.

4.2 Future Work

In this thesis, mostly the work is done for biaxial stresses as well as considering only one two-dimensional circular inclusion. Here are few suggestions for future work

1. Inclusion of shear stress.
2. Introduction of an elliptical hole as a two dimensional inclusion.
3. Multiple circular inclusions.

APPENDIX A

MATHEMATICA CODE FOR INFINITE PLATE WITH A HOLE SUBJECTED TO BIAXIAL LOADING

$$\gamma' = \sum_{n=1}^{3} a_n z^{-n}$$

$$\frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3}$$

$$\gamma = \frac{S + T}{4} z + \gamma'$$

$$\frac{1}{4} (S + T) z + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3}$$

$$\psi' = \sum_{n=1}^{3} b_n z^{-n}$$

$$\frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3}$$

$$\psi = \frac{T - S}{2} z + \psi'$$

$$\frac{1}{2} (-S + T) z + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3}$$

$$\chi = \int \psi \, dz$$

$$\frac{1}{4} (-S + T) z^2 + \text{Log}[z]\, b_1 - \frac{b_2}{z} - \frac{b_3}{2 z^2}$$

$$z = r\, e^{i\theta}$$

$$E^{I\theta} r$$

$$\overline{z} = \text{ComplexExpand}[\text{Conjugate}[z]]$$

$$r\, \text{Cos}[\theta] - I\, r\, \text{Sin}[\theta]$$

$$\phi^- = \bar{z}\,\gamma + \chi$$

$$\frac{1}{4} E^{2 I \theta} r^2 (-S + T) + (r \operatorname{Cos}[\theta] - I\, r \operatorname{Sin}[\theta]) \left(\frac{1}{4} E^{I\theta} r (S + T) + \frac{E^{-I\theta} a_1}{r} + \frac{E^{-2I\theta} a_2}{r^2} + \frac{E^{-3I\theta} a_3}{r^3}\right) + \operatorname{Log}[E^{I\theta} r]\, b_1 - \frac{E^{-I\theta} b_2}{r} - \frac{E^{-2I\theta} b_3}{2 r^2}$$

$$\phi^+ = \frac{1}{4 r^2} (r^4 (S - E^{2I\theta} (S - T) + T) + 4 E^{-3I\theta} r\, a_2 + 4 E^{-4I\theta} a_3 + 4 r^2 \operatorname{Log}[E^{I\theta} r]\, b_1 - 4 E^{-I\theta} r\, b_2 + 2 E^{-2I\theta} (2 r^2 a_1 - b_3))$$

$$\frac{1}{4 r^2} (r^4 (S - E^{2I\theta} (S - T) + T) + 4 E^{-3I\theta} r\, a_2 + 4 E^{-4I\theta} a_3 + 4 r^2 \operatorname{Log}[E^{I\theta} r]\, b_1 - 4 E^{-I\theta} r\, b_2 + 2 E^{-2I\theta} (2 r^2 a_1 - b_3))$$

$$\phi^+ = \frac{1}{4 r^2} (-(-1 + E^{2I\theta}) r^4 S + 4 E^{-3I\theta} r\, a_2 + 4 E^{-4I\theta} a_3 + 4 r^2 \operatorname{Log}[E^{I\theta} r]\, b_1 - 4 E^{-I\theta} r\, b_2 + 2 E^{-2I\theta} (2 r^2 a_1 - b_3))$$

$$\frac{1}{4 r^2} ((1 - E^{2I\theta}) r^4 S + 4 E^{-3I\theta} r\, a_2 + 4 E^{-4I\theta} a_3 + 4 r^2 \operatorname{Log}[E^{I\theta} r]\, b_1 - 4 E^{-I\theta} r\, b_2 + 2 E^{-2I\theta} (2 r^2 a_1 - b_3))$$

$$\phi = \text{FullSimplify}[\text{ComplexExpand}[\text{Re}[\phi^+]]]$$

$$\frac{1}{4 r^2} (r^4 (S + T + (-S + T) \operatorname{Cos}[2\theta]) + 4 \operatorname{Cos}[4\theta]\, a_3 + 2 r (2 r \operatorname{Cos}[2\theta]\, a_1 + 2 \operatorname{Cos}[3\theta]\, a_2 + r \operatorname{Log}[r^2]\, b_1 - 2 \operatorname{Cos}[\theta]\, b_2) - 2 \operatorname{Cos}[2\theta]\, b_3)$$

$$\phi_{rr} = \frac{r^2 S}{4} + \frac{r^2 T}{4} - \frac{1}{4} r^2 S \operatorname{Cos}[2\theta] + \frac{1}{4} r^2 T \operatorname{Cos}[2\theta] + \operatorname{Cos}[2\theta]\, a_1 + \frac{\operatorname{Cos}[3\theta]\, a_2}{r} + \frac{\operatorname{Cos}[4\theta]\, a_3}{r^2} + \frac{1}{2} \operatorname{Log}[r^2]\, b_1 - \frac{\operatorname{Cos}[\theta]\, b_2}{r} - \frac{\operatorname{Cos}[2\theta]\, b_3}{2 r^2}$$

$$\frac{r^2 S}{4} + \frac{r^2 T}{4} - \frac{1}{4} r^2 S \operatorname{Cos}[2\theta] + \frac{1}{4} r^2 T \operatorname{Cos}[2\theta] + \operatorname{Cos}[2\theta]\, a_1 + \frac{\operatorname{Cos}[3\theta]\, a_2}{r} + \frac{\operatorname{Cos}[4\theta]\, a_3}{r^2} + \frac{1}{2} \operatorname{Log}[r^2]\, b_1 - \frac{\operatorname{Cos}[\theta]\, b_2}{r} - \frac{\operatorname{Cos}[2\theta]\, b_3}{2 r^2}$$

$\phi_{rr} =$

$$\frac{1}{2\,r^2}\left(r^4\,S\,\mathrm{Sin}[\theta]^2 + 2\,\mathrm{Cos}[4\,\theta]\,a_3 + 2\,r^2\,\mathrm{Cos}[2\,\theta]\,a_1 + 2\,r\,\mathrm{Cos}[3\,\theta]\,a_2 + r^2\,\mathrm{Log}[r^2]\,b_1 - 2\,r\,\mathrm{Cos}[\theta]\,b_2 - \mathrm{Cos}[2\,\theta]\,b_3\right)$$

$$\frac{1}{2\,r^2}\left(r^4\,S\,\mathrm{Sin}[\theta]^2 + 2\,r^2\,\mathrm{Cos}[2\,\theta]\,a_1 + 2\,r\,\mathrm{Cos}[3\,\theta]\,a_2 + 2\,\mathrm{Cos}[4\,\theta]\,a_3 + r^2\,\mathrm{Log}[r^2]\,b_1 - 2\,r\,\mathrm{Cos}[\theta]\,b_2 - \mathrm{Cos}[2\,\theta]\,b_3\right)$$

σ_{rr} = ((D[ϕ_{rr}, r] * 1 / r) + (D[ϕ_{rr}, {θ, 2}] * 1 / r^2)) /. r → a /. Cos′[θ] → −Sin[θ] /. Cos″[θ] → −Cos[θ] // TrigReduce

$$\frac{1}{2\,a^4}\left(a^4\,S + a^4\,T + a^4\,S\,\mathrm{Cos}[2\,\theta] - a^4\,T\,\mathrm{Cos}[2\,\theta] - 8\,a^2\,\mathrm{Cos}[2\,\theta]\,a_1 - 20\,a\,\mathrm{Cos}[3\,\theta]\,a_2 - 36\,\mathrm{Cos}[4\,\theta]\,a_3 + 2\,a^2\,b_1 + 4\,a\,\mathrm{Cos}[\theta]\,b_2 + 6\,\mathrm{Cos}[2\,\theta]\,b_3\right)$$

$\sigma_{\theta\theta}$ = (D[ϕ_{rr} , {r, 2}]) /. r → a // TrigReduce

$$\frac{1}{2\,a^4}\left(a^4\,S + a^4\,T - a^4\,S\,\mathrm{Cos}[2\,\theta] + a^4\,T\,\mathrm{Cos}[2\,\theta] + 4\,a\,\mathrm{Cos}[3\,\theta]\,a_2 + 12\,\mathrm{Cos}[4\,\theta]\,a_3 - 2\,a^2\,b_1 - 4\,a\,\mathrm{Cos}[\theta]\,b_2 - 6\,\mathrm{Cos}[2\,\theta]\,b_3\right)$$

$\tau_{r\theta}$ = −D[D[ϕ_{rr}, θ] * 1 / r, r] /. r → a // TrigReduce

$$\frac{1}{2\,a^4}\left(-a^4\,S\,\mathrm{Sin}[2\,\theta] + a^4\,T\,\mathrm{Sin}[2\,\theta] - 4\,a^2\,\mathrm{Sin}[2\,\theta]\,a_1 - 12\,a\,\mathrm{Sin}[3\,\theta]\,a_2 - 24\,\mathrm{Sin}[4\,\theta]\,a_3 + 4\,a\,\mathrm{Sin}[\theta]\,b_2 + 6\,\mathrm{Sin}[2\,\theta]\,b_3\right)$$

γ' = D[γ, θ]

$$\frac{1}{4}\,I\,E^{I\theta}\,r\,(S+T) - \frac{I\,E^{-I\theta}\,a_1}{r} - \frac{2\,I\,E^{-2I\theta}\,a_2}{r^2} - \frac{3\,I\,E^{-3I\theta}\,a_3}{r^3}$$

$\overline{\gamma'}$ = ComplexExpand[Conjugate[γ']]

$$-\frac{1}{4}\,I\,E^{-I\theta}\,r\,S - \frac{1}{4}\,I\,E^{-I\theta}\,r\,T + \frac{I\,E^{I\theta}\,a_1}{r} + \frac{2\,I\,E^{2I\theta}\,a_2}{r^2} + \frac{3\,I\,E^{3I\theta}\,a_3}{r^3}$$

$\overline{\psi}$ = ComplexExpand[Conjugate[ψ]]

$$-\frac{1}{2}E^{-I\theta}\,r\,S+\frac{1}{2}E^{-I\theta}\,r\,T+\frac{E^{I\theta}\,b_1}{r}+\frac{E^{2I\theta}\,b_2}{r^2}+\frac{E^{3I\theta}\,b_3}{r^3}$$

$z1 = \overline{z}\,\gamma'$

$$(r\,\mathrm{Cos}[\theta]-I\,r\,\mathrm{Sin}[\theta])\left(\frac{1}{4}I\,E^{I\theta}\,r\,(S+T)-\frac{I\,E^{-I\theta}\,a_1}{r}-\frac{2\,I\,E^{-2I\theta}\,a_2}{r^2}-\frac{3\,I\,E^{-3I\theta}\,a_3}{r^3}\right)$$

$z2 = \kappa\,\gamma - z1 - \overline{\psi}$

$$\frac{1}{2}E^{-I\theta}\,r\,S-\frac{1}{2}E^{-I\theta}\,r\,T-(r\,\mathrm{Cos}[\theta]-I\,r\,\mathrm{Sin}[\theta])\left(\frac{1}{4}I\,E^{I\theta}\,r\,(S+T)-\frac{I\,E^{-I\theta}\,a_1}{r}-\frac{2\,I\,E^{-2I\theta}\,a_2}{r^2}-\frac{3\,I\,E^{-3I\theta}\,a_3}{r^3}\right)+\kappa\left(\frac{1}{4}E^{I\theta}\,r\,(S+T)+\frac{E^{-I\theta}\,a_1}{r}+\frac{E^{-2I\theta}\,a_2}{r^2}+\frac{E^{-3I\theta}\,a_3}{r^3}\right)-\frac{E^{I\theta}\,b_1}{r}-\frac{E^{2I\theta}\,b_2}{r^2}-\frac{E^{3I\theta}\,b_3}{r^3}$$

$z3 = \dfrac{z2}{2\,\mu\,e^{i\,\theta}}$

$$\frac{1}{2\mu}\left(E^{-I\theta}\left(\frac{1}{2}E^{-I\theta}\,r\,S-\frac{1}{2}E^{-I\theta}\,r\,T-(r\,\mathrm{Cos}[\theta]-I\,r\,\mathrm{Sin}[\theta])\left(\frac{1}{4}I\,E^{I\theta}\,r\,(S+T)-\frac{I\,E^{-I\theta}\,a_1}{r}-\frac{2\,I\,E^{-2I\theta}\,a_2}{r^2}-\frac{3\,I\,E^{-3I\theta}\,a_3}{r^3}\right)+\kappa\left(\frac{1}{4}E^{I\theta}\,r\,(S+T)+\frac{E^{-I\theta}\,a_1}{r}+\frac{E^{-2I\theta}\,a_2}{r^2}+\frac{E^{-3I\theta}\,a_3}{r^3}\right)-\frac{E^{I\theta}\,b_1}{r}-\frac{E^{2I\theta}\,b_2}{r^2}-\frac{E^{3I\theta}\,b_3}{r^3}\right)\right)$$

$z4 =$

$$\frac{1}{8\,r^3\,\mu}\big(r^4\,S\,\kappa+r^4\,T\,\kappa-I\,r^5\,S\,\mathrm{Cos}[\theta]-I\,r^5\,T\,\mathrm{Cos}[\theta]+2\,r^4\,S\,\mathrm{Cos}[2\theta]-2\,r^4\,T\,\mathrm{Cos}[2\theta]-r^5\,S\,\mathrm{Sin}[\theta]-r^5\,T\,\mathrm{Sin}[\theta]-2\,I\,r^4\,S\,\mathrm{Sin}[2\theta]+2\,I\,r^4\,T\,\mathrm{Sin}[2\theta]+4\,r^2\,\kappa\,\mathrm{Cos}[2\theta]\,a_1+4\,I\,r^3\,\mathrm{Cos}[3\theta]\,a_1-4\,I\,r^2\,\kappa\,\mathrm{Sin}[2\theta]\,a_1+4\,r^3\,\mathrm{Sin}[3\theta]\,a_1+4\,r\,\kappa\,\mathrm{Cos}[3\theta]\,a_2+8\,I\,r^2\,\mathrm{Cos}[4\theta]\,a_2-4\,I\,r\,\kappa\,\mathrm{Sin}[3\theta]\,a_2+8\,r^2\,\mathrm{Sin}[4\theta]\,a_2+4\,\kappa\,\mathrm{Cos}[4\theta]\,a_3+12\,I\,r\,\mathrm{Cos}[5\theta]\,a_3-4\,I\,\kappa\,\mathrm{Sin}[4\theta]\,a_3+12\,r\,\mathrm{Sin}[5\theta]\,a_3-4\,r^2\,b_1-4\,r\,\mathrm{Cos}[\theta]\,b_2-4\,I\,r\,\mathrm{Sin}[\theta]\,b_2-4\,\mathrm{Cos}[2\theta]\,b_3-4\,I\,\mathrm{Sin}[2\theta]\,b_3\big)$$

$$\frac{1}{8 r^3 \mu} (r^4 S \kappa + r^4 T \kappa - I r^5 S \text{Cos}[\theta] - I r^5 T \text{Cos}[\theta] + 2 r^4 S \text{Cos}[2\theta] - 2 r^4 T \text{Cos}[2\theta] - r^5 S \text{Sin}[\theta] - r^5 T \text{Sin}[\theta] - 2 I r^4 S \text{Sin}[2\theta] + 2 I r^4 T \text{Sin}[2\theta] + 4 r^2 \kappa \text{Cos}[2\theta] a_1 + 4 I r^3 \text{Cos}[3\theta] a_1 - 4 I r^2 \kappa \text{Sin}[2\theta] a_1 + 4 r^3 \text{Sin}[3\theta] a_1 + 4 r \kappa \text{Cos}[3\theta] a_2 + 8 I r^2 \text{Cos}[4\theta] a_2 - 4 I r \kappa \text{Sin}[3\theta] a_2 + 8 r^2 \text{Sin}[4\theta] a_2 + 4 \kappa \text{Cos}[4\theta] a_3 + 12 I r \text{Cos}[5\theta] a_3 - 4 I \kappa \text{Sin}[4\theta] a_3 + 12 r \text{Sin}[5\theta] a_3 - 4 r^2 b_1 - 4 r \text{Cos}[\theta] b_2 - 4 I r \text{Sin}[\theta] b_2 - 4 \text{Cos}[2\theta] b_3 - 4 I \text{Sin}[2\theta] b_3)$$

u_r = FullSimplify[ComplexExpand[Re[z4]]]

$$\frac{1}{8 r^3 \mu} (r^4 ((S+T)\kappa + 2(S-T)\text{Cos}[2\theta] - r(S+T)\text{Sin}[\theta]) + 4(r^2(\kappa \text{Cos}[2\theta] + r \text{Sin}[3\theta]) a_1 + r(\kappa \text{Cos}[3\theta] + 2 r \text{Sin}[4\theta]) a_2 + \kappa \text{Cos}[4\theta] a_3 + 3 r \text{Sin}[5\theta] a_3 - r^2 b_1 - r \text{Cos}[\theta] b_2 - \text{Cos}[2\theta] b_3))$$

u_θ = FullSimplify[ComplexExpand[Im[z4]]]

$$-\frac{1}{8 r^3 \mu} (r^4 \text{Cos}[\theta] (r(S+T) + 4(S-T)\text{Sin}[\theta]) + 4(r^2(-r \text{Cos}[3\theta] + \kappa \text{Sin}[2\theta]) a_1 + r(-2 r \text{Cos}[4\theta] + \kappa \text{Sin}[3\theta]) a_2 - 3 r \text{Cos}[5\theta] a_3 + \kappa \text{Sin}[4\theta] a_3 + r \text{Sin}[\theta] b_2 + \text{Sin}[2\theta] b_3))$$

$$u_{\theta\theta} = -\frac{1}{8 r^3 \mu} (r^5 S \text{Cos}[\theta] + 4 r^4 S \text{Cos}[\theta] \text{Sin}[\theta] - 4 r^3 \text{Cos}[3\theta] a_1 + 4 r^2 \kappa \text{Sin}[2\theta] a_1 - 8 r^2 \text{Cos}[4\theta] a_2 + 4 r \kappa \text{Sin}[3\theta] a_2 - 12 r \text{Cos}[5\theta] a_3 + 4 \kappa \text{Sin}[4\theta] a_3 + 4 r \text{Sin}[\theta] b_2 + 4 \text{Sin}[2\theta] b_3)$$

$$-\frac{1}{8 r^3 \mu} (r^5 S \text{Cos}[\theta] + 4 r^4 S \text{Cos}[\theta] \text{Sin}[\theta] - 4 r^3 \text{Cos}[3\theta] a_1 + 4 r^2 \kappa \text{Sin}[2\theta] a_1 - 8 r^2 \text{Cos}[4\theta] a_2 + 4 r \kappa \text{Sin}[3\theta] a_2 - 12 r \text{Cos}[5\theta] a_3 + 4 \kappa \text{Sin}[4\theta] a_3 + 4 r \text{Sin}[\theta] b_2 + 4 \text{Sin}[2\theta] b_3)$$

σ_{rr}

$$\frac{1}{2 a^4} (a^4 S + a^4 T + a^4 S \text{Cos}[2\theta] - a^4 T \text{Cos}[2\theta] - 8 a^2 \text{Cos}[2\theta] a_1 - 20 a \text{Cos}[3\theta] a_2 - 36 \text{Cos}[4\theta] a_3 + 2 a^2 b_1 + 4 a \text{Cos}[\theta] b_2 + 6 \text{Cos}[2\theta] b_3)$$

$$g1=\frac{S}{2}+\frac{1}{4}E^{-2I\theta}S+\frac{1}{4}E^{2I\theta}S+\frac{T}{2}-\frac{1}{4}E^{-2I\theta}T-\frac{1}{4}E^{2I\theta}T-\frac{2E^{-2I\theta}a_1}{a^2}-\frac{2E^{2I\theta}a_1}{a^2}-\frac{5E^{-3I\theta}a_2}{a^3}-\frac{5E^{3I\theta}a_2}{a^3}-\frac{9E^{-4I\theta}a_3}{a^4}-\frac{9E^{4I\theta}a_3}{a^4}+\frac{b_1}{a^2}+\frac{E^{-I\theta}b_2}{a^3}+\frac{E^{I\theta}b_2}{a^3}+\frac{3E^{-2I\theta}b_3}{2a^4}+\frac{3E^{2I\theta}b_3}{2a^4}$$

$$\frac{S}{2}+\frac{1}{4}E^{-2I\theta}S+\frac{1}{4}E^{2I\theta}S+\frac{T}{2}-\frac{1}{4}E^{-2I\theta}T-\frac{1}{4}E^{2I\theta}T-\frac{2E^{-2I\theta}a_1}{a^2}-\frac{2E^{2I\theta}a_1}{a^2}-\frac{5E^{-3I\theta}a_2}{a^3}-\frac{5E^{3I\theta}a_2}{a^3}-\frac{9E^{-4I\theta}a_3}{a^4}-\frac{9E^{4I\theta}a_3}{a^4}+\frac{b_1}{a^2}+\frac{E^{-I\theta}b_2}{a^3}+\frac{E^{I\theta}b_2}{a^3}+\frac{3E^{-2I\theta}b_3}{2a^4}+\frac{3E^{2I\theta}b_3}{2a^4}$$

$$g1=\frac{S}{2}+\frac{T}{2}+\frac{1}{2}S\,\mathrm{Cos}[2\theta]-\frac{1}{2}T\,\mathrm{Cos}[2\theta]-\frac{4\,\mathrm{Cos}[2\theta]a_1}{a^2}-\frac{10\,\mathrm{Cos}[3\theta]a_2}{a^3}-\frac{18\,\mathrm{Cos}[4\theta]a_3}{a^4}+\frac{b_1}{a^2}+\frac{2\,\mathrm{Cos}[\theta]b_2}{a^3}+\frac{3\,\mathrm{Cos}[2\theta]b_3}{a^4}$$

$$\frac{S}{2}+\frac{T}{2}+\frac{1}{2}S\,\mathrm{Cos}[2\theta]-\frac{1}{2}T\,\mathrm{Cos}[2\theta]-\frac{4\,\mathrm{Cos}[2\theta]a_1}{a^2}-\frac{10\,\mathrm{Cos}[3\theta]a_2}{a^3}-\frac{18\,\mathrm{Cos}[4\theta]a_3}{a^4}+\frac{b_1}{a^2}+\frac{2\,\mathrm{Cos}[\theta]b_2}{a^3}+\frac{3\,\mathrm{Cos}[2\theta]b_3}{a^4}$$

$$g2=i\,\tau_{r\theta}$$

$$\frac{1}{2a^4}\left(I\left(-a^4\,S\,\mathrm{Sin}[2\theta]+a^4\,T\,\mathrm{Sin}[2\theta]-4a^2\,\mathrm{Sin}[2\theta]a_1-12a\,\mathrm{Sin}[3\theta]a_2-24\,\mathrm{Sin}[4\theta]a_3+4a\,\mathrm{Sin}[\theta]b_2+6\,\mathrm{Sin}[2\theta]b_3\right)\right)$$

$$g3=\frac{1}{4}E^{-2I\theta}S-\frac{1}{4}E^{2I\theta}S-\frac{1}{4}E^{-2I\theta}T+\frac{1}{4}E^{2I\theta}T+\frac{E^{-2I\theta}a_1}{a^2}-\frac{E^{2I\theta}a_1}{a^2}+\frac{3E^{-3I\theta}a_2}{a^3}-\frac{3E^{3I\theta}a_2}{a^3}+\frac{6E^{-4I\theta}a_3}{a^4}-\frac{6E^{4I\theta}a_3}{a^4}-\frac{E^{-I\theta}b_2}{a^3}+\frac{E^{I\theta}b_2}{a^3}-\frac{3E^{-2I\theta}b_3}{2a^4}+\frac{3E^{2I\theta}b_3}{2a^4}$$

$$\frac{1}{4}E^{-2I\theta}S-\frac{1}{4}E^{2I\theta}S-\frac{1}{4}E^{-2I\theta}T+\frac{1}{4}E^{2I\theta}T+\frac{E^{-2I\theta}a_1}{a^2}-\frac{E^{2I\theta}a_1}{a^2}+\frac{3E^{-3I\theta}a_2}{a^3}-\frac{3E^{3I\theta}a_2}{a^3}+\frac{6E^{-4I\theta}a_3}{a^4}-\frac{6E^{4I\theta}a_3}{a^4}-\frac{E^{-I\theta}b_2}{a^3}+\frac{E^{I\theta}b_2}{a^3}-\frac{3E^{-2I\theta}b_3}{2a^4}+\frac{3E^{2I\theta}b_3}{2a^4}$$

Eq1 = Coefficient[g1, $e^{i\theta}$] – Coefficient[g3, $e^{i\theta}$]

0

Eq2 = Coefficient[g1, $e^{-i\theta}$] – Coefficient[g3, $e^{-i\theta}$]

$$\frac{2\, b_2}{a^3}$$

Eq3 = Coefficient[g1, $e^{2i\theta}$] – Coefficient[g3, $e^{2i\theta}$]

$$\frac{S}{2} - \frac{T}{2} - \frac{a_1}{a^2}$$

Eq4 = Coefficient[g1, $e^{-2i\theta}$] – Coefficient[g3, $e^{-2i\theta}$]

$$-\frac{3\, a_1}{a^2} + \frac{3\, b_3}{a^4}$$

Eq5 = Coefficient[g1, $e^{3i\theta}$] – Coefficient[g3, $e^{3i\theta}$]

$$-\frac{2\, a_2}{a^3}$$

Eq6 = Coefficient[g1, $e^{-3i\theta}$] – Coefficient[g3, $e^{-3i\theta}$]

$$-\frac{8\, a_2}{a^3}$$

Eq7 = Coefficient[g1, $e^{4i\theta}$] – Coefficient[g3, $e^{4i\theta}$]

$$-\frac{3\, a_3}{a^4}$$

Eq8 = Coefficient[g1, $e^{-4i\theta}$] – Coefficient[g3, $e^{-4i\theta}$]

$$-\frac{15\, a_3}{a^4}$$

$$\mathrm{Eq9} = \frac{S}{2} + \frac{T}{2} + \frac{b_1}{a^2}$$

$$\frac{S}{2} + \frac{T}{2} + \frac{b_1}{a^2}$$

Solve[{Eq1 == 0 , Eq2 == 0, Eq3 == 0, Eq4 == 0, Eq5 == 0, Eq6 == 0, Eq7 == 0,
Eq8 == 0 , Eq9 == 0}, $\{a_1, a_2, a_3, b_1, b_2, b_3\}$]

$$\left\{\left\{b_3 \to \frac{1}{2} a^4 (S - T), b_1 \to -\frac{1}{2} a^2 (S + T), b_2 \to 0, a_1 \to \frac{1}{2} a^2 (S - T), a_2 \to 0, a_3 \to 0\right\}\right\}$$

ϕ_{rr}

$$\frac{r^2 S}{4} + \frac{r^2 T}{4} - \frac{1}{4} r^2 S \,\mathrm{Cos}[2\theta] + \frac{1}{4} r^2 T \,\mathrm{Cos}[2\theta] + \mathrm{Cos}[2\theta]\, a_1 + \frac{\mathrm{Cos}[3\theta]\, a_2}{r} + \frac{\mathrm{Cos}[4\theta]\, a_3}{r^2} + \frac{1}{2} \mathrm{Log}[r^2]\, b_1 - \frac{\mathrm{Cos}[\theta]\, b_2}{r} - \frac{\mathrm{Cos}[2\theta]\, b_3}{2 r^2}$$

$\phi_{rr} =$

$$\frac{r^2 S}{4} + \frac{r^2 T}{4} - \frac{1}{4} r^2 S \,\mathrm{Cos}[2\theta] + \frac{1}{4} r^2 T \,\mathrm{Cos}[2\theta] + \mathrm{Cos}[2\theta]\, a_1 + \frac{\mathrm{Cos}[3\theta]\, a_2}{r} + \frac{\mathrm{Cos}[4\theta]\, a_3}{r^2} + \frac{1}{2} \mathrm{Log}[r^2]\, b_1 - \frac{\mathrm{Cos}[\theta]\, b_2}{r} - \frac{\mathrm{Cos}[2\theta]\, b_3}{2 r^2} \;/.\; b_3 \to \frac{1}{2} a^4 (S - T) \;/.\; b_1 \to -\frac{1}{2} a^2 (S + T) \;/.\; b_2 \to 0 \;/.\; a_1 \to \frac{1}{2} a^2 (S - T) \;/.\; a_2 \to 0 \;/.\; a_3 \to 0$$

$$\frac{r^2 S}{4} + \frac{r^2 T}{4} - \frac{1}{4} r^2 S \,\mathrm{Cos}[2\theta] + \frac{1}{2} a^2 (S - T) \,\mathrm{Cos}[2\theta] - \frac{a^4 (S - T)\, \mathrm{Cos}[2\theta]}{4 r^2} + \frac{1}{4} r^2 T \,\mathrm{Cos}[2\theta] - \frac{1}{4} a^2 (S + T)\, \mathrm{Log}[r^2]$$

$$\phi_{rrfinal} = \frac{1}{4} \left(-\frac{(a^2 - r^2)^2 (S - T)\, \mathrm{Cos}[2\theta]}{r^2} + (S + T)\left(r^2 - a^2 \,\mathrm{Log}[r^2]\right) \right)$$

$$\frac{1}{4} \left(-\frac{(a^2 - r^2)^2 (S - T)\, \mathrm{Cos}[2\theta]}{r^2} + (S + T)\left(r^2 - a^2 \,\mathrm{Log}[r^2]\right) \right)$$

$$\phi_{r\theta} = -\frac{1}{4\,r^2}\,(a^2\,(a^2 - 2\,r^2)\,S\,Cos[2\,\theta] + a^2\,r^2\,S\,Log[r^2] - 2\,r^4\,S\,Sin[\theta]^2)$$

$$-\frac{a^2\,(a^2 - 2\,r^2)\,S\,Cos[2\,\theta] + a^2\,r^2\,S\,Log[r^2] - 2\,r^4\,S\,Sin[\theta]^2}{4\,r^2}$$

$$\phi_{r\theta} = -\frac{1}{4\,r^2}\,(a^2\,(a^2 - 2\,r^2)\,S\,Cos[2\,\theta] + a^2\,r^2\,S\,Log[r^2] - 2\,r^4\,S\,Sin[\theta]^2)$$

$$-\frac{a^2\,(a^2 - 2\,r^2)\,S\,Cos[2\,\theta] + a^2\,r^2\,S\,Log[r^2] - 2\,r^4\,S\,Sin[\theta]^2}{4\,r^2}$$

$$\sigma_{rrfinal} = ((D[\phi_{rr}, r] * 1/r) + (D[\phi_{rr}, \{\theta, 2\}] * 1/r\verb|^|2))\ /.\ Cos'[\theta] \rightarrow -Sin[\theta]\ /.\ Cos''[\theta] \rightarrow -Cos[\theta]\ //\ TrigReduce$$

$$\frac{1}{2\,r^4}\,(-a^2\,r^2\,S + r^4\,S - a^2\,r^2\,T + r^4\,T + 3\,a^4\,S\,Cos[2\,\theta] - 4\,a^2\,r^2\,S\,Cos[2\,\theta] + r^4\,S\,Cos[2\,\theta] - 3\,a^4\,T\,Cos[2\,\theta] + 4\,a^2\,r^2\,T\,Cos[2\,\theta] - r^4\,T\,Cos[2\,\theta])$$

$$\sigma_{rrfinal} = \frac{(a - r)\,(a + r)\,(-r^2\,(S + T) + (3\,a^2 - r^2)\,(S - T)\,Cos[2\,\theta])}{2\,r^4}$$

$$\frac{(a - r)\,(a + r)\,(-r^2\,(S + T) + (3\,a^2 - r^2)\,(S - T)\,Cos[2\,\theta])}{2\,r^4}$$

$$\sigma_{\theta\theta final} = (D[\phi_{rr}, \{r, 2\}])\ //\ TrigReduce$$

$$\frac{a^2\,r^2\,S + r^4\,S + a^2\,r^2\,T + r^4\,T - 3\,a^4\,S\,Cos[2\,\theta] - r^4\,S\,Cos[2\,\theta] + 3\,a^4\,T\,Cos[2\,\theta] + r^4\,T\,Cos[2\,\theta]}{2\,r^4}$$

$$\tau_{r\theta final} = -D[D[\phi_{rr}, \theta] * 1/r, r]\ //\ TrigReduce$$

$$\frac{1}{2\,r^4}\,(3\,a^4\,S\,Sin[2\,\theta] - 2\,a^2\,r^2\,S\,Sin[2\,\theta] - r^4\,S\,Sin[2\,\theta] - 3\,a^4\,T\,Sin[2\,\theta] + 2\,a^2\,r^2\,T\,Sin[2\,\theta] + r^4\,T\,Sin[2\,\theta])$$

$$u_{rrfinal} = u_r /. \left\{b_3 \to \frac{a^4\, S}{2},\, b_1 \to -\frac{a^2\, s}{2},\, b_2 \to 0,\, a_1 \to \frac{a^2\, S}{2},\, a_2 \to 0,\, a_3 \to 0\right\} /.\, a \to r$$

$$\frac{1}{8\, r^3\, \mu} \left(r^4\, S\, (\kappa + 2\, \mathrm{Cos}[2\, \theta] - r\, \mathrm{Sin}[\theta]) + 4 \left(\frac{r^4\, s}{2} - \frac{1}{2}\, r^4\, S\, \mathrm{Cos}[2\, \theta] + \frac{1}{2}\, r^4\, S\, (\kappa\, \mathrm{Cos}[2\, \theta] + r\, \mathrm{Sin}[3\, \theta])\right)\right)$$

$$\frac{r\, (2\, s + S\, \kappa + 2\, S\, \kappa\, \mathrm{Cos}[2\, \theta] - r\, S\, \mathrm{Sin}[\theta] + 2\, r\, S\, \mathrm{Sin}[3\, \theta])}{8\, \mu}$$

$$u_{\theta\theta final} = u_{\theta\theta} /. \left\{b_3 \to \frac{a^4\, S}{2},\, b_1 \to -\frac{a^2\, s}{2},\, b_2 \to 0,\, a_1 \to \frac{a^2\, S}{2},\, a_2 \to 0,\, a_3 \to 0\right\} /.\, a \to r$$

$$-\frac{1}{8\, r^3\, \mu} \left(r^5\, S\, \mathrm{Cos}[\theta] - 2\, r^5\, S\, \mathrm{Cos}[3\, \theta] + 4\, r^4\, S\, \mathrm{Cos}[\theta]\, \mathrm{Sin}[\theta] + 2\, r^4\, S\, \mathrm{Sin}[2\, \theta] + 2\, r^4\, S\, \kappa\, \mathrm{Sin}[2\, \theta]\right)$$

$\mathrm{EquilibriumCondition1} = \sigma_{rrfinal}\ /.\ r \to a$

0

$\mathrm{EquilibriumCondition2} = \tau_{r\theta final}\ /.\ r \to a$

0

APPENDIX B

MATHEMATICA CODE FOR INFINITE PLATE WITH A TWO DIMENSIONAL CIRCULAR INCLUSION

$$\gamma = \sum_{n=0}^{2} a_n z^n$$

$$a_0 + z a_1 + z^2 a_2$$

$$\psi = \sum_{n=0}^{2} b_n z^n$$

$$b_0 + z b_1 + z^2 b_2$$

$$\gamma' = \sum_{n=1}^{1} m_n z^{-n}$$

$$\frac{m_1}{z}$$

$$\gamma 1 = \frac{S + T}{4} z + \gamma'$$

$$\frac{1}{4} (S + T) z + \frac{m_1}{z}$$

$$\psi' = \sum_{n=1}^{3} f_n z^{-n}$$

$$\frac{f_1}{z} + \frac{f_2}{z^2} + \frac{f_3}{z^3}$$

$$\psi 1 = \frac{T - S}{2} z + \psi'$$

$$\frac{1}{2} (-S + T) z + \frac{f_1}{z} + \frac{f_2}{z^2} + \frac{f_3}{z^3}$$

$$d = D[\gamma, z]$$

$$a_1 + 2 z a_2$$

$d1 = D[\gamma 1, z]$

$$\frac{S+T}{4} - \frac{m_1}{z^2}$$

$$\chi = \int \psi \, dz$$

$$z\,b_0 + \frac{z^2\,b_1}{2} + \frac{z^3\,b_2}{3}$$

$$\chi 1 = \int \psi 1 \, dz$$

$$\frac{1}{4}(-S+T)\,z^2 + \mathrm{Log}[z]\,f_1 - \frac{f_2}{z} - \frac{f_3}{2\,z^2}$$

$z = r\,e^{i\,\theta}$

$E^{I\theta}\,r$

$\bar{z} = \mathrm{ComplexExpand}[\mathrm{Conjugate}[z]]$

$r\,\mathrm{Cos}[\theta] - I\,r\,\mathrm{Sin}[\theta]$

$h = \mathrm{FullSimplify}[\mathrm{ComplexExpand}[(\bar{z}\ \gamma) + \chi]]$ // TrigReduce

$$\frac{1}{6}\,E^{-I\theta}\,r\,(6\,a_0 + 6\,E^{I\theta}\,r\,a_1 + 6\,E^{2I\theta}\,r^2\,a_2 + 6\,E^{2I\theta}\,b_0 + 3\,E^{3I\theta}\,r\,b_1 + 2\,E^{4I\theta}\,r^2\,b_2)$$

$h1 = \mathrm{FullSimplify}[\mathrm{ComplexExpand}[(\bar{z}\ \gamma 1) + \chi 1]]$ // TrigReduce

$$-\frac{1}{4\,r^2}\,(E^{-2I\theta}\,(-E^{2I\theta}\,r^4\,S + E^{4I\theta}\,r^4\,S - E^{2I\theta}\,r^4\,T - E^{4I\theta}\,r^4\,T - 4\,I\,E^{2I\theta}\,r^2\,\mathrm{Arg}[E^{I\theta}\,r]\,f_1 - 2\,E^{2I\theta}\,r^2\,\mathrm{Log}[r^2]\,f_1 + 4\,E^{I\theta}\,r\,f_2 + 2\,f_3 - 4\,r^2\,m_1))$$

$\phi_{cirinc} = \mathrm{FullSimplify}[\mathrm{ComplexExpand}[\mathrm{Re}[h]]]$

$$\frac{1}{6}\,r\,(6\,r\,a_1 + 6\,\mathrm{Cos}[\theta]\,(a_0 + r^2\,a_2 + b_0) + 3\,r\,\mathrm{Cos}[2\theta]\,b_1 + 2\,r^2\,\mathrm{Cos}[3\,\theta]\,b_2)$$

ϕ_{plate} = FullSimplify [ComplexExpand [Re[h1]]]

$$\frac{1}{4\,r^2}\left(r^4\,(S + T + (-S + T)\,\text{Cos}[2\,\theta]) + 2\,r^2\,\text{Log}[r^2]\,f_1 - 4\,r\,\text{Cos}[\theta]\,f_2 - 2\,\text{Cos}[2\,\theta]\,(f_3 - 2\,r^2\,m_1)\right)$$

$\sigma_{\text{rrlnicirinc}}$ = ((D[ϕ_{cirinc}, r] * 1 / r) + (D[ϕ_{cirinc}, {θ, 2}] * 1 / r^2)) /. Cos'[θ] → − Sin[θ] /.
Cos''[θ] → − Cos[θ] // TrigReduce

$$2\,a_1 + 2\,r\,\text{Cos}[\theta]\,a_2 - \text{Cos}[2\theta]\,b_1 - 2\,r\,\text{Cos}[3\theta]\,b_2$$

$\sigma_{\text{rrlniplate}}$ = ((D[ϕ_{plate}, r] * 1 / r) + (D[ϕ_{plate}, {θ, 2}] * 1 / r^2)) /. Cos'[θ] → − Sin[θ] /.
Cos''[θ] → − Cos[θ] // TrigReduce

$$\frac{1}{2\,r^4}\left(r^4\,S + r^4\,T + r^4\,S\,\text{Cos}[2\theta] - r^4\,T\,\text{Cos}[2\,\theta] + 2\,r^2\,f_1 + 4\,r\,\text{Cos}[\theta]\,f_2 + 6\,\text{Cos}[2\,\theta]\,f_3 - 8\,r^2\,\text{Cos}[2\,\theta]\,m_1\right)$$

σ_{rrcirinc} = FullSimplify [((D[ϕ_{cirinc}, r] * 1 / r) + (D[ϕ_{cirinc}, {θ, 2}] * 1 / r^2))] /. r → a /.
Cos'[θ] → − Sin[θ] /. Cos''[θ] → − Cos[θ] // TrigReduce

$$2\,a_1 + 2\,a\,\text{Cos}[\theta]\,a_2 - \text{Cos}[2\,\theta]\,b_1 - 2\,a\,\text{Cos}[3\,\theta]\,b_2$$

σ_{rrplate} = FullSimplify [((D[ϕ_{plate}, r] * 1 / r) + (D[ϕ_{plate}, {θ, 2}] * 1 / r^2))] /. r → a /.
Cos'[θ] → − Sin[θ] /. Cos''[θ] → − Cos[θ] // TrigReduce

$$\frac{1}{2\,a^4}\left(a^4\,S + a^4\,T + a^4\,S\,\text{Cos}[2\,\theta] - a^4\,T\,\text{Cos}[2\,\theta] + 2\,a^2\,f_1 + 4\,a\,\text{Cos}[\theta]\,f_2 + 6\,\text{Cos}[2\,\theta]\,f_3 - 8\,a^2\,\text{Cos}[2\,\theta]\,m_1\right)$$

$\sigma_{\theta\theta\text{lnicirinc}}$ = D[ϕ_{cirinc} , {r, 2}] // TrigReduce

$$2\,a_1 + 6\,r\,\text{Cos}[\theta]\,a_2 + \text{Cos}[2\,\theta]\,b_1 + 2\,r\,\text{Cos}[3\,\theta]\,b_2$$

$\sigma_{\theta\theta\text{cirinc}} = (\text{D}[\phi_{\text{cirinc}}, \{r, 2\}])$ /. r → a // TrigReduce

$2\,a_1 + 6\,a\,\text{Cos}[\theta]\,a_2 + \text{Cos}[2\,\theta]\,b_1 + 2\,a\,\text{Cos}[3\,\theta]\,b_2$

$\sigma_{\theta\theta\text{plate}} = \text{D}[\phi_{\text{plate}}, \{r, 2\}]$ /. r → a // TrigReduce

$$\frac{1}{2\,a^4}\left(a^4\,S + a^4\,T - a^4\,S\,\text{Cos}[2\,\theta] + a^4\,T\,\text{Cos}[2\,\theta] - 2a^2\,f_1 - 4\,a\,\text{Cos}[\theta]\,f_2 - 6\,\text{Cos}[2\,\theta]\,f_3\right)$$

$\tau_{r\theta\text{Inicirinc}} = \text{FullSimplify}[-\text{D}[\text{D}[\phi_{\text{cirinc}}, \theta] * 1/r, r]]$ /. Cos′[θ] → −Sin[θ] /.
Cos″[θ] → −Cos[θ] // TrigReduce

$2\,r\,\text{Sin}[\theta]\,a_2 + \text{Sin}[2\,\theta]\,b_1 + 2\,r\,\text{Sin}[3\,\theta]\,b_2$

$\tau_{r\theta\text{cirinc}} = \text{FullSimplify}[-\text{D}[\text{D}[\phi_{\text{cirinc}}, \theta] * 1/r, r]]$ /. r → a /. Cos′[θ] → −Sin[θ] /.
Cos″[θ] → −Cos[θ] // TrigReduce

$2\,a\,\text{Sin}[\theta]\,a_2 + \text{Sin}[2\,\theta]\,b_1 + 2\,a\,\text{Sin}[3\,\theta]\,b_2$

$\tau_{r\theta\text{Iniplate}} = \text{FullSimplify}[-\text{D}[\text{D}[\phi_{\text{plate}}, \theta] * 1/r, r]]$ /. Cos′[θ] → −Sin[θ] /.
Cos″[θ] → −Cos[θ] // TrigReduce

$$\frac{-r^4\,S\,\text{Sin}[2\,\theta] + r^4\,T\,\text{Sin}[2\,\theta] + 4\,r\,\text{Sin}[\theta]\,f_2 + 6\,\text{Sin}[2\,\theta]\,f_3 - 4\,r^2\,\text{Sin}[2\,\theta]\,m_1}{2\,r^4}$$

$\tau_{r\theta\text{plate}} = \text{FullSimplify}[-\text{D}[\text{D}[\phi_{\text{plate}}, \theta] * 1/r, r]]$ /. r → a /. Cos′[θ] → −Sin[θ] /.
Cos″[θ] → −Cos[θ] // TrigReduce

$$\frac{-a^4\,S\,\text{Sin}[2\,\theta] + a^4\,T\,\text{Sin}[2\,\theta] + 4\,a\,\text{Sin}[\theta]\,f_2 + 6\,\text{Sin}[2\,\theta]\,f_3 - 4\,a^2\,\text{Sin}[2\,\theta]\,m_1}{2\,a^4}$$

$\overline{\gamma'}$ = ComplexExpand[Conjugate[d]]

$a_1 + 2\,E^{-I\,\theta}\,r\,a_2$

$\overline{\gamma 1'}$ = ComplexExpand[Conjugate[d1]]

$$\frac{S}{4} + \frac{T}{4} - \frac{E^{2\,I\,\theta}\,m_1}{r^2}$$

$\overline{\psi}$ = ComplexExpand [Conjugate [ψ]]

$$b_0 + E^{-I\theta}\, r\, b_1 + E^{-2I\theta}\, r^2\, b_2$$

$\overline{\psi 1}$ = ComplexExpand [Conjugate [ψ1]]

$$-\frac{1}{2}\, E^{-I\theta}\, r\, S + \frac{1}{2}\, E^{-I\theta}\, r\, T + \frac{E^{I\theta}\, f_1}{r} + \frac{E^{2I\theta}\, f_2}{r^2} + \frac{E^{3I\theta}\, f_3}{r^3}$$

$z' = z\,\overline{\gamma'}$

$$(r\,\mathrm{Cos}[\theta] - I\, r\,\mathrm{Sin}[\theta])\,(a_1 + 2\, E^{-I\theta}\, r\, a_2)$$

$z1' = z\,\overline{\gamma 1'}$

$$(r\,\mathrm{Cos}[\theta] - I\, r\,\mathrm{Sin}[\theta])\left(\frac{S}{4} + \frac{T}{4} - \frac{E^{2I\theta}\, m_1}{r^2}\right)$$

$z2 = \kappa\gamma - z' - \overline{\psi}$

$$-(r\,\mathrm{Cos}[\theta] - I\, r\,\mathrm{Sin}[\theta])\,(a_1 + 2\, E^{-I\theta}\, r\, a_2) + \kappa\,(a_0 + E^{I\theta}\, r\, a_1 + E^{2I\theta}\, r^2\, a_2) - b_0 - E^{-I\theta}\, r\, b_1 - E^{-2I\theta}\, r^2\, b_2$$

$z2' = \kappa 1\, \gamma 1 - z1' - \overline{\psi 1}$

$$\frac{1}{2}\, E^{-I\theta}\, r\, S - \frac{1}{2}\, E^{-I\theta}\, r\, T - \frac{E^{I\theta}\, f_1}{r} - \frac{E^{2I\theta}\, f_2}{r^2} - \frac{E^{3I\theta}\, f_3}{r^3} - (r\,\mathrm{Cos}[\theta] - I\, r\,\mathrm{Sin}[\theta])\left(\frac{S}{4} + \frac{T}{4} - \frac{E^{2I\theta}\, m_1}{r^2}\right) + \kappa 1\left(\frac{1}{4}\, E^{I\theta}\, r\,(S + T) + \frac{E^{-I\theta}\, m_1}{r}\right)$$

$$z3 = \frac{z2}{2\,\mu\, e^{i\,\theta}}$$

$$\frac{1}{2\,\mu}\,(E^{-I\theta}\,(-(r\,\mathrm{Cos}[\theta] - I\, r\,\mathrm{Sin}[\theta])\,(a_1 + 2\, E^{-I\theta}\, r\, a_2) + \kappa\,(a_0 + E^{I\theta}\, r\, a_1 + E^{2I\theta}\, r^2\, a_2) - b_0 - E^{-I\theta}\, r\, b_1 - E^{-2I\theta}\, r^2\, b_2))$$

$$z4 = \frac{1}{2\,\mu}\,(-\mathrm{I}\,\kappa\,\mathrm{Sin}[\theta]\,a_0 - r\,a_1 + r\,\kappa\,a_1 - 2r^2\,\mathrm{Cos}[\theta]\,a_2 + r^2\,\kappa\,\mathrm{Cos}[\theta]\,a_2 + 2\,\mathrm{I}\,r^2\,\mathrm{Sin}[\theta]\,a_2 + \mathrm{I}\,r^2\,\kappa\,\mathrm{Sin}[\theta]\,a_2 - \mathrm{Cos}[\theta]\,b_0 + \mathrm{I}\,\mathrm{Sin}[\theta]\,b_0 - r\,\mathrm{Cos}[2\,\theta]\,b_1 + \mathrm{I}\,r\,\mathrm{Sin}[2\,\theta]\,b_1 - r^2\,\mathrm{Cos}[3\,\theta]\,b_2 + \mathrm{I}\,r^2\,\mathrm{Sin}[3\,\theta]\,b_2 + a_0\,\kappa\,\mathrm{Cos}[\theta])$$

$$\frac{1}{2\,\mu}\,(\kappa\,\mathrm{Cos}[\theta]\,a_0 - \mathrm{I}\,\kappa\,\mathrm{Sin}[\theta]\,a_0 - r\,a_1 + r\,\kappa\,a_1 - 2\,r^2\,\mathrm{Cos}[\theta]\,a_2 + r^2\,\kappa\,\mathrm{Cos}[\theta]\,a_2 + 2\,\mathrm{I}\,r^2\,\mathrm{Sin}[\theta]\,a_2 + \mathrm{I}\,r^2\,\kappa\,\mathrm{Sin}[\theta]\,a_2 - \mathrm{Cos}[\theta]\,b_0 + \mathrm{I}\,\mathrm{Sin}[\theta]\,b_0 - r\,\mathrm{Cos}[2\theta]\,b_1 + \mathrm{I}\,r\,\mathrm{Sin}[2\,\theta]\,b_1 - r^2\,\mathrm{Cos}[3\,\theta]\,b_2 + \mathrm{I}\,r^2\,\mathrm{Sin}[3\,\theta]\,b_2)$$

$$\frac{1}{2\,\mu}\,(-\mathrm{I}\,\kappa\,\mathrm{Sin}[\theta]\,a_0 - r\,a_1 + r\,\kappa\,a_1 - 2\,r^2\,\mathrm{Cos}[\theta]\,a_2 + r^2\,\kappa\,\mathrm{Cos}[\theta]\,a_2 + 2\,\mathrm{I}\,r^2\,\mathrm{Sin}[\theta]\,a_2 + \mathrm{I}\,r^2\,\kappa\,\mathrm{Sin}[\theta]\,a_2 - \mathrm{Cos}[\theta]\,b_0 + \mathrm{I}\,\mathrm{Sin}[\theta]\,b_0 - r\,\mathrm{Cos}[2\,\theta]\,b_1 + \mathrm{I}\,r\,\mathrm{Sin}[2\theta]\,b_1 - r^2\,\mathrm{Cos}[3\theta]\,b_2 + \mathrm{I}\,r^2\,\mathrm{Sin}[3\theta]\,b_2 + a_0\,\kappa\,\mathrm{Cos}[\theta])$$

$$\frac{1}{2\,\mu}\,(\kappa\,\mathrm{Cos}[\theta]\,a_0 - \mathrm{I}\,\kappa\,\mathrm{Sin}[\theta]\,a_0 - r\,a_1 + r\,\kappa\,a_1 - 2\,r^2\,\mathrm{Cos}[\theta]\,a_2 + r^2\,\kappa\,\mathrm{Cos}[\theta]\,a_2 + 2\,\mathrm{I}\,r^2\,\mathrm{Sin}[\theta]\,a_2 + \mathrm{I}\,r^2\,\kappa\,\mathrm{Sin}[\theta]\,a_2 - \mathrm{Cos}[\theta]\,b_0 + \mathrm{I}\,\mathrm{Sin}[\theta]\,b_0 - r\,\mathrm{Cos}[2\theta]\,b_1 + \mathrm{I}\,r\,\mathrm{Sin}[2\,\theta]\,b_1 - r^2\,\mathrm{Cos}[3\,\theta]\,b_2 + \mathrm{I}\,r^2\,\mathrm{Sin}[3\,\theta]\,b_2)$$

$$z3' = \frac{z2'}{2\,\mu 1\,e^{i\,\theta}}$$

$$\frac{1}{2\,\mu 1}\left(\mathrm{E}^{-\mathrm{I}\theta}\left(\frac{1}{2}\,\mathrm{E}^{-\mathrm{I}\theta}\,r\,S - \frac{1}{2}\,\mathrm{E}^{-\mathrm{I}\theta}\,r\,T - \frac{\mathrm{E}^{\mathrm{I}\,\theta}\,f_1}{r} - \frac{\mathrm{E}^{2\,\mathrm{I}\,\theta}\,f_2}{r^2} - \frac{\mathrm{E}^{3\,\mathrm{I}\,\theta}\,f_3}{r^3} - (r\,\mathrm{Cos}[\theta] - \mathrm{I}\,r\,\mathrm{Sin}[\theta])\left(\frac{S}{4} + \frac{T}{4} - \frac{\mathrm{E}^{2\,\mathrm{I}\,\theta}\,m_1}{r^2}\right) + \kappa 1\left(\frac{1}{4}\,\mathrm{E}^{\mathrm{I}\,\theta}\,r\,(S + T) + \frac{\mathrm{E}^{-\mathrm{I}\,\theta}\,m_1}{r}\right)\right)\right)$$

$$z4' = \frac{1}{8\,r^3\,\mu 1}\,(-r^4\,(T - S) + r^4\,(T - S)\,\kappa 1 + 2\,r^4\,(T - S)\,\mathrm{Cos}[2\theta] - 2\,\mathrm{I}\,r^4\,(T - S)\,\mathrm{Sin}[2\theta] - 4\,r^2\,f_1 - 4\,r\,\mathrm{Cos}[\theta]\,f_2 - 4\,\mathrm{I}\,r\,\mathrm{Sin}[\theta]\,f_2 - 4\,\mathrm{Cos}[2\,\theta]\,f_3 - 4\,\mathrm{I}\,\mathrm{Sin}[2\theta]\,f_3 + 4\,r^2\,\mathrm{Cos}[2\,\theta]\,m_1 + 4\,r^2\,\kappa 1\,\mathrm{Cos}[2\,\theta]\,m_1 + 4\,\mathrm{I}\,r^2\,\mathrm{Sin}[2\theta]\,m_1 - 4\,\mathrm{I}\,r^2\,\kappa 1\,\mathrm{Sin}[2\,\theta]\,m_1)$$

$$\frac{1}{8\,r^3\,\mu 1}\,(-r^4\,(-S + T) + r^4\,(-S + T)\,\kappa 1 + 2\,r^4\,(-S + T)\,\mathrm{Cos}[2\theta] - 2\,\mathrm{I}\,r^4\,(-S + T)\,\mathrm{Sin}[2\,\theta] - 4\,r^2\,f_1 - 4\,r\,\mathrm{Cos}[\theta]\,f_2 - 4\,\mathrm{I}\,r\,\mathrm{Sin}[\theta]\,f_2 - 4\,\mathrm{Cos}[2\,\theta]\,f_3 - 4\,\mathrm{I}\,\mathrm{Sin}[2\,\theta]\,f_3 + 4\,r^2\,\mathrm{Cos}[2\theta]\,m_1 + 4\,r^2\,\kappa 1\,\mathrm{Cos}[2\theta]\,m_1 + 4\,\mathrm{I}\,r^2\,\mathrm{Sin}[2\,\theta]\,m_1 - 4\,\mathrm{I}\,r^2\,\kappa 1\,\mathrm{Sin}[2\,\theta]\,m_1)$$

$u_{rcirinc}$ = FullSimplify [ComplexExpand [Re[z4]]]

$$\frac{1}{2\,\mu}\left(\kappa\,\mathrm{Cos}[\theta]\,a_0 + r\,(-1+\kappa)\,a_1 - 2\,r^2\,\mathrm{Cos}[\theta]\,a_2 + r^2\,\kappa\,\mathrm{Cos}[\theta]\,a_2 - \mathrm{Cos}[\theta]\,b_0 - r\,\mathrm{Cos}[2\,\theta]\,b_1 - r^2\,\mathrm{Cos}[3\,\theta]\,b_2\right)$$

u_{rplate} = FullSimplify [ComplexExpand [Re[z4′]]]

$$-\frac{1}{8\,r^3\,\mu 1}\left(r^4\,(S-T)\,(-1+\kappa 1+2\,\mathrm{Cos}[2\,\theta]) + 4\,r^2\,f_1 + 4\,r\,\mathrm{Cos}[\theta]\,f_2 + 4\,\mathrm{Cos}[2\,\theta]\,(f_3 - r^2\,(1+\kappa 1)\,m_1)\right)$$

$u_{\theta cirinc}$ = FullSimplify [ComplexExpand [Im[z4]]]

$$\frac{-\kappa\,\mathrm{Sin}[\theta]\,a_0 + r^2\,(2+\kappa)\,\mathrm{Sin}[\theta]\,a_2 + \mathrm{Sin}[\theta]\,b_0 + r\,\mathrm{Sin}[2\,\theta]\,b_1 + r^2\,\mathrm{Sin}[3\,\theta]\,b_2}{2\,\mu}$$

$u_{\theta plate}$ = FullSimplify [ComplexExpand [Im[z4′]]]

$$\frac{\mathrm{Sin}[\theta]\,(-r\,f_2 + \mathrm{Cos}[\theta]\,(r^4\,(S-T) - 2\,f_3 - 2\,r^2\,(-1+\kappa 1)\,m_1))}{2\,r^3\,\mu 1}$$

$u_{rfinicirinc}$ = FullSimplify [$u_{rcirinc}$] // TrigReduce

$$\frac{1}{2\,\mu}\left(\kappa\,\mathrm{Cos}[\theta]\,a_0 - r\,a_1 + r\,\kappa\,a_1 - 2\,r^2\,\mathrm{Cos}[\theta]\,a_2 + r^2\,\kappa\,\mathrm{Cos}[\theta]\,a_2 - \mathrm{Cos}[\theta]\,b_0 - r\,\mathrm{Cos}[2\,\theta]\,b_1 - r^2\,\mathrm{Cos}[3\,\theta]\,b_2\right)$$

$u_{rfiniplate}$ = FullSimplify [u_{rplate}] // TrigReduce

$$\frac{1}{8\,r^3\,\mu 1}\left(r^4\,S - r^4\,T - r^4\,S\,\kappa 1 + r^4\,T\,\kappa 1 - 2\,r^4\,S\,\mathrm{Cos}[2\,\theta] + 2\,r^4\,T\,\mathrm{Cos}[2\,\theta] - 4\,r^2\,f_1 - 4\,r\,\mathrm{Cos}[\theta]\,f_2 - 4\,\mathrm{Cos}[2\,\theta]\,f_3 + 4\,r^2\,\mathrm{Cos}[2\,\theta]\,m_1 + 4\,r^2\,\kappa 1\,\mathrm{Cos}[2\,\theta]\,m_1\right)$$

$u_{rrcirinc}$ = FullSimplify [$u_{rfinicirinc}$] /. r → a // TrigReduce

$$\frac{1}{2\,\mu}\left(\kappa\,\mathrm{Cos}[\theta]\,a_0 - a\,a_1 + a\,\kappa\,a_1 - 2\,a^2\,\mathrm{Cos}[\theta]\,a_2 + a^2\,\kappa\,\mathrm{Cos}[\theta]\,a_2 - \mathrm{Cos}[\theta]\,b_0 - a\,\mathrm{Cos}[2\,\theta]\,b_1 - a^2\,\mathrm{Cos}[3\,\theta]\,b_2\right)$$

$u_{\text{rrplate}} = \text{FullSimplify}\,[u_{\text{rIniplate}}]\,/.\ r \to a\ //\ \text{TrigReduce}$

$$\frac{1}{8\,a^3\,\mu 1}\,(a^4\,S - a^4\,T - a^4\,S\,\kappa 1 + a^4\,T\,\kappa 1 - 2\,a^4\,S\,\text{Cos}[2\,\theta] + 2\,a^4\,T\,\text{Cos}[2\,\theta] - 4\,a^2\,f_1 - 4\,a\,\text{Cos}[\theta]\,f_2 - 4\,\text{Cos}[2\,\theta]\,f_3 + 4\,a^2\,\text{Cos}[2\,\theta]\,m_1 + 4\,a^2\,\kappa 1\,\text{Cos}[2\,\theta]\,m_1)$$

$u_{\theta\text{Inicirinc}} = \text{FullSimplify}\,[u_{\theta\text{cirinc}}]\,/.\ \text{Cos}^2\,\theta + \text{Sin}^2\,\theta \to 1\ //\ \text{TrigReduce}$

$$\frac{1}{2\,\mu}\,(-\kappa\,\text{Sin}[\theta]\,a_0 + 2\,r^2\,\text{Sin}[\theta]\,a_2 + r^2\,\kappa\,\text{Sin}[\theta]\,a_2 + \text{Sin}[\theta]\,b_0 + r\,\text{Sin}[2\,\theta]\,b_1 + r^2\,\text{Sin}[3\,\theta]\,b_2)$$

$u_{\theta\text{Iniplate}} = \text{FullSimplify}\,[u_{\theta\text{plate}}]\,/.\ \text{Cos}^2\,\theta + \text{Sin}^2\,\theta \to 1\ //\ \text{TrigReduce}$

$$\frac{1}{4\,r^3\,\mu 1}\,(r^4\,S\,\text{Sin}[2\theta] - r^4\,T\,\text{Sin}[2\,\theta] - 2\,r\,\text{Sin}[\theta]\,f_2 - 2\,\text{Sin}[2\theta]\,f_3 + 2\,r^2\,\text{Sin}[2\theta]\,m_1 - 2\,r^2\,\kappa 1\,\text{Sin}[2\theta]\,m_1)$$

$u_{\theta\theta\text{cirinc}} = \text{FullSimplify}\,[u_{\theta\text{Inicirinc}}]\,/.\ r \to a\ //\ \text{TrigReduce}$

$$\frac{1}{2\,\mu}\,(-\kappa\,\text{Sin}[\theta]\,a_0 + 2\,a^2\,\text{Sin}[\theta]\,a_2 + a^2\,\kappa\,\text{Sin}[\theta]\,a_2 + \text{Sin}[\theta]\,b_0 + a\,\text{Sin}[2\,\theta]\,b_1 + a^2\,\text{Sin}[3\,\theta]\,b_2)$$

$u_{\theta\theta\text{plate}} = \text{FullSimplify}\,[u_{\theta\text{Iniplate}}]\,/.\ r \to a\ //\ \text{TrigReduce}$

$$\frac{1}{4\,a^3\,\mu 1}\,(a^4\,S\,\text{Sin}[2\theta] - a^4\,T\,\text{Sin}[2\,\theta] - 2\,a\,\text{Sin}[\theta]\,f_2 - 2\,\text{Sin}[2\,\theta]\,f_3 + 2\,a^2\,\text{Sin}[2\,\theta]\,m_1 - 2\,a^2\,\kappa 1\,\text{Sin}[2\,\theta]\,m_1)$$

$\text{Eq1} = \text{FullSimplify}\,[\text{Coefficient}[\sigma_{\text{rrcirinc}}\ //\ \text{Expand}\,,\ \text{Cos}[\theta]] - \text{Coefficient}\,[\sigma_{\text{rrplate}}\ //\ \text{Expand}\,,\ \text{Cos}[\theta]]]$

$$2\,a\,a_2 - \frac{2\,f_2}{a^3}$$

$\text{Eq2} = \text{FullSimplify}\,[\text{Coefficient}[\sigma_{\text{rrcirinc}}\ //\ \text{Expand}\,,\ \text{Cos}[2\,\theta]] - \text{Coefficient}[\sigma_{\text{rrplate}}\ //\ \text{Expand}\,,\ \text{Cos}[2\theta]]]$

$$\frac{1}{2}\left(-S + T - 2\,b_1 - \frac{6\,f_3}{a^4} + \frac{8\,m_1}{a^2}\right)$$

$$Eq3 = \text{FullSimplify}[\text{Coefficient}[\sigma_{\text{rrcirinc}}\ //\ \text{Expand}\ ,\ \text{Cos}[3\,\theta]] - \text{Coefficient}[\sigma_{\text{rrplate}}\ //\ \text{Expand}\ ,\ \text{Cos}[3\,\theta]]]$$

$$-2\,a\,b_2$$

$$Eq4 = \text{FullSimplify}[\text{Coefficient}[\sigma_{\text{rrcirinc}}\ //\ \text{Expand}\ ,\ \text{Cos}[4\,\theta]] - \text{Coefficient}[\sigma_{\text{rrplate}}\ //\ \text{Expand}\ ,\ \text{Cos}[4\,\theta]]]$$

$$0$$

$$Eq5 = \text{FullSimplify}[\text{Coefficient}[\tau_{\text{r}\theta\text{cirinc}}\ //\ \text{Expand}\ ,\ \text{Sin}[\theta]] - \text{Coefficient}[\tau_{\text{r}\theta\text{plate}}\ //\ \text{Expand}\ ,\ \text{Sin}[\theta]]]$$

$$2\,a\,a_2 - \frac{2\,f_2}{a^3}$$

$$Eq6 = \text{FullSimplify}[\text{Coefficient}[\tau_{\text{r}\theta\text{cirinc}}\ //\ \text{Expand}\ ,\ \text{Sin}[2\,\theta]] - \text{Coefficient}[\tau_{\text{r}\theta\text{plate}}\ //\ \text{Expand}\ ,\ \text{Sin}[2\,\theta]]]$$

$$b_1 + \frac{1}{2}\left(S - T - \frac{6\,f_3}{a^4} + \frac{4\,m_1}{a^2}\right)$$

$$Eq7 = \text{FullSimplify}[\text{Coefficient}[\tau_{\text{r}\theta\text{cirinc}}\ //\ \text{Expand}\ ,\ \text{Sin}[3\,\theta]] - \text{Coefficient}[\tau_{\text{r}\theta\text{plate}}\ //\ \text{Expand}\ ,\ \text{Sin}[3\,\theta]]]$$

$$2\,a\,b_2$$

$$Eq8 = \text{FullSimplify}[\text{Coefficient}[\tau_{\text{r}\theta\text{cirinc}}\ //\ \text{Expand}\ ,\ \text{Sin}[4\,\theta]] - \text{Coefficient}[\tau_{\text{r}\theta\text{plate}}\ //\ \text{Expand}\ ,\ \text{Sin}[4\,\theta]]]$$

$$0$$

$$Eq9 = \text{Expand}[\text{FullSimplify}[\text{Coefficient}[u_{\text{rrcirinc}}\ //\ \text{Expand}\ ,\ \text{Cos}[\theta]] - \text{Coefficient}[u_{\text{rrplate}}\ //\ \text{Expand}\ ,\ \text{Cos}[\theta]]]]$$

$$\frac{\kappa\,a_0}{2\,\mu} - \frac{a^2\,a_2}{\mu} + \frac{a^2\,\kappa\,a_2}{2\,\mu} - \frac{b_0}{2\,\mu} + \frac{f_2}{2\,a^2\,\mu 1}$$

Eq10 = Expand[FullSimplify[
    Coefficient[$u_{rrcirinc}$ // Expand , Cos[2 θ]] – Coefficient[$u_{rrplate}$ // Expand , Cos[2 θ]]]]

$$\frac{a\,S}{4\,\mu 1}-\frac{a\,T}{4\,\mu 1}-\frac{a\,b_1}{2\,\mu}+\frac{f_3}{2\,a^3\,\mu 1}-\frac{m_1}{2\,a\,\mu 1}-\frac{\kappa 1\,m_1}{2a\,\mu 1}$$

Eq11 = Expand[FullSimplify[
    Coefficient[$u_{rrcirinc}$ // Expand , Cos[3 θ]] – Coefficient[$u_{rrplate}$ // Expand , Cos[3 θ]]]]

$$-\frac{a^2\,b_2}{2\,\mu}$$

Eq12 = Expand[FullSimplify[
    Coefficient[$u_{rrcirinc}$ // Expand , Cos[4 θ]] – Coefficient[$u_{rrplate}$ // Expand , Cos[4 θ]]]]

0

Eq13 = Expand[FullSimplify[
    Coefficient[$u_{\theta\theta cirinc}$ // Expand , Sin[θ]] – Coefficient[$u_{\theta\theta plate}$ // Expand , Sin[θ]]]]

$$-\frac{\kappa\,a_0}{2\,\mu}+\frac{a^2\,a_2}{\mu}+\frac{a^2\,\kappa\,a_2}{2\,\mu}+\frac{b_0}{2\,\mu}+\frac{f_2}{2\,a^2\,\mu 1}$$

Eq14 = Expand[FullSimplify[
    Coefficient[$u_{\theta\theta cirinc}$ // Expand , Sin[2 θ]] – Coefficient[$u_{\theta\theta plate}$ // Expand , Sin[2 θ]]]]

$$-\frac{a\,S}{4\,\mu 1}+\frac{a\,T}{4\,\mu 1}+\frac{a\,b_1}{2\,\mu}+\frac{f_3}{2a^3\,\mu 1}-\frac{m_1}{2\,a\,\mu 1}+\frac{\kappa 1\,m_1}{2\,a\,\mu 1}$$

Eq15 = Expand[FullSimplify[
    Coefficient[$u_{\theta\theta cirinc}$ // Expand , Sin[3 θ]] – Coefficient[$u_{\theta\theta plate}$ // Expand , Sin[3 θ]]]]

$$\frac{a^2\,b_2}{2\,\mu}$$

Eq16 = Expand[FullSimplify[
    Coefficient[$u_{\theta\theta cirinc}$ // Expand , Sin[4 θ]] – Coefficient[$u_{\theta\theta plate}$ // Expand , Sin[4 θ]]]]

0

$$\text{Eq17} = \text{Expand}[u_{\text{rrcirinc}} - \text{Coefficient}[u_{\text{rrcirinc}}, \text{Cos}[\theta]] * \text{Cos}[\theta] - \text{Coefficient}[u_{\text{rrcirinc}}, \text{Cos}[2\theta]] * \text{Cos}[2\theta] - \text{Coefficient}[u_{\text{rrcirinc}}, \text{Cos}[3\theta]] * \text{Cos}[3\theta] - \text{Coefficient}[u_{\text{rrcirinc}}, \text{Cos}[4\theta]] * \text{Cos}[4\theta] - (u_{\text{rrplate}} - \text{Coefficient}[u_{\text{rrplate}}, \text{Cos}[\theta]] * \text{Cos}[\theta] - \text{Coefficient}[u_{\text{rrplate}}, \text{Cos}[2\theta]] * \text{Cos}[2\theta] - \text{Coefficient}[u_{\text{rrplate}}, \text{Cos}[3\theta]] * \text{Cos}[3\theta] - \text{Coefficient}[u_{\text{rrplate}}, \text{Cos}[4\theta]] * \text{Cos}[4\theta])]$$

$$-\frac{a\,S}{8\,\mu 1} + \frac{a\,T}{8\,\mu 1} + \frac{a\,S\,\kappa 1}{8\,\mu 1} - \frac{a\,T\,\kappa 1}{8\,\mu 1} - \frac{a\,a_1}{2\,\mu} + \frac{a\,\kappa\,a_1}{2\,\mu} + \frac{f_1}{2\,a\,\mu 1}$$

$$\text{Eq18} = \text{Expand}[\sigma_{\text{rrcirinc}} - \text{Coefficient}[\sigma_{\text{rrcirinc}}, \text{Cos}[\theta]] * \text{Cos}[\theta] - \text{Coefficient}[\sigma_{\text{rrcirinc}}, \text{Cos}[2\theta]] * \text{Cos}[2\theta] - \text{Coefficient}[\sigma_{\text{rrcirinc}}, \text{Cos}[3\theta]] * \text{Cos}[3\theta] - \text{Coefficient}[\sigma_{\text{rrcirinc}}, \text{Cos}[4\theta]] * \text{Cos}[4\theta] - (\sigma_{\text{rrplate}} - \text{Coefficient}[\sigma_{\text{rrplate}}, \text{Cos}[\theta]] * \text{Cos}[\theta] - \text{Coefficient}[\sigma_{\text{rrplate}}, \text{Cos}[2\theta]] * \text{Cos}[2\theta] - \text{Coefficient}[\sigma_{\text{rrplate}}, \text{Cos}[3\theta]] * \text{Cos}[3\theta] - \text{Coefficient}[\sigma_{\text{rrplate}}, \text{Cos}[4\theta]] * \text{Cos}[4\theta])]$$

$$-\frac{S}{2} - \frac{T}{2} + 2\,a_1 - \frac{f_1}{a^2}$$

$$\text{Eq19} = \text{Expand}[u_{\theta\theta\text{cirinc}} - \text{Coefficient}[u_{\theta\theta\text{cirinc}}, \text{Sin}[\theta]] * \text{Sin}[\theta] - \text{Coefficient}[u_{\theta\theta\text{cirinc}}, \text{Sin}[2\theta]] * \text{Sin}[2\theta] - \text{Coefficient}[u_{\theta\theta\text{cirinc}}, \text{Sin}[3\theta]] * \text{Sin}[3\theta] - \text{Coefficient}[u_{\theta\theta\text{cirinc}}, \text{Sin}[4\theta]] * \text{Sin}[4\theta] - (u_{\theta\theta\text{plate}} - \text{Coefficient}[u_{\theta\theta\text{plate}}, \text{Sin}[\theta]] * \text{Sin}[\theta] - \text{Coefficient}[u_{\theta\theta\text{plate}}, \text{Sin}[2\theta]] * \text{Sin}[2\theta] - \text{Coefficient}[u_{\theta\theta\text{plate}}, \text{Sin}[3\theta]] * \text{Sin}[3\theta] - \text{Coefficient}[u_{\theta\theta\text{plate}}, \text{Sin}[4\theta]] * \text{Sin}[4\theta])]$$

$$0$$

$$\text{Eq20} = \text{Expand}[\tau_{r\theta\text{cirinc}} - \text{Coefficient}[\tau_{r\theta\text{cirinc}}, \text{Sin}[\theta]] * \text{Sin}[\theta] - \text{Coefficient}[\tau_{r\theta\text{cirinc}}, \text{Sin}[2\theta]] * \text{Sin}[2\theta] - \text{Coefficient}[\tau_{r\theta\text{cirinc}}, \text{Sin}[3\theta]] * \text{Sin}[3\theta] - \text{Coefficient}[\tau_{r\theta\text{cirinc}}, \text{Sin}[4\theta]] * \text{Sin}[4\theta] - (\tau_{r\theta\text{plate}} - \text{Coefficient}[\tau_{r\theta\text{plate}}, \text{Sin}[\theta]] * \text{Sin}[\theta] - \text{Coefficient}[\tau_{r\theta\text{plate}}, \text{Sin}[2\theta]] * \text{Sin}[2\theta] - \text{Coefficient}[\tau_{r\theta\text{plate}}, \text{Sin}[3\theta]] * \text{Sin}[3\theta] - \text{Coefficient}[\tau_{r\theta\text{plate}}, \text{Sin}[4\theta]] * \text{Sin}[4\theta])]$$

$$0$$

$$\text{Solve}[\{\text{Eq17} == 0, \text{Eq18} == 0\}, \{f_1, a_1\}]$$

$$\left\{\left\{f_1 \to a^2\left(-\frac{S}{2} - \frac{T}{2}\right) - \frac{a^2\,(-3\,S - T + S\,\kappa 1 - T\,\kappa 1)\,\mu}{2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)},\ a_1 \to -\frac{(-3\,S - T + S\,\kappa 1 - T\,\kappa 1)\,\mu}{4\,(2\,\mu - \mu 1 + \kappa\,\mu 1)}\right\}\right\}$$

$$\text{Solve}[\{\text{Eq1} == 0, \text{Eq2} == 0, \text{Eq3} == 0, \text{Eq5} == 0, \text{Eq6} == 0, \text{Eq7} == 0, \text{Eq9} == 0, \text{Eq10} == 0, \text{Eq11} == 0, \text{Eq13} == 0, \text{Eq14} == 0, \text{Eq15} == 0, \text{Eq17} == 0, \text{Eq18} == 0\}, \{f_1, a_1, f_2, a_2, f_3, a_0, b_1, m_1, b_2, m_2, b_0\}]$$

$$\Big\{\Big\{a_0 \to \frac{b_0}{\kappa}, f_1 \to a^2\left(-\frac{S}{2} - \frac{T}{2}\right) - \frac{a^2 (-3\,S - T + S\,\kappa 1 - T\,\kappa 1)\,\mu}{2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)},$$
$$a_1 \to -\frac{(-3\,S - T + S\,\kappa 1 - T\,\kappa 1)\,\mu}{4\,(2\,\mu - \mu 1 + \kappa\,\mu 1)}, f_2 \to 0, a_2 \to 0,$$
$$f_3 \to -\frac{-a^4\,S\,\mu + a^4\,T\,\mu - a^4\,S\,\mu 1 + a^4\,T\,\mu 1}{2\,(\kappa 1\,\mu + \mu 1)}, b_1 \to -\frac{-S\,\mu + T\,\mu + S\,\kappa 1\,\mu - T\,\kappa 1\,\mu}{2\,(\kappa 1\,\mu + \mu 1)},$$
$$m_1 \to \frac{a^2\,(S\,\mu - T\,\mu + S\,\mu 1 - T\,\mu 1)}{2\,(\kappa 1\,\mu + \mu 1)}, b_2 \to 0\Big\}\Big\}$$

$$\sigma_{\text{rcirinc}} = \sigma_{\text{rrhicirinc}} \,/.\, a_1 \to -\frac{(-3\,S - T + S\,\kappa 1 - T\,\kappa 1)\,\mu}{4\,(2\,\mu - \mu 1 + \kappa\,\mu 1)} \,/.$$
$$b_1 \to -\frac{-S\,\mu + T\,\mu + S\,\kappa 1\,\mu - T\,\kappa 1\,\mu}{2\,(\kappa 1\,\mu + \mu 1)} \,/.\, a_2 \to 0 \,/.\, b_2 \to 0$$

$$-\frac{(-3\,S - T + S\,\kappa 1 - T\,\kappa 1)\,\mu}{2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)} + \frac{(-S\,\mu + T\,\mu + S\,\kappa 1\,\mu - T\,\kappa 1\,\mu)\,\text{Cos}[2\,\theta]}{2\,(\kappa 1\,\mu + \mu 1)}$$

$$\sigma_{\text{rplate}} = \text{Expand}\Big[\sigma_{\text{rrhiplate}} \,/.\, f_2 \to 0 \,/.\, f_3 \to -\frac{-a^4\,S\,\mu + a^4\,T\,\mu - a^4\,S\,\mu 1 + a^4\,T\,\mu 1}{2\,(\kappa 1\,\mu + \mu 1)} \,/.$$
$$m_1 \to \frac{a^2\,(S\,\mu - T\,\mu + S\,\mu 1 - T\,\mu 1)}{2\,(\kappa 1\,\mu + \mu 1)} \,/.\, m_2 \to 0 \,/.$$
$$f_1 \to a^2\left(-\frac{S}{2} - \frac{T}{2}\right) - \frac{a^2\,(-3\,S - T + S\,\kappa 1 - T\,\kappa 1)\,\mu}{2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)}\Big]$$

$$\frac{S}{2} - \frac{a^2\,S}{2\,r^2} + \frac{T}{2} - \frac{a^2\,T}{2\,r^2} + \frac{3\,a^2\,S\,\mu}{2\,r^2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)} + \frac{a^2\,T\,\mu}{2\,r^2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)} -$$
$$\frac{a^2\,S\,\kappa 1\,\mu}{2\,r^2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)} + \frac{a^2\,T\,\kappa 1\,\mu}{2\,r^2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)} + \frac{1}{2}\,S\,\text{Cos}[2\,\theta] - \frac{1}{2}\,T\,\text{Cos}[2\,\theta] +$$
$$\frac{3\,a^4\,S\,\mu\,\text{Cos}[2\,\theta]}{2\,r^4\,(\kappa 1\,\mu + \mu 1)} - \frac{2\,a^2\,S\,\mu\,\text{Cos}[2\,\theta]}{r^2\,(\kappa 1\,\mu + \mu 1)} - \frac{3\,a^4\,T\,\mu\,\text{Cos}[2\,\theta]}{2\,r^4\,(\kappa 1\,\mu + \mu 1)} + \frac{2\,a^2\,T\,\mu\,\text{Cos}[2\,\theta]}{r^2\,(\kappa 1\,\mu + \mu 1)} +$$
$$\frac{3\,a^4\,S\,\mu 1\,\text{Cos}[2\,\theta]}{2\,r^4\,(\kappa 1\,\mu + \mu 1)} - \frac{2\,a^2\,S\,\mu 1\,\text{Cos}[2\,\theta]}{r^2\,(\kappa 1\,\mu + \mu 1)} - \frac{3\,a^4\,T\,\mu 1\,\text{Cos}[2\,\theta]}{2\,r^4\,(\kappa 1\,\mu + \mu 1)} + \frac{2\,a^2\,T\,\mu 1\,\text{Cos}[2\,\theta]}{r^2\,(\kappa 1\,\mu + \mu 1)}$$

$$\sigma_{\theta\text{cirinc}} = \sigma_{\theta\theta\text{Inicirinc}} \,/.\, a_1 \to -\frac{(-3\,S - T + S\,\kappa 1 - T\,\kappa 1)\,\mu}{4\,(2\,\mu - \mu 1 + \kappa\,\mu 1)} \,/.$$

$$b_1 \to -\frac{-S\,\mu + T\,\mu + S\,\kappa 1\,\mu - T\,\kappa 1\,\mu}{2\,(\kappa 1\,\mu + \mu 1)} \,/.\, a_2 \to 0 \,/.\, b_2 \to 0$$

$$-\frac{(-3\,S - T + S\,\kappa 1 - T\,\kappa 1)\,\mu}{2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)} - \frac{(-S\,\mu + T\,\mu + S\,\kappa 1\,\mu - T\,\kappa 1\,\mu)\,\mathrm{Cos}[2\,\theta]}{2\,(\kappa 1\,\mu + \mu 1)}$$

$$\sigma_{\theta\text{plate}} = \mathrm{Expand}\Big[\sigma_{\theta\theta\text{Iniplate}} \,/.\, f_2 \to 0 \,/.\, f_3 \to -\frac{-a^4\,S\,\mu + a^4\,T\,\mu - a^4\,S\,\mu 1 + a^4\,T\,\mu 1}{2\,(\kappa 1\,\mu + \mu 1)} \,/.$$

$$m_1 \to \frac{a^2\,(S\,\mu - T\,\mu + S\,\mu 1 - T\,\mu 1)}{2\,(\kappa 1\,\mu + \mu 1)} \,/.\, m_2 \to 0 \,/.$$

$$f_1 \to a^2\left(-\frac{S}{2} - \frac{T}{2}\right) - \frac{a^2\,(-3\,S - T + S\,\kappa 1 - T\,\kappa 1)\,\mu}{2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)}\Big]$$

$$\frac{S}{2} + \frac{a^2\,S}{2\,r^2} + \frac{T}{2} + \frac{a^2\,T}{2\,r^2} - \frac{3\,a^2\,S\,\mu}{2\,r^2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)} - \frac{a^2\,T\,\mu}{2\,r^2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)} +$$
$$\frac{a^2\,S\,\kappa 1\,\mu}{2\,r^2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)} - \frac{a^2\,T\,\kappa 1\,\mu}{2\,r^2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)} - \frac{1}{2}\,S\,\mathrm{Cos}[2\,\theta] + \frac{1}{2}\,T\,\mathrm{Cos}[2\,\theta] -$$
$$\frac{3\,a^4\,S\,\mu\,\mathrm{Cos}[2\,\theta]}{2\,r^4\,(\kappa 1\,\mu + \mu 1)} + \frac{3\,a^4\,T\,\mu\,\mathrm{Cos}[2\,\theta]}{2\,r^4\,(\kappa 1\,\mu + \mu 1)} - \frac{3\,a^4\,S\,\mu 1\,\mathrm{Cos}[2\,\theta]}{2\,r^4\,(\kappa 1\,\mu + \mu 1)} + \frac{3\,a^4\,T\,\mu 1\,\mathrm{Cos}[2\,\theta]}{2\,r^4\,(\kappa 1\,\mu + \mu 1)}$$

$$\tau_{r\theta\text{cirinc}} = \tau_{r\theta\text{Inicirinc}} \,/.\, a_1 \to -\frac{(-3\,S - T + S\,\kappa 1 - T\,\kappa 1)\,\mu}{4\,(2\,\mu - \mu 1 + \kappa\,\mu 1)} \,/.$$

$$b_1 \to -\frac{-S\,\mu + T\,\mu + S\,\kappa 1\,\mu - T\,\kappa 1\,\mu}{2\,(\kappa 1\,\mu + \mu 1)} \,/.\, a_2 \to 0 \,/.\, b_2 \to 0$$

$$-\frac{(-S\,\mu + T\,\mu + S\,\kappa 1\,\mu - T\,\kappa 1\,\mu)\,\mathrm{Sin}[2\,\theta]}{2\,(\kappa 1\,\mu + \mu 1)}$$

$$\tau_{r\theta\text{plate}} = \mathrm{Expand}\Big[\tau_{r\theta\text{Iniplate}} \,/.\, f_2 \to 0 \,/.\, f_3 \to -\frac{-a^4\,S\,\mu + a^4\,T\,\mu - a^4\,S\,\mu 1 + a^4\,T\,\mu 1}{2\,(\kappa 1\,\mu + \mu 1)} \,/.$$

$$m_1 \to \frac{a^2\,(S\,\mu - T\,\mu + S\,\mu 1 - T\,\mu 1)}{2\,(\kappa 1\,\mu + \mu 1)} \,/.\, m_2 \to 0 \,/.$$

$$f_1 \to a^2\left(-\frac{S}{2} - \frac{T}{2}\right) - \frac{a^2\,(-3\,S - T + S\,\kappa 1 - T\,\kappa 1)\,\mu}{2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)}\Big]$$

$$-\frac{1}{2}\,S\,Sin[2\theta]+\frac{1}{2}\,T\,Sin[2\theta]+\frac{3a^4\,S\,\mu\,Sin[2\theta]}{2\,r^4\,(\kappa 1\,\mu+\mu 1)}-\frac{a^2\,S\,\mu\,Sin[2\theta]}{r^2\,(\kappa 1\,\mu+\mu 1)}-$$
$$\frac{3\,a^4\,T\,\mu\,Sin[2\theta]}{2\,r^4\,(\kappa 1\,\mu+\mu 1)}+\frac{a^2\,T\,\mu\,Sin[2\theta]}{r^2\,(\kappa 1\,\mu+\mu 1)}+\frac{3\,a^4\,S\,\mu 1\,Sin[2\theta]}{2\,r^4\,(\kappa 1\,\mu+\mu 1)}-$$
$$\frac{a^2\,S\,\mu 1\,Sin[2\theta]}{r^2\,(\kappa 1\,\mu+\mu 1)}-\frac{3a^4\,T\,\mu 1\,Sin[2\theta]}{2\,r^4\,(\kappa 1\,\mu+\mu 1)}+\frac{a^2\,T\,\mu 1\,Sin[2\theta]}{r^2\,(\kappa 1\,\mu+\mu 1)}$$

$$-((S-T)\,(-3\,a^4\,(\mu+\mu 1)+2\,a^2\,r^2\,(\mu+\mu 1)+r^4\,(\kappa 1\,\mu+\mu 1))\,Sin[2\theta])\,/\,(2\,r^4\,(\kappa 1\,\mu+\mu 1))$$

$$u_{rcirinc}=$$
$$\text{FullSimplify}\Big[u_{r\text{Inicirinc}}\,/.\,b_0\to\kappa\,a_0\,/.\,b_2\to 0\,/.\,a_2\to 0\,/.\,a_1\to-\frac{(-3\,S-T+S\,\kappa 1-T\,\kappa 1)\,\mu}{4\,(2\,\mu-\mu 1+\kappa\,\mu 1)}\,/.$$
$$b_1\to-\frac{-S\,\mu+T\,\mu+S\,\kappa 1\,\mu-T\,\kappa 1\,\mu}{2\,(\kappa 1\,\mu+\mu 1)}\Big]$$

$$\frac{1}{8}\,r\left(-\frac{(-1+\kappa)\,(S\,(-3+\kappa 1)-T\,(1+\kappa 1))}{2\,\mu+(-1+\kappa)\,\mu 1}+\frac{2\,(S-T)\,(-1+\kappa 1)\,Cos[2\theta]}{\kappa 1\,\mu+\mu 1}\right)$$

$$u_{\theta cirinc}=$$
$$\text{FullSimplify}\Big[u_{\theta\text{Inicirinc}}\,/.\,b_0\to\kappa\,a_0\,/.\,b_2\to 0\,/.\,a_2\to 0\,/.\,a_1\to-\frac{(-3\,S-T+S\,\kappa 1-T\,\kappa 1)\,\mu}{4\,(2\,\mu-\mu 1+\kappa\,\mu 1)}\,/.$$
$$b_1\to-\frac{-S\,\mu+T\,\mu+S\,\kappa 1\,\mu-T\,\kappa 1\,\mu}{2\,(\kappa 1\,\mu+\mu 1)}\Big]$$

$$-\frac{r\,(S-T)\,(-1+\kappa 1)\,Sin[2\theta]}{4\,(\kappa 1\,\mu+\mu 1)}$$

$$u_{rplate}=\text{FullSimplify}\Big[u_{rplate}\,/.\,f_2\to 0\,/.\,f_3\to-\frac{-a^4\,S\,\mu+a^4\,T\,\mu-a^4\,S\,\mu 1+a^4\,T\,\mu 1}{2\,(\kappa 1\,\mu+\mu 1)}\,/.$$
$$m_1\to\frac{a^2\,(S\,\mu-T\,\mu+S\,\mu 1-T\,\mu 1)}{2\,(\kappa 1\,\mu+\mu 1)}\,/.\,m_2\to 0\,/.$$
$$f_1\to a^2\left(-\frac{S}{2}-\frac{T}{2}\right)-\frac{a^2\,(-3\,S-T+S\,\kappa 1-T\,\kappa 1)\,\mu}{2\,(2\,\mu-\mu 1+\kappa\,\mu 1)}\Big]$$

$$-\frac{1}{8\,r^3\,\mu 1}$$
$$\Big((2\,a^2\,r^2\,(-(S-T)\,(-1+\kappa 1)\,\mu-(S+T)\,(-1+\kappa)\,\mu 1))\,/\,(2\,\mu+(-1+\kappa)\,\mu 1)+\frac{1}{\kappa 1\,\mu+\mu 1}$$
$$(2\,a^2\,(S-T)\,(a^2-r^2\,(1+\kappa 1))\,(\mu+\mu 1)\,Cos[2\theta])+r^4\,(S-T)\,(-1+\kappa 1+2\,Cos[2\theta])\Big)$$

$$u_{\theta plate} = \text{FullSimplify}\Big[u_{\theta plate} \text{ /. } f_2 \to 0 \text{ /. } f_3 \to -\frac{-a^4\, S\, \mu + a^4\, T\, \mu - a^4\, S\, \mu 1 + a^4\, T\, \mu 1}{2\,(\kappa 1\, \mu + \mu 1)} \text{ /.}$$
$$m_1 \to \frac{a^2\,(S\,\mu - T\,\mu + S\,\mu 1 - T\,\mu 1)}{2\,(\kappa 1\, \mu + \mu 1)} \text{ /. } m_2 \to 0 \text{ /.}$$
$$f_1 \to a^2\left(-\frac{S}{2} - \frac{T}{2}\right) - \frac{a^2\,(-3\,S - T + S\,\kappa 1 - T\,\kappa 1)\,\mu}{2\,(2\,\mu - \mu 1 + \kappa\,\mu 1)}\Big]$$

$$-((S - T)\,(a^4\,(\mu + \mu 1) + a^2\,r^2\,(-1 + \kappa 1)\,(\mu + \mu 1) - r^4\,(\kappa 1\,\mu + \mu 1))\,\text{Sin}[2\,\theta]) /$$
$$(4\,r^3\,\mu 1\,(\kappa 1\,\mu + \mu 1))$$

$$\text{Simplify}\Big[D[\sigma_{rcirinc}, r] + D[\tau_{r\theta cirinc}, \theta] * \frac{1}{r} + \frac{(\sigma_{rcirinc} - \sigma_{\theta cirinc})}{r}\Big]$$

0

$$\text{Simplify}\Big[D[\sigma_{rplate}, r] + D[\tau_{r\theta plate}, \theta] * \frac{1}{r} + \frac{(\sigma_{rplate} - \sigma_{\theta plate})}{r}\Big]$$

0

$$\text{Simplify}\Big[D[\tau_{r\theta cirinc}, r] + D[\sigma_{\theta cirinc}, \theta] * \frac{1}{r} + \frac{2\,\tau_{r\theta cirinc}}{r}\Big]$$

0

$$\text{Simplify}\Big[D[\tau_{r\theta plate}, r] + D[\sigma_{\theta plate}, \theta] * \frac{1}{r} + \frac{2\,\tau_{r\theta plate}}{r}\Big]$$

0

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BIOGRAPHICAL INFORMATION

Rohan Patil graduated from Shivaji University, India with a Bachelor of Engineering degree in Mechanical Engineering in August 2002.Later he worked with Anil plastics, India for one year. He joined University of Texas at Arlington with interest in design and analysis field .He earned his Master of Science degree in Mechanical Engineering in August 2007.