NEW EQUILIBRIA FOR NONCOOPERATIVE GAMES

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ABSTRACT

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In this dissertation we present an alternative to the Nash Equilibrium (NE), which is regarded as the fundamental solution concept in game theory. An NE provides a solution in which no player can improve his payoff by unilaterally changing strategies. However, the NE has weaknesses as exemplified by the paradoxical Prisoner's Dilemma game, where the unique NE is dominated by another possible outcome. Moreover, an NE assumes all players select their strategies according to the same NE so equilibrium holds. For multiple NE's, no standard approach exists for selecting a single one, though various refinements such as perfect Nash equilibria have been suggested.

An NE is characterized by its minimizing each player's expected regret for any fixed strategies of the other players. We present a complementary equilibrium based on the notion of disappointment, where disappointing oneself is regret. Our new equilibrium is called a Disappointment Equilibrium (DE). In a DE, for every player *i*, any or all players except *i* can change strategies and possibly decrease *i*'s payoff, while certainly never making *i*'s payoff better. Remarkably, the same DE has this property for every player *i*. A DE thereby enforces equilibrium with an implicit cooperative property based on the possible loss and certain non-improvement of payoff that any player might incur from some opponent's change of strategy. Such cooperation may be better for players than an NE, as in Prisoner's Dilemma. Thus a DE demonstrates that the requirements for an NE are not necessary conditions for a rational solution.

We prove that a DE always exists, provide a method to compute one, and present examples. We also show that the DE is a dual equilibrium to the NE. In an NE, each player is assumed to act out of self-interest, while in a DE each player acts out of concern for the action of the other players. This duality is particularly useful in two-person games. We also define a Pareto Intercession Equilibrium (π) that represents a compromise between the NE and DE solution criteria. Together, the new concepts of a DE and π resolve some important issues in game theory.

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CHAPTER 1

INTRODUCTION

Game theory is the study of strategic interactions among game players. It determines the outcomes of games if players behave in a rational manner, usually interpreted as acting selfishly. In reality, however, games may not give the results predicted by current game theory. The terms “selfish” and “rational” are therefore subject to various interpretations. Hence, game theory can be considered to describe, but not necessarily predict or prescribe, human behavior.

In general, each player evaluates the outcomes of the game resulting from his and the other players’ actions, then chooses the actions that will give him a “best” reward according to his personal interpretation of “best.” When a player picks only one of his choices of actions as his strategy, he is playing a pure strategy. When a player randomly picks one of his choices of actions according to a selected probability distribution, he is playing a mixed strategy. A game could be discussed either as a one-time game or as a multiple game. A multiple game is also called a dynamic or a repeated game.

A solution to a game is based a consideration of the payoffs that all players achieve as a consequence of their individual strategies. In this research we present an alternative solution to the Nash equilibrium (NE), currently regarded as the most fundamental solution concept in noncooperative, if not all, game theory.

1.1 Description of the Problem

The NE solution to a noncooperative game requires that no player can improve his utility by unilaterally changing his strategy. However, the notion of an NE has several weaknesses as exemplified by the paradoxical game of Prisoner's Dilemma (PD) in which the unique NE is dominated by the cooperative outcome. This cooperative outcome in a noncooperative game gives a better payoff to both players than does the NE, but is often viewed as the result of irrational behavior for a one-time game.

Moreover, there are often multiple pure or mixed NE's in a game. An NE solution assumes that all players select their strategies according to the same NE for the equilibrium property to hold. It remains unresolved as to which NE strategy pair two rational players would rationally select. In other words, the players could play their NE's strategies from different NE's resulting in a strategy selection that may not yield an equilibrium. To address this problem, various refinements of an NE have been proposed in which certain NE's are eliminated from consideration. An example is the notion of a perfect equilibrium, among others, that requires further properties of an NE. However, such refinements still require that a solution be an NE.

Conceptually different solution concepts such as Correlated Equilibria (CE's) and Non-Myopic Equilibria (NME's) have also been to alleviate the difficulties associated with NE's. But CE's and NME's have their own drawbacks. For example, a CE requires an external random process to obtain joint probability or a learning behavior. And finding an NME with large payoff matrices is complicated since all scenarios must be defined by each possible initial state and all players' moves must be analyzed.

Finally, noncooperative games are probably the most type of games because of the so-called "Nash program." This current trend in game theory attempts to eliminate the distinction between cooperative and noncooperative games. Cooperative games are essentially those in which agreements between players can be enforced, where in noncooperative games only the equilibria are sustainable. John Nash of "Beautiful Mind" fame took the initial steps of including any relevant enforcement mechanisms in the model itself of the game in his study of bargaining. Hence, the Nash program is to model all games as noncooperative games. The problem is that an NE is based on self-interest, without cooperative aspects. A complementary solution is thus needed.

1.2 Objective of the Research

Our objective is to develop alternative solution concepts to the NE that

1. Explain human behavior not amenable to the Nash program,
2. Resolve some weaknesses of previous solutions,
3. Explain certain classical paradoxes of games modeling social dilemmas.

1.3 Related Work

A systematic theory of games was initially presented in 1944 by John von Neuman and Oskar Morgenstern with an emphasis on describing economic behavior. They defined two-person zero-sum games, discussed cooperation and coalition, and proved the existence of the Minimax Theorem. In 1950, John Nash defined the concept of a Nash Equilibrium (NE), which extended the von Neuman's Minimax Theorem to cover N-person, nonzero-sum games. For this discovery Nash shared the 1994 Nobel Prize in Economics with Reinhard Selten and John Hasanyi.

In 1974, Aumann defined the concept of correlated equilibrium (CE), an equilibrium in noncooperative games based on different probabilities than a mixed NE. Aumann's greatest contribution was in the area of repeated games. In 1979 Kahneman and Tversky developed Prospect Theory as a psychologically realistic alternative to expected utility theory. They also empirically studied human decision making and isolated many common errors committed in the decision process. Kahneman and Aumann were awarded a Nobel Prize in Economics in 2002 and 2004, respectively.

In 1994 Brams introduced a Non-Myopic Equilibrium (NME), different from the NE, as a result of his Theory of Moves (TOM) for noncooperative games. In 2002 Montague and Berns first published how people make decisions as revealed by medical monitoring of the human brain. In doing so, they opened field of Neuroeconomics. Despite such post-Nash advances, however, the main thrust of modern game theory remains the Nash program.

1.4 Overview of the Dissertation

The Nash Equilibrium selects choosing a player's best response against the unspecified strategies of his opponents. An NE is characterized by minimizing each individual player's regret or expected regret with regard to the other players' strategies. We refer to an NE as a Regret Equilibrium (RE) and present a dual equilibrium to the NE that based on the notion of disappointment, where disappointing oneself becomes the regret for an NE.

Our new equilibrium is called a Disappointment Equilibrium (DE). It selects a player's best strategy based on the disappointment that the responses of his opponents would cause him for each of his strategies. A DE provides an equilibrium such that for every player i, any or all players except i can change strategy and possibly decrease i's payoff, while certainly never making i's payoff better. Remarkably, the same DE has this property for every fixed player i. A DE thereby enforces equilibrium with an implicit cooperative property based on the possible loss and certain non-improvement of payoff that any player might incur from some opponent's change of strategy. As a special case, in a DE no player can unilaterally change his strategy to increase any opponent's payoff but may be able to reduce it.

We also present a solution called Pareto Intercession Equilibrium (PI Equilibrium or simply π), where "intercession" refers to an intervening between parties to reconcile differences. A π provides a compromise between the RE and DE solution criteria. The notions of a DE and π resolve some important paradoxes in game theory. In such social dilemmas as Prisoner's Dilemma, Chicken Game, and Stag Hunt, the NE and other solution concepts have not proven satisfactory. However, DE may not only improve the outcomes of games but also alleviate conflict among players.

In Chapter 2 we give basic terminology, further literature, and the classical paradoxes in game theory. Details of current game solution concepts are presented in Chapter 3. Then in Chapter 4 we explain the conversion of a payoff bimatrix for two players to a regret bimatrix, its conversion to a disappointment bimatrix, the relationship between the regret and disappointment bimatrices to NE's and DE's, respectively, and the notions of regret and disappointment dominant strategies.

In Chapter 5 we develop the RE, DE, and π for two-person games. We also discuss various bimatrix games and social dilemmas. In Chapter 6 we generalize these results to the RE, Marginal DE, and Total DE for N-person games. We also present a method for their calculation. Finally, in Chapter 7 we summarize our work and discuss future research.

CHAPTER 2

PRELIMINARIES

2.1 Terminology

The basics of game theory are summarized in [1], [2], and [3]. In particular**,** a **game** is a situation in which the outcome of the game is determined by the action of more than one player, where a **Player** is a participating agent who incurs the outcome of the game. Games can be can be classified by as either zero-sum or nonzero-sum. A **zero-sum game** is one in which the players' payoffs sum to zero. In a two-person zero-sum game, one player's gain equals to the other player's loss. In a **nonzero-sum game** the players' payoffs do not sum to zero**.** In the two-person case both players could possibly profit from their respective strategies.

Games can be classified into three categories: a person playing against nature (which some people do not call a game), a two-person game, or an n–person game. In a **game against nature**, nature is considered as a player whose strategy is independent of other players but may still be rational as a natural phenomenon obeying certain principles. The probabilities of all nature's strategies may be unknown (a game against nature under uncertainty) or known (a game against nature under risk). An example might involve an investment portfolio. The investor is a game player who will choose investments that will give them the best return under the uncertainty or risk of the financial markets. **Two-person game** theory deals with the strategic choices of the two players. **N–person game** theory involves more than two players with possible coalitions among players, with the distribution of rewards to all n players. Game players (except nature) are assumed to have the ability to evaluate the outcomes of the game. Each Player is assumed to be rational in some self–serving way and to choose a strategy for achieving a preferred outcome. Thus the player has some insight into each other's behavior. But game theory is not normative in that it does not prescribe a player's strategy. Neither is it predictive in the sense that a player can predict the strategies of the other players. Game theory is simply a discription of the player's behaviors, generally valid in the long run.

A **cooperative game** is one in which players can make binding and enforceable agreements. A **noncooperative game** may or may not allow for communication among the players. A solution for a noncooperative game is usually taken to be an equilibrium in which payoffs cannot be improved by an appropriate player or players changing strategies. Hence there would be not reason to change.

A **Pareto improvement** is a movement from one allocation of benefits to another among a group of individuals for a given set of alternative allocations. It improves at least one individual's benefit, without reducing any other individual's benefit. An allocation of resources is **Pareto efficient**, or **Pareto optimal**, when no further Pareto improvements can be made. Multiple Pareto optima are possible.

Perfect Information in game theory refers to situation where each player knows what has transpired in a game to the point where the player must take action. If not, the game is one of imperfect information game.

Rationality is the characteristic of a player acting according to his objective in the game to achieve a "best" reward as defined by the player. Rationality is embodied in personal behavioral dispositions resulting from natural, cultural, idiosyncratic, or economic factors. Therefore rationality in decision making can only be defined relative to a person's decision criteria, whatever their origin. Rationality as used here has no absolute definition except that consistency in decision making is required. Generally one player would consider the other player rational if both use the same criteria in making game theoretic decisions and irrational perhaps otherwise.

A **strategy** is an inclusive plan of action of a player for any situation that might occur during the game. A player's strategy determines the action to be taken at any stage of the game. Strategies are predetermined. The outcomes of a game are computed corresponding to the players' strategies. A player uses a **pure strategy** when he chooses precisely one action as his strategy. A player uses a **mixed strategy** when he has more than one action as his strategy. In a mixed strategy, the players assign probabilities to their possible actions. Ultimately a player must make a choice, however, and randomly selects a pure strategy according to given by his mixed strategy.

A **solution concept** is a process by which strategies of all players, with the associated payoffs, or rewards, are identified, though not enforced.

A **utility** is a numerical measure of satisfaction gained from consuming goods and services. Economists distinguish between cardinal utility and ordinal utility. In cardinal utility the relative magnitude of the number distinguishes the degree of satisfaction. Ordinal utility, on the other hand, captures only the rank and not strength of preferences.

The **expected utility hypothesis** is the assumption in economics that the utility of an agent facing uncertainty is calculated by evaluating the utility of each possible unknown state and obtaining a weighted average of these utilities. The weights are the agent's estimate of the probability of each state. The expected utility is thus an expectation in terms of probability theory.

2.2 Literature Review

We now discuss some previous relevant work in game theory. In [4] von Neuman and Morgenstern proved the Minimax Theorem (MT). The minimax model (actually a maximin model with respect to benefits despite conventional terminology) maximizes the minimum gain of a player regardless of what the other player does. To select a pure strategy, each player chooses an action by determining the worst possible result of any of his actions for the various possible actions of his opponent, then selects an action yielding the best of these worst results. The MT applies to two-person zero-sum games in which the payoffs are usually given in terms one player's gain for a strategy pair - the negative of the other's loss. So the best outcome for both players in the minimax model is a "best of worst" payoff, i.e., a conservative value. If any player does not select his minimax strategy, his payoff could be worse. Unfortunately if both players play such a minimax strategy, one can often improve his payoff with a unilateral change in his pure strategy. The MT guarantees that there exist mixed strategies for each player for which no improvement in his expected payoff by a unilateral change in his mixed strategy.

For the noncooperative games studied here, Nash [5] proposed the concept of the Nash Equilibrium (NE) extending von Neumann's MT to cover nonzero-sum games, which for two players are call bimatrix since each player's payoff is given by different payoff matrices not negatives of each other. Aumann [6] defined the concept of correlated equilibrium (CE) in noncooperative game theory, which is more flexible than the NE. A CE involves a joint probability distribution combined from all players' actions, while NE involves probability distributions that each player assigns to his own actions.

Computationally, it is unknown whether either CE's or mixed NE's can be found in polynomial time. Nau, Canovas, and Hansen [7] studied the relation of the NE and CE, concluding that all NE's lie on the boundary of the CE convex polytope. Kar, Ray, and Serrano [8] noted that a CE is difficult to determine and does not satisfy Maskin monotonicity. To find NE's, Lemke and Howson [9] developed a linear complementarity problem (LCP) for two-person games, while Porter, Nudelman, and Shoham [10] developed a constraint programming method. Other approaches include that of Sandholm, Gilpin, and Conitzer [11], who developed a mixed integer programming approach. Raghavan [12] summarized other equilibria such as perfect equilibria [13] [14] that are the refinement of NE's, quasi-strict equilibria [15] [16], and regular equilibria [14] [17].

Brams [18] introduced the concept called a Non-Myopic Equilibrium (NME) obtained by his Theory of Moves (TOM) for noncooperative games. The NME is resulted from players looking ahead and making rational calculations of where, from each of initial states, the move-countermove process will end. Brams stated that it is rational when the player picks the next best choice, among his own outcomes to avoid the game going to a non-Pareto optimum. Ghosh and Sen [19] presented a learning approach by which TOM players can learn to converge to the NME without prior knowledge of its opponent's preferences. In addition, Brams [20] discussed the conflict of solutions to Prisoner's Dilemma under the expected utility principle and the dominance principle.

Bimatrix, noncooperative, ordinal games have been classified by various researchers. Kilgour and Fraser [21] described a practical taxonomy of all the 726 ordinal 2×2 games. Rapaport and Guyer [22] presented the 78 distinctive 2×2 games with such that no single outcome has the same ordinal preference to either player. Finally, Poundstone [23] identified 4 distinct social dilemmas of 2×2 games, which we will later analyze.

2.3 Classical Paradoxes in Game Theory

2.3.1 Prisoner's Dilemma

The **Prisoner's Dilemma** (PD) is a two-person nonzero-sum game where players have the option of cooperating with the other player or defecting. If one player defects and the other cooperate, the defector receives more reward than when they both cooperate and the cooperator receives less reward than when they both defect. While the rewards of cooperating are more than that of defecting, the NE results in both players defecting to avoid a possible loss from being cheated. The Cold War military strategy, a real-life example of PD, presented this issue. The NE of the Cold War game was to mutually ensure destruction with a pre-emptive strike strategy by both the United States and the Soviet Union. And, indeed, each country spent a tremendous amount of money and effort on nuclear arms as a threat to a strike by the other.

The payoff matrix in Figure 2.1 shows an example of a PD game. It represents the situation giving this model the name "Prisoner's Dilemma." Assume two criminals committed a crime together. After arresting them with a lack of sufficient evidence for a maximum sentence, the police separate them without any communication with the other, then offer each less jail time if he confesses. Each prisoner's dilemma lies in the decision whether to defect (confess) or cooperate (not confess). Neither prisoner knows the other's decision. If both defect by confessing, they will be jailed 3 years. If they cooperate, they will be jailed 1 year. If one confesses and the other does not, the one who confesses will be free and the other will be jailed 7 years. The dominant strategy of this game is 'Defect,' which is an NE, yet both do better if they cooperate. This fact represents the most famous paradox in game theory.

Player *II*

		Defect	*Cooperate*
Player *I*	*Defect*	(–3,–3)	(0,–7)
	Cooperate	(–7,0)	(–1,–1)

Figure 2.1 Prisoner's Dilemma

2.3.2 Chicken Game

The **Chicken game** is a two-person game in which two players engage in an activity that will result in destruction unless one of them backs out. Each of the two parties care either *dare* (*D*) or *chicken out (C)*. If one dares, it is better for the other to chicken out. But if one chickens out, it is better for the other to dare. The result is a situation where each wants to dare, but only if the other chickens out. Consider the following Chicken game payoff matrix, Figure 2.2. There are two pure NE strategies: (*D*, *C*) and (*C*, *D*). A mixed NE strategy occurs when each player dares with probability 1/3 and earns expected payoff of

$$(1/3)\times(1/3)\times 0 + (2/3)\times(1/3)\times 2 + (1/3)\times(2/3)\times 7 + (2/3)\times(2/3)\times 6 = 42/9 = 4.667.$$

Obviously in this game when the players play the pure NE's (*D*, *C*) or (*C*, *D*), one does much worse than he does in (*C*, *C*) and one does only slightly better. Moreover, for the mixed NE, neither on the average can gain more than when they play (*C*, *C*) all the time. The fact that (*C*, *C*) seems the "sensible" strategy pair in both the pure and mixed NE cases has been not satisfactorily resolved.

Player *II*

Player *I*		*Dare*	*Chicken out*
	Dare	(0,0)	(7,2)
	Chicken out	(2,7)	(6,6)

Figure 2.2 Chicken Game

2.3.3 Stag Hunt

The **Stag Hunt** is a two-person game which describes a conflict between social cooperation and self assurance. Its other names are "assurance game," "coordination game," and "trust dilemma." Consider the situation where two individuals go on a hunt. Each can individually choose to hunt a stag or to hunt a hare. Each player must choose an action without knowing the choice of the other. If an individual hunts a stag, he must have the cooperation of his partner in order to succeed. An individual can hunt a hare by himself, but a hare is worth less than a stag. This game is an important analogy for social cooperation. Figure 2.3 represents an example of the payoff matrix for Stag Hunt. Obviously each player receives a better utility when both hunt stag than when both hunt hare. Yet strategy pairs are pure NE's.

Player *II*

Player *I*		*Stag*	*Hare*
	Stag	(4,4)	(0,3)
	Hare	(3,0)	(3,3)

Figure 2.3 Stag Hunt

2.3.4 Battle of Sexes

The **Battle of Sexes** is a two-person game where noncooperation does not give any player a higher reward. Imagine a couple, Alice and Bob. Alice wants to go to the opera. Bob wants to go to the football game. But both would enjoy being with each other more than attend his preferred event. Consider their payoff matrix in Figure 2.4. This game has two pure NE's, where both go to the opera and where both go to the football game. A mixed NE also exists in which Alice and Bob each play his/her favorite strategy with probability 5/6 and the other strategy with probability 1/6, earning each an expected utility of 5/6. The mixed NE gives both Alice and Bob a worse utility than the two cooperative pure NE's.

		Bob	
		Opera	*Football*
Alice	*Opera*	(5,1)	(0,0)
	Football	(0,0)	(1,5)

Figure 2.4 Battle of Sexes

CHAPTER 3

FUNDAMENTAL OF NONCOOPERATIVE GAME THEORY

3.1 Representation of Games

Games can be represented either in a normal form (also called strategic form) or extensive form. A normal form or strategic form game is a matrix that shows the players, their strategies, and payoffs as seen below in Section 3.2. An extensive form game is presented as a tree that shows sequence of players' decisions over their possible choice of actions. A node or vertex of a tree represents a point where a player making his decisions. The lines out of the vertex represent each possible action of the player. The bottom of the tree is placed with the payoffs associates with each sequence of action. An example of an extensive form of the Chicken game is shown in Figure 3.1.

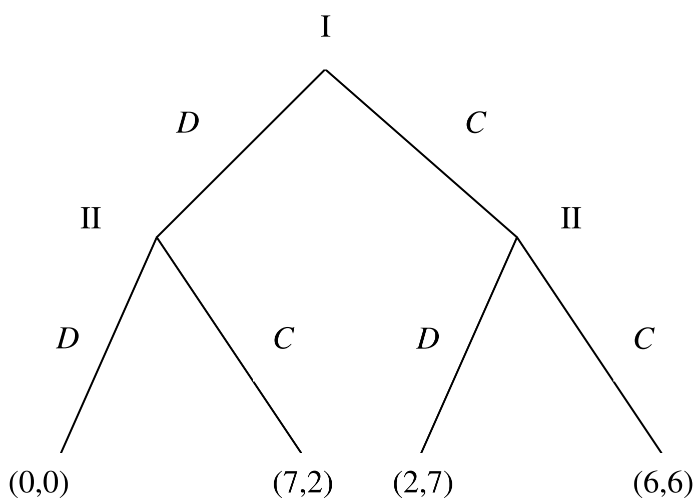

Figure 3.1 An Extensive Form of a Chicken Game

In this dissertation we focus on normal form games. The normal form of two-person games is described in the next section.

3.2 A Normal Form for Two-Person Games

The following notation and definitions are used to represent two-person normal form games and to describe such fundamental concepts as the Minimax Theorem (MT), the Nash Equilibrium (NE), the Correlated Equilibria (CE), and the Theory of Moves (TOM). Additional notation will be stated in each section as needed. Here, let

- a two-person normal form game be represented as (S_r, T_c, Q_r, Q_c) for the two players be designated as player I (row player), and Player II (column player).
- S_r denote the set of pure strategies of Player I, where $S_r \geq 2$.
- T_c denote the set of pure strategies of Player II, where $T_c \geq 2$.
- Q_r denote the set of payoffs or utilities of Player I.
- Q_c denote the set of payoffs or utilities of Player II.
- Player I has m predetermined pure strategies, with $S_r = \{s_r \mid r = 1,2,\ldots, m\}$.
- Player II has n predetermined pure strategies, with $T_c = \{t_c \mid c = 1,2,\ldots, n\}$.
- a_{rc} denote player I's payoff or utility and b_{rc} denote Player II's payoff or utility when player I plays s_r strategy and Player II plays t_c strategy.
- Player I's $m \times n$ payoff matrix be $A = [a_{rc}]$, $a_{rc} \in Q_r$.
- Player II's $m \times n$ payoff matrix be $B = [b_{rc}]$, $b_{rc} \in Q_c$.
- the mixed extension of an $m \times n$ bimatrix game be represented as (X, Y, P_r, P_c).

– X be the set of possible mixed strategies, or probability vectors for player I, where $X = \{\boldsymbol{x} \mid \boldsymbol{x} = [x_1, x_2, \ldots, x_m]^T \in \boldsymbol{R}^m$ and x_r , $r = 1, \ldots, m,$ is the probability that s_r is chosen so that $\sum_{r=1}^{m} x_r = 1$ and all $x_r \geq 0\}$.

– Y be the set of possible mixed strategies, or probabilities for Player II, where $Y = \{\boldsymbol{y} \mid \boldsymbol{y} = [y_1, y_2, \ldots, y_n]^T \in \boldsymbol{R}^n$ and y_c, $c = 1, \ldots, n,$ is the probability that t_c is chosen so that $\sum_{c=1}^{n} y_c = 1$ and all $y_c \geq 0\}$.

– P_r be the set of possible payoff or utility for player I, with

$P_r = \{ p_r(\boldsymbol{x},\boldsymbol{y}) \mid p_r(\boldsymbol{x},\boldsymbol{y}) = \sum_{r=1}^{m} \sum_{c=1}^{n} a_{rc} x_r y_c = \boldsymbol{x}^T A \boldsymbol{y} \}$.

– P_c be the set of possible payoff or utility for Player II, with

$P_c = \{ p_c(\boldsymbol{x},\boldsymbol{y}) \mid p_c(\boldsymbol{x},\boldsymbol{y}) = \sum_{r=1}^{m} \sum_{c=1}^{n} b_{rc} x_r y_c = \boldsymbol{x}^T B \boldsymbol{y} \}$. t

– $p(\boldsymbol{x},\boldsymbol{y})$ be a possible payoff or utility pair of player I and Player II, with

$P(\boldsymbol{x},\boldsymbol{y}) = (p_r(\boldsymbol{x},\boldsymbol{y}), p_c(\boldsymbol{x},\boldsymbol{y}))$.

For two-person games the payoff matrices A, B, as well as the bimatrix (A,B) are shown in Figure 3.2, 3.3 and 3.4, respectively.

Player *II*

Player *I*		t_1	t_2	...	t_n
	s_1	a_{11}	a_{12}	...	a_{1n}
	s_2	a_{21}	a_{22}	...	a_{2n}
	$\vdots$	$\vdots$	$\vdots$		$\vdots$
	s_m	a_{m1}	a_{m2}	...	a_{mn}

Figure 3.2 Player *I*'s Payoff Matrix *A*

		Player *II*			
		t_1	t_2	$\dots$	t_n
	s_1	b_{11}	b_{12}	$\dots$	b_{1n}
Player *I*	s_2	b_{21}	b_{22}	$\dots$	b_{2n}
	$\vdots$	$\vdots$	$\vdots$		$\vdots$
	s_m	b_{m1}	b_{m2}	$\dots$	b_{mn}

Figure 3.3 Player *II*'s Payoff Matrix *B*

		Player *II*			
		t_1	t_2	$\dots$	t_n
	s_1	(a_{11}, b_{11})	(a_{12}, b_{12})	$\dots$	(a_{1n}, b_{1n})
Player *I*	s_2	(a_{21}, b_{21})	(a_{22}, b_{12})	$\dots$	(a_{2n}, b_{2n})
	$\vdots$	$\vdots$	$\vdots$		$\vdots$
	s_m	(a_{m1}, b_{m1})	(a_{m2}, b_{m2})	$\dots$	(a_{mn}, b_{mn})

Figure 3.4 A Two-person Nonzero-sum Bimatrix Game

3.3 The Minimax Theorem and the Nash Equilbrium

3.3.1 Two-person Zero-sum Game Pure Strategies

In a two-person zero-sum game there is a (maximin, minimax) pair such that player *I* wins at least $\min_c a_{rc}$ by choosing $s_{c*} \in S_c$ such that

$$\min_c a_{r*c} = \max_r \min_c a_{rc} = \blacktriangledown(A,B) = \text{the lower value.}$$

Player *II* wins at least $\min_r b_{rc}$ by choosing $t_{c*} \in T_c$ such that

$$\min_r b_{rc*} = \max_c \min_r b_{rc} .$$

Since $B = -A$, then

$$\min_r \; b_{rc^*} = \min_r - a_{r^*c} = - \min_r a_{rc^*}$$

$$\min_r \; a_{rc^*} = \max_c \min_r \; a_{rc} = \blacktriangle(A,B) = \text{the upper value.}$$

It is always true a two person zero-sum game that $\blacktriangledown(A,B) \leq \blacktriangle(A,B)$, and so such games can be partitioned into two mutually exclusive classes as follows:

Type I: games with $\blacktriangledown(A,B) = \blacktriangle(A,B)$ in which case a (maximin, minimax) pair (s_{r^*}, t_{c^*}) is a pure NE;

Type II: games with $\blacktriangledown(A,B) < \blacktriangle(A,B)$ and there is no pure NE.

3.3.2 Two-person Zero-sum Game Mixed Strategies

The Minimax Theorem states that for every finite two-person zero-sum game, there exist mixed strategies $\boldsymbol{x}$ for Player I and $\boldsymbol{y}$ for Player II such that the payoff of player I, $p_r(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^T A\boldsymbol{y}$ and the payoff of Player II, $p_c(\boldsymbol{x}, \boldsymbol{y}) = -\boldsymbol{x}^T A\boldsymbol{y}$ satisfy

$$\max_{\mathrm{x}} \min_{\mathrm{y}} \; \boldsymbol{x}^T A\boldsymbol{y} = \min_{\mathrm{y}} \max_{\mathrm{x}} \; \boldsymbol{x}^T A\boldsymbol{y} = v.$$

The number v is called the value of the game. It should be noted that a pure strategy is a special case of a mixed strategy with exactly one nonzero x_r and exactly one nonzero y_c. If there is more than one such strategy pair, there are infinitely many.

3.3.3 Two-person Nonzero-sum Game Pure Strategies

In the two-person nonzero-sum game, a strategy pair (s_{r^*}, t_{c^*}), is an NE, if

$$a_{r^*c^*} \geq a_{rc^*}, \; \forall \; s_r \in S_r \text{ and } b_{r^*c^*} \geq b_{r^*c}, \forall \; t_c \in T_c .$$

A maximin criterion gives a conservative strategy guaranteeing the best of the worst possible outcomes. If Player *I* plays s_r and Player *II* plays t_c so that $\max_r \min_c a_{rc} = l_r$ and $\max_c \min_r b_{rc} = l_c$, then l_r is called the security level of player *I* and l_c the security level of Player *II*.

3.3.4 Two-person Nonzero-sum Game Mixed Strategies

In a two-person nonzero sum game, a strategy pair $(\boldsymbol{x}^*, \boldsymbol{y}^*) \in X \times Y$ is an NE if

$$p_r(\boldsymbol{x}^*, \boldsymbol{y}^*) \geq p_r(\boldsymbol{x}, \boldsymbol{y}^*),\ \forall\, \boldsymbol{x} \in X, \text{ and } p_c(\boldsymbol{x}^*, \boldsymbol{y}^*) \geq p_c(\boldsymbol{x}^*, \boldsymbol{y}),\ \forall\, \boldsymbol{y} \in Y.$$

The expected maximin value of Player *I*'s mixed strategies is $\max_x \min_y p_r(\boldsymbol{x},\boldsymbol{y})$, and the expected maximin value of Player *II* mixed strategy $\boldsymbol{y}$ is $\max_y \min_x p_c(\boldsymbol{x},\boldsymbol{y})$.

3.4 Correlated Equilibria

A Correlated Equilibrium (CE) is an extension of an NE. A strategy profile is chosen according to the probability distribution of joint strategies. If no player gains by deviating from the recommended strategy, the distribution is called a CE. To maintain a correlated equilibrium, each player will know only part of their move due to the probability distribution constraints.

As further notation, let

- G denote a finite noncooperative game.
- $\boldsymbol{S} = S_1 \times S_2 \times \ldots \times S_n$ denote the set of all joint strategies of G.
- $\boldsymbol{s}$ denote a joint strategy of all players, where $\boldsymbol{s} = (s_1, s_2, \ldots, s_n) \in \boldsymbol{S}$.

– p_k ($\boldsymbol{s}$) denote the payoff (utility) to Player k when he chooses strategy $\boldsymbol{s}$.

– $\underline{p}_k$ ($\boldsymbol{d}_k$, $\boldsymbol{s}_{-k}$) denote the payoff (utility) to Player k when he chooses strategy $\boldsymbol{d}$ and the others choose $\boldsymbol{s}$.

The game G is said to be nontrivial if $p_k(\boldsymbol{s}) \neq \underline{p}_k(\boldsymbol{d}_k, \boldsymbol{s}_{-k})$ for some player k, some $\boldsymbol{s} \in \boldsymbol{S}$, and some $\boldsymbol{d} \in \boldsymbol{S}$. A correlated equilibrium distribution of G is a vector $\boldsymbol{w}$ in R^n satisfying the following linear constraints [6]:

$$\boldsymbol{w}(\boldsymbol{s}) \geq 0 \text{ for all } \boldsymbol{s} \in \boldsymbol{S} \textit{ and } \sum_{s \in S} \boldsymbol{w}(\boldsymbol{s}) = 1$$

$$\sum_{s_{-k} \in S_{-k}} \boldsymbol{w}(\boldsymbol{s})\,(p_k(\boldsymbol{s}) - \underline{p}_k(d_k, \boldsymbol{s}_{-k})) \geq 0 \text{ for all k and }, \boldsymbol{s}_{-k} \in \boldsymbol{S}_{-k},\ \boldsymbol{d}_k \in \boldsymbol{S}_k.$$

In the Chicken game of Figure 2.2, let $\boldsymbol{w} = \{w_1, w_2, w_3, w_4\}$ (left to right, top to bottom). We solve

$w_1(0–2) + w_2(7–6) \geq 0$ and $w_3(2–0) + w_4(6–7) \geq 0$ for $k = 1$,

$W_1(0–2) + w_3(7–6) \geq 0$ and $w_2(2–0) + w_4(6–7) \geq 0$ for $k = 2$,

$W_1 + w_2 + w_3 + w_4 = 1$,

$W_n \geq 0$, $n = 1,2,3,4$.

Then the CE lies on the polytope of the linear constraints

$–2w_1 + w_2 \geq 0$, $2w_3 – w_4 \geq 0$, $–2w_1 + w_3 \geq 0$, $2w_2 – w_4 \geq 0$,

$w_1 + w_2 + 3w_3 –1 \geq 0$, $w_1 + 3w_2 + w_3 –1 \geq 0$, $w_1 + w_2 + w_3 \leq 1$.

Consider a third party that draws one of three cards labeled (C, C), (D, C), and (C, D). After drawing the card, the third party informs each player of the individual strategies assigned to him on the card but not the strategy assigned to his opponent. Suppose a player is assigned D. Then he would not want to deviate from D if he assumed the other player played his assigned strategy since he would get 7, the possible highest payoff. Suppose a Player is assigned *C*. Then the other player will play *C* with probability 1/2 and *D* with probability 1/2. The expected utility of daring is 0(1/2) + 7(1/2) = 3.5, and the expected utility of chickening out is 2(1/2) + 6(1/2) = 4. So the player would prefer to chicken out. Since neither player has an incentive to deviate, this point is a correlated equilibrium. Its expected payoff is 7(1/3) + 2(1/3) + 6(1/3) = 5.

3.5 Theory of Moves

A game in Theory of Moves (TOM) is analyzed by dynamic moves unlike classical non-repeated game theory, which is considered as one-time game. TOM analyzes how games go forward as each player responds to strategies used by the other. The players will move from an initial state to the next state by the TOM rules [10]. A player will not move from his current state if the move (i) leads to a less preferred final state or (ii) returns the play to the initial state.

Consider the following example from [10] of a two-person nonzero-sum game to be solved by the TOM. Then note the dynamic nature of TOM from each initial state.

Player *II*

		t_1	t_2
Player *I*	s_1	(2,4)	(4,2)
	s_2	(1,1)	(3,3)

Figure 3.5 Bram's TOM Example

<u>(2,4) is the initial state</u>

<table>
<tr><td></td><td>State 1</td><td>State 2</td><td>State 3</td><td>State 4</td><td></td><td>I</td><td>II</td><td>I</td><td>II</td></tr>
<tr><td>Player I starts:</td><td>(2,4)</td><td>→</td><td>(1,1)</td><td>→</td><td>(3,3)</td><td>→ |</td><td>(4,2)</td><td>→</td><td>(2,4)</td></tr>
<tr><td>Survivor:</td><td>(3,3)</td><td>(3,3)</td><td>(3,3)</td><td>(2,4)</td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

The survivor is determined by working backward, after a putative cycle has been completed.
→ indicates the move from one state to the next. → | shows the blockage for a player's move.

According to Bram, Player *I* has motivation to move from (2,4) to (1,1) by looking ahead that he could reach (3,3) since Player *II* would prefer (3,3) more than (1,1). At State 3 (3,3) player *I* will not move to (4,2) knowing that he will induce Player *II* to move to (2,4).

<table>
<tr><td></td><td>State 1</td><td>State 2</td><td>State 3</td><td>State 4</td><td></td><td>II</td><td>I</td><td>II</td><td>I</td><td></td></tr>
<tr><td>Player II starts:</td><td></td><td>(2,4)</td><td>→ |</td><td>(4,2)</td><td>→ |</td><td>(3,3)</td><td>→ |</td><td>(1,1)</td><td>→</td><td>(2,4)</td></tr>
<tr><td>Survivor:</td><td></td><td>(2,4)</td><td>(4,2)</td><td>(2,4)</td><td>(2,4)</td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

State (2,4) is the first blockage since Player *II* prefers it over any other outcomes and will decide to stay. One of Bram's rules is that a player's motivation to move takes precedence over and override another player's decision to stay. Therefore, when the initial states is (2,4), the outcome is (3,3).

<u>(4,2) is the initial state</u>

<table>
<tr><td></td><td>State 1</td><td>State 2</td><td>State 3</td><td>State 4</td><td></td><td>I</td><td>II</td><td>I</td><td>II</td><td></td></tr>
<tr><td>Player I starts:</td><td></td><td>(4,2)</td><td>→ |</td><td>(3,3)</td><td>→</td><td>(1,1)</td><td>→</td><td>(2,4)</td><td>→ |</td><td>(4,2)</td></tr>
<tr><td>Survivor:</td><td></td><td>(4,2)</td><td>(2,4)</td><td>(2,4)</td><td>(2,4)</td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

<table>
<tr><td></td><td>State 1</td><td>State 2</td><td>State 3</td><td>State 4</td><td></td><td>II</td><td>I</td><td>II</td><td>I</td><td></td></tr>
<tr><td>Player II starts:</td><td></td><td>(4,2)</td><td>→ |c</td><td>(2,4)</td><td>→</td><td>(1,1)</td><td>→</td><td>(3,3)</td><td>→</td><td>(4,2)</td></tr>
<tr><td>Survivor:</td><td></td><td>(4,2)</td><td>(4,2)</td><td>(4,2)</td><td>(4,2)</td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

→ | c shows the blockage occurred from player moving to the initial state creating a cycle.

State (4,2) is the first blockage since Player *I* prefer it over any other of his outcomes. Player *II* prefers to stay since his move will only bring them back to the initial state. Hence an initial state of (4,2) yields an outcome of (4,2).

<u>(3,3) is the initial state:</u>

	State 1	State 2	State 3	State 4		*I*	*II*	*I*	*II*	
Player *I* starts:		(3,3)	→ |	(4,2)	→	(2,4)	→	(1,1)→	(3,3)	
Survivor:		(3,3)	(3,3)	(3,3)	(3,3)					

	State 1	State 2	State 3	State 4		*II*	*I*	*II*	*I*	
Player *II* starts:		(3,3)	→	(1,1)	→	(2,4)	→ |	(4,2)	→ |	(3,3)
Survivor:		(2,4)	(2,4)	(2,4)	(4,2)					

When the initial state is (3,3), the outcome is (2,4). Although Player *I* prefers to stay at (3,3), Player *II* has motivation to move.

<u>(1,1) is the initial state:</u>

	State 1	State 2	State 3	State 4		*I*	*II*	*I*	*II*	
Player *I* starts:		(1,1)	→	(2,4)	→ |	(4,2)	→ |	(3,3)	→ |	(1,1)
Survivor:		(2,4)	(2,4)	(4,2)	(3,3)					

	State 1	State 2	State 3	State 4		*II*	*I*	*II*	*I*	
Player *II* starts:		(1,1)	→	(3,3)	→ |	(4,2)	→	(2,4)	→	(1,1)
Survivor:		(3,3)	(3,3)	(2,4)	(2,4)					

When the initial states is (1, 1), Player *I* would want to move to (2,4) and Player *II* would want to move to (3,3). The outcome is either (2,4) or (3,3) depending on which player moves first.

The outcomes of the moves described above represent Non-Myopic Equilibria (NME). TOM gives three NME in this game, (3,3), (2,4) and (4,2), depending on the initial states and the player who makes the first move.

In the next chapter we explain the Regret Bimatrix, and its relation to the NE. More importantly for our results, we introduce the Disappointment Bimatrix that leads us to our notion of a Disappointment Equilibrium (DE), and a type of dominant strategy called a Disappointment Dominant (DD) strategy. We compare the DD to the standard Regret Dominance (RD) strategies called simply dominant strategies.

CHAPTER 4

REGRET AND DISAPPOINTMENT TRANSFORMATIONS OF NORMAL FORM GAMES

4.1 A Regret Bimatrix

The regret function of any payoff function is a transformation of a player's payoff function for pure strategies to a loss function. In particular, a player's regret function gives the amount he would lose by not choosing his best response to fixed pure strategies of his opponent. For mixed strategies, the regret function has a continuous extension. It will be shown that, in effect, the regret function transforms the players' payoff functions for a game into loss functions with the same NE's. It may be thought of as a utility transformation into the negative of regret value. For bimatrix game, this regret function is completely described by a Regret Bimatrix (RM) obtained from the payoff bimatrix for the players.

In addition to the notation given in Chapter 3, let

- $s_e \in S$ and $t_f \in T$
- $a_{ef} \in A$ and $b_{ef} \in B$
- $R_r(s_e, t_c)$ denote a regret to Player *I* when he plays s_e strategy and Player *II* plays t_c strategy.
- $R_c(s_r, t_f)$ denote a regret to Player *II* when he plays t_f strategy and Player *I* plays s_r strategy.

- $R_r(s_e, t_c) = \max_r \ a_{rc} - a_{ec}$

- $R_c(s_r, t_f) = \max_c \ b_{rc} - b_{rf}$.

4.2 The Relation of an NE to an RM

Recall that for a two-person nonzero-sum game, a pure NE strategy pair (s_{r^*}, t_{c^*}) satisfies $a_{r^*c^*} \geq a_{rc^*}$ for $\forall \ s_r \in S_r$ and $b_{r^*c^*} \geq b_{r^*c}$ for $\forall \ t_c \in T_c$. Therefore $a_{r^*c^*} = \max_r \ a_{rc^*}$ and $b_{r^*c^*} = \max_c \ b_{r^*c}$. It follows that player I's regret at NE is

$$R_r(s_{r^*}, t_{c^*}) = \max_r \ a_{rc^*} - a_{r^*c^*}$$

$$= \max_r \ a_{rc^*} - \max_r \ a_{rc^*} = 0.$$

Similarly Player II's regret at an NE is

$$R_c(s_{r^*}, t_{c^*}) = \max_c \ b_{r^*c} - b_{r^*c^*} = 0$$

$$= \max_c \ b_{r^*c} - \max_c \ b_{r^*c} = 0.$$

Hence Result 4.1 below now follows.

Result 4.1. A strategy pair in a two-person game is a pure NE if and only if this strategy pair yields zero regret for both players in the RM.

Now consider the following Prisoner's Dilemma payoff matrix.

		Player *II*	
		t_1 (*Cooperate*)	t_2 (*Defect*)
Player *I*	s_1 (*Cooperate*)	(3,3)	(0,5)
	s_2 (*Defect*)	(5,0)	(1,1)

Figure 4.1 Prisoner's Dilemma Payoff Matrix

If Player *I* cooperates, Player *II* is tempted to defect and receive his best utility while Player *I* receives his worse utility, and vice versa. If both cooperate, they get the reward for mutual cooperation utility of 3 each, while if they both defect they get the mutual defection utility of 1. The NE for this PD game is where both players defect and receive the utilities (1,1). By the definition of an NE, if either player moves away from NE, he can only do worse. Each player receives the maximum reward by staying at NE strategy if the other player does not move. In other words, each player has zero regret choosing NE strategy. Figure 4.2 shows the RM of this PD game.

		Player *II*	
		t_1 (*Cooperate*)	t_2 (*Defect*)
Player *I*	s_1 (*Cooperate*)	(2,2)	(1,0)
	s_2 (*Defect*)	(0,1)	(0,0)

Figure 4.2 Prisoner's Dilemma RM

The regret calculations for Figure 4.2 are given as follows.

By inspection, $\max_r a_{r1} = a_{21} = 5$ from which

$$R_r(s_1, t_1) = \max_r a_{r1} - a_{11} = 5 - 3 = 2$$

and

$$R_r(s_2, t_1) = \max_r a_{r1} - a_{21} = 5 - 5 = 0.$$

Moreover, $\max_r \ a_{r2} = a_{22} = 1$ from which

$$R_r(s_1, t_2) = \max_r \ a_{r2} - a_{12} = 1 - 0 = 1 \text{ and}$$

$$R_r(s_2, t_2) = \max_r \ a_{r2} - a_{22} = 1 - 1 = 0.$$

Also, $\max_c \ b_{1c} = b_{12} = 5$ from which

$$R_c(s_1, t_1) = \max_c \ b_{1c} - b_{11} = 5 - 3 = 2 \text{ and}$$

$$R_c(s_1, t_2) = \max_c \ b_{1c} - b_{12} = 5 - 5 = 0.$$

Finally, $\max_c \ b_{2c} = b_{22} = 1$ from which

$$R_c(s_2, t_1) = \max_c \ b_{2c} - b_{21} = 1 - 0 = 1 \text{ and}$$

$$R_c(s_2, t_2) = \max_c \ b_{2c} - b_{22} = 1 - 1 = 0.$$

The regret pair $(R_r(s_2, t_2), R_c(s_2, t_2)) = (0,0)$, so the strategy pair (s_2, t_2) is an NE.

Since the minimum possible regret of either player in the RM is zero, the regret pair (0,0) is also a Pareto minimum for the RM.

Result 4.2. A pure NE is a Pareto optimum of the RM.

As a consequence of Results 4.1 and 4.2, which explain the relation of a pure NE to the RM, we shall henceforth refer to an NE as a Regret Equilibrium (RE) to contrast it with a DE.

4.3 A Disappointment Bimatrix

Players also hope to do well in response to the actions of the other players. We thus present the disappointment function, another transformation of a player's payoff function into losses. A player's disappointment function gives the amount he would lose for a fixed pure strategy of the player if his opponents did not choose the pure strategies yielding his maximum payoff. A disappointment function also has a continuous extension for mixed strategies. For bimatrix games this disappointment function is completely described by a Disappointment Bimatrix (DM) obtained from the payoff bimatrix for the players.

In addition to previous notation, let

- $s_g \in S$ and $t_h \in T$.
- $a_{rh} \in A$ and $b_{gc} \in B$.
- $D_r(s_r, t_h)$ denote a disappointment to Player *I* when he plays s_r strategy and Player *II* plays t_h strategy.
- $D_c(s_g, t_c)$ denote a disappointment to Player *II* when he plays t_c strategy and Player *I* plays s_g strategy
- $D_r(s_r, t_h) = \max\limits_{c} a_{rc} - a_{rh}$
- $D_c(s_g, t_c) = \max\limits_{r} b_{rc} - b_{gc}$

The DM of Figure 4.1 Prisoner's Dilemma payoff matrix is shown in Figure 4.3 with the following calculations.

		Player II	
		t_1 (*Cooperate*)	t_2 (*Defect*)
Player I	s_1 (*Cooperate*)	(0,0)	(3,0)
	s_2 (*Defect*)	(0,3)	(4,4)

Figure 4.3 Prisoner's Dilemma DM

Immediately $\max\limits_c \ a_{1c} = a_{12} = 3$ from which

$$D_r(s_1, t_1) = \max_c \ a_{1c} - a_{11} = 3 - 3 = 0 \text{ and}$$

$$D_r(s_1, t_2) = \max_c \ a_{1c} - a_{12} = 3 - 0 = 3.$$

Also $\max\limits_c \ a_{2c} = a_{21} = 5$ from which

$$D_r(s_2, t_1) = \max_c \ a_{2c} - a_{21} = 5 - 5 = 0 \text{ and}$$

$$D_r(s_2, t_2) = \max_c \ a_{2c} - a_{22} = 5 - 1 = 4.$$

Next, $\max\limits_r \ b_{r1} = b_{11} = 3$ from which

$$D_c(s_1, t_1) = \max_r \ b_{r1} - b_{11} = 3 - 3 = 0 \text{ and}$$

$$D_c(s_2, t_1) = \max_r \ b_{r1} - b_{21} = 3 - 0 = 3.$$

Finally, $\max\limits_r \ b_{r2} = b_{12} = 5$ from which

$$D_c(s_1, t_2) = \max_r \ b_{r2} - b_{12} = 5 - 5 = 0 \text{ and}$$

$$D_c(s_2, t_2) = \max_r \ b_{r2} - b_{22} = 5 - 1 = 4.$$

The strategy pair (s_1, t_1) results in zero disappointment to both players. Each player realizes that a unilateral move from this strategy pair lowers his opponent's payoff. Hence, they maintain its equilibrium in the game theory equivalent of a standoff. The idea of each possibly doing worse at the whim of the other player enforces this equilibrium in an enforced cooperation resulting in behavior reminiscent to the dictum of the Golden Rule: "Do unto others as you would have them to unto you." The DM thus provides an explaination why the 'Defect' strategy is often not chosen, especially in repeated games where the players have the opportunity to learn from past experience. The 'Defect' strategy is dominated by the 'Cooperate' strategy in the DM.

4.4 Dominant Strategies

Consider the PD payoff matrix of Figure 4.1. Strategy s_1 in row two and t_1 in column two is dominated by s_2 and t_2 for the row and column players' payoffs respectively. Thus according to a regret criterion, s_2 and t_2 are dominant strategies. We call them Regret Dominant (RD) strategies.

On the other hand, observe that strategy t_2 in column two for the row player's payoffs is dominated by strategy t_1 in column one. Also, strategy s_2 in row two for the column player's payoffs is dominated by strategy s_1 in row one. According to a disappointment criterion, we call strategies s_1 and t_1 Disappointment Dominant (DD) strategies. The column player might not play strategy t_2 since then he would not have power over the row player, and the row player might not play strategy s_2 since then he would not have power over the column player.

In a PD game, the effect of disappointment dominance is more significant than that of regret dominance, yielding (3,3) as the preferred outcome. By playing their s_1

and t_1 strategies, the row and column player have more power to maintain the DD than the RD. In addition, the DD is better. This fact is manifested by adding the regret and disappointment for the two dominated answers to give (2, 2) for s_1 and t_1 and (4, 4) for s_1 and t_1.

However, if the row player reasons by regret and the column player by disappointment we get the (R, D) Matrix as shown in Figure 4.4.

Player *II* (Column Player)

Player *I* (Row Player)		t_1 (*Cooperate*)	t_2 (*Defect*)
	s_1 (*Cooperate*)	(2,0)	(1,0)
	s_2 (*Defect*)	(0,3)	(0,4)

Figure 4.4 Prisoner's Dilemma (R, D) Matrix

With their different utility transformations, by dominance arguments the players should select the strategy pair (s_1, t_2), which is clearly better for Player *II*. The reason for the discrepancy with RE or DE equilibrium strategy pairs is that in effect each players is considered irrational by the other. Their utilities are now different. Hence, they cannot agree on what constitutes an equilibrium. Hence, the assumption of rationality is the assumption that both players consistently employ the same criterion, whatever it is.

CHAPTER 5

EQUILIBRIA OF TWO-PERSON GAMES

5.1 Regret and Disappointment Equilibria of Nonzero-sum Games

In this chapter, we present here the theoretical basis for the new Disappointment Equilibrium (DE), which is an alternative solution concept to an NE (or Regret Equilibrium RE). An RE results from each player minimizing his regret for his own responses to possible fixed actions of the other players. On the other hand, a DE results from each player minimizing his disappointment for the other players' responses to his own possible fixed actions.

5.1.1 Definitions and Theorems

In addition to previous notation, let

- a two-person normal form game be represented as (S_r, T_c, Q_r, Q_c) for the two players be designated as Player I (row player), and Player II (column player).
- S_r denote the set of pure strategies of Player I, where $S_r \geq 2$.
- T_c denote the set of pure strategies of Player II, where $T_c \geq 2$.
- Q_r denote the set of payoffs or utilities of Player I.
- Q_c denote the set of payoffs or utilities of Player II.
- Player I has m predetermined pure strategies, with $S_r = \{s_r \mid r = 1, \ldots, m\}$.

– Player II has n predetermined pure strategies, with $T_c = \{t_c \mid c = 1,\ldots, n\}$.

– a_{rc} denote Player I's payoff or utility and b_{rc} denote Player II's payoff or utility when Player I plays s_r strategy and Player II plays t_c strategy.

– Player I's $m \times n$ payoff matrix be $A = [a_{rc}]$, $a_{rc} \in Q_r$.

– Player II's $m \times n$ payoff matrix be $B = [b_{rc}]$, $b_{rc} \in Q_c$.

– the mixed extension of an $m \times n$ bimatrix game be represented as (X, Y, P_r, P_c).

– X be the set of possible mixed strategies, or probability vectors for Player I, where $X = \{\boldsymbol{x} \mid \boldsymbol{x} = [x_1,\ldots, x_m]^T \in \boldsymbol{R}^m$ and x_r, $r = 1,\ldots, m$, is the probability s_r that is chosen so that $\sum_{r=1}^{m} x_r = 1$ and all $x_r \geq 0\}$.

– Y be the set of possible mixed strategies, or probabilities for Player II, where $Y = \{\boldsymbol{y} \mid \boldsymbol{y} = [y_1,\ldots, y_n]^T \in \boldsymbol{R}^n$ and y_c, $c = 1,\ldots, n$, is the probability that t_c is chosen so that $\sum_{c=1}^{n} y_c = 1$ and all $y_c \geq 0\}$.

– P_r be the set of possible payoff or utility for Player I, with

$$P_r = \{p_r(\boldsymbol{x},\boldsymbol{y}) \mid v_r(\boldsymbol{x},\boldsymbol{y}) = \sum_{r=1}^{m} \sum_{c=1}^{n} a_{rc}\, x_r y_c = \boldsymbol{x}^T A \boldsymbol{y}\}.$$

– P_c be the set of possible payoff or utility for Player II, with

$$P_c = \{p_c(\boldsymbol{x},\boldsymbol{y}) \mid v_c(\boldsymbol{x},\boldsymbol{y}) = \sum_{r=1}^{m} \sum_{c=1}^{n} b_{rc}\, x_r y_c = \boldsymbol{x}^T B \boldsymbol{y}\}.$$

– $p(\boldsymbol{x},\boldsymbol{y})$ be a possible payoff or utility pair of Player I and Player II, with $p(\boldsymbol{x},\boldsymbol{y}) = (p_r(\boldsymbol{x},\boldsymbol{y}), p_c(\boldsymbol{x},\boldsymbol{y}))$.

Definition 5.1. An RE for the two-person game (S_r, S_c, P_r, P_c) above satisfies

$$\boldsymbol{x}^T A\boldsymbol{y}^* \leq \boldsymbol{x}^{*T} A\boldsymbol{y}^* \text{ for all } \boldsymbol{x} \in X \text{, and} \tag{5.1}$$

$$\boldsymbol{x}^{*T} B\boldsymbol{y} \leq \boldsymbol{x}^{*T} B\boldsymbol{y}^* \text{ for all } \boldsymbol{y} \in Y . \tag{5.2}$$

Definition 5.2. A DE for the two-person game (S_r, S_c, P_r, P_c) above satisfies

$$\boldsymbol{x}^{*T} A\boldsymbol{y} \leq \boldsymbol{x}^{*T} A\boldsymbol{y}^* \text{ for all } \boldsymbol{y} \in Y \text{, and} \tag{5.3}$$

$$\boldsymbol{x}^T B\boldsymbol{y}^* \leq \boldsymbol{x}^{*T} B\boldsymbol{y}^* \text{ for all } \boldsymbol{x} \in X . \tag{5.4}$$

Definition 5.3. The bimatrix game (B, A) is the dual of the primal game (A, B).

Definition 5.4. The RM for the game (A, B) is denoted by $R(A, B)$, and the DM for the game (A, B) is denoted by $D(A, B)$. Moreover, for the bimatrix (A, B), define the swap matrix of $(A, B)^S$ as (B, A). In particular the swap matrices of $R(A, B)$ and $D(A, B)$ are written $R(A, B)^S$ and $D(A, B)^S$.

The following results are immediate consequences of the above definitions.

Theorem 5.1. The dual bimatrix game of the dual game of (A, B) is (A, B).

Theorem 5.2. $R(A, B) = D^S(B, A)$, $R(B, A) = D^S(A, B)$, $D(B, A) = R^S(A, B)$, and $D(A, B) = R^S(B, A)$. Consequently, the set of RE for (A, B), or (B, A), is the set of DE for (B, A) or (A, B), respectively, when the payoffs and strategies for the row and column players are swapped.

Theorem 5.2 essentially says that any computational and existence properties of bimatrix RE's also hold for bimatrix DE's. In particular, any method for finding such an RE for (B, A) can be used to find a DE for (A, B). Hence, a DE exists for (A, B) since an RE exists for (B, A) from the work of Nash [5] specialized to $N = 2$. However, we prove this result again to gain insight into the substantially more difficult case for N > 2.

Lemma 5.1. (Brouwer Fixed-Point Theorem [24]). Let K be a non-empty compact convex set of $\boldsymbol{R}^n$, and let $f : K \to K$ be a continuous function. Then there exists $\boldsymbol{x}^* \in X$ for which $f(\boldsymbol{x}^*) = \boldsymbol{x}^*$.

Theorem 5.3. The mixed extension of every finite bimatrix game has a DE.

Proof. Let (A, B) be an $m \times n$ bimatrix game, $\boldsymbol{e}_k$ denote the column vector of with a 1 as component k and zeros elsewhere, where order will be understood from context. For each $c \in \{1, 2, \ldots, \mathrm{N}\}$, let v_c: $X \times Y \to \boldsymbol{R}$ be the continuous disappointment function defined by

$$v_c(\boldsymbol{x}, \boldsymbol{y}) = \max\{0, \boldsymbol{x}^T A \boldsymbol{e}_c - \boldsymbol{x}^T A \boldsymbol{y}\}.$$

For each $r \in \{1, 2, \ldots, m\}$, let u_r: $X \times Y \to \boldsymbol{R}$ be the continuous disappointment function defined by

$$u_r(\boldsymbol{x}, \boldsymbol{y}) = \max\{0, \boldsymbol{e}_r^T B \boldsymbol{y} - \boldsymbol{x}^T B \boldsymbol{y}\}.$$

From the maps $\boldsymbol{v} = (v_1, \ldots, v_n)$: $X \times Y \to \boldsymbol{R}^n$, and $\boldsymbol{u} = (u_1, \ldots, u_m)$: $X \times Y \to \boldsymbol{R}^m$, we define the continuous function f: $X \times Y \to X \times Y$ by

$$f(\boldsymbol{x},\boldsymbol{y}) = \left(\frac{\boldsymbol{x}+u(\boldsymbol{x},\boldsymbol{y})}{1+\sum_{r=1}^{m} u_r(\boldsymbol{x},\boldsymbol{y})}, \frac{\boldsymbol{y}+v(\boldsymbol{x},\boldsymbol{y})}{1+\sum_{c=1}^{n} v_c(\boldsymbol{x},\boldsymbol{y})} \right).$$

It is obvious that f: $X\times Y \rightarrow X\times Y$.

For example, component r of $\dfrac{\boldsymbol{x}+u(\boldsymbol{x},\boldsymbol{y})}{1+\sum_{r=1}^{m} u_r(\boldsymbol{x},\boldsymbol{y})}$ is $\dfrac{x_r + u_r(\boldsymbol{x},\boldsymbol{y})}{1+\sum_{r=1}^{m} u_r(\boldsymbol{x},\boldsymbol{y})}$.

Summing gives $\dfrac{\sum_{r=1}^{m} x_r + \sum_{r=1}^{m} u_r(\boldsymbol{x},\boldsymbol{y})}{1+\sum_{r=1}^{m} u_r(\boldsymbol{x},\boldsymbol{y})} = \dfrac{1+\sum_{r=1}^{m} u_r(\boldsymbol{x},\boldsymbol{y})}{1+\sum_{r=1}^{m} u_r(\boldsymbol{x},\boldsymbol{y})} = 1$.

So $\dfrac{\boldsymbol{x}+u(\boldsymbol{x},\boldsymbol{y})}{1+\sum_{r=1}^{m} u_r(\boldsymbol{x},\boldsymbol{y})} \in X$.

A similar argument gives $\dfrac{\boldsymbol{y}+v(\boldsymbol{x},\boldsymbol{y})}{1+\sum_{c=1}^{n} v_c(\boldsymbol{x},\boldsymbol{y})} \in Y$.

Since $X\times Y$ is a compact subset of $\boldsymbol{R}^m \times \boldsymbol{R}^n$, which can be identified with $\boldsymbol{R}^{m+n}$, Lemma 5.1 implies that there is a $(\boldsymbol{x}^*, \boldsymbol{y}^*) \in X\times Y$ with $f(\boldsymbol{x}^*, \boldsymbol{y}^*) = (\boldsymbol{x}^*, \boldsymbol{y}^*)$. We will show that $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is a DE for the mixed extension of (A, B).

Given $\boldsymbol{y}^* \in Y$ and $\boldsymbol{x}^{*T}A\boldsymbol{y} = \sum_{c=1}^{n} (\boldsymbol{x}^{*T}A\boldsymbol{e}_c)y_c$, there is a $k \in \{1, 2, \ldots, \mathrm{N}\}$ with $y_k^* > 0$ and $\boldsymbol{x}^{*T}A\boldsymbol{e}_k \leq \boldsymbol{x}^{*T}A\boldsymbol{y}^*$ such that $v_k(\boldsymbol{x}^*, \boldsymbol{y}^*) = 0$. Otherwise, if $\boldsymbol{x}^{*T}A\boldsymbol{e}_c > \boldsymbol{x}^{*T}A\boldsymbol{y}^*$ for all c with $y_c^* > 0$, then summing gives the contradiction $\boldsymbol{x}^{*T}A\boldsymbol{y}^* > \boldsymbol{x}^{*T}A\boldsymbol{y}^*$. Since $\boldsymbol{x}^{*T}A\boldsymbol{e}_k \leq \boldsymbol{x}^{*T}A\boldsymbol{y}^*$, by definition of the disappointment function, $v_k(x^*, y^*) = 0$. Hence

$$y_k^* > 0 \text{ and } y_k^* = \frac{y_k^* + v_k(\boldsymbol{x}^*, \boldsymbol{y}^*)}{1 + \sum_{c=1}^{n} v_c(\boldsymbol{x}^*, \boldsymbol{y}^*)} = \frac{y_k^*}{1 + \sum_{c=1}^{n} v_c(\boldsymbol{x}^*, \boldsymbol{y}^*)}$$

imply that $\sum_{c=1}^{n} v_c(\boldsymbol{x}^*, \boldsymbol{y}^*) = 0$. So $v_c(\boldsymbol{x}^*, \boldsymbol{y}^*) = 0$ for each $c \in \{1, 2, \ldots, n\}$.

Hence, $\boldsymbol{x}^T A \boldsymbol{e}_c \leq \boldsymbol{x}^{*T} A \boldsymbol{y}^*$ for all $c \in \{1, 2, \ldots, \mathrm{N}\}$, from which

$$\boldsymbol{x}^{*T} A \boldsymbol{y} \leq \boldsymbol{x}^{*T} A \boldsymbol{y}^* \text{ for all } \boldsymbol{y} \in Y. \tag{5.3}$$

Similarly, since $\boldsymbol{x}^* \in X$ and $\boldsymbol{x}^T B \boldsymbol{y}^* = \sum_{r=1}^{m} x_r (\boldsymbol{e}_r B \boldsymbol{y}^*)$ it follows that there is a $k \in \{1,\ldots, m\}$ with $x_k^* > 0$ and $\boldsymbol{e}_k^{\mathrm{T}} B \boldsymbol{y}^* \leq \boldsymbol{x}^{*T} B \boldsymbol{y}^*$. As before,

$$x_k^* > 0, \text{ and } x_k^* = \frac{x_k^* + u_k(\boldsymbol{x}^*, \boldsymbol{y}^*)}{1 + \sum_{r=1}^{m} u_r(\boldsymbol{x}^*, \boldsymbol{y}^*)} = \frac{x_k^*}{1 + \sum_{r=1}^{m} u_r(\boldsymbol{x}^*, \boldsymbol{y}^*)}$$

together imply that $\sum_{r=1}^{m} u_r(\boldsymbol{x}^*, \boldsymbol{y}^*) = 0$. So $u_r(\boldsymbol{x}^*, \boldsymbol{y}^*) = 0$ for each $r \in \{1,\ldots,m\}$.

Hence, $\boldsymbol{e}_r^T B \boldsymbol{y}^* \leq \boldsymbol{x}^{*T} B \boldsymbol{y}^*$ for all $r \in \{1, 2, \ldots, m\}$, from which

$$\boldsymbol{x}^T B \boldsymbol{y}^* \leq \boldsymbol{x}^{*T} B \boldsymbol{y}^* \text{ for all } \boldsymbol{x} \in X. \tag{5.4}$$

Theorem 5.3 now follows from (5.3), (5.4), and Definition 2.

5.1.2 Finding Regret Equilibria

Various computational methods to find RE's have been developed. We present a direct nonlinear programming method for finding one. From Definition 5.1, $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is an RE if

$$\boldsymbol{x}^T A \boldsymbol{y}^* \leq \boldsymbol{x}^{*T} A \boldsymbol{y}^* \text{ for all } \boldsymbol{x} \in X, \text{ and} \tag{5.1}$$

$$\boldsymbol{x}^{*T}B\boldsymbol{y} \le \boldsymbol{x}^{*T}B\boldsymbol{y}^* \text{ for all } \boldsymbol{y} \in Y. \quad (5.2)$$

Let $\boldsymbol{e}_k$ denote the column vector of with a 1 in as component k and zeros elsewhere, whose order will be understood from context. Then ($\boldsymbol{x}^*$, $\boldsymbol{y}^*$) is an RE if and only if for each $\boldsymbol{x} = \boldsymbol{e}_r$, $r = 1,\ldots, m$, $\boldsymbol{e}_r^T A\boldsymbol{y}^* \le \boldsymbol{x}^{*T}A\boldsymbol{y}^*$, $r = 1,\ldots, m,$ from (5.1) and (ii) for each $\boldsymbol{y} = \boldsymbol{e}_c$, $c = 1,\ldots,$ n, $\boldsymbol{x}^*B\boldsymbol{e}_c \le \boldsymbol{x}^{*T}B\boldsymbol{y}^*$, $c = 1,\ldots,$ n, from (5.2). We next use the facts that for $\forall\boldsymbol{x} = [x_1, x_2, \ldots, x_m]^T$,

$$\boldsymbol{x}^T A\boldsymbol{y}^* = \sum_{r=1}^{m} x_r\boldsymbol{e}_r^T A\boldsymbol{y}^* \le \sum_{r=1}^{m} x_r^*\boldsymbol{e}_r^T A\boldsymbol{y}^* = \boldsymbol{x}^{*T}A\boldsymbol{y}^*$$

and that for $\forall\boldsymbol{y} = [y_1, y_2, \ldots, y_m]^T$,

$$\boldsymbol{x}^{*T}B\boldsymbol{y} = \sum_{c=1}^{n} \boldsymbol{x}^{*T}B\boldsymbol{e}_c \le \sum_{c=1}^{n} \boldsymbol{x}^{*T}B\boldsymbol{e}_c y_c^* = \boldsymbol{x}^{*T}A\boldsymbol{y}^*.$$

It follows now that $\sum_{c=1}^{n} a_{rc}\, y_c^* \le \boldsymbol{x}^{*T}A\boldsymbol{y}^*$ if and only if (5.1) with $\sum_{c=1}^{n} y_c = 1$ and $y_c \ge 0$.

In addition, $\sum_{r=1}^{m} b_{rc}\, x_r^* \le \boldsymbol{x}^{*T}B\boldsymbol{y}^*$ if and only if (5.2) holds with $\sum_{r=1}^{m} x_r^* = 1$ and $x_r^* \ge 0$.

Therefore, ($\boldsymbol{x}^*$, $\boldsymbol{y}^*$) is a bimatrix RE if and only if ($\boldsymbol{x}^*$, $\boldsymbol{y}^*$) satisfies

$$\sum_{c=1}^{n} a_{rc}\, y_c^* - \sum_{r=1}^{\mathrm{m}} \sum_{c=1}^{n} a_{rc} x_r y_c \le 0,\ r = 1, \ldots, m,$$

$$\sum_{r=1}^{m} b_{rc}\, x_r^* - \sum_{r=1}^{\mathrm{m}} \sum_{c=1}^{n} b_{rc} x_r y_c \le 0,\ c = 1, \ldots, n,$$

$$\sum_{r=1}^{m} x_r = 1,\ \sum_{c=1}^{n} y_c = 1,\ x_r \ge 0, \text{ and } y_c \ge 0.$$

We can add slack and artificial variables to this set of inequalities and equalities to get a nonlinear program for finding ($\boldsymbol{x}$*, $\boldsymbol{y}$*). Doing so yields the following problem.

Minimize $f(x_1, \ldots, x_m, y_1, \ldots, y_n, E_1, \ldots, E_m, F_1, \ldots, F_n, U_1, \ldots, U_m, V_1, \ldots, V_n) =$

$$\sum_{r=1}^{m} U_r + \sum_{c=1}^{n} V_c$$

subject to

$$\sum_{c=1}^{n} a_{rc}\; y_c^* - \sum_{r=1}^{\mathrm{m}} \sum_{c=1}^{n} a_{rc} x_r y_c \; + E_r + U_r = 0,\ r = 1, \ldots, m$$

$$\sum_{r=1}^{m} b_{rc}\; x_r^* - \sum_{r=1}^{\mathrm{m}} \sum_{c=1}^{n} b_{rc} x_r y_c \; + F_c + V_c = 0,\ c = 1, \ldots, n$$

$$\sum_{r=1}^{m} x_r = 1$$

$$\sum_{c=1}^{n} y_c = 1$$

$$x_r \geq 0,\ y_c \geq 0.$$

For a 2×2 bimatrix game, this problem becomes

Minimize $f(x_1, x_2, y_1, y_2, E_1, E_2, F_1, F_2, U_1, U_2, V_1, V_2) = U_1 + U_2 + V_1 + V_2$

subject to

$$a_{11}y_1 + a_{12}y_2 - a_{11}x_1y_1 - a_{12}\, x_1y_2 - a_{21}x_2y_1 - a_{22}\, x_2y_2 + E_1 + U_1 = 0$$

$$a_{21}y_1 + a_{22}y_2 - a_{11}x_1y_1 - a_{12}\, x_1y_2 - a_{21}x_2y_1 - a_{22}\, x_2y_2 + E_2 + U_2 = 0$$

$$b_{11}x_1 + b_{21}x_2 - b_{11}x_1y_1 - b_{12}\, x_1y_2 - b_{21}x_2y_1 - b_{22}\, x_2y_2 + F_1 + V_1 = 0$$

$$b_{12}x_1 + b_{22}x_2 - b_{11}x_1y_1 - b_{12}\, x_1y_2 - b_{21}x_2y_1 - b_{22}\, x_2y_2 + F_2 + V_2 = 0$$

$$x_1 + x_2 = 1$$

$y_1 + y_2 = 1$

$x_1, x_2, y_1, y_2, E_1, E_2, F_1, F_2, U_1, U_2, V_1, V_2 \geq 0.$

Example 5.1. Consider the bimatrix game G_1 below. To find its RE's we solve the problem

		Player *II*	
		t_1	t_2
Player *I*	s_1	(10,3)	(4,7)
	s_2	(2,6)	(9,5)

Figure 5.1 The Bimatrix Game G_1

Minimize $f(x_1, x_2, y_1, y_2, E_1, E_2, F_1, F_2, U_1, U_2, V_1, V_2) = U_1 + U_2 + V_1 + V_2$

subject to

$10*y_1 + 4*y_2 - 10*x_1*y_1 - 4*x_1*y_2 - 2*x_2*y_1 - 9*x_2*y_2 + E_1 + U_1 = 0$

$2*y_1 + 9*y_2 - 10*x_1*y_1 - 4*x_1*y_2 - 2*x_2*y_1 - 9*x_2*y_2 + E_2 + U_2 = 0$

$3*x_1 + 6*x_2 - 3*x_1*y_1 - 7*x_1*y_2 - 6*x_2*y_1 - 5*x_2*y_2 + F_1 + V_1 = 0$

$7*x_1 + 5*x_2 - 3*x_1*y_1 - 7*x_1*y_2 - 6*x_2*y_1 - 5*x_2*y_2 + F_2 + V_2 = 0$

$x_1 + x_2 = 1$

$y_1 + y_2 = 1$

$x_1, x_2, y_1, y_2, E_1, E_2, F_1, F_2, U_1, U_2, V_1, V_2 \geq 0.$

The RE for G_1 is $\boldsymbol{x}^* = (x_1^*, x_2^*) = (0.2, 0.8)$ and $\boldsymbol{y}^* = (y_1^*, y_2^*) = (0.3846, 0.6154)$ with the optimal expected payoffs to Players *I* and *II* given by $p(\boldsymbol{x}^*, \boldsymbol{y}^*) = (6.308, 5.4)$.

5.1.3 Finding Disappointment Equilibria

From Definition 5.2, ($\boldsymbol{x}$*, $\boldsymbol{y}$*) is an DE if and only if

$$\boldsymbol{x}^{*T}A\boldsymbol{y} \le \boldsymbol{x}^{*T}A\boldsymbol{y}^* \text{ for all } \boldsymbol{y} \in Y \text{, and} \tag{5.3}$$

$$\boldsymbol{x}^{T}B\boldsymbol{y}^* \le \boldsymbol{x}^{*T}B\boldsymbol{y}^* \text{ for all } \boldsymbol{x} \in X \text{.} \tag{5.4}$$

In a derivation similar to the one of Section 5.1.2 for finding a RE, we get from (5.3) and (5.4) that ($\boldsymbol{x}$*, $\boldsymbol{y}$*) is a bimatrix DE if and only if ($\boldsymbol{x}$*, $\boldsymbol{y}$*) satisfies the set of inequalities

$$\sum_{r=1}^{m} a_{rc}\ x_r^* - \sum_{r=1}^{\mathrm{m}} \sum_{c=1}^{n} a_{rc} x_r y_c \le 0,\ c = 1, \ldots, n,$$

$$\sum_{c=1}^{n} b_{rc}\ y_c^* - \sum_{r=1}^{\mathrm{m}} \sum_{c=1}^{n} b_{rc} x_r y_c \le 0,\ r = 1, \ldots, m,$$

$$\sum_{r=1}^{m} x_r = 1,\ \sum_{c=1}^{n} y_c = 1,\ x_r \ge 0, \text{ and } x_c \ge 0.$$

As for RE's, we can add slack and artificial variables to this last set of inequalities and equalities to get a nonlinear program for finding ($\boldsymbol{x}$*, $\boldsymbol{y}$*). Doing so yields the following problem.

Minimize $f(x_1, \ldots, x_m, y_1, \ldots, y_n, E_1, \ldots, E_m, F_1, \ldots, F_n, U_1, \ldots, U_m, V_1, \ldots, V_n) =$

$$\sum_{r=1}^{m} U_r + \sum_{c=1}^{n} V_c$$

subject to

$$\sum_{r=1}^{m} a_{rc}\ x_r^* - \sum_{r=1}^{\mathrm{m}} \sum_{c=1}^{n} a_{rc} x_r y_c + F_c + V_c = 0,\ c = 1, \ldots, n$$

$$\sum_{c=1}^{n} b_{rc}\ y_c{}^* - \sum_{r=1}^{m}\ \sum_{c=1}^{n} b_{rc} x_r y_c\ + E_r + U_r = 0,\ r = 1,\ \ldots,\ m$$

$$\sum_{r=1}^{m}\ x_r = 1,\ \sum_{c=1}^{n}\ y_c = 1,\ x_r \geq 0, \text{ and } x_c \geq 0$$

For a 2×2 bimatrix game, this problem becomes

Minimize $f(x_1, x_2, y_1, y_2, E_1, E_2, F_1, F_2, U_1, U_2, V_1, V_2) = U_1 + U_2 + V_1 + V_2$

subject to

$a_{11}x_1 + a_{21}x_2 - a_{11}x_1y_1 - a_{12}\ x_1y_2 - a_{21}x_2y_1 - a_{22}\ x_2y_2 + E_1 + U_1 = 0$

$a_{12}x_1 + a_{22}x_2 - a_{11}x_1y_1 - a_{12}\ x_1y_2 - a_{21}x_2y_1 - a_{22}\ x_2y_2 + E_2 + U_2 = 0$

$b_{11}y_1 + b_{12}y_2 - b_{11}x_1y_1 - b_{12}\ x_1y_2 - b_{21}x_2y_1 - b_{22}\ x_2y_2 + F_1 + V_1 = 0$

$b_{21}y_1 + b_{22}y_2 - b_{11}x_1y_1 - b_{12}\ x_1y_2 - b_{21}x_2y_1 - b_{22}\ x_2y_2 + F_2 + V_2 = 0$

$x_1 + x_2 = 1$

$y_1 + y_2 = 1$

$x_1, x_2, y_1, y_2, E_1, E_2, F_1, F_2, U_1, U_2, V_1, V_2 \geq 0.$

Example 5.2 The DE of the game G_1 in Figure 5.1 is calculated by solving the problem

Minimize $f(x_1, x_2, y_1, y_2, E_1, E_2, F_1, F_2, U_1, U_2, V_1, V_2) = U_1 + U_2 + V_1 + V_2$

subject to

$10\times x_1 + 2\times x_2 - 10\times x_1\times y_1 - 4\times x_1\times y_2 - 2\times x_2\times y_1 - 9\times x_2\times y_2 + F_1 + V_1 = 0$

$4\times x_1 + 9\times x_2 - 10\times x_1\times y_1 - 4\times x_1\times y_2 - 2\times x_2\times y_1 - 9\times x_2\times y_2 + F_2 + V_2 = 0$

$3\times y_1 + 7\times y_2 - 3\times x_1\times y_1 - 7\times x_1\times y_2 - 6\times x_2\times y_1 - 5\times x_2\times y_2 + E_1 + U_1 = 0$

$6 \times y_1 + 5 \times y_2 - 3 \times x_1 \times y_1 - 7 \times x_1 \times y_2 - 6 \times x_2 \times y_1 - 5 \times x_2 \times y_2 + E_2 + U_2 = 0$

$x_1 + x_2 = 1$

$y_1 + y_2 = 1$

$x_1, x_2, y_1, y_2, E_1, E_2, F_1, F_2, U_1, U_2, V_1, V_2 \geq 0.$

The DE of G_1 is $\boldsymbol{x}^* = (x_1^*, x_2^*) = (0.5385, 0.4615)$ and $\boldsymbol{y}^* = (y_1^*, y_2^*) = (0.4, 0.6)$ with optimal expected payoffs to Players *I* and *II* given by $p(\boldsymbol{x}^*, \boldsymbol{y}^*) = (6.308, 5.4)$.

Example 5.3. The DE of the PD game in Figure 4.1 obtained by solving

Minimize $f(x_1, x_2, y_1, y_2, E_1, E_2, F_1, F_2, U_1, U_2, V_1, V_2) = U_1 + U_2 + V_1 + V_2$

subject to

$3 \times x_1 + 5 \times x_2 - 3 \times x_1 \times y_1 - 0 \times x_1 \times y_2 - 5 \times x_2 \times y_1 - 1 \times x_2 \times y_2 + F_1 + V_1 = 0$

$0 \times x_1 + 1 \times x_2 - 3 \times x_1 \times y_1 - 0 \times x_1 \times y_2 - 5 \times x_2 \times y_1 - 1 \times x_2 \times y_2 + F_2 + V_2 = 0$

$3 \times y_1 + 5 \times y_2 - 3 \times x_1 \times y_1 - 5 \times x_1 \times y_2 - 0 \times x_2 \times y_1 - 1 \times x_2 \times y_2 + E_1 + U_1 = 0$

$0 \times y_1 + 1 \times y_2 - 3 \times x_1 \times y_1 - 5 \times x_1 \times y_2 - 0 \times x_2 \times y_1 - 1 \times x_2 \times y_2 + E_2 + U_2 = 0$

$x_1 + x_2 = 1$

$y_1 + y_2 = 1$

$x_1, x_2, y_1, y_2, E_1, E_2, F_1, F_2, U_1, U_2, V_1, V_2 \geq 0.$

The DE for this game is $\boldsymbol{x}^* = (x_1^*, x_2^*) = (1, 0)$ and $\boldsymbol{y}^* = (y_1^*, y_2^*) = (1, 0)$ with the optimal expected payoffs to Players *I* and *II* given by $p(\boldsymbol{x}^*, \boldsymbol{y}^*) = (3, 3)$. This result is also obtained from (0,0) of the DM in Figure 4.3, where the disappointment of both players is 0.

Example 5.4. As another example, consider the following bimatrix game G_2 of Figure 5.2. We obtain the DE's by finding all solutions to the problem

Player *II*

Player *I*		t_1	t_2
	s_1	(1,1)	(100,0)
	s_2	(0,100)	(100,100)

Figure 5.2 The Bimatrix Game G_2

Minimize $f(x_1, x_2, y_1, y_2, E_1, E_2, F_1, F_2, U_1, U_2, V_1, V_2) = U_1 + U_2 + V_1 + V_2$

subject to

$$1\times x_1 + 0\times x_2 - 1\times x_1\times y_1 - 100\times x_1\times y_2 - 0\times x_2\times y_1 - 100\times x_2\times y_2 + F_1 + V_1 = 0$$

$$100\times x_1 + 100\times x_2 - 1\times x_1\times y_1 - 100\times x_1\times y_2 - 0\times x_2\times y_1 - 100\times x_2\times y_2 + F_2 + V_2 = 0$$

$$1\times y_1 + 0\times y_2 - 1\times x_1\times y_1 - 0\times x_1\times y_2 - 100\times x_2\times y_1 - 100\times x_2\times y_2 + E_1 + U_1 = 0$$

$$100\times y_1 + 100\times y_2 - 1\times x_1\times y_1 - 0\times x_1\times y_2 - 100\times x_2\times y_1 - 100\times x_2\times y_2 + E_2 + U_2 = 0$$

$$x_1 + x_2 = 1$$

$$y_1 + y_2 = 1$$

$$x_1, x_2, y_1, y_2, E_1, E_2, F_1, F_2, U_1, U_2, V_1, V_2 \geq 0$$

The DE for this game is $\boldsymbol{x}^* = (x_1^*, x_2^*) = (0, 1)$ and $\boldsymbol{y}^* = (y_1^*, y_2^*) = (0, 1)$ with the optimal expected payoffs to Players *I* and *II* given by $p(\boldsymbol{x}^*, \boldsymbol{y}^*) = (100, 100)$. This result can again be obtained from the DM of Figure 5.3.

Player *II*

	t_1	t_2
Player *I* s_1	(99,99)	(0,100)
s_2	(100,0)	(0,0)

Figure 5.3 The DM of Game G_2

Player *II*

	t_1	t_2
Player *I* s_1	(0,0)	(0,1)
s_2	(1,0)	(0,0)

Figure 5.4 The RM of game G_2

Notice that game G_2 has two pure RE's, obtained from the RM in Figure 5.4, and one DE. One RE is also a DE. We would expect such a joint RE-DE to provide a better outcome than the other RE. In this case, the conjecture is patently true.

G_2 represents games with weakly dominant strategies of both players. Strategy s_1 weakly dominates s_2 because at least one payoff from s_1 is better than s_2, but not both. Similarly, t_1 weakly dominated t_2 and t_1. The result is a non-Pareto RE. For this reason, we suggest that one should not consider dominant strategies as their first criteria in obtaining a solution to a game. Not doing so here gives the strategy pair (s_2, t_2), which is a joint RE-DE.

Example 5.5. For the example game G_3 of Figure 5.5, we obtain the DE from the DM of Figure 5.6 and the RE from the RM of Figure 5.7.

Player *II*

	t_1	t_2
Player *I* s_1	(10,0)	(5,2)
s_2	(10,11)	(2,0)

Figure 5.5 The Bimatrix Game G_3

Player *II*

	t_1	t_2
Player *I* s_1	(0,11)	(5,0)
s_2	(0,0)	(8,2)

Figure 5.6 The DM of Game G_3

Player *II*

	t_1	t_2
Player *I* s_1	(0,2)	(0,0)
s_2	(0,0)	(3,11)

Figure 5.7 The RM of Game G_3

The DE for this game is $\boldsymbol{x}^* = (x_1^*, x_2^*) = (0, 1)$ and $\boldsymbol{y}^* = (y_1^*, y_2^*) = (1, 0)$ with the expected payoffs to Players *I* and *II* given by $p(\boldsymbol{x}^*, \boldsymbol{y}^*) = (10, 11)$. The DE is also one of the RE's.

As in game G_2, Player *I* of game G_3 has a weakly dominant strategy s_1. Strategy s_1 weakly dominates s_2 because at least one payoff from s_1 is better than s_2, but not both. Again we suggest that the joint RE-DE solution is better than an equilibrium including a weakly dominant strategy.

Example 5.6. Game G_4 of Figure 5.8 is an example of a game in which one player has a strictly dominant strategy. We obtain the DE from the DM of Figure 5.9 and the RE from the RM of Figure 5.10.

Player *II*

		t_1	t_2
Player *I*	s_1	(1,10)	(3,1)
	s_2	(0,2)	(2,4)

Figure 5.8 The Bimatrix Game G_4

Player *II*

		t_1	t_2
Player *I*	s_1	(2,0)	(0,3)
	s_2	(2,8)	(0,0)

Figure 5.9 The DM of Game G_4

Player *II*

		t_1	t_2
Player *I*	s_1	(0,0)	(0,9)
	s_2	(1,2)	(1,0)

Figure 5.10 The RM of Game G_4

The DE for this game is $\boldsymbol{x}^* = (x_1^*, x_2^*) = (0, 1)$ and $\boldsymbol{y}^* = (y_1^*, y_2^*) = (0, 1)$ with the expected payoffs to Players *I* and *II* given by $p\,(\boldsymbol{x}^*, \boldsymbol{y}^*) = (2, 4)$. The RE for this game is $\boldsymbol{x}^* = (x_1^*, x_2^*) = (1, 0)$ and $\boldsymbol{y}^* = (y_1^*, y_2^*) = (1, 0)$ with the expected payoffs to Players *I* and *II* given by $p\,(\boldsymbol{x}^*, \boldsymbol{y}^*) = (1, 10)$.

Unlike games G_2 and G_3, the DE of game G_4 is not an RE. However, the DE outcome may be fairer than the RE. Moreover, the DE of this game is an NME under TOM if the initial state is either the DE or the RE.

5.1.4 Properties of Bimatrix RE's and DE's

We next define the notion of a Joint Equilibrium of a bimatrix game and derive some properties relating RE's and DE's.

Definition 5.6. For a bimatrix game a mixed strategy pair $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ for that is both an RE and DE is called a joint equilibrium (JE).

From definitions of an RE and DE, we have that

$$\text{RE:} \quad \boldsymbol{x}^T A \boldsymbol{y}^* \leq \boldsymbol{x}^{*T} A \boldsymbol{y}^* \text{ for all } \boldsymbol{x} \in X \tag{5.5.1}$$

$$\text{DE:} \quad \boldsymbol{x}^T B \boldsymbol{y}^* \leq \boldsymbol{x}^{*T} B \boldsymbol{y}^* \text{ for all } \boldsymbol{x} \in X \tag{5.5.2}$$

$$\text{RE:} \quad \boldsymbol{x}^{*T} B \boldsymbol{y} \leq \boldsymbol{x}^{*T} B \boldsymbol{y}^* \text{ for all } \boldsymbol{y} \in Y \tag{5.5.3}$$

$$\text{DE:} \quad \boldsymbol{x}^{*T} A \boldsymbol{y} \leq \boldsymbol{x}^{*T} A \boldsymbol{y}^* \text{ for all } \boldsymbol{y} \in Y. \tag{5.5.4}$$

Adding (5.5.1) and (5.5.2) gives

$$\boldsymbol{x}^T (A+B) \boldsymbol{y}^* \leq \boldsymbol{x}^{*T} (A+B) \boldsymbol{y}^* \text{ for all } \boldsymbol{x} \in X. \tag{5.6}$$

Adding (5.5.3) and (5.5.4) gives

$$\boldsymbol{x}^{*T} (A+B) \boldsymbol{y} \leq \boldsymbol{x}^{*T} (A+B) \boldsymbol{y}^* \text{ for all } \boldsymbol{y} \in Y. \tag{5.7}$$

Thus from (5.6) and (5.7) we have proved Property 1 of the following result. The remaining properties are easily established.

Theorem 5.4. An RE, DE, and JE of a two-person, zero-sum games satisfies the following properties.

Property 1. If a point ($\boldsymbol{x}^*$, $\boldsymbol{y}^*$) is a JE of the game (A,B) it is both an RE and DE of the game ((A+B), (A+B)).

Property 2. Consider the bimatrix game (A,B). If a payoff pair (a_{ij}, b_{ij}) dominates (a_{kj}, b_{kj}), and (a_{il}, b_{il}) for all k and l,then (a_{ij}, b_{ij}) is a JE.

Property 3. Let real numbers $e, f > 0$ and the $m \times n$ matrix $E = [e_{rc}]$ such that $e_{rc} = e_c$, $r = 1,\ldots, m,$ and $F = [f_{rc}]$ be such that $f_{rc} = f_r$, $c = 1,\ldots, n$. In other words, each column c of E is a vector $\begin{bmatrix} e_c \\ \vdots \\ e_c \end{bmatrix}$ and each row r of F is a vector $\begin{bmatrix} f_r \\ \vdots \\ f_r \end{bmatrix}^T$. For the bimatrix game (A,B), then (eA, fB) has the same set of RE's as (A, B), as does (A+E, B+F). Hence (eA+E, fB+F) has the same set of RE's as (A, B).

Property 4. The game (fA+F, eB+E) has the same DE's as (A, B), and (eB+E, fA+F). It also has the same RE's as (B, A), which are the same as the DE's for (A, B).

Property 5. Let (A, B) have a dominant element (a_{rc}, b_{rc}). Let E, F be constant matrices. Then for $e, f > 0$ the game (eA+E, fB+F) has the rc^{th} strategy as a JE.

Property 6. Consider the game (A, A). Then the RE's for (A, A) is the same as the DE's, so each RE is also an DE and vice versa. Hence, each RE or DE is a JE. Moreover, let E, F be constant matrices. Then the set of JE's of (eA+E, fA+F) is the set of RE's or DE's.

Example 5.7. Consider the matrices for bimatrix game G_5 of Figures 5.11 - 5.13.

Player *II*

Player *I*		t_1	t_2	t_3
	s_1	(3,3)	(2,2)	(1,1)
	s_2	(2,3)	(3,1)	(7,2)
	s_3	(2,1)	(4,7)	(5,5)

Figure 5.11 The Bimatrix Game G_5

Player *II*

Player *I*		t_1	t_2	t_3
	s_1	(0,0)	(2,1)	(6,2)
	s_2	(1,0)	(1,2)	(0,1)
	s_3	(1,4)	(0,0)	(2,2)

Figure 5.12 The RM of Game G_5

Player *II*

Player *I*		t_1	t_2	t_3
	s_1	(0,0)	(1,5)	(2,4)
	s_2	(5,0)	(4,6)	(0,3)
	s_3	(3,0)	(1,0)	(0,0)

Figure 5.13 The DM of Game G_5

The strategy pair (s_1,t_1) with the payoff pair (3,3) is a JE, Notice that this game does not exhibit row or column dominance. However, as in Prisoner's Dilemma, the JE (3,3) is dominated by (5,5), so the JE is not a Pareto optimum of the payoff matrix. Hence, contrary to the case in Example 5.4 and 5.5, a JE is not necessarily better than an RE or DE that is not a JE.

Example 5.8. We now show that the DE's of the RM and the RE's of the DM do not result in the same equilibria. Figure 5.14 shows the RM of the PD game, while Figure 5.15 shows the DM of the PD game. Comparing the right-hand matrix in each figue establishes this fact.

	Payoff Bimatrix		RM		DM of the RM	
	Cooperate	*Defect*	*Cooperate*	*Defect*	*Cooperate*	*Defect*
Cooperate	(3,3)	(0,5)	(2,2)	(1,0)	(1,1)	(0,0)
Defect	(5,0)	(1,1)	(0,1)	(0,0)	(0,0)	(0,0)

Figure 5.14 The RM of the PD Game and its Disappointment Utilities.

	Payoff Bimatrix		DM		RM of the DM	
	Cooperate	*Defect*	*Cooperate*	*Defect*	*Cooperate*	*Defect*
Cooperate	(3,3)	(0,5)	(0,0)	(3,0)	(0,0)	(0,0)
Defect	(5,0)	(1,1)	(0,3)	(4,4)	(0,0)	(1,1)

Figure 5.15 The DM of the PD Game and its Regret Utilities.

5.2 Disappointment Equilibria of Zero-sum Games

From Theorem 5.2, the DE's for a zero sum game (A, –A) can be obtained by finding the RE's of (–A, A). Hence, we may consider only the A matrix for row player. Equivalent to finding the RE of (A, –A) by looking at the maximin for the row of A and the minimax for the columns of A to find the RE of (A, –A), we find the RE of (–A, A) by looking at the minimax row strategy and maximin column strategy of A. In effect, each player ensures the best security level within his choice of strategies. If the resulting values are the same, then the result is a DE.

To show the relations between RE's and DE's for zero-sum games, we first state the standard results for zero-sum games.

Theorem 5.5. (Minimax Theorem [4]). For every finite two-person zero-sum game, there exist mixed strategies $\boldsymbol{x}$ for Player I and $\boldsymbol{y}$ for Player II forming an RE $(\boldsymbol{x}, \boldsymbol{y})$ such that the payoff to player I, $p(\boldsymbol{x}, \boldsymbol{y})$ satisfies

$$\max_{x} \min_{y} \; p(\boldsymbol{x}, \boldsymbol{y}) = \min_{y} \max_{x} \; p(\boldsymbol{x}, \boldsymbol{y}) = v.$$

The number v is called the value of the game. The payoff to Player II is $-p(\boldsymbol{x}, \boldsymbol{y})$.

Player I's probabilities $\boldsymbol{x}$ can be determined by solving the following maximin problem by [4] [25]

$$\max_{\boldsymbol{x}_r \in X_r} \{\min (\sum_{r=1}^{m} a_{r1}\, x_r, \sum_{r=1}^{m} a_{r2}\, x_r ,\ldots, \sum_{r=1}^{m} a_{rn}\, x_r)\} \;;\; \sum_{r=1}^{m} x_r = 1,\; x_r \geq 0,\; r = 1,\ldots, m,$$

which implies that $\sum_{r=1}^{m} a_{rc}\, x_r \geq v$.

Player II's probabilities $\boldsymbol{y}$ can be determined by solving the following minimax problem

$$\min_{\boldsymbol{y}_c \in Y_c} \{\max (\sum_{c=1}^{n} a_{1c}\, y_c, \sum_{c=1}^{n} a_{2c}\, y_c ,\ldots, \sum_{c=1}^{n} a_{mc}\, y_c);\; \sum_{c=1}^{n} y_c = 1,\; y_c \geq 0,\; c = 1,\ldots, n\},$$

which implies that $\sum_{c=1}^{n} a_{rc}\, y_c \leq v$. Therefore we get the following linear programs for finding $\boldsymbol{x}$ and $\boldsymbol{y}$.

Maximize v

subject to

$$\sum_{r=1}^{m} a_{rc}\, x_r \geq v$$

$$\sum_{r=1}^{m} x_r = 1$$

$$x_r \geq 0,\ r = 1, \ldots, m,$$

and

Minimize v

subject to

$$\sum_{c=1}^{n} a_{rc}\, y_c \leq v$$

$$\sum_{c=1}^{n} y_c = 1$$

$$y_c \geq 0,\ c = 1, \ldots, n.$$

From these two linear programs, Player *II*'s problem is the linear programming dual to Player *I*'s problem.

Example 5.8. Consider the following zero-sum matrix game G_6.

		Player *II*		
		t_1	t_2	t_3
Player *I*	s_1	5	2	6
	s_2	3	6	7

Figure 5.16 The Zero-sum Matrix Game G_6

The maximin and minimax pair of this game can be calculated by the following linear programming problems.

Maximize v	Minimize v
subject to	subject to
$5\times x_1 + 3\times x_2 \geq v$;	$5\times y_1 + 2\times y_2 + 6\times y_3 \leq v$;
$2\times x_1 + 6\times x_2 \geq v$;	$3\times y_1 + 6\times y_2 + 7\times y_3 \leq v$;
$6\times x_1 + 7\times x_2 \geq v$;	$y_1 + y_2 + y_3 = 1$;
$x_1 + x_2 = 1$; $x_1, x_2 \geq 0$;	$y_1, y_2, y_3 \geq 0$;

Solving these problems, we obtain the value of the game $v = 4$ with $\boldsymbol{x} = (0.5, 0.5)^T$ and $\boldsymbol{y} = (0.67, 0.33, 0)^T$.

We now state the corresponding result for DE's of a zero-sum game. Theorem 5.2 immediately gives the next theorem.

Theorem 5.6. A Disappointment Equilibrium for a two-person zero-sum game is the minimax mixed strategy $\boldsymbol{x}$ for the row Player *I* and the maximin strategy $\boldsymbol{y}$ for the column Player *II*, with the expected payoffs equal for the two players. The mixed strategies $\boldsymbol{x}$ for Player *I* and $\boldsymbol{y}$ for Player *II* exist and satisfy

$$\min_{x} \max_{y} \; p(\boldsymbol{x}, \boldsymbol{y}) \; = \; \max_{y} \min_{x} \; p(\boldsymbol{x}, \boldsymbol{y}) \; = \; w,$$

where $p(\boldsymbol{x}, \boldsymbol{y})$ is the payoff for Player *I* and $-p(\boldsymbol{x}, \boldsymbol{y})$ is the payoff for Player *II*.

A consequence of Theorem 5.6 is that a pure strategy DE is obtained when the minimax value for the row Player *I* equals the maximin value for the column Player *II*. Note that this situation is exactly opposite that for an RE. Moreover, from Theorem 5.6 and the previous linear program for RE's, it follows that Player *I*'s mixed strategy $\boldsymbol{x}$ can be determined by solving the minimax problem

$$\min_{\boldsymbol{x}_r \in X_r} \{\max\,(\sum_{r=1}^{m} a_{r1}\, x_r,\ \sum_{r=1}^{m} a_{r2}\, x_r\ ,\ldots,\sum_{r=1}^{m} a_{rn}\, x_r)\ ;\ \sum_{r=1}^{m} x_r\ = 1,\ x_r \geq\ 0,\ r = 1,\ldots,\ m\},$$

which implies that $\sum_{r=1}^{m} a_{rc}\ x_r\ \leq\ w$. Similarly, Player *II*'s mixed strategy $\boldsymbol{y}$ can be determined by solving the maximin problem

$$\max_{\boldsymbol{y}_c \in Y_c} \{\min\,(\sum_{c=1}^{n} a_{1c}\, y_c,\ \sum_{c=1}^{n} a_{2c}\, y_c\ ,\ldots,\ \sum_{c=1}^{n} a_{mc}\, y_c)\ ;\ \sum_{c=1}^{n} y_c\ = 1,\ y_c \geq\ 0,\ c = 1,\ldots,\ n\},$$

which implies that $\sum_{c=1}^{n} a_{rc}\ y_c \geq w$. Therefore we get the following linear programs for finding the DE ($\boldsymbol{x}, \boldsymbol{y}$).

Minimize w

subject to

$$\sum_{r=1}^{m}\ a_{rc}\, x_r \leq w$$

$$\sum_{r=1}^{m}\ x_r\ = 1$$

$$x_r \geq\ 0,\ r = 1,\ldots,\ m,$$

and

Maximize w

subject to

$$\sum_{c=1}^{n} a_{rc}\, y_c \geq w$$

$$\sum_{c=1}^{n} y_c = 1$$

$y_c \geq 0$, $c = 1,\ldots, n$.

Example 5.9. Consider the zero-sum matrix game G_6 in Figure 5.16.

The minimax and maximin pair of this game can be calculated by the following linear programs.

Minimize w	Maximin w
subject to	subject to
$5\times x_1 + 3\times x_2 \leq w$;	$5\times y_1 + 2\times y_2 + 6\times y_3 \geq w$;
$2\times x_1 + 6\times x_2 \leq w$;	$3\times y_1 + 6\times y_2 + 7\times y_3 \geq w$;
$6\times x_1 + 7\times x_2 \leq w$;	$y_1 + y_2 + y_3 = 1$;
$x_1 + x_2 = 1$; $x_1, x_2 \geq 0$;	$y_1, y_2, y_3 \geq 0$;

We get the DE $\boldsymbol{x} = (1, 0)^T$ and $\boldsymbol{y} = (0, 0, 1)^T$ where the payoff to Player I is 6.

If either player moves from this DE, his opponent gets a worse payoff.

5.3 Pareto Intercession Equilibria of Nonzero-sum Games

Considering RE's and DE's we define as follows a solution concept called a Pareto Intercession Equilibrium (abbreviated PI Equilibria or simply π). "Intercession" refers to an intervening between parties to reconcile differences, and the set of Pareto maxima of the possible payoff pairs for a bimatrix game is defined by

> P_o = {$\boldsymbol{p}$ ($\boldsymbol{x}_o$, $\boldsymbol{y}_o$)| ($\boldsymbol{x}_o$, $\boldsymbol{y}_o$) is a strategy pair for which there does not exist ($\boldsymbol{x}$, $\boldsymbol{y}$) satisfying either $p_r(\boldsymbol{x}, \boldsymbol{y}) \geq p_r(\boldsymbol{x}_o, \boldsymbol{y}_o)$ and $p_c(\boldsymbol{x}, \boldsymbol{y}) > p_c(\boldsymbol{x}_o, \boldsymbol{y}_o)$, or else $p_c(r,c) \geq p_c(\boldsymbol{x}_o, \boldsymbol{y}_o)$ and $p_r(\boldsymbol{x}, \boldsymbol{y}) > p_r(\boldsymbol{x}_o, \boldsymbol{y}_o)$}.

The set of Pareto minima is similarly defined with the inequalities in the opposite direction.

Definition 5.7. A π strategy is defined by the following steps.

1. Find the set of Pareto maxima P_o of expected payoffs for all DE's and RE's, obtained without eliminating any dominated strategies.
2. From P_o chose a π according to the following criteria.
   (a) For each such equilibrium select the minimum payoff among all players.
   (b) Chose all equilibria with the maximum value from (a).
   (c) For each equilibrium in (b) select the second minimum payoff among all players.
   (d) Chose all equilibria from (b) that maximize the second minimum payoff among the players.
   (e) A π strategy is a member of the set of strategies in step (d).

5.4 Pareto Intercession Equilibria and Social Dilemmas

A total of seventy-eight distinctive configuration of a 2×2 game can be formed with the ordinal preferences of players over the four possible outcomes of the games [17]. Among these games there are four symmetric games (assuming no ties) for which players have temptation to defect [18]. These four games are known as social dilemmas because of their real-life analogs. One is the PD game. The other three are Deadlock, Chicken, and Stag Hunt, which are modifications of PD by switching cells or values for the row or column players. Deadlock is the simplest among the four games. The game Deadlock is similar to the PD game, but mutual defection gives higher payoffs to both players than mutual cooperation. The PD game is the only one of the 78 games for which no RE is a Pareto maximum in payoff.

We present the RM's and DM's for PD, Deadlock, Chicken, and Stag Hunt in the following figures and discuss how regret and disappointment can explain the strategies for these games.

	Utility Matrix		Regret Matrix		Disappointment Matrix	
	Cooperate	*Defect*	*Cooperate*	*Defect*	*Cooperate*	*Defect*
Cooperate	(2,2)	(0,3)	(1,1)	(1,0)	(0,0)	(2,0)
Defect	(3,0)	(1,1)	(0,1)	(0,0)	(0,2)	(2,2)

Figure 5.17 Prisoner's Dilemma Matrices

The choice of strategies under regret and disappointment of PD was explained in Chapter 4, where a player chooses either the *Cooperate* or *Defect* strategy depending whether he uses the disappointment or regret criteria, respectively.

	Utility Matrix		Regret Matrix		Disappointment Matrix	
	Cooperate	*Defect*	*Cooperate*	*Defect*	*Cooperate*	*Defect*
Cooperate	(1,1)	(0,3)	(2,2)	(2,0)	(0,0)	(1,0)
Defect	(3,0)	(2,2)	(0,2)	(0,0)	(0,1)	(1,1)

Figure 5.18 Deadlock Matrices

In Deadlock game, the total regret is higher than disappointment. Either player has no incentive to choose the *Cooperate* strategy.

	Utility Matrix		Regret Matrix		Disappointment Matrix	
	Chicken	*Dare*	*Chicken*	*Dare*	*Chicken*	*Dare*
Chicken	(2,2)	(1,3)	(1,1)	(0,0)	(0,0)	(1,0)
Dare	(3,1)	(0,0)	(0,0)	(1,1)	(0,1)	(3,3)

Figure 5.19 Chicken Game Matrices

In Chicken, there are two pure RE's but only one pure DE. If both players *Dare*, each has a higher disappointment than regret. The pure DE is where both players *Chicken Out*.

	Utility Matrix		Regret Matrix		Disappointment Matrix	
	Stag	*Hare*	*Stag*	*Hare*	*Stag*	*Hare*
Stag	(3,3)	(0,2)	(0,0)	(1,1)	(0,0)	(3,0)
Hare	(2,0)	(1,1)	(1,1)	(0,0)	(0,3)	(1,1)

Figure 5.20 Stag Hunt Matrices

In the Stag Hunt game, neither player has regret if both players choose the same strategy. However, when both hunt hare, each player incurs disappointment.

Figure 5.22 shows the utility and transformation matrices of the Battle of Sexes paradox discussed in Chapter 2. Observe that the disappointment and regret matrices for the Battle of Sex game are symmetric.

	Utility Matrix		Regret Matrix		Disappointment Matrix	
	Opera	*Football*	*Opera*	*Football*	*Opera*	*Football*
Opera	(2,1)	(0,0)	(0,0)	(1,1)	(0,0)	(2,2)
Football	(0,0)	(1,2)	(2,2)	(0,0)	(1,1)	(0,0)

Figure 5.21 Battle of Sexes Matrices

We summarize our solutions for the above social dilemmas, as well as the Battle of Sexes.

1. PD: DE and RE give different pure strategies. The RE is dominated by DE. DE is a Pareto maximum between the two equilibria. The DE is a π.

2. Deadlock: DE and RE give different pure strategies. The DE is dominated by the RE. The RE is a Pareto maximum, as well as a π.

3. Stag Hunt: RE gives two pure strategies; DE gives one that is the same as an RE. One RE is dominated by the other RE that is also a DE. The DE is a π.

4. Chicken: RE gives two pure strategies; DE gives one. None are the same. There is also a mixed RE with $\boldsymbol{x}$ = (0.5, 0.5), $\boldsymbol{y}$ = (0.5, 0.5) yielding a utility of 1.5 for each player. However it is not a Pareto maximum. All pure equilibria are Pareto minima of their respective type. The DE is a π.

5. Battle of the Sexes: the pure RE's and pure DE's are the same. Both are Pareto optima and π's. There is also a mixed RE with $\boldsymbol{x}$ = (0.67, 0.33), $\boldsymbol{y}$ = (0.33, 0.67), yielding a payoff of 0.67 for each player. The mixed DE $\boldsymbol{x}$ = (0.5, 0.5) and $\boldsymbol{y}$ = (0.5, 0.5) yields a utility of 0.75 for each player. Neither mixed strategy is a π. However, the mixed DE gives a better expected utility than the mixed RE.

5.5 The Cournot-Nash Equilibrium and the DE

The classic game in economics called an oligopoly game is another example where the best-response solution of Cournot [26] is an RE (NE) dominated by a DE. In oligopolistic markets, the pricing and productions of every firm in industry have a significant effect on the profitability of its competitors. The Cournot model assumes that the market price per unit of output is a decreasing function of the total output produced by all firms [2]. Hence deciding on the quantity of output by a firm affects his profit as well as his opponent's profit.

In an example of McMillan [27], two firms ***A*** and ***B*** produce an identical product. The relationship between the price (ρ) and the output (τ) is $\rho = 13 - \tau$. Each unit sold costs \$1 to the firm.

The total output τ = ***A***'s output τ_a + ***B***'s output τ_b.

A's profit $\varphi_a = (13 - (\tau_a + \tau_b))\tau_a - \tau_a$. (5.8)

We can find ***A***'s best-response function by taking the partial derivative of (5.8) with respect to τ_a and finding the output level at which this derivative equals zero. Thus we have

$$\frac{\partial \varphi_a}{\partial \tau_a} = 12 - 2\,\tau_a - \tau_b = 0.$$

A's optimal output in terms of ***B***'s output gives ***A***'s best-response function as

$$\tau_a = (12 - \tau_b)/2. \tag{5.9}$$

Similarly, ***B***'s profit is $\varphi_b = (13 - (\tau_a + \tau_b))\tau_b - \tau_b$,

with ***B***'s best-response function given by $\tau_b = (12 - \tau_a)/2$.

An RE (NE) of this game is where each firm plays its best response. A assumes that B decides to produce according to its best-response function, in addition to A itself doing so. A's equilibrium output thus satisfies

$\tau_a = [12 - (12 - \tau_a)/2]/2 = 4.$

Symmetrically, B has the equilibrium output $\tau_b = 4$. Therefore, the equilibrium profit to each firm is \$16. ***A***'s and ***B***'s profits (φ_a, φ_b) at different levels of their outputs are shown in Table 5.1. Table 5.2 shows ***A***'s and ***B***'s regrets (R_a, R_b) at each level of their outputs. A firm will incur regret if its opponent produces at a Cournot NE but the firm itself does not.

Table 5.1 Profits to Firm ***A*** and Firm ***B***

		τ_b									
		0	1	2	3	4	5	6	7	8	9
τ_a	0	(0,0)	(0,11)	(0,20)	(0,27)	(0,32)	(0,35)	(0,36)	(0,35)	(0,32)	(0,27)
	1	(11,0)	(10,10)	(9,18)	(8,24)	(7,28)	(6,30)	(5,30)	(4,28)	(3,24)	(2,18)
	2	(20,0)	(18,9)	(16,16)	(14,21)	(12,24)	(10,25)	(8,24)	(6,21)	(4,16)	(2,9)
	3	(27,0)	(24,8)	(21,14)	(18,18)	(15,20)	(12,20)	(9,18)	(6,14)	(3,8)	(0,0)
	4	(32,0)	(28,7)	(24,12)	(20,15)	(16,16)	(12,15)	(8,12)	(4,7)	(0,0)	(-4,-9)
	5	(35,0)	(30,6)	(25,10)	(20,12)	(15,12)	(10,10)	(5,6)	(0,0)	(-5,-8)	(-10,-18)
	6	(36,0)	(30,5)	(24,8)	(18,9)	(12,8)	(6,5)	(0,0)	(-6,-7)	(-12,-16)	(-18,-27)
	7	(35,0)	(28,4)	(21,6)	(14,6)	(7,4)	(0,0)	(-7,-6)	(-14,-14)	(-21,-24)	(-28,-36)
	8	(32,0)	(24,3)	(16,4)	(8,3)	(0,0)	(-8,-5)	(-16,-12)	(-24,-21)	(-32,-32)	(-40,-45)
	9	(27,0)	(18,2)	(9,2)	(0,0)	(-9,-4)	(-18,-10)	(-27,-18)	(-36,-28)	(-45,-40)	(-54,-54)

Table 5.2 The Regret Matrix of Firm ***A*** and Firm ***B***

		τ_b									
		0	1	2	3	4	5	6	7	8	9
τ_a	0	(36,36)	(30,25)	(25,16)	(20,9)	(16,4)	(12,1)	(9,0)	(6,1)	(4,4)	(2,9)
	1	(25,30)	(20,20)	(16,12)	(12,6)	(9,2)	(6,0)	(4,0)	(2,2)	(1,6)	(0,12)
	2	(16,25)	(12,16)	(9,9)	(6,4)	(4,1)	(2,0)	(1,1)	(0,4)	(0,9)	(0,16)
	3	(9,20)	(6,12)	(4,6)	(2,2)	(1,0)	(0,0)	(0,2)	(0,6)	(1,12)	(2,20)
	4	(4,16)	(2,9)	(1,4)	(0,1)	(0,0)	(0,1)	(1,4)	(2,9)	(4,16)	(6,25)
	5	(1,12)	(0,6)	(0,2)	(0,0)	(1,0)	(2,2)	(4,6)	(6,12)	(9,20)	(12,30)
	6	(0,9)	(0,4)	(1,1)	(2,0)	(4,1)	(6,4)	(9,9)	(12,16)	(16,25)	(20,36)
	7	(1,6)	(2,2)	(4,0)	(6,0)	(9,2)	(12,6)	(16,12)	(20,20)	(25,30)	(30,42)
	8	(4,4)	(6,1)	(9,0)	(12,1)	(16,4)	(20,9)	(25,16)	(30,25)	(36,36)	(42,49)
	9	(9,2)	(12,0)	(16,0)	(20,2)	(25,6)	(30,12)	(36,20)	(42,30)	(49,42)	(56,56)

Now consider ***A***'s or ***B***'s disappointment matrix in Table 5.3. If A fixes its level of output, then A has an increasing disappointment as B's output increases since the price of the product decreases by its level of output, regardless of who produces it.

Table 5.3 The Disappointment Matrix of Firm ***A*** and Firm ***B***

		τ_b									
		0	1	2	3	4	5	6	7	8	9
τ_a	0	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
	1	(0,0)	(1,1)	(2,2)	(3,3)	(4,4)	(5,5)	(6,6)	(7,7)	(8,8)	(9,9)
	2	(0,0)	(2,2)	(4,4)	(6,6)	(8,8)	(10,10)	(12,12)	(14,14)	(16,16)	(18,18)
	3	(0,0)	(3,3)	(6,6)	(9,9)	(12,12)	(15,15)	(18,18)	(21,21)	(24,24)	(27,27)
	4	(0,0)	(4,4)	(8,8)	(12,12)	(16,16)	(20,20)	(24,24)	(28,28)	(32,32)	(36,36)
	5	(0,0)	(5,5)	(10,10)	(15,15)	(20,20)	(25,25)	(30,30)	(35,35)	(40,40)	(45,45)
	6	(0,0)	(6,6)	(12,12)	(18,18)	(24,24)	(30,30)	(36,36)	(42,42)	(48,48)	(54,54)
	7	(0,0)	(7,7)	(14,14)	(21,21)	(28,28)	(35,35)	(42,42)	(49,49)	(56,56)	(63,63)
	8	(0,0)	(8,8)	(16,16)	(24,24)	(32,32)	(40,40)	(48,48)	(56,56)	(64,64)	(72,72)
	9	(0,0)	(9,9)	(18,18)	(27,27)	(36,36)	(45,45)	(54,54)	(63,63)	(72,72)	(81,81)

From Table 5.1 we need only consider only A's and B's profits from the output decision levels of 3 and 4 units, as shown Figure 5.23. The resulting game resembles Prisoner's Dilemma.

	Player *II*	
	$\tau_b = 3$	$\tau_b = 4$
Player *I* $\tau_a = 3$	(18,18)	(15,20)
$\tau_a = 4$	(20,15)	(16,16)

Figure 5.22 The 2 × 2 Bimatrix Profits for Firms ***A*** and ***B***

The 2 × 2 RM and DM of this game are shown in Figures 5.24 and 5.25. While the output level of 4 units per firm is the Cournot RE with a profit of \$16 per firm, an output of 3 units per firm is the DE with a profit of \$18 per firm. The firms would thus improve on the classical Cournot classical solution with the DE.

	Player *II*	
	$\tau_b = 3$	$\tau_b = 4$
Player *I* $\tau_a = 3$	(2,2)	(1,0)
$\tau_a = 4$	(0,1)	(0,0)

Figure 5.23 The 2 × 2 RM for Firms ***A*** and ***B***

	Player *II*	
	$\tau_b = 3$	$\tau_b = 4$
Player *I* $\tau_a = 3$	(0,0)	(3,0)
$\tau_a = 4$	(0,3)	(3,3)

Figure 5.24 The 2 × 2 DM for Firms ***A*** and ***B***

CHAPTER 6

EQUILIBRIA OF N-PERSON GAMES

6.1 Nash or Regret Equilibria

The notion of a Disappointment Equilibrium (DE) is now extended to N-person noncooperative games. We begin by defining the Nash Equilibrium (or Regret Equilibrium RE). Again, an RE results from each player responding to the possible actions of the other players by choosing a strategy to minimizing his regret.

Definition 6.1. For a finite N-person game, $N \geq 2$, let

– i denote a player,

– $m(i)$ denote a finite number of player i's pure strategies,

– $\boldsymbol{x}_i$ denote the individual mixed strategy of Player i,

– $\alpha(i)$ denote the α^{th} component of Player i's mixed or pure strategy $\boldsymbol{x}_i$,

– $x_{i,\alpha(i)}$ denote the α^{th} component of the individual mixed strategy $\boldsymbol{x}_i$,

– $\boldsymbol{e}_{i,\alpha(i)}$ denote the α^{th} pure strategy of Player i's $m(i)$ pure strategies, where $\boldsymbol{e}_{i,\alpha(i)} =$ $(0,\ldots, 0, 1, 0,\ldots, 0)$ with 1 as component $\alpha(j)$ and 0 elsewhere for the $m(i)$ components.

– X_i, be a set of mixed strategies of any player i, where

$$X_i = \{\boldsymbol{x}_i \mid \boldsymbol{x}_i = (x_{i,1}, \ldots, x_{i,\alpha(i)},\ldots, x_{i,m(i)})^T, x_{i,\alpha(i)} \geq 0, \ \sum_{\alpha(i)=1}^{m(i)} x_{\alpha(i)} = 1\}.$$

Definition 6.2 (Nash [5]). The point $(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*)$ is an RE for a finite N-person game if and only if the payoff $p_i(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*)$ for player $i = 1,\ldots, N$ satisfies

$$p_i(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*) = \max_{\boldsymbol{x}_i \in X_i} p_i(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_{i-1}^*, \boldsymbol{x}_i, \boldsymbol{x}_{i+1}^*,\ldots, \boldsymbol{x}_N^*). \tag{6.1}$$

Theorem 6.1 (Nash [5]). Every finite N-person game has an RE.

Lemma 6.1 (Nash [5]). For a finite N-person game, for $i = 1,\ldots,$ N,

$$p_i(\boldsymbol{x}_1, \boldsymbol{x}_2,\ldots ,\boldsymbol{x}_N) = \sum_{\alpha(1)=1}^{m(1)} \sum_{\alpha(2)=1}^{m(2)} \cdots \sum_{\alpha(N)=1}^{m(N)} x_{1,\alpha(1)} \ldots, x_{, N,\alpha(N)}\, p_i[\boldsymbol{e}_{1,\alpha(1)}, \ldots ,\boldsymbol{e}_{N,\alpha(N)}]$$

$$= \sum_{\alpha(i)=1}^{m(i)} x_{i,\alpha(i)}\, p_{i,\alpha(i)}[\boldsymbol{x}_1,\ldots, \boldsymbol{x}_{i-1}, \boldsymbol{e}_{i,\alpha(i)}, \boldsymbol{x}_{i+1},\ldots, \boldsymbol{x}_N].$$

Lemma 6.2 (Nash [5]). A mixed strategy $(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*)$ is anRE for a finite N-person game if and only if for any player i

$$p_i[\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_{i-1}^*, \boldsymbol{e}_{i,\alpha(i)}, \boldsymbol{x}_{i+1}^*,\ldots, \boldsymbol{x}_N^*] \leq p_i(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*) \tag{6.2}$$

and

$$\sum_{\alpha(1)=1}^{m(1)} \cdots \sum_{\alpha(i-1)=1}^{m(i-1)} \sum_{\alpha(i+1)=1}^{m(i+1)} \cdots \sum_{\alpha(N)=1}^{m(N)} x_{1,\alpha(1)} \ldots x_{i-1,\alpha(i-1)}\, x_{i+1,\alpha(i+1)} \ldots x_{N,\alpha(N)}\, p_i\, [\boldsymbol{e}_{1,\alpha(1)}, \ldots, \boldsymbol{e}_{N,\alpha(N)}] \leq$$

$$\sum_{\alpha(1)=1}^{m(1)} \cdots \sum_{\alpha(N)=1}^{m(N)} x_{1,\alpha(1)} \ldots x_{N,\alpha(N)}\, p_i\, [\boldsymbol{e}_{1,\alpha(1)}, \ldots, \boldsymbol{e}_{N,\alpha(N)}] \text{ for } \alpha(i) = 1,\ldots, m(i);\ i = 1,\ldots, N.$$

6.2 Disappointment Equilibria

A DE results from each player choosing a strategy to minimize his disappointment as a consequence of the other players' response to his action. We first the notion of marginal DE's to show the generality of the Disappointment Equilibrium and possible use in coalitions of players as future work.

6.2.1 Marginal Disappointment Equilibria

Definition 6.3. For a finite N-person game, $N \geq 2$, let

- $<p_1,\ldots, p_N>$ represent a finite N-person game, where

  p_i denote a payoff function for Player i,
- i, j denote a player,
- $\alpha(j)$ denote Player j's α^{th} pure strategy,
- $m(j)$ denote a finite number of Player j's pure strategies,
- $\boldsymbol{x}_j$ denote the individual mixed strategy of Player j,
- $x_{j,\alpha(j)}$ denote the α^{th} component of the individual mixed strategy $\boldsymbol{x}_j$,
- X_j, be a set of mixed strategies of any player j, where

$$X_j = \{\boldsymbol{x}_j \mid \boldsymbol{x}_j = (x_{j,1}, \ldots, x_{j,\alpha(j)}, \ldots, x_{j,m(j)})^T \in \boldsymbol{R}^{m(j)},\ x_{j,\alpha(j)} \geq 0,\ \sum_{\alpha(j)=1}^{m(j)} x_{\alpha(j)} = 1\},$$

– $\boldsymbol{e}_{j,\alpha(j)}$ denote the α^{th} pure strategy of Player j's $m(j)$ pure strategies, where $\boldsymbol{e}_{j,\alpha(j)} =$ $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 as component $\alpha(j)$ and 0 elsewhere for the $m(j)$ components.

Definition 6.4. A mixed strategy $(\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_N^*) \in X_1 \times \ldots \times X_N$ is a j^{th} Marginal Disappointment Equilibrium (MDE) for a finite N-person game if and only if the payoff $p_i(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N)$ for $i = 1, \ldots, N;\ i \neq j.$ satisfies

$$p_i(\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_{j-1}^*, \boldsymbol{x}_j, \boldsymbol{x}_{j+1}^*, \ldots, \boldsymbol{x}_N^*) \leq p_i(\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_N^*) \text{ for } \boldsymbol{x}_j \in X_j. \qquad (6.3)$$

According to Definition 6.4, for $i = 1, \ldots, N$, exactly one player $j \neq i$ could unilaterally change his strategy and possibly make Player *i's* payoff worse but could not make it better, thereby gaining some control over the other players, who know this fact. Observe further in Definition 6.5 observe that if Player i were allowed to be identical to Player j (something not allowed), then inequality (6.3) would be precisely the definition of an RE. Hence, Disappointment Equilibria represent a generalization of Nash Equilibria. In effect, disappointment with respect to oneself is regret.

Definition 6.5. A mixed strategy $(\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_N^*) \in X_1 \times \ldots \times X_N$ is a Marginal Disappointment Equilibrium (MDE) for a finite N-person game if and only if the payoff $p_i(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N)$ for $i = 1, \ldots, N$ satisfies

$$p_i\,[\boldsymbol{x}_1{}^*,\ldots, \boldsymbol{x}_{j-1}{}^*, \boldsymbol{x}_j, \boldsymbol{x}_{j+1}{}^*,\ldots, \boldsymbol{x}_N{}^*] \;\le\; p_i\,(\boldsymbol{x}_1{}^*,\ldots, \boldsymbol{x}_N{}^*) \text{ for}$$

$$\boldsymbol{x}_j \in X_j,\, j = 1,\ldots, N,\, j \neq i. \tag{6.4}$$

In Definition 6.5, for $i = 1,\ldots,$ N, any player $j \neq i$ could unilaterally change his strategy and possibly make Player *i's* payoff worse but could not make it better.

Theorem 6.2. Every finite N-person game has a MDE $(\boldsymbol{x}_1{}^*,\ldots, \boldsymbol{x}_N{}^*)$.

Proof. Let $i \in \{1,\ldots, N\}$. For player $j = 1,\ldots, N$ and mixed strategy $(\boldsymbol{x}_1,\ldots, \boldsymbol{x}_N)$ for players $1,\ldots, N$ define the continuous marginal disappointment function

$$d_i\,(\boldsymbol{x}_1,\ldots, \boldsymbol{x}_N;\, \alpha(j)) = \max\,\{0,\, p_i[\boldsymbol{x}_1,\ldots, \boldsymbol{x}_{j-1},\, \boldsymbol{e}_{j,\alpha(j)}, \boldsymbol{x}_{j+1},\ldots, \boldsymbol{x}_N] - p_i\,(\boldsymbol{x}_1,\ldots, \boldsymbol{x}_N)\},$$

where $p_i\;[\boldsymbol{x}_1,\ldots, \boldsymbol{x}_{j-1},\, \boldsymbol{e}_{j,\alpha(j)}, \boldsymbol{x}_{j+1},\ldots, \boldsymbol{x}_N]$ is the payoff for $p_i\,(\boldsymbol{x}_1,\ldots, \boldsymbol{x}_N)\}$ when player $j \neq i$ chooses his $\alpha(j)^{\text{th}}$ strategy.

Now define the continuous map

$$f(\boldsymbol{x}_1,\ldots, \boldsymbol{x}_N) = (f_1(\boldsymbol{x}_1,\ldots, \boldsymbol{x}_N),\ldots,f_N(\boldsymbol{x}_1,\ldots, \boldsymbol{x}_N)) \colon X_1\times\ldots\times X_N \to X_1\times\ldots\times X_N\,, \tag{6.5}$$

a compact subset of $\boldsymbol{R}^{m(1)}\times\ldots\times\boldsymbol{R}^{m(N)}$ identified with $\boldsymbol{R}^{\sum_{j=1}^{N} m(j)}$. Also, denote component $\alpha(j)$, $\alpha(j) = 1,\ldots, m(j)$, of $f_j(\boldsymbol{x}_1,\ldots, \boldsymbol{x}_N)$, $j = 1,\ldots, N, j \neq i$, in (4) as $f_j(\boldsymbol{x}_1,\ldots, \boldsymbol{x}_N\,;\, \alpha(j))$ given by

$$f_j\,(\boldsymbol{x}_1,\ldots, \boldsymbol{x}_N\,;\, \alpha(j)) = \frac{x_{j,\alpha(j)} + d_j(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N)}{1+\sum_{j=1}^{N} d_j(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N)},\quad j = 1,\ldots, N. \tag{6.6}$$

Note that $f_j(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N;\alpha(j)) \geq 0$, $\alpha(j) = 1,\ldots, m(j)$, $j = 1,\ldots, N$, from (6.4), (6.5), (6.6), and the fact that $\boldsymbol{x}_i = (x_{i,1},\ldots, x_{i,\alpha(i)},\ldots, x_{i,m(i)})^T$ is a mixed strategy. Moreover from (6.6), $\sum_{\alpha(j)}^{m(j)} f_j(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N;\alpha(j)) = 1$, $j = 1,\ldots, N$, so in fact $f\colon X_1\times\ldots\times X_N \rightarrow X_1\times\ldots\times X_N$.

Consider now $f(\boldsymbol{x}_1, \boldsymbol{x}_2,\ldots,\boldsymbol{x}_N) = (f_1(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N),\ldots,f_N(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N))$, and define component $f_j(\boldsymbol{x}_1,\ldots, \boldsymbol{x}_N)$ by (6.6). By Brouwer's fixed point theorem there exists a fixed point $(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*) \in X_1\times\ldots\times X_N$ for which component $\alpha(j)$ of $f_j(\boldsymbol{x}_1,\ldots, \boldsymbol{x}_N)$ satisfies

$$f_j(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*;\ \alpha(j)) = x_{j,\alpha(j)}^* = \frac{x_{j,\alpha(j)}^* + d_i(\boldsymbol{x}_1^*,\ldots,\boldsymbol{x}_N^*;\ \alpha(\mathrm{j}))}{1+\sum_{\alpha(j)=1}^{m(j)} d_i(\boldsymbol{x}_1^*,\ldots,\boldsymbol{x}_N^*;\ \alpha(\mathrm{j}))},\quad j = 1,\ldots, N.$$

(6.7)

For $j = 1,\ldots, N$; $j \neq i$, suppose that

$p_i[\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_{j-1}^*, \boldsymbol{e}_{j,\alpha(j)}, \boldsymbol{x}_{j+1}^*, \ldots, \boldsymbol{x}_N^*] = p_i(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*,\ldots, \boldsymbol{x}_N^*)$ for all $\alpha(j)$ with $x_{j,\alpha(j)}^* > 0$.

Then $\sum_{\alpha(j)=1}^{m(j)} x_{j,\alpha(j)}^*\, p_i[\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_{j-1}^*, \boldsymbol{e}_{j,\alpha(j)}, \boldsymbol{x}_{j+1}^*, \ldots, \boldsymbol{x}_N^*] = p_i(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*,\ldots, \boldsymbol{x}_N^*) >$

$$\sum_{\alpha(j)=1}^{m(j)} x_{j,\alpha(j)}^*\, p_i[\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*] = p_i(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*,\ldots, \boldsymbol{x}_N^*).$$

But this result is a contradiction. Hence for $j = 1,\ldots, N$; $j \neq i$, there exist $\hat{\alpha}(j)^i$ with $x_{j,\alpha(j)}^* > 0$, for which

$$p_i[\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_{j-1}^*, \boldsymbol{e}_{j,\hat{\alpha}(j)^i}, \boldsymbol{x}_{j+1}^*, \ldots, \boldsymbol{x}_N^*] \leq p_i(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*,\ldots, \boldsymbol{x}_N^*),\ i = 1,\ldots, N;\ i \neq j.$$

Thus,

$$d_i(\boldsymbol{x}_1^*,\ldots,\boldsymbol{x}_N^*;\ \hat{\alpha}(j)^i) = 0,\ j = 1,\ldots, N;\ j \neq i, \tag{6.8}$$

where $\hat{\alpha}(j)^i$ denotes dependence on i.

Observe that (6.8) holds for the same fixed point ($\boldsymbol{x}_1$*, $\boldsymbol{x}_2$*,…, $\boldsymbol{x}_N$*) for each $i = 1,\ldots, N$; $i \neq j$. Hence from (6.7)

$$x_{j,\hat{\alpha}(j)}{}^* = \frac{x_{j,\hat{\alpha}(j)^i}{}^*}{1+\sum_{\alpha(j)=1}^{m(j)} d_i(\boldsymbol{x}_1{}^*,\ldots,\boldsymbol{x}_N{}^*;\ \alpha(\mathrm{j}))},\quad j = 1,\ldots, N; j \neq i, \text{ from which}$$

$$\sum_{\alpha(j)=1}^{m(j)} d_i(\boldsymbol{x}_1{}^*,\ldots,\boldsymbol{x}_N{}^*;\ \alpha(\mathrm{j})) = 0, j = 1,\ldots, N; j \neq i, \text{ since}$$

$$d_i(\boldsymbol{x}_1{}^*,\ldots,\boldsymbol{x}_N{}^*;\ \hat{\alpha}(j)) = 0, j = 1,\ldots, N; j \neq i. \text{ So from (6.3)}$$

$$p_i\,[\boldsymbol{x}_1{}^*,\ldots, \boldsymbol{x}_{j-1}{}^*\ ,\ \boldsymbol{e}_{j,\alpha(j)},\ \boldsymbol{x}_{j+1}{}^*,\ldots, \boldsymbol{x}_N{}^*] \ \leq\ p_i\,(\boldsymbol{x}_1{}^*,\ldots, \boldsymbol{x}_N{}^*), i = 1,\ldots, N, i \neq j.$$

Thus by definition ($\boldsymbol{x}_1$*,…, $\boldsymbol{x}_N$*) is an MDE.

Notice that an MDE ($\boldsymbol{x}_1$*,…, $\boldsymbol{x}_N$*) maximizes $p_i\,(\boldsymbol{x}_1{}^*,\ldots, \boldsymbol{x}_{j-1}{}^*\ , \boldsymbol{x}_j,\ \boldsymbol{x}_{j+1}{}^*,\ldots, \boldsymbol{x}_N{}^*)$ for $i = 1,\ldots, N$, $j = 1,\ldots, N$, $j \neq i$. If it were allowed that $j = i$, we get the definition of an RE by considering maximizes $p_i\,(\boldsymbol{x}_1{}^*,\ldots, \boldsymbol{x}_{i-1}{}^*\ , \boldsymbol{x}_i,\ \boldsymbol{x}_{i+1}{}^*,\ldots, \boldsymbol{x}_N{}^*)$ for $i = 1,\ldots, N$. An RE would then be considered as the i^{th} marginal DE of <$p_1,\ldots, p_N$> for $p_i\,(\boldsymbol{x}_1,\ldots, \boldsymbol{x}_N)$, $i = 1,\ldots, N$. The definition of an MDE precluded this interpretation, but it can be said that MDE's in some sense generalize RE's.

6.2.2 Total Disappointment Equilibria

The notion of an MDE generalizes that for a DE when N = 2. Another generalization is now given.

Definition 6.6. For a finite N-person game, a mixed strategy $(\boldsymbol{x}_1^*,\ldots,\boldsymbol{x}_N^*) \in X_1 \times \ldots \times X_N$ is a Total Disappointment Equilibrium (TDE) if and only if the payoff $p_i(\boldsymbol{x}_1, \boldsymbol{x}_2,\ldots,\boldsymbol{x}_N)$, $i = 1,\ldots, N,$ satisfies

$$p_i(\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_N^*) = \max_{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{i-1},\boldsymbol{x}_{i+1},\ldots\boldsymbol{x}_N} p_i(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i-1}, \boldsymbol{x}_i^*, \boldsymbol{x}_{i+1}, \ldots, \boldsymbol{x}_N),$$

$$\boldsymbol{x}_j \in X_j;\ j = 1,\ldots, N, j \neq i, i = 1,\ldots, N. \qquad (6.9)$$

In Definition 6.6, no player i will not be disappointed with $(\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_N^*)$ if any number of the other players $j \neq i$ change strategies. Thus a TDE characterized by (6.9) is an extremely strong concept, much stronger than an RE characterized by (6.1), where the maximization of $p_i(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_{i-1}^*, \boldsymbol{x}_i, \boldsymbol{x}_{i+1}^*,\ldots, \boldsymbol{x}_N^*)$ is only the single variable $\boldsymbol{x}_i \in X_i$. Moreover, the TDE $(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*)$ may be considered a complementary equilibrium, even a dual equilibrium, to an RE for the game $<p_1,\ldots,p_N>$. For an RE, no individual player i can unilaterally improve his payoff by changing from $\boldsymbol{x}_i^*$ to $\boldsymbol{x}_i$. A TDE represents the complementary situation where for each player i any or all other players can possibly make Player i's payoff worse and can certainly make it no better by changing from $\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_{i-1}^*, \boldsymbol{x}_{i+1}^*, \ldots, \boldsymbol{x}_N^*$ to some $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i-1}, \boldsymbol{x}_{i+1}, \ldots, \boldsymbol{x}_N$. Again a TDE requires much more than a RE. However, a TDE always exists.

Theorem 6.3. Every finite N-person game $<p_1,\ldots,p_N>$ has a TDE $(\boldsymbol{x}_1^*,\ldots,\boldsymbol{x}_N^*)$.

Proof. Let $i \in \{1,\ldots, N\}$. Define the continuous nonnegative disappointment function

$d_i(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N; \alpha(1),\ldots,\alpha(i\text{-}1), \alpha(i+1),\ldots,\alpha(N))$

$$= \max\{0, p_i[\boldsymbol{e}_{1,\alpha(1)},\ldots,\boldsymbol{e}_{i-1,\alpha(i-1)}, \boldsymbol{x}_i, \boldsymbol{e}_{i+1,\alpha(i+1)},\ldots, \boldsymbol{e}_{N,\alpha(N)}] - p_i(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N)\}, \tag{6.10}$$

where $p_i[\boldsymbol{e}_{1,\alpha(1)},\ldots,\boldsymbol{e}_{i-1,\alpha(i-1)}, \boldsymbol{x}_i, \boldsymbol{e}_{i+1,\alpha(i+1)},\ldots, \boldsymbol{e}_{N,\alpha(N)}]$ is the payoff for player i when player $j \neq i$, $j = 1,\ldots, N,$ chooses j's $\alpha(j)^{\text{th}}$ pure strategy.

Now define the continuous map

$$f(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N) = (f_1(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N),\ldots, f_N(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N)): X_1\times\ldots\times X_N \rightarrow X_1\times\ldots\times X_N, \tag{6.11}$$

a compact subset of $\boldsymbol{R}^{m(1)}\times\ldots\times\boldsymbol{R}^{m(N)}$ identified with $\boldsymbol{R}^{\sum_{j=1}^{N} m(j)}$, where $X_j \subset \boldsymbol{R}^{m(j)}$ is the set of mixed strategies $X_j = \{\boldsymbol{x}_j \mid \boldsymbol{x}_j = (x_{j,1},\ldots, x_{j,\alpha(j)},\ldots, x_{j,m(j)})^T \in \boldsymbol{R}^{m(j)}, x_{j,\alpha(j)} \geq 0, \sum_{\alpha(j)=1}^{m(j)} x_{\alpha(j)} = 1\}$ for player j.

Denote component $\alpha(j)$, $\alpha(j) = 1,\ldots, m(j)$ of $f_j(\boldsymbol{x}_1,\ldots, \boldsymbol{x}_N)$, $j = 1,\ldots, N$ as $f_j(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N; \alpha(j))$ given by

$f_j(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N; \alpha(j)) =$

$$\begin{cases} \dfrac{x_{j,\alpha(j)} + \sum\limits_{\alpha(1)=1}^{m(1)} \cdots \sum\limits_{\substack{\alpha(k)=1 \\ k\neq i,j}}^{m(k)} \cdots \sum\limits_{\alpha(N)}^{m(N)} d_i(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N; \alpha(1),\ldots,\alpha(i-1),\alpha(i+1),\ldots,\alpha(N))}{1+\sum\limits_{\alpha(1)=1}^{m(1)} \cdots \sum\limits_{\substack{\alpha(k)=1 \\ k\neq i}}^{m(k)} \cdots \sum\limits_{\alpha(N)}^{m(N)} d_i(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N; \alpha(1),\ldots,\alpha(i-1),\alpha(i+1),\ldots,\alpha(N))}, & j=1,\ldots,N. \\ x_{i,\alpha(i)}, & j=i \end{cases} \tag{6.12}$$

Note that $f_j(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N; \alpha(j)) \geq 0$, $\alpha(j) = 1,\ldots, m(j)$, $j = 1,\ldots, N$ from (6.10), (6.11), (6.12).

Moreover from (6.12), $\sum_{\alpha(j)=1}^{m(j)} f_j(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N; \alpha(j)) = 1$ since $f_j(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N; \alpha(j)) \in \boldsymbol{R}^{m(j)}$ is a mixed strategy, $j = 1,\ldots,$ N. Thus in fact, $X_1\times\ldots\times X_N \rightarrow X_1\times\ldots\times X_N$.

Consider $f(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N) = (f_1(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N),\ldots, f_N(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N))$ with component defined for $f_j(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N)$ by (6.12). By Brouwer's fixed point theorem there exists a fixed point fixed point $(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*) \in X_1\times\ldots\times X_N$ for which component $\alpha(j)$ of $f_j(\boldsymbol{x}_1,\ldots, \boldsymbol{x}_N)$ satisfies

$$f_i(\boldsymbol{x}_1^*,\ldots,\boldsymbol{x}_N^*; \alpha(i)) = x_{i,\alpha(i)}^*$$

$$f_j(\boldsymbol{x}_1^*,\ldots,\boldsymbol{x}_N^*; \alpha(j)) =$$

$$\frac{x_{j,\alpha(j)}^* + \sum_{\alpha(1)=1}^{m(1)} \ldots \sum_{\substack{\alpha(k)=1 \\ k\neq i,j}}^{m(k)} \ldots \sum_{\alpha(N)}^{m(N)} d_i(\boldsymbol{x}_1^*,\ldots,\boldsymbol{x}_N^*; \alpha(1),\ldots,\ \alpha(i-1),\ \alpha(i+1),\ldots,\ \alpha(N))}{1+\sum_{\alpha(1)=1}^{m(1)} \ldots \sum_{\substack{\alpha(k)=1 \\ k\neq i}}^{m(k)} \ldots \sum_{\alpha(N)}^{m(N)} d_i(\boldsymbol{x}_1^*,\ldots,\boldsymbol{x}_N^*; \alpha(1),\ldots,\ \alpha(i-1),\ \alpha(i+1),\ldots,\ \alpha(N))},\ j \neq i. \quad (6.13)$$

Now for $j \neq i$ in (6.13), suppose for all $\alpha(j)$ with $x_{j,\alpha(j)}^* > 0$, there exists, $\hat{\alpha}(1),\ldots,\hat{\alpha}(k),\ldots,\hat{\alpha}(N)$, $k = 1,\ldots, N$, $k \neq i, j$, for which

$$p_i[\boldsymbol{e}_{1,\hat{\alpha}(1)},\ldots,\ \boldsymbol{e}_{k,\hat{\alpha}(k)},\ldots,\ \boldsymbol{e}_{j-1,\hat{\alpha}(j-1)},\ \boldsymbol{e}_{j,\alpha(j)},\ \boldsymbol{e}_{j+1,\hat{\alpha}(j+1)},\ldots,\ \boldsymbol{e}_{i-1,\hat{\alpha}(i-1)},\ \boldsymbol{x}_i^*,\ \boldsymbol{e}_{i+1,\hat{\alpha}(i+1)},\ldots,\ \boldsymbol{e}_{N,\hat{\alpha}(N)}] >$$

$$p_i[\boldsymbol{e}_{1,\hat{\alpha}(1)},\ldots,\ \boldsymbol{e}_{j-1,\hat{\alpha}(j-1)},\ \boldsymbol{x}_j^*,\ \boldsymbol{e}_{j+1,\hat{\alpha}(j+1)},\ldots,\ \boldsymbol{e}_{i-1,\hat{\alpha}(i-1)},\ \boldsymbol{x}_i^*,\ \boldsymbol{e}_{i+1,\hat{\alpha}(i+1)},\ldots,\ \boldsymbol{e}_{N,\hat{\alpha}(N)}] \quad (6.14)$$

Then multiplying each side of (6.14) by $x_{j,\alpha(j)}^*$ and summing over $\alpha(j) = 1,\ldots, m(j)$ gives the contradiction $p_i[\boldsymbol{e}_{1,\hat{\alpha}(1)},\ldots,\ \boldsymbol{e}_{j-1,\hat{\alpha}(j-1)},\ \boldsymbol{x}_j^*,\ \boldsymbol{e}_{j+1,\hat{\alpha}(j+1)},\ldots,\ \boldsymbol{e}_{i-1,\hat{\alpha}(i-1)},\ \boldsymbol{x}_i^*,\ \boldsymbol{e}_{i+1,\hat{\alpha}(i+1)},\ldots,\ \boldsymbol{e}_{N,\hat{\alpha}(N)}]$

is greater than itself. Hence there exist $\alpha(j)$ with $x_{j,\alpha(j)}^* > 0$ for which $\alpha(k)$, $k = 1,\ldots, N$; $k \neq i, j$, and $\hat{\alpha}(j)^i$ satisfy

$$p_i\,[\boldsymbol{e}_{1,\alpha(1)},\ldots,\ \boldsymbol{e}_{j-1,\alpha(j-1)},\ \boldsymbol{e}_{j,\hat{\alpha}(j)^i},\ \boldsymbol{e}_{j+1,\alpha(j+1)},\ldots,\ \boldsymbol{e}_{i-1,\alpha(i-1)},\ \boldsymbol{x}_i^*,\ \boldsymbol{e}_{i+1,\alpha(i+1)},\ldots,\ \boldsymbol{e}_{N,\alpha(N)}] \leq$$

$$p_i\,[\boldsymbol{e}_{1,\alpha(1)},\ldots,\ \boldsymbol{e}_{j-1,\alpha(j-1)},\ \boldsymbol{x}_j^*,\ \boldsymbol{e}_{j+1,\alpha(j+1)},\ldots,\ \boldsymbol{e}_{i-1,\alpha(i-1)},\ \boldsymbol{x}_i^*,\ \boldsymbol{e}_{i+1,\alpha(i+1)},\ldots,\ \boldsymbol{e}_{N,\alpha(N)}] \qquad (6.15)$$

But from (6.15) $\displaystyle\sum_{\alpha(1)=1}^{m(1)} \cdots \sum_{\substack{\alpha(k)=1\\ k\neq i,j}}^{m(k)} \cdots \sum_{\alpha(N)}^{m(N)} x_{1,\alpha(1)}\ldots x_{j-1,\alpha(j-1)}x_{j+1,\alpha(j+1)}\ldots x_{i-1,\alpha(i-1)}x_{i+1,\alpha(i+1)}\ldots x_{N,\alpha(N)}\times$

$$p_i\,[\boldsymbol{e}_{1,\alpha(1)},\ldots,\boldsymbol{e}_{j-1,\alpha(j-1)},\ \boldsymbol{e}_{j,\hat{\alpha}(j)^i},\ \boldsymbol{e}_{j+1,\alpha(j+1)},\ldots,\ \boldsymbol{e}_{i-1,\alpha(i-1)},\ \boldsymbol{x}_i^*,\ \boldsymbol{e}_{i+1,\alpha(i+1)},\ldots,\ \boldsymbol{e}_{N,\alpha(N)}] \leq$$

$$p_i\,(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*,\ldots, \boldsymbol{x}_N^*).$$

Hence,

$$p_i\,(\boldsymbol{x}_1^*,\ \ldots, \boldsymbol{x}_{j-1}^*, \boldsymbol{e}_{j,\hat{\alpha}(j)^i},\ \boldsymbol{x}_{j+1}^*\ldots, \boldsymbol{x}_N^*) \leq p_i\,(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*) \qquad (6.16)$$

Then from (6.16 by definition

$$d_i\,(\boldsymbol{x}_1^*,\ \ldots, \boldsymbol{x}_N^*;\ \alpha(1),\ldots, \alpha(j\text{-}1),\ \hat{\alpha}(j)^i,\ \alpha(j+1),\ldots, \alpha(N)) = 0 \qquad (6.17)$$

Note that (6.17) helds for the same fixed point $(\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_N^*)$ for $i = 1,\ldots, N$, where $\hat{\alpha}(j)^i$, $j \neq i$, depends on i.

From (6.13) it follows that

$$\sum_{\alpha(1)=1}^{m(1)} \cdots \sum_{\substack{\alpha(k)=1\\ k\neq i,j}}^{m(k)} \cdots \sum_{\alpha(N)}^{m(N)} d_i\,(\boldsymbol{x}_1^*,\ \ldots, \boldsymbol{x}_N^*;\ \alpha(1),\ldots, \alpha(j\text{-}1),\ \hat{\alpha}(j)^i,\ \alpha(j+1),\ldots, \alpha(N)) = 0,$$

$$i = 1,\ldots, N,\ j = 1,\ldots, N;\ j \neq i.$$

Then also from 6.13, for $i = 1,\ldots, N$, $j \neq i$

$$x_{j,\hat{\alpha}(j)^i} = \frac{x_{j,\hat{\alpha}(j)}{}^*}{1+\sum_{\alpha(1)=1}^{m(1)} \cdots \sum_{\substack{\alpha(k)=1\\k\neq i}}^{m(k)} \cdots \sum_{\alpha(N)}^{m(N)} d_i(\boldsymbol{x}_1^*,\ldots,\boldsymbol{x}_N^*;\alpha(1),\ldots,\alpha(i-1),\alpha(i+1),\ldots,\alpha(N))}. \tag{6.18}$$

It follows from (6.18) that

$d_i\,(\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_N^*; \alpha(1),\ldots, \alpha(i\text{-}1), \alpha(i+1),\ldots, \alpha(N)) = 0,\ i = 1,\ldots, N.$

Thus

$$p_i\,[\boldsymbol{e}_{1,\alpha(1)},\ldots, \boldsymbol{e}_{i-1,\alpha(i-1)}, \boldsymbol{x}_i^*, \boldsymbol{e}_{i+1,\alpha(i+1)},\ldots, \boldsymbol{e}_{N,\alpha(N)}] \le p_i\,(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*,\ldots, \boldsymbol{x}_N^*),$$

$$i = 1,\ldots, N;\ \alpha(j) = 1,\ldots, m(j), j \neq i. \tag{6.19}$$

Multiply both sides of (6.19) by

$x_{1,\alpha(1)}\ldots x_{i\text{-}1,\alpha(i\text{-}1)}x_{i+1,\alpha(i+1)}\ldots x_{N,\alpha(N)}$ for any $\boldsymbol{x}_j \in X_j, j \neq i$. Then summing over

$\sum_{\alpha(1)=1}^{m(1)} \cdots \sum_{\alpha(k)=1}^{m(k)} \cdots \sum_{\alpha(N)}^{m(N)}$, $k \neq i$, gives that $(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*,\ldots, \boldsymbol{x}_N^*)$ is a TDE.

We next show that TDE's and MDE's are actually the same, a fact not apparent by definition.

Theorem 6.4. The N-tuple of strategies $(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*)$ for the finite N-person game $<p_1,\ldots,p_N>$ is an MDE and only if it is a TDE. Hence, both can simply be called a Disappointment Equilibrium (DE) without ambiguity.

Proof. Let $(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*)$ be a TDE. Then it maximizes $p_i\,(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i-1}, \boldsymbol{x}_i^*, \boldsymbol{x}_{i+1}, \ldots, \boldsymbol{x}_N)$ over $\boldsymbol{x}_j \in \boldsymbol{X}_j, j = 1, \ldots, N, j \neq i$. In particular, $p_i\,[\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_{j-1}^*, \boldsymbol{x}_j, \boldsymbol{x}_{j+1}^*,\ldots, \boldsymbol{x}_N^*] \le$ $p_i\,(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*)$ for $\boldsymbol{x}_j \in X_j, j = 1,\ldots, N, j \neq i$. By definition, $(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*)$ is a MDE.

Next let $(\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_N^*)$ be an MDE. Suppose it's not a TDE. Then there exists $i \in \{1, \ldots, N\}$ and $j \neq i$ for which $p_i\,[\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_{j-1}^*, \boldsymbol{x}_j, \boldsymbol{x}_{j+1}^*, \boldsymbol{x}_{i-1}^*, \boldsymbol{x}_i^*, \boldsymbol{x}_{i+1}^*, \ldots, \boldsymbol{x}_N^*] <$ $p_i\,(\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_N^*)$ for all $\boldsymbol{x}_j \in \boldsymbol{X}_j$. Letting $\boldsymbol{x}_j = \boldsymbol{x}_j^*$ gives the contradiction that $p_i\,(\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_N^*) < p_i\,(\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_N^*)$. It follows that $(\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_N^*)$ is a TDE to complete the proof.

6.3 Properties of RE's and DE's

We next establish some properties of RE's and DE's. Two definitions are needed.

Definition 6.7. Let $(p_1, \ldots, p_N)$ be a finite game. Define the pure regret incurred by player $i = 1, \ldots, N$ when player i choose pure strategy $\boldsymbol{e}_{i,\alpha(i)}$ and the other $N-1$ players choose pure strategies $\boldsymbol{e}_{j,\alpha(j)}$, $j = 1, \ldots, N$, $j \neq i$ by

$$r_i\,[\boldsymbol{e}_{1,\alpha(1)}, \ldots, \boldsymbol{e}_{i-1,\alpha(i-1)}, \boldsymbol{e}_{i,\alpha(i)}, \boldsymbol{e}_{i+1,\alpha(i+1)}, \ldots, \boldsymbol{e}_{N,\alpha(N)}] =$$

$$\max_{\alpha(i)=1,\ldots,m(i)} p_i\,[\boldsymbol{e}_{1,\alpha(1)}, \ldots, \boldsymbol{e}_{i-1,\alpha(i-1)}, \boldsymbol{e}_{i,\alpha(i)}, \boldsymbol{e}_{i+1,\alpha(i+1)}, \ldots, \boldsymbol{e}_{N,\alpha(N)}] -$$

$$p_i\,[\boldsymbol{e}_{1,\alpha(1)}, \ldots, \boldsymbol{e}_{i-1,\alpha(i-1)}, \boldsymbol{e}_{i,\alpha(i)}, \boldsymbol{e}_{i+1,\alpha(i+1)}, \ldots, \boldsymbol{e}_{N,\alpha(N)}],\ i = 1, \ldots, N. \tag{6.20}$$

Definition 6.8. For the game of Definition 6.7, defined the continuous extension r_i: $X_1 \times \ldots \times X_N \to \boldsymbol{R}^1$ of the pure regret function (6.19) for any player i, $i = 1, \ldots, N$ when Player i chooses mixed strategy $\boldsymbol{x}_i$ and the other player $N-1$ choose mixed strategy

$\boldsymbol{x}_j$, $j = 1,\ldots, N$, $j \neq i$, by

$$r_i(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N) =$$

$$\sum_{\alpha(1)=1}^{m(1)} \cdots \sum_{\substack{\alpha(k)=1\\ k\neq i}}^{m(k)} \cdots \sum_{\alpha(N)}^{m(N)} x_{1,\alpha(1)}\ldots x_{i-1,\alpha(i-1)}x_{i+1,\alpha(i+1)}\ldots x_{N,\alpha(N)}\, c_i(\alpha(1),\ldots,\alpha(i\text{-}1),\alpha(i+1),\ldots,\alpha(N)) -$$

$$p_i(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N),\ i = 1,\ldots, N, \tag{6.21}$$

where

$$c_i(\alpha(1),\ldots,\alpha(i\text{-}1),\alpha(i+1),\ldots,\alpha(N)) =$$

$\max\limits_{\alpha(i)} p_i\,[\boldsymbol{e}_{1,\alpha(1)},\ldots, \boldsymbol{e}_{i-1,\alpha(i-1)}, \boldsymbol{e}_{i,\alpha(i)}, \boldsymbol{e}_{i+1,\alpha(i+1)},\ldots, \boldsymbol{e}_{N,\alpha(N)}]$, $i = 1,\ldots, N$, for all the $\alpha(j)^{\text{th}}$ pure strategies of the other players $j = 1,\ldots, N$, $j \neq i$.

We now have the following result.

Theorem 6.5. The set of RE's for the finite game $<p_1,\ldots, p_N>$ is the same as the set of RE's for the finite game $<r_1,\ldots, r_N>$.

Proof. The strategies $(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*,\ldots, \boldsymbol{x}_N^*)$ is an RE for $<p_1,\ldots, p_N>$ if and only if

$$p_i(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_{i-1}^*, \boldsymbol{x}_i, \boldsymbol{x}_{i+1}^*,\ldots, \boldsymbol{x}_N^*) \leq p_i(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*),\ i = 1,\ldots, N.$$

But this inequality is true from (6.21) if and only if

$$r_i[\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_{i-1}^*, \boldsymbol{x}_i, \boldsymbol{x}_{i+1}^*,\ldots, \boldsymbol{x}_N^*] \leq r_i(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*),\ i = 1,\ldots, N,$$

in which case $(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*)$ is an RE for $<r_1,\ldots, r_N>$.

Corollary 6.1 The N-tuple of pure strategies $(\boldsymbol{e}_{1,\alpha(1)^*},\ldots,\boldsymbol{e}_{N,\alpha(N)^*})$ is a pure RE for $<p_1,\ldots, p_N>$ if and only if $(r_1(\boldsymbol{e}_{1,\alpha(1)^*},\ldots,\boldsymbol{e}_{N,\alpha(N)^*}), \ldots, r_N(\boldsymbol{e}_{1,\alpha(1)^*},\ldots,\boldsymbol{e}_{N,\alpha(N)^*}))$ is the Pareto minimum $(0,\ldots,0)$ for $(r_1,\ldots, r_N)$.

Similar results hold for DE's with the following definitions.

Definition 6.9. Let $<p_1,\ldots, p_N>$ be a finite game. Define the pure disappointment incurred by player $i = 1,\ldots, N$ when player i chooses pure strategy $\boldsymbol{e}_{i,\alpha(i)}$ and the other $N\!-\!1$ players choose pure strategies $\boldsymbol{e}_{j,\alpha(j)}$, $j = 1,\ldots, N, j \neq i$ by

$$d_i\,[\boldsymbol{e}_{1,\alpha(1)},\ldots, \boldsymbol{e}_{i-1,\alpha(i-1)}, \boldsymbol{e}_{i,\alpha(i)}, \boldsymbol{e}_{i+1,\alpha(i+1)},\ldots, \boldsymbol{e}_{N,\alpha(N)}] =$$

$$\max_{\alpha(1),\ldots,\alpha(i-1),\alpha(i+1),\ldots,\alpha(N)} p_i\,[\boldsymbol{e}_{1,\alpha(1)},\ldots, \boldsymbol{e}_{i-1,\alpha(i-1)}, \boldsymbol{e}_{i,\alpha(i)}, \boldsymbol{e}_{i+1,\alpha(i+1)},\ldots, \boldsymbol{e}_{N,\alpha(N)}] -$$

$$p_i\,[\boldsymbol{e}_{1,\alpha(1)},\ldots, \boldsymbol{e}_{i-1,\alpha(i-1)}, \boldsymbol{e}_{i,\alpha(i)}, \boldsymbol{e}_{i+1,\alpha(i+1)},\ldots, \boldsymbol{e}_{N,\alpha(N)}],\ i = 1,\ldots, N. \qquad (6.22)$$

Definition 6.10. For the game of Definition 6.9, define the continuous extension d_i: $X_1\times\ldots\times X_N \to \boldsymbol{R}^1$ of the pure disappointment function (6.22) for player i, $i = 1,\ldots, N$, when player i chooses mixed strategy $\boldsymbol{x}_i$ and the other player $N\!-\!1$ choose mixed strategy $\boldsymbol{x}_j$, $j = 1,\ldots, N, j \neq i$, by

$$d_i\,(\boldsymbol{x}_1,\ldots, \boldsymbol{x}_N) = \sum_{\alpha(i)=1}^{m(i)} x_{i,\alpha(i)}c_i\,(\alpha(i)) - p_i\,(\boldsymbol{x}_1,\ldots, \boldsymbol{x}_N),\ i = 1,\ldots, N, \qquad (6.23)$$

where $c_i\,(\alpha(i)) = \max\limits_{\alpha(1),\ldots,\alpha(i-1),\alpha(i+1),\ldots,\alpha(N)} p_i\,[\boldsymbol{e}_{1,\alpha(1)},\ldots,\ \boldsymbol{e}_{i-1,\alpha(i-1)},\ \boldsymbol{e}_{i,\alpha(i)},\ \boldsymbol{e}_{i+1,\alpha(i+1)},\ldots,\ \boldsymbol{e}_{N,\alpha(N)}]$ for all the $\alpha(i)^{\text{th}}$ pure strategies of player i.

Much as the proof of Theorem 6.5, Theorem 6.6 and its corollary follow immediately.

Theorem 6.6. The set of DE's for the finite game $<p_1,\ldots, p_N>$ is the same as the set of DE's for the finite game $<d_1,\ldots, \mathrm{d}_N>$.

Corollary 6.2 The N-tuple of pure strategies $(\boldsymbol{e}_{1,\alpha(1)^*},\ldots,\boldsymbol{e}_{N,\alpha(N)^*})$ is a pure DE for $<p_1,\ldots, p_N>$ if and only if $(d_1(\boldsymbol{e}_{1,\alpha(1)^*},\ldots,\boldsymbol{e}_{N,\alpha(N)^*}),\ \ldots,\ d_N(\boldsymbol{e}_{1,\alpha(1)^*},\ldots,\boldsymbol{e}_{N,\alpha(N)^*}))$ is the Pareto minimum $(0,\ldots,0)$ for $(d_1,\ldots, d_N)$.

Observe two points about the previous two theorems. First, comparing (6.21) and (6.23) shows that regret can be interpreted as disappointment with respect to oneself as previously mentioned. Second, the two corollaries provide a simple approach for determining any pure RE's or DE's for the game $<p_1,\ldots, p_N>$. Simply calculate the pure regrets and pure disappointments. Then the pure RE's and pure DE's are exactly the pure strategies corresponding to pure regrets and disappointments, respectively, of $(0,\ldots,0)$.

Since these pure RE's and DE's of $(0,\ldots,0)$ are obviously Pareto minima of the pure regrets and disappointments, one may conjecture that any RE $(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*)$ or DE

($\boldsymbol{x}_1$*,…, $\boldsymbol{x}_N$*) gives a Pareto minimum in (r_1,…, r_N) or (d_1,…, d_N), respectively. This conjecture is false, however.

Example 6.1 Consider the bimatrix game of Figure 6.1

		Player *II*	
		t_1	t_2
Player *I*	s_1	(-2,-2)	(2,0)
	s_2	(0,2)	(1,1)

Figure 6.1 The Bimatrix Game G_7

For this game, (2,0) and (0,2) are the payoffs for the two pure RE's, while (1,1) is the payoff for the only pure DE. So both the RM and DM have (0,0)'s. But there is also a mixed RE and DE of (1/3, 2/3) yielding (2/3, 2/3) in both regret and disappointment that is dominated by the pure RE and DE with (0,0)'s.

6.4 Finding Equilibria of N-person games

6.4.1 Finding Regret Equilibria

Generalizing our results for N = 2, we next write a nonlinear programming to find a RE ($\boldsymbol{x}_1^*$, …, $\boldsymbol{x}_N^*$) as

$$\text{Minimize } \sum_{i=1}^{N} \sum_{\alpha(i)=1}^{m(i)} U_{i,\alpha(i)} = f(\boldsymbol{x}_i, E_i, U_i)$$

subject to

$$\sum_{\alpha(1)=1}^{m(1)} \cdots \sum_{\alpha(i-1)=1}^{m(i-1)} \sum_{\alpha(i+1)=1}^{m(i+1)} \cdots \sum_{\alpha(N)=1}^{m(N)} x_{1,\alpha(1)} \ \ldots \ x_{i-1,\alpha(i-1)} \ x_{i+1,\alpha(i+1)} \ \ldots \ x_{N,\alpha(N)} \ p_i \ [\boldsymbol{e}_{1,\alpha(1)}, \ldots, \boldsymbol{e}_{i-1,\alpha(i-1)},$$

$$\boldsymbol{x}_i, \boldsymbol{e}_{i+1,\alpha(i+1)}, \ldots, \boldsymbol{e}_{N,\alpha(N)}] - \sum_{\alpha(1)=1}^{m(1)} \cdots \sum_{\alpha(N)=1}^{m(N)} x_{1,\alpha(1)} \ldots x_{N,\alpha(N)} \, p_i \, [\boldsymbol{e}_{1,\alpha(1)}, \ldots, \boldsymbol{e}_{N,\alpha(N)}] + E_{i,\alpha(i)} + U_{i,\alpha(i)} = 0,$$

$\alpha(i) = 1, \ldots, m(i)$ and $i = 1, \ldots, N$

$$\sum_{\alpha(i)=1}^{m(i)} x_{i,\alpha(i)} = 1, \ i = 1, \ldots, N$$

$x_{i,\alpha(i)}, E_{i,\alpha(i)}, U_{i,\alpha(i)} \geq 0.$

Consider an example for a three-person game where Players I, II, and III have pure strategies α, β, and γ and mixed strategies $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$, respectively. Let $\boldsymbol{x} = (x_1, x_2)^T$, $\boldsymbol{y} = (y_1, y_2)^T$, and $\boldsymbol{z} = (z_1, z_2)^T$. We then obtain

Minimize $f(x_1, x_2, y_1, y_2, z_1, z_2, E_{I,1}, E_{I,2}, E_{II,1}, E_{II,2}, E_{III,1}, E_{III,2}, U_{I,1}, U_{I,2}, U_{II,1}, U_{II,2}, U_{III,1}, U_{III,2}) = U_{I,1} + U_{I,2} + U_{II,1} + U_{II,2} + U_{III,1} + U_{III,2}$

subject to

$$\sum_{\beta=1}^{2} \sum_{\gamma=1}^{2} y_\beta z_\gamma p_I(\alpha,\beta,\gamma) - \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \sum_{\gamma=1}^{2} x_\alpha y_\beta z_\gamma p_I(\alpha,\beta,\gamma) + E_{I,\alpha} + U_{I,\alpha} = 0, \ \alpha = 1, 2$$

$$\sum_{\alpha=1}^{2} \sum_{\gamma=1}^{2} x_\alpha z_\gamma p_{II}(\alpha,\beta,\gamma) - \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \sum_{\gamma=1}^{2} x_\alpha y_\beta z_\gamma p_{II}(\alpha,\beta,\gamma) + E_{II,\beta} + U_{II,\beta} = 0, \ \beta = 1, 2$$

$$\sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} x_\alpha y_\beta p_{III}(\alpha,\beta,\gamma) - \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \sum_{\gamma=1}^{2} x_\alpha y_\beta z_\gamma p_{III}(\alpha,\beta,\gamma) + E_{III,\gamma} + U_{III,\gamma} = 0, \ \gamma = 1, 2$$

$x_1 + x_2 = 1$

$y_1 + y_2 = 1$

$z_1 + z_2 = 1$

$x_1, x_2, y_1, y_2, z_1, z_2, E_{I,1}, E_{I,2}, E_{II,1}, E_{II,2}, E_{III,1}, E_{III,2}, U_{I,1}, U_{I,2}, U_{II,1}, U_{II,2}, U_{III,1}, U_{III,2} \geq 0$

Example 6.2 Assuming two vendors, B and C, are competitors selling the substituted materials. Company A is currently buying the material from B that cost \$100. A must make a decision whether to change its vendor since it may create customer dissatisfaction and cost \$10 for its own administration. B must make a decision whether to maintain its price of material or reduce its price by half. If B reduces the price of its material to A, it will cost B at least \$10 for discounts that B may have to give to other customers.

In addition, C must make a decision whether to try to earn their businesses with A by offer its material for \$20. C will only support A's decision if A agrees to pay C's administration fee of \$10 in case that A does not buy C's material.

We form a normal-form game G_8 of this situation with payoffs to each company according their decisions follows.

Company A has two pure strategies:

- The first is α_1, buying the material from B that cost \$100 if vendor B does not reduce its price of material. If B lowers its price, it will cost A \$50. In either case, A will pay C \$10 to have C as its alternative.
- The second is α_2, buying material from C and pay \$100 for the materials if C does not compete with B or \$20 if C does. In either case, A pay additional \$10 for its own administrations.

Similarly, vendor B has two pure β strategies:

- Its first is β_1, keeping the same material's price and earn \$100 if A does not change vendor or earn nothing if A changes its vendor.
- The second is β_2, reducing its material's price to \$50. However it will cost B \$10 for the discounts that B may have to give to other customers.

Vendor C has also two pure γ strategies:

- The first is γ_1, not competing on material price with B.
- The second is γ_2, offering materials to A at \$20 and it will cost C \$10 for the effort.

The payoffs of A, B, and C are ($a_{\alpha\beta\gamma}$, $b_{\alpha\beta\gamma}$, $c_{\alpha\beta\gamma}$) are shown in Figure 6.2.

	γ_1			γ_2	
	β_1	β_2		β_1	B_2
α_1	(–110, 100, 10)	(–60, 40, 10)		(–110, 100, 10)	(–60, 40, 10)
α_2	(–110, 0, 100)	(–110, –10, 100)		(–30, 0, 10)	(–30, –10, 10)

Figure 6.2 The Three-Person Game G_8

Let x_1 , x_2, y_1 , y_2, z_1 , and z_2, be probabilities that players chose their strategies α_1, α_2, β_1, β_2, γ_1, and γ_2 respectively.

We can find an RE of this game by the following nonlinear programming problem.

Minimize $f(x_1, x_2, y_1, y_2, z_1, z_2, E_1, E_2, F_1, F_2, G_1, G_2, U_1, U_2, V_1, V_2, W_1, W_2) =$

$U_1 + U_2 + V_1 + V_2 + W_1 + W_2$

subject to

$(-110)\times y_1\times z_1 + (-60)\times y_2\times z_1 + (-110)\times y_1\times z_2 + (-60)\times y_2\times z_2 - (-110)\times x_1\times y_1\times z_1 - (-60)\times x_1\times y_2\times z_1 - (-110)\times x_2\times y_1\times z_1 - (-110)\times x_2\times y_2\times z_1 - (-110)\times x_1\times y_1\times z_2 - (-60)\times x_1\times y_2\times z_2 - (-30)\times x_2\times y_1\times z_2 - (-30)\times x_2\times y_2\times z_2 + E_1 + U_1 = 0$

$(-110)\times y_1\times z_1 + (-110)\times y_2\times z_1 + (-30)\times y_1\times z_2 + (-30)\times y_2\times z_2 - (-110)\times x_1\times y_1\times z_1 - (-60)\times x_1\times y_2\times z_1 - (-110)\times x_2\times y_1\times z_1 - (-110)\times x_2\times y_2\times z_1 - (-110)\times x_1\times y_1\times z_2 - (-60)\times x_1\times y_2\times z_2 - (-30)\times x_2\times y_1\times z_2 - (-30)\times x_2\times y_2\times z_2 + E_2 + U_2 = 0$

$(100)\times x_1\times z_1 + (0)\times x_2\times z_1 + (100)\times x_1\times z_2 + (0)\times x_2\times z_2 - (100)\times x_1\times y_1\times z_1 - (40)\times x_1\times y_2\times z_1 - (0)\times x_2\times y_1\times z_1 - (-10)\times x_2\times y_2\times z_1 - (100)\times x_1\times y_1\times z_2 - (40)\times x_1\times y_2\times z_2 - (0)\times x_2\times y_1\times z_2 - (-10)\times x_2\times y_2\times z_2 + F_1 + V_1 = 0$

$(40)\times x_1\times z_1 + (-10)\times x_2\times z_1 + (40)\times x_1\times z_2 + (-10)\times x_2\times z_2 - (100)\times x_1\times y_1\times z_1 - (40)\times x_1\times y_2\times z_1 - (0)\times x_2\times y_1\times z_1 - (-10)\times x_2\times y_2\times z_1 - (100)\times x_1\times y_1\times z_2 - (40)\times x_1\times y_2\times z_2 - (0)\times x_2\times y_1\times z_2 - (-10)\times x_2\times y_2\times z_2 + F_2 + V_2 = 0$

$(0)\times x_1\times y_1 + (100)\times x_2\times y_1 + (0)\times x_1\times y_2 + (100)\times x_2\times y_2 - (10)\times x_1\times y_1\times z_1 - (10)\times x_1\times y_2\times z_1 - (100)\times x_2\times y_1\times z_1 - (100)\times x_2\times y_2\times z_1 - (10)\times x_1\times y_1\times z_2 - (10)\times x_1\times y_2\times z_2 - (10)\times x_2\times y_1\times z_2 - (10)\times x_2\times y_2\times z_2 + G_1 + W_1 = 0$

$(10)\times x_1\times y_1 + (10)\times x_2\times y_1 + (10)\times x_1\times y_2 + (10)\times x_2\times y_2 - (10)\times x_1\times y_1\times z_1 - (10)\times x_1\times y_2\times z_1 - (100)\times x_2\times y_1\times z_1 - (100)\times x_2\times y_2\times z_1 - (10)\times x_1\times y_1\times z_2 - (10)\times x_1\times y_2\times z_2 - (10)\times x_2\times y_1\times z_2 - (10)\times x_2\times y_2\times z_2 + G_2 + W_2 = 0$

$x_1 + x_2 = 1$

$y_1 + y_2 = 1$

$z_1 + z_2 = 1$

$x_1, x_2, y_1, y_2, z_1, z_2, E_1, E_2, F_1, F_2, G_1, G_2, U_1, U_2, V_1, V_2, W_1, W_2 \geq 0.$

We obtain two RE's where $(y_1, y_2) = (1,0)$, $(z_1, z_2) = (1,0)$, along with either $(x_1, x_2) = (1,0)$ or $(x_1, x_2) = (0,1)$. For these RE's, A either changes his vendor or not, B does not reduce the price of its materials, and C does not compete with B. Since A does not factor its cost of customer dissatisfaction to change the vendor in this matrix, A is likely to pick α_1. The Regret Matrix of this game is shown in Figure 6.3. Note that we have computationally obtained the obvious pure RE's with regret (0,0,0). The payoff for the RE (α_1, β_1, γ_1) is (– 110, 100, 10).

	γ_1			γ_2	
	β_1	β_2		β_1	B_2
α_1	(0, 0, 0)	(0, 60, 0)		(80, 0, 0)	(30, 60, 0)
α_2	(0, 0, 0)	(50, 10, 0)		(0, 0, 90)	(0, 10, 90)

Figure 6.3 The Regret Matrix of Game G_8

6.4.2 Finding Marginal Disappointment Equilibria

We can characterize a MDE using Definition 6.4 and Lemma 6.2 to get

$$p_i\,[\boldsymbol{x}_1^*,\ldots,\boldsymbol{x}_{j-1}^*\,,\ \boldsymbol{e}_{j,\alpha(j)},\ \boldsymbol{x}_{j+1}^*,\ldots,\boldsymbol{x}_N^*] \le p_i\,(\boldsymbol{x}_1^*,\ldots,\boldsymbol{x}_N^*),$$

$$i = 1,\ldots, N,\ i \neq j,\ \alpha(j),\ \alpha(j) = 1,\ldots, m(j)\ . \tag{6.24}$$

From (6-24)

$$\sum_{\alpha(1)=1}^{m(1)} \cdots \sum_{\alpha(j-1)=1}^{m(j-1)} \sum_{\alpha(j+1)=1}^{m(j+1)} \cdots \sum_{\alpha(N)=1}^{m(N)} x_{1,\alpha(1)} \ldots x_{j-1,\alpha(j-1)}\, x_{j+1,\alpha(j+1)} \ldots x_{N,\alpha(N)}\, p_i\,[\boldsymbol{e}_{1,\alpha(1)},\ldots,\boldsymbol{e}_{N,\alpha(N)}] \le$$

$$\sum_{\alpha(1)=1}^{m(1)} \cdots \sum_{\alpha(N)=1}^{m(N)} x_{1,\alpha(1)} \ldots x_{N,\alpha(N)}\, p_i\,[\boldsymbol{e}_{1,\alpha(1)},\ldots,\boldsymbol{e}_{N,\alpha(N)}],$$

$$i = 1,\ldots, N;\ j = 1,\ldots, N;\ j \neq i;\ \alpha(j) = 1,\ldots, m(j)\ . \tag{6.25}$$

We can thus find a marginal DE from (6-25) by finding $x_{k,\alpha(k)}^*$, $\alpha(k) = 1,\ldots, m(k)$, $k = 1,\ldots, N$, which solve the following nonlinear programming problem.

$$\text{Minimize } g\,(x_{i,\alpha(i)}, E_{i,j,\alpha(i)}, U_{i,j,\alpha(i)},) = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \sum_{\alpha(j)=1}^{m(j)} U_{i,j,\alpha(j)}$$

subject to

$$\sum_{\alpha(1)=1}^{m(1)} \cdots \sum_{\alpha(j-1)=1}^{m(j-1)} \sum_{\alpha(j+1)=1}^{m(j+1)} \cdots \sum_{\alpha(N)=1}^{m(N)} x_{1,\alpha(1)} \ldots x_{j-1,\alpha(j-1)}\, x_{j+1,\alpha(j+1)} \ldots x_{N,\alpha(N)}\, p_i[\boldsymbol{e}_{1,\alpha(1)},\ldots,\boldsymbol{e}_{N,\alpha(N)}] -$$

$$\sum_{\alpha(1)=1}^{m(1)} \cdots \sum_{\alpha(N)=1}^{m(N)} x_{1,\alpha(1)} \ldots x_{N,\alpha(N)}\, p_i[\boldsymbol{e}_{1,\alpha(1)},\ldots,\boldsymbol{e}_{N,\alpha(N)}] + E_{i,j,\alpha(j)} + U_{i,j,\alpha(j)} = 0,\ i = 1,\ldots, N;$$

$j = 1,\ldots, N;\ j \neq i;\ \alpha(j) = 1,\ldots, m(j).$

$$\sum_{\alpha(i)=1}^{m(i)} x_{i,\alpha(i)} = 1,\ i = 1,\ldots, N$$

$x_{i,\alpha(i)},\ E_{i,j,\alpha(i)},\ U_{i,j,\alpha(i)} \geq 0,\ i = 1,\ldots, N;\ j = 1,\ldots, N;\ j \neq i;\ \alpha(j) = 1,\ldots, m(j).$

As a special case, we specialize the above problem to a three-person game. Suppose Players *I*, *II,* and *III* have pure strategies α, β, and γ and mixed strategies $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$, where $\boldsymbol{x} = (x_1, x_2)^T$, $\boldsymbol{y} = (y_1, y_2)^T$, $\boldsymbol{z} = (z_1, z_2)^T$. Any marginal DE must therefore solve the problem

Minimize $g(x_1, x_2, y_1, y_2, z_1, z_2, E_{I,III,1}, E_{I,III,2}, E_{I,II,1}, E_{I,II,2}, E_{II,I,1}, E_{II,I,2},$

$E_{II,III,1}, E_{II,III,2}, E_{III,I,1}, E_{III,I,2}, E_{III,II,1}, E_{III,II,2}, U_{I,III,1}, U_{I,III,2}, U_{I,II,1}, U_{I,II,2},$

$U_{II,I,1}, U_{II,I,2,}, U_{II,III,1}, U_{II,III,2}, U_{III,I,1}, U_{III,I,2,}, U_{III,II,1}, U_{III,II,2}) =$

$U_{I,III,1} + U_{I,III,2} + U_{I,II,1} + U_{I,II,2} + U_{II,I,1} + U_{II,I,2,} + U_{II,III,1} + U_{II,III,2} + U_{III,I,1} +$

$U_{III,I,2,} + U_{III,II,1} + U_{III,II,2}$

subject to

$$\sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} x_\alpha y_\beta p_I(\alpha,\beta,\gamma) - \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \sum_{\gamma=1}^{2} x_\alpha y_\beta z_\gamma p_I(\alpha,\beta,\gamma) + E_{I,III,\gamma} + U_{I,III,\gamma} = 0,\ \gamma = 1,2$$

$$\sum_{\alpha=1}^{2} \sum_{\gamma=1}^{2} x_\alpha z_\gamma p_I(\alpha,\beta,\gamma) - \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \sum_{\gamma=1}^{2} x_\alpha y_\beta z_\gamma p_I(\alpha,\beta,\gamma) + E_{I,II,\beta} + U_{I,II,\beta} = 0,\ \beta = 1,2$$

$$\sum_{\beta=1}^{2}\sum_{\gamma=1}^{2} y_\beta z_\gamma p_{II}(\alpha,\beta,\gamma) - \sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\sum_{\gamma=1}^{2} x_\alpha y_\beta z_\gamma p_{II}(\alpha,\beta,\gamma) + E_{II,I,\alpha} + U_{II,I,\alpha} = 0,\ \alpha = 1,2$$

$$\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2} x_\alpha y_\beta p_{II}(\alpha,\beta,\gamma) - \sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\sum_{\gamma=1}^{2} x_\alpha y_\beta z_\gamma p_{II}(\alpha,\beta,\gamma) + E_{II,III,\gamma} + U_{II,III,\gamma} = 0,\ \gamma = 1,2$$

$$\sum_{\beta=1}^{2}\sum_{\gamma=1}^{2} y_\beta z_\gamma p_{III}(\alpha,\beta,\gamma) - \sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\sum_{\gamma=1}^{2} x_\alpha y_\beta z_\gamma p_{III}(\alpha,\beta,\gamma) + E_{III,I,\alpha} + U_{III,I,\alpha} = 0,\ \alpha = 1,2$$

$$\sum_{\alpha=1}^{2}\sum_{\gamma=1}^{2} x_\alpha z_\gamma p_{III}(\alpha,\beta,\gamma) - \sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\sum_{\gamma=1}^{2} x_\alpha y_\beta z_\gamma p_{III}(\alpha,\beta,\gamma) + E_{III,II,\beta} + U_{III,II,\beta} = 0,\ \beta = 1,2$$

$x_1 + x_2 = 1$

$y_1 + y_2 = 1$

$z_1 + z_2 = 1$

$x_1, x_2, y_1, y_2, z_1, z_2, E_{I,III,1}, E_{I,III,2}, E_{I,II,1}, E_{I,II,2}, E_{II,I,1}, E_{II,I,2}, E_{II,III,1}, E_{II,III,2}, E_{III,I,1}, E_{III,I,2}, E_{III,II,1}, E_{III,II,2}, U_{I,III,1}, U_{I,III,2}, U_{I,II,1}, U_{I,II,2}, U_{II,I,1}, U_{II,I,2}, U_{II,III,1}, U_{II,III,2}, U_{III,I,1}, U_{III,I,2}, U_{III,II,1}, U_{III,II,2} \geq 0.$

Example 6.3 Consider the three-person game G_8 of Figure 6.2. Each MDE must solve

Minimize g ($x_1, x_2, y_1, y_2, z_1, z_2, E_1, E_2, E_3, E_4, F_1, F_2, F_3, F_4, G_1, G_2, G_3, G_4, U_1, U_2, U_3, U_4, V_1, V_2, V_3, V_4, W_1, W_2, W_3, W_4$) =

$U_1 + U_2 + U_3 + U_4 + V_1 + V_2 + V_3 + V_4 + W_1 + W_2 + W_3 + W_4$

subject to

$(-110)\times x_1\times z_1 + (-110)\times x_1\times z_2 + (-110)\times x_2\times z_1 + (-30)\times x_2\times z_2 - (-110)\times x_1\times y_1\times z_1 - (-60)\times x_1\times y_2\times z_1 - (-110)\times x_2\times y_1\times z_1 - (-110)\times x_2\times y_2\times z_1 - (-110)\times x_1\times y_1\times z_2 - (-60)\times x_1\times y_2\times z_2 - (-30)\times x_2\times y_1\times z_2 - (-30)\times x_2\times y_2\times z_2 + E_1 + U_1 = 0$

$(-60)\times x_1\times z_1 + (-60)\times x_1\times z_2 + (-110)\times x_2\times z_1 + (-30)\times x_2\times z_2 - (-110)\times x_1\times y_1\times z_1 - (-60)\times x_1\times y_2\times z_1 - (-110)\times x_2\times y_1\times z_1 - (-110)\times x_2\times y_2\times z_1 - (-110)\times x_1\times y_1\times z_2 - (-60)\times x_1\times y_2\times z_2 - (-30)\times x_2\times y_1\times z_2 - (-30)\times x_2\times y_2\times z_2 + E_2 + U_2 = 0$

$(-110)\times x_1\times y_1 + (-60)\times x_1\times y_2 + (-110)\times x_2\times y_1 + (-110)\times x_2\times y_2 - (-110)\times x_1\times y_1\times z_1 - (-60)\times x_1\times y_2\times z_1 - (-110)\times x_2\times y_1\times z_1 - (-110)\times x_2\times y_2\times z_1 - (-110)\times x_1\times y_1\times z_2 - (-60)\times x_1\times y_2\times z_2 - (-30)\times x_2\times y_1\times z_2 - (-30)\times x_2\times y_2\times z_2 + E_3 + U_3 = 0$

$(-110)\times x_1\times y_1 + (-60)\times x_1\times y_2 + (-30)\times x_2\times y_1 + (-30)\times x_2\times y_2 - (-110)\times x_1\times y_1\times z_1 - (-60)\times x_1\times y_2\times z_1 - (-110)\times x_2\times y_1\times z_1 - (-110)\times x_2\times y_2\times z_1 - (-110)\times x_1\times y_1\times z_2 - (-60)\times x_1\times y_2\times z_2 - (-30)\times x_2\times y_1\times z_2 - (-30)\times x_2\times y_2\times z_2 + E_4 + U_4 = 0$

$(100)\times y_1\times z_1 + (40)\times y_2\times z_1 + (100)\times y_1\times z_2 + (40)\times y_2\times z_2 - (100)\times x_1\times y_1\times z_1 - (40)\times x_1\times y_2\times z_1 - (0)\times x_2\times y_1\times z_1 - (-10)\times x_2\times y_2\times z_1 - (100)\times x_1\times y_1\times z_2 - (40)\times x_1\times y_2\times z_2 - (0)\times x_2\times y_1\times z_2 - (-10)\times x_2\times y_2\times z_2 + F_1 + V_1 = 0$

$(0)\times y_1\times z_1 + (-10)\times y_2\times z_1 + (0)\times y_1\times z_2 + (-10)\times y_2\times z_2 - (100)\times x_1\times y_1\times z_1 - (40)\times x_1\times y_2\times z_1 - (0)\times x_2\times y_1\times z_1 - (-10)\times x_2\times y_2\times z_1 - (100)\times x_1\times y_1\times z_2 - (40)\times x_1\times y_2\times z_2 - (0)\times x_2\times y_1\times z_2 - (-10)\times x_2\times y_2\times z_2 + F_2 + V_2 = 0$

$(100)\times x_1\times y_1 + (0)\times x_2\times y_1 + (40)\times x_1\times y_2 + (-10)\times x_2\times y_2 - (100)\times x_1\times y_1\times z_1 - (40)\times x_1\times y_2\times z_1 - (0)\times x_2\times y_1\times z_1 - (-10)\times x_2\times y_2\times z_1 - (100)\times x_1\times y_1\times z_2 - (40)\times x_1\times y_2\times z_2 - (0)\times x_2\times y_1\times z_2 - (-10)\times x_2\times y_2\times z_2 + F_3 + V_3 = 0$

$(100)\times x_1\times y_1 + (0)\times x_2\times y_1 + (40)\times x_1\times y_2 + (-10)\times x_2\times y_2 - (100)\times x_1\times y_1\times z_1 - (40)\times x_1\times y_2\times z_1 - (0)\times x_2\times y_1\times z_1 - (-10)\times x_2\times y_2\times z_1 - (100)\times x_1\times y_1\times z_2 - (40)\times x_1\times y_2\times z_2 - (0)\times x_2\times y_1\times z_2 - (-10)\times x_2\times y_2\times z_2 + F_4 + V_4 = 0$

$(10)\times x_1\times z_1 + (100)\times x_2\times z_1 + (10)\times x_1\times z_2 + (10)\times x_2\times z_2 - (10)\times x_1\times y_1\times z_1 - (10)\times x_1\times y_2\times z_1 - (100)\times x_2\times y_1\times z_1 - (100)\times x_2\times y_2\times z_1 - (10)\times x_1\times y_1\times z_2 - (10)\times x_1\times y_2\times z_2 - (10)\times x_2\times y_1\times z_2 - (10)\times x_2\times y_2\times z_2 + G_1 + W_1 = 0$

$(10)\times x_1\times z_1 + (100)\times x_2\times z_1 + (10)\times x_1\times z_2 + (10)\times x_2\times z_2 - (10)\times x_1\times y_1\times z_1 - (10)\times x_1\times y_2\times z_1 - (100)\times x_2\times y_1\times z_1 - (100)\times x_2\times y_2\times z_1 - (10)\times x_1\times y_1\times z_2 - (10)\times x_1\times y_2\times z_2 - (10)\times x_2\times y_1\times z_2 - (10)\times x_2\times y_2\times z_2 + G_2 + W_2 = 0$

$(10)\times y_1\times z_1 + (10)\times y_2\times z_1 + (10)\times y_1\times z_2 + (10)\times y_2\times z_2 - (10)\times x_1\times y_1\times z_1 - (10)\times x_1\times y_2\times z_1 - (100)\times x_2\times y_1\times z_1 - (100)\times x_2\times y_2\times z_1 - (10)\times x_1\times y_1\times z_2 - (10)\times x_1\times y_2\times z_2 - (10)\times x_2\times y_1\times z_2 - (10)\times x_2\times y_2\times z_2 + G_3 + W_3 = 0$

$(100)\times y_1\times z_1 + (100)\times y_2\times z_1 + (10)\times y_1\times z_2 + (10)\times y_2\times z_2 - (10)\times x_1\times y_1\times z_1 - (10)\times x_1\times y_2\times z_1 - (100)\times x_2\times y_1\times z_1 - (100)\times x_2\times y_2\times z_1 - (10)\times x_1\times y_1\times z_2 - (10)\times x_1\times y_2\times z_2 - (10)\times x_2\times y_1\times z_2 - (10)\times x_2\times y_2\times z_2 + G_4 + W_4 = 0$

$x_1 + x_2 = 1$

$y_1 + y_2 = 1$

$z_1 + z_2 = 1$

$x_1, x_2, y_1, y_2, z_1, z_2, E_1, E_2, E_3, E_4, F_1, F_2, F_3, F_4, G_1, G_2, G_3, G_4, U_1, U_2, U_3, U_4, V_1, V_2, V_3, V_4, W_1, W_2, W_3, W_4 \geq 0.$

We get an MDE of this game to be $(x_1, x_2) = (1/2,1/2)$, $(y_1, y_2) = (0,1)$ and $(z_1, z_2) = (0,1)$, where A does not change his vendor, B decides to lower the price of its materials, and C offers the material to A with a cheaper price. The DM of this game is shown in Figure 6.4. Note that the MDE is a TDE as previously proved in Theorem 6.4.

	γ_1			γ_2	
	B_1	β_2		β_1	β_2
α_1	(50, 0, 90)	(0, 0, 90)		(50, 0, 0)	(0, 0, 0)
α_2	(80, 100, 0)	(80, 50, 0)		(0, 100, 0)	(0, 50, 0)

Figure 6.4 The Disappointment Matrix of Game G_8

6.4.3 Finding Total Disappointment Equilibria

As before, a TDE can be found by solving the following nonlinear programming problem

$$\text{Minimize } g(x_{i,\alpha(i)}, E_{i,j,\alpha(i)}, U_{i,j,\alpha(i)}) = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} U_{i,j,\alpha(j)}$$

subject to

$$\sum_{\alpha(i)=1}^{m(i)} x_{i,\alpha(i)}\, p_i\,[\boldsymbol{e}_{1,\alpha(1)},\ldots,\boldsymbol{e}_{N,\alpha(N)}] - \sum_{\alpha(1)=1}^{m(1)} \cdots \sum_{\alpha(N)=1}^{m(N)} x_{1,\alpha(1)} \ldots x_{N,\alpha(N)}\, p_i\,[\boldsymbol{e}_{1,\alpha(1)},\ldots,\boldsymbol{e}_{N,\alpha(N)}] + E_{i,j} + U_{i,j}$$

$= 0,\ i = 1,\ldots, N;\ j = 1,\ldots, N;\ j \neq i;\ \alpha(j) = 1,\ldots,m(j)$

$$\sum_{\alpha(i)=1}^{m(i)} x_{i,\alpha(i)} = 1,\ i = 1,\ldots, N$$

$x_{i,\alpha(i)}, E_{i,j}, U_{i,j} \geq 0, i = 1,\ldots, N; j = 1,\ldots, N; j \neq i; \alpha(j) = 1,\ldots, m(j).$

Again, we specialize the previous nonlinear program for a TDE or just DE to a three-person game. Let Players *I*, *II,* and *III* have pure strategies α, β, γ and mixed strategies $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ respectively, where $\boldsymbol{x} = (x_1, x_2)^T$. $\boldsymbol{y} = (y_1, y_2)^T$. $\boldsymbol{z} = (z_1, z_2)^T$. Each DE must solve

Minimize $(x_1, x_2, y_1, y_2, z_1, z_2, E_{I,11}, E_{I,12}, E_{I,21}, E_{I,22}, E_{II,11}, E_{II,12},$

$E_{II,21}, E_{II,22}, E_{III,11}, E_{III,12}, E_{III,21}, E_{III,22}, U_{I,11}, U_{I,12}, U_{I,21}, U_{I,22}, U_{II,11}, U_{II,12},$

$U_{II,21}, U_{II,22}, U_{III,11}, U_{III,12}, U_{III,21}, U_{III,22}) =$

$U_{I,11} + U_{I,12} + U_{I,21} + U_{I,22} + U_{II,11} + U_{II,12} + U_{II,21} + U_{II,22} + + U_{III,11} + U_{III,12} +$

$U_{III,21} + U_{III,22}$

subject to

$$\sum_{\alpha=1}^{2} x_\alpha p_I(\alpha,\beta,\gamma) - \sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\sum_{\gamma=1}^{2} x_\alpha y_\beta z_\gamma p_I(\alpha,\beta,\gamma) + E_{I,\beta\gamma} + U_{I,\beta\gamma} = 0,\ \beta = 1,2,\ \gamma = 1,2$$

$$\sum_{\beta=1}^{2} y_\beta p_{II}(\alpha,\beta,\gamma) - \sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\sum_{\gamma=1}^{2} x_\alpha y_\beta z_\gamma p_{II}(\alpha,\beta,\gamma) + E_{II,\alpha\gamma} + U_{II,\alpha\gamma} = 0,\ \alpha = 1,2,\ \gamma = 1,2$$

$$\sum_{\gamma=1}^{2} z_\gamma p_{III}(\alpha,\beta,\gamma) - \sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\sum_{\gamma=1}^{2} x_\alpha y_\beta z_\gamma p_{III}(\alpha,\beta,\gamma) + E_{III,\alpha\beta} + U_{III,\alpha\beta} = 0,\ \alpha = 1,2,\ \beta = 1,2$$

$x_1 + x_2 = 1$

$y_1 + y_2 = 1$

$z_1 + z_2 = 1$

$x_1, x_2, y_1, y_2, z_1, z_2, E_{I,11}, E_{I,12}, E_{I,21}, E_{I,22}, E_{II,11}, E_{II,12}, E_{II,21}, E_{II,22},, E_{III,11}, E_{III,12}, E_{III,21},$

$E_{III,22,} U_{I,11}, U_{I,12}, U_{I,21}, U_{I,22}, U_{II,11}, U_{II,12}, U_{II,21}, U_{II,22},, U_{III,11}, U_{III,12}, U_{III,21}, U_{III,22} \geq 0$

Example 6.4. Again consider the three-person game G_5 of Figure 6.2. We now get the nonlinear program for its DE's to be

Minimize $g(x_1, x_2, y_1, y_2, z_1, z_2, E_1, E_2, E_3, E_4, F_1, F_2, F_3, F_4, G_1, G_2, G_3, G_4, U_1,$

$U_2, U_3, U_4, V_{1,} V_2, V_{3,} V_4, W_1, W_2, W_3, W_4) =$

$U_1 + U_2 + U_3 + U_4 + V_1 + V_2 + V_3 + V_4 + W_1 + W_2 + W_3 + W_4$

subject to

$(-110)\times x_1 + (-110)\times x_2 - (-110)\times x_1\times y_1\times z_1 - (-60)\times x_1\times y_2\times z_1 - (-110)\times x_2\times y_1\times z_1 -$
$(-110)\times x_2\times y_2\times z_1 - (-110)\times x_1\times y_1\times z_2 - (-60)\times x_1\times y_2\times z_2 - (-30)\times x_2\times y_1\times z_2 -$
$(-30)\times x_2\times y_2\times z_2 + E_1 + U_1 = 0$

$(-60)\times x_1 + (-110)\times x_2 - (-110)\times x_1\times y_1\times z_1 - (-60)\times x_1\times y_2\times z_1 - (-110)\times x_2\times y_1\times z_1 -$
$(-110)\times x_2\times y_2\times z_1 - (-110)\times x_1\times y_1\times z_2 - (-60)\times x_1\times y_2\times z_2 - (-30)\times x_2\times y_1\times z_2 -$
$(-30)\times x_2\times y_2\times z_2 + E_2 + U_2 = 0$

$(-110)\times x_1 + (-30)\times x_2 - (-110)\times x_1\times y_1\times z_1 - (-60)\times x_1\times y_2\times z_1 - (-110)\times x_2\times y_1\times z_1 -$
$(-110)\times x_2\times y_2\times z_1 - (-110)\times x_1\times y_1\times z_2 - (-60)\times x_1\times y_2\times z_2 - (-30)\times x_2\times y_1\times z_2 -$
$(-30)\times x_2\times y_2\times z_2 + E_3 + U_3 = 0$

$(-60)\times x_1 + (-30)\times x_2 - (-110)\times x_1\times y_1\times z_1 - (-60)\times x_1\times y_2\times z_1 - (-110)\times x_2\times y_1\times z_1 -$
$(-110)\times x_2\times y_2\times z_1 - (-110)\times x_1\times y_1\times z_2 - (-60)\times x_1\times y_2\times z_2 - (-30)\times x_2\times y_1\times z_2 -$
$(-30)\times x_2\times y_2\times z_2 + E_4 + U_4 = 0$

$(100)\times y_1 + (40)\times y_2 - (100)\times x_1\times y_1\times z_1 - (40)\times x_1\times y_2\times z_1 - (0)\times x_2\times y_1\times z_1 -$
$(-10)\times x_2\times y_2\times z_1 - (100)\times x_1\times y_1\times z_2 - (40)\times x_1\times y_2\times z_2 - (0)\times x_2\times y_1\times z_2 - (-10)\times x_2\times y_2\times z_2 +$
$F_1 + V_1 = 0$

$(0)\times y_1 + (-10)\times y_2 - (100)\times x_1\times y_1\times z_1 - (40)\times x_1\times y_2\times z_1 - (0)\times x_2\times y_1\times z_1 - (-10)\times x_2\times y_2\times z_1$
$- (100)\times x_1\times y_1\times z_2 - (40)\times x_1\times y_2\times z_2 - (0)\times x_2\times y_1\times z_2 - (-10)\times x_2\times y_2\times z_2 + F_2 + V_2 = 0$

$(100)\times y_1 + (40)\times y_2 - (100)\times x_1\times y_1\times z_1 - (40)\times x_1\times y_2\times z_1 - (0)\times x_2\times y_1\times z_1 -$
$(-10)\times x_2\times y_2\times z_1 - (100)\times x_1\times y_1\times z_2 - (40)\times x_1\times y_2\times z_2 - (0)\times x_2\times y_1\times z_2 - (-10)\times x_2\times y_2\times z_2 +$
$F_3 + V_3 = 0$

$(0)\times y_1 + (-10)\times y_2 - (100)\times x_1\times y_1\times z_1 - (40)\times x_1\times y_2\times z_1 - (0)\times x_2\times y_1\times z_1 - (-10)\times x_2\times y_2\times z_1 - (100)\times x_1\times y_1\times z_2 - (40)\times x_1\times y_2\times z_2 - (0)\times x_2\times y_1\times z_2 - (-10)\times x_2\times y_2\times z_2 + F_4 + V_4 = 0$

$(10)\times z_1 + (10)\times z_2 - (10)\times x_1\times y_1\times z_1 - (10)\times x_1\times y_2\times z_1 - (100)\times x_2\times y_1\times z_1 - (100)\times x_2\times y_2\times z_1 - (10)\times x_1\times y_1\times z_2 - (10)\times x_1\times y_2\times z_2 - (10)\times x_2\times y_1\times z_2 - (10)\times x_2\times y_2\times z_2 + G_1 + W_1 = 0$

$(100)\times z_1 + (10)\times z_2 - (10)\times x_1\times y_1\times z_1 - (10)\times x_1\times y_2\times z_1 - (100)\times x_2\times y_1\times z_1 - (100)\times x_2\times y_2\times z_1 - (10)\times x_1\times y_1\times z_2 - (10)\times x_1\times y_2\times z_2 - (10)\times x_2\times y_1\times z_2 - (10)\times x_2\times y_2\times z_2 + G_2 + W_2 = 0$

$(10)\times z_1 + (10)\times z_2 - (10)\times x_1\times y_1\times z_1 - (10)\times x_1\times y_2\times z_1 - (100)\times x_2\times y_1\times z_1 - (100)\times x_2\times y_2\times z_1 - (10)\times x_1\times y_1\times z_2 - (10)\times x_1\times y_2\times z_2 - (10)\times x_2\times y_1\times z_2 - (10)\times x_2\times y_2\times z_2 + G_3 + W_3 = 0$

$(100)\times z_1 + (10)\times z_2 - (10)\times x_1\times y_1\times z_1 - (10)\times x_1\times y_2\times z_1 - (100)\times x_2\times y_1\times z_1 - (100)\times x_2\times y_2\times z_1 - (10)\times x_1\times y_1\times z_2 - (10)\times x_1\times y_2\times z_2 - (10)\times x_2\times y_1\times z_2 - (10)\times x_2\times y_2\times z_2 + G_4 + W_4 = 0$

$x_1 + x_2 = 1$

$y_1 + y_2 = 1$

$z_1 + z_2 = 1$

$x_1, x_2, y_1, y_2, z_1, z_2, E_1, E_2, E_3, E_4, F_1, F_2, F_3, F_4, G_1, G_2, G_3, G_4, U_1, U_2, U_3, U_4, V_1, V_2, V_3, V_4, W_1, W_2, W_3, W_4 \geq 0.$

We get the DE $(x_1, x_2) = (1,0)$, $(y_1, y_2) = (0,1)$ and $(z_1, z_2) = (0,1)$, where again A does not change his vendor, B decides to lower the price of its materials, and C offers the material to A with a cheaper price. Obviously in Examples 6.2 and 6.3 the MDE and TDE, or simply DE's, are the same.

6.5 Duality of N-person Games

For a game <p$_1$,…, p$_N$>, let $P = \{1,\ldots,N\}$, J_i be a strict subset of P, $i = 1,\ldots, N$ and $K_i = P \setminus J_i$, $i = 1,\ldots, N$. Consider the inequalities

$$\max_{X_j, j \in J_i} p_i(\boldsymbol{x}_j, \boldsymbol{x}_k^*) \le p_i(\boldsymbol{x}_j, \ldots, \boldsymbol{x}_N^*),\ k \in K_i,\ i = 1,\ldots, N, \tag{6.26}$$

where $p_i(\boldsymbol{x}_j, \boldsymbol{x}_k^*)$, $j \in J_i$, $k \in K_i$, denotes a function $p_i(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ with $\boldsymbol{x}_k^*$, $k \in K_i$, is fixed and $\boldsymbol{x}_j$, $j \in J_i$, is not fixed. Hence $p_i(\boldsymbol{x}_j, \boldsymbol{x}_k^*)$ is a function only of $\boldsymbol{x}_j$, $j \in J_i$.

Definition 6.11 Let $J_i \subseteq P$ be fixed, $i = 1,\ldots,N$. Then $(\boldsymbol{x}_1^*,\ldots, \boldsymbol{x}_N^*)$ is a (J_i , K_i) equilibrium for if (6.26) holds. An (J_i , K_i) equilibrium is said to be a dual equilibrium of a (K_i , J_i)equilibium. In particular, if $J_i = \{i\}$, then an RE and DE are dual equilibrium for $<p_1,\ldots,p_N>$ analogous to the bimatrix case.

Note that J_i and K_i represent a partition of P. Definition 6.11 could be generalized by considering nonempty partitions of P into more than two subsets. Such a construction may have relevance to some notion of coalitions for the game <p$_1$,…, p$_N$>. However, duality will not be explored further in this dissertation.

CHAPTER 7

CONCLUSIONS

7.1 Summary and Contributions

In game theory a rational strategy for player should be one that gives him the "best" expected reward according to his definition of "best." In this dissertation we have provided an alternate concept of "best" in developing new equilibria for one-time noncooperative games. Equally important, we have provided justification for their use.

We have shown the existence of our new Disappointment Equilibrium, developed a direct computational procedure to find one, and related Disappointment Equilibria (DE's) to Nash, or Regret Equilibria (RE's). We have also specialized DE's to two-person zero and nonzero-sum games, where the relation between RE's is particularly elegant. In addition, we have presented numerous examples to illustrate the usage and theoretical aspects of DE's

Our new solutions DE and π are not refinements of RE's, as other equilibria have been. For example, DE's may be considered both a generalization of an RE, as well as its dual. In the RE scenario, a player regrets his action for fixed strategies of all his opponents. A player is more concerned by what his opponents do and judges his decision by what they did. It is a reactive judgment in which the player decides it is futile to move from an RE since he cannot improve his own payoff. In the DE scenario, however, a player is disappointed by what his opponents do for his own choice of strategy. A player is more concerned by what he himself does and judges his decision by what he himself does. He makes a proactive decision in which there is a standoff between the players resulting in no move. Furthermore, it has been noted [28] that people have a tendency to take risks more readily to avoid losses, than to seek gains. A DE can also be construed to invoke this psychological observation. An RE is based on an equilibrium in which no one can get a better payoff. A DE avoids disappointment for the players.

In addition, the DE offers insight into some classical paradoxes of one-time noncooperative game theory not explained by previous solution concepts. The prototypic example is PD. Recall that for PD the DE results from a cooperative strategy of each player, while the RE results from a defective strategy for each. Thus the DE has cooperative implications. Indeed, it captures the philosophy of the moral dictum,"Do unto others as you would have them do to you," albeit by enforcement.

The DE strategy for PD also coincides with an NME resulting from sequential reasoning as follows. In the first round of reasoning, each player would pick his strategy. In the second round, each player would evaluate his opponent's first strategy and decide his own strategy as if it were a sequential move, with the process continuing. If initially each player chooses to cooperate, each might be tempted to cheat in the next round. However by looking ahead at the consequence of both being defective, each player will not want to change his strategy unless his opponent does. The incentive to defect is overcome by the threat of punishment, which is incorporated in the notion of a DE.

A DE also coincides with an Iterated Prisoner's Dilemma (IPD) competition, where the PD game is played repeatedly. A computer decision-making tournament was organized by Axelrod and Hamilton, where each player's computer program made the decisions in a series of PD games against each other entrant [29]. The strategy "Tit for Tat" in which a player starts with a cooperative strategy and then duplicates his opponent's previous strategy won the game with the best average payoff. According to Axelrod [30], a player in such a series of PD games with the same opponent should not be first to defect. A cooperative DE strategy results in a type of repeated equilibrium for such games. Thus from both a TOM and IPD perspective, the DE is better than the RE. Hence, the notion of a DE may be construed to have long-term implications.

For games without such dilemmas, however, neither an RE nor a DE is always better than the other. For that reason, the Pareto Intercession Equilibrium could alleviate current difficulties. It considers both and seeks as a solution strategies for the players that result in a Pareto maximum for their payoffs. It further injects a measure of fairness in the sense that a player will be less likely to be satisfied with a payoff substantially worse than that of other players. In other words, a π attempts to alleviate another tendency of human nature – the intense aversion to being treated unfairly.

In summary, the notion of a Disappoint Equilibrium, defined and analyzed here, provides an additional solution concept for noncooperative games that may sometimes provide a superior result to the players of a game. It refutes the belief that the defining requirements for a Nash equilibrium are necessary conditions for a rational solution. Indeed, the Disappointment Equilibrium explains why humans sometimes behave in ways that are not explainable by previous work.

7.2 Future Work

Future research will be attempt to apply DE's to other areas of game theory, which involve the idea of RE's. In particular, the "Nash program" attempts to solve all games, both cooperative and noncooperative, as noncooperative games via the Nash equilibrium as initiated by Nash in [5], [32] and [33]. Cooperative games are essentially those in which agreements can be enforced, where in noncooperative games only the equilibria are sustainable. The current trend is to include any relevant enforcement mechanisms in the model itself of the game, so that every game would become noncooperative with the Nash Equilibria taken as the candidate solutions.

The Disappointment Equilibria actually seems better suited for this purpose, with its intrinsic cooperation enforced by players wanting to cooperate so as not to be further disappointed. In addition, the j^{th} marginal disappointment equilibria could be generalized for any number of players, not just one or all, to inject a coalitional aspect.

Finally, future research should be directed at developing efficient computational algorithms for solving the nonlinear programs proposed for finding both RE's and DE's. Their special structure offer several possibilities, including transformations making them convex programs.

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BIOGRAPHICAL INFORMATION

Phantipa Insuwan is a Ph.D. student in the Department of Industrial and Manufacturing Systems Engineering at The University of Texas at Arlington (UTA) with a concentration in Operations Research and Finance. Her particular areas of interest are game theory, optimization, stochastic and financial modeling, and financial derivatives. She obtained the degrees Master of Science in Industrial Engineering from UTA in December, 1998, and Bachelor of Engineering in Production Engineering at King Mongkut Institute located in Thonburi, Thailand, in March 1994. After her M.S. degree, Phantipa worked in the telecommunications industry for such companies as Nokia and T-Mobile. In Fall, 2006, she returned to full-time study to complete her Ph.D. dissertation.

In addition to her technical pursuits, Phantipa enjoys museums, art shows, and painting, as well as competitive ballroom dancing, gardening, traveling, reading, movies, and her pet dogs.