# MODE TRUNCATION ANALYSIS IN FORCED RESPONSE OF STRUCTURAL DYANAMICS

by

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ABSTRACT

MODE TRUNCATION ANALYSIS IN FORCED RESPONSE

OF STRUCTURAL DYANAMICS

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Mode truncation in Modal analysis is an important problem and should be addressed with an effective truncation criterion. The truncation errors are considered as error forces and these error limits can be used to select the number of modes in the solution. Usually frequency based approach is used in selection of modes. But this method ignores the error forces in truncation and not effective if the cutoff frequency is greater than highest natural mode. An effective criterion would limit error forces, thereby minimizing errors in the response solution. To overcome the inaccuracies due to truncation, Mode Truncation Augmentation was used to represent the response due to non-retained modes due to its demonstrated superiority over other algorithms.

This work proposes a mode truncation criteria based on modal force contribution to estimate the number of modes required to approximate the response. The proposed algorithm is implemented with MTA. Several numerical examples were carried out to demonstrate the effectiveness of the proposed algorithm using MATLAB .

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# CHAPTER 1

# INTRODUCTION

## 1.1. Introduction to Mode Truncation Analysis

Modal analysis is a popular technique for structural dynamic analysis [9]. This is based on the expansion theorem of eigen vectors, thus, response of the structure to the summation of excitations is the sum of responses to each excitation. Hence it is possible to analyze each excitation separately and then superpose them. This is popularly known as modal response or mode superposition method which is widely used both in analytical and structural dynamics this provides a simple and efficient way to represent dynamic behavior of structure.

Mode Superposition method has two basic variants; the mode displacement method and mode acceleration method. In Mode displacement (MD) method the displacements are calculated from modal coordinates by superposition. The accuracy is satisfactory if sufficient modes are retained in the solution. The mode acceleration (MA) method was introduced by Williams [7] and has been widely used for several decades. MA is more accurate than MD method. The improved accuracy is due to the static correction term included in the solution to represent the contribution due to non-retained modes. Variants in acceleration methods have been proposed which have been concluded to be equivalent by suitable transformation [1].An alternative approach referred as Mode truncation augmentation method is much newer and is only beginning to be implemented in all fields of structural vibration analysis. This method appends a pseudo Eigen or MT vectors to include in the modal set [10] which when included in the modal solution was proposed to have good convergences with exact solution.

The truncation criterion is a popular subject while performing modal analysis to compute dynamic responses structures. All Modal based response algorithms start with choosing the number of modes as a preliminary step before carrying out the analysis. It is always possible to choose lower frequency modes in the modal solution and ignore the response of a structure in higher frequencies as the effective mass is so high at these frequencies that it is not possible to generate energy required to excite the structure to obtain a considerable response. In linear analysis techniques the response is proportional to excitation, the ratio between excitation and response can be determined independently of excitation which is one of second primary properties of Modal analysis. The truncation errors can be considered as error forces which can be an important quantity while determining acceptability of modes. This is true from the second property to say that these error forces are responsible for the response missing from the modal solution. Quantification of response not represented by the proportional error forces in to response solution is basis on which Mode truncation augmentation method was proposed [10].

This study proposes a truncation criteria based on allowable error forces using modal force participation. Numerical examples on configurations of 6 DOF system was discussed to as a part of development of criteria. The proposed criteria was extended to 20 DOF spring mass system using MTA algorithm to evaluate its effectiveness by comparing it with exact solution.

## 1.2. Outline of Thesis

In Chapter 2 a detailed description was given on modal analysis, Properties of Eigen value problem its application to Response analysis was discussed.

In chapter 3 various algorithms available and their procedures for dynamic response analysis were discussed. These algorithms were compared numerically and MTA method was shown to be the best among them.

In chapter 4, A modal truncation criterion is proposed. The procedure and algorithm was discussed using modal expansion of force vector with a 6 DOF spring mass system.

In chapter 5 the proposed criteria was evaluated using a 20 DOF spring mass system the response was compared to exact solution.

Chapter 6 contains conclusions and future work.

# CHAPTER 2

# DYNAMIC ANALYSIS WITH MODAL APPROACH

## 2.1. Modal Analysis Procedure

The main goal of modal analysis is to calculate the dynamic response of structure. Modal analysis is based on the expansion theorem of eigen vectors. Hence, to perform modal analysis, we must first solve the eigen value problem to obtain natural mode shapes and natural frequencies of a structure during free vibration. To avoid resonance, the forcing frequency must not coincide with the natural frequencies of the structure. Modes are the inherent properties of structure which are determined by the properties of structure which are Mass, damping, stiffness and also boundary conditions. They are independent on the load applied or loading function.

Modes in modal analysis can be real or complex. For the modal analysis to possess real modes the damping term is neglected to develop fundamental results which in turn can be extended to damped systems.

Modal analysis is a linear technique applied to linear systems. A brief procedure to carry out modal analysis is as follows:

- Solving the Eigen value problem, which results in Eigen values and eigen vectors which are natural frequencies and mode shapes of the vibrating system.
- These Eigen vectors can be used to decouple equations and make them N decoupled SDOF system.
- Apply load data and calculate the response from each of these SDOF systems to know the contribution of each mode to the overall response.
- Elimination of modes which do not contribute in the response i.e. modes which cannot be excited.

## 2.2. Equations of Motion (EOM)

Equations of motion are equations that describe the behavior of a system in structural dynamics it defines the behavior of the lumped masses in the system with given loading conditions and loading function as a function of time

A simple single degree of freedom system with mass, stiffness and damping will have following equation of motion

$$M\ddot{u} + C\dot{u} + Ku = P(t) \qquad 2.1$$

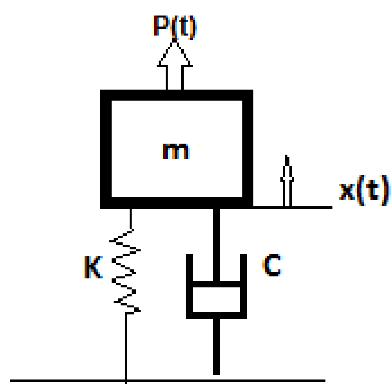


Figure 2.1 Spring mass system

For a MDOF system with N degree of freedom the M, C and K will be a N x N matrices. U and P will be Nx1 vectors. The EOM can be represented as:

$$[M]\ddot{u} + [C]\dot{u} + [K]u = P(t) \qquad 2.2$$

For an undamped case where $[C]$ = 0 and undergoing free vibration $P(t)$ = 0.

Thus the system reduces to

$$[M]\ddot{u} + [K]u = 0 \qquad 2.3$$

Solving the equation with these given conditions will result in determining the natural frequencies and modes of vibration of the structure. Thus the EOM results in the Eigen value problem of MDOF systems.

2.2.1. Lumped Systems

A continuous vibrating structure can be discretized into small elements. Mass of such each element is assumed to concentrate as one rigid object. Thus the inertia and stiffness of a vibrating structure can be represented as an interconnected spring mass system. The number of independent motions of this system is the number of degrees of freedom. The advantage of this approach is the equations of systems become ordinary differential equations which can be solved by Modal analysis technique.

For the 3-DOF model of cantilever beam shown in Figure 2.2 the lumped mass matrix is given by

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

Where $m_r$ is the lumped mass of the individual discretized element.

The same applies to stiffness and can be expressed as

$$K = \begin{bmatrix} K_1 + K_2 & -K_2 & 0 \\ -K_2 & K_2 + K_3 & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix}$$

The mass matrix M is a diagonal matrix. And the stiffness matrix K is said to have stiffness coupling.

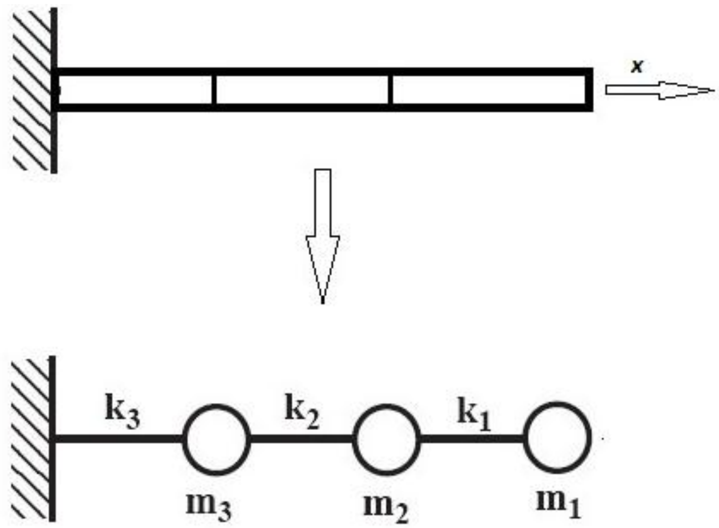


Figure 2.2 An example of a lumped mass system. Transformation of a cantilever beam to lumped mass system.

2.2.2. Degrees of Freedom

The discrete independent coordinates after discretizing a vibrating structure into elements in modal analysis is termed as degrees of freedom (DOF) of the system. Based on the system it can be described as Single and Multi degree of freedom systems (SDOF and MDOF).An MDOF system may be treated as N SDOF systems.

2.2.3. SDOF Systems

Let’s consider a simple structure undergoing vibration. This kind of structure has a mass and stiffness such a system is given as follows

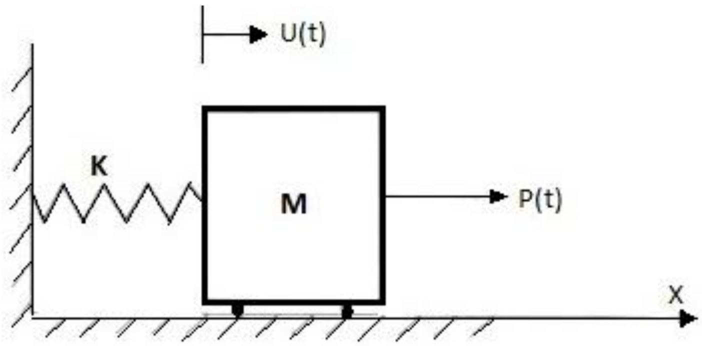


Figure 2.3 Single Degree of freedom system

Here m is the mass, k is the stiffness p(t) is the load applied u(t) is the displacement.

The mathematical model of such system is obtain by applying Newton's second law

$$\sum F_x = ma_x \qquad 2.4$$

Where m is the mass a is the acceleration

The free body diagram of the system is given in Figure 2.4 Free body diagram of mass in spring mass system

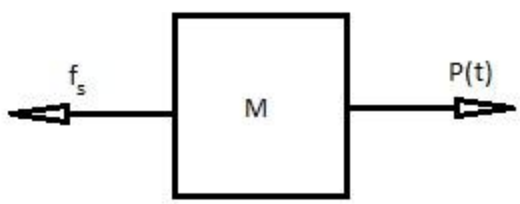


Figure 2.4 Free body diagram of mass in spring mass system

The equation of the following system in dynamic equilibrium is given as

$$-f_s + p(t) = ma_x \qquad 2.5$$

The term $a_x$ is the second derivative of displacement which is given as

$$a_x = \ddot{u}(t) \qquad 2.6$$

The force displacement relation in terms of stiffness is given as

$$f_s = ku \qquad 2.7$$

Hence the equation from the FBD becomes

$$-ku + p(t) = m\ddot{u} \qquad 2.8$$

or

$$m\ddot{u} + ku = p(t) \qquad 2.9$$

Equation (2.9) is the Equation of motion for an undamped single degree of freedom system.

If damping is considered the EOM becomes

$$m\ddot{u} + c\dot{u} + ku = p(t) \qquad 2.10$$

<u>2.2.4. MDOF Systems</u>

For MDOF systems the dynamics can be represented a matrix form of equation of motion. The system can be represented by matrices –mass, damping and stiffness matrices.

$$M\ddot{u} + C\dot{u} + Ku = p(t) \qquad 2.11$$

Where M is the mass matrix

K is the stiffness matrix

C is the viscous damping matrix

For an N-DOF systems the size these matrices are of N x N

The term u(t) is the displacement vector of size N x 1

p(t) is the loading vector of size N x 1

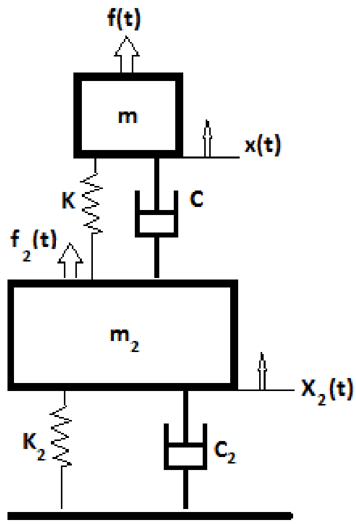


Figure 2.5 An example of MDOF system

For the above example $M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \; ; \; C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}$$

$$p(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \quad ; \quad u(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Hence EOM the matrix equation is given by

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{2.12}$$

These matrices are obtained by discretising the structure using finite element methods. Further these equations of system can be used in modal analysis.

## 2.3. Eigen Value Problem

Modal analysis is initialized by determining the natural frequencies and modes of the undamped system subjected to free vibration which are extracted from formulated eigen value problem.

The linearized EOM for a free undamped system

$$[M]\ddot{u} + [K]\mathrm{u} = 0 \qquad 2.13$$

The free vibration of an undamped in one of its natural vibration modes is described as [5]

$$u(t) = \phi_n \eta_n(t) \qquad 2.14$$

In Eq (2.14) $\phi_n$ is not a time variant. The time variation of displacement is given by a simple harmonic function

$$\eta_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t \qquad 2.15$$

Combining Eq (2.14) and (2.15) gives

$$u(t) = \phi_n (A_n \cos \omega_n t + B_n \sin \omega_n t) \qquad 2.16$$

Substituting this into the EOM gives

$$(-\omega_n^2[M] + [K]\,)\phi_n \eta_n(t) = 0 \qquad 2.17$$

Hence the equation reduces to

$$(\,[K]\; - \omega_n^2[M])\phi_n = 0 \qquad 2.18$$

The above eq represents Eigenvalue problem.

This equation has non trivial solution if

$$|[K]\; - \omega_n^2[M]| = 0 \qquad 2.19$$

This equation has real and positive roots for $\omega_n^2$ if M and K are symmetric and positive definite.

2.3.1. Natural frequencies and Modes of vibration

The solution of Eigen value problem is the eigen pairs $(\omega^2, \emptyset_r)$. The eigen values will be the square of natural frequencies of the system.

From the solution eigen values can be put in matrix form as eigen value matrix

$$[\omega_r^2] = \begin{bmatrix} \omega_1^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \omega_N^2 \end{bmatrix} \quad 2.20$$

Or diag($\omega_r^2$)=$[\omega_1^2, \omega_2^2, \omega_3^2 \dots . \omega_N^2]$ $\phi_r$ from eigen value problem are the eigen vectors or modes of vibration .

Modal matrix:

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & & \phi_{1r} \\ \vdots & \vdots & \vdots & \dots\dots & \vdots \\ \vdots & \vdots & \vdots & \dots\dots & \vdots \\ \vdots & \vdots & \vdots & \dots\dots & \vdots \\ \phi_{N1} & \phi_{N2} & \phi_{N3} & & \phi_{Nr} \end{bmatrix} \quad 2.21$$

Note that each column of the modal matrix is a eigen vector.

Consider a spring mass system below

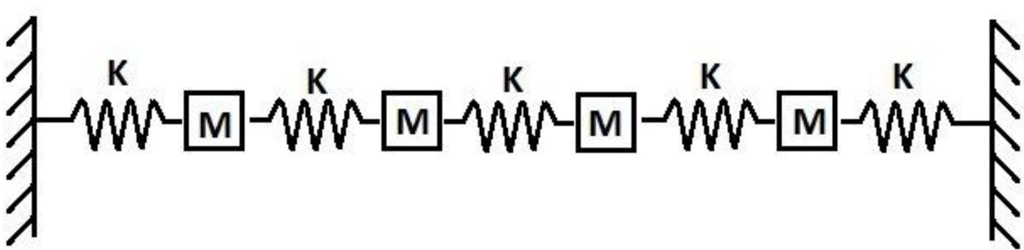


Figure 2.6 A 4 DOF spring mass system with both ends fixed.

The system mass and stiffness matrices are given below.

$$M=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0.5\end{bmatrix}$$

$$K = 10{,}000 \times \begin{bmatrix}2 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & 1\\ 0 & 0 & -1 & 2\end{bmatrix}$$

Solving the eigen value problem we get the following

Natural frequencies of the system:

$$\omega_n = diag[64 \;\; 127 \; 178 \; 220]$$

Eigen vectors or Modes of vibration of the system:

$$\phi = \begin{bmatrix}0.62 & 1 & 1 & 0.14\\ 1 & 0.38 & -1.15 & -0.4\\ 0.96 & -0.85 & 0.32 & 1\\ 0.53 & -0.7138 & 0.77 & -2.42\end{bmatrix}$$

Each individual column of matrix is a mode of vibration and the modes were normalized. It can be graphically represented in Figure 2.7.

Mode shapes are important for analysis as they determine how many modes should be considered in the modal solution.

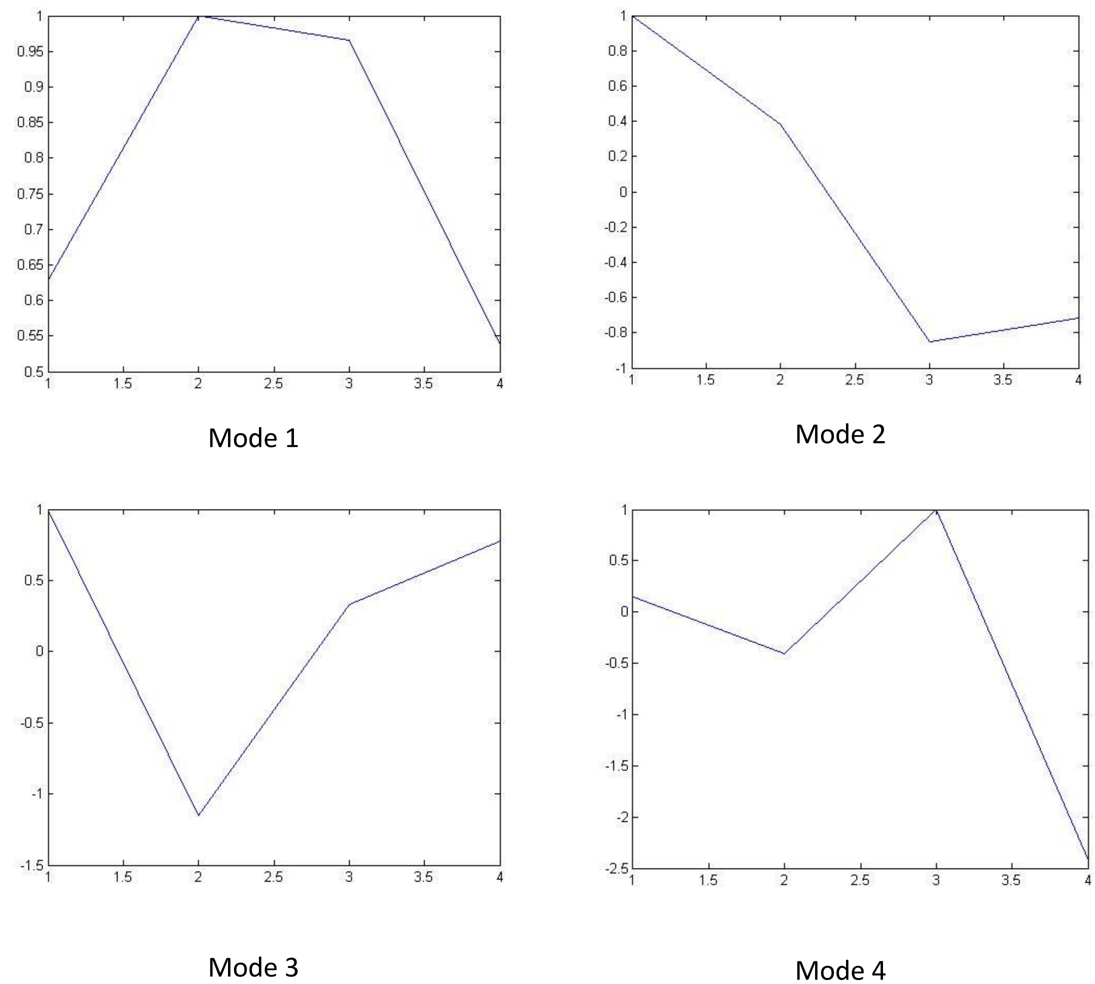


Figure 2.7 Natural modes of vibration of 4 DOF spring mass system

2.3.2. Properties of Eigen vectors

2.3.2.1 Orthogonality of modes

Orthogonality properties of eigen vectors are given by

$$\phi_r^T M \phi_s = 0$$

$$\phi_r^T K \phi_s = 0$$

When $r \neq s$.

In matrix form these properties can be written as

$M = \phi^T M \phi$ =diagonal

$K = \phi^T K \phi$ = diagonal

And the diagonal elements of the matrices are

$M_r = \phi_r^T M \phi_r$ and stiffness $K_r = \phi_r^T K \phi_r = \omega_r^2 M_r$ $[r = 1,2 \ldots\ldots n]$

2.3.2.2 Normalization of Eigen vectors

Eigen vectors are solution of a system of homogenous equation. Hence it can be normalized arbitrarily.

There are two common methods of normalization

1. Normalizing for maximum component to be unity
2. Normalizing with respect to mass.

Normalizing with respect to unity is to make largest element in each eigen vector equal to unity by dividing each columns by largest absolute value in that column.

Normalizing with respect to mass such that $\phi_n^T M \phi_n = 1$.

In this study normalization with respect to unity is followed used.

2.3.2.3 Expansion theorem

The response of MDOF system can be expressed as the liner combination independent set of N-dimensional Eigen vectors

$$u = \sum_{r=1}^{N} C_r \phi_r$$

Where $\phi_r$ are the normal modes, r=1, 2, …..N. This equation states the Expansion theorem in eigen value problem or this is the Principle of Mode superposition in structural dynamic analysis.

2.3.3. Transformation of EOM to Modal Equations

The mode superposition starts with transforming the equation of motion to modal coordinates $\eta(t)$, assume the physical response is given by

$$u(t) = \phi\, \eta(t) = \sum_{r=1}^{N} \phi_r\, \eta_r\,(t) \qquad 2.22$$

This is substituted into the Equation of motion and multiplied with $\phi^T$ to give

$$M\ddot{\eta} + C\dot{\eta} + K\eta = f(t) \qquad 2.23$$

Where, M, C, K and f (t) are modal mass, damping, stiffness matrices and force vector respectively.

$$M = \phi^T \mathrm{M}\, \phi \qquad 2.24$$

$$C = \phi^T \mathrm{C}\, \phi = diag(2\xi_r \omega_r M_r) \qquad 2.25$$

$$K = \phi^T \mathrm{K}\, \phi = diag(\omega_r^2\, M_r) \qquad 2.26$$

$$f(t) = \phi^T p(t) \qquad 2.27$$

Note that C is diagonal only for proportionally damped systems.

CHAPTER 3

# DYNAMIC RESPONSE ANALYSIS ALGORITHMS

## 3.1. Direct Method

The equation of motion of a damped forced vibration with harmonic excitation is given by

$$M\ddot{U} + C\dot{U} + KU = P\, e^{i\Omega t} \quad 3.1$$

For harmonic motion the solution for is of the form

$$U = (u\,\Omega)\, e^{i\Omega t} \quad 3.2$$

The first and second derivatives of this equation is

$$\dot{U} = i\omega\,(u\,\Omega)e^{i\Omega t} \quad 3.3$$

$$\ddot{U} = \omega^2\,(u\,\Omega)e^{i\Omega t} \quad 3.4$$

Substituting these back into the equation of motion and dividing by $e^{i\omega t}$ simplifies to

$$[K + i\Omega\, C - \Omega^2 M]\, u = P \quad 3.5$$

and

$$u = [K + i\Omega\, C - \Omega^2 M]^{-1}\, P \quad 3.6$$

For an un damped system

$$u = [K - \Omega^2 M]^{-1}\, P \quad 3.7$$

Direct method has the advantage of calculating the response the natural frequencies known. However the equation indicates that this method requires calculating the inverse of dynamic stiffness matrix for each excitation frequency. The computation may be excessive for very large degree of freedom systems. Hence direct method can only be applied to systems with few DOFs and few excitation frequencies. For Transient loading, direct method involves the numerical integration of equation of motion [6].

## 3.2. Mode Superposition Method

Modal analysis is an efficient way to carry out dynamic analysis for large systems. It usually employs mode superposition method to carry out the analysis by solving uncoupled equations which can be treated as N SDOF systems. The use of mode shapes gives an advantage to reduce the size of problem and make the modal solution more efficient. The procedure starts by transforming the equation of motion to modal co-ordinates. The transformed N uncoupled modal equations of motion can be written as

$$M_r\ddot{\eta} + C_r\dot{\eta} + K_r\eta = f_r(t) \tag{3.8}$$

Or

$$M_r\ddot{\eta}_r + 2\xi_r\omega_r M_r\dot{\eta}_r + \omega_r^2\, M_r\eta_r = f_r(t) \tag{3.9}$$

This can be written as

$$\ddot{\eta}_r + 2\xi_r\omega_r\dot{\eta}_r + \omega_r^2\eta_r = f_r(t)/\, M_r \tag{3.10}$$

$$\eta_r(t) = \frac{1}{M_r(\omega_d)_r} \int_0^t f_r(\tau)e^{-\xi_r\omega_r(t-\tau)} \sin\omega_{dr}(t-\tau)d\tau \tag{3.11}$$

where the damped modal natural frequency is given by

$$\omega_{dr} = \omega_r\sqrt{1-\xi_r^2} \tag{3.12}$$

and r is the mode number. r = 1, 2, ….N.

For harmonic loading, the transient term dissipates in time and later only the steady state response sustains

For $f(t) = P\ cos\Omega t$ the modal response for a steady state condition is given as

$$\eta_r(t) = \frac{F_r}{K_r}\frac{1}{\sqrt{(1-r_r^2)^2 \quad + (2\zeta_r r_r)^2}}\cos(\Omega t - \alpha_r) \qquad 3.13$$

$r_R$ is the modal frequency ratio given by

$$r_r = \frac{\Omega}{\omega_r} \qquad 3.14$$

$\alpha_r$ is the phase angle is given by

$$tan\alpha_r = \frac{2\zeta_r r_r}{1 - r_r^2} \qquad 3.15$$

For a steady state undamped model C=0

$$M_r\ddot{\eta} + K_r\eta = f_r(t) \qquad 3.16$$

The response is given by

$$u(t) = \sum_{r=1}^{N} \phi_r \frac{F_r}{K_r}\left[\frac{1}{1-\left(\frac{\Omega}{\omega}\right)^2}\right]\cos \Omega t \qquad 3.17$$

3.2.1. Mode Displacement (MD) Method

If M modes are used for a mode displacement method the solution of the transformed equation is given by

$$\hat{u}(t) = \hat{\phi}\ \hat{\eta}(t) = \sum_{r=1}^{m} \phi_r\ \eta_r\ (t) \qquad 3.18$$

Where m indicates the number of truncated modes.

And the truncated modal matrix is given by $\hat{\phi} = [\phi_1 \ \phi_2 \ \phi_3 \ \ldots \ldots \phi_m]$

$$\widehat{\eta_r}(t) = \frac{1}{M_r(\omega_d)_r} \int_0^t f_r(\tau) e^{-\xi_r \omega_r (t-\tau)} \sin \omega_{dr}(t-\tau) d\tau \qquad 3.19$$

3.2.2. Mode Acceleration (MA) Method

The Mode displacement method ignores the contribution of non-retained modes. This often leads to slow convergence and accuracy problems. Mode acceleration method overcomes this problem by adding a quasistaic response to its solution. This method suggests the modes retained in the solution accurately spans the frequency range of interest; any loading represented by the non-retained modes will produce a quasistaic response. Therefore it can be said that non retained modes will have no dynamic amplification or causes no considerable velocity or acceleration.

The following is a brief derivation of the mode acceleration method

Recall the EOM

$$M\ddot{u} + C\dot{u} + Ku = P(t) \qquad 3.20$$

Solve for u from equation 3.18 to get equation 3.21

$$u = K^{-1}[P(t) - C\dot{u} - M\ddot{u}] \qquad 3.21$$

Using modal coordinates to represent $\dot{u}$ and $\ddot{u}$ right hand side of the equation 3.19

We get

$$u = K^{-1}[P(t) - C\phi\dot{\eta} - M\phi\ddot{\eta}] \quad 3.22$$

Or

$$\hat{u}(t) = K^{-1}P(t) - \sum_{r=1}^{m} \frac{2\zeta_r}{\omega_r}\phi_r\,\dot{\eta}_r(t) - \sum_{r=1}^{m}\frac{1}{\omega_r^2}\phi_r\ddot{\eta}_r(t) \quad 3.23$$

For steady state undamped system

$$\hat{u}(t) = K^{-1}P(t) - \sum_{r=1}^{m}\frac{1}{\omega_r^2}\phi_r\ddot{\eta}_r(t) \quad 3.24$$

The first term in the solution is the static response, and the second term represents response due to acceleration. This solution improves the accuracy and convergence compared to MD method.

3.2.3. Mode Truncation Augmentation (MTA) Method

Usually truncation response solution has inaccuracies occur due to truncation of load vector. Pseudo Eigen vectors or Mode truncation vectors are created in the modal set for the response analysis in order to compensate for the losses due to truncation. These truncation vectors are created using Rayleigh-Ritz approximation and appended to the retained eigen vectors. The modal response analysis will be carried with these new set of eigen vectors.[1]

This can be represented as

$$[\phi] \rightarrow [\tilde{\phi}] = [\phi_m \quad P] \quad 3.25$$

$[\tilde{\phi}]$ is the new eigen vector set

$\phi_m$ is the retained set of eigen vectors.

$P$ is the calculated pseudo eigen vector

The algorithm for MTA is carried out by first calculating the truncated force vector

$$\{R_t\} = \{R_o\} - \{R_s\} \quad 3.26$$

Where $\{R_s\}$ is modal spatial load vector given by $[M][\phi]\{\alpha\}$

$$\alpha = \{\phi_m\}^T R_o \quad 3.27$$

And $R_o$ is the physical load vector

Next to solve for the pseudo static response

$$[K]\{u\} = \{R_t\}. \quad 3.28$$

To form

$$[\bar{K}] = \{u\}^t [K]\{u\} \quad 3.29$$

$$[\bar{M}] = \{u\}^t [M]\{u\} \quad 3.30$$

The eigen value problem of the reduced set is given by

$$([\bar{K}] - [\bar{M}]\ \omega_p^2)Q = 0. \quad 3.31$$

and

$$\{P\} = \{x\}Q. \quad 3.30$$

The response analysis is carried out as in standard MD method, with pseudo modal set of vectors.[10]

## 3.3. Numerical Comparison of Superposition Methods

A study to compare the results of various superposition methods is presented in this section. For analysis all methods retained same number of modes from the solution. The mass stiffness, force distribution is same throughout the analysis. Mode displacement, Mode acceleration, Mode truncation augmentation algorithms are compared with exact solution i.e. all modes were retained in the solution.

The harmonic response analysis was carried out for the configuration shown in Figure 3.1 Degree spring mass configuration for comparison of response algorithms

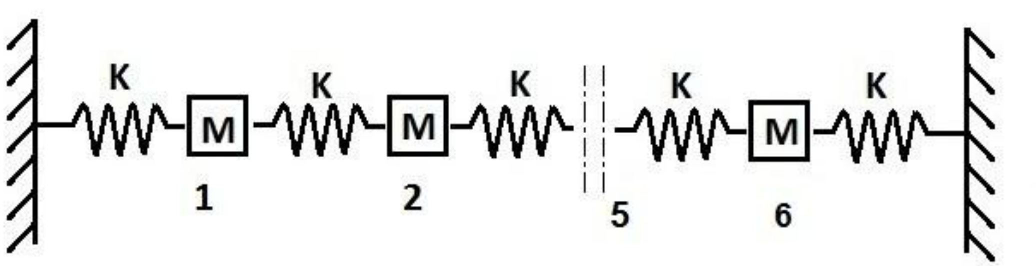


Figure 3.1 Degree spring mass configuration for comparison of response algorithms

The assembled mass and stiffness matrices are

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}$$

$$K = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The spatial load vector with a single force at DOF 4

$$f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The natural frequencies and mode shapes are given in Table 3.1

The displacements were compared in a frequency range from 0 to 300 rad/sec for a 6 degree of freedom system excited at 4$^{th}$ DOF and two modes were retained in the solution for all algorithms. The modes of the Eigen solution were normalized w.r.t unity before carrying out the analysis.

Table 3.1 Complete eigen solution

| Freq rad/sec | 45.0730 | 90.0701 | 131.9224 | 166.6723 | 190.7592 | 219.7352 |
|---|---|---|---|---|---|---|
| DOF no | Mode 1 | Mode2 | Mode3 | Mode 4 | Mode5 | Mode 6 |
| 1 | 1.09217 | 0.84122 | -1.07229 | 0.92158 | 0.59311 | -0.01010 |
| 2 | 1.96246 | 1 | -0.27841 | -0.71696 | -0.97205 | 0.028575 |
| 3 | 2.43406 | 0.34750 | 1 | -0.36381 | 1 | -0.07071 |
| 4 | 2.41116 | -0.58690 | 0.53806 | 1 | -0.66685 | 0.17144 |
| 5 | 1.89842 | -1.04518 | -0.86029 | -0.41415 | 0.09290 | -0.41417 |
| 6 | 1 | -0.65554 | -0.76143 | -0.67780 | 0.514585 | 1 |

All three algorithms were compared with exact solution with all modes retained in the solution.

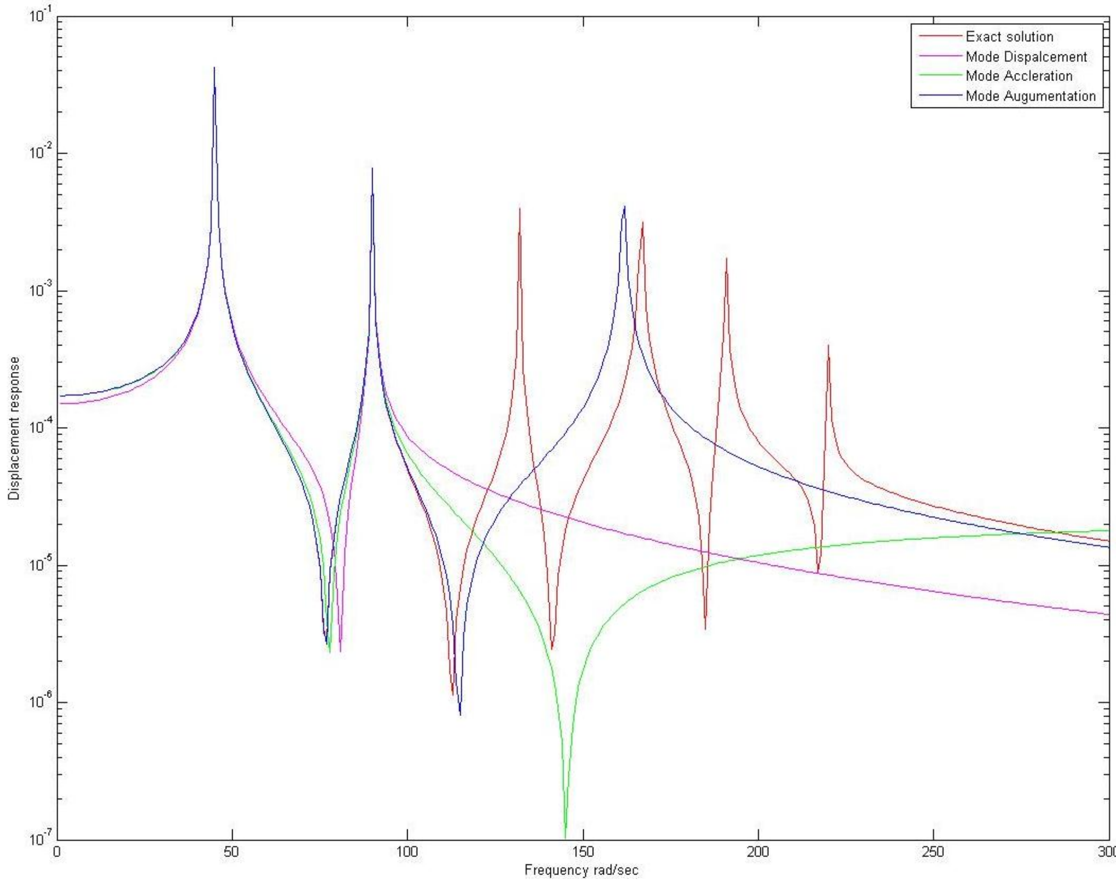


Figure 3.2 Displacement amplitude response for DOF 4 using Exact, MD, MA and MTA methods.

For displacements Figure 3.2 show that all three algorithms produced nearly same results for frequency range of 0 to 90 rad/sec .later after the truncation region Mode truncation had good accuracy over all other algorithms because of the pseudo modal set. The pseudo frequency from pseudo modal set acts as if it is natural frequency for not retained terms. The pseudo modes contribute to the response not represented by non-retained modes.

A comparison of absolute errors was calculated at 100 rad/sec the errors induced by each algorithm is listed in Table 3.2 for DOF 4.

Table 3.2 Comparison of Displacement by MD, MA and MTA methods

| Displacement | At 4th DOF $\times 10^{-5}$ | Error % |
|---|---|---|
| Exact | -5.70 | 0 |
| MD-method | -8.73 | 53 |
| MA-method | -6.51 | 14 |
| MTA-method | -5.13 | 10 |

From this comparison one can come to the conclusion that mode truncation augmentation is superior among all discussed algorithms.

# CHAPTER 4

# MODE TRUNCATION CRITERIA

## 4.1. Development of the Criterion

The error induced by the truncation of spatial load vector is the primary reason for inaccuracies in the response calculations. During the transformation of physical equations to the reduced set of modal equation, the choice of number of modes retained is based on the frequency range of interest. The frequency of the highest mode retained should sufficiently exceed the frequency content of the applied time history loading by a determined range. [6,10 ]. This criterion does not address the time dependent portion of he applied loading and ignores the truncation effects of spatial loading. This problem is well addressed by method of residual modes or residual vectors. This approach is advantageous over traditional displacement and acceleration methods where the response due to load truncation vectors represents the portion of the overall response that is lost [6,10]. Reduction in the spatial load truncation would address the problem of inaccurate responses and improve its quality. This can be achieved by retaining optimum number of modes, less than N modes (full model), before calculating the response.

Mode shapes can be a deciding factor while predicting the sufficiency of modes by calculating the generalized force for a particular mode to the total force vector. If the difference between the two consecutive modal force vectors approaches zero, then including any modes further would not affect the response significantly causing. Furthermore any spatial load not represented due to this truncation can be improved by residual mode methods such as mode truncation augmentation. This is the basis of the proposed criterion in this study.

## 4.2. Algorithm for Mode Truncation Criterion

A modal truncation criterion is developed in this section. Figure 4.2 is a flow chart to implement this criterion. The first step to calculate modal force contributions is calculating modal expansion of excitation vector. Considering common loading condition $P(t)$ has the same time variation as $p(t)$ which is a scalar function and spatial distribution is given as $S$. Thus the external loading vector can be represented as [5]

$$P(t) = S\, p(t) \tag{4.1}$$

The expansion of vector $S$ is given as

$$S = \sum_{n=1}^{N} S_n = \sum_{n=1}^{N} \Gamma_n\, m\phi_n \tag{4.2}$$

Multiplying on both sides by $\phi^T$gives

$$\Gamma_n = \frac{\phi_n^T\, S}{M_r} \tag{4.3}$$

$M_r$ is the mass vector in modal co-ordinates given by $M_r = \phi^T M \phi_n$

Equation 4.2 decomposes the force load vector as the summation of individual modal force contributions and $S$ is independent of method of normalization. The expansion of equation 4.1 has a useful property [5] that the force vector $S_n\, p(t)$ produces response only in the nth mode but no response in any other mode. This property can be demonstrated from generalized force of $r^{th}$ mode as

$$P_r(t) = \phi_r^t\, S_n\, p(t) \tag{4.4}$$

If j (j $< N$) modes are chosen for analysis the total modal force vector acting is the summation of partial load vectors from 1 to j to represent the load acting at $j^{th}$ mode such that the $S_j$ is

$$S_J = \sum_{n=1}^{j} S_j \qquad 4.5$$

Hence it can be concluded that if the $j^{th}$ force load vector is representing almost the entire excitation load then only j modes are required in the modal solution to approximate the response. Then error force vector is given as

$$S_e = abs\left[S_N - S_J\right] \qquad 4.6$$

Where $S_e$ is the error force vector to the total load. One can control the error by limiting the amount of allowable error in to the summation. This can be done by first judging the desired level of accuracy.

4.2.1. Criterion for Acceptable Modes

Let m = number of modes required

And $S_m$ = mode representation of spatial load vector S

Then m = smallest integer such that $|S_{im}| \leq ARF$

ARF = allowable residual force

$= (1 - E)max(|S|)$

E = a preselected factor (0< E <1)

Algorithm to implement mode truncation criterion in step by step process is given as:

Step 1: calculate allowable residual force

$$ARf = (1 - E)max(|S|) \qquad 4.7$$

E =maximum allowable error.

Step 2: Calculate $S_j$ for j = 1

$$S_j = \sum_{n=1}^{j} \Gamma_n \, m\phi_n \qquad 4.8$$

Step 3: Check mode truncation criteria

$$|S_{ij}| \leq ARF \qquad 4.9$$

For all i.

Step 4: if condition in step 2 is true then j = m = number of modes to be used

If false then go to step 2 with j=j+1.

This is repeated until the truncation criterion is satisfied.

Consider a system below

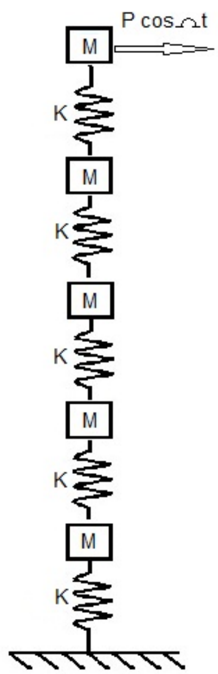


Figure 4.1 Fix free spring mass system with load acting on free end

The stiffness is given as

K=800 * [1 -1 0 0 0 ; -1 3 -2 0 0 ; 0 -2 5 -3 0 ; 0  0 -3 7 -4 ; 0  0 0 -4 9];

Mass distribution as

M=diag [1,2,2,3,3]

The force distribution vector is

Pt = [1 0 0 0 0 ]' cos Ω t

The expansion of this vector $S_n$ in modal coordinates is given in Table 4.1.

Table 4.1 Modal expansion of force vectors

| $S_1$ | $S_2$ | $S_3$ | $S_3$ | $S_4$ |
|---|---|---|---|---|
| 0.268467 | 0.389731 | 0.317525 | 0.02307 | 0.001207 |
| 0.447968 | 0.139677 | -0.48153 | -0.09773 | -0.00838 |
| 0.329259 | -0.29486 | -0.19317 | 0.135086 | 0.023683 |
| 0.320623 | -0.63481 | 0.338252 | 0.014219 | -0.03828 |
| 0.150829 | -0.38841 | 0.363202 | -0.16034 | 0.034717 |

And the summation of modal expansion vectors is given in Table 4.2.

Table 4.2 Summation of modal forces

| $\sum S_1$ | $\sum S_2$ | $\sum S_3$ | $\sum S_4$ | $\sum S_5$ |
|---|---|---|---|---|
| 0.268467 | 0.6582 | 0.9757 | 0.9988 | 1.0000 |
| 0.447968 | 0.5876 | 0.1061 | 0.0084 | 0.0000 |
| 0.329259 | 0.0344 | -0.1588 | -0.0237 | 0.0000 |
| 0.320623 | -0.3142 | 0.0241 | 0.0383 | 0.0000 |
| 0.150829 | -0.2376 | 0.1256 | -0.0347 | 0.0000 |

It can be seen at $\sum S_j = \sum S_3$ indicates that at mode j=3 it is sufficiently equal to the force vector $Pt$ and the algorithm stops as max error is less than the maximum allowable error, hence we can ignore the last two modal terms in the series as they contribute a little in the response solution.

From Table 4.2 the error is 0 if j=5 at this point $S_J = S_N = Pt$.

One can choose the allowable error force into the analysis and predict at what mode number the respective modal force vector is representing load within the error limit. These predicted modes will be used in mode superposition method. For the above example m=3 for response analysis.

The effect and inaccuracies due to truncation can be further reduced using earlier discussed superior summation methods which can efficiently represent the response due to non-retained modes.

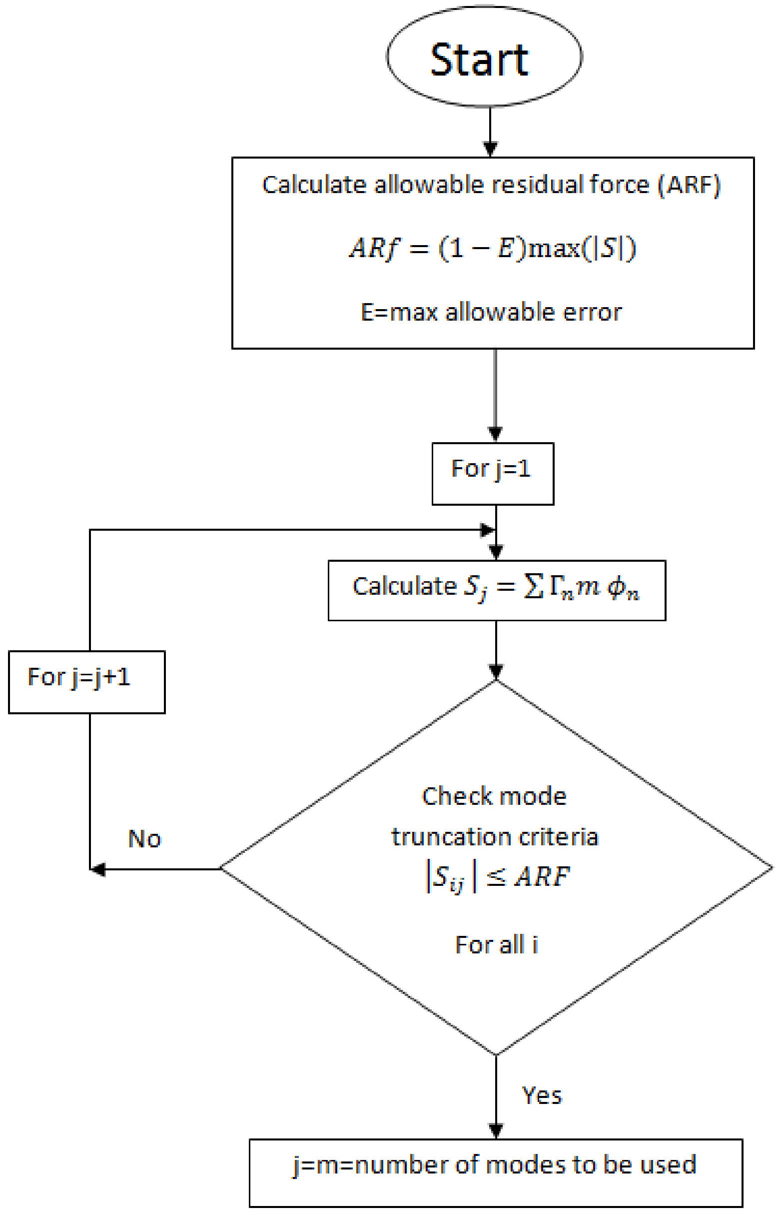


Figure 4.2 Flow chart of the proposed truncation algorithm

CHAPTER 5

# COMPARISION AND RESULTS OF RESPONSE WITH TRUNCATION CRITERIA

From a preliminary analysis in chapter 3 the MTA method was found to be superior upon all algorithms. In this chapter, the truncation criterion developed in chapter 4 is tested using MTA method. The predicted mode number was the number of retained modes in the Eigen solution.

## 5.1. Comparison with a 20 DOF system

For numerical comparison of MTA method a 20 DOF spring mass system Figure 5.1 was with following stiffness and mass distribution.

$$M = \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0.5 \end{bmatrix}_{20\times 20}$$

The last mass was set to 0.5 to have distinct modes

$$K = \begin{bmatrix} 2 & -1 & \dots & 0 & 0 \\ -1 & 2 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 2 & -1 \\ 0 & 0 & \dots & -1 & 2 \end{bmatrix}_{20\times 20}$$

This system was excited at 10$^{th}$ DOF with a single force

$$f = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{20\times 1}$$

The initial condition for $\dot{u}(0)$ and $\ddot{u}(0)$ are set to 0 throughout the analysis. The modal damping was set to 2% for all modes. Maximum allowable error force $\max(S_{e\,all})$ was set to 0.25 of the force vector. The predicted number of modes from the Truncation criteria for these values was found to be 14 of the maximum 20 modes. The further analysis was carried out with these many number of modes retained in the solution.

A comparison of modal force load distribution at retained modal number with the actual force distribution is given in Table 5.1.

Table 5.1 Comparison of modal force load distribution to actual force at j=15

| DOF no | $\sum_{j}^{15} S_j$ | $f$ |
|---|---|---|
| 1 | -0.01901 | 0 |
| 2 | 0.033532 | 0 |
| 3 | -0.03893 | 0 |
| 4 | 0.030823 | 0 |
| 5 | -0.00629 | 0 |
| 6 | -0.03456 | 0 |
| 7 | 0.087211 | 0 |
| 8 | -0.1426 | 0 |
| 9 | 0.188623 | 0 |
| 10 | 0.7844 | 1 |
| 11 | 0.207902 | 0 |
| 12 | -0.17158 | 0 |
| 13 | 0.111241 | 0 |
| 14 | -0.04101 | 0 |
| 15 | -0.02178 | 0 |
| 16 | 0.061726 | 0 |
| 17 | -0.07028 | 0 |
| 18 | 0.048656 | 0 |
| 19 | -0.00758 | 0 |
| 20 | -0.018 | 0 |

The % error at DOF 10 to actual force applied was 21.56 % with E=0.25 of force distribution in modal coordinates at j=15 is shown in Figure 5.2. It is shown in previous analysis that any increment to the modal number would not affect the response much. This is also proven in further analysis using MTA method with these many retained modes.

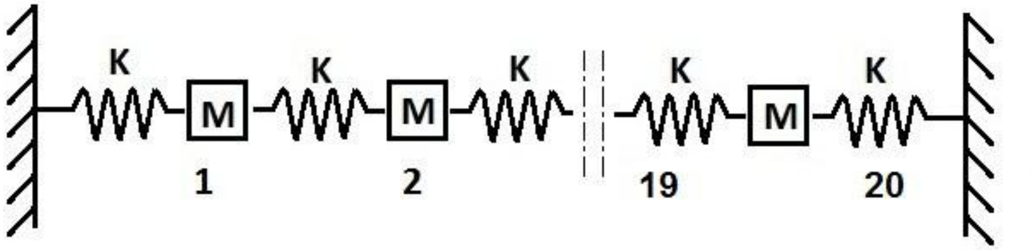


Figure 5.1 A 20 DOF spring mass configuration.

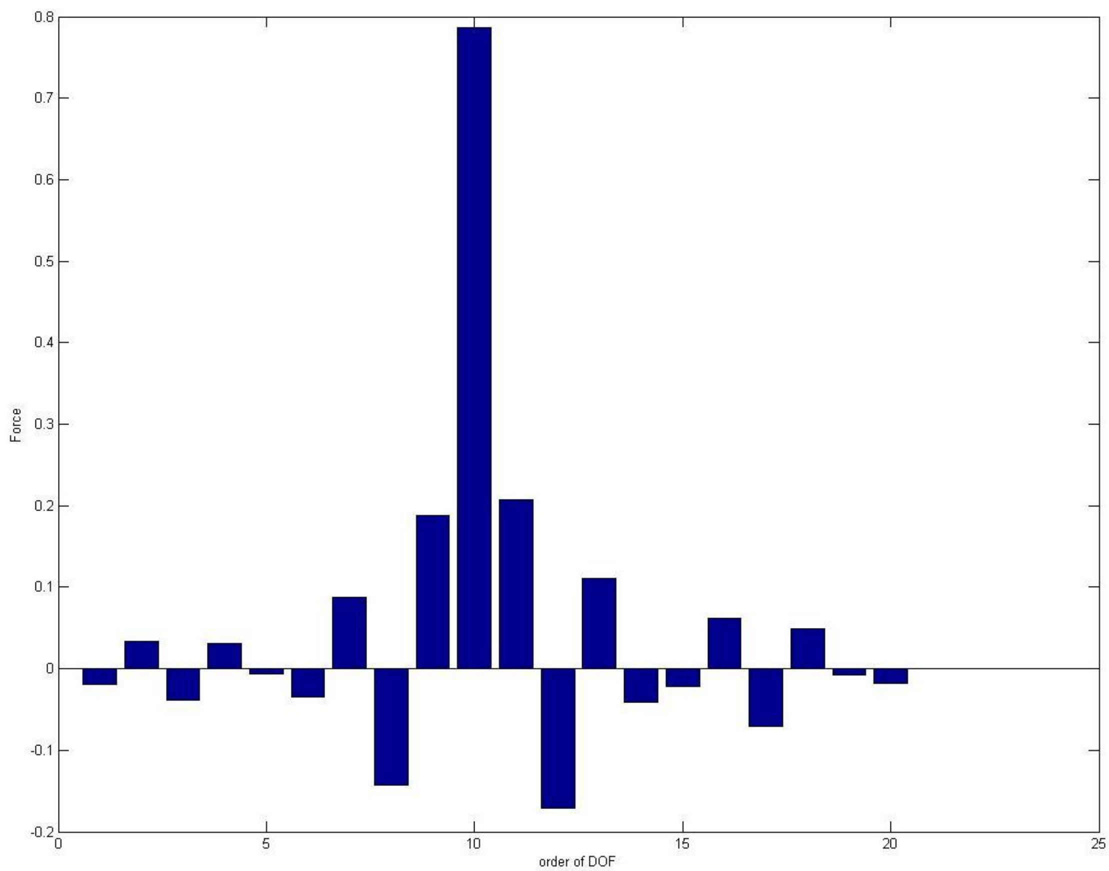


Figure 5.2 Force distribution of $S_j$ at j=15

5.1.1. Harmonic Response Analysis

The Harmonic response analysis was performed on the spring mass system in Figure 5.1 as a function of frequency of applied harmonic force with a frequency range of 0 to 250 rad/sec. MTA was compared with exact solution using all modes. The displacement response of MTA was in good convergence till the point of truncation with minimal errors seen in frequency range after truncation considering MTA to be the best modal analysis algorithm.

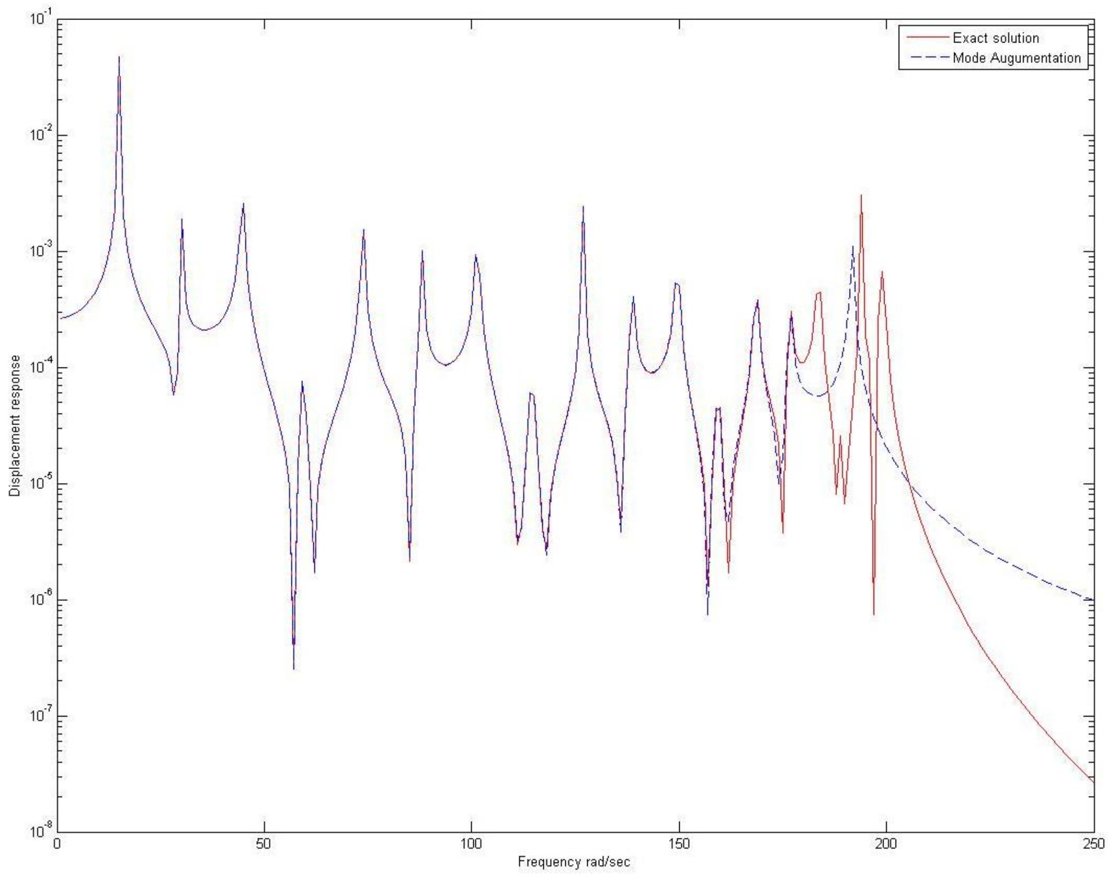


Figure 5.3 Displacement amplitude response for DOF 10 using exact solution and MTA.

Comparison of this system was excited with distributed force under similar conditions

$$f = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix}_{20\times 1}$$

A comparison of modal force load distribution at retained modal number with the actual force given in Table 5.2.

Table 5.2 Comparison of modal force load distribution to actual force at j=12

| DOF no | $\sum_{j}^{12} S_j$ | $f$ |
|---|---|---|
| 1 | 0.1989 | 0 |
| 2 | 0.7528 | 1 |
| 3 | 1.1321 | 1 |
| 4 | 1.0217 | 1 |
| 5 | 0.9177 | 1 |
| 6 | 1.0319 | 1 |
| 7 | 1.0398 | 1 |
| 8 | 0.9528 | 1 |
| 9 | 0.9964 | 1 |
| 10 | 1.0408 | 1 |
| 11 | 0.9787 | 1 |
| 12 | 0.9772 | 1 |
| 13 | 1.0319 | 1 |
| 14 | 1.0029 | 1 |
| 15 | 0.9691 | 1 |
| 16 | 1.0099 | 1 |
| 17 | 1.0306 | 1 |
| 18 | 0.9925 | 1 |
| 19 | 0.8532 | 1 |
| 20 | 0.2349 | 0 |

The % error at DOF 10 to actual force applied was 21.56 % with E=0.25 of force distribution in modal coordinates at j=15 is shown in Figure 5.4. It is shown in previous analysis that any increment to the modal number would not affect the response much. This is also proven in further analysis using MTA method with these many retained modes.

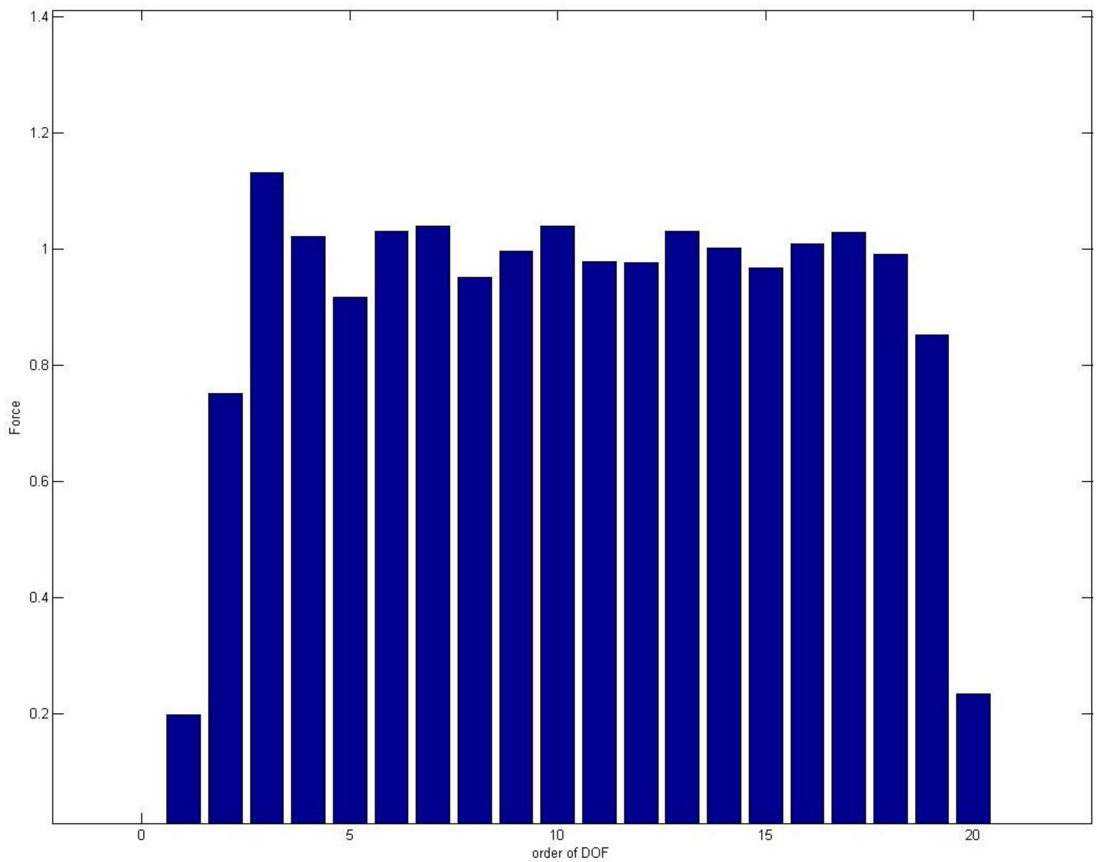


Figure 5.4 Force distribution of $S_j$ at j=12

The Harmonic response analysis was performed on the spring mass system with distributed load in Figure 5.5 as a function of frequency of applied harmonic force with a frequency range of 0 to 250 rad/sec. MTA was compared with exact solution using all modes.

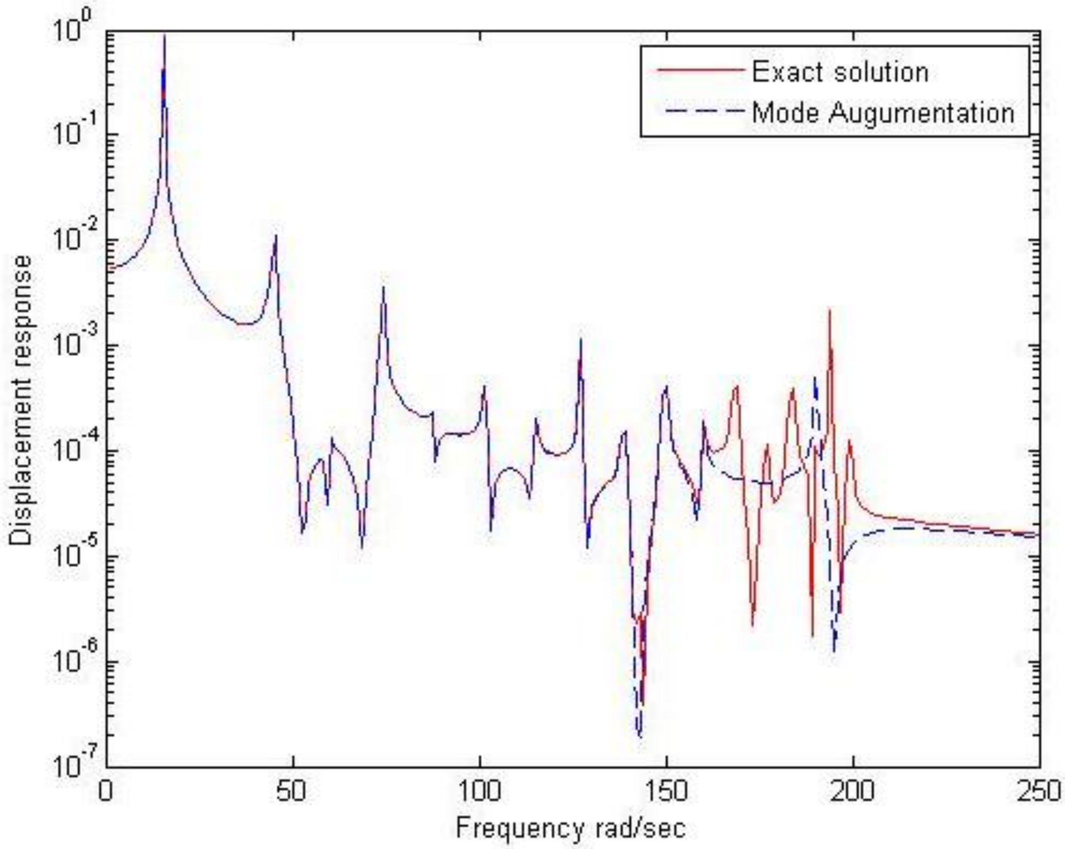


Figure 5.5 Displacement amplitude response for DOF 10 using exact solution and MTA.

5.1.2. Response to Periodic Excitation

For periodic excitation $p(t) = 3\cos(180\,\mathrm{t})$ is applied for the same configuration and initial conditions at DOF 10. The displacement response is shown in Figure 5.6. MTA was compared with exact solution using all modes.

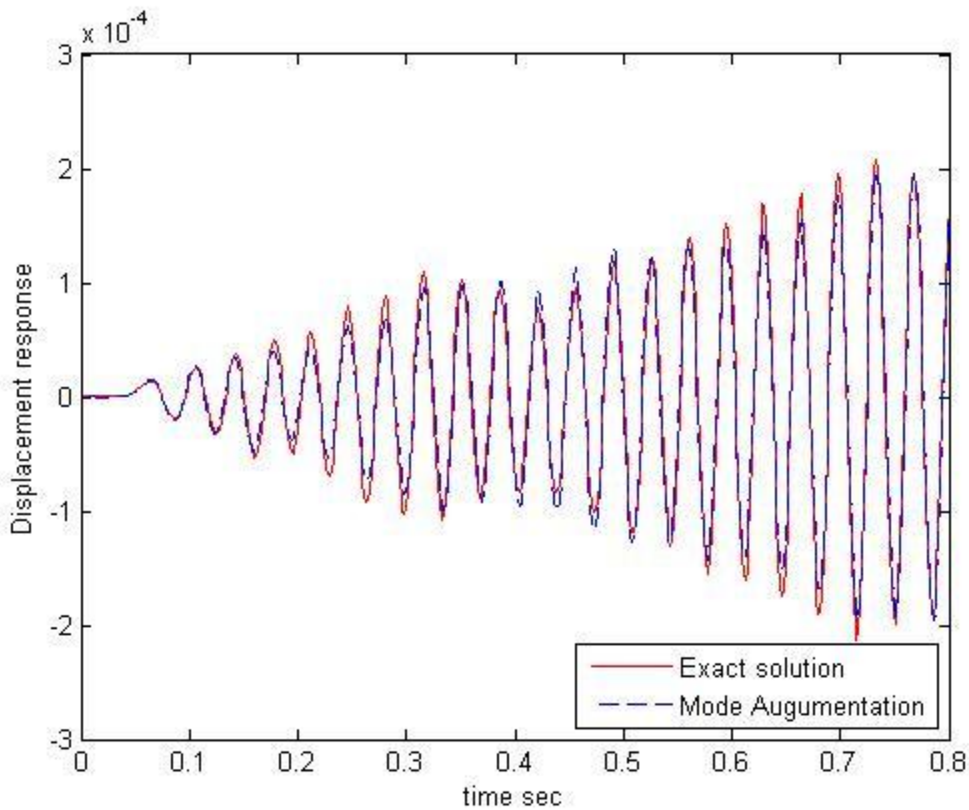


Figure 5.6 Displacement response for periodic excitation.

5.1.3. Response to Step Loading

For step loading $p(t) = p_o = 3$ is applied for the same configuration and initial conditions at DOF 10. The displacement response is shown in Figure 5.7 this kind of input tends to excite all the modes of a structure as step function has all frequencies. MTA was compared with exact solution using all modes.

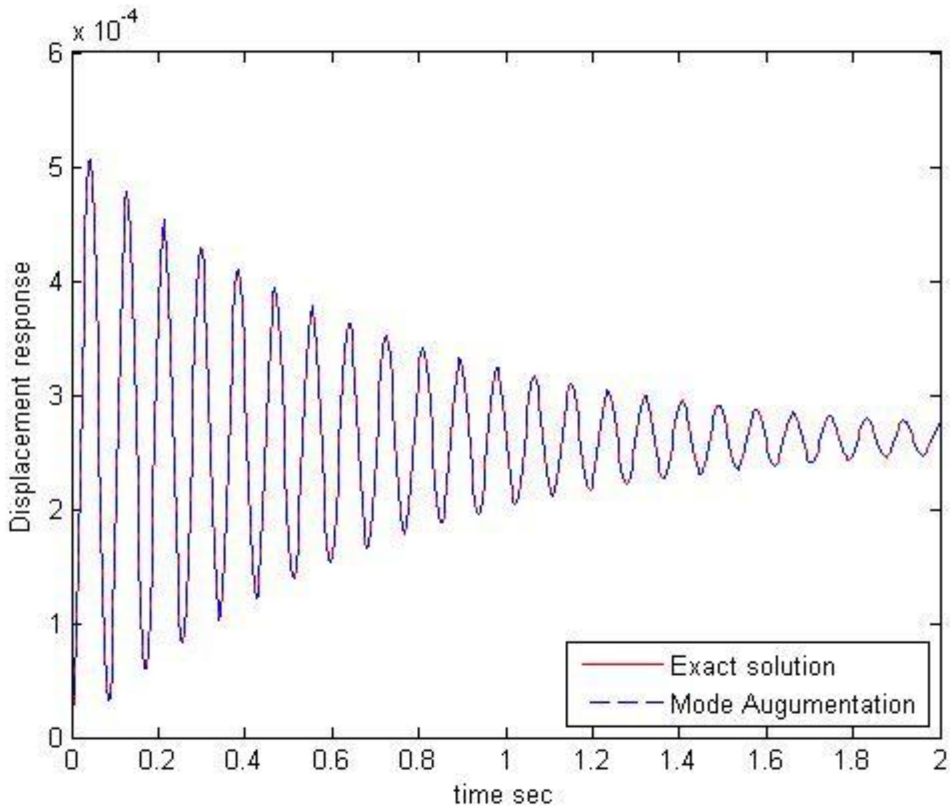


Figure 5.7 Displacement response for step loading.

The time domain analysis shown in sections would not add anything new to the conclusion on accuracy of MTA method as it is not designed for analysis frequency above highest retained mode. That is the reason MTA method shows good accuracy in time domain analysis.

## 5.2. Summary of Comparison

A summary of the entire analysis is listed in Table 5.3 to demonstrate Truncation criterion with MTA method for the spring mass configuration in Figure 5.1. The computation time taken for these methods for the analysis was observed on a computer with Intel Pentium(R) Dual-core CPU 2.20 GHz processor with 4.00GB RAM and 80GB hard disk.

Table 5.3 Summary of comparison of MTA method with truncation criteria

| | Exact solution | MTA |
|---|---|---|
| modes retained | 20 | 15 |
| modal reduction | -- | 5 |
| % of modes reduced | -- | 25 |
| computation time | 0.043 | 0.033 |
| CPU time | 1 | 0.77 |
| displacement | 4.53E-04 | 4.51E-04 |
| Accuracy % | 100 | 99.54 |

A comparison of MTA method with exact solution with the described truncation criteria for 20 DOF system for $10^{th}$ DOF at $\Omega$ = 183 rad/sec (truncation region) for frequency range of 1-250.

Another comparison for 1000 DOF model for all dynamic response algorithms is listed in Table 5.4 for $500^{th}$ DOF for frequency range of 0-250 at truncation region $\Omega$=180.

Table 5.4 A system with 1000 DOF Ω = 180 for DOF 500 for frequency range 0-250.

| | Computation time sec | CPU Time | Accuracy % | Modes | Modes reduced % |
|---|---|---|---|---|---|
| Direct method | 13.8 | 4.8 | 100 | N A | NA |
| Exact solution | 2.38 | 1 | 100 | 1000 | 0 |
| M D Method | 1.59 | 0.33 | 88.8 | 750 | 25 |
| M A Method | 1.93 | 0.14 | 90.3 | 750 | 25 |
| M T A Method | 1.62 | 0.32 | 98.8 | 750 | 25 |

# CHAPTER 6

# CONCLUSION AND FUTURE WORK

## 6.1. Discussion of Analysis With Proposed Truncation Criteria

The proposed truncation criteria was shown to predict the number of modes sufficiently enough to carry out a response analysis using MTA method, which was proven to be superior of all Response analysis algorithms. MTA method used the predicted number of modes to approximate a response in agreement with exact solution which was demonstrated using spring mass systems. The reduction in number of modes and response approximation with MTA method was proven to be a time reducing technique.

## 6.2. Future Work

The proposed criterion uses error force vector method than a conventional frequency cutoff method. The proposed criteria can be further tested on frames with various loading conditions using very large DOF systems to further test its validity.

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## BIOGRAPHICAL INFORMATION

Sravan Neela was born in Hyderabad India on June 10th, 1987. He received his bachelor's degree in Mechanical engineering from Jawaharlal Nehru Technological University, Hyderabad in year 2008. He came to USA in 2009 to pursue his master's education in Mechanical engineering at the University of Texas at Arlington. His interests include Finite element methods, Structural analysis and scientific programming.