QUALITATIVE BEHAVIOR OF DYNAMICAL VECTOR FIELDS

by

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The world of the twenty-first century is considerably more complicated than the world into which I was born. From the beginning I always had a fondness for Mathematics and Science.

This dissertation may be a minor addendum to the Library of Knowledge. It represents to me and my supporters a major personal result. I have been fortunate to have worked closely with several fine mentors.

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I wish he were here.

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My mother Helen Lucille Colgrove Kirby encouraged me to be organized, and never to stop learning.

I am proud to offer this work to be available to my children and to the children of children, as an example of perseverance.

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ABSTRACT

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Differential Equations come in two classes, deterministic and stochastic. The first part of this document analyzes some of the stable properties of the set of all trajectories in the real plane converging on a critical point defined by two distinct negative eigenvalues_ a so-called *node.*

Secondly, also in the deterministic class, we offer a *new* method for finding closed-form primitives for a great variety of differential forms, through a reduction process faciltated by a Lyapunov-type Energy function. Many of these forms lie in classes which heretofore have not been shown to be solvable in closed form.

In the stochastic section, the third part of this work outlines the appropriate procedures for calculating differentials and solutions for fields perturbed by random processes.

For the final chapter, we present the development of a theory of Laplace Transforms for stochastic calculations. The resulting Table of Transforms has been initiated, and shall eventually be enlarged. Applications are offered to demonstrate the utility of this approach.

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CHAPTER 1

QUALITATIVE PROPERTIES OF TWO-DIMENSIONAL DYNAMICAL SYSTEMS UNDER NONLINEAR DETERMINISTIC PERTUBATIONS

A mathematical model of dynamical processes in biological, chemical and physical sciences can be described by vector fields. In this chapter, we consider the class of two-dimensional vector fields described by a system of nonlinear autonomous differential equations. We shall concentrate on classifying the character of the phase plane around a linear node when perturbed by various nonlinear autonomous fields with rotational and/or radial components.

1.1 Two-dimensional Dynamical Fields : Problem Formulation

Each vector field we consider shall be specified by a pair of autonomous differential equations of the following form **:**

$$\begin{cases} x' = F(x,\, y) \\ y' = G(x,\, y) \end{cases} \tag{1.1.1}$$

where $F,\, G \in C^1[E,\, R], \quad E \subseteq R \times R.$

The mathematical analysis of the behavior of the trajectories of (1.1.1) in the neighborhood of a critical point may in general be quite complex. By employing a coordinate transformation of the critical point in question, one can move the analysis to the origin; i.e. without loss in generality, one can assume that the origin is the critical point of (1.1.1). Our approach to the analysis of the general system (1.1.1) is to decompose and transform (1.1.1) into simpler sub-vector fields (perturbed) described by

$$\begin{cases} x' = \lambda\, x + f(x,y) \\ y' = \mu\, y + g(x,y). \end{cases} \tag{1.1.2}$$

where $f,\, g \in C^1[\, E,\, R],$ and $\lambda,\, \mu$ are the eigenvalues of the linear part of the vector field < F(x,y) , G(x,y) >. The interactions (perturbations) among the 2-dimensional sub-systems of (1.1.2) are described by f and g.

The isolated sub-vector field (unperturbed linear system) corresponding to (1.1.2) is **:**

$$\begin{cases} x' = \lambda x \\ y' = \mu y\ . \end{cases} \qquad (1.1.3)$$

Let us first examine the qualitative behavior of a particular type (attractor node) of critical point of (1.1.3), and then discuss the corresponding behavior of perturbed system (1.1.2). We begin with a special case of perturbed system (1.1.2) of the following form:

$$\begin{cases} x' = \lambda x \ - y\phi(x,y) \\ y' = \mu y + \ x\phi(x,y). \end{cases} \qquad (1.1.4)$$

where $\phi \ \in \ C^1[E,\ R]$ represents the rotational pertubation of the flow ,
and $\lambda,\ \mu \ \in \ R$;

Here we see

$$\begin{cases} f(x,y) = \ - y\phi(x,y) \\ g(x,y) = \quad x\phi(x,y) \end{cases} \qquad (1.1.5)$$

(or notation $< \ - y\phi(x,y), x\phi(x,y) >$) is the rotational component of the field.

Also we make note that the perturbing field $< f(x,y)\ ,\ g(x,y) >$ shall of course have no linear terms (else they would be included in the linear field $< \lambda x, \mu y \ >$)**;** and thus

$\phi(x,y)$ must vanish as $r \rightarrow 0$.

1.2 Definitions, Notations, Results

For reference, a few preliminary results, definitions, and notation to be used in subsequent discussions are presented.

First, whenever we shall employ the polar coordinate system :

$$x \ = r\cos\theta\ ; \qquad y \ = \ r\sin\theta\ ;$$
$$r^2 = \ x^2 + y^2\ ; \quad \theta = \arctan\left(\frac{y}{x}\right) \qquad (1.2.1)$$

systems (1.1.1), (1.1.2) and (1.1.4) shall be transformed into the following forms **:**

$$\begin{cases} r' = \cos\theta F(r\cos\theta,\, r\sin\theta) + \sin\theta\, G(r\cos\theta,\, r\sin\theta) \\ \theta' = \frac{1}{r}\,[\,\cos\theta\, G(r\cos\theta,\, r\sin\theta) - \sin\theta\, F(r\cos\theta,\, r\sin\theta)\,] \end{cases} \tag{1.2.2}$$

$$r' = r\Big(\lambda\cos^2\theta + \mu\sin^2\theta\Big) + \cos\theta f(r\cos\theta,\, r\sin\theta) + \sin\theta\, g(r\cos\theta,\, r\sin\theta)$$

$$\theta' = (\mu-\lambda)\cos\theta\,\sin\theta + \frac{1}{r}\,[\cos\theta\, g(r\cos\theta,\, r\sin\theta) - \sin\theta\, f(r\cos\theta,\, r\sin\theta)] \tag{1.2.3}$$

$$\begin{cases} r' = r\Big(\lambda\cos^2\theta + \mu\sin^2\theta\Big) \\ \theta' = (\mu-\lambda)\cos\theta\,\sin\theta + \phi(\,r\cos\theta,\, r\sin\theta\,) \end{cases} \tag{1.2.4}$$

Definition 1.2.5: The set of all points $(x,\, y)$ on a specific solution of (1.1.1) without reference to a parametrization is called an *orbit* of (1.1.1). Furthermore, parts of an orbit determined by the parameter $t \geq t_0$ and $t \leq t_0$ for some $t_0 \in R$ are termed a *positive* half-orbit and a *negative* half-orbit, respectively.
Let us denote orbit by upper case Roman and positive half-orbit, and negative half-orbit by the following superscripts : O, O^+ and O^-, respectively.

Definition 1.2.6 : The isolated critical point (0,0) of dynamical system (1.1.1) is said to be an *attractor* in positive time if :

all solutions $(x(t),\, y(t\,)) = (x(t,t_0,x_0,y_0),\, y(t,t_0,x_0,y_0))$ exist

(for $t \geq t_0$ and sufficiently small $|x_0| + |y_0|$)

and if $|x(t)| + |\,y(t\,)| \rightarrow 0$ as $t \rightarrow \infty$.

Definition 1.2.7 : The isolated attractor $(0,0)$ of dynamical system (1.1.1) is said to be a *node* if :

all orbits $(x(t),\, y(t\,))$ have tangents at (0, 0); that is, if a continuous determination of $\theta(t) = \arctan\left(\frac{y(t)}{x(t)}\right)$ tends to a real limit θ_{lm} .

We now present a theorem without proof which summarizes well-known results of linear system (1.1.3).

Theorem 1.2.8 : Assume that the coefficients λ and μ defined in (1.1.3) satisfy

$$\lambda < \mu < 0. \tag{1.2.9}$$

Then

L1. Each positive half orbit is attracted to the origin with a well-defined tangency angle (characteristic direction) θ_{lm}

L2. There is precisely one orbit (denoted X^+) for which $\theta_{lm} = 0$ and precisely one orbit (denoted X^-) for which $\theta_{lm} = \pi$;

L3. There exists an (infinite) family of orbits with $\theta_{lm} = \frac{\pi}{2}$ and a family of orbits with $\theta_{lm} = -\frac{\pi}{2}$;

L4. The family of orbits with $\theta_{lm} = \frac{\pi}{2}$ can be decomposed into two subfamilies and one *separator* orbit Y^+ according to those orbits which satisfy :

$$\begin{cases} 0 < \theta(t) < \frac{\pi}{2} & \text{for all } t \geq t_0 \text{ or} \\ \pi > \theta(t) > \frac{\pi}{2} & \text{for all } t \geq t_0 \end{cases}$$

Similarly, the other family for which $\theta_{lm} = -\frac{\pi}{2}$ can be separated as orbits for which $-\frac{\pi}{2}$ is either an upper or lower bound for $\theta(t)$, separated by orbit Y^- .

Remark 1.2.8a : When $\mu < \lambda < 0$, the roles played by X^+ , X^-, Y^+, and Y^- become interchanged; and similarly when $\mu, \lambda > 0$ the origin becomes a *source.*

Remark 1.2.8b : For the purely linear field above all four separator orbits X^+ , X^-, Y^+, and Y^- lie exactly on the coordinate axes; we intend to show that under any higher order (than linear) perturbation, separators exist corresponding to X^+ and X^-; and under many higher order perturbations, separators exist corresponding to Y^+, and Y^-.

1.3 Rotational Perturbations on Linear Nodes

In this section, we investigate the limiting behavior of trajectories with respect to the node at the origin perturbed by a purely rotational dynamic field (1.1.5) in system (1.1.4). These results will provide auxiliary tools to investigate the more general nonlinear perturbed system (1.1.2). In fact, sufficient conditions are given on the rotational component of the fields in (1.1.5) that guarantees the preservation of the limiting behavior of unperturbed system (1.1.3), in particular, Properties L1 and L2.

We now present a lemma which establishes the end behavior of all positive half-orbits of dynamical system (2.4) for which $\theta(t)$ is bounded.

Lemma 1.3.1 : Assume that $\lambda < \mu < 0$ for the dynamical system (1.1.4). $Then$

(i) the isolated critical point, (0, 0) of (1.1.4) is an attractor;

(ii) for any bounded positive half-orbit, $\lim_{t\to\infty} \theta(t) = \theta_{lm}$ exists ; and $\theta_{lm} \in \{-\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi\}$.

Proof: Recall polar representation

$$r' = r\,(\lambda\cos^2\theta + \mu\sin^2\theta)$$
$$\theta' = (\mu - \lambda)\cos\theta\,\sin\theta \;+\; \phi(r\cos\theta, r\sin\theta) \qquad (1.2.4)$$

From (1.3.1) and $\lambda < \mu < 0$, it is clear that :

$$r' \le \mu\, r < 0, \text{ for all } r > 0 \text{ and } t \in R\,. \qquad (1.3.2)$$

Hence the origin is an attractor; in fact, it is a global attractor. Thus (i) is proven.

In order to establish the validity of (ii) , we assume that $\theta(t)$ is bounded for future t; that is, assume the existence of a real number T_1, and angles θ_* and θ^* such that,

$$\text{for all } t \ge T_1, \qquad \theta_* \le \theta(t) \le \theta^*. \qquad (1.3.3)$$

Such bounds will indeed produce the existence of the limit θ_{lm} of $\theta(t)$ as $t\to\infty$, and will further restrict tangency angle θ_{lm} to satisfy :

$$\sin\theta_{lm}\cos\theta_{lm} = 0. \qquad (1.3.4)$$

Proceeding contrapositively, assume $\lim_{t\to\infty}\theta(t)$ fails to exist.

Utilizing the Heine-Borel Theorem , there must then exist real sequences

$\{t_n\}_{n=1}^{\infty}$ and $\{\tau_n\}_{n=1}^{\infty}$ such that

$t_n \to \infty$ and $\tau_n \to \infty$ as $n \to \infty$;

$\{\theta(t_n)\}_{n=1}^{\infty} \in [\theta_*, \theta^*]$, and $\{\theta(\tau_n)\}_{n=1}^{\infty} \in [\theta_*, \theta^*]$,

$\lim_{n\to\infty}\theta(t_n) = \theta_1$; $\lim_{n\to\infty}\theta(\tau_n) = \theta_2$ for distinct $\theta_1, \theta_2 \in [\theta_*, \theta^*]$.

Without loss in generality, assume $\theta_1 < \theta_2$. This implies the existence of $\overline{\theta}$ such

that $$\theta_1 < \overline{\theta} < \theta_2 \quad \text{and} \quad \cos\overline{\theta}\sin\overline{\theta} \neq 0. \qquad (1.3.5)$$

(The periodic $\sin\theta$ and $\cos\theta$ can have only finitely many zeros between θ_1 and θ_2.)

We now proceed to show that (1.3.5) is in fact impossible.

Consider case 1: Suppose $\cos\overline{\theta}\sin\overline{\theta} > 0$. Here we note 3 facts :

$$\phi \in C^1[E, R] \ , \ r(t) \text{ is monotonic, and } \lim_{r\to 0}\phi(r\cos\theta, r\sin\theta) = 0 \qquad (1.3.6)$$

which together imply the existence of $T_2 \geq T_1$ (recall (1.3.3)) such that for $t \geq T_2$

$$|\phi(r\cos\theta, r\sin\theta)| < \tfrac{1}{2}(\mu - \lambda)\cos\overline{\theta}\sin\overline{\theta}. \qquad (1.3.7)$$

From (1.3.7) and $\theta'(t)$ from (1.2.4), we have forced $\theta'(t) > 0$ for $t \geq T_2$.

Thus $\theta(t)$, and also $\{\theta(t_n)\}_{n=1}^{\infty}$ and $\{\theta(\tau_n)\}_{n=1}^{\infty}$ are monotonic eventually.

But $\lim_{n\to\infty}\theta(t_n) = \theta_1 < \lim_{n\to\infty}\theta(\tau_n) = \theta_2$ contradicts this monotonicity of $\theta(t)$,

because $\theta(t)$ must contain both sequences. Explicitly, N_2 can be found large enough to make both $\tau_{N_2} \geq T_2$ and $\overline{\theta} < \theta(\tau_{N_2}) \leq \theta_2$; and subsequently N_1 can be found such that $t_{N_1} \geq \tau_{N_2}$ and $\theta_1 \leq \theta(t_{N_1}) < \overline{\theta} < \theta(\tau_{N_2})$, a direct violation of the monotonicity of $\theta(t)$.

Similarly case 2: $\cos\overline{\theta}\sin\overline{\theta} < 0$ can be invalidated. Thus one cannot choose a $\overline{\theta}$ such that $\theta_1 < \overline{\theta} < \theta_2$ and $\cos\overline{\theta}\sin\overline{\theta} \neq 0$. $\theta_1 = \theta_2$ follows and therefore $\theta_{\rm lm}$ exists (i..e. $\theta_1 = \theta_2 = \theta_{\rm lm}$).

But also the same facts (1.3.6) allow one to establish that $\theta_{\rm lm}$ must satisfy (1.3.4). Otherwise $\theta'(t)$ from (1.3.1) would be *finitely bounded from vanishing,* which would violate (1.3.3) . This completes the proof of statement (ii) and the Lemma. □

We are now prepared to present the result analogous to property L.1 in Theorem 1.2.1.

Theorem 1.3.8 : For any dynamic system of the form (1.1.4) such that $\lambda < \mu < 0$, all positive half orbits are attracted to the origin, each with a well-defined tangency angle

$$\theta_{lm} \in \{-\tfrac{\pi}{2},\, 0,\, \tfrac{\pi}{2},\, \pi\}\,.$$

Proof For some $\epsilon : 0 < \epsilon < \frac{\pi}{2}$, consider the unbounded ϵ-wedge consisting of the origin together with all rays $\frac{\pi}{2} - \epsilon \leq \theta \leq \frac{\pi}{2} + \epsilon,\ r > 0$.

Since $\lim\limits_{(x,y)\to(0,0)} \phi(r\cos\theta, r\sin\theta) = 0$, we can see that on the ray $\theta = \frac{\pi}{2} - \epsilon$ there exists $r_{\epsilon^-} > 0$ such that for $0 < r < r_{\epsilon^-}$

$$0 \leq |\phi(r\cos\theta, r\sin\theta)| < (\mu-\lambda)|\cos\theta\sin\theta|. \qquad (1.3.9)$$

Similarly there exists $r_{\epsilon^+} : 0 < r < r_{\epsilon^+}$, for which (1.3.7) holds on ray $\theta = \frac{\pi}{2} + \epsilon$.

Defining $r_\epsilon = min\{r_{\epsilon^-},\, r_{\epsilon^+}\}$,

and using $\theta' = (\mu-\lambda)\cos\theta\sin\theta + \phi\,(r\cos\theta, r\sin\theta)$ we obtain

$$\theta' \begin{cases} > 0 \ \text{ on radial segment } (\,0 < r < r_\epsilon\,,\ \theta = \frac{\pi}{2} - \epsilon\,) \\ < 0 \ \text{ on segment } (\,0 < r < r_\epsilon\,,\ \theta = \frac{\pi}{2} + \epsilon\,) \end{cases}$$

and

$$r' < 0\,,\ \text{ on arc } (\,r = r_\epsilon, \quad \tfrac{\pi}{2} + \epsilon \leq \theta \leq \tfrac{\pi}{2} + \epsilon\,).$$

Thus the radial wedge bounded by $r = r_\epsilon, \theta = \frac{\pi}{2} - \epsilon,$ and $\theta = \frac{\pi}{2} + \epsilon$

including the origin, is *positively invariant* with respect to the flow of field (1.1.4).

Let us denote this bounded wedge by

$$\mathcal{W}_\epsilon(\tfrac{\pi}{2}) = \{(r,\theta)\colon\ 0 \leq r \leq r_\epsilon,\ \tfrac{\pi}{2} - \epsilon \leq \theta \leq \tfrac{\pi}{2} + \epsilon\}.$$

From the application of Lemma 1.3.1, it is clear that any orbit intersecting

$\mathcal{W}_\epsilon(\frac{\pi}{2})$ is of course attracted to the origin.

Moreover, each orbit in $\mathcal{W}_\epsilon(\frac{\pi}{2})$ does indeed have

a well-defined tangency $\theta_{lm} \in \{-\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi\}$.

Thus $\theta_{lm} = \frac{\pi}{2}$ is the only possibility for any orbit meeting wedge $\mathcal{W}_\epsilon(\frac{\pi}{2})$.

But also we see that the existence of any such wedge, together with the negative character of $r'(t)$ does indeed imply that each positive half orbit is in fact bounded; therefore all conclusions of the Lemma hold. □

Corollary 1.3.10 : A similar argument can be used to prove that wedge

$$\mathcal{W}_\epsilon(\text{-}\tfrac{\pi}{2}) = \{(r,\theta)\colon\ 0 \leq r \leq r_\epsilon,\ -\tfrac{\pi}{2} - \epsilon \leq \theta \leq -\tfrac{\pi}{2} + \epsilon\}$$

is also a non-empty positively invariant set within the flow of the field, so that each positive half-orbit which intersects this wedge must eventually lead to $\theta_{lm} = -\frac{\pi}{2}$.

Next we shall present a result establishing the existence of two *unique separator* orbits X^+ and X^- within the flow of any node possessing a rotational component of the form (1.1.4) .

This pair of trajectories separate the vector field and are unique with respect to satisfying :

given any ϵ ($0 < \epsilon < \frac{\pi}{2}$), there exists a positive $r(\epsilon)$ such that :

neither X^+ nor X^- ever intersects either wedge

$\mathcal{W}_\epsilon(\frac{\pi}{2})$ or $\mathcal{W}_\epsilon(-\frac{\pi}{2})$ defined above;

This property shall lead to tangency angles $\theta_{lm} = 0$ for orbit X^+

and $\theta_{lm} = \pi$ for orbit X^-.

Theorem 1.3.11 shall represent the analogue to property L.2 possesed by the purely linear vector fields of Theorem 1.2.1.

Theorem 1.3.11 : Under the hypotheses of Theorem 1.3.8, within the field of system (1.1.4), there exist :

a unique orbit X^+ corresponding to tangency angle $\theta_{lm} = 0$,

and

a unique orbit X^- corresponding to tangency angle $\theta_{lm} = \pi$.

Proof: We establish the existence and uniqueness of the orbit X^+ .

With reference to $r_\epsilon = min\{r_{\epsilon^-}, r_{\epsilon^+}\}$ defined above, suppose that *for the specific* value $\epsilon = \pi/4$ we have determined a number $\rho > 0$ to serve as r_ϵ for both wedges $\mathcal{W}_{\pi/4}(\frac{\pi}{2})$ and $\mathcal{W}_{\pi/4}(-\frac{\pi}{2})$.

Explicitly, on the four rays $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$,

$$|\phi(r\cos\theta, r\sin\theta)| < (\mu-\lambda)\cos(\tfrac{\pi}{4})\sin(\tfrac{\pi}{4}) = \tfrac{\mu-\lambda}{2} \quad \text{for } 0 < r < \rho \qquad (1.3.12)$$

Now consider the set S of points of the ¾-circle : $r = \rho$; $-\frac{3\pi}{4} < \theta < \frac{3\pi}{4}$.

We now define two (disjoint) subsets S^+ and S^- of S thus :

$\mathrm{S}^{+} \;=\; \{ (\rho\,,\theta) \in S \,|\, \text{the orbit passing through } (\rho\,,\theta) \text{ has } \theta_{lm} = \frac{\pi}{2} \}$

$\mathrm{S}^{-} \;=\; \{ (\rho\,,\theta) \in S \,|\, \text{the orbit passing through } (\rho\,,\theta) \text{ has } \theta_{lm} = -\frac{\pi}{2} \}.$

We note that these sets are indeed nonempty, and in fact set S^{+} contains the *arc boundary* of wedge $\mathcal{W}_{\pi/4}(\frac{\pi}{2})$; while set S^{-} contains the arc boundary of wedge $\mathcal{W}_{\pi/4}(-\frac{\pi}{2})$.

Next we note that the uniqueness of trajectories/orbits of (1.1.4) through any particular polar point (ρ, θ_0) , combined with the *positive invariance* of the wedges, indeed induce the following order relations on the points of S^{+} and on the points of S^{-} :

$(\rho, \theta_0) \in \mathrm{S}^{+}$ implies $(\rho, \theta) \in \mathrm{S}^{+}$ for all $\frac{3\pi}{4} \geq \theta \geq \theta_0$.

$(\rho, \theta_0) \in \mathrm{S}^{-}$ implies $(\rho, \theta) \in \mathrm{S}^{-}$ for all $-\frac{3\pi}{4} \leq \theta \leq \theta_0$. (1.3.13)

Define $\omega_* = \inf \{\theta < \frac{3\pi}{4} \mid (\rho, \theta) \in \mathrm{S}^{+} \}$
and (1.3.14)
$\omega^* = \sup \{ \theta > -\frac{3\pi}{4} \mid (\rho, \theta) \in \mathrm{S}^{-} \}$.

Now since S^{+} and S^{-} are disjoint, we *must* have $-\frac{\pi}{4} \leq \omega^* \leq \omega_* \leq \frac{\pi}{4}$. (1.3.15)

<u>Lemma 1.3.16</u> : $i)$ The point $\mathrm{P}\,(\rho, \omega_*) \notin \mathrm{S}^{+}$ and $ii)$ the point $\mathrm{Q}\,(\rho, \omega^*) \notin \mathrm{S}^{-}$.

<u>Proof of Lemma</u> : $i)$ For contradiction, we suppose (ρ, ω_*) to be in set S^{+}.

Consider the orbit K through (ρ, ω_*) . This orbit K must eventually cross into the wedge $\mathcal{W}_{\pi/4}(\frac{\pi}{2})$ in order to satisfy $\theta_{lm} = \frac{\pi}{2}$, and must therefore cross the lateral boundary of wedge $\mathcal{W}_{\pi/4}(\frac{\pi}{2})$ at some distinct point A $(\eta , \frac{\pi}{4})$ such that $0 < \eta \leq \rho$.

But now we consider a point B (for instance) midway between the pole (origin) and A.

Now the orbit through point B$(\frac{\eta}{2}, \frac{\pi}{4})$, touching the wedge , must also satisfy $\theta_{lm} = \frac{\pi}{2}$.

But then retracing this orbit back we must inevitably cross the circle $r = \rho$ discretely clockwise of orbit K , at some point (ρ , θ_*). Further we must have $-\frac{\pi}{4} < \theta_* < \omega_*$ (the former bound coming from the lower wedge $\mathcal{W}_{\pi/4}(-\frac{\pi}{2})$) . But such considerations have produced the contradiction $(\rho , \theta_*) \in$ S⁺ while $\theta_* < \omega_*$, the infimum.

Thus $i)$ $(\rho, \omega_*) \notin$ S⁺ has been established.

Now a companion argument involving $\theta_{lm} = -\frac{\pi}{2}$ and the boundaries of the wedges will produce $ii)$ $(\rho, \omega^*) \notin$ S⁻. $lemma$□

<u>Returning to Theorem 1.3.11</u> , consider again the orbit K passing throug;h the point P (ρ, ω_*). Refering to (1.3.13) , (1.3.14) , (1.3.15) , Lemma 1.3.16, and Theorem 1.3.8, we have established :

a) any orbit *counterclockwise of P* must enter wedge $\mathcal{W}_{\pi/4}(\frac{\pi}{2})$ and produce $\theta_{lm} = \frac{\pi}{2}$.

b) any orbit *clockwise of Q* must enter wedge $\mathcal{W}_{\pi/4}(-\frac{\pi}{2})$ and produce $\theta_{lm} = -\frac{\pi}{2}$.

c) *every* orbit through set S must satisfy Theorem 4.1 : $\theta_{lm} \in \{-\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi\}$.

These constraints imply :

d) orbit K through point P must have $\theta_{lm} = 0$;

e) the orbit J through point Q must also have $\theta_{lm} = 0$;

f) any orbit clockwise of P and counterclockwise of Q must have $\theta_{lm} = 0$;

g) $\theta_{lm} = 0$ can *only* be achieved by d) , e) , or f) .

The next and longest Lemma will establish $\omega^* = \omega_*$, implying points P and Q are one, and therefore orbit K *is* orbit J, and therefore is indeed the unique orbit X⁺ satisfying $\theta_{lm} = 0$.

Lemma 1.3.17 : $\omega^* = \omega_*$

Proof of Lemma : First we note any orbit satisfying d), e), f) or g) above must remain disjoint from both closed wedges $\mathcal{W}_{\pi/4}(\frac{\pi}{2})$ and $\mathcal{W}_{\pi/4}(-\frac{\pi}{2})$, and therefore such an orbit is confined to the (laterally open) wedge W $= \{(r, \theta)\colon\ 0 \leq r \leq \rho\ ,\ -\frac{\pi}{4} < \theta < \frac{\pi}{4}\}$.

Now an orbit of system (1.1.4) $\begin{cases} x' = \lambda x - y\phi(x,y) \\ y' = \mu y + x\phi(x,y). \end{cases}$ $(\lambda < \mu < 0)$

or

polar companion (1.2.4) $\begin{cases} r' = r\Big(\lambda\cos^2\theta + \mu\sin^2\theta) \\ \theta' = (\mu - \lambda)\cos\theta\,\sin\theta + \phi. \end{cases}$

even though confined within wedge W , and even though $\frac{dr}{dt}(t)$ is strictly negative, may in fact oscillate within wedge W in such a manner that the trajectory fails to yield a *function* $y = y(x)$.

However $\frac{dr}{dt}(t) < 0$ *does* yield a function $y = y(r)$.

Therefore we shall convert (1.1.4) into *yet another* equivalent system to facillitate the final steps of the proof.:

$$\begin{cases} r' = \lambda r + p(r,y) \\ y' = \mu y + q(r,y) \end{cases} \qquad (1.3.18)$$

where $p(r,\ y) = (\mu - \lambda)\frac{y^2}{r}$

and $q(r,\ y) = \pm\sqrt{r^2 - y^2}\phi(\pm\sqrt{r^2 - y^2}, y)$.

We note that $p, q \in C^1[E,\ R]$ and $\frac{p}{r}, \frac{q}{r} \to 0$ as $r \to 0$;

Also $\frac{dr}{dt}(t) < 0$ continues to ensure that $r(t)$ is monotonic decreasing in t.

Now suppose $w^* < w_*$.

Then the (supposedly) distinct orbits K through point (ρ, ω_*) and J through point (ρ, w^*) must traverse distinct curves $y_*(r)$ and $y^*(r)$ each having tangency angle $\theta_{lm} = 0$ at the origin.

Since orbits must not intersect,

these *functions of* r must satisfy $y^*(r) < y_*(r)$ for all $0 < r < \rho$.

Now since $\frac{\mathrm{p}(r,y)}{r} < (\mu - \lambda)$, there must exist $t_* = t_*(r)$ and $t^* = t^*(r)$

such that trajectory K : $(x_*(t), y_*(t))$ through point P (ρ, ω_*) and

trajectory J : $(x^*(t), y^*(t))$ through point Q (ρ, ω^*) satisfy

$$|x_*(t_*)|^2 + |y_*(t_*)|^2 = r^2 = |x^*(t^*)|^2 + |y^*(t^*)|^2. \tag{1.3.19}$$

Thus if indeed $w^* < w_*$, then $y^*(t^*(r)) < y_*(t_*(r))$ at each r.

We now define function

$$Z(r) = y_*(t_*(r)) - y^*(t^*(r)) = y_*(r) - y^*(r) > 0 \tag{1.3.20}$$

Paralleling a proof by Sansone and Conti [9] , we will show that $Z(r) > 0$ yields the following contradiction :

$$A) \ \frac{Z(r)}{r} \to 0 \ \text{as} \ r \to 0, \ \text{and}$$
$$B) \ \frac{dZ}{dr}(r) = \frac{Z(r)}{r}\delta(1 + \mathcal{E}(r)); \quad 0 < \delta < 1; \ \ \mathcal{E}(r) \to 0 \ \text{as} \ r \to 0. \tag{1.3.21}$$

Thereby $Z(r) > 0$ shall have been proven impossible; hence $w^* = w_*$ follows.

To establish the validity of A), we note $\lim_{t\to\infty}\left[\frac{y_*(t)}{x_*(t)}\right] = 0$ since $\theta_{lm} = 0$.

This, together with $\frac{|y_*(r)|}{r} \leq \frac{|y_*(t)|}{|x_*(t)|}$ implies $\frac{y_*(r)}{r} \to 0$ as $r \to 0$.

Similarly $\frac{y^*(r)}{r} \to 0$ as $r \to 0$. Thus $\frac{Z(r)}{r} = \frac{y_*(r)) - y^*(r)}{r} \to 0$ as $r \to 0$.

A) is proven.

To establish B), let us *simplify notation* .

Let $\mathrm{p}_*(r) = p(r(t), y_*(t))$ and $q_*(r) = q(r(t), y_*(t))$.

Similarly let $p^*(r) = p(r(t), y^*(t))$; $q^*(r) = q(r(t), y^*(t))$.

Since p and q are C^1, for each r the MeanValue Theorem yields two values

$y^{\diamond}(r)$ and $y_{\diamond}(r)$, such that $y^*(r) \leq y^{\diamond}(r)$, $y_{\diamond}(r) \leq y_*(t)$ and

such that : $$\mathrm{p}_*(r) - p^*(r) = \frac{\partial}{\partial y} p(r, y_{\diamond}(r)) Z(r)\,;$$

$$q_*(r) - q^*(r) = \frac{\partial}{\partial y} q(r, y^{\diamond}(r)) Z(r). \tag{1.3.22}$$

We now compute the following expressions for $\frac{d}{dr} Z(r)$

$$\frac{d}{dr} Z(r) \;=\; \frac{d}{dr} y_*(r) - \frac{d}{dr} y^*(r) \;=\; \frac{\frac{d}{dt} y_*}{\frac{dr}{dt}} \;-\; \frac{\frac{d}{dt} y^*}{\frac{dr}{dt}}$$

$$= \frac{[\mu y_*(r) + \; q_*(r)]}{[\lambda r + \mathrm{p}_*(r)]} - \frac{[\mu y^*(r) + q^*(r)]}{[\lambda r + p^*(r)]}$$

$$= \frac{[\mu Z(r) + [\, q_*(r) - q^*(r)\,] - [\mathrm{p}_*(r) - p^*(r)] \frac{[(\mu y^*(r) + q^*(r)]}{[\lambda r + p^*(r)\,]}\;]}{\lambda r + \mathrm{p}_*(r)}$$

$$= \frac{[\mu Z(r) + [\, \frac{\partial}{\partial y} q(r, y^{\diamond}(r)) Z(r)] - [\frac{\partial}{\partial y} p(r, y_{\diamond}(r)) Z(r)] \frac{[(\mu y^*(r) + q^*(r)]}{[\lambda r + p^*(r)\,]}\;]}{\lambda r + \mathrm{p}_*(r)}$$

$$= \frac{\mu Z(r) \left[1 + \frac{1}{\mu} \frac{\partial}{\partial y} q(r, y^{\diamond}(r)) - \frac{1}{\mu} \frac{\partial}{\partial y} p(r, y_{\diamond}(r) \frac{[(\mu y^*(r) + q^*(r)]}{[\lambda r + p^*(r)]}\right]}{\left[\lambda r + \mathrm{p}_*(r)\right]} \tag{1.3.23}$$

Now as $r \to 0$, each of the following functions of r has limit zero :

$$\frac{\mathrm{p}_*(r)}{r}\;, \quad \frac{\partial}{\partial y} q(r, y^{\diamond}(r))\,, \quad \frac{\partial}{\partial y} p(r, y_{\diamond}(r))\,, \quad \frac{y^*(r)}{r}\;, \quad \frac{q^*(r)}{r}\;, \quad \frac{p^*(r)}{r}$$

Hence, there exists $\mathcal{E}(r)$ such that $\mathcal{E}(r) \to 0$ as $r \to 0$ and (1.3.23) reduces to :

$$\frac{d}{dr} Z(r) = \frac{Z(r)}{r} \frac{\mu}{\lambda} \left(1 + \mathcal{E}(r) \right) \qquad \text{and } B \text{ is established} \tag{1.3.24}$$

Finally, to see the contradiction produced, define $\delta = \frac{\mu}{\lambda}$ and define $k = \frac{1}{3}\delta + \frac{2}{3}$.

Then $0 < \delta < 1$ implies $\delta < k < 1$. So then as $\mathcal{E}(r) \to 0$, from $B)$ we get the

differential inequality $\frac{r}{Z} \frac{dZ}{dr} = \delta\,(\,1 + \mathcal{E}\,) < k < 1$ or simply $\frac{dZ}{Z} < k \frac{dr}{r}$ (1.3.25)

which implies $\frac{Z(r)}{r} > C r^{k-1}$ for some $C > 0$. But $r^{k-1} \to \infty$ as $r \to 0$,

which contradicts $A)$ $\frac{Z(r)}{r} \to 0$ as $r \to 0$ and so $Z(r) > 0$ is false.

This completes the proof of the existence and uniqueness of orbit X^+ with respect to tangency angle $\theta_{lm} = 0$. The proof of the existence and uniqueness of orbit X^- with respect to tangency angle $\theta_{lm} = \pi$ can be formulated analogously. □

1.4 General Nonlinear Perturbations on Stable Linear Nodes

In this section, by employing the results of Section 1.3 concerning the rotational vector field perturbations (1.1.5) in (1.1.4), the more general nonlinear vector field system (1.1.2) is investigated. In fact, sufficient conditions are given on the nonlinear perturbations in (1.1.2) that guarantee the preservation of the limiting behavior of linear system (1.1.3). The following lemma illustrates the validity of Properties L1 and L2 described in Theorem 1.2.8, with regard to dynamical system (1.1.2).

Lemma 1.4.1: Let $< f, g >$ be any perturbation vector field in (1.1.2) which satisfies:

(i) $\lambda < \mu < 0$;

(ii) $f,\, g \in C^1[E,\, R]$;

(iii) $\frac{f}{r} \to 0$ and $\frac{g}{r} \to 0$ as $r \to 0$.

Then there exists rotational component ϕ corresponding to system (1.1.2) satisfying

$a)$ $\phi \in C^1[E,\, R]$;

$b)$ $\lim\limits_{(x,y)\to(0,0)} \phi(x,y) = 0$; and

$c)$ ϕ forms a rotational field of the form (1.1.5) but having the same limiting behavior at the node as system (1.1.2) described by :

$$\begin{cases} x' = \overline{\lambda}x - y\phi(x,y) \\ y' = \overline{\mu}y + x\phi(x,y), \end{cases} \quad \text{for some real numbers } \overline{\lambda} \text{ and } \overline{\mu} \qquad (1.4.2)$$

dependent on λ and μ, also satisfying $\overline{\lambda} < \overline{\mu} < 0$.

Proof: Assumption (i) guarantees that θ is bounded in the future (Theorem 1.3.1). Let us define the rotational component of the flow with respect to any given nonlinear vector field $< f, g >$ in (1.1.2) as follows :

$$\phi(x,y) = \tfrac{1}{r^2}[xg(x,y) - yf(x,y)]. \tag{1.4.3}$$

Now choose α such that $\lambda < \mu < \alpha < 0$ and define $\overline{\mu} = \mu - \alpha$ and $\overline{\lambda} = \lambda - \alpha$. Note that $\overline{\lambda} < \overline{\mu} < 0$. Under these notations and definition of ϕ in (1.4.3), a system corresponding to system (1.1.2) can be written as:

$$\begin{cases} x' = \overline{\lambda}\, x + \overline{f}(x,y) \\ y' = \overline{\mu}\, y + \overline{g}(x,y), \end{cases} \tag{1.4.4}$$

where

$$\begin{cases} \overline{f}(x,y) = -\, y\phi(x,y) \\ \overline{g}(x,y) = x\phi(x,y). \end{cases} \tag{1.4.5}$$

Now we need to verify that ϕ in (1.4.3) satisfies all conclusions of the lemma. It is obvious that $\phi \in C^1[E,\, R]$, moreover, from $\frac{f}{r}$ and $\frac{g}{r} \to 0$ as $r \to 0$, one can easily conclude that $\phi(x,y) \to 0$ as $(x,y) \to (0,0)$. In view of these statements, the conclusions of Theorems 1.3.8 and 1.3.11 are valid with regard to system (1.4.4). It remains to prove that this terminal behavior at the node is also true for system (1.1.2). For this purpose, we *partially* rewrite systems (1.4.4) and (1.1.2) in polar coordinate form as :

$$\begin{cases} rr' = \overline{\mu}\, r^2 + (\lambda - \mu)\, x^2 \\ r^2\theta' = -\,(\lambda - \mu)\, xy + r^2\phi(x,y), \end{cases} \tag{1.4.6}$$

and

$$\begin{cases} rr' = \overline{\mu}\, r^2 + (\lambda - \mu)\, x^2 + \alpha r^2 + x f(x,y) + y\, \mathrm{g}(x,y) \\ r^2\theta' = -\,(\lambda - \mu)\, xy + r^2\phi(x,y), \end{cases} \tag{1.4.7}$$

respectively.

Now for each $\alpha < 0$ assumption (iii) implies there exists a positive real number $r_\alpha > 0$ such that for any $r < r_\alpha$,

$$|x\, f(x,y) + y\, g(x,y)| < -\alpha r^2. \tag{1.4.8}$$

This, together with (1.4.7) yields the following system of differential inequality:

$$\begin{cases} rr' \leq \overline{\mu}\, r^2 + (\lambda - \mu)\, x^2 \\ r^2\theta' = -(\lambda - \mu)\, xy + r^2\phi(x,y), \end{cases} \tag{1.4.9}$$

It is obvious that system (1.4.6) is a comparison system for (1.4.7). Hence, by the application of comparison theorem the validity of conclusions of Theorems 1.3.8 and 1.3.11 with regard to system (1.1.2) follows immediately.

In the following, we present an example that illustrates the scope of Lemma 1.4.1.

<u>Example 1.4.10</u> : We consider the following system of differential equations

$$\begin{cases} x' = -2x - x^2 + xy \\ y' = -y - 2xy + y^2 \end{cases} \tag{1.4.10}$$

In this example, ϕ in (1.4.3), $\overline{\lambda}$, $\overline{\mu}$, $\overline{f}$, and $\overline{g}$ in (1.4.4) are as follows :

$$\phi(x,y) = \tfrac{1}{r^2}[xg(x,y) - yf(x,y)] = \tfrac{-x^2y}{r^2},$$

$$\overline{\lambda} = -(2+\alpha), \quad \overline{\mu} = -(1+\alpha),$$

$$\overline{f}(x,y) = -\ y\phi(x,y) = \tfrac{-x^2y^2}{r^2}$$

and $\overline{g} = x\phi(x,y) = \tfrac{-x^3y}{r^2}$,

where α satisfies the relation: $-2 < -1 < \alpha < 0$. Hence the system corresponding to (1.4.4) with respect to (1.4.8) can be written as :

$$\begin{cases} x' = -(2+\alpha)x + \frac{x^2y^2}{r^2} \\ y' = -(1+\alpha)\, y + \frac{-x^3y}{r^2}, \end{cases} \tag{1.4.11}$$

The (mixed) coordinate representation of (1.4.11) and (1.4.10) are :

$$\begin{cases} r' = -r - \frac{x^2}{r} - \alpha r \\ \theta' = \cos\theta \, \sin\theta - r(\cos\theta)^2 \sin\theta \,, \end{cases} \tag{1.4.12}$$

and

$$\begin{cases} r' = -r - \frac{x^2}{r} + \frac{1}{r}(-x^3 + x^2 y - 2xy^2 + y^3) \\ \theta' = \cos\theta \, \sin\theta - r(\cos\theta)^2 \sin\theta \,, \qquad \text{respectively} \end{cases} \tag{1.4.13}$$

We now choose radius R(α) small so that for $r \leq$ R(α),

$\frac{1}{r}|(-x^3 + x^2 y - 2xy^2 + y^3)| < -\alpha r$ and thus r′ (1.4.13) $<$ r′ (1.4.12) $<$ 0,

while θ' (1.4.13) $=$ θ' (1.4.12) , which implies that the boundedness of $\theta(t)$ for (1.4.12) (which results from Theorems 1.3.8 and 1.3.11) must thereby control the boundedness of $\theta(t)$ for (1.4.13), which in turn implies the conclusions of Theoerems 1.3.8 and 1.3.11 . This argument explains why the *radial* component of the perturbation field $< f, g >$ can be considered negligible (near the origin).

Lemma 1.4.1 and the example are sufficient to provide results analogous to Theorem 1.3.8 and Theorem 1.3.11 with respect to system (1.1.2). The detailed proofs are omitted.

<u>Theorem 1.4.11</u> : Under the hypotheses of Lemma 1.4.1, all positive half-orbits of system (1.1.2) are attracted to the origin with well-defined tangency angles θ_{lm}.

<u>Theorem 1.4.12</u>: Under the hypotheses of Theorem 1.4.11, system (1.1.2) has unique orbits corresponding to tangency angles $\theta_{lm} = 0$ and $\theta_{lm} = \pi$.

We are now interested in investigating the existence of two other separator orbits corresponding to tangency angles $\theta_{lm} = \frac{\pi}{2}$ and $\theta_{lm} = -\frac{\pi}{2}$.

These orbits shall be denoted Y^+ and Y^-. In order to study the existence of these orbits, we shall need to impose an additional condition on the rotational component ϕ in the flow of (1.1.4) .

We now define a *limited oscillation property* (LOP) on the function $\phi\,(x, y)$ or $\phi\,(r, \theta)$ of the system

$$x' = \lambda\, x - y\, \phi \qquad\qquad r' = \mu\, r + (\lambda - \mu) r \cos^2 \theta$$

$$y' = \mu\, y + x\, \phi \qquad \theta' = (\mu - \lambda) \cos\,\theta \sin\,\theta + \phi.$$

We shall require a positive radius, within which, *on each axis,* ϕ either *vanishes* identically, or else *never* vanishes. For example, polynomials, analytic functions, and even many non-analytic functions, such as $\phi = (lnr)^{-1}$ possess this property LOP. However $\phi = r \sin \frac{1}{r}$ or $f = y^{1+\epsilon} \cos \frac{1}{r}$ do *not* satisfy LOP, instead oscillate wavelike near the origin, yet still terminate with well-defined tangency at the node. Such flow is *not* separable into the subregions described below.

1.5 Limited Oscillation

Definition 1.5.1: We say $\phi\,(x, y)$ satisfies a *limited oscillation property* (denoted LOP) on any ray $\theta = \omega$ if there exists a radius $\rho > 0$ such that either

$$\phi\,(r \cos \omega,\, r \sin\, \omega) = 0 \quad \text{for} \quad 0 < r < \rho$$

$$\text{or else} \quad \phi\,(r \cos\, \omega,\, r \sin\, \omega) \neq 0 \quad \text{for} \quad 0 < r < \rho$$

We now develop the nonlinear analog of the unique linear straight line orbits which coincide with the y-axis in a simple linear field having distinct negative eigenvalues. For nonlinear system (1.1.2) consider now the set, which we shall denote F^+, of *all* orbits having tangency $\theta_{lm} = \frac{\pi}{2}$.

Theorem 1.5.2 For system (1.1.2) or (1.1.4), whenever f and g (or ϕ) satisfies the LOP defined above on rays $\theta = \pm\frac{\pi}{2}$, then there exists a separator orbit $Y^+ \in F^+$, which separates the set F^+ into either : the pair of subsets A_1 and A_2; or else into the pair of subsets B_1 and B_2, characterized respectively by :

a1) $\theta(t) < \frac{\pi}{2}$; a2) $\theta(t) \geq \frac{\pi}{2}$ somewhere.

b1) $\theta(t) > \frac{\pi}{2}$; b2) $\theta(t) \leq \frac{\pi}{2}$ somewhere.

In each case either $Y^+ \in A_1$ or $Y^+ \in B_1$.

Similarly the family F^- composed of all orbits with $\theta_{lm} = -\frac{\pi}{2}$ can be decomposed into $F^- = C_1 \cup C_2$ or $F^- = D_1 \cup D_2$ characterized by:

c1) $\theta(t) > -\frac{\pi}{2}$; c2) $\theta(t) \leq -\frac{\pi}{2}$ somewhere.

d1) $\theta(t) < -\frac{\pi}{2}$ d2) $\theta(t) \geq -\frac{\pi}{2}$ somewhere.

Moreover there exists a separator orbit $Y^- \in C_1$ or $Y^- \in D_1$.

Proof of Theorem 1.5.2:

Suppose $\phi\,(x, y)$ satisfies LOP on ray $\theta = \frac{\pi}{2}$.

Thus there exists radius ρ such that $\phi\,(x, y)$ is either always positive or always negative for $0 < y < \rho$. Without loss of generality, suppose $\phi\,(0, y) > 0$.

We note that on ray $\theta = \frac{\pi}{2}$, $\theta' = \phi > 0 \quad (r < \rho)$

Recall orbit X^+. X^+ passes through point (ρ, ω_*) and thereafter remains within wedge $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$, $0 \leq r \leq \rho$. Consider the future of the set of orbits through arc $r = \rho$, $\omega_* \leq \theta \leq \frac{\pi}{2}$. The orbit through (ρ, ω_*) we call X^+ has $\theta_{\lim} = 0$.

All other orbits through (ρ, θ) for $\omega_* < \theta \leq \frac{\pi}{2}$ have $\theta_{\lim} = \frac{\pi}{2}$.

Now we define $U := \{\theta \in (\,\omega_*,\, \frac{\pi}{2}\,] \mid$ orbit through $(\,\rho,\,\theta)$ crosses y^+ axis $\}$.

Note U is not empty. Let $u_* =$ $\inf U$. We claim $u_* \notin U$. Suppose $u_* \in U$.

Then the orbit through (ρ, u_*) meets the y^+axis at $(0,\,\eta)$ with $0 < \eta \leq \rho$.

We argue as in Lemma 1.3.3 that the orbit through $(0,\,\frac{\eta}{2})$ must intersect the circle of radius ρ at some (polar) point $(\,\rho,\,\theta_*)$ with $w_* < \theta_* < u_*$. This result contradicts u_* as infimum of U. This orbit through $(\,\rho, u_*)$ we call Y^+.

Now we note any orbit counterclockwise of Y^+ must touch the $y+$ axis.

Thus we may define

$$A_1 = \{\text{orbits in } F^+ \text{ clockwise of } Y^+\} \qquad A_2 = F^+ - A_1 \text{ (set difference)}$$

Similarly the case $\phi\,(0, y) < 0,\quad 0 < y < \rho$ produces a separator orbit Y^+ which is the orbit through point $(\,\rho,\,u^*)$ where

$u^* =$ sup $\{\theta \in [\,\frac{\pi}{2},\,\frac{3}{2}\pi) \mid$ the orbit through $(\,\rho,\,\theta)$ meets the y^+ axis$\}$.

In this case we define

$B_1 = \{$orbits in F^+ clockwise of $Y^+\}$ and $B_2 = F^+ - B_1$, and note $Y^+ \in B_1$.

Finally, and analogously, we argue the decomposition of

$F^- =$ $C_1 \cup C$ or $F^- =$ $D_1 \cup D_2$ with separator orbit $Y^- \in C_1$ or $Y^- \in C_2$,

depending on $\phi < 0$ on y^- axis , or $\phi > 0$ there, respectively.

This completes the proof of Theorem 1.5.1. $\square$

For completeness we should add that any of the four separator orbits X^+, X^-, Y^+, Y^- could indeed be coincidental with the axes themselves, as in the case of the simple linear node (1.1.3).

One final discussion will demonstrate that the X^+ and X^- orbits are each contained within a single quadrant.

For this, we assume the LOP on the x^+ axis, and without loss of generality specify that ϕ and therefore θ' take only negative values at all points $(x,0)$ such that $0 < x \leq \rho$. (Recall X^+ is the unique orbit with $\theta_{\text{lim}} = 0$.)

We claim : X^+ is disjoint from set $\{ (x, 0) \mid 0 < x \leq \rho \}$.

For indeed suppose $(a, 0)$ were a point of X^+ for some $0 < a \leq \rho$.

$Then$:

1) X^+ must pass through $(a, 0)$ with $r' \leq 0$ and $\theta' < 0$ so that in future time X^+ remains within quadrant IV. That is, either : $i)$ tangency at any point $(x, 0),\ 0 < x \leq a$, or $ii)$ returning to quadrant I, would indeed violate $\theta' < 0$ there.

But then,

2) The orbit through $(\frac{a}{2}, 0)$ must likewise pass into quadrant IV, and in the future *also never* again meet the x^+ axis. But this latter orbit must be distinct from X^+, never again enter quadrant I, yet have $\theta_{\text{lim}} \geq 0$, since only X^+ has $\theta_{\text{lim}} = 0$. These are clearly contradictory. Thus X^+ must not pass into quadrant IV within disk $r \leq \rho$. The other option is for $\phi\,(x, 0) \equiv 0$, in which case clearly X^+ is the ray $\theta = 0$.

Similarly, within disk $r \leq \rho$, the special orbit X^-, with $\theta_{\text{lim}} = \pi$, must behave in exactly one of three possible ways :

1) X^- is coincidental to the x^- axis when $\phi \equiv 0$ on ray $\theta = \pi$;

2) X^- is contained within Quadrant II when $\phi < 0$ for all $-\rho \leq x < 0$: or

3) X^- is contained in Quad III for $\phi > 0$ there.

Thus, when the rotational component ϕ has the LOP on ray $\theta = \frac{\pi}{2}$, there does exist a subset of those orbits with $\theta_{\text{lim}} = \frac{\pi}{2}$ having boundary orbit Y^+; and Y^+ (within disk $r \leq \rho$) either misses quadrant I or misses quandrant II.
Similarly Y^- must exist and be disjoint from either quadrant III or from quadrant IV.
But further, when ϕ has LOP on ray $\theta = 0$, then (for $r \leq \rho$) the special orbit X^+ also must have no point in common with quadrant I or else no point in quadrant IV.
Likewise X^- either misses quadrant II or misses quadrant III.

In summary, when a two-dimensional autonomous non-linear vector field produces distinct negative eigenvalues, and when its non-linear rotational component has limited oscillation on the rays $\theta = 0, \pm\frac{\pi}{2}, \pi,$ within some positive radius ρ;
then the local phase plane must resemble one 16 possible flow arrangements, from combinations of 1), 2), 3) above, and Theorem 1.5.2 a1, a2, b1, b2, c1, c2, d1, d2.

Remark Whenever the vector field perturbation $< f, g >$ can be expanded as *polynomial* terms in x and y, there is a very efficient algorithm for determining which of the 16 possible phase diagrams is applicable.

For the X^+ and X^- separators, we merely locate the lowest degree term in variable x (only) for $g(x, 0)$, and estimate the *direction* of $\frac{d\theta}{dt}$ at $(x, 0)$.

To demonstrate this analysis, consider the following specific polynomial field.

Example 1.5.3 : Let $< f,\, g > \;=\; < xy + y^2,\; -x^2 - xy - y^3 >$.

We compute $r^2\theta' \;=\; xg - yf \;=\; -x^3 - x^2y - xy^3 - x^2y - y^3$.

Thus $r^2\, \theta'\, (x,\, 0) \;=\; -x^3$,

so $\theta'\, (x,\, 0)$ is negative for $x > 0$,

and $\theta'\, (x,\, 0)$ is positive for $x < 0$.

Similarly, $r^2\, \theta'(0,\, y) = -y^3$ is negative for $y > 0$,

and $\theta'(0,\, y)$ is positive for $y < 0$.

These calculations identify the flow crossing each arm of each axis. From this determination, we can then identify the cases with respect to Theorem 1.5.2.

CHAPTER 2

ENERGY METHOD FOR SOLVING TWO-DIMENSIONAL NONLINEAR DYNAMICAL SYSTEMS

We present a general conceptual algorithm as an alternative approach to solving a first order nonlinear differential equation. We make use of an Energy function associated with a given dynamical system. This method, which is basically a search with conditions, will include as special cases exact forms and integrating factors; *but* will extend beyond these known method, producing solutions to equations which are neither exact nor reducible to exact by integrating factors.

The procedure utilizes an Energy /Lyaponov type formulation, in order to create a new and simpler (reduced) differential equation, whose solution will in turn produce an *implicit primitive* for the original differential equation.

2.1 General Problem; Basics

Consider the first order equation : $dx = f(t,\, x)\, dt,$ (2.1.1)

where f is continuous on $J \times \mathbb{R}$ into $\mathbb{R}$ for some interval $J = [a,\, b] \subset \mathbb{R}$.

Definition 2.1.2 A function $x(t) : J \to \mathbb{R}$ is a *solution* of (2.1.1) if $x(t)$ and its differential $dx(t)$ satisfy (2.2.1) on J.

The following well-known result provides sufficient conditions on the rate function f which ensure the existence and uniqueness of solution $x(t)$.

Theorem 2.1.3 If $f(t, x)$ is continuous on $J \times \mathbb{R}$ into $\mathbb{R}$ for interval $J = [a, b] \subset \mathbb{R}$, and if there exist positive K, L such that:

$$|f(t, x)| \leq K(1 + |x|^2) \text{ (growth) and}$$

$$|f(t, x)| - f(t, y)| \leq L|x - y| \text{ for all } (t, x), (t, y) \in J \times \mathbb{R},$$

then the initial value problem $dx = f(t, x)dt, \; x(t_0) = x_0$ (2.1.4)

has unique solution $x(t) = x(t, t_0, x_0)$ for $t \geq t_0$.

2.2 Procedure

Summarized description: We impose conditions on an (unknown) Energy function $V(t, x)$; we then conduct a search for a suitable $V(t, x)$ with the goal of eventually producing a reduced (solvable) differential equation, which in turn shall provide a closed form implicit/explicit solution or primitive for the non-linear equation (2.1.1).

Step 1: We assume the existence of $V(t, x)$ satisfying :

a) $V(t, x)$ is continuous on $J \times \mathbb{R}$.

b) $V(t, x)$ is monotonic in x, for each $t \in J$

c) V is continuously differentiable with respect to t and twice continuously differentiable with respect to x.

d) For each $t \in J$, V has an "inverse" $E(t, x)$ such that

$$V(t, E(t, x)) = x = E(t, V(t, x)).$$

Step 2: Define differential operator L associated with (2.1.1) :

$L = \frac{\partial}{\partial t} + f(t, x)\frac{\partial}{\partial x}$ and apply L to V thus :

$$dV(t, x(t)) = LV(t, x(t))\,dt \quad (2.2.1)$$

or simply $dV = V_t\,dt + fV_x dt$.

Step 3: Define composite $m(t) = V(t, x(t))$. Study the structure of (2.2.1) and select a useful form or class of rate function $F(t, m)$ for which the reduced differential equation $dm = F(t, m)\,dt$ (2.2.2)

can be readily solved.

Step 4: Combine (2.2.1) and (2.2.2) to produce

$$F(t,\, V(t,\, x)) \;=\; \tfrac{\partial}{\partial t}V(t,\, x) \,+\, f(t,\, x)\,\tfrac{\partial}{\partial x}\,V(t,\, x)\ . \qquad (2.2.3)$$

Next analyze and search for such a $V(t,\, x)$ whose associated composite $m(t)$ solves the reduced (2.2.2).

Step 5: Recover the solution $x(t)$ of (2.1.1) from the (usually implicit) equation :

$$V(t,\, x) \;=\; m(t) + C \qquad (2.2.4)$$

Let us approach the analysis of this Method by considering various classes of the resulting *reduced* form $F(t, m)$. A starting place is the simple class of explicit integrable functions. We begin by considering the class :

2.3 Integrable Reduced Forms

In this section we demonstrate the Method for the class of differential equations (2.1.1) which can be reduced to an explicitly integrable rate function $F(t,\, m) \;=\; p(t)$ in (2.2.2) . The simpler $p(t)$ which results from the Method will be continuous and therefore integrable.

This resulting $p(t)$, or F, shall have been required to satisfy (2.2.3) .

The original ODE (2.1.1) shall be reduced to $(2.2.2)$ in the form

$$dm \;=\; p(t)\, dt \qquad (2.3.1)$$

which will in turn produce the implicit primitive (2.2.4) for (2.1.1).

Procedure :

Perform the general steps 1 and 2 above.

Now, using the chosen class $F(t,\, m) \;=\; p(t)$, (2.2.3) becomes

$$p(t) \;=\; \tfrac{\partial}{\partial t}V(t,\, x) \,+\, f(t,\, x)\,\tfrac{\partial}{\partial x}\,V(t,\, x) \qquad . \qquad (2.3.2)$$

Step 4 : If $x(t)$ is to be a solution of (2.1.1) , then (2.3.1) imposes condition (2.2.1) on energy $V(t,x)$: $dV \,=\, V_t\,dt + f V_x dt \,=\, p(t)\,dt$ (2.3.3)

which in turn produces :

Step 5 : $V(t,\, x(t)) \,=\, \int dV(t, x(t)) \,=\, \int p(t)\,dt \,+\, C$ (2.3.4)

where C is a constant of integration.

Observation 2.3.5 **:** Differential equation (2.1.1) is the most general type of explicit nonlinear ODE. Let us consider the application of the Method to the subclass of the form

$$f(t,\, x) \,=\, -\frac{M(t,x)}{N(t,x)} \;;\quad dx = f(t,\, x)\,dt \qquad (2.3.5)$$

and choice of reduced ODE rate $F(t,\, m) \,=\, p(t)$ in (2.3.3).

Note (2.3.5) also has form $N\,dx \,+\, M\,dt \,=\, 0$. M , N , and p are continuous.

Here (2.3.2) becomes $p(t) \,=\, \frac{\partial}{\partial t}V(t,x) \,-\, \frac{M(t,x)}{N(t,x)}\,\frac{\partial}{\partial x}\,V(t,x)$. (2.3.6)

We proceed to search for a useful combination pair $p(t)$ and $V(t,x)$.

For form (2.3.5) we approach the search by considering a choice of energy function of the form

$$V(t,x) \,=\, \int u(t,x)\,N(t,x)\;dx \qquad (2.3.7)$$

where now the nonzero factor $u(t,x)$ becomes our search goal **.**

For the sake of clarity, and using suppressed notation where feasible, we note

$V_x \,=\, u\,N$; condition (2.3.6) now becomes

$p\,(t \text{ only}) \,=\, V_t + f V_x = V_t - \frac{M}{N} V_x \,=\, V_t + f V_x = V_t - u\,M$ **.** (2.3.8)

This implies $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\left(\int u(t,x)\,N(t,x)\;dx\right) - u(t,\,x)\;M(t,\,x)\right) = 0$ (2.3.9)

Thus the given functions $M(t,x)$, $N(t,x)$, and the unknown target $u(t,x)$ must together satisfy (2.3.9) in order to reduce the ODE (2.3.5) to the integrable class (2.3.1).

Observation 2.3.10 **:** Manipulating (2.3.8) with (2.3.7) and (2.3.5) as follows, we find

$$\frac{\partial}{\partial t}\int u\,N\,dx \;-\; u\,M$$

$= \int \frac{\partial}{\partial t}(u\,N)dx - \int \frac{\partial}{\partial x}(\,u\,M)dx \;+\; q(t)$, where q results from partial antiderivative. Thus the condition (2.4.9) can be guaranteed by the vanishing of

$$\int \frac{\partial}{\partial t}(u\,N)dx - \int \frac{\partial}{\partial x}(\,u\,M)dx \;=\; \int \left[\frac{\partial}{\partial t}(u\,N) - \frac{\partial}{\partial x}(\,u\,M)\right]\,dx$$

or the vanishing of the integrand $\frac{\partial}{\partial t}(u\,N) - \frac{\partial}{\partial x}(\,u\,M)$.

Thus, in this case, the factor $u(t,x)$ plays the role of an integrating factor, and our Method reduces to the *Generalized Method of Integrating Factor.*

Further if $u(t,x) \;\equiv\; u(t)$ or if $u(t,x) \;\equiv\; u(x)$, then the Energy procedure is equivalent to the usual *Method of Integrating Factor***.**

Observation 2.3.11 **:** In the further case where $u(t,x) \;\equiv\; 1$, we see the Energy Method reduces to the usual *Method of Exact Differential Equation***.**

Continuing our attempts to reduce to *integrable* equations of the form

$dm = p(t)\,dt$, we consider :

Example 2.3.12 $\quad dx = \;-\left(2\sec(tx) + \frac{x}{t}\right)\,dt.$

Note $\quad f(t,\,x) = \;-\,\dfrac{2t + x\cos(tx)}{t\cos(tx)}$ in the $-\,\dfrac{M(t,x)}{N(t,x)}$ form.

Assuming there exists a $V(t,\,x)$, we formally write

$dV = V_t\,dt + V_x\,dx$ and make a

choice for $dm = F(t,\, m)\, dt.$ Suppose we choose reduced form $dm = p(t)\, dt.$
We are now seeking a $V = V(t,\, m)$ such that

$$V_t + fV_x = V_t - \tfrac{M}{N} V_x \;=\; p\ (t\,\text{only}) \qquad (2.3.13)$$

One approach is to transfer the search for $V(t,\, x)$ to a search for some $u(t,\, x)$
such that $V = \int uN\, dx$ or formally $V(t,\, x) = \int_a^x u(t,\, y)\, N(t,\, y)\, dy.$
Also note $V_x = uN = u(t,\, x)\, t\cos(tx).$ (2.3.14)
Denoting $\int N\, dx$ by $\tilde{N}$; here $\tilde{N} = \sin(tx)$ and $\frac{\partial \tilde{N}}{\partial x} = N.$ We compute by parts

$V = \int uN\, dx = u\tilde{N} - \int u\tilde{N}\, dx$ and

$$V_t = u_t\, \tilde{N} + u\tilde{N}_t - \tfrac{\partial}{\partial t} \int u\, \tilde{N}\, dx \qquad (2.3.15)$$

Thus (2.4.2) becomes $V_t - \frac{M}{N} V_x = p(t)$ or

$$u_t \sin(tx) + ux\cos(tx) - \tfrac{\partial}{\partial t} \int u\, \tilde{N}\, dx - uM \;=\; p(t)$$

[Recall $M = 2t + x\cos(tx)$]

Thus we seek $u(t,\, x)$ to satisfy :

$$u_t \sin(tx) + ux\cos(tx) - \tfrac{\partial}{\partial t} \int u\, \tilde{N}\, dx - ux\cos tx - 2ut \;=\; p(t)$$

or simply $$u_t \sin(tx) - 2ut - \tfrac{\partial}{\partial t} \int u\, \tilde{N}\, dx \;=\; p(t).$$

Setting $u \equiv 1$ will reduce the solution to $p(t) \;=\; -2t$. We have reached :

Step 5**:** Reduced ODE $m(t) = -2t\, dt$ implies $m(t) = -t^2 + C.$

The key step now is to recall $m(t)$ is defined as the composite $V(t,\, x(t)).$
And energy $V = \int u\, N\, dx = \int N\, dx = \tilde{N} = \sin(tx)$

Thus we have the implicit primitive $-t^2 + C = \sin(tx)$ solving (2.3.12).

Note: This example is actually an exact form; but we shall see further cases.

Example 2.3.16 $$x' = -\frac{2t\,\tan x + 2x\,t^2 + x - 2t}{t + \sec^2 x}$$

Here $f(t,x) = -\frac{M(t,x)}{N(t,x)}$ is *not* *exact*.

Again set $V = \int u\,N\,dx = u\tilde{N} - \int u_x\,\tilde{N}\,dx$

where $\tilde{N} = \int N\,dx = tx + \tan x$.

Thus, $V_t = u_t\,\tilde{N} + ux - \frac{\partial}{\partial t}\int u_x\,\tilde{N}\,dx$. Also note again $V_x = uN$

We are attempting to reach the reduced form

$$dm = p(t)\,dt \quad \text{where} \quad m(t) = V(t,\, x(t)).$$

Thus we seek $V = \int u\,N\,dx$ to satisfy

$$m' = V_t + fV_x = p(t \text{ only}); \quad i.e,$$

$$u_t(tx + \tan x) + ux - \frac{\partial}{\partial t}\int u_x\,\tilde{N}\,dx$$
$$- u[2t\,\tan x + 2xt^2 + x - 2t] = p(t) \qquad (2.3.17)$$

Here $u \equiv$ constant does not work. Try $u(t,\, x) = u(t)$; $u_x = 0$.

Condition (2.3.17) becomes

$$(u' - 2tu)\,(tx + \tan x) + 2tu = p(t\text{only})$$

Setting $u' - 2tu = 0$ produces

$$2tu = p(t). \quad \text{Note } u = e^{t^2} \text{ suffices.}$$

We have $m' = p(t) = 2te^{t^2}$ which gives $m(t) = e^{t^2} + C$.

But $m(t) = V(t,\, x(t))$ and $V = \int u\,N\,dx = e^{t^2}\tilde{N} = e^{t^2}[tx + \tan x]$

Thus implicity we have $e^{t^2} + C = e^{t^2}[tx + \tan x]$ or primitive

$$\boxed{tx + \tan x - 1 = C\,e^{-t^2}} \qquad (2.3.18)$$

We remark that this example, while not exact, can be made exact by introduction of integrating factor e^{t^2}. Now other methods would indeed have developed the same integrating factor. However it is interesting to note how our accommodating factor $u(t)$ produced the energy function $V(t,x) = \int u(t)\, N(tx)\, dx$, which produced $u' - 2tu = 0$, which generated the integrating factor. Thus this general Energy Method does incorporate exactness and integrating factors, and as we shall see, other classes of equations.

2.4 Linear Nonhomogenous Forms :

We now consider the Energy Method approach to the problem of reducing nonlinear equations (2.1.1) into the class of linear equations of the form

$$dm = F(t,m)\, dt = [\,\mu(t)\, m + p(t)\,]\, dt \qquad (2.4.1)$$

here $\mu(t)$ and $p(t)$ are continuous real-valued variable coefficients.

Preliminary Steps 1 and 2 are parallel.

Having chosen the class of forms $F(t,\, V(t,\, x)) = \mu(t)\;\; V(t,x) + p(t)$,

Step 3 is to compute the differential of $m(t) = V(t, x(t))$ along $x(t)$.

Step 4 becomes $\mu(t)\, V(t,x) + p(t) = \frac{\partial}{\partial t} V(t,\, x) + f(t,\, x)\, \frac{\partial}{\partial x} V(t,\, x)$.

which Energy $V(t,x)$ must satisfy in order to reduce (2.2.1) to (2.4.1).

Since the solution of (2.4.1) is

$$m(t) = C\,\text{exp}[\, \int^t \mu(r)\, dr + \int^t \text{exp}[\, \int_s^t \mu(r) dr] p(s) ds$$

we have

Step 5 : $V(t,\, x(t)) = C\,\text{exp}[\, \int^t \mu(r)\, dr + \int^t \text{exp}[\, \int_s^t \mu(r) dr] p(s) ds$

forms the implicit solution of (2.1.1).

Observation 2.4.2 **:**

Let us consider the application of the Method to the subclass of the form

$$f(t,\, x) \;=\; -\,\frac{M(t,x)+R(t,x)}{N(t,x)} \;\;;$$

$$dx = f(t,\, x)\, dt \qquad\qquad (2.4.2)$$

and choice of reduced ODE rate

$F(t,\, m) \;=\; \mu(t)\, m \,+\, p(t)$ in (2.4.1).

Note (2.4.2) also has form $N\, dx \,+\, (M+R)\, dt \;=\; 0$.

M , N , R and p are continuous. (2.4.2) becomes

$$\mu(t)\, V +\, p(t) \;\;=\; \frac{\partial}{\partial t} V(t,x) \,-\, \frac{M(t,x)+R(t,x)}{N(t,x)}\, \frac{\partial}{\partial x}\, V(t,x) \qquad . \;\; (2.4.3)$$

We search for a useful combination triple $\mu(t)$, $p(t)$ and $V(t,x)$ to satisfy (2.4.3). As before, we approach the search by considering a choice of energy function of the form

$$V(t,x) \;=\; \int u(t,x)\, N(t,x)\; dx \qquad (2.4.4)$$

where again the nonzero function u becomes our goal.

Since $V_x \;=\; u\, N$, condition (2.4.3) becomes

$$p\,(t \text{ only}) = \; V_t + f V_x = \; V_t - \tfrac{M}{N} V_x \;=\; V_t - u\, M .$$

which implies

$$\frac{\partial}{\partial x} \left(\frac{\partial}{\partial t} \left(\int u(t,x)\, N(t,x)\; dx \right) - u(t,\, x)\; M(t,\, x) \right) = \;\; 0 \qquad (2.4.5)$$

and also we seek to have $-\; R(t,x)\, u(t,x) \;=\; \mu(t)\, V(t,x)$ (2.4.6)

Thus, given functions $M(t,x)$, $N(t,x)$, $R(t,x)$ and the unknown $u(t,x)$ together must satisfy (2.4.5) and (2.4.6) in order to reduce the ODE (2.4.2) to linear form (2.4.1).

Example 2.4.7 Consider $x' = -\frac{2t + x\cos tx + 2\sin tx}{t\cos tx}$.

We shall attempt to use the Energy Method to reduce (2.4.7) to the simpler

linear form (2.4.1) $$m'(t) = \mu(t)\, m(t) + p(t)$$

where again $m(t)$ is to be the composite of our Energy function $V(t,\, x(t))$.

Form (2.4.1) is suggested by writing (2.4.2) as $Nx' = -M - R$

where $N(t,\, x) = t\cos tx$, $M(t,\, x) = 2t + x\cos tx$,

and $R(t,\, x) = 2\sin tx$ and noting $R_x = 2N$.

Thus (2.4.1) arranged $m' = p + \mu m$ has a pattern such that a derivative

of the last expression on the RHS resembles the first term on the LHS.

Proceeding, we seek energy $V = \int uN\, dx$ for some useful u.

[Also note the association $uN = V_x$; thus N is a "derivative" of V; while in

original (2.4.2), N is a derivative of R; while in (2.4.1) m' is derivative of the last

term $\mu\, m = \mu V(t,\, x(t))$; which is a sort of transitive identification.]

Continuing, as before, we formulate V by parts:

$$V = \int uN\, dx = u\widetilde{N} - \int u_x \widetilde{N}\, dx$$

and differentiate $V_t = u_t \widetilde{N} + u\, \widetilde{N}_t - \frac{\partial}{\partial t}\int u_x \widetilde{N}\, dx.$

Here; $\widetilde{N} = \int N\, dx = \int t\cos tx\, dx = \sin tx = \frac{1}{2}R.$

And again $V_x = uN$ gives us

$$x'V_x = uNx' = u(-M - R)$$

which in turn produces

$$V_t + x' V_x \;=\; u_t \tfrac{1}{2} R + ux \cos tx - \tfrac{\partial}{\partial t}\int u_x \widetilde{N}\ dx - uM - uR$$

$$= \; u_t \sin tx + ux \cos tx - \tfrac{\partial}{\partial t}\int u_x \widetilde{N}\ dx - u(2t + x \cos tx) - u\, 2\sin tx$$

$$= \; (u_t - 2u) \sin tx - 2ut - \tfrac{\partial}{\partial t}\int u_x \widetilde{N}\ dx \qquad (2.4.8)$$

At this point, interestingly, we can proceed in **2** ways **:**

a) Suppose we let $u(t,\, x) \equiv 1$. Then (2.4.8) simplifies to

$$m' = V_t + x' V_x = \; 2\sin tx - 2t,$$

making $m = V = \; \widetilde{N} \; = \sin tx;$ in other terms

$\boxed{m' = \; -2m - 2t}$ has the chosen linear form. (2.4.9)

But also we can approach (2.4.8) by trying :

b) Let $u(t,\, x) = u(t)$ *while* setting $u' - 2u = \; 0.$

Thus $u = e^{2t}$, and

$$m \;=\; V = \; e^{2t}\, \widetilde{N} \;=\; e^{2t} \sin tx, \quad \text{and (2.4.8) becomes}$$

$$m' = \; -2t\, e^{2t} \qquad \text{, simple integrable class.}$$

Now reduction a) has solution

$$e^{2t}\, m \;=\; C - t\, e^{2t} + \tfrac{1}{2}\, e^{2t}$$

Together with $m(t) = \; V(t,\, x(t)) \;=\; \sin tx$,

we have the implicit primitive

$$e^{2t}\, [2\sin tx + 2t - 1] = C.S \qquad (2.4.10)$$

Similarly, reduction b) has solution

$$m = e^{2t}\, (\tfrac{1}{2} - t\,) + C.$$

which leads to (2.4.10) also (of course).

2.5 Reduction to Separable Forms

The following examples illustrate the scope of this approach beyond the *linear reducible* differential equations.

Example 2.5.1 Suppose $x' = \frac{(t^2x+\sin x)^2-2t\,y}{t^2+\cos x}$.

Let $N = t^2 + \cos x.$ Let $J = t^2x + \sin x.$

Then x' has the form $-\frac{J^2+2t\,x}{J_x}$ so that we might try for

separable reduced form $m' = \mu(t)m^2.$

Next we compute the defining conditions on the Energy function .

$V = \int uN\,dx = uJ - \int u_x J\,dx.$ V must satisfy **:**

$$V_t = u_t\,J + u\,J_t - \tfrac{\partial}{\partial t}\int u_x J\,dx$$

$$= u_t\,J + u\,2tx - \tfrac{\partial}{\partial t}\int u_x J\,dx$$

while $V_x = uN = uJ_x.$ These imply

$$m' = V_t + x'\,V_x$$

$$= u_t\,J + 2u\,tx - u\,J^2 - 2u\,tx - \tfrac{\partial}{\partial t}\int u_x J\,dx$$

$$= u_t[t^2x + \sin x] - u\,J^2 - \tfrac{\partial}{\partial t}\int u_x J\,dx$$

We see $u \equiv 1$ works here, producing $m' = -J^2$

and making energy function $V = J.$

Our reduced ODE is $m' = -m^2$, not only separable but also *autonomous,* (2.5.2)

whose solution is $m(t)\,[\,t + C] = 1.$ Thus the general solution for (2.5.1) is

$$(t^2x + \sin x)\,(t + C) = 1. \qquad (2.5.3)$$

Example 2.5.4: $dx = -\frac{1}{2}\frac{(tx+\tan x)^3+2x}{\sec^2 x+t}dt.$

We set $M(t,x) = -\frac{1}{2}(tx+\tan x)^3 - x$

and $N(t,x) = \sec^2 x + t.$ We note that

$$\frac{\partial}{\partial x} M(t,x) = -\frac{3}{2}(tx+\tan x)^2(\sec^2 x + t) - 1$$

and $\frac{\partial}{\partial t}N(t,x) = 1.$

Thus (2.5.4) is neither exact nor reducible to exact by an integrating factor**.** However, we initiate the Energy method procedure . Following the argument used in Example (2.3.12), we arrive at:

$$V(t,x) = \int u(\sec^2 x + t)dx$$

$$= u(tx+\tan x) - \int_a^x (tx+\tan x)\frac{\partial}{\partial x}u dx \quad (\text{integration by parts})$$

$$\frac{\partial}{\partial x}\Big(V(t,x) - V(t,a)\Big) = u\frac{\partial}{\partial x}(tx+\tan x) = u(\sec^2 x + t),$$

$$\frac{\partial}{\partial t}[V(t,x) - V(t,a)] = \frac{\partial}{\partial t}[u(tx+\tan x)] - \frac{\partial}{\partial t}[\int_a^x (tx+\tan x)\frac{\partial}{\partial x}u dx].$$

$$\frac{\partial}{\partial t}V(t,x(t)) + f(t,x(t))\frac{\partial}{\partial x}V(t,x(t))$$

$$= \frac{\partial}{\partial t}\Big(\int uN(t,x)dx\Big) - uM(t,x)$$

$$- u\frac{1}{2}(tx+\tan x)^3 - ux.$$

Again we $choose$ $u(t) = 1.$ $Thus\ V(t,x) = tx + \tan x.$ (2.5.5)

$$\frac{dm}{dt} = \frac{\partial}{\partial t}V(t,x(t)) + f(t,x(t))\frac{\partial}{\partial x}V(t,x(t))$$

$$= \frac{\partial}{\partial t}[(tx+\tan x)] - \frac{1}{2}(tx+\tan x)^3 - x$$

$$= -\frac{1}{2}(tx+\tan x)^3 = -\frac{1}{2}[V(t,x(t))]^3.$$

The reduced autonomous separable form is $dm = -\frac{1}{2}m^3 dt$ (2.5.6)

Combining (2.5.5) , (2.5.6) , and of course definition $m(t) = V(t,x(t))$

the general implicit solution to (2.5.4) is $\boxed{1 = (t + \mathrm{C})(tx+\tan x)^2}$

2.6 Separable Differential Equations.

This class of differential equations are easily reducible to integrable differential equations. Each *separable* differential equation is characterized by a rate function $f(t, x)$ which is in fact decomposible into a product of two functions, one of which is a function of the independent variable t, and the other is a function of the dependent variable $x(t)$. *Thus* we assume that $f(t, x)$ is a separable function in t and x variables **:**

$f(t, x) = a(t)b(x)$; a and b are continuous,

and $G(x) = \int_c^x \frac{ds}{b(s)}$ is invertible.

2.6.1 Problem Formulation The original problem structure now becomes

$$dx = f(t, x)\, dt = a(t)b(x)dt. \tag{2.6.1}$$

2.6.2 Calculations The basic calculations of our so-called Energy method become:

$$F(t, m) = p(t) = V_t + f(t, x)\, V_x$$

$$V_t + ab\, V_x = p(t).$$

Since the RHS is independent of x, we consider a choice of energy function $V(t, x)$ to make the LHS independent of x. Here we see that we can satisfy this condition by choosing $V(t, x) = V(x)$. In other words, one such choice is $V_t = 0$ together with $a(t)\, b(x)\, V_x\,(x) = p(t)$.

But this would mean $b(x)\, V_x\,(x)$ is a constant. We choose 1 for simplicity.

Altogether we now have reduced the situation to $p(t) = a(t)$. Also we have $V(t, x) = V(x) = \int_c^x \frac{1}{b(u)} du$. In other words, as we have seen several times, the energy function V is in the form of an i*ntegral* over x.

And we finish by solving both $m'(t) = p(t) = a(t)$ which is the *reduced* ODE.

Also we solve the energy integral $V(x) = \int_c^x \frac{1}{b(u)} du$; and this produces the

solution $m(t) = V(t, x(t))$. *i.e.* $\int_q^t a(\tau) d\tau = \int_c^x \frac{1}{b(u)} du + C$.

Remark : Of course we already knew this was to be the solution to a separable ODE, but it is still interesting to see this approach .

2.7 Homogeneous Differential Equations

The class of equations referred to as *homogeneous* are reducible to the separable class by known methods; here we shall analyze them with respect to the Energy procedure.

Definition 2.7.1: A differential equation (2.1.1) is said to be *homogeneous*

if the rate function $f(t, x)$ in (2.1.1) is a homogeneous function of degree zero,

that is, $f(\mathrm{k}t, \mathrm{k}x) = f(t, x)$ for any nonzero k..

Procedure **:**

We assume that rate function $f(t, x)$ in (2.1.1) is homogeneous of degree zero.

Further we shall need to assume that $(f(1, u) - u)$ does not vanish;

and that the indefinite integral $G(u) := \int^u \frac{ds}{f(1,s)-s}$ is invertible.

Let $v = \frac{x}{t}$ $Note \; f(t, x) = f(1, v)$ by homogeneity.

Also note $\frac{\partial v}{\partial t} = \frac{-x}{t^2} \;\; and \;\; \frac{\partial v}{\partial x} = \frac{1}{t}$ and $dv = \frac{x(x-t)}{t^3} dt$

Consider the type of energy function $V(t, x)$ $also \;\; homogeneous$:

$$V(t, x) = P(v) = P(\tfrac{x}{t}) \tag{2.7.2}$$

where P has yet to be determined.

The problem of seeking unknown energy function $V(t, x)$

is equivalent to the problem of seeking unknown function P.

Compute dV as follows:

$$\frac{\partial}{\partial t}V(t,x) = \frac{\partial}{\partial t}P(\tfrac{x}{t}) \;=\; P'(v)(-\tfrac{x}{t^2})$$

and

$$\frac{\partial}{\partial x}V(t,x) = \frac{\partial}{\partial x}P(\tfrac{x}{t}) \;=\; P'(v)\,(\tfrac{1}{t})$$

$Now \qquad dx \,=\, f(t,x)$ and $\quad dt \,=\, f(1,v)\,dt.$

Hence, $\qquad dV(t,x) = \; P'(v)[\,-\frac{x}{t^2}dt \,+\, \frac{1}{t}dx]$

$$= P'(v)[\,-\tfrac{x}{t^2}dt \,+\, \tfrac{1}{t}dx]$$

$$= P'(v)[\,-\tfrac{v}{t} \,+\, \tfrac{1}{t}f(1,v)\,]dt$$

We try the indefinite integral function $\;G(v) \;:=\; \int^{v}\frac{ds}{f(1,s)-s} \;\; for \;\; P(v).$

Here we have Energy $\;V(t,x) \,=\, G(\,\frac{x}{t}\,) \,=\, \int^{\frac{x}{t}}\frac{ds}{f(1,s)-s}$

We compute $\;\frac{dG}{dt} \;=\; \frac{dG}{dv}[\,v_t + v_x x'\,]$

$$=\; \frac{1}{[f(1,v)-v]}[\tfrac{-x}{t^2} + f(1,v)\tfrac{1}{t}]$$

$$= \tfrac{1}{t}\;.$$

Thus the reduced form is $\;m' \;=\; \frac{1}{t}\;.$

Altogether we have $\boxed{\log t = \int^{\frac{x}{t}}\frac{ds}{f(1,s)-s} + C}$ for the general solution. (2.7.3)

<u>Example 2.7.4</u> $\quad dx = \frac{x^2}{t^2}dt.\quad$ We note the homogeneity.

Further we see $\;G(u) \;:=\; \int^{u}\frac{ds}{f(1,s)-s} \;=\; \int^{u}\frac{ds}{s^2-s}\;$ is invertible

by partial fractions. $\;i.e.\;\; z = G(u) = \log(1-\frac{1}{u})\;$ has inverse

$$\boldsymbol{u} \;=\; \frac{1}{1-e^{z}}$$

Altogether we have $dm(t) = dV(t,x) = \frac{dG}{dv}dv = \frac{1}{v^2-v}\frac{x(x-t)}{t^3}dt = \frac{1}{t}dt.$

Solve for $m(t) = \log(t) + C$. Equate to $V = G(\frac{x}{t}) = \log(1 - \frac{t}{x})$.

The general solution is $\boxed{c^2 t = 1 - \frac{t}{x}}$ or $\boxed{k\,tx = x - t \ , \ k > 0}$

Remark. Of course being separable, this result is also obtainable through classic methods.

2.8. Bernoulli Equations

In this section, we present another subclass of differential equations reducible to (2.4.1). This class of equations are referred to as *Bernoulli* differential equations. First, we introduce a definition of Bernoulli differential equation.

Definition 2.8.1: A differential equation is said to be a *Bernoulli* differential equation, if the rate function $f(t,x)$ in $dx = f(t,x)dt$ is of the following form:

$$f(t,x) = K(t)x + Q(t)x^n \text{ for } some \text{ real number } n \neq 0{,}1$$

Reduction : Start with $dx = [K(t)x + Q(t)x^n]dt$ (2.8.1)

It is assumed that K and Q are continuous nonzero functions.

We initiate the procedure to reduce the Bernoulli type equation using the energy method. we associate a suitable natural energy/Lyapunov function in a unified and coherent way. We propose a differential of the form :

$$dV(t,x) = \mu(t)V(t,x) + p(t)$$

$$= \frac{\partial}{\partial t}V(t,x) + [K(t)x + Q(t)x^n]\frac{\partial}{\partial x}V(t,x), \quad (2.8.2)$$

In minimal notation

$$dV = \mu V + p = V_t + KxV_x + Qx^nV_x \quad (2.8.2)$$

We attempt $\mu V = KxV_x$ with $\mu(t) = \delta K(t)$, which gives

$$\frac{V_x}{V} = \frac{\delta}{x} \tag{2.8.3}$$

From this, it is clear that the quotient of $\frac{\partial}{\partial x}V(t,x)$ with $V(t,x)$

is *independent of* t.

Therefore, we can assume that $V(t,x) \equiv V(x)$,

that is, $V(t,x)$ is *independent of* t.

This means that $V_t = 0$ and $p(t) = Qx^n V_x$ from (2.8.2).

Solving (2.8.3)

$$V(x) = Cx^{\delta}\ ,\ C > 0. \tag{2.8.4}$$

We compute $\frac{d}{dx}V(x) = V_x = \delta C x^{\delta-1}$ and

$$p(t) = \delta C Q(t) x^{n+\delta-1}\,. \tag{2.8.7}$$

Separating, we have

$$x^{n+\delta-1} = \frac{p(t)}{\delta Q(t) C}. \tag{2.8.8}$$

We note that the right-hand side of (2.8.8) is a function of t only.

Therefore, we let $\delta = 1 - n$. Thus (2.8.7) becomes

$$p(t) = (1-n)C\,Q(t)$$

$$dV(t,x) = \mu(t)V(t,x) + p(t)$$

$$= (1-n)K(t)\,Cx^{1-n} + (1-n)CQ(t)$$

Letting C=1 the reduced form is linear :

$$m'(t) = (1-n)K(t)\,m(t) + (1-n)Q(t) \tag{2.8.9}$$

which is a linear solvable form in $m(t)$. Thus the general solution becomes

$$\boxed{x^{1-n} = m(t) + \mathbf{C}}$$

where of course $m(t)$ solves (2.8.9) through classic procedures.

CHAPTER 3

CRITICAL POINT THEORY UNDER RANDOM PERTURBATIONS

In this chapter we present a few preliminary results and definitions. We also discuss a method of stochastic integration.. Finally a few elementary examples illustrating the random perturbation effects in the theory of critical points of autonomous two-dimensional dynamical vector fields. These examples illustrate the role and scope of mathematics in real world problems. These examples also exhibit the significance of two-dimensional dynamical vector fields under environmental random perturbations. Moreover, the chapter provides a motivation to undertake further study in the critical point theory for two-dimensional dynamical vector fields under random perturbations. In addition, we generate several issues in the modeling of dynamic processes, namely, the effects of random perturbations [5]. We propose to undertake this study in the future.

3.1 The Wiener Process

We shall analyze perturbations involving the *Wiener process* type of Stochastic variables.

Properties 3.1.1. A *Wiener process* $w(t) : [0, T] \to \mathbb{R}$ shall satisfy :

a) $w(t)$ is continuous and $w(0) = 0$,

b) $E\,[\,w(t + \triangle\, t) - w(t)] = 0$, where $\triangle\, t > 0$,

c) $E\,[\,(w(t + \triangle\, t) - w(t))^2] = \triangle\, t > 0$.

where E is $expected\ value.$

Definition 3.1.2 The *Stochastic Integral* of $f(t)$ is defined by

$$\int_0^T f(t)\,dw(t) = \lim_{\mu(P)\to 0} \sum_{k=0}^{n-1} f(t_k)[\,w(t_{kt1}) - w(t_k)\,]$$

where P partitions $[0, T]$ thus: $0 = t_0 < t_1 < .... < t_n = T$

and μ is the mesh/norm of P, $\mu\,(P) = \max\ \triangle_k t.$

Definition 3.1.3 Let $x : [0, T] \to \mathbb{R}$ be a stochastic function.

We say $x(t)$ has a *stochastic differential* if

$$x(t+h) - x(t) = \int_t^{t+h} a(s)\, ds + \int_t^{t+h} \sigma(s)\, dw(s) \qquad (3.1.3)$$

where $a(t)$, $\sigma(t)$ are continuous and $0 \le t < t+h \le T$.

We denote $dx(t) = a(t)\, dt + \sigma(t)\, dw(t)$. (3.1.4)

We note the first integral in (3.1.3) is deterministic while the second is the Ito - Doob stochastic integral.

Theorem 3.1.5 Let $V(t, x) : [0, T] \times \mathbb{R} \to \mathbb{R}$ be class C^1 with respect to t, and class C^2 w.r.t. x. If $x = w(t)$, the Wiener process, then the differential

$$dV(t, w(t)) = [V_t(t, w(t)) + \tfrac{1}{2} V_{xx}(t, w(t))]\, dt + V_x(t, w)\, dw \qquad (3.1.5)$$

Proof. We shall apply the Taylor series expansion to V, and because of Properties 3.1.1, we shall retain only the terms $\triangle t$, $\triangle w$, and $(\triangle w)^2$.

$$\begin{aligned} \triangle V &= V(t + \triangle t, w + \triangle w) - V(t, w(t)) \\ &= V_t(t, w) \triangle t + V_x(t, w) \triangle w + \tfrac{1}{2} V_{xx}(t, w) (\triangle w)^2. \end{aligned} \qquad (3.1.6)$$

All higher terms, such as $\triangle t$, $\triangle w$, $(\triangle t)^2$, $(\triangle w)^3$, etc. are $o(\triangle t)$; that is, $\frac{(\triangle t)(\triangle w)}{\triangle t} \to 0$, $\frac{(\triangle w)^3}{\triangle t} \to 0$ as $\triangle t \to 0$; *except* that $\frac{(\triangle w)^2}{\triangle t} \to 1$ as $\triangle t \to 0$ because of Property 3.1.1c. Thus, when $\triangle t \to 0$, we have: $\triangle t \to dt$, the deterministic differential, $\triangle w \to dw$, the stochastic differential; and $(\triangle w)^2 \to dt$. (3.1.5) follows. □

We shall refer to (3.1.5) as the

natural Ito-Doob Stochastic Differential (or just Ito differential)

Example 3.1.7. Consider $V(t,\,x) \;=\; \exp[(a - \frac{1}{2}\sigma^2)t \,+\, \sigma x]$.

Denote $z \,=\, (a - \frac{1}{2}\,\sigma^2)t + \sigma\, x$ and set $x = w(t)$.

Then $dV = (a - \frac{1}{2}\,\sigma^2)\, e^z\, dt + \sigma\, e^z\, dw + \frac{1}{2}\,\sigma^2\, e^z\, dt$

$$= \quad (\exp[(a - \tfrac{1}{2}\,\sigma^2)\, t + \sigma\, w(t)]\,)[adt + \sigma\, dw(t)].$$

Consider now the more general case $V(t,\,x)$, where x has both a linear and a stochastic term:

$$x(t,\,w(t)) \;=\; at + \sigma\, w(t).$$

Theorem 3.1.8 Let $V(t,\,x(t)) : [0,\,T] \times \mathbb{R} \to \mathbb{R}$ be class C^1 w.r.t. t, and class C^2 w.r.t. x. Let $x = at + \sigma\, w(t)$, where $w(t)$ is the Wiener process.

Then the total stochastic differential $dV(t,\,x(t))$ is given by **:**

$$[\, V_t\,(t,\,x) + aV_x(t,\,x) + \tfrac{1}{2}\,\sigma^2\, V_{xx}(t,\,x)]\, dt + \sigma\, V_x(t,\,x)\, dw\,(t) \qquad (3.1.8)$$

Proof**:** We note $(dx)^2 \;=\; (adt + \sigma\, dw)^2 \;=\; a^2(dt)^2 + 2a\,\sigma\, dtdw + \sigma^2(dw)^2$,

but only the last term is not $o(\triangle\, t)$. Thus

$$dV \,=\, V_t dt + V_x dx + \tfrac{1}{2}\, V_{xx}(dx)^2 = V_t dt + V_x\,(adt + \sigma\, dw) + \tfrac{1}{2}\, V_{xx}\, \sigma^2 (dw)^2\;,$$

and (3.1.8) follows from the limit $(\triangle\, w)^2 \;\to dt$. ☐

Note: When computing the Ito-Doob differential of $V(t,\,x(t))$ with $x \,=\, at + \sigma\, w(t)$ we shall sometimes combine the dt terms as

$$DV(t,\,x(t)) = \;\; V_t(t,\,x) + a\, V_x\,(t,\,x) + \tfrac{1}{2}\,\sigma^2\, V_{xx}(t,\,x) \qquad (3.1.9)$$

$$i.e. \quad dV = DV dt + \sigma\, V_x dw \qquad (3.1.10)$$

Example 3.1.11 Consider $dV(t,\,x)$ for $V(t,\,x) \;=\; x^2$. Then

$$DV \,=\, 0 + 2ax + \sigma^2 \text{ so that } dV = (2ax + \sigma^2)\, dt + 2\sigma x dw.$$

Let x and y be stochastic processes with differentials

$$dx = \;\; adt + \sigma dw \quad \text{and} \quad dy = \;\; bdt + \lambda dw.$$

Consider functions $U_1\,(t,\,x)$ and $U_2(t,\,x)$.

Linear combinations, products, and quotients will follow the usual deterministic rules of Calculus (interpreting dU_1, dU_2 according to (3.1.8) above).

We now formulate the various differential rules of Ito-Doob stochastic functions of the form $U = U(t,\, x(t,w))$ and $V = V(t,\, y(t,w))$

where $dx = adt + \sigma dw$ and $dy = bdt + \lambda dw$.

Theorem 3.1.12 Let $U(t,\,z)$ and $V(t,\,z)$ be continuously differentiable with respect to t and twice continuously differentiable with respect to z.

Define $Q(t,\,x,\,y) = U(t,\,x) + kV(t,\,y)$ for some $k \in \mathbb{R}$.

Then the Ito differential of Q is $dQ = dU + kdV$; in other words, linearity holds.

The Proof is straightforward.

Example 3.1.13 Let $U(t,\,z) = \sin^2 z$, $V(t,\,z) = e^z$.

Let $Q(t,\,x,\,y) = U(t,\,x) + 3V\,(t,\,y)$. Compute :

$$dQ = Q_t\,dt + Q_x\,dx + Q_y\,dy + \tfrac{1}{2}[\,Q_{xx}\,(dx)^2 + 2Q_{xy}\,dx\,dy + Q_{yy}\,(dy)^2]$$

$$= (U_t + 3V_t)\,dt + U_x\,d_x + 3V_y\,dy + \tfrac{1}{2}\,U_{xx}\,\sigma^2\,dt + 0 + 3V_{yy}\,\lambda^2\,dt$$

$$= 0 + \sin 2x(adt + \sigma dw) + 3e^y\,(bdt + \lambda dw) + \sigma^2\cos 2x\,dt + 3\lambda^2 e^y\,dt$$

$$= [\,a\sin 2x + 3be^y + \sigma^2\cos 2x + 3\lambda^2\,e^y\,]\,dt + [\,\sigma\sin 2x + 3\lambda e^y\,]\,dw.$$

In other words, differentials of linear combinations are normal.

3.1.14 Independent Products Consider now computing the Ito differential

dQ for $Q(t,\,x,\,y) = U(t,\,x)\;V(t,\,y)$

with $dx = adt + \sigma dw$ and $dy = bdt + \lambda dw$.

Theorem 3.1.14 The stochastic product total differential is given by :

$$\boxed{d(UV) \, = \, V dU + U dV + \sigma\lambda \, U_x V_y \, dt}$$

Proof. We compute $d(UV) =$

$$(UV)_t \, dt + (UV)_x \, dx + (UV)_y \, dy + \tfrac{1}{2} \, (UV)_{xx} \, dx^2 + (UV)_{xy} \, dx \, dy + \tfrac{1}{2} \, (UV)_{yy} \, dy^2$$

$$= (VU_t + UV_t) \, dt + VU_x \, dx \, + UV_y \, dy + \tfrac{1}{2} \, VU_{xx} \, dx^2 + U_x V_y \, dx \, dy + \tfrac{1}{2} \, UV_{yy} \, dy^2$$

$$= (VU_t + UV_t) \, dt + VU_x \, (adt + \sigma dw) + UV_y \, (bdt + \lambda dw)$$

$$+ \, [\tfrac{1}{2} \, \sigma^2 VU_{xx} + \sigma\lambda \, U_x V_y + \tfrac{1}{2} \, \lambda^2 \, UV_{yy}] \, dt.$$

However $V du = V(U_t + aU_x + \frac{1}{2} \, \sigma^2 U_{xx}) \, dt \, + \sigma \, VU_x \, dw$

and $U d \, V = U(V_t + bV_y + \frac{1}{2} \, \lambda^2 \, V_{yy}) \, dt + \lambda U V_y dw.$

This verifies the formula. □

Example 3.1.15 Compute $d(x^2 y^2)$: $U(t,x) = x^2$; $V(t,y) = y^2$;

$$dU \, = \, 2xdx + dx^2 = \, 2axdt + 2 \, \sigma x dw + \sigma^2 \, dt. \qquad dV \, = 2bydt + 2\lambda y dw + \lambda^2 \, dt.$$

Thus $d(x^2 y^2) = [2axy^2 + \sigma^2 \, y^2 + 2b \, x^2 \, y + \lambda^2 \, x^2 + 4 \, \sigma\lambda xy] \, dt + 2[\sigma \, xy^2 + \lambda x^2 \, y] \, dw.$

The next result gives the Stochastic Dfferential Quotient Formula . We have omitted the calculations, which can be computed from the Product Formula (3.1.14) applied to

$$U(t, x(t,w)) \, V(t, y(t,w))^{-1}.$$

Theorem 3.1.16 For the Wiener process $w(t)$ and stochastic variables

$$x = at + \sigma \, w(t), \quad \text{and} \quad y = bt + \lambda \, w(t).$$

the Ito-Doob Stochastic differential for $\frac{U(t,x)}{V(t,y)}$ can be computed from :

$$\boxed{V^2 d \, (\tfrac{U}{V}) = V dU - U dV - \sigma \, \lambda \, U_x V_y \, dt + \lambda^2 \, \tfrac{U}{V} \, (V_y)^2 \, dt}$$

Example 3.1.17 Compute $d(\frac{x^3}{y})$ for $dx = adt + \sigma dw$ and $dy = bdt + \lambda\, dw.$

$$d(x^3) = 3x^2 dx + \tfrac{1}{2}\, 6x (dx)^2 = 3x\,[\, ax\, dt + \sigma x\, dw + \sigma^2\, dt]\,\textbf{.}$$ And

$$y^2 d\,(\frac{x^3}{y}) = 3xy[\, ax\, dt + \sigma x\, dw + \sigma^2\, dt] - x^3\,(bdt + \lambda\, dw) - \sigma\,\lambda\,(3x^2)\, dt + \frac{\lambda^2 x^3}{y}\, dt\textbf{.}$$

$$= \left[3a\, x^2 y + 3\,\sigma^2 xy - bx^3 - 3\,\sigma\,\lambda\, x^2 + \lambda^2\,\frac{x^3}{y}\,\right] dt + \left[\,3\,\sigma\, x^2 y - \lambda\, x^3\,\right] dw.$$

3.2 Integration of Stochastics

We shall use a combination of the total stochastic differential (3.1.5) and integration procedures. Consider computing an Ito-Doob integral of the form

$$\int f\,(w(t))\, dw(t) \tag{3.2.1}$$

for some continuously differentiable $f : R \to R$ with $w(t)$ being the Wiener process. We often simply write w for $w(t)$. Let $F(u)$ be a usual antiderivative for $f(u)$.

Step 1**:** Compute the Ito-Doob Stochastic differential (3.1.5)

$$dF(w) = \tfrac{1}{2}\, F_{uu}(w)\; dt + F_u\,(w)\, dw = \tfrac{1}{2}\, f_u(w)\, dt + f(w)\, dw \tag{3.2.2}$$

Step 2**:** Compute the formal Ito-Doob integral.

$$F(w) = \tfrac{1}{2}\int f_u\,(w)\, dt + \int f(w)\, dw. \tag{3.2.3}$$

Thus
$$\boxed{\int f(w(t))\, dw(t) = F(w(t)) - \tfrac{1}{2}\int f_w(w(t))\, dt + C} \tag{3.2.4}$$

Note the result appears *in terms of* a *Riemann* integral.

Example 3.2.5 $\int w^2(t)\, dw\,(t).$ Set $F(u) = \tfrac{1}{3}u^3.$

Noting $\frac{d}{du}\tfrac{1}{3}u^3 = u^2$ we compute $dF\,(w) = \tfrac{1}{2}\, 2w\, dt + w^2\; dw.$

Thus $F(w) = \tfrac{1}{3}w^3 = \int w(t)\, dt + \int w^2(t)\, dw\,(t)$ or

$$\int w^2(t)\, dw\,(t) = \tfrac{1}{3}w^3(t) - \int w(t)\, dt + C.$$

Example 3.2.6 $\int e^{w(t)}\, dw\,(t)$. Set $F(u) = e^u$.

$$dF(w) = \tfrac{1}{2}e^{w}dt + e^{w}dw. \qquad Thus,$$

$$\int e^{w(t)}\, dw\,(t) = e^{w(t)} - \tfrac{1}{2}\int e^{w(t)}\, dw\,(t) + C. \qquad (3.2.6)$$

Example 3.2.7 Compute $\int e^{x(t,\, w)}\, dw$ where again $x = at + \sigma\, w(t)$.

Now $d(e^x) = e^x\, dx + \frac{1}{2}\, e^x\, dx^2 = e^x\,(adt + \sigma\, dw) + \frac{\sigma^2}{2}\; e^x\, dt.$

Thus $e^x = a\int e^x\, dt + \sigma\int e^x\, dw(t) + \frac{\sigma^2}{2}\int e^x\, dt$.

Therefore, $\int exp\,[\,at + \sigma\, w\,(t)]\, d\, w\,(t)$

$$= \frac{1}{\sigma}\, exp\,[\,at + \sigma\, w] - \left(\frac{a}{\sigma} + \frac{\sigma}{2}\right)\int exp\,[\,at + \sigma\, w\,(t)]\, dt + C. \qquad (3.2.7)$$

3.2.8 Integration By Parts

In order to compute a stochastic integral of the form $\int f(t,\, w)\, dw\,(t)$,

we outline the following procedure :

Step 1: Differentiate. Set $V(t,\, w(t)) = w\,(t)\, f\,(t,\, w(t))$ and use tools from section 3.1 to compute

$$dV = V_t\, dt + V_w\, dw + \tfrac{1}{2}\, V_{ww}\, dt \qquad \text{from (3.1.5)}$$

$$= wf_t dt + (f + wf_w)\, dw + \tfrac{1}{2}[\,2f_w + w\, f_{ww}]\, dt \qquad \text{from (3.1.14)}$$

Step 2: Integrate: $V(t,\, w)$

$$= \int wf_t\, dt + \int f\, dw + \int w\, f_w\, dw + \int f_w\, dt + \tfrac{1}{2}\int w\, f_{ww}\, dt + C$$

Thus $\int f\, dw = wf - \int[\,wf_t + f_w + \frac{1}{2}\, wf_{ww}]\, dt - \int wf_w\, dw + C \qquad (3.2.9)$

Step 3: Approach the last integral $\int wf_w\, dw$ analogously.

Set $U = w^2 f_w$ and perform steps 1 and 2.

1. $dU = w^2 f_{wt}\, dt + 2wf_w\, dw + w^2 f_{ww}\, dw$
$$+ f_w\, dt + \tfrac{1}{2} w^2 f_{www}\, dt$$

2. $w^2 f_w = 2\int wf_w\, dw + \int w^2 f_{ww}\, dw$ (3.2.10)
$$+ \int [\, w^2 f_{wt} + f_w + \tfrac{1}{2} w^2 f_{www}]\, dt$$

Step 4**:** Use (3.2.10) within (3.2.9) to yield

$$\int f dw = wf + \tfrac{1}{2} w^2 f_w - \tfrac{1}{2}\int w^2 f_{ww}\, dw -$$

$$\int [\, wf_t + \tfrac{3}{2} f_w + \tfrac{1}{2} wf_{ww} + \tfrac{1}{2} w^2 f_{wt} + \tfrac{1}{4} w^2 f_{www}]\, dt.$$

Continue until the *stochastic* integral term *repeats* *or* *terminates*.

Example 3.2.11 $\int w(t)\, dw\,(t)$. Compute the differential

$$d(w^2(t)) = 2w\, dw + (dw)^2 = 2\, wdw + dt.$$

Thus $w^2(t) = 2\int wdw + t + C$

The result : $$\boxed{\int w(t)\, dw = \tfrac{1}{2}\,(w^2(t) - t) + C} \qquad (3.2.11)$$

Example 3.2.12 $\int t^4 w^5(t)\, dw\,(t)$ We differentiate

$$d(t^4 w^6) = 4t^3 w^6\, dt + 6t^4 w^5\, dw + \tfrac{1}{2}\, 30 t^4 w^4 dt$$

Integrate: $t^4 w^6 = 4\int t^3 w^6\, dt + 6 \boldsymbol{\int t^4 w^5\, dw} + 15\int t^4 w^4\, dt$

$$Thus \quad \int t^4 w^5\, dw$$
$$= \tfrac{1}{6} t^4 w^6 - \tfrac{2}{3}\int t^3 w^6\, dt - \tfrac{5}{2}\int t^4 w^4\, dt + C$$

Example 3.2.13 Compute the definite integral $\int_a^b g(t)\, dw(t)$

Differentiate $d(wg) = g'\, wdt + gdw$

Integrate $w(t)\, g(t)\,|_a^b = \int_a^b g'\,(t)\, w(t)\, dt + \int_a^b g(t)\, dw(t)$

$$\boxed{\int_a^b g(t)\, dw(t) = w(b)\, g(b) - w(a)\, g(a) - \int_a^b g'\,(t)\, w(t)\, dt}$$

3.3 Points Versus Focal Points

In the following, we consider a degenerate two-dimensional deterministic dynamical vector field.

$$dy = Aydt, \tag{3.3.1}$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

In this case, the fundamental matrix solution of (3.3.1) is

$$\Phi_{\mathrm{d}}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.3.2}$$

In this case, all points are critical points and all trajectories reduce to points.

Next we consider the Ito-Doob type stochastic perturbed system relative to (3.3.1).

$$dy = Aydt + Bydw(t), \tag{3.3.3}$$

where dy stands for the Ito-Doob type differential of y ; the matrix A is as defined in (3.3.1), and the matrix B is defined by

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3.3.4}$$

In this case, by following the solution procedure [], the fundamental matrix solution process of (3.3.3) is

$$\Phi(t) \equiv \Phi_{\mathrm{s}}(t) = \exp[\tfrac{1}{2}t] \begin{bmatrix} \cos w(t) & \sin w(t) \\ -\sin w(t) & \cos w(t) \end{bmatrix}. \tag{3.3.5}$$

From this, it is easy to conclude that the critical point is still the origin, but it is now a focus.

From this, we conclude that the Ito-Doob type stochastic perturbation has caused the creation of the focal point.

3.4 Node Versus Focus

In the following, we consider a two-dimensional deterministic dynamical vector field whose unperturbed flow is that of a source, an unstable node.

Consider an unperturbed simple linear system given by :

$$dx = Axdt, \tag{3.4.1}$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this case, the fundamental matrix solution of (3.4.1) is

$$\Phi_d(t) = \exp[t] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.4.2}$$

Here the critical point is the origin, which is a node (unstable).

Now, we consider the Ito-Doob type stochastic perturbed system relative to (3.4.1) .

$$dy = Aydt + Bydw(t), \tag{3.4.3}$$

where dy stands for the Ito-Doob type differential of y ; the matrix A is as defined in (3.4.1), and the matrix B is defined in (3.3.4).

In this case, by following the solution procedure [], the fundamental matrix solution process of (3.4.3) is

$$\Phi(t) = \exp[\tfrac{3}{2}t] \begin{bmatrix} \cos w(t) & \sin w(t) \\ -\sin w(t) & \cos w(t) \end{bmatrix}. \tag{3.4.4}$$

From this, it is easy to conclude that the critical point is still the origin, but it is now a focus.

From this, we conclude that the Ito-Doob type stochastic perturbation has caused the change of the critical point "node" to the critical point focus.

3.5 Node Versus Center

In the following, we consider a two-dimensional deterministic dynamical vector field whose flow is in the form of a simple linear stable node. Consider the unperturbed system given by :

$$dx = Axdt, \tag{3.5.1}$$

where

$$A = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

In this case, the fundamental matrix solution of (3.5.1) is

$$\Phi_d(t) = \exp[-\tfrac{1}{2}t]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.5.2}$$

Therefore, in this case, the critical point is the origin, and the origin is a node (stable). Now, we consider the Ito-Doob type stochastic perturbed system relative to (3.5.1)

$$dy = Aydt + Bydw(t), \tag{3.5.3}$$

where dy stands for $I$to-Doob type differential of y; the matrix A is as defined in (3.5.1), and the matrix B is defined in (3.3.4).

In this case, by following the solution procedure [], the fundamental matrix solution process of (3.5.3) is

$$\Phi(t) = \begin{bmatrix} \cos w(t) & \sin w(t) \\ -\sin w(t) & \cos w(t) \end{bmatrix}. \tag{3.5.4}$$

From this, it is easy to conclude that the critical point is the origin, and it is a center. From this, we conclude that the Ito-Doob type stochastic perturbation has caused the creation of the critical point center out of the stable "node".

3.6 Center Versus Focus

In the following, we consider a two-dimensional deterministic dynamical vector field whose trajectories are orbits about the origin.

$$dx = Axdt, \tag{3.6.1}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

In this case, the fundamental matrix solution of (3.6.1) is

$$\Phi_d(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}. \tag{3.6.2}$$

Therefore, in this case, the critical point is the origin, and the origin is a center. Now, we consider the Ito-Doob type stochastic perturbation affecting this system (3.6.1) .

$$dy = Aydt + Bydw(t), \tag{3.6.3}$$

where dy stands for the $I$to-Doob type differential of y; the matrix A is as defined in (3.6.1), and the matrix B is defined by

.

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.6.4}$$

In this case, by following the solution procedure [], the fundamental matrix solution process of (3.6.3) is

$$\Phi(t) = \exp[-\tfrac{1}{2}t + w(t)] \begin{bmatrix} \cos t) & \sin t \\ -\sin t & \cos t \end{bmatrix}. \tag{3.6.5}$$

From this, it is easy to conclude that the critical point is the origin, but it is now a focal point (focus).

From this, we conclude that the Ito-Doob type stochastic pertiurbation has caused the qualitative change from the critical point "node" to the critical point focus

CHAPTER 4

STOCHASTIC LAPLACE TRANSFORMS

This chapter deals with the Laplace transforms of stochastic processes with respect to both the Cauchy-Reimann and the Itô-Doob improper integrals .

Laplace transforms are applied for finding the closed-form solutions of initial value problems . Many results will be provided to illustrate the methods and the necessary adaptations for stochastics. The definitions and methods will parallel the classic deterministic results as much as possible.

4.1 The Laplace Transform

In this section, we present the concept of Laplace transform and its applications to higher order linear nonhomogeneous differential equations with constant coefficients.

Let f be a real-valued function of two variables $(t, w(t))$ defined for all real numbers $t \geq 0$, and let $w(t)$ be the Wiener process.

Definition 4.1.1 **:** The *Laplace transform of* f, *in the sense of the Cauchy-Riemann* integral, is defined by

$$\mathfrak{L}(f)(s) = \int_0^\infty e^{-st} f(t, w(t))dt = \lim_{T\to\infty} [\int_0^T e^{-st} f(t,\ w(t))dt], \tag{4.1.1}$$

for all values of s for which this improper integral exists.

Definition 4.1.2 **:** The *Laplace transform of* f *in the sense of the Itô-Doob integral* with respect to the Wiener process $w(t)$ (or simply the *Ito-transform*) denoted by $\mathfrak{L}^w(f)(s)$, is defined by **:**

$$\mathfrak{L}^w(f)(s) = \int_0^\infty e^{-st} f(t, w(t))dw(t)$$
$$= \lim_{T\to\infty} [\int_0^T e^{-st} f(t,\ w(t))dw(t)]$$

for all values of s for which this improper integral exists.

Example 4.1.3 **:** For any real $c \neq 0$, recall $\mathfrak{L}(c)(s) = \frac{c}{s}$ for $s > 0$.
We now find the Laplace transform of $f(t, w(t)) = c$
in the sense of the Itô-Doob integral.

Calculation. The Itô-Doob improper integral

$$\mathfrak{L}^w(c)(s) = \int_0^\infty e^{-st} c\, dw(t) = c \lim_{T\to\infty} [\int_0^T e^{-st} dw(t)]$$

$$= c \lim_{T\to\infty} [\, e^{-st} w(t)|_0^T + s \int_0^T e^{-st} w(t)\, dt \qquad (\textit{integration by parts})$$

$$= c \lim_{T\to\infty} [\, e^{-st} w(t)|_0^T] + c\, s \lim_{T\to\infty} \int_0^T e^{-st} w(t)\, dt$$

$$= s\, c \lim_{T\to\infty} \int_0^T e^{-st} w(t)\, dt = cs\mathfrak{L}(w(t))(s)\,.$$

$$Thus \qquad \boxed{\mathfrak{L}^w(1)(s) = s\mathfrak{L}(w(t))(s)} \qquad (4.1.4)$$

Example 4.1.5: Recall the Laplace transform (deterministic) of $f(t) = t$ **.**

$$\mathfrak{L}(f)(s) = \int_0^\infty e^{-st} t\, dt = \frac{1}{s^2} \quad (\, s > 0\,)$$

$$i.e. \quad \mathfrak{L}(t)(s) = \frac{1}{s^2}. \qquad (4.1.5)$$

Let us find the transform $\mathfrak{L}^w$ for $w(t)$, the $Wiener$ process,
in the sense of Itô-Doob.

Calculation. In order to calculate the Itô-Doob improper integral

$$\mathfrak{L}^w(w(t))(s) = \int_0^\infty e^{-st} w(t)\, dw(t)$$

we shall use the procedure of Integration by Parts for stochastics outlined in Section 3.2**.**
We recall the formula

$$\int f\, dw = wf - \int [\, wf_t + f_w + \tfrac{1}{2}\, wf_{ww}]\, dt - \int wf_w\, dw + C \qquad (3.2.9)$$

Here $f(t, w) = e^{-st} w(t)$. We compute (minimal notation) **:**

$$f_t = -sf; \qquad f_w = e^{-st}; \qquad f_{ww} = 0$$

Thus $\int f\, dw = wf + s\int wf\, dt - \int e^{-st}\, dt - \int f\, dw$

is repeating type.

$$\text{Thus} \qquad 2\int f\, dw = wf + s\int wf\, dt - \int e^{-st}\, dt$$

$$= we^{-st} + s\int w^2 e^{-st}\, dt - \int e^{-st}\, dt\,.$$

Therefore $2\int_0^T e^{-st} w\,dw = \; we^{-st}|_0^T + \; s\int_0^T w^2 e^{-st}\,dt \;-\; \int_0^T e^{-st}\,dt\,.$

But $w(0) = 0$, and as $T \to \infty$, we get

$$2\,\mathfrak{L}^w(w)\,(s) = s\,\mathfrak{L}(w^2)(s) - \mathfrak{L}(1)(s). \tag{4.1.6}$$

It can be shown that the Stochastic Laplace transform inherits the *linearity* properties of deterministic Laplace transforms.

Theorem 4.1.7: For any real a, and functions $f(t, w(t))$, $g(t, w(t))$,

$$\mathfrak{L}(af + g)(s) = a\,\mathfrak{L}(f)(s) + \mathfrak{L}(g)(s), \quad and$$
$$\mathfrak{L}^w(af + g)(s) = a\,\mathfrak{L}^w(f)(s) + \mathfrak{L}^w(g)(s),$$

whenever they exist.

Proof**:** Omitted.

We now proceed to build a table of the Laplace transforms in the sense of the Ito-Doob stochastic integrals for many basic stochastic forms; introduce the inverse forms; and demonstrate applications.

So far we have $\mathfrak{L}^w(1)$, $\mathfrak{L}^w(w)$, and linear combinations of $\mathfrak{L}(f)$ and $\mathfrak{L}^w(g)$.

Recall the simple linear stochastic form $x(t, w) = at + \sigma w$.

Example 4.1.8: Find the Ito-Laplace transform $\mathfrak{L}^w$ of

$$f(t, w(t)) = e^{at+\sigma w(t)} = e^x$$

Now $\mathfrak{L}^w(\sigma e^{at+\sigma w(t)})(s) = \sigma\int_0^\infty e^{-st} e^{at+\sigma w(t)}\,dw(t)\,.$

Let $\mathrm{E} = \exp[x - st] = \exp[(a-s)t + \sigma w]\,.$

So $d\mathrm{E} = \mathrm{E}_t dt + \mathrm{E}_w dw + \frac{1}{2}\mathrm{E}_{ww} dt$

$$= (a-s)\mathrm{E}\,dt + \sigma\mathrm{E}\,dw + \tfrac{1}{2}\sigma^2\mathrm{E}\,dt.$$

Integrating, $$\mathrm{E} = (a-s)\int\mathrm{E}\,dt + \sigma\int\mathrm{E}\,dw + \tfrac{1}{2}\sigma^2\int\mathrm{E}\,dt \tag{4.1.9}$$

Now for $s > a$, as $T \to \infty$, $\mathrm{E}(T)$ vanishes, $\int \mathrm{E}\, dt$ becomes $\mathfrak{L}(e^x)(s)$, and $\int \mathrm{E}\, dw$ becomes $\mathfrak{L}^w(e^x)(s)$ and (4.1.9) becomes

$$-1 = (a-s)\mathfrak{L}(e^x)(s) + \sigma\mathfrak{L}^w(e^x)(s) + \tfrac{1}{2}\sigma^2\mathfrak{L}(e^x)(s) \quad (s > a)$$

Thus,

$$\mathfrak{L}(e^{at+\sigma w(t)})(s) = \frac{1+\sigma\mathfrak{L}^w(e^{at+\sigma w(t)})(s)}{(s-a-\frac{1}{2}\sigma^2)} \quad (s > a)$$

$$(s - a - \tfrac{1}{2}\sigma^2)\,\mathfrak{L}(e^{at+\sigma w(t)})(s) - 1 \qquad (4.1.10)$$
$$= \sigma\mathfrak{L}^w(e^{at+\sigma w(t)})(s)$$

Note: In the case $a = 0$, (4.1.10) reduces to

$$\boxed{(s - \tfrac{1}{2}\sigma^2)\,\mathfrak{L}(e^{\sigma w(t)})(s) - 1 = \sigma\mathfrak{L}^w(e^{\sigma w(t)})(s)} \qquad (s > 0)$$

Examples 4.1.11: Recall the classic deterministic Laplace transforms:

$$\mathfrak{L}(e^{at})(s) = \frac{1}{s-a} \qquad (s > a)$$
$$\mathfrak{L}(\sin at)(s) = \frac{a}{s^2+a^2} \quad (s > 0)$$
$$\mathfrak{L}(\cos at)(s) = \frac{a}{s^2+a^2} \quad (s > 0)$$

The note (4.1.10) above serves to compute $\mathfrak{L}^w(e^{\sigma w(t)})(s)$.

Let us compute the analogous Ito-Laplace transforms $\mathfrak{L}^w$

for $\sin\sigma w(t)$ and $\cos\sigma w(t)$.

It is useful here to define $U(t,w) = e^{-st}\sin\sigma w(t)$; $V(t,w) = e^{-st}\cos\sigma w(t)$.

Differentiating U $\quad dU = U_t dt + U_w dw + \frac{1}{2}U_{ww}dt$

$$-sU\,dt + \sigma e^{-st}\cos\sigma w\, dw - \tfrac{1}{2}\sigma^2\, U\,\mathrm{dt}\,.$$

Integrating, $\quad U = -s\int U\,dt + \sigma\int e^{-st}\cos\sigma w\, dw - \frac{1}{2}\sigma^2\int U\,\mathrm{dt}$

$$U = -s\int e^{-st}\sin\sigma w\;\; dt + \sigma\int e^{-st}\cos\sigma w\;\; \mathrm{d}w - \tfrac{1}{2}\sigma^2\int e^{-st}\sin\sigma w dt.$$As

As $T \to \infty$

$$0 = - s\mathfrak{L}(\sin \sigma w)(s) + \sigma \mathfrak{L}^w(\cos \sigma w)(s) - \tfrac{1}{2}\sigma^2 \mathfrak{L}(\sin \sigma w)(s)$$

or $$\sigma \mathfrak{L}^w(\cos \sigma w)(s) - (s + \tfrac{1}{2}\sigma^2)\mathfrak{L}(\sin \sigma w)(s) = 0 \quad (4.1.15)$$

Similarly,

$$dV = V_t dt + V_w dw + \tfrac{1}{2} V_{ww} dt$$

$$= - sV dt - \sigma e^{-st} \sin \sigma w \, dw - \tfrac{1}{2}\sigma^2 V dt .$$

Integrating, $$V = - s\int V dt - \sigma \int e^{-st} \sin \sigma w \, dw - \tfrac{1}{2}\sigma^2 \int V dt$$

$$V = - s\int e^{-st} \cos \sigma w \, dt - \sigma \int e^{-st} \sin \sigma w \, dw - \tfrac{1}{2}\sigma^2 \int e^{-st} \cos \sigma w dt.$$

Let $T \to \infty$.

$$- 1 = - s\mathfrak{L}(\cos \sigma w)(s) - \sigma \mathfrak{L}^w(\sin \sigma w)(s) - \tfrac{1}{2}\sigma^2 \mathfrak{L}(\cos \sigma w)(s)$$

or $$\boxed{\sigma \mathfrak{L}^w(\sin \sigma w)(s) + (s + \tfrac{1}{2}\sigma^2)\mathfrak{L}(\cos \sigma w)(s) = 1} \quad (4.1.16)$$

4.2 Derivatives, Integrals, and Convolutions

Analogous to the Riemann integrals, we present:

Theorem 4.2.1 (Transform of Derivative)

If $f(t, w(t))$ has a derivative $f'(t, w(t))$ which is piecewise-continuous for $t \geq 0$, then

$$\mathfrak{L}[\, f'(t, w(t))]\,(s) \;=\; s\mathfrak{L}[f(t, w(t))] - f(0,0). \tag{4.2.1}$$

Proof **:** (Integration by parts)

$$\int_0^T e^{-st} f'(t, w(t))dt \;= e^{-st} f(t, w(t))|_0^T s \int_0^T e^{-st} f(t, w(t))dt$$

$$= e^{-sT} f(T, w(T)) - f(0,0) + s\int_0^T e^{-st} f(t, w(t))dt \quad Let\ T \to \infty$$

Assuming $f(t, w)$ does not grow as quickly as e^{-st}, (4.2.1) follows .□

Naturally, higher order derivatives will require repeated applications of (4.2.1).

Theorem 4.2.2 (Transform of Indefinite Integral) **:**

Assume $\mathfrak{L}(f(t,w(t))(s)$ exists.

Denote $C(t) \;=\; \int_0^t f(u, w(u))du$, the Cauchy-Riemann integral of f and

$I(t) \;=\; \int_0^t f(u, w(u))dw(u)$, the Itô-Doob integral of f.

Then

$$s\,\mathfrak{L}(C(t))(s) \;=\; \mathfrak{L}(f(t))(s) \quad \text{and}$$

$$s\,\mathfrak{L}(I(t))(s) \;=\; \mathfrak{L}^w(f(t))(s) \tag{4.2.2}$$

Proof **:** Now $dI \;= d[\int_0^t f(u, w(u))dw(u)] \;=\; f(t, w(t))dw(t)$.

Therefore $s\,\mathfrak{L}(I(t))(s) \;=\; s\int_0^\infty e^{-st} I(t)\,dt = s\lim_{T\to\infty}[\int_0^T e^{-st} I(t)\,dt]$

$$= -\lim_{T\to\infty}[\int_0^T -se^{-st} I(t)\,dt] \;=\; -\lim_{T\to\infty}[\int_0^T -se^{-st} I(t)\,dt]$$

$$= -\lim_{T\to\infty}[\,I(t)e^{-st}|_0^T \;-\; \int_0^T e^{-st}dI(t)\,]$$

$$= -\lim_{T\to\infty}[\,I(t)e^{-st}|_0^T\,] + \lim_{T\to\infty}[\,\int_0^T e^{-st}dI(t)\,]$$

$$= 0 + \lim_{T\to\infty}\int_0^T e^{-st}f(t,w(t))dw(t) \;=\; \mathfrak{L}^w(f(t))(s).$$

The proof for $\int_0^t f(u,w(u))du$ is straightforward. □

<u>Example 4.2.3:</u> We compute $\mathfrak{L}^w(e^{at})$

We apply the Itô-Doob differential formula to $w(t)e^{at}$ **:**

$$d(w(t)e^{at}) \;=\; a\,w(t)e^{at}dt + e^{at}dw(t)$$

Integrate : $\int_0^t d(w(u)e^{au}) = \int_0^t aw(u)e^{au}du + \int_0^t e^{au}dw(u)$

$$w(t)e^{at} = \int_0^t aw(u)e^{au}du + \int_0^t e^{au}dw(u)$$

Apply $\mathfrak{L}$: $\mathfrak{L}[w(t)e^{at}] = \mathfrak{L}\int_0^t aw(u)e^{au}du + \mathfrak{L}\int_0^t e^{au}dw(u)$

Multiply by s : $s\mathfrak{L}[w(t)e^{at}] = s\mathfrak{L}\int_0^t aw(u)e^{au}du + s\mathfrak{L}\int_0^t e^{au}dw(u)$

Apply theorem : $s\mathfrak{L}[w(t)e^{at}] \;=\; a\mathfrak{L}[w(t)e^{at}] + \mathfrak{L}^w(e^{at})$

$Thus$
$$\boxed{\mathfrak{L}^w(e^{at}) = (s-a)\,\mathfrak{L}[w(t)e^{at}]} \qquad (4.2.3)$$

$Note$: For $a=0$ (4.2.3) reduces to (4.1.4) $\mathfrak{L}^w(1)(s) = \mathfrak{L}(w(t))(s)$

<u>Example 4.2.4</u>: Compute $\mathfrak{L}^w(te^{\sigma w(t)})$

We apply the Itô-Doob differential formula to $te^{\sigma w}$ **:**

$$d(te^{\sigma w}) \;=\; e^{\sigma w}dt + \sigma te^{\sigma w}dw(t) + \tfrac{1}{2}\,\sigma^2 te^{\sigma w}dt$$

Integrate : $te^{\sigma w} = \int_0^t e^{\sigma u}du + \sigma\int_0^t ue^{\sigma w}dw(u) + \frac{1}{2}\sigma^2\int_0^t ue^{\sigma w}du$

$$\mathfrak{L}(te^{\sigma w}) = \mathfrak{L}(\int_0^t e^{\sigma u}du) + \sigma\mathfrak{L}(\int_0^t ue^{\sigma w}dw(u)) + \tfrac{1}{2}\sigma^2\mathfrak{L}(\int_0^t ue^{\sigma w}du)$$

$$s\mathfrak{L}(te^{\sigma w}) = s\mathfrak{L}(\int_0^t e^{\sigma u}du) + \sigma s\mathfrak{L}(\int_0^t ue^{\sigma w}dw(u)) + \tfrac{1}{2}\sigma^2 s\mathfrak{L}(\int_0^t ue^{\sigma w}du)$$

Apply theorem: $s\mathfrak{L}(te^{\sigma w}) = \mathfrak{L}(e^{\sigma t}) + \sigma\mathfrak{L}^w(te^{\sigma w}) + \frac{1}{2}\sigma^2\mathfrak{L}(te^{\sigma w})$

$Thus$

$$\boxed{\sigma\mathfrak{L}^w(te^{\sigma w}) = (s - \tfrac{1}{2}\sigma^2)\mathfrak{L}(te^{\sigma w}) - \mathfrak{L}(e^{\sigma t})} \qquad (4.2.4)$$

$Note$: For $\sigma = 0$ (4.2.4) reduces to classic result

$$\mathfrak{L}(t) = \tfrac{1}{s}$$

Continuing to parallel the classic tools, we now adopt and adapt characteristic step functions and convolutions for stochastics.

Let κ denote the characteristic step function for the interval $[0, \infty)$.

Specifically,

$$\kappa(t) \equiv \begin{cases} 0, \text{ if } t < 0 \\ 1, \text{if } t \geq 0 \end{cases}, \quad \text{for any real } c$$

(or translated)

$$\kappa(t - c) = \begin{cases} 0, \text{ if } t < c \\ 1, \text{if } t \geq c \end{cases} \qquad (4.2.5)$$

Theorem 4.2.6: $Given:$ any $c \geq 0$ and real-valued $g(t, w(t))$. $Let\ v = t - c.$

$$Then \qquad \mathfrak{L}[g(v, w(v))\kappa(v)](s) = e^{-sc}\mathfrak{L}(g(t, w))(s). \qquad (4.2.6)$$

Proof: (analogous to the classic case)

$$\int_0^\infty e^{-st} g(t - c, w(t - c))\kappa(t - c)dt = \int_c^\infty e^{-st} g(t - c, w(t - c))dt$$

$$= \int_0^\infty e^{-s(v+c)} g(v, w(v))dv = e^{-sc}\mathfrak{L}(g(t, w))(s). \qquad \square$$

$Corollary:$ $\mathfrak{L}(\kappa(t - c))(s) = e^{-sc}\mathfrak{L}(1)(s) = \frac{1}{s}e^{-sc}$.

Example 4.2.7: Consider $f(t, w(t)) = \begin{cases} 3, & \text{if } 0 < t < 1 \\ w(t), \text{if } t \geq 1 \end{cases}$

Let $u = t - 1.$ Rewrite $f(t, w(t))$ in terms of κ and v as follows:

$$f(t, w(t)) = 3 - 3\kappa(v) + [w(v + \kappa(v))]\kappa(v)$$

$Apply\ \ tranform\ operator\ \ \mathfrak{L}\ \ and\ \ theorem\ to\ each\ term\ \ of\ \ expansion.$

$$\mathfrak{L}f(t, w(t)) = \mathfrak{L}(3) - 3\,(\mathfrak{L}\kappa(v)) + \mathfrak{L}[w(v + \kappa\,(v))\,\kappa(v)]$$

$$\mathfrak{L}(f(t, w(t))(s) = \tfrac{3}{s} - \tfrac{3}{s}e^{-s} + e^{-s}\mathfrak{L}(w(t))(s)$$

$$= \tfrac{3}{s} - \tfrac{3}{s}e^{-s} + e^{-s}\,\tfrac{\mathfrak{L}^{w}(1)}{s}$$

<u>Definition 4.2.8</u> Let $f(t, w)$, $g(t, w)$ be real-valued piecewise continuous functions defined for $t \geq 0$, and let $w(t)$ be the Wiener process.

The *Itô-Doob convolution integral of* f *and* g

$$f{*}g(t, w(t)) = \int_0^t g(t-u, w(t-u)) f(u, w(u)) du.$$

<u>Remark</u>**.** Commutativity $f{*}g = g{*}f$ holds for these functions as well.

The Laplace operator $\mathfrak{L}$ applied to Ito-Doob convolutions of stochastic functions behaves normally.

<u>Theorem 4.2.9. Laplace Transform of Convolution Integral.</u>

Let $f(t, w)$, $g(t, w)$ be real-valued piecewise continuous functions defined on $t \geq 0$, and let $w(t)$ be the Wiener process. $Then$

$$\mathfrak{L}(\, f{*}g)(s) = \; = \mathfrak{L}(f)(s)\mathfrak{L}(g)(s) \qquad (4.2.9)$$

<u>Proof</u>: $Let\ v = t - u.$

$$\mathfrak{L}(\, f{*}g)(s) = \int_0^\infty e^{-st}[\int_0^t g(t-u, w(t-u)) f(u, w(u)) du] dt$$

$$= \int_0^\infty \int_u^\infty e^{-st} g(t-u, w(t-u)) f(u, w(u)) dt\, du$$

$$= \int_0^\infty [\, \int_0^\infty \kappa(v) e^{-st} g(v, w(v))\, dt] f(u, w(u))\, du$$

$$= \int_0^\infty \mathfrak{L}[\, \kappa(v)\, g(v, w(v))\,](s) f(u, w(u))\, du$$

$$= \int_0^\infty e^{-su} \mathfrak{L}(g(t,w))(s) f(u,w(u))\, du$$

$$= \mathfrak{L}(g(t,w))(s) \int_0^\infty e^{-su} f(u,w(u))\, du$$

$$= \mathfrak{L}(g(t,w))(s)\, \mathfrak{L}(f(t,w))(s) \qquad \square$$

<u>Example</u> <u>4.2.10</u>

Solve the *integral* equation: $g(t) = w(t) + \int_0^t \sin(t-u) g(u) du.$

Note $g(t) = w(t) + (\sin t) * g(t).$

$$Recall \quad \mathfrak{L}(t)(s) = \frac{1}{s^2} \qquad (4.1.5)$$

$$and \quad \mathfrak{L}(\sin t)(s) = \frac{1}{1+s^2} \qquad (4.1.11)$$

Compute $\mathfrak{L}(g)(s) = \mathfrak{L}(w(t))(s) + \mathfrak{L}(\sin(t) * g(t))(s)$

$$= \mathfrak{L}(w(t))(s) + \mathfrak{L}(\sin(t))(s)\ \mathfrak{L}(g)(s)$$

$$= \mathfrak{L}(w(t))(s) + \frac{1}{1+s^2}\ \mathfrak{L}(g)(s)$$

$Thus$ $\mathfrak{L}(g)(s) = \mathfrak{L}(w(t))(s) + \frac{1}{s^2} \mathfrak{L}(w(t))(s)$

$$= \mathfrak{L}(w(t))(s) + \mathfrak{L}(t)(s)\, \mathfrak{L}(w(t))(s)$$

$$= \mathfrak{L}(w(t))(s) + \mathfrak{L}(t * w(t))(s)$$

$Therefore$ $g(t) = w(t) + t * w(t)$ is the solution.

<u>Example 4.2.11</u>: Compute $\mathfrak{L}^w$ transform of $\cos(at + \sigma w(t))$ $\sigma \neq 0$

Compute the Itô-Doob differential of $e^{-st} \sin(at + \sigma w(t)).$

$$d[e^{-st} \sin(at + \sigma w(t))]$$

$$= -se^{-st} \sin(at + \sigma w(t)) dt + e^{-st} d[\sin(at + \sigma w(t))]$$

$$= -se^{-st} \sin(at + \sigma w(t)) dt + e^{-st} a \cos(at + \sigma w(t)) dt$$

$$+ \sigma e^{-st} \cos(at + \sigma w(t)) dw - \tfrac{1}{2} e^{-st} \sin(at + \sigma w(t))\, \sigma^2 dt$$

(by Theorem 3.1.5)

$$= \quad -(s+\tfrac{1}{2}\sigma^2)e^{-st}\sin(at+\sigma w(t))dt$$

$$+ae^{-st}\cos(at+\sigma w(t))dt$$

$$+\sigma e^{-st}\cos(at+\sigma w(t))dw.$$

Integrate

$$[e^{-st}\sin(at+\sigma w(t))]\Big|_0^T$$

$$= \quad -(s+\tfrac{1}{2}\sigma^2)\int_0^{\mathrm{T}} e^{-st}\sin(at+\sigma w(t))dt$$

$$+a\int_0^{\mathrm{T}} e^{-st}\cos(at+\sigma w(t))dt$$

$$+\ \sigma\int_0^{\mathrm{T}} e^{-st}\cos(at+\sigma w(t))dw$$

This implies

$$\lim_{T\to\infty}[e^{-sT}\sin(aT+\sigma w(T))] - e^{-s0}\sin(a0+\sigma w(0))$$

$$= -(s+\tfrac{1}{2}\sigma^2)\lim_{T\to\infty}\int_0^{\mathrm{T}} e^{-st}\sin(at+\sigma w(t))dt$$

$$+a\lim_{T\to\infty}\int_0^{\mathrm{T}} e^{-st}\cos(at+\sigma w(t))dt$$

$$+\sigma\lim_{T\to\infty}\int_0^{\mathrm{T}} e^{-st}\cos(at+\sigma w(t))dw$$

$i.e.$

$$0 \ = \ -(s+\tfrac{1}{2}\sigma^2)\mathfrak{L}[\sin(at+\sigma w(t))](s)$$

$$+a\mathfrak{L}[\cos(at+\sigma w(t))](s)$$

$$+\sigma\mathfrak{L}^w[\cos(at+\sigma w(t))](s)\ .$$

Thus

$$\sigma\mathfrak{L}^w[\cos(at+\sigma w(t))](s)$$

$$=(s+\tfrac{1}{2}\sigma^2)\mathfrak{L}[\sin(at+\sigma w(t))](s)$$

$$-a\mathfrak{L}[\cos(at+\sigma w(t))](s)$$

which is equivalent to

$$\mathfrak{L}[\sin(at+\sigma w(t))](s) \quad =$$

$$\frac{a\mathfrak{L}[\cos(at+\sigma w(t))](s)+\sigma\mathfrak{L}^w[\cos(at+\sigma w(t))](s)}{(s+\frac{1}{2}\sigma^2)}.$$

Example 4.2.12: Find the Laplace transform $\mathfrak{L}^w$ of $\sin(at+\sigma w(t))$

$$
\begin{aligned}
&d[e^{-st}\cos(at+\sigma w(t))] \\
&\quad= -se^{-st}\cos(at+\sigma w(t))dt + e^{-st}d[\cos(at+\sigma w(t))] \\
&\quad= -se^{-st}\cos(at+\sigma w(t))dt - ae^{-st}\sin(at+\sigma w(t))\,dt \\
&\qquad - e^{-st}\sin(at+\sigma w(t))\,\sigma dw \\
&\qquad - \tfrac{1}{2}e^{-st}\cos(at+\sigma w(t))\,\sigma^2 dt \\
&\quad= -(s+\tfrac{1}{2}\sigma^2)e^{-st}\cos(at+\sigma w(t))dt \\
&\qquad - ae^{-st}\sin(at+\sigma w(t))dt \\
&\qquad - \sigma e^{-st}\sin(at+\sigma w(t))dw.
\end{aligned}
$$

From the Itô-Doob integral, we have

$$
\begin{aligned}
&[e^{-st}\cos(at+\sigma w(t))]\Big|_0^T \\
&\quad= -(s+\tfrac{1}{2}\sigma^2)\int_0^T e^{-st}\cos(at+\sigma w(t))dt \\
&\qquad - a\int_0^T e^{-st}\sin(at+\sigma w(t))dt \\
&\qquad - \sigma\int_0^T e^{-st}\sin(at+\sigma w(t))dw
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\lim_{T\to\infty}[e^{-sT}\cos(aT+\sigma w(T))] - e^{-s0}\cos[a0+\sigma w(0)] \\
&\quad= -(s+\tfrac{1}{2}\sigma^2)\lim_{T\to\infty}\int_0^T e^{-st}\cos(at+\sigma w(t))dt \\
&\qquad - a\lim_{T\to\infty}\int_0^T e^{-st}\sin(at+\sigma w(t))dt \\
&\qquad - \sigma\lim_{T\to\infty}\int_0^T e^{-st}\sin(at+\sigma w(t))dw \\
&-1 = -(s+\tfrac{1}{2}\sigma^2)\,\mathfrak{L}[\cos(at+\sigma w(t))](s) \\
&\qquad - a\mathfrak{L}[\sin(at+\sigma w(t))](s) \\
&\qquad - \sigma\mathfrak{L}^w[\sin(at+\sigma w(t))](s)
\end{aligned}
$$

From the above expession and simplifications, we have

$$\sigma\mathfrak{L}^w(\sin(at+\sigma w(t)))(s)$$

$$= \quad 1-(s+\tfrac{1}{2}\sigma^2)\mathfrak{L}[\cos(at+\sigma w(t))](s)$$

$$-\, a\mathfrak{L}[\sin(at+\sigma w(t))](s)$$

which is equivalent to

$$\mathfrak{L}[\cos(at+\sigma w(t))](s)$$

$$= \frac{1-a\mathfrak{L}[\sin(at+\sigma w(t))](s)-\sigma\mathfrak{L}^w[\sin(at+\sigma w(t))](s)}{(s+\frac{1}{2}\sigma^2)}$$

Formulas 4.2.13 Combining results (4.2.12) and (4.2.14) we get **:**

Let $x = at+\sigma w(t)$. *Then*

$i)$ $$\mathfrak{L}(\cos x) = \frac{(s+\frac{1}{2}\sigma^2)-a\sigma\mathfrak{L}^w(\cos x)-\sigma(s+\frac{1}{2}\sigma^2)\mathfrak{L}^w(\sin x)}{(s+\frac{1}{2}\sigma^2)^2+a^2}$$

$ii)$ $$\mathfrak{L}(\sin x) = \frac{a-a\sigma\mathfrak{L}^w(\sin x)+\sigma(s+\frac{1}{2}\sigma^2)\mathfrak{L}^w(\cos x)}{(s+\frac{1}{2}\sigma^2)^2+a^2}$$

4.3 Applications of Laplace Transforms

The Laplace transform will be used to solve initial value problems. The Laplace transform transforms a linear differential equation with constant coefficients into an algebraic equation. The techniques for solving the algebraic equation may be easier than the methods of solving the initial value problems.

Example 4.3.1: Use the Laplace transform to solve the following initial value problem **:**

$$dy' + y\,dt = \sigma dw, \quad y(0)=0, \;\; y'(0)=1, \quad \sigma\neq 0.$$

We note that the Itô-Doob differential equation is equivalent to the following integral equation

$$y'(t) = y'(0) - \int_0^t y(u)du + \sigma\int_0^t dw(u).$$

We apply the Laplace transform to both sides,

$$\mathfrak{L}(y'(t)) \ = \ \mathfrak{L}[\ 1 - \int_0^t y(u)du \ + \ \sigma \int_0^t dw(u)]$$

$$= \mathfrak{L}\,(1) - \mathfrak{L}(\int_0^t y(u)du) + \sigma\mathfrak{L}(\int_0^t dw(u))$$

$$= \tfrac{1}{s} - \tfrac{\mathfrak{L}(y(t))}{s} + \tfrac{\sigma\mathfrak{L}^w(1)}{s}$$

Thus $$s\mathfrak{L}(y(t)) = \tfrac{1}{s} - \tfrac{\mathfrak{L}(y(t))}{s} + \tfrac{\sigma\mathfrak{L}^w(1)}{s}.$$

$$\mathfrak{L}(y(t)) = \tfrac{1+\sigma\mathfrak{L}^w(1)}{1+s^2}$$

By applying the inverse Laplace transform both sides, we get

$$y(t) = \ \sin(t) \ + \ \cos(t) * w(t)$$

Thus the solution of the initial value problem is given by

$$y(t) = \sin t + \int_0^t \cos(t-u)w(u)du).$$

<u>Example 4.3.2 (Langevin-Equation)</u>: Use the Laplace transform to solve the IVP:

$$dy' + \beta y' dt = \sigma dw, \quad y(0) = y_0, \quad y'(0) \ = \ v_0 \ ,$$

$$[\text{ for } \sigma \neq 0 \ \text{ and } \ \beta > 0.]$$

Convert to the integral equation

$$y'(t) = y'(0) - \beta\int_0^t y'(u)du + \sigma\int_0^t dw(u).$$

Now, we apply the Laplace transform both sides, and obtain

$$\mathfrak{L}(y'(t)) = \mathfrak{L}\ [y'(0) - \beta\int_0^t y'(u)du + \sigma\int_0^t dw(u)]$$

$$= \tfrac{v_0}{s} - \beta\mathfrak{L}[\int_0^t y'(u)du] + \sigma\mathfrak{L}[\int_0^t dw(u)\,]$$

$$= \tfrac{v_0}{s} - \tfrac{\beta\mathfrak{L}[y'(t)]}{s} + \tfrac{\sigma\mathfrak{L}^w(1)}{s}$$

Also $\mathfrak{L}[y'(t)] \ = \ s\mathfrak{L}[y(t)] - y(0).$

$$s\mathfrak{L}(y(t)) - y_0 = \frac{v_0}{s} - \frac{s\beta\mathfrak{L}[y(t)] - \beta\, y_0}{s} + \frac{\sigma\mathfrak{L}^w(1)}{s}.$$

Now, we solve for $\mathfrak{L}[y(t)]$, and have

$$\mathfrak{L}(y(t)) = \frac{v_0}{\beta s+s^2} + \frac{(\beta+s)y_0}{\beta s+s^2} + \frac{\sigma\mathfrak{L}^w(1)}{\beta s+s^2}$$

$$= \frac{v_0}{s(\beta+s)} + \frac{y_0}{s} + \frac{\sigma\mathfrak{L}^w(1)}{s(\beta+s)}$$

$$= \frac{v_0}{\beta}\left[\frac{1}{s} - \frac{1}{\beta+s}\right] + \frac{y_0}{s} + \frac{1}{\beta+s}\frac{\sigma\mathfrak{L}^w(1)}{s}$$

By applying the inverse Laplace transform both sides, we get

$$y(t) = y_0 + \frac{v_0}{\beta}(1 - e^{-\beta t}) + w(t) * e^{-\beta t}$$

<u>Example 4.3.3 (Chandrasekhar Equation)</u>:

Use the Laplace transform to solve the IVP:

$$dy' + (\beta\, y' + \lambda^2 y)dt = \sigma dw\,,$$

$$y(0) = y_0, \quad y'(0) = v_0, \quad \sigma \neq 0\ , \quad \beta > 0.$$

We note that the Itô-Doob differential equation is equivalent to the following integral equation

$$y'(t) = v_0 - \beta\int_0^t y'(u)du - \lambda^2\int_0^t y(u)du + \sigma\int_0^t dw(u).$$

Now, we apply the Laplace transform both sides, and obtain

$$\mathfrak{L}(y'(t))$$

$$= \mathfrak{L}[\, v_0 - \beta\int_0^t y'(u)du - \lambda^2\int_0^t y(u)du + \sigma\int_0^t dw(u)]$$

$$= \frac{v_0}{s} - \beta\mathfrak{L}[\int_0^t y'(u)du] - \lambda^2\mathfrak{L}[\int_0^t y(u)du] + \sigma\mathfrak{L}\int_0^t dw(u)$$

$$= \frac{v_0}{s} - \frac{\beta\mathfrak{L}(y'(t))}{s} - \frac{\lambda^2\mathfrak{L}(y(t))}{s} + \sigma\,\mathfrak{L}(w(t))$$

$$\mathfrak{L}(y'(t)) = s\mathfrak{L}(y(t)) - y_0\,,$$

$$s^2\mathfrak{L}[y(t)] - sy_0$$

$$= v_0 - \beta s\mathfrak{L}[y(t)] + \beta y_0 - \lambda^2\mathfrak{L}[y(t)] + \sigma s\mathfrak{L}(w(t))$$

$$\mathfrak{L}(y(t)) = \frac{v_0 + \beta\, y_0 + \sigma s\mathfrak{L}[w(t)]}{\lambda^2 + \beta s + s^2}$$

Depending on the magnitudes of λ^2 and β^2 , computing inverses,

the representation of the solution of Chandrasekhar's equation varies.

The details are left to the reader.

TABLE 1. LAPLACE TRANSFORMS

$f(t,w)$	$\mathfrak{L}(f\ )(s)$	$\mathfrak{L}^w(f\ )(s)$
c	$\frac{c}{s}$ $s>0$	$cs\,\mathfrak{L}(w)$
t	$\frac{1}{s^2}$	
t^n	$\frac{n!}{s^{n+1}}$	
$t^n\, e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	
$w(t)$	$\frac{\mathfrak{L}^w(1)}{s}$	$\frac{s}{2}\mathfrak{L}(w^2)\ -\ \frac{1}{2s}$
w^2	$\frac{1}{s^2}+\frac{2}{s}\ \mathfrak{L}^w(w)$	$\frac{s}{3}\mathfrak{L}(w^3)-\frac{\mathfrak{L}^w(1)}{s}$
$w^n(t)$		$s\,\mathfrak{L}(w^n)$=$n\mathfrak{L}^w(w^{n-1})$+$\frac{n(n-1)}{2}\ \mathfrak{L}(w^{n-2})$
e^{at}	$\frac{1}{s-a}$ $s>a$	$(s-a)\ \mathfrak{L}[w(t)e^{at}]$
$e^{\sigma w}$	$\frac{1+\sigma\mathfrak{L}^w(e^{\ \sigma w})(s)}{(s-\ \frac{1}{2}\sigma^2)}$	
$e^{at+\sigma w(t)}$	$\frac{1+\sigma\mathfrak{L}^w(e^{at+\sigma w(t)})(s)}{(s-a-\frac{1}{2}\sigma^2)}$	
$\sin at$	$\frac{a}{s^2+a^2}$	
$\cos at$	$\frac{s}{s^2+a^2}$	
$\sin\sigma w(t)$		$\sigma\mathfrak{L}^w(\cos\sigma w)(s)$-$(s+\frac{1}{2}\ \sigma^2)\mathfrak{L}(\sin\sigma w)(s)$=0
$\cos\sigma w(t)$		$\sigma\mathfrak{L}^w(\sin\sigma w)(s)$+$(s+\frac{1}{2}\sigma^2)\mathfrak{L}(\cos\sigma w)(s)$=1
$t\ \sin at$	$\frac{2as}{(s^2+a^2)^2}$	
$t\ \cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$	
$\sinh at$	$\frac{a}{s^2-a^2}$	
$\cosh at$	$\frac{s}{s^2-a^2}$	
$e^{bt}\sin at$	$\frac{a}{(s-b)^2+a^2}$ $s>b$	
$e^{bt}\cos at$	$\frac{s-b}{(s-b)^2+a^2}$ $s>b$	

BIBLIOGRAPHY

[1] Coduneanu, C., "Principles of Differential and Integral Equations", Allyn and Bacon, Boston, Mass., 1971.

[2] Doob, J. L., "Stochastic Processes", John Wiley and Sons, New York, N. Y. , 1953.

[3] Ladde, A. G. and Ladde, G. S., "An Introduction to Differential Equations I : Deterministic Modeling, Methods, and Analysis" (in Press).

[4] Ladde, A. G. and Ladde, G. S., "An Introduction to Differential Equations II : Stochastic Modeling, Methods, and Applications" (in Preparation).

[5] Ladde, A. G. and Ladde, G. S., "Dynamic Processes Under Random Environment", The Bulletin of Maratswada Mathematical Society, (in Press), 2007.

[6] Ladde, G. S. and Sambandham, M., "Stochastic Versus Deterministic Systems of Differential Equations", Marcel Dekker, Inc., New York, N. Y., 2004.

[7] Nemytskii, V. V. and Stepanov, V. V. , "Qualitative Theory of Differential Equations", Princeton University Press, Princeton, New Jersey, 1960.

[8] Perko, Lawrence, "Differential Equations and Dynamical Systems" Springer-Verlag, New York, N. Y., 1991.

[9] Sansone, G. and Conti, R. , "Non-Linear Differential Equations", MacMillan, New York, N. Y., 1964.

[10] Zubov, V. I., "Methods of A. M. Lyapunov and their Application", P. Noordhoff, Ltd., Groningen, The Netherlands, 1964.

BIOGRAPHICAL INFORMATION

Roger D. Kirby was born in Dallas, Texas in the United States of America. He attended Rice University, on an academic scholarship, graduating in the year 1970. He also received a Master of Science Degree in Mathematics from the University of Texas at Arlington in 1997. He currently resides in Fort Worth, Texas, and is a lecturer in Mathematics at the University of Texas at Arlington.