BUCKLING ANALYSIS OF THIN PLATES WITH OR WITHOUT A HOLE UNDER ARBITRARY BOUNDARY CONDITIONS USING THE GALERKIN METHOD

by

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Presented to the Faculty of the Graduate School of

The University of Texas at Arlington in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE IN MECHANICAL ENGINEERING

THE UNIVERSITY OF TEXAS AT ARLINGTON

DECEMBER 2011

ACKNOWLEDGEMENTS

I am deeply indebted to my supervising professor, Dr. Seiichi Nomura for his guidance, patience, and motivation. I would like to sincerely thank him for his help and guidance throughout my thesis work. I would also like to thank Dr. Dereje Agonafer and Dr. A. Haji-Sheikh for serving on my thesis committee. Finally, I would like to thank my parents, sisters, and all my friends for their immense support and encouragement throughout my Master's program.

November 16, 2011

ABSTRACT

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The University of Texas at Arlington, 2011

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This thesis demonstrates how to find the critical buckling load value of thin plates with or without a hole under different boundary conditions using the Galerkin method. The use of symbolic software is essential due to the lengthy computations involved because of the complexity of the problems.

Firstly, the lateral deflection of the plate is expressed in a series of polynomials each of which satisfies the given boundary conditions. Then by using the Galerkin method, the coefficients of these polynomials are found and with the help of symbolic algebra system, the matrices for the corresponding eigenvalue problem are built from which the buckling loads (eigenvalues) are determined. Since this analysis involves very complex calculations, it is almost impossible to carry out all the computations involved without the aid of symbolic software.

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CHAPTER 1

INTRODUCTION

1.1 Overview

Elasticity is a captivating subject that deals with the determination of stresses and displacements in a body in the presence of external forces. Elasticity is governed by Hooke's law i.e., the deformation is proportional to the load that produces them. The material exhibiting elasticity has the property for complete recovery to its natural shape upon the removal of the applied external load.

It all began in the early 1800s when Cauchy, Poisson, Lagrange, Kirchhoff, and Navier did some significant research on the analysis of plates. Euler in 1766 was the first to define a mathematical approach to the membrane theory of plates by solving the problem of free vibrations of rectangular and circular elastic membranes using two systems of stretched strings normal to each other [1]. It can be said that Navier (1785-1836) was the founder of the modern theory of elasticity and his numerous research work includes solutions of various plate problems. It was Navier who derived the correct differential equation of rectangular plates with flexural resistance and later Poisson (1829) extended it to the lateral vibration of circular plates which was applicable only to thick plates. But it was Kirchhoff (1824-1887) whose intense research work on plate theory defines a method which takes into account the combined bending and stretching [2]. The great techniques defined by these engineers were very significant which are still used in most of the engineering analysis done today.

Since the coming of the digital computer i.e., in the last 40 years other methods such as finite differences and finite elements have come into existence. It was Topp, Martin, Tuner, and Clough [3] in 1956 who introduced the concept of the finite element method to solve complex plate and shell problems. The use of the finite element method required high speed computers with high storage capacity.

1.2 Method of Weighted Residuals

The Method of Weighted Residual (MWR) is an approximation technique for solving differential equations and it's been in use even before the finite element method came into existence. The basic idea of MWR is to drive a residual error to zero through a set of orthogonality conditions. The idea is to use a polynomial involving the parameters to approximate the differential equation that satisfies the boundary conditions involved.

There are three different methods under MWR,

1.2.1 Collocation Method

Choose u_i so that the residual error vanishes at N selected points, i.e.

$$R(X_i) = 0, \quad i = 1, \dots \dots \dots N \tag{3.7}$$

Although this method gives the exact values at the selected points, there is no guarantee that the approximation behaves nicely between the selected points.

1.2.2 Least Square Method

Choose u_i so that the magnitude of residual error becomes the minimum i.e.

$$\|R(x)\| \rightarrow min. \tag{3.8}$$

The least square method is by far the oldest of the methods of weighted residuals.

1.2.3 Galerkin Method

The Galerkin method is credited to the great Russian mathematician Boris Galerkin. This method can be applied in various areas of engineering and science such as acoustics, neutron transport, fluid mechanics, fracture mechanics, electromagnetics, and dynamics. The Galerkin method was introduced by Galerkin in 1915 and it was used extensively from 1950 onwards for various mechanical and aerospace applications.

The Galerkin method can be used for converting a continuous operator problem such as a differential equation to a discrete problem. The concept of the Galerkin method is to apply the method of variation of parameters to a function space, by converting the equation to a weak formulation. The next step is to apply the boundary conditions on the function space to characterize the space with a finite set of base functions. The Galerkin method and the finite element method (which is a special case of Galerkin method) are currently the most commonly used numerical technique to solve various nonlinear problems.

1.3 Use of Symbolic Algebra Software

Since the development of hardware and software of computers many software packages such as MATHEMATICA and MAPLE has come into existence. During the 1970's software packages such as MACSYMA and REDUCE were used which were written in LISP and it required high memory and processing capacity for performing routine mathematical calculations. Packages such as MATHEMATICA and MAPLE are written in the C language and its variations and are the most commonly used packages nowadays.

The coming of MATHEMATICA in 1988 marked the beginning of the modern technical computing. But the visionary concept of MATHEMATICA was to create once and for all a single system that could handle all the various aspects of technical computing--and beyond--in a coherent and unified way. The key advancement that made this possible was the invention of a new kind of symbolic computer language that could, for the first time, manipulate the very wide range of objects needed to achieve the generality required for technical computing, using only a fairly small number of basic primitives.

The major features of symbolic algebra systems are performing integrations, differentiations, expansions, and solving equations exactly. The feature that makes this software stand out from the rest is its ability to deal with both symbolic formulae and numbers.

This research demonstrates how to find the critical buckling load value of a square plate with or without a hole under different boundary conditions using the Galerkin method. This research stands out from the rest as one of the unique method to have dealt with the buckling analysis of a plate with a hole using the Galerkin method. In this research work, a series of polynomial is assumed which represents the lateral deflection of the plate and each of these polynomials satisfies the associated boundary conditions. The Galerkin method is used to find the coefficients of these polynomials. The use of the symbolic software (MATHEMATICA) in this thesis is essential due to the rigorous calculations involved due to the complexity of the geometry.

CHAPTER 2

THEORY OF BUCKLING

2.1 Definition of Buckling

When a thin structure is loaded in compression, for small loads it deforms with hardly any noticeable change in geometry and load carrying ability. Once it reaches a critical load value, the structure tends to suddenly experience a large deformation and it may lose its ability to carry the applied load. At this juncture, the structure is considered to have buckled. For example, when an axial compressive force is applied to a rod, it first shortens slightly but once it reaches the critical load the rod bows out, and it can be stated that the rod has buckled.

Buckling is also termed as structural instability and is classified into two categories:

2.1.1 Bifurcation Buckling

Bifurcation buckling is the one in which the deflection under compressive load changes from one direction to a different direction i.e., from axial shortening to lateral deflection.

2.1.2 Limit Load Buckling

Limit load buckling is the one in which the structure attains a maximum load without any previous bifurcation i.e., with only one mode of deflection.

Other classifications of buckling are made with respect to the displacement magnitude (i.e., small or large), or metal behavior such as elastic buckling or inelastic buckling, or static versus dynamic buckling.

2.2 Governing Equation Derivation

Here we consider the classical plate theory (CPT) which is based on the Kirchhoff hypothesis [4]:

(a) Straight lines perpendicular to the mid-surface (i.e., transverse normals) before deformation remain straight after deformation.

(b) The transverse normals do not experience elongation (i.e., they are inextensible).

(c) The transverse normals rotate such that they remain perpendicular to the mid-surface after deformation.

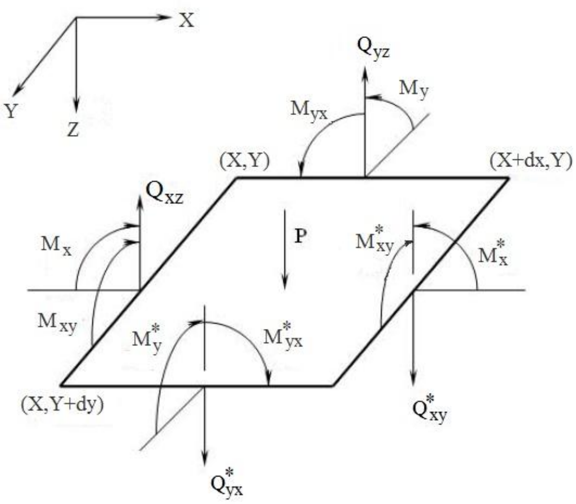

Figure 2.1 Differential plate elements with stress resultants

The equilibrium condition is satisfied by taking a rectangular differential element of dimensions dx, dy and h as shown in Figure 2.1. Only the middle surface of the plate is shown in the figure for simplicity.

Considering that the sum of moments of all forces around the Y axis is zero we get,

$$(m_x + \frac{\partial m_x}{\partial x} dx)dy - m_x dy + \left(m_{yx} + \frac{\partial m_{yx}}{\partial y} dy\right) dx - m_{yx} dx - \left(q_x + \frac{\partial q_x}{\partial x} dx\right) dy \frac{dx}{2} - q_x dy \frac{dx}{2} = 0 \quad (2.1)$$

After simplification Eq. (2.1) becomes,

$$\frac{\partial m_x}{\partial x} dx\, dy + \frac{\partial m_{yx}}{\partial y} dy\, dx - q_x\, dx\, dy = 0 \quad (2.2)$$

and, after division by $dx\ dy$, we get

$$\frac{\partial m_x}{\partial x} + \frac{\partial m_{yx}}{\partial y} = q_x \tag{2.3}$$

Similarly the sum of moments around X axis is,

$$\frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x} = q_y \tag{2.4}$$

The summation of all forces in the z direction yields the third equilibrium equation:

$$\frac{\partial q_x}{\partial x} dx\ dy + \frac{\partial q_y}{\partial y} dy\ dx + p_z\ dx\ dy = 0 \tag{2.5}$$

after division by $dx\ dy$, we get

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = -p_z \tag{2.6}$$

Substituting Eqs. (2.3) and (2.4) into Eq. (2.6) and assuming $m_{yx} = m_{xy}$, we get

$$\frac{\partial^2 m_x}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x\,\partial y} + \frac{\partial^2 m_y}{\partial y^2} = -p_z(x, y) \tag{2.7}$$

The bending and twisting moments in Eq. (2.7) depends on the strains and the strains are functions of the displacement components.

Now we have,

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) \tag{2.8}$$

Similarly,

$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) \tag{2.9}$$

Next, we consider the geometry of the deflected plate to express strains in terms of displacement.

$$\epsilon_x = -z\frac{\partial^2 w}{\partial x^2} \tag{2.10}$$

and

$$\epsilon_y = -z\frac{\partial^2 w}{\partial y^2} \tag{2.11}$$

The stress components σ_x and σ_y produce bending moments in the plate element in a manner similar to that in elementary beam theory. Hence, by integrating the normal stress components, the bending moments, acting on the plate element are obtained:

$$m_x = \int_{-h/2}^{+h/2} \sigma_x \ z \, dz \quad \text{and} \quad m_y = \int_{-h/2}^{+h/2} \sigma_y \ z \, dz \tag{2.12}$$

Similarly, the twisting moments produced by the shear stresses $\tau = \tau_{xy} = \tau_{yx}$ can be calculated from

$$m_{xy} = \int_{-h/2}^{+h/2} \tau_{xy} \ z \, dz \quad \text{and} \quad m_{yx} = \int_{-h/2}^{+h/2} \tau_{yx} \ z \, dz \tag{2.13}$$

but $\tau = \tau_{xy} = \tau_{yx}$, and hence $m_{xy} = m_{yx}$.

By substituting Eqs. (2.10) and (2.11) into Eqs. (2.8) and (2.9) we can express the stresses σ_x and σ_y in terms of lateral deflection w. Hence we can write

$$\sigma_x = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \tag{2.14}$$

and

$$\sigma_y = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \tag{2.15}$$

Integrating Eq. (2.12) after substituting Eqs. (2.14) and (2.15), we get

$$m_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \tag{2.16}$$

$$m_y = -D\left(\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \tag{2.17}$$

$$m_{yx} = m_{xy} = D(1 - \nu)\frac{\partial^2 w}{\partial x \, \partial y} \tag{2.18}$$

The stress resultant expressions acting on the transverse faces at (x+dx, y) and (x, y+dy) are determined by expanding each into a Taylor' series about (x,y). The higher order terms can be neglected since dx and dy are infinitesimally small values.

$$Q_{xz}^{*} = Q_{xz} + \frac{\partial \mathrm{Qxz}}{\partial x}\, dx \tag{2.19}$$

$$Q_{yz}^{*} = Q_{yz} + \frac{\partial \mathrm{Qyz}}{\partial y}\, dy \tag{2.20}$$

$$m_x^{*} = m_x + \frac{\partial m_x}{\partial x}\, dx \tag{2.21}$$

$$m_y^{*} = m_y + \frac{\partial m_y}{\partial y}\, dy \tag{2.22}$$

$$m_{xy}^{*} = \ m_{xy} + \frac{\partial m_{xy}}{\partial x}\, dx \tag{2.23}$$

$$m_{yx}^{*} = \ m_{yx} + \frac{\partial m_{yx}}{\partial y}\, dy \tag{2.24}$$

The condition of a vanishing resultant force in the 3-direction results in the equation

$$\frac{\partial \mathrm{Q_{xz}}}{\partial \mathrm{x}} + \frac{\partial \mathrm{Q_{yz}}}{\partial \mathrm{y}} + \mathrm{p} = 0 \tag{2.25}$$

If the resultant moment about an edge parallel to the x-axis is set to zero and by neglecting higher order terms the resulting equation becomes

$$\frac{\partial m_{xy}}{\partial x} - \frac{\partial m_y}{\partial y} + Q_{yz} = 0 \tag{2.26}$$

The equilibrium equation with respect to rotation about an edge parallel to the y-axis is

$$\frac{\partial m_x}{\partial x} - \frac{\partial m_{yx}}{\partial y} - Q_{xz} = 0 \tag{2.27}$$

By substituting Eq. (2.26) and Eq. (2.27) into Eq. (2.25) we get the resulting equation as

$$\frac{\partial^2 m_x}{\partial x^2} - 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_y}{\partial y^2} = \ -p \tag{2.28}$$

By substituting the Eqs. (2.16), (2.17) and (2.18) into Eqs. (2.26) and (2.27) we can get the expressions for Q_{xz} and Q_{yz} in terms of the deflection of the middle surface

$$Q_{xz} = -D\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right) = -D\frac{\partial}{\partial x}(\nabla^2 w) \qquad (2.29)$$

$$Q_{yz} = -D\left(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y}\right) = -D\frac{\partial}{\partial y}(\nabla^2 w) \qquad (2.30)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad (2.31)$$

2.2.1 Governing Equation

2.2.1.1 Rectangular Coordinates

By substituting Eqs. (2.16), (2.17), and (2.18) into the equilibrium equation of (2.28) the governing partial differential equation defining the lateral deflection of the middle surface in terms of the applied transverse load is obtained.

$$\frac{\partial^4 \mathrm{w}}{\partial \mathrm{x}^4} + 2\frac{\partial^4 \mathrm{w}}{\partial \mathrm{x}^2 \partial \mathrm{y}^2} + \frac{\partial^4 \mathrm{w}}{\partial \mathrm{y}^4} = \frac{\mathrm{p}}{\mathrm{D}}\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2} \qquad (2.32)$$

or
$$\nabla^4 w = \frac{\mathrm{p}}{\mathrm{D}}\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2} \qquad (2.33)$$

or
$$\nabla^2\nabla^2 w = \frac{\mathrm{p}}{\mathrm{D}}\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2} \qquad (2.34)$$

where,
$$\nabla^4 = \frac{\partial^4}{\partial \mathrm{x}^4} + 2\frac{\partial^4}{\partial \mathrm{x}^2 \partial \mathrm{y}^2} + \frac{\partial^4}{\partial \mathrm{y}^4} \qquad (2.35)$$

The fourth-order partial differential equation of (2.34) can be reduced to two separate second-order partial differential equations which are preferred sometimes based upon the method of solution to be used and it's done in the following way,

Adding Eq. (2.18) and Eq. (2.19),
$$M_x + M_y = -D(1+\nu)\left(\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{w}}{\partial \mathrm{y}^2}\right) \qquad (2.36)$$

or
$$\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{w}}{\partial \mathrm{y}^2} = \frac{M_x + M_y}{-D(1+\nu)} \qquad (2.37)$$

or
$$\nabla^2 w = \frac{M}{D} \qquad (2.38)$$

where,
$$\mathrm{M} = -\frac{m_x + m_y}{(1+\nu)} \qquad (2.39)$$

Substituting Eq. (2.38) in Eq. (2.34)

$$\nabla^2 \left(\frac{M}{D}\right) = \frac{p}{D} \tag{2.40}$$

or

$$\nabla^2 M = p \tag{2.41}$$

Hence, the fourth order equation of (2.32) has been reduced to two separate second-order equation of (2.38) and (2.41). The quantity m*(x, y)* can be found by solving Eq. (2.41) by using the proper boundary condition and by substituting a value for the transverse load p and Eq. (2.38) can be solved for *w(x,y)*.

2.2.1.2 Polar Coordinates

The governing equations for circular plates can be determined by using the transformation relations (x = r cosθ , y = r sinθ) between the polar coordinates (r, θ) and the rectangular coordinates (x,y) (see Fig. 2.3). The equation of equilibrium is,

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial y \partial x} + \frac{\partial^2 M_{yy}}{\partial y^2} = \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x} + N_{yy}\frac{\partial w}{\partial y}\right) \tag{2.42}$$

where (M_{xx},M_{yy}) are the bending moments per unit length, M_{xy} is the twisting moment per unit length, and (N_{xx}, N_{yy}, N_{xy}) are the applied inplane compressive and shear forces measure per unit length(see Fig. 2.2)

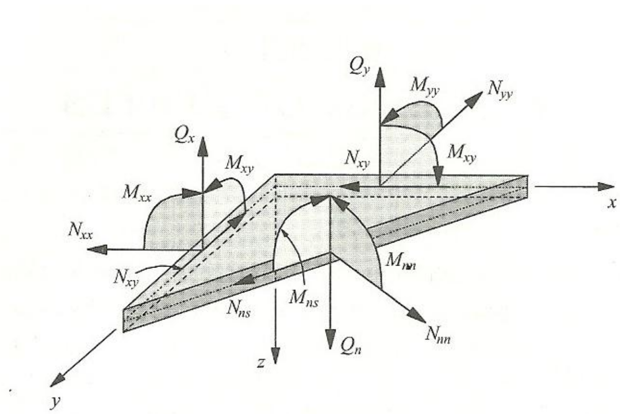

Figure 2.2 Applied inplane forces and moments in a plate element [4]

By using the transformation relations one can write the equation of equilibrium (2.25) governing the buckling of a circular plate as

$$\frac{1}{r}\left(\frac{\partial}{\partial r}(rQ_r) + \frac{\partial Q_\theta}{\partial \theta} - \frac{\partial}{\partial r}\left(rN_{rr}\frac{\partial w}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial \theta}\left(N_{\theta\theta}\frac{\partial w}{\partial \theta}\right)\right) = 0 \qquad (2.43)$$

where (Q_r,Q_θ) are the shear forces, (M_r, $M_\theta, M_{r\theta}$) are the bending moments, and (N_{rr}, $N_{\theta\theta}, N_{r\theta}$) are the inplane compressive forces(see Fig. 4.3).

$$Q_r = \frac{1}{r}(\frac{\partial}{\partial r}(rM_{rr}) + \frac{\partial}{\partial \theta}M_{r\theta} - M_{\theta\theta}) \qquad (2.44)$$

$$Q_\theta = \frac{1}{r}(\frac{\partial}{\partial r}(rM_{r\theta}) + \frac{\partial}{\partial \theta}M_{\theta\theta} - M_{r\theta}) \qquad (2.45)$$

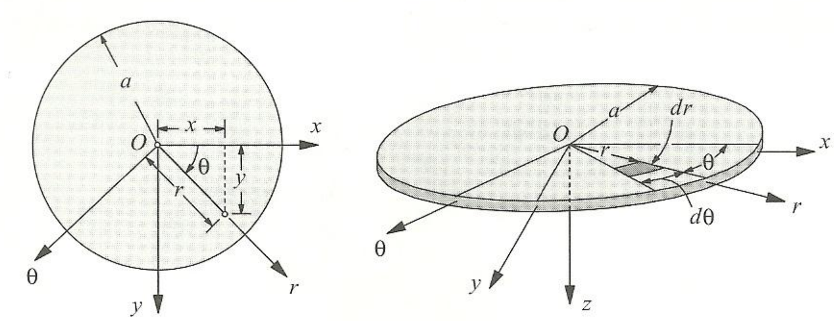

Figure 2.3 Transformation between rectangular and polar coordinate systems [4]

$$M_{rr} = -(D\frac{\partial^2 w}{\partial r^2} + \nu\ \frac{1}{r}(\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial^2 w}{\partial \theta^2})) \qquad (2.46)$$

$$M_{\theta\theta} = -(\nu\frac{\partial^2 w}{\partial r^2} + D\ \frac{1}{r}(\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial^2 w}{\partial \theta^2})) \qquad (2.47)$$

$$M_{r\theta} = -(1-\nu)D\frac{1}{r}(\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial w}{\partial \theta}) \qquad (2.48)$$

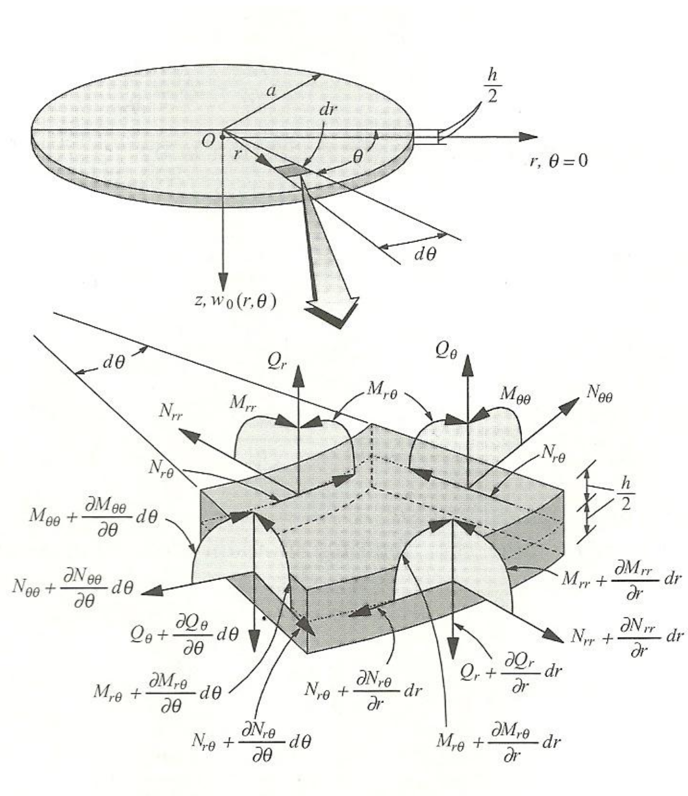

Figure 2.4 Moments and shear forces on an element of a circular plate [4]

The natural (force) boundary conditions can be written as

$$V_n^* = V_n - (N_{rr}\frac{\partial w}{\partial r}n_r + \frac{1}{r}N_{\theta\theta}\frac{\partial w}{\partial\theta}n_\theta + N_{r\theta}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}n_r + \frac{\partial w}{\partial r}n_\theta\right)) \qquad (2.49)$$

where

$$V_n = Q_n + \frac{\partial M_{ns}}{\partial s}, \quad V_r = Q_r + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta}, \quad V_\theta = Q_\theta + \frac{\partial M_{r\theta}}{\partial r} \qquad (2.50)$$

The boundary conditions for circular plates involve defining one quantity in each of the following pairs on positive r- and θ- planes:

At r = r^*, constant:

$$w = w^* \qquad \text{or} \qquad r^*V_r = r^*V_r^* \qquad (2.51)$$

$$\frac{\partial w}{\partial r} = \frac{\partial w^*}{\partial r} \qquad \text{or} \qquad r^*M_{rr} = r^*M_{rr}^* \qquad (2.52)$$

At $\theta = \theta^*$, constant:

$$w = w^* \qquad \text{or} \qquad V_\theta = V_\theta^* \tag{2.53}$$

$$\frac{1}{r}\frac{\partial w}{\partial \theta} = \frac{1}{r}\frac{\partial w^*}{\partial \theta} \qquad \text{or} \qquad M_{\theta\theta} = M_{\theta\theta}^* \tag{2.54}$$

where

$$V_r = Q_r + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta} - N_{rr}\frac{\partial w}{\partial r} - \frac{1}{r}N_{r\theta}\frac{\partial w}{\partial \theta} \tag{2.55}$$

$$V_\theta = Q_\theta + \frac{\partial M_{r\theta}}{\partial r} - N_{\theta\theta}\frac{\partial w}{\partial \theta} - N_{r\theta}\frac{\partial w}{\partial r} \tag{2.56}$$

The moments are related to the deflection w by

$$M_{rr} = -D(\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\left(\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}\right)) \tag{2.57}$$

$$M_{\theta\theta} = -D(\nu\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\left(\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}\right)) \tag{2.58}$$

$$M_{r\theta} = -(1-\nu)D\frac{1}{r}(\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial w}{\partial \theta}) \tag{2.59}$$

Now the equation of equilibrium for an isotropic plate can be written in terms of the displacement with the help of Eqs. (2.57), (2.58), (2.59) as

$$D\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = -\frac{1}{r}\frac{\partial}{\partial r}\left(rN_{rr}\frac{\partial w}{\partial r}\right) - \frac{1}{r^2}\frac{\partial}{\partial \theta}(N_{\theta\theta}\frac{\partial w}{\partial \theta}) \tag{2.60}$$

Using the Laplace operator,

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{2.61}$$

Equation (2.60) can be written simply as

$$D\nabla^2\nabla^2 w + \frac{1}{r}\frac{\partial}{\partial r}\left(rN_{rr}\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(N_{\theta\theta}\frac{\partial w}{\partial \theta}\right) = 0 \tag{2.62}$$

For axisymmetric case all variables are independent of the angular coordinate θ, and they are functions of the radial coordinate r only. Hence the moment-deflection relationships for this case becomes,

$$M_{rr} = -D(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}) \tag{2.63}$$

$$M_{\theta\theta} = -D(\nu\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}) \tag{2.64}$$

$$Q_r = -D\frac{d}{dr}(\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)) \tag{2.65}$$

and the equation of equilibrium simplifies to

$$\frac{D}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right)\right) + \frac{1}{r}\frac{d}{dr}\left(rN_{rr}\frac{dw}{dr}\right) = 0 \tag{2.66}$$

2.3 Boundary Conditions

A complete solution of the governing equation (2.34) is based upon the knowledge of the conditions of the plate at the boundaries in terms of the lateral deflection of the middle surface w(x,y). Hence, expressions for these conditions must be developed. We consider three types of boundary conditions: clamped, simply supported, and free.

2.3.1 Rectangular Coordinates

2.3.1.1 Clamped edge condition

A clamped edge is one which is geometrically fully restrained i.e., the deflection and the slope of the middle surface must vanish at the boundary. The boundary conditions on a clamped edge parallel to the y-axis at x=a are,

$$w\,\Big|_{x=a} = 0 \tag{2.67}$$

$$\frac{\partial w}{\partial x}\,\Big|_{x=a} = 0 \tag{2.68}$$

The boundary conditions on a clamped edge parallel to the x-axis at y=b are

$$w\,\Big|_{y=b} = 0 \tag{2.69}$$

$$\frac{\partial w}{\partial x}\,\Big|_{y=b} = 0 \tag{2.70}$$

2.3.1.2 Simply supported edge condition

A simply supported edge is the one in which the transverse deflection and normal bending moment are zero. The conditions on a simply supported edge parallel to the y-axis at x=a are

$$w\,\Big|_{x=a} = 0 \tag{2.71}$$

$$M_x\Big|_{x=a} = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\Big|_{x=a} = 0 \tag{2.72}$$

The boundary conditions on a simply supported edge parallel to the x-axis at y=b are

$$w\,\Big|_{y=b} = 0 \tag{2.73}$$

$$M_y\Big|_{y=b} = -D\left(\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\Big|_{y=b} = 0 \tag{2.74}$$

2.3.1.3 Free edge Condition

A free edge condition is defined as the one which is geometrically not restrained in any manner. Hence, we have

$$M_y\Big|_{y=b} = -D\left(\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\Big|_{y=b} = 0 \tag{2.75}$$

$$V_y\Big|_{y=b} = -D\left(\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right)\Big|_{y=b} = 0 \tag{2.76}$$

2.3.2 Polar Coordinates

2.3.2.1 Clamped edge condition

The boundary condition equations for a clamped edge condition in polar coordinates are

$$w\Big|_{r=a} = 0 \tag{2.77}$$

$$\frac{\partial w}{\partial r}\Big|_{r=a} = 0 \tag{2.78}$$

2.3.2.2 Simply Supported edge condition

The boundary condition equations for a simply supported edge condition in polar coordinates are

$$w\Big|_{r=a} = 0 \tag{2.79}$$

$$\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\left(\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}\right)\Big|_{r=a} = 0 \tag{2.80}$$

2.3.2.3 Free edge condition

The boundary condition equations for a free edge condition in polar coordinates are

$$\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\left(\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}\right)\Big|_{r=a} = 0 \tag{2.81}$$

$$\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r} + \frac{2-\nu}{r^2}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial \theta^2} - \frac{3-\nu}{r^3}\frac{\partial^2 w}{\partial \theta^2}\Big|_{r=a} = 0 \tag{2.82}$$

2.4 Galerkin Method

The Galerkin method is a member of the large class of methods known as the method of weighted residuals (MWR) and the concept of weighted residuals was introduced by Crandall.

Consider a differential equation that can be represented in the following form

$$L\,u = c \tag{2.87}$$

in a domain D(x,y), with boundary conditions is given as

$$S(u) = 0 \tag{2.88}$$

on ∂D the boundary of D. The Galerkin method assumes that u can be accurately represented by an approximate solution

$$\tilde{u} = \sum_{j=1}^{N} a_j \emptyset_j(x,y) \tag{2.89}$$

where the $\emptyset_j{}'s$ are known analytic functions, and $a_j{}'s$ are coefficients to be determined. By substituting Eq. (2.89) into Eq. (2.87), we get a nonzero residual, R, given by

$$R\big(a_1, a_{2,\dots\dots\dots} a_N\big) = L(\tilde{u}) - c = \sum_{j=1}^{N} a_j L(\emptyset_j) - c \qquad (2.90)$$

If we can define an inner product (f, g) between two functions $\mathrm{f}(\mathrm{x}, \mathrm{y})$ and $\mathrm{g}(\mathrm{x}, \mathrm{y})$ as

$$(f, g) = \iint_D f\ g\ dx\ dy \qquad (2.91)$$

then the unknown coefficients a_j can be determined by solving the following systems of equations,

$$(R, \emptyset_k) = 0 \quad k = 1,2, \dots \dots . N \qquad (2.92)$$

where $\emptyset_k{}'s$ are the same analytic functions. By solving the above equation we can find out a_j and by substituting a_j in Eq. (2.89) we can obtain $\tilde{u}(x, y)$.

The conditions that are required in implementing the Galerkin method are:

(a) The functions $\emptyset_k$ are chosen from the same set of trial functions $\emptyset_j$.

(b) The trial functions should exactly satisfy the homogeneous boundary conditions.

(c) The trial function must be linearly independent.

The accuracy of the Galerkin method depends upon the selection of trial functions and the order of polynomials.

CHAPTER 3

METHOD OF WEIGHTED RESIDUALS

3.1 Introduction

Prior to the development of the finite element method, there existed an approximation technique for solving the differential equations called the method of weighted residuals (MWR). The basic idea of MWR is to use a trial function with a number of unknown parameters to approximate the solution. Then a weighted average over the interior and boundary is set to zero. The basic idea is to approximate the solution with a polynomial involving a set of parameters. The polynomial is chosen in such a way that it satisfies both the differential equation and the associated boundary conditions.

The method of weighted residuals can be described in the following way. Let us assume a differential equation of the form

$$L\,u = c \tag{3.1}$$

where *L* is a linear operator, *u* is the unknown function and *c* is a given function.

An approximate solution to eq. (3.1) is derived by a linear combination of N base vectors in the linear space as

$$\bar{U} = \sum_{i=1}^{N} u_i \emptyset_i \tag{3.2}$$

where u_i is the unknown coefficient and $\emptyset_i$ is the base function in a linear function space.

The residual error, R, between the approximate solution and the exact solution is defined as

$$R = Lu - c \tag{3.3}$$

$$= L\sum_{i=1}^{N} u_i \emptyset_i - c \tag{3.4}$$

$$= \sum_{i=1}^{N} u_i L\emptyset_i(x) - c(x) \tag{3.5}$$

Also for a function space, R is a function of the position, i.e.

$$R(x) = \sum_{i=1}^{N} u_i L\emptyset_i(x) - c(x) \tag{3.6}$$

The Method of Weighted Residual can be broadly classified into the following methods

3.1.1 Collocation method

Choose u_i so that the residual error vanishes at N selected points, i.e.

$$R(X_i) = 0, \quad i = 1, \ldots \ldots \ldots N \tag{3.7}$$

Although this method gives the exact values at the selected points, there is no guarantee that the approximation behaves nicely between the selected points.

3.1.2 Least Square method

Choose u_i so that the magnitude of residual error becomes the minimum i.e.

$$\|R(x)\| \to min. \tag{3.8}$$

The least square method is by far the oldest of the methods of weighted residuals. This method has a natural appeal for steady problems, where it might be expected that a minimization of the square of the equation residual would imply a small value. This method is expected to give an overall well-behaved approximation.

3.1.3 Galerkin method

Choose u_i such that R is orthogonal to N base functions (e_i), i.e.

$$(R, e_i) = 0 \qquad i = 1, \ldots \ldots \ldots \ldots . N \tag{3.9}$$

The idea of the Galerkin's method is that if $e_i{}'s$ span the entire linear space, a vector that is perpendicular to all the base vectors must be a zero vector.

The Galerkin method has been used to solve many problems in structural mechanics, dynamics, fluid flow, hydrodynamic stability, magneto hydrodynamics, heat and mass transfer, acoustics, microwave theory, etc. Problems governed by ordinary differential equation, partial differential equations, and integral equations have been investigated via the Galerkin formulation. Steady, unsteady, and eigen value problems have proved to be equally amenable to the Galerkin treatment. Most importantly, any problem whose governing equations can be written down is a candidate for the Galerkin method.

3.2 Application of Galerkin method

Most applications of the Galerkin method prior to 1972 were the traditional Galerkin method. In applying the Galerkin method, the trial functions must be chosen from a complete set to ensure convergence and from the lowest members of the complete set to achieve high accuracy with few terms in the trial solution. This aspect of the method highlights the need to choose the trial functions to take advantage of prior knowledge of the expected solution.

Consider a rectangular plate as shown in the figure below with sides a and b.

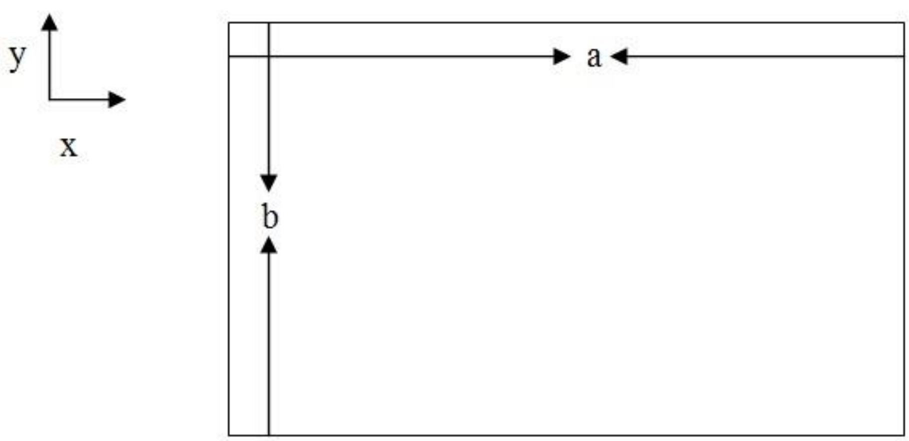

Figure 3.1 Rectangular plate

The governing differential equation for a plate can be written as:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p_z(x,y)}{D}\frac{\partial^2 w}{\partial x^2} \tag{3.10}$$

where p_z is the external load acting on the plate surface and D is the bending or flexural rigidity of the plate. For any given plate problem the exact solution of the governing equation (3.10) must simultaneously satisfy the differential equation and the boundary conditions. Since the governing equation (3.10) is a fourth-order differential equation, two boundary conditions either for the displacements or for the internal forces are required at each boundary.

The displacements components to be used in development of the boundary conditions are lateral deflections and slope. For instance, at clamped edges the deflection and the slope of the deflected plate surface are zero.

$$(\mathrm{w})_{\mathrm{x}} = 0 \qquad (\frac{\partial \mathrm{w}}{\partial \mathrm{x}})_{\mathrm{x}} = 0 \qquad (\mathrm{x} = 0 \text{ or } \mathrm{a}) \tag{3.11}$$

$$(\mathrm{w})_{\mathrm{y}} = 0 \qquad (\frac{\partial \mathrm{w}}{\partial \mathrm{x}})_{\mathrm{y}} = 0 \qquad (\mathrm{y} = 0 \text{ or } \mathrm{a}) \tag{3.12}$$

Using the two-dimensional Laplace operator ($\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$) equation (3.10) can be represented as

$$\nabla^2 \nabla^2 w = \frac{p_z}{D} \frac{\partial^2 w}{\partial x^2} \tag{3.13}$$

Now, consider a differential operator L,

$$L = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \tag{3.14}$$

Now Eq.(3.10) can be rewritten as

$$Lw = \frac{p_z}{D} \tag{3.15}$$

First step in finding the eigenvalues is to select the base functions $\emptyset_i(x, y)$ properly such that it satisfies the boundary conditions and the above equation can be expressed in the following way.

$$\sum_i^N c_i \, L\emptyset_i(x, y) = \sum_i^N \lambda \, c_i \, L\emptyset_i(x, y) \tag{3.16}$$

where λ is the eigenvalue to be determined.

Now multiplying the above eqn. with another base function $\emptyset_j(x, y)$ that also satisfies the boundary condition, we get

$$\sum_{i}^{N}\sum_{j}^{N} c_i\, L\emptyset_i(x,y)\emptyset_j(x,y) = \sum_{i}^{N}\sum_{j}^{N} \lambda\, c_i\, L\emptyset_i(x,y)\emptyset_j(x,y) \tag{3.17}$$

Now the above equation can be written as,

$$A\,\hat{c} = \lambda\, B\,\hat{c} \tag{3.18}$$

where,

$$\begin{aligned} a_{ij} &= \int_0^a \int_0^b \Delta\Delta\emptyset_i\,(x,y)\emptyset_j(x,y)dx\,dy \\ b_{ij} &= \int_0^a \int_0^b \Delta\emptyset_i\,(x,y)\emptyset_j(x,y)dx\,dy \end{aligned} \tag{3.19}$$

The quantities are A and B are N x N square matrices as shown below:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{NN} & \cdots & a_{NN} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1N} \\ \vdots & \ddots & \vdots \\ b_{NN} & \cdots & b_{NN} \end{bmatrix}$$

where λ is the eigenvalue and $\hat{c}$ is the corresponding eigenvectors.

By solving as an eigenvalue problem we can determine the eigenvalues and eigenvectors for the given system.

The definition of the eigenvalue method is the following. If A is any square matrix and

$$A\,\hat{c} = \lambda\, B\,\hat{c} \tag{3.20}$$

where λ is the eigenvalue of A and $\hat{c}$ is the corresponding eigenvector.

For our problem Eq. (3.20) is in the form

$$(A - \lambda\, B)v = 0 \tag{3.21}$$

We can calculate the eigenvalues λ and corresponding eigenvectors v with the aid of symbolic algebra software *Mathematica*. Eigenvalue method is a straight-forward and fast method to solve linear system equations.

For our perfect plate problem we will be considering only two types of boundary conditions – simply supported and free edge condition.

3.2.1 Simply Supported Boundary Condition

We need to begin with the governing differential equation of the plate before we study each of the boundary conditions stated before. First, we consider the simply supported boundary condition for a square plate subjected to lateral loads.

The governing differential equation of the plate is,

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p_z(x,y)}{D}\frac{\partial^2 w}{\partial x^2} \tag{3.22}$$

where,

p_z is the lateral load being applied

D is the bending or flexural rigidity of the plate

A simply supported edge is the one in which the transverse deflection and normal bending moment are zero. The conditions on a simply supported edge parallel to the y-axis at x=a are

$$w\,|_{x=a} = 0 \tag{3.23}$$

$$M_x|_{x=a} = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)|_{x=a} = 0 \tag{3.24}$$

Since the change of w with respect to the y coordinate vanishes along the edge, the above conditions become

$$w\,|_{x=a} = 0 \tag{3.25}$$

$$M_x|_{x=a} = \frac{\partial^2 w}{\partial x^2}|_{x=a} = 0 \tag{3.26}$$

Similarly on a simply supported edge parallel to the x-axis at y=a, the change of w with respect to the x-coordinate vanishes, thus the conditions become,

$$w\,|_{y=a} = 0 \tag{3.27}$$

$$M_y|_{y=a} = \frac{\partial^2 w}{\partial y^2}|_{y=a} = 0 \tag{3.28}$$

Now we assume a solution of the form

$$w(x,y) = \sum_i c_i e_i(x,y) \tag{3.29}$$

Substituting Eq. (3.29) into Eq. (3.22) and then we integrate it over the entire plate to produce the following eigenvalue problem

$$[A]v = \lambda[B]v \tag{3.30}$$

Now we introduce the notation

$$\overline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \tag{3.31}$$

also, $$\Delta = \overline{\nabla}.\overline{\nabla} \tag{3.32}$$

By using the Galerkin method, the elements of [A] and [B] matrices can be represented as,

$$a_{ij} = \int_0^b \int_0^a \Delta\Delta\emptyset_i\emptyset_j \, dx \, dy \tag{3.33}$$

$$b_{ij} = \int_0^b \int_0^a \Delta\emptyset_i\emptyset_j \, dx \, dy \tag{3.34}$$

The above expressions can be solved by integration by parts in the following way,

Distributing the operator over both $\emptyset_i$ and $\emptyset_j$

$$a_{ij} = \int_0^a \int_0^b \Delta\Delta\emptyset_i\emptyset_j \, dx \, dy \tag{3.35}$$

$$= \int_0^a \int_0^b (\overline{\nabla}.\overline{\nabla})\overline{\nabla}.\overline{\nabla}\emptyset_i\emptyset_j \, dx \, dy \tag{3.36}$$

And by applying the surface integral over the entire surface of the plate the homogeneous boundary conditions a_{ij} becomes,

$$a_{ij} = -\int_0^b \int_0^a \overline{\nabla}(\overline{\nabla}.\overline{\nabla})\emptyset_i\overline{\nabla}\emptyset_j \, dx \, dy \tag{3.37}$$

By using another homogeneous boundary condition yields the expression,

$$a_{ij} = \int_0^b \int_0^a (\overline{\nabla}.\overline{\nabla})\emptyset_i(\overline{\nabla}.\overline{\nabla})\emptyset_j \, dx \, dy \tag{3.38}$$

By substituting eq. (3.31) into eq. (3.37),the final expression for elements a_{ij} becomes,

$$a_{ij} = \int_0^b \int_0^a \left[\frac{\partial^2\emptyset_i}{\partial x^2} + \frac{\partial^2\emptyset_i}{\partial y^2}\right]\left[\frac{\partial^2\emptyset_j}{\partial x^2} + \frac{\partial^2\emptyset_j}{\partial y^2}\right] dx \, dy \tag{3.39}$$

Similarly by deriving the expression for b_{ij} in the same way by starting with equation (3.39) and by using integration by parts, we get

$$b_{ij} = -\int_0^b \int_0^a \Delta\emptyset_i\emptyset_j \, dx \, dy \tag{3.40}$$

The first important step in solving this problem is to choose a base function ∅ that satisfies the boundary condition. A polynomial approximating function will be used to represent the lateral displacement of the plate. The trial function $\emptyset_i(x,y)$will be represented as,

$$\emptyset_i(x,y) = \sum_{j=1}^{N} c_j u_j(x,y) \tag{3.41}$$

where,

$$u_j(x,y) = x^g y^h \tag{3.42}$$

g and h are positive integers and c_j are coefficients to be determined.

For the simply supported boundary condition problem, it is found that an eight order polynomial is the lowest order possible to satisfy the boundary conditions.

$$\begin{aligned}\emptyset_i &= c[1] + c[2]x + c[3]y + c[4]xy \\ &+c[5]x^2 + \dots\dots\dots\dots\dots\dots..c[44]xy^7 + c[45]y^8\end{aligned} \tag{3.43}$$

where $\mathrm{c}[1]$ through $\mathrm{c}[45]$ are unknown coefficients of $\emptyset_i$.

Once the eighth order polynomial is defined as mentioned in eq. (3.43), the next step is to apply the boundary conditions and solve for the unknown coefficients. This is done by developing eight new equations using *Mathematica*.

Each equation represents a different boundary condition.

For Deflection

$$\text{BC-1: } \emptyset(0, y) = 0 \tag{3.44}$$

$$\text{BC-2: } \emptyset(a, y) = 0 \tag{3.45}$$

$$\text{BC-3: } \emptyset(x, 0) = 0 \tag{3.46}$$

$$\text{BC-4: } \emptyset(a, y) = 0 \tag{3.47}$$

For Moments

$$\text{BC-5: } \left(\frac{\partial^2\emptyset}{\partial x^2} + \nu\frac{\partial^2\emptyset}{\partial y^2}\right)|_{x=0} = 0 \tag{3.48}$$

$$\text{BC-6: } \left(\frac{\partial^2\emptyset}{\partial x^2} + \nu\frac{\partial^2\emptyset}{\partial y^2}\right)|_{x=a} = 0 \tag{3.49}$$

$$\text{BC-7: } \left(\nu\frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2}\right)|_{y=0} = 0 \tag{3.50}$$

$$\text{BC-8: } \left(\nu\frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2}\right)|_{y=b} = 0 \tag{3.51}$$

Once these boundary conditions are applied we get an undetermined system of equations with forty-five unknowns. The next step is to flatten out all the eight equations by using the Flatten[Table] command in *Mathematica*. By tabulating all the eight equations and solving the equations we get values for the coefficients c_j.

By substituting these values of the coefficients back into the eq. (3.43) yields one independent equation.

$$\emptyset_i(x, y) = (-1 + x)x(-1 - x + x^2)(-1 + y)y(-1 - y + y^2) \tag{3.52}$$

Using *Mathematica* the elements of matrices of [A] and [B] are found by performing integration over the polynomial. The eigenvalue and corresponding eigenvectors are determined by using the Eigensystem[] command in *Mathematica*.

In order to get a good convergence on the critical buckling load value λ, a higher order approximating polynomial must be considered. A ninth order polynomial considered is of the following form:

$$\begin{aligned}\emptyset_i = c[1] + c[2]x + c[3]y + c[4]x^2 + c[5]xy + c[6]y^2 + \ \ldots\ldots\ldots \\ +c[42]x^3y^5 + c[43]x^2y^6 + c[44]xy^7 + c[45]y^8 + \ \ldots\ldots\ldots \\ +c[49]x^6y^3 + c[50]x^5y^4 + c[51]x^4y^5 + c[52]x^3y^6 + c[53]x^2y^7 \\ + c[54]xy^8 + c[55]y^9\end{aligned} \tag{3.53}$$

Now by applying the boundary conditions on the ninth order polynomial equation and then solving for all the coefficients of the ninth order using *Mathematica*, we get a system of three independent trial functions $\emptyset_1$, $\emptyset_2$, and $\emptyset_3$ as shown below.

$$\begin{aligned}\emptyset_1 &= (-1+x)x(-1-x+x^2)(-1+y)y(-1-y+y^2) \\ \emptyset_2 &= \frac{1}{3}(-1+x)x(1+x)(-7+3x^2)(-1+y)y(-1-y+y^2) \\ \emptyset_3 &= \frac{1}{3}(-1+x)x(-1-x+x^2)(-1+y)y(1+y)(-7+3y^2)\end{aligned} \tag{3.54}$$

The first order eigenvalue system for the eighth order approximating polynomial was solved by simply multiplying the single element matrix [A] by the single element inverse matrix [B] but, this system of third order is solved using *Mathematica*. For the third order system the elements of the matrices [A] and [B] is found by integrating over the polynomial using the definite integral function in *Mathematica* and then using the eigensystem function we can evaluate the eigenvalues and corresponding eigenvectors.

One of the important properties of the Galerkin method is the convergence of the critical buckling load λ from an overestimated value towards the exact value. This property occurs when the order of the approximating function is increased. We can observe that in the current example $\emptyset_1$ is the same for both eighth and ninth order polynomials. If not for this, the Galerkin system would not be satisfied and the calculated value λ would be incorrect.

3.2.2 Plate with a hole

Here we consider a simply supported plate with a hole having the free edge boundary condition. The boundary conditions for the plate with simply supported edge parallel to the y-axis at x=a are

$$w\,|_{x=a} = 0 \tag{3.55}$$

$$M_x|_{x=a} = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)|_{x=a} = 0 \tag{3.56}$$

The boundary conditions for the hole with free edge are

$$\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}(\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial^2 w}{\partial \theta^2})|_{r=a} = 0 \tag{3.57}$$

$$\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r} + \frac{2-\nu}{r^2}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial \theta^2} - \frac{3-\nu}{r^3}\frac{\partial^2 w}{\partial \theta^2}|_{r=a} = 0 \tag{3.58}$$

We need to begin with the governing differential equation of a plate Eq. (3.22) in order to find the critical buckling load of a plate. Polynomial approximating functions will be used to represent the lateral displacement of the plate. The first step to solve this problem is to systematically choose a trial function that satisfies the plate's boundary conditions. The trial function $\emptyset_i(x, y)$will be represented as,

$$\emptyset_i(x, y) = \sum_{j=1}^{N} c_j u_j(x, y) \tag{3.59}$$

where,

$$u_j(x, y) = x^g y^h \tag{3.60}$$

g and h are positive integers and c_j are coefficients to be determined.

For this problem it is found that a ninth order polynomial is the lowest order possible to satisfy the boundary conditions.

$$\emptyset_i = c[1] + c[2]x + c[3]y + c[4]x^2 + c[5]xy + c[6]y^2 + \ \dots\dots\dots$$
$$+c[42]x^3y^5 + c[43]x^2y^6 + c[44]xy^7 + c[45]y^8 + \ \dots\dots\dots$$
$$+c[49]x^6y^3 + c[50]x^5y^4 + c[51]x^4y^5 + c[52]x^3y^6 + c[53]x^2y^7 \tag{3.61}$$
$$+ c[54]xy^8 + c[55]y^9$$

Once we define the polynomial approximating function the next step is to apply the boundary conditions and solve for unknown coefficients. This is carried out by generating ten new equations using *Mathematica*. Each equation represents a different boundary condition.

$$\text{BC-1: } \emptyset(0,y) = 0 \tag{3.62}$$

$$\text{BC-2: } \emptyset(a,y) = 0 \tag{3.63}$$

$$\text{BC-3: } \emptyset(x,0) = 0 \tag{3.64}$$

$$\text{BC-4: } \emptyset(a,y) = 0 \tag{3.65}$$

$$\text{BC-5: } \left(\frac{\partial^2\emptyset}{\partial x^2} + \nu\frac{\partial^2\emptyset}{\partial y^2}\right)|_{x=0} = 0 \tag{3.66}$$

$$\text{BC-6: } \left(\frac{\partial^2\emptyset}{\partial x^2} + \nu\frac{\partial^2\emptyset}{\partial y^2}\right)|_{x=a} = 0 \tag{3.67}$$

$$\text{BC-7: } \left(\nu\frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2}\right)|_{y=0} = 0 \tag{3.68}$$

$$\text{BC-8: } \left(\nu\frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2}\right)|_{y=b} = 0 \tag{3.69}$$

$$\text{BC-9: } \frac{\partial^2\emptyset}{\partial r^2} + \frac{\nu}{r}\left(\frac{\partial\emptyset}{\partial r} + \frac{1}{r}\frac{\partial^2\emptyset}{\partial\theta^2}\right)|_{r=a} = 0 \tag{3.70}$$

$$\text{BC-10: } \frac{\partial^3\emptyset}{\partial r^3} + \frac{1}{r}\frac{\partial^2\emptyset}{\partial r^2} - \frac{1}{r^2}\frac{\partial\emptyset}{\partial r} + \frac{2-\nu}{r^2}\frac{\partial\emptyset}{\partial r}\frac{\partial^2\emptyset}{\partial\theta^2} - \frac{3-\nu}{r^3}\frac{\partial^2\emptyset}{\partial\theta^2}|_{r=a} = 0 \tag{3.71}$$

By solving all these ten boundary condition equations and tabulating them using the Table[] command in *Mathematica*, we get values of coefficients c_{ij}. Substituting these values back into eq. (3.61) we get one independent equation.

$$\emptyset_1 = \frac{1}{3}(x^2-4)x(-28+3x^2)(y^2-4)(-20+y^2) \tag{3.72}$$

Then we determine the elements of matrices [A] and [B] and by using the Eigensystem command we can find the eigenvalue and corresponding eigenvectors using *Mathematica*.

In order to get a good convergence on the critical buckling load value λ, a higher order approximating polynomial must be considered. A tenth order polynomial considered is of the following form:

$$\begin{gathered}\emptyset_i = c[1] + c[2]x + c[3]y + c[4]x^2 + c[5]xy + c[6]y^2 + \ \ldots\ldots\ldots \\ +c[42]x^3y^5 + c[43]x^2y^6 + c[44]xy^7 + c[45]y^8 + \ \ldots\ldots\ldots \\ +c[49]x^6y^3 + c[50]x^5y^4 + c[51]x^4y^5 + \ \ldots\ldots.. \\ +c[61]x^5y^5 + c[62]x^4y^6C[62] + c[63]x^3y^7 \\ +c[64]x^2y^8 + c[65]xy^9 + c[66]y^{10}\end{gathered} \tag{3.73}$$

Now, by applying the boundary conditions to Eq. (3.73) and solving for all the coefficients using *Mathematica* and by substituting the coefficient values back in to Eq. (3.73) we get a system of three independent trial function equations $\emptyset_1$ and $\emptyset_2$.

$$\emptyset_1 = \frac{1}{3}(x^2 - 4)x(-28 + 3x^2)(y^2 - 4)(-20 + y^2) \tag{3.74}$$

$$\emptyset_2 = -(x^2 - 4)(y^2 - 4)(x^2 - y^2)(144 - 20x^2 - 20y^2 + x^2y^2) \tag{3.75}$$

The next step is to calculate the elements of matrices [A] and [B] by using the definite integral function in *Mathematica* and then applying Eigensystem command we can find the eigenvalues and eigensystem for this system.

CHAPTER 4

NUMERICAL RESULTS

In this Chapter, the numerical results achieved for the problems discussed in Chapter 3 are presented. All of the necessary computations was carried out with the help of a symbolic algebra software, *Mathematica* [14].

4.1 Simply Supported Plate

We calculate the buckling load of a square plate for the simply supported boundary condition. A simply supported edge is the one in which the transverse direction and normal bending moment are zero.

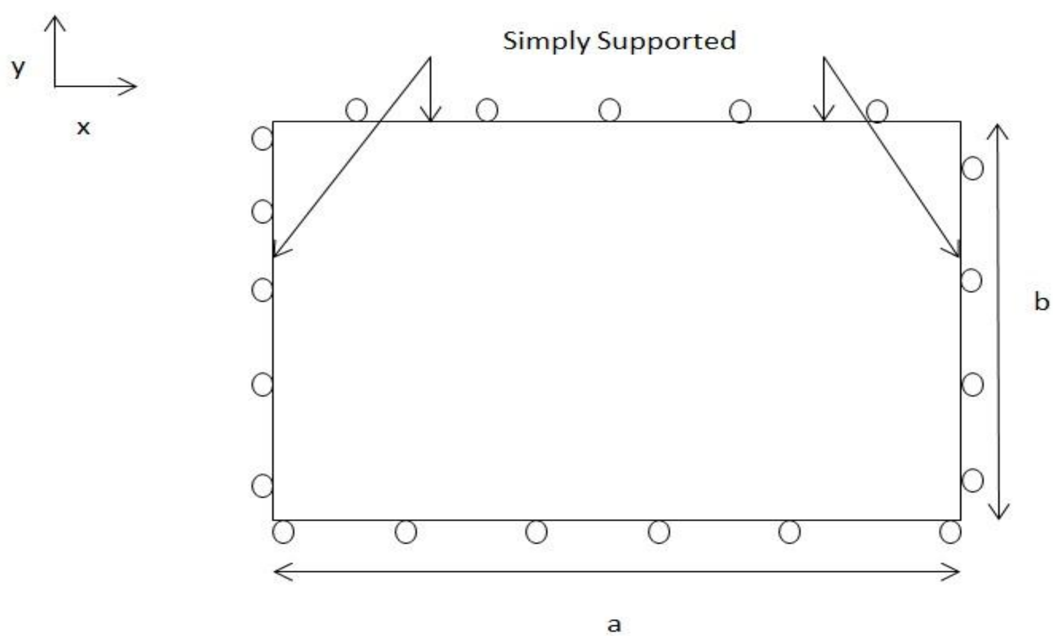

Figure 4.1 Simply Supported Plate

A simply supported edge is the one in which the transverse deflection and normal bending moment are zero. The conditions on a simply supported edge parallel to y-axis at x=a are

$$w\,|_{x=a} = 0 \tag{4.1}$$

$$M_x|_{x=a} = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)|_{x=a} = 0 \tag{4.2}$$

Similarly on a simply supported edge parallel to the x-axis at *y=a*, the change of w with respect to the x-coordinate vanishes, thus the conditions become,

$$w\,|_{y=a} = 0 \tag{4.3}$$

$$M_y|_{y=a} = \frac{\partial^2 w}{\partial y^2}|_{y=a} = 0 \tag{4.4}$$

The first important step in solving this problem is to choose a trial function $\emptyset$ that satisfies the boundary condition. A polynomial approximating function will be used to represent the lateral displacement of the plate. The trial function $\emptyset_i(x,y)$will be represented as,

$$\emptyset_i(x,y) = \sum_{j=1}^{N} c_j u_j(x,y) \tag{4.5}$$

where,

$$u_j(x,y) = x^g y^h \tag{4.6}$$

g and h are positive integers and c_j are coefficients to be determined.

For the simply supported boundary condition problem, it is found that an eighth order polynomial is the lowest order possible to satisfy the boundary conditions.

$$\begin{aligned}\emptyset_i &= c[1] + c[2]x + c[3]y + c[4]xy \\ &+c[5]x^2 + \ldots\ldots\ldots\ldots\ldots\ldots c[44]xy^7 + c[45]y^8\end{aligned} \tag{4.7}$$

where $\mathrm{c}[1]$ through $\mathrm{c}[45]$ are unknown coefficients of $\emptyset_i$.

Table 4.1 Eigenvalues for simply supported plate

Order	Eigenvalues
8	19.7533
9	19.7533, 49.79
10	19.7392, 49.79, 79.6
11	19.7392, 49.3498, 79.6
12	19.7392, 49.3498, 78.96

Once we define the polynomial approximating function as in Eq. (4.7), the next step is to apply the boundary conditions and solve for unknown coefficients The eigenvalues were obtained by forming the elements of matrices of [A] and [B] and then by using the Eigensystem command in *Mathematica*.

4.2 Plate with a hole

This example is the most important of all. Here we consider a simply supported plate with a hole having a free edge boundary condition as shown in Figure 4.2

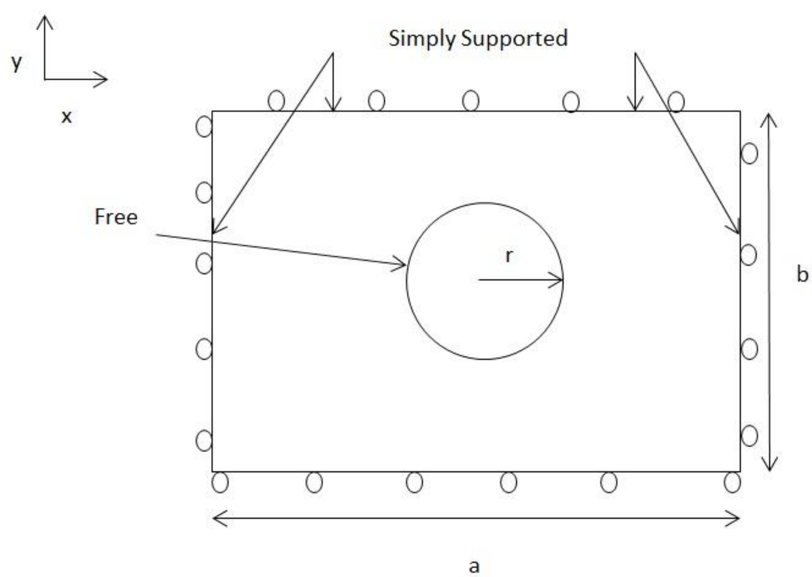

Figure 4.2 Plate with a hole

The boundary conditions for the plate with simply supported edge parallel to the y-axis at x=a are

$$w\,|_{x=a} = 0 \tag{4.8}$$

$$M_x|_{x=a} = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)|_{x=a} = 0 \tag{4.9}$$

The boundary conditions for the hole with free edge are

$$\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\left(\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}\right)|_{r=a} = 0 \tag{4.10}$$

$$\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r} + \frac{2-\ \nu}{r^2}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial \theta^2} - \frac{3-\ \nu}{r^3}\frac{\partial^2 w}{\partial \theta^2}|_{r=a} = 0 \tag{4.11}$$

The first step to solve this problem is to systematically choose a trial function that satisfies the plate's boundary conditions. The trial function $\emptyset_i(x,y)$will be represented as,

$$\emptyset_i(x,y) = \sum_{j=1}^{N} c_j u_j(x,y) \tag{4.12}$$

where,

$$u_j(x,y) = x^g y^h \tag{4.13}$$

g and h are positive integers and c_j are coefficients to be determined.

For this problem it is found that a ninth order polynomial is the lowest order possible to satisfy the boundary conditions and for instance we assume an eleventh order polynomial

$$\begin{aligned}\emptyset_i = c[1] + c[2]x + c[3]y + c[4]x^2 + c[5]xy + c[6]y^2 + \ \ldots\ldots\ldots \\ +c[42]x^3y^5 + c[43]x^2y^6 + c[44]xy^7 + c[45]y^8 + \ \ldots\ldots\ldots \\ +c[49]x^6y^3 + c[50]x^5y^4 + c[51]x^4y^5 + c[52]x^3y^6 + \ \ldots\ldots. \\ +c[63]x^3y^7 + c[64]x^2y^8 + c[65]xy^9 + c[66]y^{10} + \ \ldots\ldots \\ +c[74]x^4y^7 + c[75]x^3y^8 + c[76]x^2y^9 + c[77]xy^{10} + c[78]y^{11}\end{aligned} \tag{4.14}$$

Once the approximating function is defined as in Eq. (4.14) the next step is to apply the boundary conditions and solve for the unknown coefficients. This is done by generating ten new equations using *Mathematica.* Each equation represents a different boundary condition.

$$\text{BC-1: } \emptyset(0, y) = 0 \tag{4.15}$$

$$\text{BC-2: } \emptyset(a, y) = 0 \tag{4.16}$$

$$\text{BC-3: } \emptyset(x, 0) = 0 \tag{4.17}$$

$$\text{BC-4: } \emptyset(a, y) = 0 \tag{4.18}$$

$$\text{BC-5: } \left(\frac{\partial^2\emptyset}{\partial x^2} + \nu\frac{\partial^2\emptyset}{\partial y^2}\right)|_{x=0} = 0 \tag{4.19}$$

$$\text{BC-6: } \left(\frac{\partial^2\emptyset}{\partial x^2} + \nu\frac{\partial^2\emptyset}{\partial y^2}\right)|_{x=a} = 0 \tag{4.20}$$

$$\text{BC-7: } \left(\nu\frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2}\right)|_{y=0} = 0 \tag{4.21}$$

$$\text{BC-8: } \left(\nu\frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2}\right)|_{y=b} = 0 \tag{4.22}$$

$$\text{BC-9: } \frac{\partial^2\emptyset}{\partial r^2} + \frac{\nu}{r}\left(\frac{\partial\emptyset}{\partial r} + \frac{1}{r}\frac{\partial^2\emptyset}{\partial\theta^2}\right)|_{r=a} = 0 \tag{4.23}$$

$$\text{BC-10: } \frac{\partial^3\emptyset}{\partial r^3} + \frac{1}{r}\frac{\partial^2\emptyset}{\partial r^2} - \frac{1}{r^2}\frac{\partial\emptyset}{\partial r} + \frac{2-\nu}{r^2}\frac{\partial w}{\partial r}\frac{\partial^2\emptyset}{\partial\theta^2} - \frac{3-\nu}{r^3}\frac{\partial^2\emptyset}{\partial\theta^2}|_{r=a} = 0 \tag{4.24}$$

By solving all these ten boundary condition equations and tabulating them using the Table command in Mathematica, we get the values of coefficients for a eleventh order approximating polynomial as

$$c[2] \to \frac{8960}{3}c[50] + 30720\, c[71] + \frac{100352}{3}c[73],$$

$$c[4] \to -2304\, c[62], \qquad c[6] \to 2304\, c[62],$$

$$c[7] \to -\frac{3200}{3}c[50] - 8960c[71] - \frac{35840}{3}\, c[73]$$

$$c[9] \to -896c[50] - 9216c[71] - 8960\, c[73], \qquad c[11] \to 896\, c_{62},$$

$$c[15] \to -896\, c[62], \qquad c[16] \to 80\, c[50] + 896c[73]$$

$$c[18] \to 320c[50] + 2688c[71] + 3200\, c[73],$$

$$c[20] \to \frac{112}{3}c[50] + 384\, c[71]$$

$$c[22] \to -80c[62], c[24] \to -240c[62], c[26] \to 240c[62]$$

$$c[28] \to 80c[62], c[29] \to 80c[71], c[31] \to -24c[50] - 240c[73]$$

$$c[33] \to -\frac{40}{3}c[50] - 112c[71], c[35] \to \frac{112}{3}c[73], c[39] \to 24c[62]$$

$$c[43] \to -24c[62], c[48] \to -24c[71], c[52] \to -\frac{40}{3}c[73],$$

$$c[60] \to -c[62], c[1] \to 0, c[3] \to 0, c[5] \to 0, c[8] \to 0,$$

$$c[10] \to 0, c[12] \to 0, c[13] \to 0, c[14] \to 0, c[17] \to 0, c[19] \to 0,$$

$$c[21] \to 0, c[23] \to 0, c[25] \to 0, c[27] \to 0, c[30] \to 0, c[32] \to 0,$$

$$c[34] \to 0, c[36] \to 0, c[37] \to 0, c[38] \to 0, c[40] \to 0, c[41] \to 0,$$

$$c[42] \to 0, c[44] \to 0, c[45] \to 0, c[46] \to 0, c[47] \to 0, c[49] \to 0,$$

$$c[51] \to 0, c[55] \to 0, c[56] \to 0, c[57] \to 0, c[58] \to 0, c[59] \to 0,$$

$$c[61] \to 0, c[63] \to 0, c[64] \to 0, c[65] \to 0, c[66] \to 0, c[67] \to 0,$$

$$c[68] \to 0, c[69] \to 0, c[70] \to 0, c[72] \to 0, c[74] \to 0, c[75] \to 0,$$

$$c[76] \to 0, c[77] \to 0, c[78] \to 0$$

By substituting all these values back into Eq. (4.21) we get four independent equations

$$\begin{aligned}
\emptyset_1 &= \frac{1}{3}(x^2-4)x(-28+3x^2)(y^2-4)(-20+y^2) \\
\emptyset_2 &= -(x^2-4)(y^2-4)(x^2-y^2)(144-20x^2-20y^2+x^2y^2) \\
\emptyset_3 &= x(x^2-4)(-8+x^2)(12+x^2)(y^2-4)(-20+y^2) \\
\emptyset_4 &= \frac{1}{3}(x^2-4)x(-28+3x^2)(y^2-4)(-224+4y^2+y^4)
\end{aligned} \tag{4.25}$$

Then by applying the Galerkin method and integrating over the polynomial using Mathematica, we can evaluate the elements of matrices [A] and [B]. Then by using the Eigensystem command in *Mathematica* we can find out the eigenvalues. The eigenvalues determined for various order of polynomials is tabulated in Table 4.2.

Table 4.2 Eigenvalues for a plate with a hole

Order	Eigenvalues
9	3.11209
10	3.11209, 6.38816
11	3.08436, 6.38816, 11.3495
12	3.08436, 6.17167, 11.3495

One of the important properties of the Galerkin method is the convergence of the critical buckling load λ from an overestimated value towards the exact value. This property occurs when the order of the approximating function is increased. It was observed that for all the examples mentioned above $\emptyset_1$ is the same for any order of approximating polynomial considered. If not for this property, the Galerkin system would not be satisfied and the calculated value λ would be incorrect.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

The Galerkin method was used to solve the governing differential equation of the plate with or without a hole for different boundary conditions. The Galerkin method and the procedure used for solving for the coefficients of the approximating polynomials would be impossible if not for the use of symbolic software. *Mathematica*, a symbolic software was extensively used to achieve the desired results.

One of the biggest advantages of using this system is its ability to deal with both symbolic characters and numbers. It is this feature which makes it possible to solve both algebra and calculus. The approximating polynomial is chosen in such a way that it satisfies all the boundary conditions involved and hence the result obtained is very accurate and greater convergence can be achieved.

Recommended future work on this thesis topic would be

- Considering triangular plates with or without a hole.
- Considering plates with more than one hole.
- Considering plates with eccentric holes.
- Considering the hole to be of elliptical geometry.

REFERENCES

[1] EULER, L; “De motu vibratorio tympanorum”, Novi Commentari Acad. Petropolit. 10 , 243-260, 1766

[2] KIRCHOFF, G; Vorlesungen uber mathematische Physik, Vol 1, B.G Teubner, Leipzig, 1876

[3] Turner, MJ; Clough, R.W; Martin, G.C; Topp, L.J; Stiffness and deflection analysis of Complex Structures, J. Aero. Sci., 23, pp. 805–824, 1956.

[4] C.M. Wang, C.Y. Wang, and J.N. Reddy (2004). Exact Solutions for Buckling of Structural Members (1st Ed.). CRC Press

[5] S. Timoshenko and S. Woinowsky-Krieger (1964). Theory of Plates and Shells (2nd Ed.). McGraw-Hill Publishing Company.

[6] Timoshenko S.P. and Goodier J.N.(1970).Theory of Elasticity (3rd Ed.) N.Y.: McGraw-Hill.

[7] Jawad, Maan H. (1994). Theory and Design of Plate and Shell Structures (1st Ed.) Springer.

[8] Fletcher, C.A.J. (1984). Computational Galerkin Methods (1st Ed.) Springer-Verlag Telos.

[9] Trahair, N.S. (1993). Flexural - Torsional Buckling of Structures (1st Ed.) CRC Press

[10] Szilard, Rudolph (1974). Theory and Analysis of Plates: Classical and Numerical Methods. Prentice Hall

[11] Bazant, Zdenek P. (2010). Stability of Structures: Elastic, Inelastic, Fracture and Damage Theories. World Scientific Publishing Company

[12] David McFarland, Bert L. Smith and Walter D. Bernhart (1972). Analysis of Plates. (1st Ed.) Spartan Books.

[13] Farshad, Mehdi (1994). Stability of Structures. Elsevier Science Ltd.

[14] Wolfram, Stephen (1988). Mathematica: A System for Doing Mathematics by Computer. Redwood City, CA.: Addison-Wesley.

[15] Wolfram Research. Mathematica: The Smart Approach to Engineering Education, published in 2001 located at http://www.wolfram.com/solutions/

BIOGRAPHICAL INFORMATION

Srider Thirupachoor Comerica was born in Chennai, Tamil Nadu, India, in 1986. He received his Bachelor's degree in Mechanical Engineering from Anna University, Chennai, Tamil Nadu, India, in 2008. He joined The University of Texas at Arlington to pursue his Master's in the field of Mechanical Engineering in 2009. His research interests include Solid Mechanics, Composites, and Mechanical Design.