PLATE ANALYSIS WITH DIFFERENT GOEMETRIES AND ARBITRARY BOUNDARY CONDITIONS

by

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ABSTRACT

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This thesis work deals with the study of plates with various geometries and different boundary conditions. The method of study is carried out mainly using the Galerkin method combined with the help of the symbolic algebraic software, *Mathematica*. In this thesis, flat plates with rectangular and triangular geometries are subjected to uniform load acting normal to their surfaces. The lateral deflection of the plates is expressed in a series of polynomials which satisfy the homogenous boundary conditions. *Mathematica* is used in handling the algebraic operations to solve for the coefficients and generating the trial functions. The maximum deflection of the plate for various geometries and boundary conditions is determined.

Then the results obtained are compared with the exact solution which is carried out with the use of the finite element analysis software, Ansys. The results obtained from the present method show good agreement with those from Ansys.

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CHAPTER 1

INTRODUCTION

1.1 Introduction to Plates

Plates are straight, flat and non-curved surface structures whose thickness is slight compared to their other dimensions. Generally plates are subjected to load conditions that cause deflections transverse to the plate. Geometrically they are bound either by straight or curved lines. Plates have free, simply supported or fixed boundary conditions. The static or dynamic loads carried by plates are predominantly perpendicular to the plate surface. The load carrying action of plates resembles that of beams or cables to a certain extent. Hence plates can be approximated by a grid work of beams or by a network of cables, depending on the flexural rigidity of the structures. Plates are of wide use in engineering industry. Many structures such as ships and containers require complete enclosure of plates without use of additional covering which consequently saves the material and labor. Nowadays, plates are generally used in architectural structures, bridges, hydraulic structures, pavements, containers, airplanes, missiles, ships, instruments and machine parts. Plates are usually subdivided based on their structural action as

1. *Stiff Plates*, which are thin plates with flexural rigidity and carry the loads two dimensionally. In engineering practice, a plate is understood as a stiff plate unless specified
2. *Membranes*, which are thin plates without flexural rigidity and carry the lateral loads by axial shear forces. This load carrying action is approximated by a network of stressed cables since their moment resistance is of a negligible order of magnitude.

3. *Flexible Plates,* which represent a combination of stiff plates and membranes. They carry external loads by the combined action of internal moments and transverse shear forces.
4. *Thick Plates*, whose internal stress condition resembles that of three dimensional structures.

1.2 Introduction to Plate Analysis

The analysis of plates first started in the 1800s. Euler [1] was responsible for solving free vibrations of a flat plate using a mathematical approach for the first time. Then it was the German physicist Chladni [2] who discovered the various modes of free vibrations. Then later on the theory of elasticity was formulated. Navier [3] can be considered as the originator of the modern theory of elasticity. Navier's numerous scientific activities included the solution of various plate problems. He was also responsible for deriving the exact differential equation for rectangular plates with flexural resistance. For the solution to certain boundary value problems Navier introduced exact methods which transformed differential equations to algebraic equations. Poisson in 1829 [4] extended the use of governing plate equation to lateral vibration of circular plates.

Later, the theory of elasticity was extended as there were many researchers working on the plate and the extended plate theory was formulated. Kirchoff [5] is considered as the one who formulated the extended plate theory.

In the late 1900s, the theory of finite elements was evolved which is the basis for all the analysis on complex structures. However the analyses using finite elements are now being carried out using comprehensive software which requires high CPU resources to compute the results. Another method for analysis of plates statically and dynamically was later developed for arbitrary shapes using advanced finite elements. Actually there was a method called the weighted residual method which was used in analysis of plate even before the finite element method of analyzing the plate was formulated.

1.3 Symbolic Software

Symbolic software packages such as *MATHEMATICA* are useful for solving algebraic and symbolic systems. Most of the older symbolic packages which were previously developed were written in LISP but *Mathematica* is based on the C language to solve problems. *Mathematica* was first released in 1988. It has had a profound effect on the way computers are used in technical and other fields. *Mathematica* uses a generic way of writing codes and thereby it is widely used in various fields. A program written in *Mathematica* is simple, robust and it can be easily understood thereby making it simple for anyone to use it.

The framework of *Mathematic*a is such that it is split into two parts, the kernel and the front end. The kernel interprets expressions (*Mathematica* code) and returns the result. The front end of *Mathematica* is Graphical User Interface (GUI) which allows us to create edit and format the notebook. More advanced features include 3D picturing, indexing and slide show creation.

The advantage of using *Mathematica* lies in its built-in functions. It has the largest database of algorithms. It is also helpful in numerical computation, symbolic computation, data interpretation etc. Symbolic software also addresses the finite element method and is useful in finding shape functions, creating different types of meshes and can solve problems for different materials.

1.4 Analysis Software

Ansys Inc. has developed many different software packages and amongst those is the ANSYS workbench platform. It is the framework upon which advanced engineering simulations are built. It is the advanced version developed in recent years which has the schematic view and drag drop option thereby making the complex process of the user much simple. Ansys Workbench combines the strength of core problem solvers with project management tools necessary to manage project workflow. In Ansys, Workbench Analysis is built as systems which can be combined together into a project. The project is driven by a schematic workflow that manages the connections between the systems.

1.5 Galerkin Method

The Galerkin method was invented by a Russian mathematician, Boris Grigoryevich Galerkin [5]. The Galerkin method can be used to approximate the solution to ordinary differential equations and partial differential equations. It is useful in solving almost all engineering problems with prescribed boundary conditions. The Galerkin method uses the governing equation of the system and boundary conditions to solve the problem. The Galerkin method uses trial functions with a number of unknown parameters. Then a polynomial formed as a trial function and the unknown parameters are determined. The Galerkin method is the one used widely in various fields such as heat and mass transfer, fluid flow and mechanics.

The objective of this thesis is to analyze various geometries (rectangular and triangular) of flat plates using the Galerkin method under arbitrary boundary conditions and then compare the results with the use of Ansys Workbench thereby validating the result derived analytically. In this thesis, three different cases are considered and analyzed.

First a flat rectangular plate is considered with arbitrary boundary conditions. It is clamped on all edges and it is subjected to uniform loading on the top and the maximum deformation is determined by considering trial functions and applying the boundary conditions. This is done using the symbolic algebraic software, *Mathematica*. The results are then compared for the same problem and the same boundary conditions using the analysis software, Ansys. Both of the results are compared and analyzed.

In the next case the geometry of the plate is considered as being triangular such that the boundary conditions are arbitrarily chosen and satisfying the equation $X + Y = 1$. The plate is clamped on all sides and is subjected to uniform loading such that we get the maximum deflection by taking a trial function and analyzing it using *Mathematica*. Then the same case is taken and the maximum deflection is calculated using Ansys. Then both the results are compared and studied.

In the final case the geometry of the plate is considered triangular and the side of the triangle is expressed by $X + Y = 1$. The plate is clamped on two sides and simply supported on the other side. It is subjected to uniform loading and the maximum deflection is calculated both using *Mathematica* and Ansys. Then all the results are tabulated and conclusions and recommendations are arrived.

CHAPTER 2

THEORY AND ANALYSIS OF PLATES

2.1 Plate Equation

There were many plate theories formulated after the Euler–Bernoulli beam theory was proposed. The Euler–Bernoulli beam theory also known as the engineer's beam theory is a simplification of the linear theory which provides a means of calculating the load-carrying and deflection characteristics of beams. Of the numerous plate theories that have been developed since the late 19th century, two are widely accepted and used in engineering. They are:

- the Kirchhoff–Love theory of plates (classical plate theory)
- The Mindlin–Reissner theory of plates (first-order shear plate theory)

According to Kirchoff, the assumptions were made considering a mid-surface plane which helps in representing a three dimensional plate in two dimensional form. The basic assumptions according to Kirchoff are:

1. The normal lines (straight lines perpendicular to the flat surface) remain straight after deformation.
2. The normals remain the same length (unstretched).
3. The normals always remain at right angles to the mid surface after deformation

The plate equation is derived by assuming that plate is subjected to lateral forces and the following three equilibrium equations are used.

$$\sum M_x = 0 \qquad (2.1)$$

$$\sum M_y = 0 \tag{2.2}$$

$$\sum P_z = 0 \tag{2.3}$$

where M_x and M_y are the bending moments and $P_z$is the external load. The external load P_z is carried by the transverse shear forces Q_x , Q_y and bending moments M_x, M_y. The plates generally have significant deviation from the beams and it is due to the presence of twisting moment M_{xy}. In general in the theory of plates it is necessary to deal with the internal forces and moments per unit length of the middle surface. The procedure involved in forming the differential equation of the plate in equilibrium is selecting the coordinate system and draw the sketch of the plate element and showing all the internal forces by positive and negative thereby expressing them in Taylor's series.

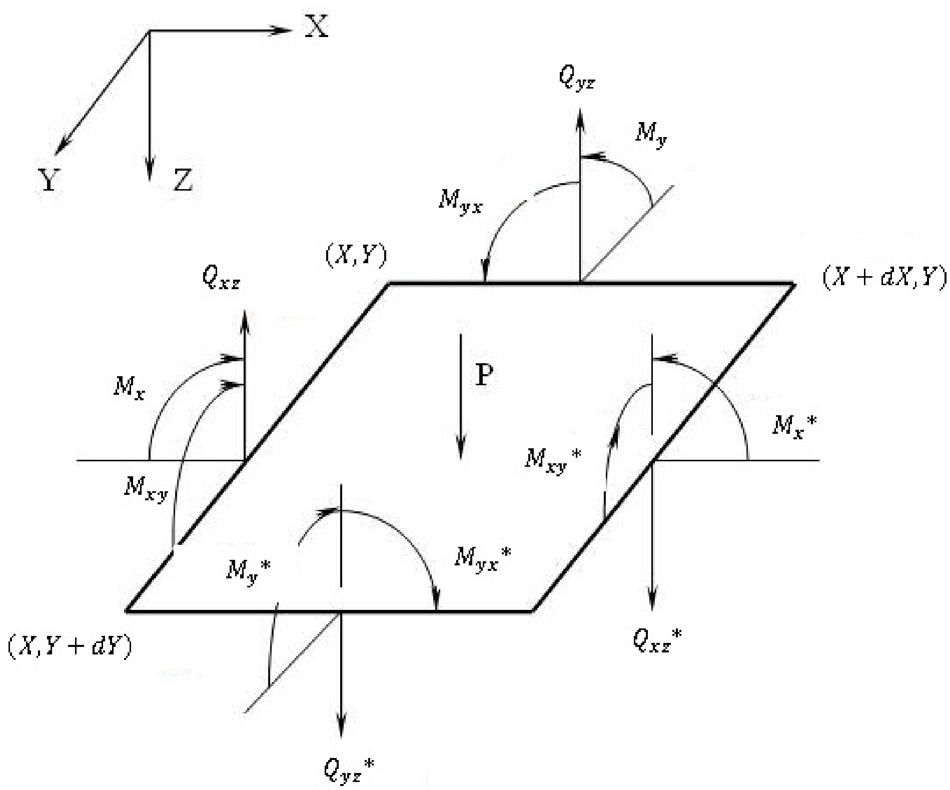

Figure 2.1 Differential Plate with Stress Resultants

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \tag{2.4}$$

$$M_y = -D\left(\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \tag{2.5}$$

$$M_{xy} = M_{yx} = -D(1-\nu)\left(\nu\frac{\partial^2 w}{\partial x\,\partial y}\right) \tag{2.6}$$

$$Q_{yz^*} = Q_{yz} + \frac{\partial Q_{yz}}{\partial y}dy \tag{2.7}$$

$$Q_{xz^*} = Q_{xz} + \frac{\partial Q_{xz}}{\partial x}dx \tag{2.8}$$

$$M_{y^*} = M_y + \frac{\partial M_y}{\partial y}\partial y \tag{2.9}$$

$$M_{x^*} = M_x + \frac{\partial M_x}{\partial x}\partial x \tag{2.10}$$

$$M_{xy^*} = M_{xy} + \frac{\partial M_{xy}}{\partial x}\partial x \tag{2.11}$$

$$M_{yx^*} = M_{yx} + \frac{\partial M_{yx}}{\partial y}\partial y \tag{2.12}$$

The condition of a vanishing resultant force in the z direction results in the following equation

$$-Q_{xz} - Q_{yz} + Q_{yz^*} + Q_{xz^*} + Pdxdy = 0$$

$$-Q_{yz}dy - Q_{yz}dx + \left(Q_{yz} + \frac{\partial Q_{yz}}{\partial y}dy\right)dx + \left(Q_{xz} + \frac{\partial Q_{xz}}{\partial x}dx\right)dy + Pdxdy = 0$$

$$\frac{\partial Q_{xz}}{\partial x} + \frac{\partial Q_{yz}}{\partial y} + P_z = 0 \tag{2.13}$$

$$\frac{\partial Q_{xz}}{\partial x} + \frac{\partial Q_{yz}}{\partial y} = -P_z \tag{2.14}$$

If the resultant moment about an edge parallel to the x and y axes is set to zero then the resulting equation after neglecting the higher order terms gives

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} + Q_{xz} = 0 \tag{2.15}$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} + Q_{yz} = 0 \tag{2.16}$$

Substituting Eq. (2.15) and Eq. (2.16) in Eq. (2.14) gives

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\ \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -P_z(x, y) \tag{2.17}$$

Now substituting Eq. (2.4), Eq. (2.5) and Eq. (2.6) in the above equation yields the differential equation of the plate subjected to lateral loads

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2\ \partial y^2} + \frac{\partial^4 W}{\partial y^4} = \frac{P_z}{D} \tag{2.18}$$

$$\nabla^4 w = \frac{P}{D} \tag{2.19}$$

where $$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\ \partial y^2} + \frac{\partial^4}{\partial y^4} \tag{2.20}$$

and $D = \frac{Eh^3}{12(1-\nu^2)}$,is called the flexural rigidity of the plate

where E- Young's modulus of the plate

h- Height of the plate

ν- Poisson's ratio

2.2 Boundary Conditions

Generally, there are different types of boundaries considered for a plate in terms of lateral deflection of the middle surface of the plate and they are:

1. Clamped edge Conditions
2. Simply Supported edge Conditions
3. Mixed edge Conditions
4. Free edge Conditions

2.2.1 Clamped Edge Conditions

If a plate is clamped at the boundary, then the deflection and the slope of the middle surface must vanish at the boundary. On a clamped edge parallel to the y axis at $x = a$, the boundary conditions are

$$w|_{\ x=a} = 0 \tag{2.24}$$

$$\frac{\partial w}{\partial X}|_{\ x=a} = 0 \tag{2.25}$$

The boundary conditions on the clamped edge parallel to the x axis at $y = b$ are

$$w|_{\ y=b} = 0 \tag{2.26}$$

$$\frac{\partial w}{\partial y}|_{\ y=b} = 0 \tag{2.27}$$

2.2.2 Simply Supported Edge Conditions

A plate boundary that is prevented from deflecting but free to rotate about a line along the boundary edge, such as a hinge, is defined as a simply supported edge. The conditions on a simply supported edge parallel to the y axis at $x = a$ are

$$w|_{\ x=a} = 0 \tag{2.28}$$

$$M_x|_{\ x=a} = -D\left(\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2} + \nu \frac{\partial^2 \mathrm{w}}{\partial \mathrm{y}^2}\right)_{\ x=a} = 0 \tag{2.29}$$

Since the change of w with respect to the y coordinate vanishes along this edge, the conditions become

$$w|_{\ x=a} = 0 \tag{2.30}$$

$$\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2}|_{\ x=a} = 0 \tag{2.31}$$

On a simply supported edge parallel to the x axis at $y = b$, the change of w with respect to the x coordinate vanishes, thus the conditions along this boundary are

$$w|_{\ y=b} = 0 \tag{2.32}$$

$$M_y|_{\ y=b} = -D\left(\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)_{\ y=b} = 0 \tag{2.33}$$

$$= -D\ \frac{\partial^2 w}{\partial y^2}|_{\ y=b} = 0 \tag{2.34}$$

2.2.3 Mixed Edge Conditions

Consider the plate to be simply supported on two opposite side and clamped on the other two sides at $y = 0$ and $y = b$. The boundary conditions for such a type of mixed edges is,

$$w|_{\ y=b} = 0 \tag{2.35}$$

$$\frac{\partial w}{\partial X}|_{\ y=b} = 0 \tag{2.36}$$

On the simply supported edges parallel to the y axis the boundary condition at $x = 0$ and $x = a$

$$w|_{\ x=a} = 0 \tag{2.37}$$

$$M\,|_{\ x=a} = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)_{\ x=a} = 0 \tag{2.38}$$

2.2.4 Free Edge Conditions

In the most general case, a twisting moment, a bending moment, and a transverse shear force act on an edge of a plate. An edge with all three of these stress resultants vanishing is defined as a free edge. The boundary conditions on a free edge parallel to the x axis at $y = b$ is

$$M_y|_{\ y=b} = -D\left(\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)_{\ y=b} = 0 \tag{2.39}$$

$$V_{yz}\,|_{\ y=b} = -D\left(\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2\,\partial y}\right)_{\ y=b} = 0 \tag{2.40}$$

The boundary conditions on a free edge parallel to the y axis at $x = a$

$$M_x|_{\ x=a} = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)_{\ x=a} = 0 \tag{2.41}$$

$$V_{yz}\,|_{\,x=a} = -D\left(\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right)_{x=a} = 0 \qquad (2.42)$$

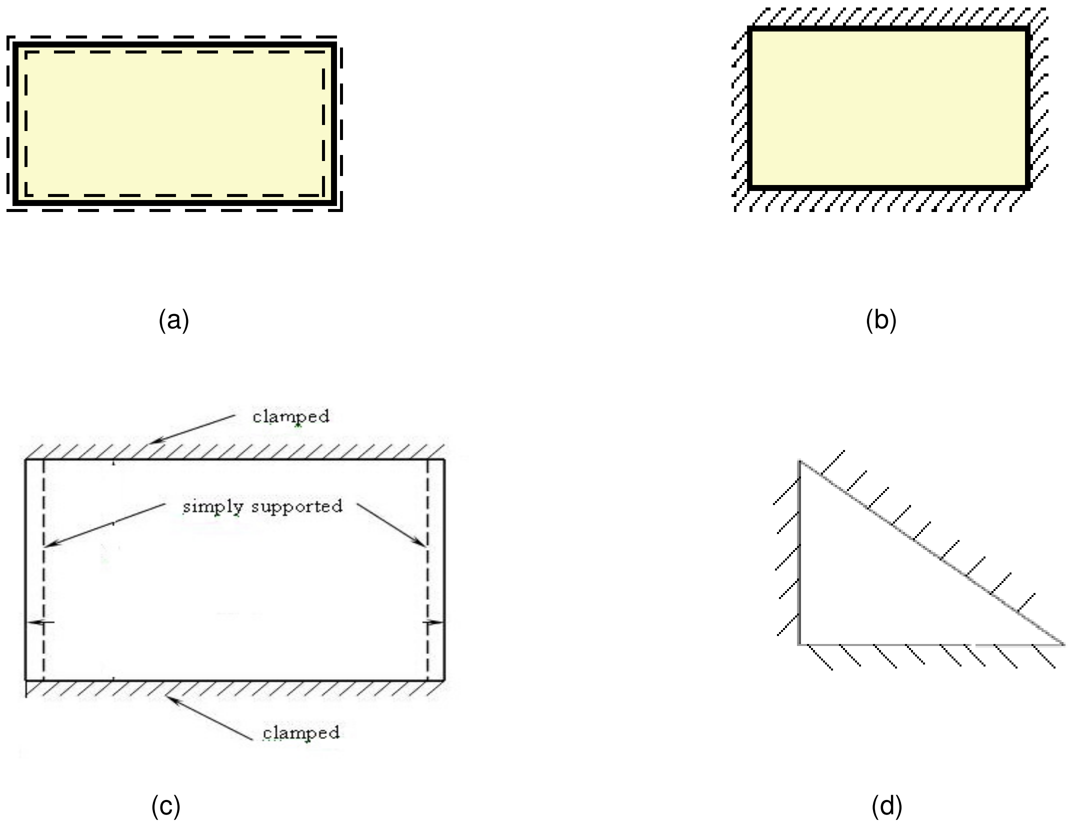

Figure 2.2 Plate with different geometries and boundaries (a) Rectangular plate simply supported all sides, (b) Rectangular plate clamped all sides, (c) Rectangular plate with mixed boundaries,(d) Triangular plate clamped all sides

2.3 Method of Weighted Residuals

Prior to the development of the finite element method, there existed an approximation technique for solving differential equations called the Method of Weighted Residuals (MWR). The basic idea of the Method of the Weighted Residuals is to use a trial function with a number of unknown parameters to approximate the solution. Then a weighted average over the boundary is set to zero. A polynomial with a set of parameters is considered and the solution is approximated accordingly. The polynomial is made to satisfy the boundary conditions. Suppose we have a linear differential operator D acting on a function 'u' to produce a function 'p' then

$$D\,(u(x)) \;=\; p(x) \qquad (2.43)$$

The function u is approximated by considering it as a linear combination of trial functions 'ũ'

$$u = \;\tilde{u} \;= \textstyle\sum_{i=1}^{n} a_i\, e_i \qquad (2.43)$$

When used into the differential operator, D, the result of the operations is not in general p(x) because an error residual will be imparted

$$E(x) \;=\; R(x) \;=\; D\,(\tilde{u}\,(x)) \;-\; p(x) \;\neq 0 \qquad (2.44)$$

The Method of Weighted Residuals is where the residual over the domain is forced to be zero

$$\int_X R(x)Widx \;=\; 0 \qquad (2.45)$$

where Wi is equal to number of unknown constants a_i.

There are three types of Method of Weighted Residuals.

1. Point Collocation Method

2. Method of Least squares

3. Galerkin Method

2.3.1 Point Collocation Method

The Point Collocation Method is one of the types of the Method of Weighted Residuals (MWR). In this method the weighting functions are taken from the Dirac delta function in the domain. That is

$$Wi(x) \;=\; \delta(x\text{–}x_i) \qquad (2.46)$$

Hence the integration of the weighted residual results in forcing of the residual to zero at specific points in the domain. Therefore

$$R(x_i) \;=\; 0 \qquad (2.47)$$

2.3.2 Method of Least Squares

The second method of the weighted residuals is called as the Method of Least Squares.

$$S = \int_X R(x)R(x)dx$$

$$= \int_X R^2(x)dx \qquad (2.48)$$

In order to achieve the minimum of this scalar function, the derivatives of S with respect to all the unknown parameters must be zero. Therefore the weight functions for the Least Square Method are just the derivatives of the residual with respect to the unknown constants.

$$Wi = \frac{\partial R}{\partial W} \qquad (2.49)$$

2.3.3 Galerkin Method

The next type of the weighted residuals is the Galerkin Method.

$$(R, e_i) = 0 \qquad (2.50)$$

where $i = 1,2, \dots \dots N$ and R is defined as the residual between the approximate solution and the exact solution. The Galerkin method is widely used in the field of engineering such as structural dynamics, acoustics and heat and mass transfer.

CHAPTER 3

GALERKIN METHOD AND ITS APPLICATIONS

3.1 Galerkin Method

The Galerkin method is used in approximating the solutions of ordinary differential equations, partial differential equations and integral equations. The main aim of the Galerkin method is to solve the differential equations. Let us consider an equation of the form

$$Lu = c \tag{3.1}$$

where L is the differential operator, u is the unknown function and c is the given function. An approximate solution to Eq. (3.1) is given by a linear combination of N base functions in the form

$$\tilde{u}(x,y) = \sum_{i=1}^{n} u_i \, e_i \, (x,y) \tag{3.2}$$

where u_i is the unknown coefficient and e_i is the base function in the function space.

The residual between the exact and approximate solution can be defined as

$$R = L\tilde{u} - c \tag{3.3}$$

where $\tilde{u}$ is the approximation of u .

We know that $\tilde{u} = \sum_{i=1}^{n} u_i e_i$ so substituting it in Eq. (2.51) we get

$$R = L\sum_{i=1}^{n} u_i e_i - c$$

$$= \sum_{i=1}^{n} u_i \, Le_i - c \tag{3.4}$$

The quantity, R, is a function of position and hence the residual equation becomes

$$R(x,y) = \sum_{i=1}^{n} u_i \, Le_i \, (x,y) - c(x,y) \tag{3.5}$$

According to the Galerkin method the unknown coefficients are determined by

$$(R, e_i) = 0 \tag{3.6}$$

which is computed as

$$\iint_V e_i \, R \, dv = 0 \tag{3.7}$$

3.1.1 Solving the plate equation using Galerkin Method

Now the Galerkin method is extended to solving the plate equation. From Eq. (2.19) we get,

$$\nabla^4 w = \frac{P}{D}$$

The approximate solution is of the form

$$w(x,y) = \sum_{i=1}^{n} a_i \, \varphi_i \, (x,y) \tag{3.8}$$

where the trial functions, φ_i's, are linearly independent and have the same boundary conditions as w.

The residual or error is in the form

$$R \equiv \nabla^4 w - \frac{P(x,y)}{D} \tag{3.9}$$

$$R \equiv \sum_{i=1}^{n} a_i \nabla^4 \, \varphi_i \, (x,y) - \frac{P(x,y)}{D} \tag{3.10}$$

Then calculating the weighted integral, we get

$$\iint \left(\sum_{i=1}^{n} a_i \nabla^4 \varphi_i - \frac{P(x,y)}{D}\right) \varphi_j \; dx\, dy = 0 \qquad (3.11)$$

or it can be rewritten as

$$\sum_{i=1}^{n} a_i \iint \nabla^4 \varphi_i \, \varphi_j \, dx\, dy = \frac{P(x,y)}{D} \iint \varphi_j \; dx\, dy$$

where the integral is carried out over the plate surface.

Let $F_j = \iint \varphi_j \; dx\, dy$ and $K_{ij} = \iint \nabla^4 \varphi_j \, \varphi_i \, dx\, dy$ then the above equation can be expressed as

$$\sum_{i=1}^{n} a_i \, K_{ij} = \frac{P(x,y)}{D} F_j \qquad (3.12)$$

3.2 Analysis of plates with different geometries

The Galerkin method is used to solve differential equations and it can be extended to plates with different geometries and arbitrary boundary conditions. The three different examples taken and analyzed are

i. Flat rectangular plate clamped on all sides
ii. Flat triangular plate clamped on all sides
iii. Flat triangular plate clamped on two sides and simply supported on the other side

The analysis is made easy and faster only because of the use of the symbolic software, *Mathematica* which is used to solve all the complex polynomials and in evaluating the coefficients which is not feasible by hand.

3.2.1 Numerical solution of a rectangular plate clamped on all sides

Consider a rectangular plate as in Figure 3.1 which is clamped on all the sides and it is subjected to uniform loading.

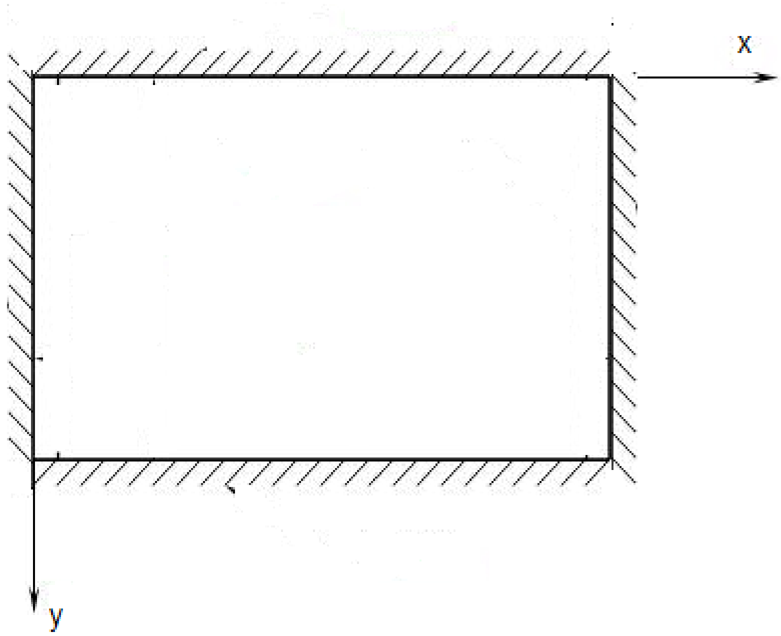

Figure 3.1 Flat rectangular plate clamped on all sides

Let the dimensions of the plate be ‘a’ and ‘b’ respectively. The governing equation of a plate in the lateral displacement w is given, from Eq. (2.18)

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2\, \partial y^2} + \frac{\partial^4 W}{\partial y^4} = \frac{P_z}{D} \tag{3.13}$$

where P_z is the load acting on the surface of the plate and D is the flexural rigidity of the plate defined as $D = \frac{Eh^3}{12(1-\nu^2)}$

where E is Young’s modulus of the plate and ν is the Poisson ratio. The exact solution of the plate equation should satisfy all the boundary conditions of any plate problem. In the analysis, the values of a and b of the plate are taken as 1. If the plate is clamped on all the edges at $x = 0, x = a, y = 0$ and $y = b$ then the boundary conditions are

At $x = 0, x = a$

$$w = 0 \tag{3.14}$$

$$\frac{\partial w}{\partial x} = 0 \tag{3.15}$$

At $y = 0, y = b$

$$w = 0 \tag{3.16}$$

$$\frac{\partial w}{\partial y} = 0 \tag{3.16}$$

In order to find the deflection of the plate using the Galerkin method, first we need to begin by considering the base function which satisfies the plate's boundary conditions. The base function is represented in the general form as

$$\mathrm{w}(x, y) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{c_i}\, \varphi_{\mathrm{i}}\,(x, y) \tag{3.17}$$

The trial function for the rectangular plate is evaluated using the symbolic software, *Mathematica*. For $n = 8$, it is given by

$$\begin{aligned}
\varphi(x,y) = {} & a[1] + xa[2] + ya[3] + x^2a[4] + xya[5] + y^2a[6] + x^3a[7]x^2ya[8] + \\
& xy^2a[9] + y^3a[10]x^4a[11] + x^3ya[12] + x^2y^2a[13] + xy^3a[14] + y^4a[15] + \\
& x^5a[16] + x^4ya[17] + x^3y^2a[18] + x^2y^3a[19] + xy^4a[20] + y^5a[21] + x^6a[22] + \\
& x^5ya[23] + x^4y^2a[24] + x^3y^3a[25] + x^2y^4a[26] + xy^5a[27] + y^6a[28] + x^7a[29] + \\
& x^6ya[30] + x^5y^2a[31] + x^4y^3a[32] + x^3y^4a[33] + x^2y^5a[34] + xy^6a[35] + y^7a[36] + \\
& x^8a[37]x^7ya[38] + x^6y^2a[39] + x^5y^3a[40] + x^4y^4a[41] + x^3y^5a[42] + x^2y^6a[43] + \\
& xy^7a[44] + y^8a[45]
\end{aligned} \tag{3.18}$$

It is found that the eighth order polynomial is the lowest possible polynomial to satisfy the boundary conditions. Once the general N^{th} order polynomial is defined as in Eq. (3.18), the boundary conditions are applied to solve for the unknown coefficients. This is done by generating eight equations for each boundary condition using *Mathematica*. In this analysis the value of 'a' and 'b' are considered to be $a = b = 1$.

Table 3.1 Boundary conditions of flat rectangular plate clamped on all sides

BC (1)	$x = 0$	$w(0, y) = 0$
BC (2)	$x = 0$	$\frac{\partial w}{\partial x} = 0$
BC (3)	$x = 1$	$w(1, y) = 0$
BC (4)	$x = 1$	$\frac{\partial w}{\partial x} = 0$
BC (5)	$y = 0$	$w(x, 0) = 0$
BC (6)	$y = 0$	$\frac{\partial w}{\partial y} = 0$
BC (7)	$y = 1$	$w(x, 1) = 0$
BC (8)	$y = 1$	$\frac{\partial w}{\partial y} = 0$

Having solved for all the boundary conditions and tabulating them using the Table[] command in *Mathematica*, we get the values of the coefficients a[i]. Substituting the values of the coefficients in the general polynomial we end up in the following trial functions,

$$\varphi(x, y) = \{x^2y^2 - 2x^3y^2 + x^4y^2 - 2x^2y^3 + 4x^3y^3 - 2x^4y^3 + x^2y^4 - 2x^3y^4 + x^4y^4\} \qquad (3.19)$$

Increasing the order of polynomial for better convergence, we get the ninth order polynomial for $n = 9$

$$
\begin{aligned}
\varphi(x,y) = \ & a[1] + xa[2] + ya[3] + x^2a[4] + xya[5] + y^2a[6] + x^3a[7] + x^2ya[8] \\
& + xy^2a[9]y^3a[10]x^4a[11] + x^3ya[12] + x^2y^2a[13] + xy^3a[14] \\
& + y^4a[15] + x^5a[16] + x^4ya[17] + x^3y^2a[18] + x^2y^3a[19] \\
& + xy^4a[20] + y^5a[21] + x^6a[22] + x^5ya[23] + x^4y^2a[24] \\
& + x^3y^3a[25] + x^2y^4a[26] + xy^5a[27] + y^6a[28] + x^7a[29] \\
& + x^6ya[30] + x^5y^2a[31] + x^4y^3a[32] + x^3y^4a[33] + x^2y^5a[34] \\
& + xy^6a[35] + y^7a[36] + x^8a[37] + x^7ya[38] + x^6y^2a[39] \\
& + x^5y^3a[40] + x^4y^4a[41] + x^3y^5a[42] + x^2y^6a[43] + xy^7a[44] \\
& + y^8a[45] + x^9a[46] + x^8ya[47] + x^7y^2a[48] + x^6y^3a[49] \\
& + x^5y^4a[50] + x^4y^5a[51] + x^3y^6a[52] + x^2y^7a[53] + xy^8a[54] \\
& + y^9a[55]
\end{aligned} \qquad (3.20)
$$

Now applying the boundary conditions to the polynomial in Eq. (3.20) and solving for the unknown coefficients, we get the following trial functions

$$\varphi 1(x,y) = x^2y^2 - 2x^3y^2 + x^4y^2 - 2x^2y^3 + 4x^3y^3 - 2x^4y^3 + x^2y^4 - 2x^3y^4 + x^4y^4 \qquad (3.21)$$

$$\varphi_2(x,y) = 2x^2y^2 - 3x^3y^2 + x^5y^2 - 4x^2y^3 + 6x^3y^3 - 2x^5y^3 + 2x^2y^4 - 3x^3y^4 + x^5y^4 \qquad (3.22)$$

$$
\begin{aligned}
\varphi_3(x,y) = \ & 2x^2y^2 - 4x^3y^2 + 2x^4y^2 - 3x^2y^3 + 6x^3y^3 - 3x^4y^3 + x^2y^5 - 2x^3y^5 \\
& + x^4y^5
\end{aligned} \qquad (3.23)
$$

Having determined all the trial functions the next step is to find the residual function. From Eq. (3.5)

$$\mathrm{R(x,y)} = \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{u_i\ Le_i}\ (x,y) - c(x,y) \qquad (3.23\ a)$$

The residual function for these trial functions is determined using *Mathematica*.

$$R \equiv \sum_{i=1}^{n} c_i \nabla^4 \varphi_i(x,y) - \frac{P(x,y)}{D} \tag{3.24}$$

where c_i is the unknown coefficient and φ_i is the trial function.

Here the value of P/D is considered to be 1 for convenience and it is later modified. Therefore the residual equation becomes

$$R \equiv \sum_{i=1}^{n} c_i \nabla^4 \varphi_i(x,y) - 1 \tag{3.25}$$

Using *Mathematica* the residual for the order n=8 is determined as

$$\begin{aligned} R = \; & -1 + c1(24x^2 - 48x^3 + 24x^4) + 2c1(4 - 24x + 24x^2 - 24y + 144xy \\ & - 144x^2y + 24y^2 - 144xy^2 + 144x^2y^2) + c1(24y^2 - 48y^3 + 24y^4) \end{aligned} \tag{3.26}$$

Using *Mathematica* the residual for the order n=9 is determined as

$$\begin{aligned} R = \; & -1 + c1(24x^2 - 48x^3 + 24x^4) + c2(48x^2 - 72x^3 + 24x^5) \\ & + c3(120x^2y - 240x^3y + 120x^4y) + c1(24y^2 - 48y^3 + 24y^4) \\ & + c2(120xy^2 - 240xy^3 + 120xy^4) + c3(48y^2 - 72y^3 + 24y^5) \\ & + 2(c1(4 - 24x + 24x^2 - 24y + 144xy - 144x^2y + 24y^2 - 144xy^2 \\ & + 144x^2y^2) + c2(8 - 36x + 40x^3 - 48y + 216xy - 240x^3y + 48y^2 \\ & - 216xy^2 + 240x^3y^2) + c3(8 - 48x + 48x^2 - 36y + 216xy \\ & - 216x^2y + 40y^3 - 240xy^3 + 240x^2y^3)) \end{aligned} \tag{3.27}$$

Then the values of all the coefficients are found out by determining the weighted integral with *Mathematica* and the values are determined as

$$c1 \rightarrow \frac{49}{144}$$

The deflection equation of the plates in the general form is

$$w = \sum_{i=1}^{n} c_i \varphi_i(x, y) \tag{3.26}$$

where φ_i is the trial function.

In this case further computing the deflection equation we get

$$\mathrm{w} = \frac{49}{144}(x^2y^2 - 2x^3y^2 + x^4y^2 - 2x^2y^3 + 4x^3y^3 - 2x^4y^3 + x^2y^4 - 2x^3y^4 + x^4y^4) \tag{3.27}$$

Further plotting the values and generating a plot to determine the maximum value of deflection using *Mathematica* we get the plot as shown.

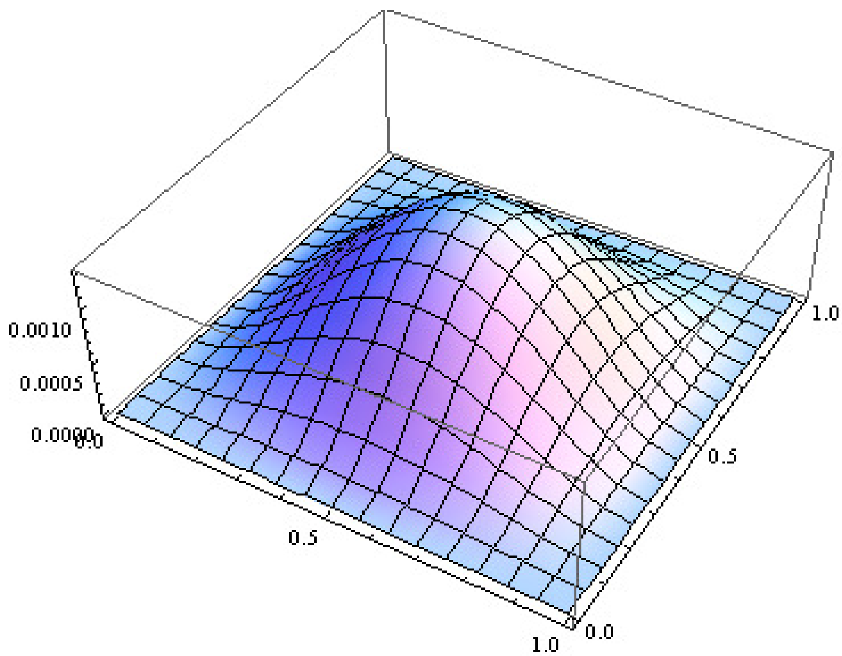

Figure 3.2 3-D plot of maximum deflection of a rectangular plate clamped on all sides

The maximum deflection w_{max} is determined to be 0.0013 for a square plate clamped on all sides. This value of w_{max} determined is for the equation $\nabla^4 \mathrm{w} = 1$ assuming the flexural rigidity to be unity (P/D=1).But the classical plate equation is $\nabla^4 \mathrm{w} = \frac{\mathrm{P}}{\mathrm{D}}$ .Hence determining the value of

P/D, taking all the values to be arbitrary and calculating the value of D, using the formula $D = \frac{Eh^3}{12(1-\nu^2)}$, we get the value of w_{max} = 14.195 for a square plate clamped on all the sides.

3.2.2 Numerical solution of a triangular plate clamped on all sides

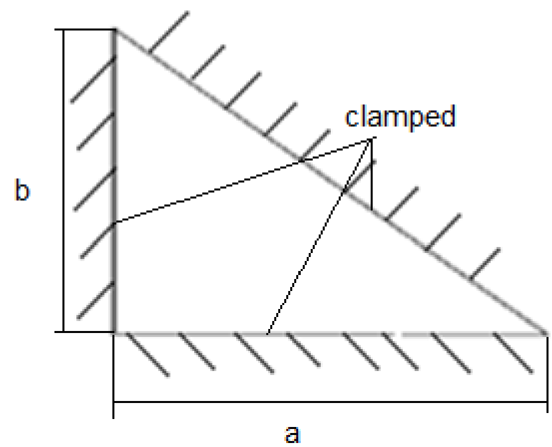

Figure 3.3 Triangular plate clamped on all sides

The governing equation of a plate in the lateral displacement w is given as, from Eq. (2.18)

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2\ \partial y^2} + \frac{\partial^4 W}{\partial y^4} = \frac{P_z(x, y)}{D}$$

where P_z is the load acting on the surface of the plate and

D is the flexural rigidity of the plate.

The plate is clamped on all the edges at $x = 0, y = 0\ , y = 1 - x$ then the boundary conditions are

At $x = 0$,

$$w = 0 \tag{3.28}$$

$$\frac{\partial w}{\partial x} = 0 \tag{3.29}$$

At $y = 0,$

$$w = 0 \tag{3.30}$$

$$\frac{\partial w}{\partial y} = 0 \tag{3.31}$$

At $y = 1 - x,$

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 0 \tag{3.32}$$

The trial function is represented in the general form as

$$\mathrm{w}(x, y) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{c_i}\ \varphi_{\mathrm{i}}\ (x, y) \tag{3.27}$$

The trial function for the plate is evaluated using the symbolic software, *Mathematica*.

$$\begin{aligned}\varphi(x,y) = \quad & a[1] + xa[2] + ya[3] + x^2a[4] + xya[5] + y^2a[6] + x^3a[7] + x^2ya[8] \\ & + xy^2a[9] + y^3a[10] + x^4a[11] + x^3ya[12] + x^2y^2a[13] + xy^3a[14] \\ & + y^4a[15] + x^5a[16] + x^4ya[17] + x^3y^2a[18] + x^2y^3a[19] \\ & + xy^4a[20] + y^5a[21] + x^6a[22] + x^5ya[23] + x^4y^2a[24] \\ & + x^3y^3a[25] + x^2y^4a[26] + xy^5a[27] + y^6a[28]\end{aligned} \tag{3.28}$$

It is found that the sixth order polynomial is the lowest possible polynomial to satisfy the boundary conditions. Once the general Nth order polynomial is defined, the boundary conditions are applied to solve the unknown coefficients. Having solved all the boundary conditions and tabulating them using the Table[ ] command in Mathematica, we get the values of the coefficients a[i]. Substituting the values of the coefficients in the general polynomial we end up with one trial function,

$$\varphi(x,y) = \{x^2y^2 - 2x^3y^2 + x^4y^2 - 2x^2y^3 + 2x^3y^3 + x^2y^4\} \tag{3.28}$$

Increasing the order of polynomial to get the convergence and evaluating the ninth order polynomial, $n = 9$ we get the following trial functions

$$\varphi_1(x,y) = x^2y^2 - 2x^3y^2 + x^4y^2 - 2x^2y^3 + 2x^3y^3 + x^2y^4 \tag{3.29}$$

$$\varphi_2(x,y) = x^3y^2 - 2x^4y^2 + x^5y^2 - 2x^3y^3 + 2x^4y^3 + x^3y^4 \tag{3.30}$$

$$\varphi_3(x,y) = 2x^2y^2 - 6x^3y^2 + 6x^4y^2 - 2x^5y^2 - 3x^2y^3 + 6x^3y^3 - 3x^4y^3 + x^2y^5 \tag{3.31}$$

$$\varphi_4(x,y)x^4y^2 - 2x^5y^2 + x^6y^2 - 2x^4y^3 + 2x^5y^3 + x^4y^4 \tag{3.32}$$

$$\varphi_5(x,y) = 2x^3y^2 - 6x^4y^2 + 6x^5y^2 - 2x^6y^2 - 3x^3y^3 + 6x^4y^3 - 3x^5y^3 + x^3y^5 \tag{3.33}$$

$$\begin{aligned}\varphi_6(x,y) = 3x^2y^2 - 12x^3y^2 + 18x^4y^2 - 12x^5y^2 + 3x^6y^2 - 4x^2y^3 + 12x^3y^3 - 12x^4y^3 \\ + 4x^5y^3 + x^2y^6\end{aligned} \tag{3.34}$$

$$\varphi_7(x,y) = x^5y^2 - 2x^6y^2 + x^7y^2 - 2x^5y^3 + 2x^6y^3 + x^5y^4 \tag{3.35}$$

$$\varphi_8(x,y) = 2x^4y^2 - 6x^5y^2 + 6x^6y^2 - 2x^7y^2 - 3x^4y^3 + 6x^5y^3 - 3x^6y^3 + x^4y^5 \tag{3.36}$$

$$\begin{aligned}\varphi_9(x,y) = 3x^3y^2 - 12x^4y^2 + 18x^5y^2 - 12x^6y^2 + 3x^7y^2 - 4x^3y^3 + 12x^4y^3 - 12x^5y^3 \\ + 4x^6y^3 + x^3y^6,\end{aligned} \tag{3.37}$$

$$\begin{aligned}\varphi_{10}(x,y) = 4x^2y^2 - 20x^3y^2 + 40x^4y^2 - 40x^5y^2 + 20x^6y^2 - 4x^7y^2 - 5x^2y^3 + 20x^3y^3 \\ - 30x^4y^3 + 20x^5y^3 - 5x^6y^3 + x^2y^7\}\end{aligned} \tag{3.38}$$

Having determined all the trial functions, the next step is to find the residual function. From Eq. (3.5)

$$\mathrm{R(x,y)} = \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{u_i\ Le_i}\ (x,y) - c(x,y) \tag{3.39}$$

The residual function for these trial functions is determined using *Mathematica* as

$$\mathrm{R} \equiv \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{c_i}\nabla^4\ \varphi_{\mathrm{i}}\ (x,y) - \frac{\mathrm{P(x,y)}}{\mathrm{D}} \tag{3.40}$$

Here the value of P/D is considered to be 1 for convenience which can be later modified. Therefore the residual equation becomes

$$R \equiv \sum_{i=1}^{n} c_i \nabla^4 \varphi_i (x, y) - 1 \tag{3.41}$$

Using M*athematica* the residual for the order of polynomial $n = 6$ is determined as

$$R = -1 + 24c1x^2 + 24c1y^2 + 2c1(4 - 24x + 24x^2 - 24y + 72xy + 24y^2) \tag{3.42}$$

Since $n = 9$ gives better convergence, the residual is given as

$$\begin{aligned}
R = {} & -1 + 24c1x^2 + 24c2x^3 + 24c4x^4 + 24c7x^5 + 120c3x^2y + 120c5x^3y + 120c8x^4y + 24c1y^2 \\
& + 360c6x^2y^2 + 360c9x^3y^2 + 840c10x^2y^3 + c3(144y^2 - 240xy^2 - 72y^3) \\
& + c2(-48y^2 + 120xy^2 + 48y^3) + c5(-144y^2 + 720xy^2 - 720x^2y^2 + 144y^3 \\
& - 360xy^3) + c6(432y^2 - 1440xy^2 + 1080x^2y^2 - 288y^3 + 480xy^3) + c10(960y^2 \\
& - 4800xy^2 + 7200x^2y^2 - 3360x^3y^2 - 720y^3 + 2400xy^3 - 1800x^2y^3) \\
& + c9(-288y^2 + 2160xy^2 - 4320x^2y^2 + 2520x^3y^2 + 288y^3 - 1440xy^3 \\
& + 1440x^2y^3) + c4(24y^2 - 240xy^2 + 360x^2y^2 - 48y^3 + 240xy^3 + 24y^4) \\
& + c7(120xy^2 - 720x^2y^2 + 840x^3y^2 - 240xy^3 + 720x^2y^3 + 120xy^4) + c8(48y^2 \\
& - 720xy^2 + 2160x^2y^2 - 1680x^3y^2 - 72y^3 + 720xy^3 - 1080x^2y^3 + 24y^5) \\
& + 2(c1(4 - 24x + 24x^2 - 24y + 72xy + 24y^2) + c2(12x - 48x^2 + 40x^3 - 72xy \\
& + 144x^2y + 72xy^2) + c4(24x^2 - 80x^3 + 60x^4 - 144x^2y + 240x^3y + 144x^2y^2) \\
& + c7(40x^3 - 120x^4 + 84x^5 - 240x^3y + 360x^4y + 240x^3y^2) + c3(8 - 72x \\
& + 144x^2 - 80x^3 - 36y + 216xy - 216x^2y + 40y^3) + c5(24x - 144x^2 + 240x^3 \\
& - 120x^4 - 108xy + 432x^2y - 360x^3y + 120xy^3) + c8(48x^2 - 240x^3 + 360x^4 \\
& - 168x^5 - 216x^2y + 720x^3y - 540x^4y + 240x^2y^3) + c6(12 - 144x + 432x^2 \\
& - 480x^3 + 180x^4 - 48y + 432xy - 864x^2y + 480x^3y + 60y^4) + c9(36x - 288x^2 \\
& + 720x^3 - 720x^4 + 252x^5 - 144xy + 864x^2y - 1440x^3y + 720x^4y + 180xy^4) \\
& + c10(16 - 240x + 960x^2 - 1600x^3 + 1200x^4 - 336x^5 - 60y + 720xy \\
& - 2160x^2y + 2400x^3y - 900x^4y + 84y^5))
\end{aligned}$$

According to the Galerkin method the unknown coefficient is determined by the inner product of the functions. Therefore,

$$(R, e_i) = 0 \tag{3.43}$$

The unknown coefficients determined are as follows:

$$c1 \to \frac{27853150}{8849181}, \mathrm{c}2 \to -\frac{543004735}{53095086}, \mathrm{c}3 \to -\frac{15213653}{5899454}, \mathrm{c}4 \to \frac{1142864905}{106190172}, \mathrm{c}5 \to \frac{342934319}{53095086},$$

$$\mathrm{c}6 \to \frac{228872371}{212380344}, \mathrm{c}7 \to -\frac{97859216}{26547543}, \mathrm{c}8 \to -\frac{97859216}{26547543}, \mathrm{c}9 \to -\frac{122816876}{79642629}, \mathrm{c}10 \to -\frac{1663844}{8849181}$$

We know the deflection equation of the plates in the general form is

$$w = \sum_{i=1}^{n} c_i \varphi_i(x, y)$$

In this case, by further computing we get the deflection as

$$\mathrm{w} = \frac{5}{48}(x^2y^2 - 2x^3y^2 + x^4y^2 - 2x^2y^3 + 2x^3y^3 + x^2y^4)\} \tag{3.44}$$

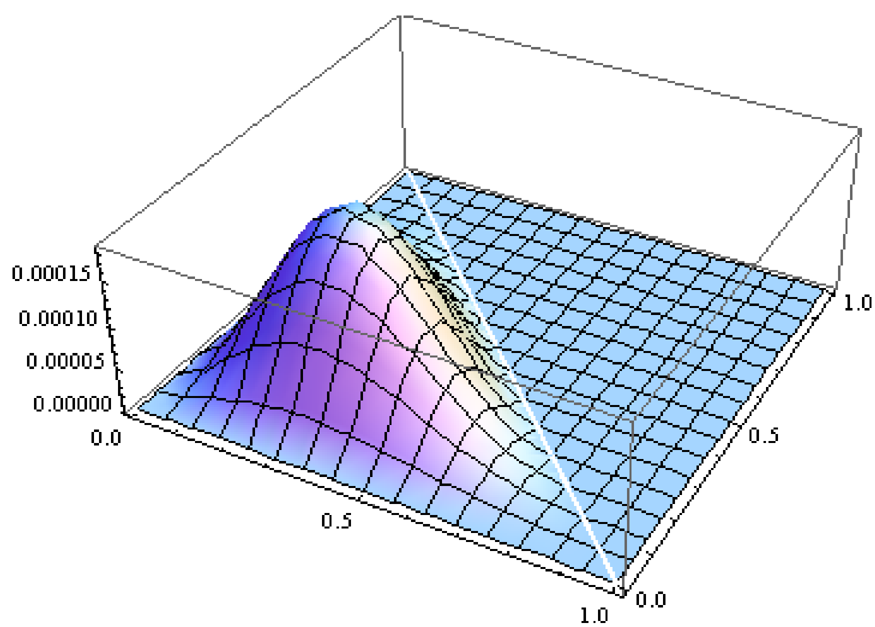

Figure 3.4 3-D plot of maximum deflection of a triangular plate clamped on all sides

The maximum deflection w_{max} is determined to be 0.000185 for a triangular plate clamped on all sides. This value of w_{max} determined is for the equation $\nabla^4 w = 1$ assuming the flexural rigidity to be unity (P/D=1).But the classical plate equation is $\nabla^4 w = \frac{P}{D}$ .Hence determining the value of P/D, taking all the values to be arbitrary and calculating the value of D, using the formula $D = \frac{Eh^3}{12(1-\nu^2)}$, we get the value of w_{max} = 2.132 for a triangular plate clamped on all the sides.

3.2.3 Numerical solution of a triangular plate clamped on two sides and simply supported on the other side

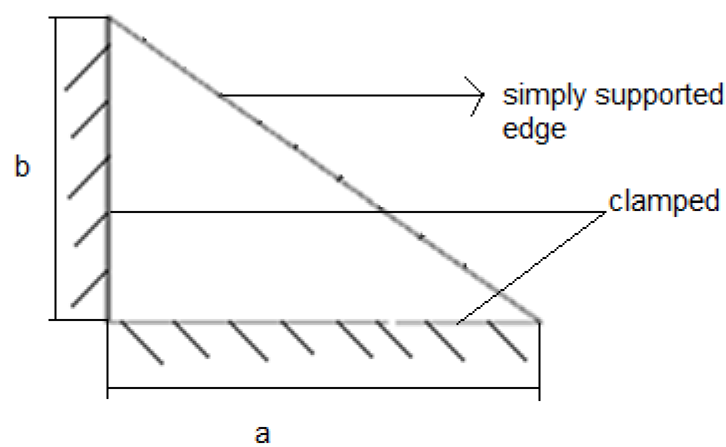

Figure 3.5 Triangular plate clamped on two sides and simply supported on the other side

Now consider a triangular plate which is clamped on two sides and simply supported along the side $x + y = 1$ .The same procedure as the above two cases is adopted. First the plate equation is

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2\ \partial y^2} + \frac{\partial^4 W}{\partial y^4} = \frac{P_z}{D}$$

The plate is clamped on the edges at $x = 0, y = 0$ and simply supported along $y = 1 - x$ then the boundary conditions are

At $x = 0,$

$$w = 0 \tag{3.45}$$

$$\frac{\partial w}{\partial x} = 0 \tag{3.46}$$

At $y = 0$,

$$w = 0 \tag{3.47}$$

$$\frac{\partial w}{\partial y} = 0 \tag{3.48}$$

At $y = 1 - x$,

$$(1 + \nu)\left(\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{w}}{\partial \mathrm{y}^2}\right) - (1 - \nu)\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}\,\partial \mathrm{y}} = 0 \tag{3.49}$$

It is found that the eighth order polynomial is the lowest possible polynomial to satisfy the boundary conditions. Once the general N^{th} order polynomial is defined, the boundary conditions are applied to solve the unknown coefficients. Having solved all the boundary conditions and tabulating them using the Table[ ] command in *Mathematica*, we get the values of the coefficients a[i]. Substituting the values of the coefficients in the general polynomial we end up with the equation,

$$\varphi_1(x,y) = -\frac{3}{2}x^2y^2 + 3x^3y^2 - \frac{3x^4y^2}{2} + 3x^2y^3 - 4x^3y^3 + x^4y^3 - \frac{3x^2y^4}{2} + x^3y^4 \tag{3.50}$$

$$\varphi_2(x,y) = \frac{7x^2y^2}{2} - 6x^3y^2 + \frac{3x^4y^2}{2} + x^5y^2 - 6x^2y^3 + 6x^3y^3 + \frac{3x^2y^4}{2} + x^2y^5 \tag{3.51}$$

$$\begin{aligned}\varphi_3(x,y) = &-2x^2y^2 + \frac{8x^3y^2}{3} + x^4y^2 - 2x^5y^2 + \frac{x^6y^2}{3} + 4x^2y^3 - \frac{8x^3y^3}{3} - \frac{8x^4y^3}{3} \\ &+ \frac{4x^5y^3}{3} - 2x^2y^4 + x^4y^4,\end{aligned} \tag{3.52}$$

Having determined all the trial functions the next step is to find the residual function. From Eq. (3.5)

$$R(x,y) = \sum_{i=1}^{n} u_i \, Le_i \, (x,y) - c(x,y)$$

The residual function for these trial functions is determined using *Mathematica*

$$R \equiv \sum_{i=1}^{n} c_i \nabla^4 \, \varphi_i \, (x,y) - \frac{P(x,y)}{D}$$

Here the value of P/D is considered to be 1 that can be modified. Therefore the residual equation becomes

$$R \equiv \sum_{i=1}^{n} c_i \nabla^4 \, \varphi_i \, (x,y) - 1$$

The residual for n=8 is evaluated as

$$\begin{aligned} R = {} & -1 + c1(-36x^2 + 24x^3) + c3(-48x^2 + 24x^4) + 360c5x^2y^2 + c2(36x^2 + 120x^2y) \\ & + c4(36x^2 + 120x^3y) + c2(36y^2 + 120xy^2) + c5(144y^2 - 360x^2y^2 \\ & - 96y^3) + c1(-36y^2 + 24y^3) + c4(-108y^2 + 360xy^2 + 120y^3 \\ & - 120xy^3) + c3(24y^2 - 240xy^2 + 120x^2y^2 - 64y^3 + 160xy^3 \\ & + 24y^4) + 2(c1(-6 + 36x - 36x^2 + 36y - 144xy + 72x^2y - 36y^2 \\ & + 72xy^2) + c3(-8 + 32x + 24x^2 - 80x^3 + 20x^4 + 48y - 96xy \\ & - 192x^2y + 160x^3y - 48y^2 + 144x^2y^2) + c2(14 - 72x + 36x^2 \\ & + 40x^3 - 72y + 216xy + 36y^2 + 40y^3) + c4(6 - 108x^2 + 120x^3 \\ & - 36y - 36xy + 360x^2y - 120x^3y + 36y^2 + 120xy^3) + c5(12 - 96x \\ & + 144x^2 - 60x^4 - 48y + 288xy - 288x^2y + 60y^4) \end{aligned} \tag{3.53}$$

Then the values of all the coefficients are found out by determining the weighted integral. It is also carried out using *Mathematica* and the values are determined.

$$c1 \to -\frac{4633321}{1614156}, c2 \to -\frac{497981}{538052}, c3 \to \frac{717717}{538052}, c4 \to \frac{239239}{269026}, c5 \to \frac{239239}{1076104}$$

In this case further computing the deflection equation we get w as

$$
\begin{aligned}
w &= -\frac{4633321\left(-\frac{3}{2}x^2y^2+3x^3y^2-\frac{3x^4y^2}{2}+3x^2y^3-4x^3y^3+x^4y^3-\frac{3x^2y^4}{2}+x^3y^4\right)}{1614156} \\
&+\frac{717717\left(\begin{array}{c}-2x^2y^2+\frac{8x^3y^2}{3}+x^4y^2-2x^5y^2+\frac{x^6y^2}{3}+4x^2y^3 \\ -\frac{8x^3y^3}{3}-\frac{8x^4y^3}{3}+\frac{4x^5y^3}{3}-2x^2y^4+x^4y^4\end{array}\right)}{538052} \\
&-\frac{497981\left(\frac{7x^2y^2}{2}-6x^3y^2+\frac{3x^4y^2}{2}+x^5y^2-6x^2y^3+6x^3y^3+\frac{3x^2y^4}{2}+x^2y^5\right)}{538052} \\
&+\frac{239239\left(\frac{3x^2y^2}{2}-\frac{9x^4y^2}{2}+3x^5y^2-3x^2y^3-x^3y^3+5x^4y^3-x^5y^3+\frac{3x^2y^4}{2}+x^3y^5\right)}{269026} \\
&+\frac{239239(3x^2y^2-8x^3y^2+6x^4y^2-x^6y^2-4x^2y^3+8x^3y^3-4x^4y^3+x^2y^6)}{1076104}
\end{aligned} \tag{3.54}
$$

Further plotting the values using *Mathematica* and generating a plot we determine the maximum deflection of the plate.

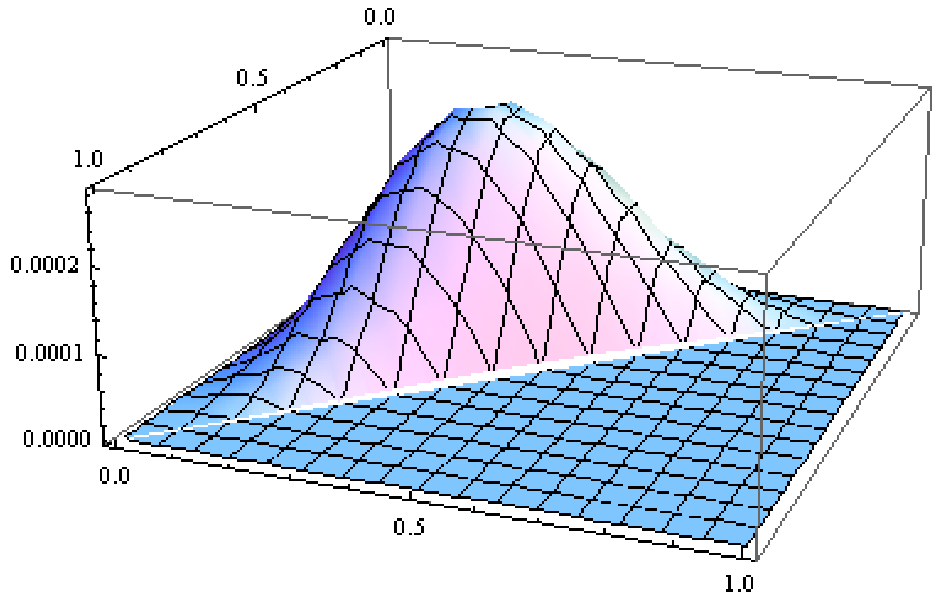

Figure 3.6 3-D plot of maximum deflection of a triangular plate clamped on two sides and simply supported on the other side

The maximum deflection w_{max} is determined to be 0.00028 for a triangular plate clamped on two sides and simply supported on the other side. This value of w_{max} determined is for the equation $\nabla^4 w = 1$ assuming the flexural rigidity to be unity (P/D=1).But the classical plate equation is $\nabla^4 w = \frac{P}{D}$ .Hence determining the value of P/D, taking all the values to be arbitrary and calculating the value of D, using the formula $D = \frac{Eh^3}{12(1-\nu^2)}$, we get the value of w_{max} = 3.057 for a triangular plate clamped on two sides and simply supported on the other side.

CHAPTER 4

RESULTS AND ANALYSIS

4.1 Analysis using Ansys

Ansys Workbench has native work spaces which include Project Schematic, Engineering Data and Design. The application of Ansys Workbench includes Mechanical APDL, Fluent, and CFX. A structural Ansys analysis is done by the following procedure.

The Ansys system has six different states

1. Engineering Data
2. Geometry
3. Model
4. Setup
5. Solution
6. Results

4.1.1 Engineering Data

The Engineering Data cell gives access to the material models for the use in the analysis. A double click on the Engineering Data tab will take to a page where the Edit menu have to be chosen to define the Engineering Material data.

4.1.2 Geometry

The Geometry option is selected to import, create or update the geometry of the model used in the analysis. There are many different options under the geometry

1. New Geometry
2. Import Geometry
3. Edit
4. Replace Geometry

5. Update from CAD
6. Refresh
7. Properties

The Properties option is used to select basic and advanced geometry properties

4.1.3 Model

The Model feature in the Mechanical application systems or the Mechanical model component system is associated with the Model branch in the Mechanical application and affects the geometry, coordinate systems, connections and mesh branches of the model. The Mesh option is used to create a mesh either being coarse or fine so that the whole model is being meshed.

4.1.4 Setup

The setup option is used to launch the appropriate application for the system. Here there is always necessity to define the boundary conditions and configure the analysis in the system. The data which are specified here will be incorporated in the project in Ansys Workbench.

4.1.5 Solution

From the solution option we can access the branch of the application. The solution option actually solves for all possible results for the analysis and it is ready to be output.

4.1.6 Results

The Results option is used to generate all the required outputs of the analysis for example the deformation, stress acting and various other output parameter.

4.2 Rectangular plate clamped along all sides

The first case that was considered is a flat rectangular plate clamped on all sides. The deflection is now evaluated using the finite element software, Ansys. First the model is generated with all the dimensions being unity. The generated model using Ansys is

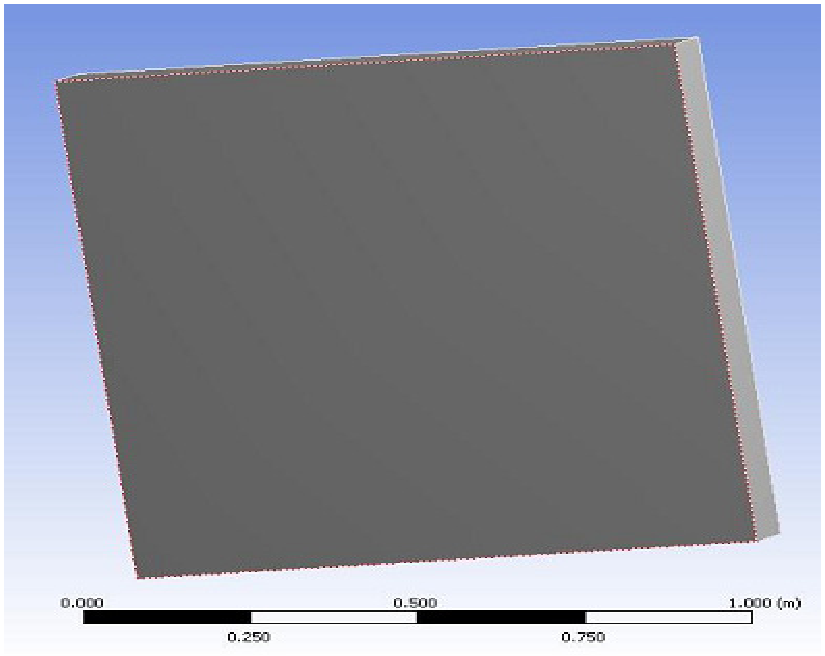

Figure 4.1 Model of a flat rectangular plate

Once the model is generated the next step is to mesh the whole geometry so that further loading and the boundary conditions are specified. The geometry after being meshed looks as in Figure 4.2

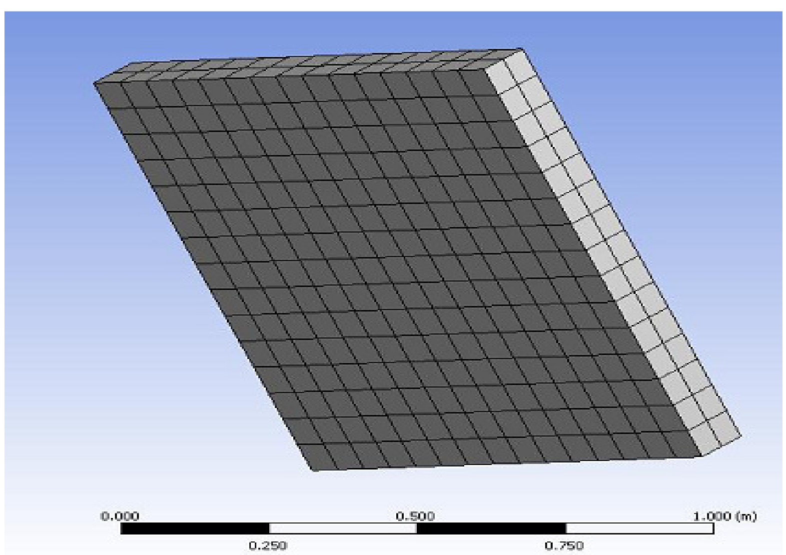

Figure 4.2 Plate with mesh areas

Once the model is meshed, the next is specifying the boundary conditions. The displacements on all sides of the geometry are specified as zero since the plate is clamped on all edges. Then the pressure is given on the top surface of the plate and the value is 1 Pa. Once the boundary conditions are specified the model looks as in Figure 4.3. Once the boundary conditions are specified, the model is selected and the solution option solves for all possible results. Then all the results including the stresses and deformations are viewed along with the maximum deformation of the rectangular plate clamped on all the edges as in Figure 4.4.

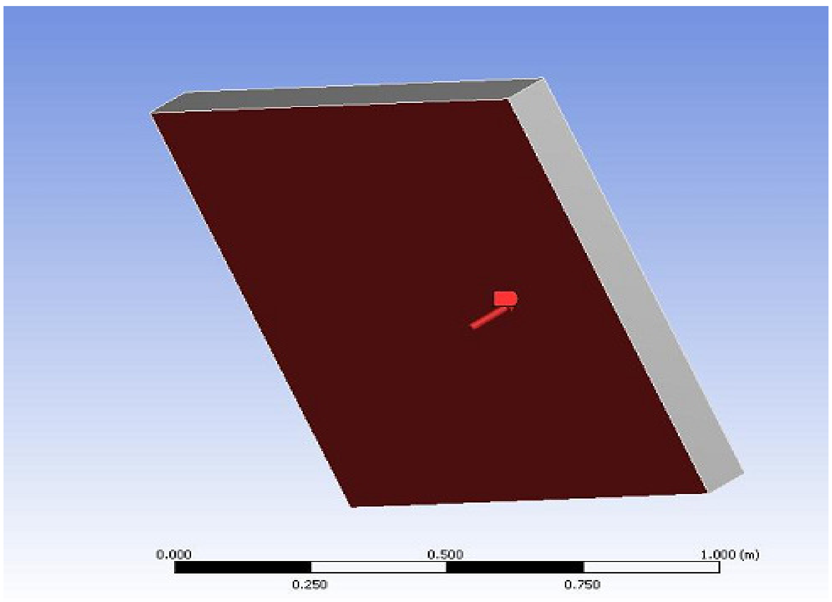

Figure 4.3 Plate with load and boundary conditions

Thus the value of the maximum deflection of the flat rectangular plate is 14.384, which shows good agreement with the value determined by the analytical method.

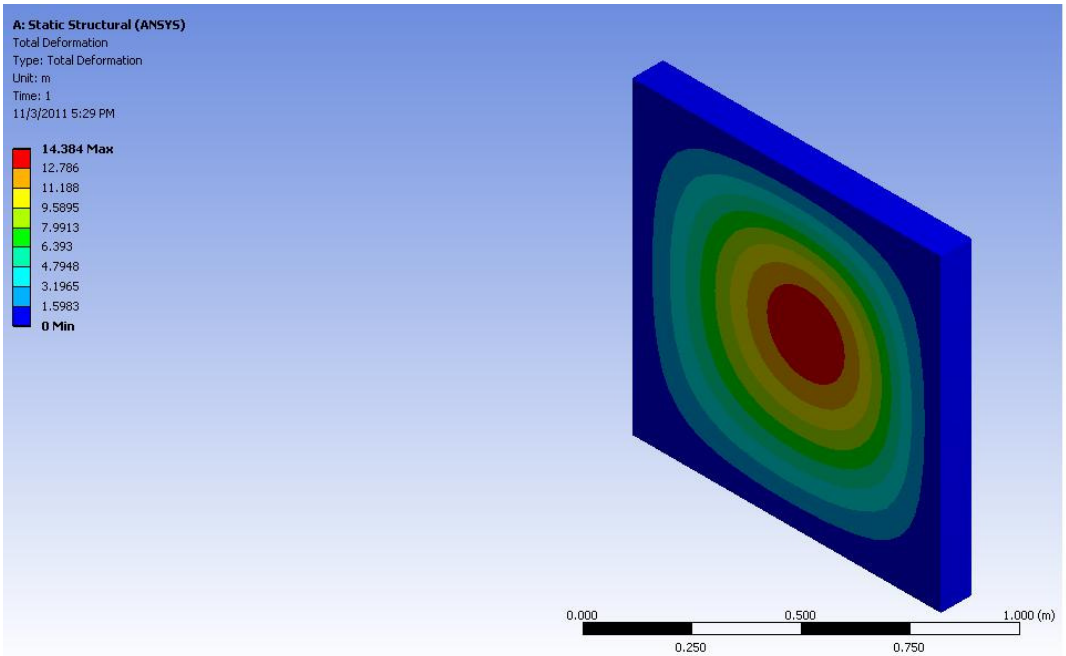

Figure 4.4 Rectangular plate after deformation

4.3 Triangular plate clamped on all sides

The second case that was considered is a triangular plate clamped on all sides. First the model is generated with all the dimensions being unity. The generated model using Ansys is shown in Figure 4.5 .The geometry is created such that the length and breadth are unity. It is also made sure that the geometry satisfies the equation $x + y = 1$

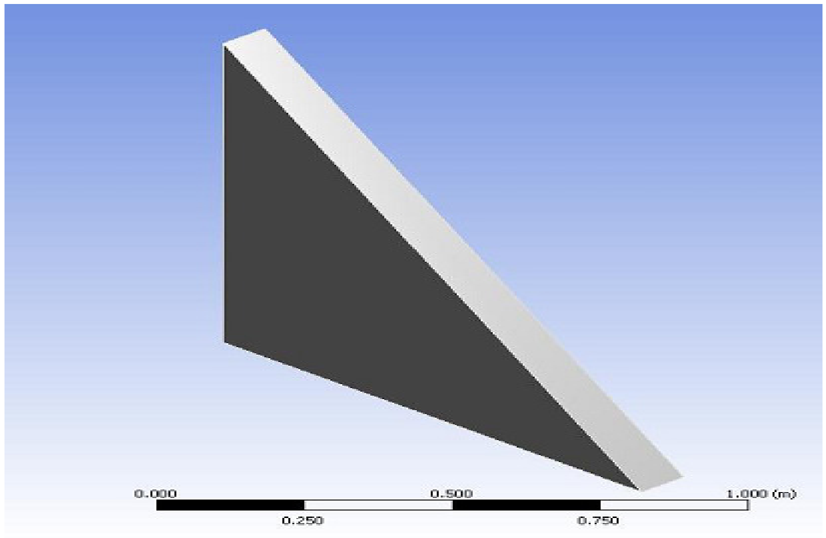

Figure 4.5 Model of a flat triangular plate

Once the model is generated the next step is to mesh the whole geometry so that further loading and the boundary conditions are specified. The geometry after being meshed is shown in Figure 4.6.

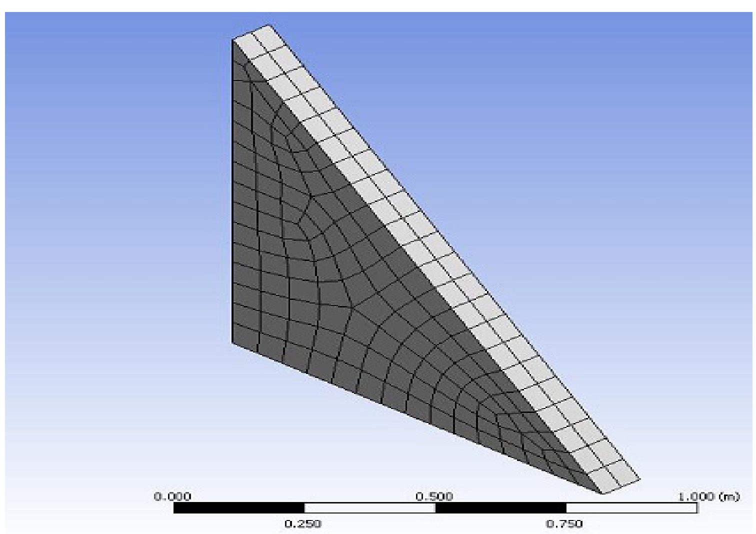

Figure 4.6 Triangular plate with mesh areas

Once the model is meshed, the next step is to specify the boundary conditions. The displacements on all sides of the geometry are specified as zero since the plate is clamped on all the edges. Then the pressure is given on the top surface of the plate and the value is 1 Pa. Once the boundary conditions are specified, the model is selected and the solution option is used to solve for all possible results. Then all results including the stresses and deformations are viewed along with the maximum deformation of the triangular plate clamped on all the edges as in Figure 4.7.

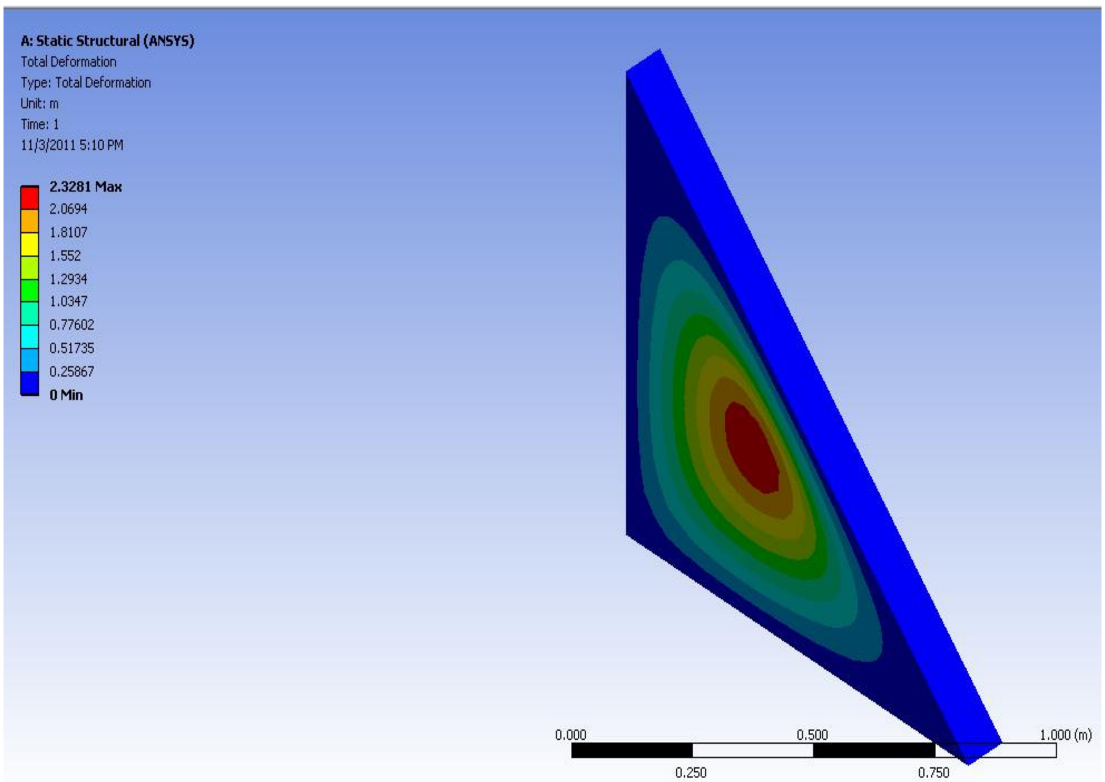

Figure 4.7 Triangular plate with deformation

Thus, the value of maximum deflection obtained using Ansys for a triangular plate clamped on all the edges is 2.3281, which shows good agreement with the value derived using the Galerkin method.

4.4 Triangular plate clamped on two sides and simply supported on the other side

In the final case of analysis a flat triangular plate with mixed boundary conditions is considered. The triangular plate is clamped on two sides and it is simply supported on the third side. The analysis is repeated as it was done for the above two cases but the only difference is that the boundary condition on the third side is changed to simply-supported. The whole procedure is repeated to get the maximum deflection. The value of the maximum deflection is compared to the value determined using Galerkin Method and it is noted that the value shows good agreement with the analytical method.

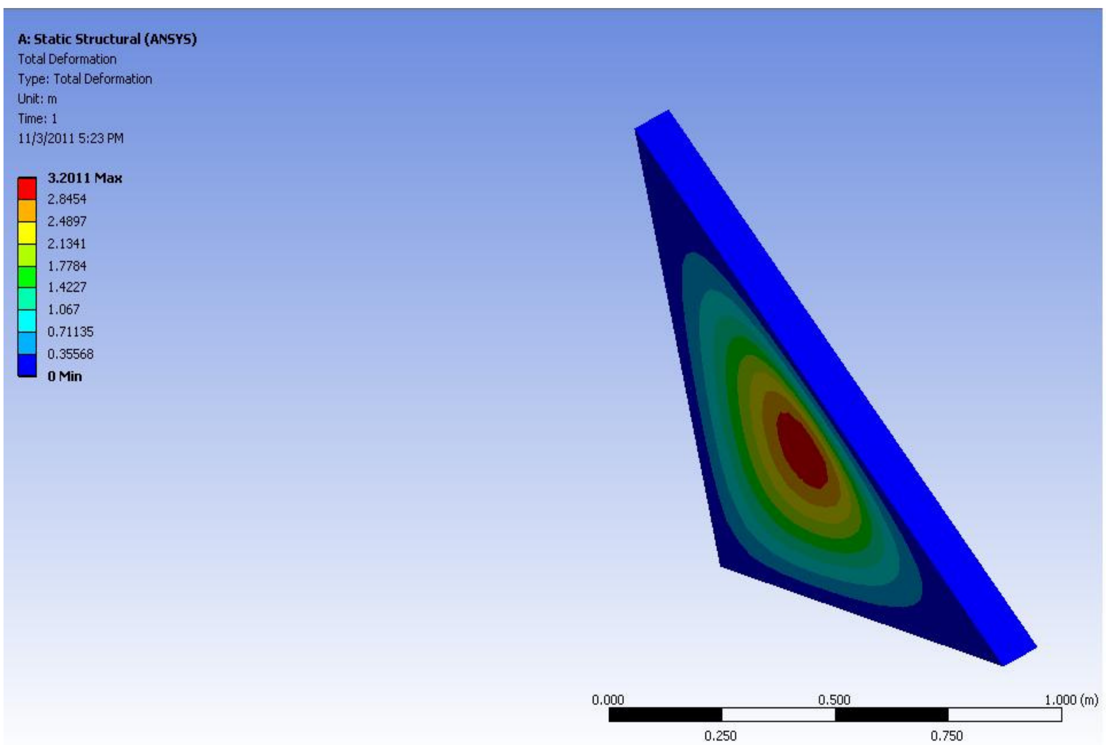

Figure 4.8 Triangular plate two sides clamped and other side simply supported with deformations

Thus, the value of the maximum deflection obtained using Ansys for a triangular plate clamped on two sides and simply supported on the other side is 3.201, which shows good agreement with the value derived using the Galerkin method.

Table 4.1 Comparison of the deflection values in various cases

Cases	Maximum Deflection (w_{max}) using Galerkin method	Maximum Deflection (w_{max}) using Ansys
Rectangular plate clamped all sides	14.195	14.384
Triangular plate clamped all sides	2.132	2.328
Triangular plate clamped on two sides and simply supported on the third side	3.057	3.201

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

The Galerkin method was used to solve the governing equation of the plate. Completing the involved algebra would have been impossible without the usage of the symbolic software, *Mathematica*. The use of symbolic software is useful for the study because it manipulates the given expressions and symbolically retains the variables. The method of selecting the polynomial which satisfies all the boundary conditions provides better accuracy and faster convergence. The use of the symbolic software provides much faster and accurate way of analyzing the engineering problems. With the examples illustrated in this thesis it is evident that the results generated by the symbolic software shows good agreement with those of the finite element software, Ansys.

The effort taken in this thesis to solve and analyze different geometrical shapes other than regular geometries should pave a way for more research in the future. With much more attributes and realistic boundary conditions the triangular geometry could be modified into a shape of an aircraft wing. With the help of powerful routines such as *Mathematica*, the research should be feasible in the near future. Further applications of the symbolic software are encouraged and the applications will lead to a better understanding of the classical analytical procedures. More complex geometries can be taken for analysis and can be analyzed with the help of the Galerkin method and the symbolic software. The geometries to be analyzed can be selected in such a way that it could be used in real time engineering in the future.

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BIOGRAPHICAL INFORMATION

Ashwin Balasubramanian received his Bachelors of Engineering degree in Mechanical Engineering from Anna University, Chennai, India. He completed his Master of Science degree in Mechanical Engineering in December 2011 at the University of Texas at Arlington. His research interests include applying the finite element method, solving complex engineering problems using the Galerkin method, engineering design and structures. His master's thesis was entitled “Plate analysis with different geometries and arbitrary boundary conditions”.